THE TENSOR PRODUCT OF TWO ABELIAN GROUPS

by

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ABSTRACT

The concept of a free group is discussed first in Chapter 1 and in Chapter 2 the tensor product of two groups for which we write $A \otimes B$ is defined by "factoring out" an appropriate subgroup of the free group on the Cartesian product of the two groups. The existence of a unique homomorphism $h : A \otimes B \rightarrow H$ is assured by the existence of a bilinear map $f : A \times B \rightarrow H$, where H is any group (Lemma 2 - 2) and this property of the tensor product is used extensively throughout the thesis. In Chapter 3 the complete characterization is given for the tensor product of two arbitrary finitely generated groups. In the last chapter we discuss the structure of $A \otimes B$ for arbitrary groups. Essentially, the only complete characterizations are for those cases where one of the two groups is torsion. Many theorems from the theory of Abelian Groups are assumed but some considered interesting are proved herein.

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TABLE OF CONTENTS

NOTATION

The following notation will be used throughout this thesis. If $f : S \to T$ is a function and $s \in S$ (function and map are used interchangeably) then for the image of s under f in T we shall write sf ; for the image of S under f we write Sf. If $g : T \to W$ is another function, then the composition of the functions f and g will be written $fg : S \to W$. The symbols of set inclusion, $\subseteq$ and $\supseteq$ and intersection and union, $\cap$ and $\cup$, are standard, as is the symbol of summation $\sum$. The expression $\langle a \rangle$ will refer to the subgroup generated by the element of a group G. The isomorphism between two groups A and B will be written as $A \cong B$.

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CHAPTER 1 : The Free Abelian Group Over a Set

The subject of this paper, the tensor product of two Abelian groups, involves intrinsically the concept of a free group (the tensor product in our case might better be termed the "free bilinear product", as by Fuchs) and so this topic will be dealt with first in its own right. In this thesis by group we shall mean Abelian group throughout.

Let S be any set. We say a group G together with a mapping ψ of S into G constitutes a free group on S providing the following condition is satisfied : if φ is any other mapping from S into any other group H, then there exists a unique homomorphism h of G into H such that the following diagram commutes :

$$\begin{tikzcd} & H \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & G \arrow[u, dashed, "h"'] \end{tikzcd}$$

The initial step from this definition would naturally be to show the existence of a free group (G, ψ) for an arbitrary set S, but first let us show that if free groups on S do exist, then any two on the same set are equal up to isomorphism :

<u>Lemma 1 - 1</u> Any two free groups on the same set S are isomorphic , in fact, between (G, ψ) and (G', ψ') there exists a unique isomorphism $i : G \rightarrow G$ such that $\psi' = \psi i$

<u>Proof :</u> By our definition of (G, ψ) and (G', ψ') as free groups on S we have the following doubly commutative diagram :

$$\begin{tikzcd} & G' \arrow[d, shift left, dotted, "h'"] \\ S \arrow[ur, "\psi'"] \arrow[r, "\psi"'] & G \arrow[u, shift left, dotted, "h"] \end{tikzcd}$$

i.e. $\psi' = \psi h$

and $\psi = \psi' h'$

Thus $(\psi h)\, h' = \psi' h' = \psi$, but also $\psi I_G = \psi$, where I_G is the identity map of G, and we conclude $h\ h' = I_G$. In an entirely analagous fashion, we also find $h'\ h = I_{G'}$, and thus h and h' are both isomorphisms and h will fit the statement of the lemma, q. e. d.

We shall now show the existence of a free group on an arbitrary set S by actually constructing one :

Theorem 1 - 2 A free group (G, ψ) exists for any set S.

Proof : Let Z denote the ring of integers and let $F(S) = \{ f \mid f : S \to Z$ and $sf = 0$ for all but a finite number of $s \in S \}$

Define $\psi : S \to F(S)$ by $s\psi = f_s$, where $tf_s = \delta_{st}$ (Kroenecker function) for $t \in S$. Then firstly $F(S)$ forms a group under the following rule of composition of functions :

$$s(f+g) = sf + sg \text{ where } s \in S,\ f, g \in F(S)$$

Associativity and commutativity are trivial under this rule since Z forms a group. Let f_0 be the zero map on S to Z i.e. $sf_0 = 0$ for all $s \in S$; then $s(f+f_0) = sf + sf_0 = sf$ for all $s \in S$ and hence $f + f_0 = f_0 + f = f$ i.e. there exists a neutral element $f_0 \in F(S)$. For any $f \in F(S)$ define $-f$ as follows : $s(-f) = -sf$ for all $s \in S$; then $s(f + (-f)) = sf - sf = 0 = sf_0$ for all $s \in S$ and hence for any $f \in F(S)$ there exists an inverse element $-f$ such that $f + (-f) = f_0$.

We show $(F(S),\Psi)$ is free. Let H be any group and let there be defined a map $\varphi : S \to H$. If $f \in F(S)$ define a map $h : F(S) \to H$ as follows : $fh = \sum_{s\in S} (sf)(s\varphi)$ if $f \neq f_0$, and $f_0 h = g_0$, the neutral element of H. We see then that such an h satisfies the requirement $s\varphi = s\Psi h$, and it remains to prove h is in fact a unique homomorphism. Let $f_1, f_2 \in F(S)$; then

$$(f_1 + f_2)\, h = \sum_{s\in S} s(f_1 + f_2)(s\varphi)$$
$$= \sum_{s\in S} (sf_1 + sf_2)(s\varphi)$$
$$= \sum_{s\in S} (sf_1)(s\varphi) + \sum_{s\in S} (sf_2)(s\varphi)$$

by definition of composition in F(S) and by the associativity of the group H. To show h is unique suppose there is another homomorphism $h' : F(S) \to H$ such that

$$\begin{tikzcd} & H \\ S \arrow[ur, "\varphi"] \arrow[r, "\Psi"'] & F(S) \arrow[u, dotted, "h'"'] \end{tikzcd}$$

commutes. Thus $s\varphi = s\Psi h = s\Psi h'$ for all $s \in S$. But then for any $f \in F(S)$ we have :

$$fh = \sum_{s\in S} (sf)(s\varphi)$$
$$= \sum_{s\in S} (sf)(s\Psi h)$$
$$= \sum_{s\in S} (sf)(s\Psi h')$$
$$= \sum_{s\in S} (sf)(s\Psi)h'$$
$$= fh'$$

and we conclude $h = h'$ and thus the pair $(F(S),\Psi)$ we have constructed forms a free group.

We can now deduce a few easy lemmas to show the actual composition of the group F(S).

<u>Lemma 1 - 3</u> If $(F(S),\Psi)$ is a free group on a set S, then Ψ is an injection.

<u>Proof :</u> Let s_1 and s_2 be elements of S, $s_1 \neq s_2$. Let $f_{s_1} : S \to Z$ be as before, and let h be the homomorphism of F(S) into Z produced by f_{s_1}, i.e. $f_{s_1} = \Psi h$. Then we have $s_1 \Psi h = 1$

and $s_2 \Psi h = 0$; thus $s_1\Psi \neq s_2\Psi$

.../since

since h is well defined as a homomorphism, q. e. d.

Lemma 1 - 4 $S\Psi$ forms a set of generators for $F(S)$.

Proof : Let $F'(S)$ be the subgroup of $F(S)$ generated by $S\Psi$. Let Ψ' be the map of $S \to F'(S)$ such that $s\Psi' = s\Psi$ for all $s \in S$. Then there is an $f : F(S) \to F'(S)$ such that $\Psi f = \Psi'$. Let $g : F'(S) \rightarrowtail F(S)$ be the injection such that $xg = x$ for all $x \in F'(S)$. We have then the diagram :

$$\begin{tikzcd} & F'(S) \arrow[d, dotted, shift right, "g"'] \\ S \arrow[ur, "\Psi'"] \arrow[r, "\Psi"'] & F(S) \arrow[u, dotted, shift right, "f"'] \end{tikzcd}$$

Then $fg : F(S) \to F(S)$ and $s\Psi fg = s\Psi f = s\Psi' = s\Psi$ for all $s \in S$. Hence fg is the identity on $F(S)$ and therefore g is onto, and so $F'(S) = F(S)$, q. e. d.

