

PERIODIC ORBITS OF THE SECOND GENUS
FOR THE
CROSSED ORBIT PROBLEM OF THE HELIUM ATOM

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I. PRELIMINARY.

§1. Introduction.

It is proposed here to construct the second genus orbits for a special case of the problem discussed by Dr. Buchanan in his paper¹⁾ "Crossed Orbits in the Restricted Problem of Three Bodies with Repulsive and Attractive Forces." The case dealt with is that designated in the latter as Part II, Case I.

The problem considered deals with the motion of two infinitesimal bodies which are attracted by a finite body but repelled by each other, the nature of the forces involved being Newtonian (i.e., obeying the inverse square law). For simplicity, the two infinitesimal bodies will be called "electrons" and the finite body the "nucleus".

A particular solution of the problem is that in which the electrons revolve in circles with the nucleus as centre and remain diametrically opposite. Two types of orbit are obtained when the electrons are displaced from their circular motion. In part I the electrons remain diametrically opposite and equidistant from the nucleus. In part II the distances of the electrons from the nucleus are equal, their longitudes differ by 180° , but their latitudes are the same. The particular case which is common to parts I and II — that in

which the latitudes are zero — is considered in part II and it is designated as case I. It is in the vicinity of these orbits that we shall make our construction of the second genus orbits.

In sections 2 and 3 a brief outline of the results obtained in "Crossed Orbits...." will be made, showing the method used in constructing the first genus orbits upon which the work of this thesis is to be based.

§2. The Differential Equations.

Taking a rectangular system of axes with origin at the nucleus and designating the coordinates of the electrons by x_1, y_1, z_1 and x_2, y_2, z_2 , we have for the force function U , of the system:

$$U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{k^2}{r_{12}} \quad (1)$$

where

k^2 = ratio of repulsion to attraction,

$$r_1 = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}}$$

$$r_2 = (x_2^2 + y_2^2 + z_2^2)^{\frac{1}{2}}$$

$$r_{12} = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$$

and the units of space and time have been so chosen that the gravitational constant equals unity.

The equations of the motion are thus

$$x_i'' = \left(\frac{\partial U}{\partial x_i} \right); \quad y_i'' = \left(\frac{\partial U}{\partial y_i} \right); \quad z_i'' = \left(\frac{\partial U}{\partial z_i} \right), \quad (i=1, 2) \quad (2)$$

From these equations and a consideration of the constraints on the motion of the electrons as outlined in the

introduction, we obtain the two divisions of the problem, viz:

Part I: $x_2 = -x_1,$

$y_2 = -y_1, \quad r_1 = r_2 = \frac{1}{2}r_{12}.$

$z_2 = -z_1.$

Part II: $x_2 = -x_1,$

$y_2 = -y_1, \quad (3)$

$z_2 = z_1.$

As we are not concerned with the development of part I we shall consider it no further.

By virtue of relations (3), we need consider the motion of one electron only. On substituting these relations in equations (2), and transforming the resulting equations and the vis-viva integral to cylindrical coordinates by the substitutions:

$x_1 = r \cos \theta, \quad y_1 = r \sin \theta, \quad z_1 = z$ (4)

we obtain:

$r'' - r\theta'^2 = -\frac{r}{(r^2 + z^2)^{3/2}} + \frac{k^2}{4r^2}$

$r\theta'' + 2r'\theta' = 0$

$z'' = -\frac{z}{(r^2 + z^2)^{3/2}}$ (5)

the integral of (5 b) being $r^2\theta' = c.$

Using this last relation to eliminate θ' from (5),

we get: $r'' = \frac{c^2}{r^3} - \frac{r}{(r^2 + z^2)^{3/2}} + \frac{k^2}{4r^2}$

$z'' = -\frac{z}{(r^2 + z^2)^{3/2}}$ (6)

These equations have the particular solution

$r = \frac{c^2}{\mu}, \quad z = 0$ (7)

where

$\mu = 1 - \frac{k^2}{4}, \quad 0 < \mu < 1$

§3. The Equations of Displacement and their Solutions.