Recalling the construction of the unique homomorphism connected with a free group, we can now, in fact, represent any element of $F(S)$ in terms of the generators, $f_s \in S\Psi$:

$$\begin{aligned} fh &= \sum_{s\in S} (sf)(s\varphi) \\ &= \sum_{s\in S} (sf)(s\Psi h) \\ &= \sum_{s\in S} (sf)(s\Psi)\, h \end{aligned}$$

and thus $f = \sum_{s\in S} (sf)(s\Psi) \equiv \sum_{s\in S} (sf)(f_s)$. Indeed, for $t \in S$ we have

$$\begin{aligned} t \sum_{s\in S} (sf)(f_s) &= \sum_{s\in S} (sf)(tf_s) \\ &= (tf)(tf_t) = tf \end{aligned}$$

Now suppose $\sum_{s\in S} \mu_s f_s = f_0$, $\mu_s \in Z$, $f_s \in S\Psi$, and $t \in S$ is arbitrary ;

$$\begin{aligned} &\text{then } t \sum_{s\in S} \mu_s f_s = 0 \\ \Rightarrow\ & \sum_{s\in S} \mu_s (tf_s) = 0 \\ \Rightarrow\ & \mu_t = 0 \end{aligned}$$

Thus there exist no relations between the generators of $F(S)$, and hence the use of the word "free" to describe $(F(S), \Psi)$. As an example of a free

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group we might cite the direct sum of an arbitrary number of infinite cyclic groups, which clearly forms a free group over the index set of the direct sum.

CHAPTER 2 : The Tensor Product of Two Groups

Let A and B be two groups, let $S = A \times B$ be their Cartesian product and let the pair $(F(A\times B), \Psi)$ be a free group over $A\times B$. We then define the tensor product of A and B as follows. In $F(A\times B)$ we can consider elements of the form :

(1) $(a_1 + a_2, b)\Psi - (a_1, b)\Psi - (a_2, b)\Psi$

(2) $(a, b_1 + b_2)\Psi - (a, b_1)\Psi - (a, b_2)\Psi$ where $a_1, a_2 \in A$

and $b_1, b_2 \in B$. Let Ω denote the subgroup of $F(A\times B)$ generated by the set of elements of the form (1) and (2) ; we now define the Tensor Product of A and B to be the factor group $\frac{F(A\times B)}{\Omega}$, for which we write $A\otimes B$. The map $\overline{\Psi} : A\times B \to \frac{F(A\times B)}{\Omega}$ given by the composition of the maps $\Psi : A\times B \to F(A\times B)$, the free map, and $\eta : F(A\times B) \to \frac{F(A\times B)}{\Omega}$, the natural homomorphism, is called the free bilinear map, or the tensor map ; for $(a,b)\overline{\Psi}$ we write $a\otimes b$.

Definition 2 - 1 A map $f : S_1 \times S_2 \to T$ from the Cartesian product of any two groups S_1 and S_2 into a third, T, (all groups having the law of composition denoted by +) is called bilinear if $(s_{11} + s_{12}, s_{21})f = (s_{11}, s_{21})f + (s_{12}, s_{21})f$ and $(s_{11}, s_{21} + s_{22})f = (s_{11}, s_{21})f + (s_{11}, s_{22})f$ for all $s_{11}, s_{12} \in S_1$, and $s_{21}, s_{22} \in S_2$.

Clearly the tensor map $\overline{\Psi} : A\times B \to A\otimes B$ is bilinear since by (1) $(a_1 + a_2, b)\overline{\Psi} - (a_1, b)\Psi - (a_2, b)\overline{\Psi} = 0$ (or $(a_1 + a_2)\otimes b = a_1 \otimes b + a_2 \otimes b$) and by (2) $(a, b_1 + b_2)\overline{\Psi} - (a, b_1)\overline{\Psi} - (a, b_2)\overline{\Psi} = 0$ (or $a\otimes(b_1 + b_2) = a\otimes b_1 + a\otimes b_2$).

Lemma 2 - 2 Let $f : A \times B \to H$ be a bilinear map from the Cartesian product of two groups A and B to a group H. Then if $\overline{\Psi} : A \times B \to A \otimes B$ is the tensor map, there exists a unique homomorphism $h : A \otimes B \to H$ such that the following diagram commutes :

$$\begin{tikzcd} & & H \\ A \times B \arrow[urr, "f"] \arrow[rr, "\overline{\Psi}"'] & & A \otimes B \arrow[u, dotted, "h"'] \end{tikzcd}$$

Proof : Let $(F(A \times B), \Psi)$ be a free group on $A \times B$ and let $h' : F(A \times B) \to H$ be the unique homomorphism associated with $(F(A \times B), \Psi)$ and the map f ;

$$\begin{tikzcd} & H & \\ A \times B \arrow[ur, "f"] \arrow[r, "\Psi"'] \arrow[rr, bend right, "\overline{\Psi}"'] & F(A \times B) \arrow[u, "h'"'] \arrow[r, "\eta"'] & A \otimes B \end{tikzcd}$$

Then, f being bilinear, we have $(a+a', b)f - (a, b)f - (a', b)f = 0$

and $(a, b+b')f - (a, b)f - (a, b')f = 0$

for all $a, a' \in A$ and $b, b' \in B$, hence

$$(a+a', b)\Psi h' - (a, b)\Psi h' - (a', b)\Psi h' = 0$$

$$\text{and } (a, b+b')\Psi h' - (a, b)\Psi h' - (a, b')\Psi h' = 0$$

which implies $(\ker h') \supseteq \Omega$; therefore, if we define, for $f \in F(A \times B)$, $(f + \Omega)h = fh'$, where $f + \Omega = f\eta$ the coset of f in $A \otimes B$, we have asserted the existence of a homomorphism h which fits the lemma. The uniqueness of such an h follows from the requirement that the diagram in the lemma should commute, q. e. d.

It is clear from the properties of the free group (Lemma 1 - 1) that the tensor product $A \otimes B$ is unique up to isomorphism ; we can now also prove the following :

Lemma 2 - 3 $A\otimes B \cong B\otimes A$ for any groups A and B.

Proof : Define the following maps : $f : A\times B \to B\times A$ and $g : B\times A \to A\times B$ by $(a,b)f = (b,a)$

and $(b,a)g = (a,b)$

Let $\overline{\Psi} : A\times B \to A\otimes B$ and $\overline{\varphi} : B\times A \to B\otimes A$ be tensor maps and we therefore have the following diagram :

$$\begin{tikzcd}
A\times B \arrow[r, "\overline{\Psi}"'] & A\otimes B \\
B\times A \arrow[u, "g"] \arrow[r, "\overline{\varphi}"'] & B\otimes A \arrow[u, dashed, "g'"'] \\
A\times B \arrow[u, "f"] \arrow[r, "\overline{\Psi}"'] & A\otimes B \arrow[u, dashed, "f'"']
\end{tikzcd}$$

f' and g' are homomorphisms whose existences are assured by the bilinearity of the compositions $f\overline{\varphi}$ and $g\overline{\Psi}$. Now fg is the identity map of $A\times B$, hence $f'g'$ is the identity of $A\otimes B$; similarly $g'f'$ is the identity of $B\otimes A$, thus $A\otimes B\cong B\otimes A$, q. e. d.

The following will give us now a set of generators for the tensor product of two groups, the sets of generators of the latter being known :

Lemma 2 - 4 If A_0 is a set of generators for A and B_0 a set for B, then $\{\alpha\otimes\beta \mid \alpha\in A_0 \text{ and } \beta\in B_0\}$ forms a set of generators for $A\otimes B$.

Proof : We know already (page 4) that any element of $F(A\times B)$ may be written as $\sum_{i=1}^{n} \lambda_i (a_i, b_i)\Psi$, where $\lambda_i \in Z$ and $(a_i, b_i)\Psi \in (A\times B)\Psi$, and at the same time the set $\mathcal{F} = \{(a,b)\Psi \mid a\in A , b\in B\}$ generates $F(A\times B)$. Then since η , the natural map, is <u>onto</u> , $\{(a,b)\Psi\eta \mid a\in A,\ b\in B\} = \{a\otimes b \mid a\in A, b\in B\}$

.../generates

generates $A \otimes B$. Now $a \in A$ implies $a = \sum \xi_i a_i$, $\xi_i \in Z$ and $a_i \in A_0$; similarly $b \in B$ implies $b = \sum \zeta_j b_j$, $\zeta_j \in Z$ and $b_j \in B_0$. Thus the bilinearity of the tensor product implies $\{\alpha \otimes \beta \mid \alpha \in A_0 \text{ and } \beta \in B_0\}$ generates $\{a \otimes b \mid a \in A, b \in B\}$, q. e. d.

We can now prove some easy lemmas based mainly on the bilinearity intrinsic to the tensor product.

Lemma 2 - 5 (a) If either $a = 0$ or $b = 0$, then $a \otimes b = 0$, the neutral element of the group $A \otimes B$.

Proof : Assume $a = 0$; then $a = a + a$ and $a \otimes b = (a + a) \otimes b = a \otimes b + a \otimes b$, hence $a \otimes b = 0$. Similarly $b = 0$ implies $a \otimes b = 0$, q. e. d.

2 - 5 (b) $(-a) \otimes b = -(a \otimes b) = a \otimes (-b)$ for any $a \in A$, $b \in B$.

Proof : $(-a) \otimes b + a \otimes b = (a - a) \otimes b = 0$, thus $(-a) \otimes b = -(a \otimes b)$; similarly $a \otimes (-b) = -(a \otimes b)$, q. e. d.

Corollary : $a \otimes b = (-a) \otimes (-b)$

2 - 5 (c) $na \otimes b = n(a \otimes b) = a \otimes nb$ for any $n \in Z$, $a \in A$ and $b \in B$.

Proof : By induction.

CHAPTER 3 : The Tensor Product of Finitely Generated Groups

In this chapter we shall determine the structure of the tensor product of two finitely generated groups. The concept of a direct sum of Abelian groups will be used extensively and therefore a workable definition will first be formulated.

Definition 3 - 1 Let H be a set and for each $\alpha \in H$, let B_α be a group. By the direct sum of the B_α, for which we write $\sum_{\alpha \in H}^{\bullet} B_\alpha$ or $B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \cdots$, we mean the following :

$\underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha = \{ f \; : \; H \to \bigcup_{\alpha\in H} B_\alpha \mid \alpha f \in B_\alpha$ and $\alpha f \neq 0$ for at most finitely many $\alpha \in H \}$ where addition of elements is performed as follows : let f_1 and $f_2 \in \underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha$; then $f_1 + f_2 \; : \; H \to \bigcup_{\alpha\in H} B_\alpha$ is defined by $\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$.

This definition covers the internal concept of a direct sum in which a group G is said to be the direct sum of subgroups B_α if $g \in G$ implies $g = \sum_{i=1}^{n} b_i$ where $b_i \in B_i$ and such a representation is unique. The first main theorem, which we shall use extensively throughout this paper, is of interest in its own right :

Theorem 3 - 2 Let **A** be any group and let $B = \underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha$ be a direct sum : then $A \otimes B \cong \underset{\alpha\in H}{\Sigma_1^{\bullet}} (A \otimes B_\alpha)$.

Proof : Consider the following diagram :

$$\begin{tikzcd}
A \times B_\gamma \arrow[r, "\bar{\Psi}_\gamma"'] & A \otimes B_\gamma \\
A \times \underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha \arrow[u, "p_\gamma"] \arrow[r, "\bar{\Psi}"'] & A \otimes \underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha \arrow[u, "\eta_\gamma"'] \\
A \times B_\beta \arrow[u, "i_\beta"] \arrow[r, "\bar{\Psi}_\beta"'] & A \otimes B_\beta \arrow[u, "\mu_\beta"']
\end{tikzcd}$$

where B_β , B_γ are direct summands of $\underset{\alpha\in H}{\Sigma_1^{\bullet}} B_\alpha$ and the maps $\bar{\Psi}_\beta$, $\bar{\Psi}_\gamma$, $\bar{\Psi}$ are the appropriate tensor maps. The maps i_β and p_γ are defined as follows :

$\iota_\beta : \quad A \times B_\beta \to A \times \sum_{\alpha \in H}^{\bullet} B_\alpha$ where $(a, b)i_\beta = (a, f_\beta)$ and $\left\{\begin{array}{l} \beta f_\beta = b \\ \alpha f_\beta = 0 \quad \text{if } \alpha \neq \beta \end{array}\right\}$ for all $a \in A$, $b \in B_\beta$.

$\rho_\gamma : \quad A \times \sum_{\alpha \in H}^{\bullet} B_\alpha \to A \times B_\gamma$ where $(a, f)\rho_\gamma = (a, \gamma f)$ for all $a \in A$ and $f \in \sum_{\alpha \in H}^{\bullet} B_\alpha$. It is clear then that the compositions $i_\beta \bar{\Psi}$ and $\rho_\gamma \bar{\Psi}_\gamma$ are bilinear so the maps μ_β and η_γ indicated on the diagram do exist and are in fact homomorphisms, so that $i_\beta \bar{\Psi} = \bar{\Psi}_\beta \mu_\beta$ and $\rho_\gamma \bar{\Psi}_\gamma = \bar{\Psi} \eta_\gamma$. Thus $(a \otimes b)\mu_\beta = a \otimes f_\beta$ for $a \in A$ and $b \in B_\beta$ and also $(a \otimes f)\eta_\gamma = a \otimes b$ for $f \in \sum_{\alpha \in H}^{\bullet} B_\alpha$ arbitrary except that $\beta f = b$. Next consider the map $h : \sum_{\alpha \in H}^{\bullet}(A \otimes B_\alpha) \to A \otimes \sum_{\alpha \in H}^{\bullet} B_\alpha$ defined as follows : let $f \in \sum_{\alpha \in H}^{\bullet}(A \otimes B_\alpha)$ be an element such that $\beta f = x_\beta \in A \times B_\beta$ for all $\beta \in H$; then let $fh = \sum_\beta x_\beta \mu_\beta$, where β ranges over those elements of H such that $\beta f \neq 0$; the sum is thus finite. (N. B. h <u>is</u> a homomorphism ; the fact that $(f_1 + f_2)h = f_1 h + f_2 h$ is clear from the definition, as is the fact that $0h = 0$). Now it is evident that $A \otimes \sum_{\alpha \in H}^{\bullet} B_\alpha$ is generated by :

$\left\{ a \otimes f \;\middle|\; a \in A \text{ and } f \in \sum_{\alpha \in H}^{\bullet} B_\alpha \text{ where f has the form : for some } \beta \in H \text{ and } b \in B_\beta \;\; \alpha f = \begin{array}{ll} b & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{array} \text{ for all } \alpha \in H \right\}$. Thus h is <u>onto</u> $A \otimes \sum_{\alpha \in H}^{\bullet} B_\alpha$, so for the desired isomorphism we need only show $\ker(h) = 0$. Suppose, then, that $fh = 0 = \sum_\beta x_\beta \mu_\beta$; but η_γ is a homomorphism for any $\gamma \in H$, and thus $(\sum_\beta x_\beta \mu_\beta)\eta_\gamma = 0$ also. However, we know from construction that :

$$i_\beta \rho_\gamma = \begin{cases} \text{identity of } A \times B_\beta \text{ if } \beta = \gamma \\ \text{annihilator of } B_\beta \text{ component if } \beta \neq \gamma \end{cases}$$

and thus
$$\mu_\beta \eta_\gamma = \begin{cases} \text{identity of } A \otimes B_\beta \text{ if } \beta = \gamma \\ \text{zero map of } A \otimes B_\beta \text{ if } \beta \neq \gamma \end{cases}$$

Therefore $\left(\sum_{\beta}' x_\beta \mu_\beta\right)\eta_\gamma = 0 = \sum_{\beta}' x_\beta \mu_\beta \eta_\gamma$
$= x_\gamma \mu_\gamma \eta_\gamma = x_\gamma$

and we conclude h is an isomorphism , q. e. d.

Corollary 3 - 3 $\sum_{\alpha\in H}^{\bullet} A_\alpha \otimes \sum_{\beta\in J}^{\bullet} B_\beta \cong \sum_{\substack{\alpha\in H\\ \beta\in J}}^{\bullet} (A_\alpha \otimes B_\beta)$

Now let G be a finitely generated group ; then it is well known that $G \cong T \oplus F$ where T is the torsion subgroup of G and F is torsion free. With this information, then, we know that : $G_1 \otimes G_2 \cong$ $(T_1 \oplus F_1) \otimes (T_2 \oplus F_2) \cong (T_1 \otimes T_2) \oplus (T_1 \otimes F_2) \oplus (F_1 \otimes T_2) \oplus (F_1 \otimes F_2)$ where G_1, G_2 are finitely generated groups and $G_i \cong T_i \oplus F_i$ describes the decomposition into torsion and torsion free summands.

To completely describe the tensor product we must analyze each of the four direct summands above ; the second and third are essentially the same ($A \otimes B \cong B \otimes A$) so that it remains to examine three classes of tensor products. The first class is merely the restricted case where finitely generated is replaced with finite, since finitely generated torsion and finite are synonomous.

Definition 3 - 4 A torsion group T is called a p - group if every element has order p^α for some $\alpha \in Z$, where p is a prime.

There is the basic theorem for the decomposition of an arbitrary torsion group into its p - subgroups as follows :

Theorem 3 - 5 A torsion group T is isomorphic to the direct sum of its p - subgroups, where p ranges over all primes.

Proof : Let $T_p = \{t \in T \mid t \text{ has order } p^\alpha, \text{ some } \alpha \in Z\}$.

Then $T_p \subseteq T$ is a subgroup , for if $a, b \in T_p$ then $p^\alpha a = p^\beta b = 0$ for some $\alpha, \beta \in Z$ whence $p^{\max(\alpha,\beta)}(a-b) = 0$. Now $T_p \cap \sum_{q \neq p}' T_q = 0$ since any element of this intersection must have orders p_1 and p_2 , two relatively prime numbers and the neutral element is the only such element. Thus the sum $\sum_p^{!} T_p$ is direct and to complete theorem 3 - 5 we need the following :

Lemma 3 - 6 Let $x \in T$ have order $n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$ where the n_i are relatively prime in pairs ; then x has a representation as $x = x_1 + x_2 + \cdots + x_k$ where (order x_i) $= n_i$,

Proof : We prove the lemma for the case $k = 2$, the induction to general k being easy. We then know there exist integers a, b such that $an_1 + bn_2 = 1$ and hence $x = an_1x + bn_2x$. If we let $x_1 = bn_2x$ and $x_2 = an_1x$ then it is easily seen that (order x_1) $= n_1'$ and (order x_2) $= n_2'$, where $n_1' | n_1$ and $n_2' | n_2$, ($s | t$ means, as usual, $t = ss_1$ for some s_1) and also that $x = x_1 + x_2$. But $n_1' n_2' x = n_1' n_2' x_1 + n_1' n_2' x_2 = 0$ which implies $n_1 n_2 | n_1' n_2'$, and thus $n_1 = n_1'$ and $n_2 = n_2'$, q. e. d.

Now any $x \in T$ must have order $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \cdots \cdot p_k^{a_k}$ where $k, a_i \in Z$ so by the lemma there exist $x_1, \cdots, x_k \in T$ such that $x = \sum_{i=1}^{k} x_i$, $x_i \in T_{p_i}$, and the theorem is proved.

Let T_1 and T_2 be as before ; then $T_1 \otimes T_2 \cong \sum^{!\bullet} T_p^1 \otimes \sum^{!\bullet} T_q^2$ where each T_p^1 and T_q^2 is a prime p -group and q - group respectively. Hence by corollary 3 - 3 $T_1 \otimes T_2 \cong \sum^{!\bullet} (T_p^1 \otimes T_q^2)$. The following lemma will enable us to eliminate many of the cross tensor products in this expression :

Lemma 3 - 7 If A is a p - group and B is a q - group for $p \neq q$ primes , then $A \otimes B = 0$.

Proof : Consider a generator of $A \otimes B$ of the form $a \otimes b$, any $a \in A$, $b \in B$. Since $A \otimes B$ is generated by $\{ a \otimes b \mid a \in A, b \in B \}$

it suffices to show that $a \otimes b = 0$; but $p^\alpha a = q^\beta b = 0$ for some $\alpha, \beta \in Z$ and since p^α and q^β have g. c. d. equal to 1 , there exist integers s, t such that $sp^\alpha + tq^\beta = 1$. Then

$$\begin{aligned} a \otimes b &= (sp^\alpha + tq^\beta)(a \otimes b) \\ &= (sp^\alpha + tq^\beta)\, a \otimes b \\ &= tq^\beta a \otimes b \\ &= a \otimes tq^\beta b = 0 \quad , \text{ q. e. d.} \end{aligned}$$

Therefore, in the tensor product $\sum^{\bullet}(T^1_p \otimes T^2_q) = T_1 \otimes T_2$ we need consider only those summands of the form $T^1_p \otimes T^2_p$ for the same prime p . Now each finite p - group may be written as a direct sum of cyclic groups of order p^α , say $T^i_p \cong \sum^{\bullet} C^i_{p^\alpha}$, $i = 1, 2$, and thus

$$\begin{aligned} T^1_p \otimes T^2_p &\cong \sum{}^{\bullet} C^1_{p^\beta} \otimes \sum{}^{\bullet} C^2_{p^\gamma} \\ &\cong \sum{}^{\bullet} (C^1_{p^\beta} \otimes C^2_{p^\gamma}). \end{aligned}$$

We shall now prove that any summand of the latter can be further simplified :

<u>Lemma 3 - 8</u> $C_{p^\beta} \otimes C_{p^\gamma} \cong C_{p^{\min(\beta,\gamma)}}$ where C_{p^α} are cyclic groups of order p^α , p prime.

<u>Proof :</u> Let $C_{p^\beta} = \langle a \rangle$ and $C_{p^\gamma} = \langle b \rangle$ and assume without loss of generality that $\beta < \gamma$. Consider the following diagram :

$$\begin{tikzcd} & C_{p^\beta} \\ C_{p^\beta} \times C_{p^\gamma} \arrow[ur, "f"] \arrow[r, "\psi"'] & C_{p^\beta} \otimes C_{p^\gamma} \arrow[u, dashed, "h"'] \end{tikzcd}$$

where $f : C_{p^\beta} \times C_{p^\gamma} \to C_{p^\beta}$ is defined by $(na, mb)f = nma$, $n, m \in Z$.

Then (1) f is well defined : suppose $n_1 \equiv n \ (p^\beta)$ and $m_1 \equiv m \ (p^\gamma)$

i. e. $n_1 = n + sp^\beta$ and $m_1 = m + tp^\gamma$ for some $s, t \in Z$.

Then $(n_1 a, m_1 b)f = n_1 m_1 a = (n+sp^\beta)(m+tp^\gamma)\, a$

$$= (nm + ntp^\gamma + smp^\beta + stp^\beta p^\gamma)\, a$$

$$\equiv nma \ (p^\beta)$$

N.B. If we had defined $f' : C_{p^\beta} \times C_{p^\gamma} \to C_{p^\gamma}$ in the same fashion, the map f' would not be well defined here!

(2) f is bilinear : $(na + n'a,\ mb)f = ((n+n')a, mb)f$

$$= (n+n')\, m\, a$$

$$= nma + n'ma$$

$$= (na, mb)f + (n'a, mb)f$$

Similarly, $(na, mb + m'b)f = (na, mb)f + (na, m'b)\,f$ and thus the homomorphism $h : C_{p^\beta} \otimes C_{p^\gamma} \to C_{p^\beta}$ exists.

We must show now (1) h is onto : clear, since $(na, b)h = na$, a general element of C .

(2) h is one to one : Let $\sum_{i=1}^{n} (n_i a \otimes m_i b)$ represent an arbitrary element of $C_{p^\beta} \otimes C_{p^\gamma}$; but then

$$\sum_{i=1}^{n} n_i a \otimes m_i b = \sum_{i=1}^{n} a \otimes n_i m_i b = a \otimes \sum_{i=1}^{n} n_i m_i b$$

$$= Na \otimes b \text{ , some } N \in Z$$

Suppose therefore $(Na \otimes b)h = 0 = Na$; hence $N \equiv 0 \ (p^\beta)$ and $Na \otimes b = 0$, and h is an isomorphism, q.e.d.

We finally have, then, $T_1 \otimes T_2 \cong \sum_1' T_p^1 \otimes \sum_1' T_q^2$

$$\cong \sum_1' (T_p^1 \otimes T_q^2)$$

$$\cong \sum_1' (T_p^1 \otimes T_p^2)$$

$$\cong \sum_1' (C^1_{p^\beta} \otimes C^2_{p^\gamma})$$

$$\cong \sum_1' C_{p^\alpha} \text{ i.e. a finite direct sum}$$

of cyclic groups of prime power order.

The second class of tensor product to examine is of the form $T_i \otimes F_j$, $i, j = 1, 2$, $i \neq j$. Now any finitely generated torsion free group

.../may be

may be represented as a finite direct sum of copies of the integers, and thus $T_i \otimes F_j \cong T_i \otimes \sum_1^{\cdot} Z \cong \sum_1^{\cdot}(T_i \otimes Z)$. We need at this point the following lemma :

<u>Lemma 3 - 9</u> $G \otimes Z \cong G$ for any group G.

<u>Proof :</u> We may again prove this by the basic lifting property of the tensor product ; consider the following diagram :

$$\begin{tikzcd}
G \times Z \arrow[r, "\bar{\Psi}"'] & G \otimes Z \\
G \arrow[u, "f_2"] \arrow[r, "I"'] & G \arrow[u, dashed, "\hat{f}_2"'] \\
G \times Z \arrow[u, "f_1"] \arrow[r, "\bar{\Psi}"'] & G \otimes Z \arrow[u, dashed, "\hat{f}_1"']
\end{tikzcd}$$

where $f_1 : G \times Z \to G$ is defined by $(g, n)f_1 = ng$ and $f_2 : G \to G \times Z$ by $gf_2 = (g, 1)$, $\bar{\Psi}$ is the tensor map and $I : G \to G$ is the identity map. Now the existence of the homomorphisms $\hat{f}_1$, $\hat{f}_2$ is assured by the bilinearity of the composites $f_1 I$ and $f_2 \bar{\Psi}$. Let $(g, n) \in G \times Z$; then $(g, n)f_1 f_2 = (ng, 1)$ so that $(g \otimes n)\hat{f}_1 \hat{f}_2 = ng \otimes 1 = g \otimes n$. Hence $\hat{f}_1 \hat{f}_2$ is the identity of $G \otimes Z$ and similarly $\hat{f}_2 \hat{f}_1$ the identity of G , which shows $G \otimes Z \cong G$, q.e.d.

We have now, then, that
$$\begin{aligned} T_i \otimes F_j &\cong T_i \otimes \textstyle\sum_1^{\cdot} Z \\ &\cong \textstyle\sum_1^{\cdot}(T_i \otimes Z) \\ &\cong \textstyle\sum_1^{\cdot} T_i \end{aligned}$$

The final class was typified by $F_1 \otimes F_2$, the tensor product of two finitely generated torsion free groups, which can now easily be expressed as $\sum_1^{\cdot} Z \otimes \sum_1^{\cdot} Z$

$$\cong \sum_{1}^{\bullet}(Z \otimes Z)$$
$$\cong \sum_{1}^{\bullet} Z$$

We have thus shown that the tensor product of two finitely generated groups can be expressed as a direct sum of cyclic groups. The latter may be explicitly calculated by determining the decomposition factors of the two given groups.

CHAPTER 4 : Structure Theory to Date of $A \otimes B$ for Arbitrary Groups

We now turn to the examination of the structure of the tensor product of two groups at least one of which is a torsion group (not necessarily finitely generated).

§ 1. Let us first deal with the case of two torsion groups. By Corollary 3 - 3 and Theorem 3 - 5, there is no restriction of generality in considering the tensor product of two p - groups, and, in fact, p - groups for the same prime p, by Lemma 3 - 7. However, we are not restricting ourselves to finitely generated groups, so that we may not, in general, represent them as direct sums of cyclic groups. To circumvent this problem we introduce the concept of a basic subgroup which is, amongst other things, the direct sum of cyclic groups, and prove eventually that the tensor product of two p - groups is essentially the tensor product of their respective basic subgroups.

Definition 4 - 1 A subgroup G_o of a group G is said to be a pure subgroup if $x \in G_o$ and $x = nx_1$ for some $x_1 \in G$ and $n \in Z$, then $x = nx_o$ for some $x_o \in G_o$; symbolically we may express this by the equation $nG_o = G_o \cap nG$.

Definition 4 - 2 A group G is divisible if for every $x \in G$ and $n \in Z$, $x = nx_n$ for some $x_n \in G$.

Definition 4 - 3 A subgroup B of G is said to be a BASIC subgroup if the following three conditions are satisfied :

(1) B is a direct sum of cyclic groups.

(2) B is a pure subgroup of G.

(3) The factor group G/B is divisible.

A fundamental theorem for basic subgroups of p - groups is the following of Kulikov ; the proof will only be sketched here and it may be found in detail in Fuchs, Abelian Groups, Chapter 5.

Theorem 4 - 4 Every p - group contains a basic subgroup.

Proof : A pure independent subset $\{x_\lambda\}_{\lambda\in\Lambda}$, that is an independent subset which generates a pure subgroup of G , exists and may be extended to a maximal set in G by Zorn's lemma. Then if B is the subgroup generated by the maximal pure independent set $\{x_\lambda\}_{\lambda\in\Lambda}$, (2) in the above definition is true and (1) is an easy consequence of independence , (3) requires the maximality of the independent set $\{x_\lambda\}_{\lambda\in\Lambda}$.

Now if A_1 and B_1 are subgroups of A and B respectively then it is not generally true that $A_1 \otimes B_1$ forms a subgroup of $A \otimes B$ but merely that the subgroup of $A \otimes B$ generated by $\{a \otimes b \mid a \in A_1, b \in B_1\}$ is a homomorphic image of $A_1 \otimes B_1$. If, however, A_1 and B_1 are pure subgroups we have the following :

Theorem 4 - 5 If A_1 and B_1 are pure subgroups of A and B respectively then the subgroup $\mathfrak{M}$ of $A \otimes B$ generated by the set $\{a \otimes b \mid a \in A_1, b \in B_1\}$ is isomorphic to $A_1 \otimes B_1$.

Proof : Consider the diagram :

$$\begin{tikzcd} & & \mathfrak{A} \\ A_1 \times B_1 \arrow[urr, "f"] \arrow[rr, "\tilde{\Psi}"'] & & A_1 \otimes B_1 \arrow[u, dashed, "h"'] \end{tikzcd}$$

where $(a_1, b_1)f = a_1 \otimes b_1$, then h is a homomorphism such that $f = \tilde{\Psi} h$. Thus we need to show that if an element $\sum_{i=1}^{n} a_i \otimes b_i$, $a_i \in A_1$, $b_i \in B_1$, of $A \otimes B$ vanishes then it also vanishes as an element of $A_1 \otimes B_1$. We need the following :

Lemma 4 - 6 If $\sum_{i=1}^{n} a_i \otimes b_i = 0$ in $A \otimes B$ then there exist finitely generated subgroups A^* and B^* such that $\sum_{i=1}^{n} a_i \otimes b_i$ vanishes as an element of $A^* \otimes B^*$.

Proof : $\sum_{i=1}^{n} a_i \otimes b_i = 0$ only if $\sum_{i=1}^{n} (a_i, b_i)\Psi$ belongs to the subgroup Ω generated by elements of the form (1) $(a_1 + a_2, b)\Psi - (a_1, b)\Psi - (a_2, b)\Psi$

and (2) $(a, b_1 + b_2)\Psi - (a, b_1)\Psi - (a, b_2)\Psi$.

Define A^* to be the subgroup of A generated by $a_1, a_2, \cdots, a_n$ and all a_j occurring in the expression of $\sum_{i=1}^{n} (a_i, b_i)\Psi$ by means of elements of the forms (1) and (2) above. Describing B^* similarly, we have the desired results.

We can now see that $\sum_{i=1}^{n} a_i \otimes b_i$ vanishes as an element of $\hat{A} \otimes \hat{B}$ where $\hat{A}$ is that subgroup of A generated by the pure subgroup A_1 and the finitely generated subgroup A^* ($\hat{B}$ is defined analagously) ; but it is known that since A^* and B^* are finitely generated and A_1 and B_1 are pure, the latter are direct summands of $\hat{A}$ and $\hat{B}$ and hence by Corollary 3 - 3 $\sum_{i=1}^{n} a_i \otimes b_i$ vanishes as an element of $A_1 \otimes B_1$, q. e. d.

Thus $A_1 \otimes B_1$ may be considered a subgroup of $A \otimes B$ and this prepares us to prove the following :

Theorem 4-7 If A_1 and B_1 are basic subgroups of the p-groups A and B, then $A_1 \otimes B_1 \cong A \otimes B$.

Proof : Since $A_1 \otimes B_1 \subseteq A \otimes B$ we need only show any element of $A \otimes B$ of the form $a \otimes b$ belongs to $A_1 \otimes B_1$. Now $a \in A$ implies $a = a_1 + p^k x$ where $a_1 \in A_1$, $k \in Z$ and $x \in A$, since A/A_1 divisible implies $a \equiv p^k x \pmod{A_1}$; similarly $b = b_1 + p^\ell y$, where $b_1 \in B_1$, $\ell \in Z$, $y \in B$. Now choosing $p^k \geq$ (order b) and $p^\ell \geq$ (order a_1) we have

$$\begin{aligned} a \otimes b = (a_1 + p^k x) \otimes b &= a_1 \otimes b \\ &= a_1 \otimes (b_1 + p^\ell y) \\ &= a_1 \otimes b_1 \end{aligned}$$

This shows $A \otimes B \subseteq A_1 \otimes B_1$ and the proof is complete.

We are now in a position to prove the main theorem of tensor products for two torsion groups :

Theorem 4 - 8 The tensor product of two torsion groups is a direct sum of prime power order cyclic groups.

Proof : By the last theorem it suffices to prove the proposition for the tensor product of the basic subgroups of two p - groups , but these are direct sums of cyclic groups and thus by Corollary 3-3 and Lemma 3-8 the statement of the theorem is true.

§ 2. The case when one of the factors is a torsion group and the other is torsion free we shall examine now. Again we may assume the torsion group T is a p-group, and let F be any torsion free group, i.e. for any $g \in F$, $ng = 0$ implies $n = 0$, $n \in Z$. Choose a maximal independent set in F modulo pF , say $\{x_\lambda\}_{\lambda \in \Lambda}$

Lemma 4 - 9 Such a set $\{x_\lambda\}_{\lambda \in \Lambda}$ forms a basis of F modulo $p^n F$ $n = 1, 2, 3, \cdots$

Proof : First if $\{x_\lambda\}$ is a maximal independent set in F/pF then for any $x \in F$, $x \notin pF$ we must have
$mx + n_1x_1 + n_2x_2 + \cdots + n_tx_t \equiv 0 \pmod{pF}$ for $m, n_\iota \in Z$ not all zero. Now $p|m$ implies $\sum_{\iota=1}^{n} n_\iota x_\iota \equiv 0 \pmod{pF}$ and not all n_ι zero, which contradicts the independence of the x_ι, hence $(p,m)=1$ and there exist integers a,b such that $ap+bm=1$. But $bmx \equiv n_1'x_1 + \cdots + n_t'x_t \pmod{pF}$ and $bmx = (1-ap)x \equiv x \pmod{pF}$ and thus $x \equiv n_1'x_1 + \cdots + n_t'x_t \pmod{pF}$. Consider an element $y \in p^kF$, $y \notin p^{k+1}F$. We have then that $y = p^kf$, $f \in F$ but $f \notin pF$ and thus $f \equiv \sum_{\lambda=1}^{s} n_\lambda x_\lambda \pmod{pF}$ which implies $y \equiv \sum_{\lambda=1}^{s} m_\lambda x_\lambda \pmod{p^{k+1}F}$. Note also that if an element y of F has finite height k then y has a representation $y \equiv \sum_{\lambda=1}^{s} m_\lambda' x_\lambda \pmod{p^\ell F}$ for arbitrary $\ell > k$, for let $y \equiv \sum_{\lambda=1}^{s} m_\lambda x_\lambda \pmod{p^{k+1}F}$ be the representation already proved. Then suppose, without loss of generality, that $y - \sum_{\lambda=1}^{s} m_\lambda x_\lambda \notin p^{k+2}F$. But then $(y - \sum_{\lambda=1}^{s} m_\lambda x_\lambda) - \sum_{\mu=1}^{t} n_\mu x_\mu \in p^{k+2}F$ for some $n_\mu \in Z$, thus $y - \sum_{\lambda=1}^{s} m_\lambda' x_\lambda \equiv 0 \pmod{p^{k+2}F}$ and in general the congruence $y - \sum m_\lambda x_\lambda \equiv 0 \pmod{p^\ell F}$ is solvable for arbitrary $\ell > k$, q.e.d.

Consider now any $t \in T$ and any $x \in F$ of height k. Note that if $f \in F$ has infinite height (i.e. $p^n|f$ for every $n \in Z$) then $t \otimes f = 0$. There exists the relation $x = n_1x_1 + \cdots + n_sx_s + p^\ell f$ for $x_\iota \in \{x_\lambda\}$ and arbitrarily large $\ell \in Z$, choosing $p^\ell \geq (\text{order } t)$ we obtain :

$$\begin{aligned} t \otimes x &= \sum_{\iota=1}^{s} t \otimes n_\iota x_\iota + t \otimes p^\ell f \\ &= \sum_{\iota=1}^{s} t_\iota \otimes x_\iota \qquad \text{where } t_\iota = n_\iota t \end{aligned}$$

We see therefore that an arbitrary element of $T \otimes F$, being a finite sum of generators of the form $t \otimes x$, may be written also as a finite sum $\sum_{\iota=1}^{n} t_\iota \otimes x_\iota$

for $t_i \in T$ and different x_i selected from the independent set $\{x_\lambda\}_{\lambda\in\Lambda}$, since addition in the generators may be carried out on the $t_i \in T$.

Suppose next that a sum such as $\sum_{i=1}^{n} t_i \otimes x_i$ vanishes as an element of $T \otimes F$. We shall show that this implies $t_i = 0$, $i = 1, \cdots, n$, and a general element of $T \otimes F$ may be expressed as an element of a direct sum of copies of T. By Lemma 4 - 6 $\sum_{i=1}^{n} t_i \otimes x_i$ vanishes also as an element of $T \otimes F_1$ where F_1 is finitely generated torsion free and contains $x_1, \cdots, x_n$; but this means F_1 is a finite direct sum of infinite cyclic groups greater than or equal to n in number. However, $x_1, \cdots, x_n$ being independent, F_1 must contain a direct summand F_1' containing $x_1, \cdots, x_n$ of rank n and then $\sum_{i=1}^{n} t_i \otimes x_i$ must vanish also in $T \otimes F_1'$ (Theorem 3 - 2). Suppose, then, $F_1' = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle$. This means $x_i = \sum_{j=1}^{n} m_{ij} a_j$ and thus $\sum_{i=1}^{n} t_i \otimes x_i = \sum_{i=1}^{n} t_i \otimes \sum_{j=1}^{n} m_{ij} a_j = \sum_{j=1}^{n} (\sum_{i=1}^{n} m_{ij} t_i \otimes a_j) = 0$ hold in the tensor product $T \otimes F_1'$ which implies $\sum_{i=1}^{n} m_{ij} t_i \otimes a_j = 0$; this in turn, by Lemma 3 - 9 implies $\sum_{i=1}^{n} m_{ij} t_i = 0$.

Let A be the matrix $(m_{ij})_{\substack{i=1\cdots n\\ j=1\cdots n}}$. If A is a singular matrix (mod p) then there exist $\beta_i \in Z$ not all zero such that $\sum_{i=1}^{k_1} \beta_i m_{ij} \equiv 0 \pmod p$, and therefore $\sum_{j=1}^{n} (\sum_{i=1}^{k_1} \beta_i m_{ij}) a_j \equiv 0 \pmod p$. But $\sum_{j=1}^{n} (\sum_{i=1}^{k_1} \beta_i m_{ij}) a_j = \sum_{i=1}^{k_1} \beta_i (\sum_{j=1}^{n} m_{ij} a_j) = \sum_{i=1}^{k_1} \beta_i x_i$ and thus $\sum_{i=1}^{k_1} \beta_i x_i \equiv 0 \pmod p$. This, however, contradicts the independence of the x_i, and hence $(p, \det A) = 1$. Let $\det A = K$ and the matrix $(\beta_{ij})_{\substack{i=1\cdots n\\ j=1\cdots n}}$ be defined as $K \cdot A^{-1}$. Now $\sum_{i=1}^{n} m_{ij} t_i = 0$ for each $j = 1, \cdots, n$ implies $\beta_{jk} \sum_{i=1}^{n} m_{ij} t_i = 0$ for each $k = 1, \cdots, n$ which in turn implies

$$\begin{aligned}\sum_{j=1}^{n} \beta_{jk} \sum_{i=1}^{n} m_{ij} t_i &= 0 \\ &= \sum_{i=1}^{n} (\sum_{j=1}^{n} m_{ij} \beta_{jk}) t_i \\ &= \sum_{i=1}^{n} (K \delta_{ik}) t_i \\ &= K t_k = 0 \quad \text{which implies } t_k = 0.\end{aligned}$$

We have thus proved the following :

Theorem 4 - 10 If T is a p - group and F is torsion free, then

$T\otimes F \cong \sum_{m}{}' T$, where m denotes the rank of the factor group F/pF

§ 3 Let us now consider the case when the non-torsion factor is an arbitrary mixed group M. We may assume the torsion group T is a p - group, and by the following lemma it is sufficient to assume the torsion subgroup of M is a p - group for the same prime p.

Lemma 4 - 11. Let A_0 and B_0 be subgroups of A and B respectively , then $\frac{A}{A_0}\otimes\frac{B}{B_0} \cong \frac{A\otimes B}{\Gamma(A_0,B_0)}$ where $\Gamma(A_0,B_0)\subseteq A\otimes B$ is generated by $\{m\otimes n \mid m\in A_0 \text{ or } n\in B_0\}$

Proof : Let $\eta : A\otimes B \to \frac{A\otimes B}{\Gamma(A_0,B_0)}$ be the natural homomorphism.

Then $(a,b)f = (a\otimes b)\eta$ is a bilinear map vanishing whenever $a\in A_0$ or $b\in B_0$; this means f depends only on $\frac{A}{A_0}$ and $\frac{B}{B_0}$,

i.e. $f : \frac{A}{A_0}\times\frac{B}{B_0} \to \frac{A\otimes B}{\Gamma(A_0,B_0)}$ is bilinear , clearly

$\{(a\otimes b)\eta \mid a\in\bar a\in\frac{A}{A_0} , b\in\bar b\in\frac{B}{B_0}\}$ generates $\frac{A\otimes B}{\Gamma(A_0,B_0)}$ and thus there exists a homomorphism h from $\frac{A}{A_0}\otimes\frac{B}{B_0}$ onto $\frac{A\otimes B}{\Gamma(A_0,B_0)}$:

$$\begin{tikzcd} & \frac{A\otimes B}{\Gamma(A_0,B_0)} \\ \frac{A}{A_0}\times\frac{B}{B_0} \arrow[ur, "f"] \arrow[r, "\bar\Psi"'] & \frac{A}{A_0}\otimes\frac{B}{B_0} \arrow[u, dashed, "h"'] \end{tikzcd}$$

i.e. $(a\otimes b)\eta = (\bar a,\bar b)\bar\Psi h$.

Now $(\sum_i(\bar a_i,\bar b_i)\bar\Psi)h = 0$ only if $\sum_i a_i\otimes b_i \in \Gamma(A_0,B_0)$ i.e. $\sum_i a_i\otimes b_i$ belongs to the subgroup of $A\otimes B$ generated by all $a\otimes b$ where either $\bar a = 0$ or $\bar b = 0$ N.B. For $x\in A$ $\bar x$ denotes the coset $x + A_0$, then certainly $\sum_i(\bar a_i,\bar b_i)\bar\Psi = 0$ and h is indeed an isomorphism q. e. d.

This lemma shows, using $A = T$, $A_0 = \{0\}$, $B = M$ and $B_0 =$ (sum of q - subgroups of M for all primes $q \neq p$) , that $T \otimes M/B_0 \cong T \otimes M$, since $\Gamma(A_0, B_0) = \{0\}$, and M/B_0 has only a p - group as torsion subgroup.

We now state and prove the main theorem of this section :

Theorem 4 - 12 Let T be a p - group and M a mixed group whose torsion subgroup M_0 is a p - group. Let B be a basic subgroup of T which is represented by $\sum_{i=1}^{\infty}{}^{\bullet} \sum_{m_i}^{\bullet} C(p^i)$ where $C(p^i)$ are prime power order cyclic groups. Then $T \otimes M \cong \sum_{i=1}^{\infty}{}^{\bullet} \sum_{m_i}^{\bullet} T/p^i T \oplus \sum_{r}^{\bullet} T$,where r denotes the rank of $\frac{M/M_0}{p(M/M_0)}$

Proof : Let $\{a_\lambda\}_{\lambda \in \Lambda}$ be a basis of the basic subgroup B and $\{\bar{x}_\mu\}_{\mu \in N}$ be a maximal independent set in $\frac{M/M_0}{p(M/M_0)}$. Then $\{a_\lambda\}_{\lambda \in \Lambda}$ is a basis of $T/p^k T$ for every $k \geq 1$, since T/B divisible implies $t = p^k t_k + b$ for any $k \in Z$ and some $b \in B$; also $\{\bar{x}_\mu\}_{\mu \in N}$ is a basis of $\frac{M/M_0}{p^k(M/M_0)}$ for every $k \geq 1$ as we know from § 2. Hence if $x_\mu \in \bar{x}_\mu$ is an arbitrary element for all $\mu \in N$, then the set $\{a_\lambda, x_\mu\}_{\lambda \in \Lambda, \mu \in N}$ forms a basis for $M/p^k M$ for all $k \geq 1$. Again, as in § 2 , we need only consider elements of M of finite height k , so if $v \in M$ is such an element there exists the equation :

$v = \sum_{\lambda=1}^{s} m_\lambda a_\lambda + \sum_{\mu=1}^{t} n_\mu x_\mu + p^\ell v'$, where $m_\lambda, n_\mu \in Z$ and $\ell > k$ is arbitrary in Z, and $v' \in M$. Choosing ℓ appropriately large , we have :

$$t \otimes v = \sum t \otimes m_\lambda a_\lambda + \sum t \otimes n_\mu x_\mu + t \otimes p^\ell v'$$
$$= \sum t_\lambda \otimes a_\lambda + \sum t_\mu \otimes x_\mu$$

Thus again an arbitrary element of $T \otimes M$ may be written as $\sum t_\lambda \otimes a_\lambda + \sum t_\mu \otimes x_\mu$ since addition may be carried out on the t_λ and t_μ , using the bilinearity of the tensor product. We must show that such an element of $T \otimes M$ may be written as an element of a direct sum, i.e. if $\sum_{\lambda=1}^{n} t_\lambda \otimes a_\lambda + \sum_{\mu=1}^{m} t_\mu \otimes x_\mu = 0$ then each summand vanishes also. But any finite direct summand of B , $\langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$, is a direct summand also of M since

$B \subseteq M$, say $M = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle \oplus M'$; by the choice of the $\{a_\lambda\}$ and $\{x_\mu\}$, M' may be assumed to contain $x_1, \cdots, x_n$. Therefore by Lemma 3 - 2 each $t_\lambda \otimes a_\lambda = 0$ and from §2 $t_\mu \otimes x_\mu = 0$ and thus $t_\mu = 0$ for each μ. The above shows that $T \otimes M \cong \sum_{\lambda \in \Lambda}^{\bullet} (T \otimes \langle a_\lambda \rangle) \oplus \sum_{r}^{\bullet} T$. We give the following lemma which completes the proof of Theorem 4 - 12, since the $\langle a_\lambda \rangle$ are finite cyclic groups :

<u>Lemma 4 - 13</u> If $C(n)$ represents a cyclic group of order n and V is any group, then $C(n) \otimes V \cong V/nV$.

<u>Proof :</u> Consider the diagram

$$\begin{tikzcd} & \frac{V}{nV} \\ C(n) \times V \arrow[ur, "f"] \arrow[r, "\bar{\Psi}"'] & C(n) \otimes V \arrow[u, dashed, "h"'] \end{tikzcd}$$

Let $C(n) = \langle a \rangle$

where f is defined as follows : $(ka, v)f = kv + nV$ for $k \in Z$, $v \in V$. Then f is clearly bilinear so that the map h exists and is a homomorphism satisfying $f = \bar{\Psi} h$. h is onto all of V/nV, for if $v + nV$ is a general element of V/nV, then $(a \otimes v)h = v + nV$; h is one to one for if $\sum_{i=1}^{n} k_i a \otimes v_i$ represents an element of $C(n) \otimes V$ such that $(\sum_1 k_i a \otimes v_i)h = 0$, then $(a \otimes v')h = 0$ (where $v' = \sum_1 k_i v_i \in V$) which implies $(a, v')\bar{\Psi} h = 0 = (a, v')f = v' + nV$; this in turn implies $v' \in nV$, and thus $a \otimes v' = 0$ and the isomorphism is established.

We can now state a corollary of Theorem 4 - 12 based on the information derived in §1 and §2 :

Corollary 4 - 14 If T is a torsion group and M a mixed group whose torsion subgroup is denoted by M_0, then $T \otimes M \cong T \otimes \frac{M}{M_0} \oplus T \otimes M_0$

§ 4 In the general case, when we consider the tensor product of two arbitrary groups, very little can be said. We can, however, determine the structure of the torsion subgroup of the tensor product of two arbitrary groups M and N with the knowledge of Lemma 4 - 11. Let M_0 and N_0 be the torsion subgroups of M and N respectively. Now the fact that the subgroup $\Gamma(M_0, N_0)$ in $M \otimes N$ is a torsion subgroup is clear ; the following lemma will show that $\Gamma(M_0, N_0)$ is the maximal torsion subgroup of $M \otimes N$:

Lemma 4 - 15 $A \otimes B$ is torsion free if both A and B are.

Proof : The proof is clear when it is noted that if a generator $a \otimes b = 0$, $a \in A$, $b \in B$, in $A \otimes B$ then $a \otimes b = 0$ also in $A_1 \otimes B_1$ where A_1 and B_1 are finitely generated ; the decomposition of finitely generated torsion free groups into direct sums of infinite cyclic groups and Lemma 3 - 9 are used for the final result.

We note next that M_0 and N_0 are pure subgroups of M and N and thus, as in Theorem 4 - 5, $M_0 \otimes N$ and $M \otimes N_0$ form subgroups of $M \otimes N$; they clearly generate $\Gamma(M_0, N_0)$. By the decomposition of M_0 and N_0 into their p-summands we also see that the p-component of $\Gamma(M_0, N_0)$ is generated by the subgroups $M_{0_p} \otimes N$ and $M \otimes N_{0_p}$ of $M \otimes N$. It suffices to consider the case, then, when the torsion subgroups of M and N are p-groups for the same prime p. The theorem is as follows :

Theorem 4 - 16 Let M_0 and N_0, the torsion subgroups of two arbitrary groups M and N, be p-groups for the same prime p ; let $B = \sum_{i=1}^{\infty}{}' \sum_{m_i}{}' C(p^i)$

be a basic subgroup of N_0 and denote the ranks of $\dfrac{N/N_0}{p(N/N_0)}$ and $\dfrac{M/M_0}{p(M/M_0)}$ by α and β respectively. Then the maximal torsion subgroup of $M \otimes N$ is isomorphic to $\sum_{i=1}^{\infty}{}' \sum_{m_i}{}' M_0/p^i M_0 \oplus \sum_{\alpha}{}' M_0 \oplus \sum_{\beta}{}' N_0$

Proof : The proof parallels that of Theorem 4 - 12 so will not be given here. It will be noted that the torsion subgroup of $M \otimes N$ is generated by tensor products of the exact type dealt with in § 3.

This theorem obviously tells nothing of the tensor product of two arbitrary torsion free groups , and, indeed, very little of the nature of this type of tensor product is known to date.

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