By means of the substitutions

$$r = \frac{c^2}{\mu} (1 + \epsilon \rho)$$

$$z = \frac{c^2}{\mu} (\epsilon \rho)$$

$$t - t_0 = \frac{c^3}{\mu^2} (1 + \delta)^{1/2} \tau$$

we obtain from equations (6) the "equations of displacement", that is, the equations giving the displacements from the plane orbit given by solutions (7). They are:

$$\begin{aligned} \ddot{\rho} + (1 + \delta) \rho &= (1 + \delta) \left[3\epsilon \left(\rho^2 + \frac{1}{2\mu} \delta^2 \right) - 6\epsilon^2 \left(\rho^3 + \frac{1}{\mu} \rho \delta^2 \right) + \dots \right] \\ \ddot{\zeta} + \frac{1}{\mu} (1 + \delta) \zeta &= \frac{1}{\mu} (1 + \delta) \left[\epsilon (3\rho\zeta) - \epsilon^2 \left(6\rho^2\zeta - \frac{3}{2}\delta^3 \right) + \dots \right] \end{aligned} \quad (9)$$

The solutions of these equations will give the periodic orbits of the first genus of which there are three types, each type being characterized by its period. The periods are determined by a consideration of the equations of variation

of (9): $\ddot{\rho} + \rho = 0 \quad ; \quad \ddot{\zeta} + \frac{1}{\mu} \zeta = 0$

which have the three sets of generating solutions, viz:

Case I: $\rho = A \cos \tau + B \sin \tau$

Period: 2π

Case II: $\zeta = D \cos \frac{1}{\sqrt{\mu}} \tau + E \sin \frac{1}{\sqrt{\mu}} \tau$

Period: $2\pi \sqrt{\mu}$, $\sqrt{\mu}$ irrational.

Case III:

$$\rho = A \cos \tau + B \sin \tau$$

$$\zeta = D \cos \frac{1}{\sqrt{\mu}} \tau + E \sin \frac{1}{\sqrt{\mu}} \tau$$

Period: $T = n_1 (2\pi) = n_2 (2\pi \sqrt{\mu})$.

Corresponding to case I, a solution of equations (9) is found to exist only when $\xi \equiv 0$. This solution forms the "generating solution" for the second genus orbits to be constructed.

The first genus orbits for case I are obtained by setting

$$\rho = \rho_0 + \rho_1 \epsilon + \rho_2 \epsilon^2 + \dots$$

$$\xi \equiv 0$$

$$\delta = \delta_1 \epsilon + \delta_2 \epsilon^2 + \dots$$

where the ρ_j are variables and the δ_j are constants.

On substituting in equations (9) and equating coefficients of ϵ^j on each side of the resulting equation, a series of differential equations of the form :

$$\ddot{\rho}_j + \rho_j = R_j(\rho_i, \delta_j), \quad \begin{cases} i = 1, 2, \dots, j-1 \\ j = 1, 2, \dots \end{cases}$$

will arise, which, when integrated sequentially, determine the various ρ_j and δ_j .

The initial conditions

$$\rho(0) = 1 \quad ; \quad \dot{\rho}(0) = 0$$

serve to evaluate the constants arising from the integrations.

The results are:

$$\begin{aligned} \rho = & \cos \tau + \epsilon \left(\frac{3}{2} - \cos \tau - \frac{1}{2} \cos 2\tau \right) \\ & + \epsilon^2 \left(-3 + \frac{13}{8} \cos \tau + \cos 2\tau + \frac{3}{8} \cos 3\tau \right) + \dots \end{aligned}$$

$$\xi \equiv 0$$

(10)

$$\delta = 0 \cdot \epsilon + 3\epsilon^2 - 6\epsilon^3 + \dots$$

II. THE SECOND GENUS ORBITS.

§4. Definition of Second Genus Orbits.²⁾

Suppose we have a set of differential equations

$$\frac{dx_i}{dt} = X_i(x_j, \varepsilon; t)$$

in which the X_i are analytic in the arguments, do not contain t explicitly, and are periodic with period T . The period is, in general, a function of the parameter ε . If these equations admit the periodic solutions

$$x_i = \theta_i(\varepsilon; t)$$

having the period T , then such solutions are said to be of the first genus.

Now let

$$\varepsilon = \varepsilon_0(1 + \lambda)$$

$$x_i = \theta_i(\varepsilon_0; t) + y_i$$

where ε_0 is considered as a fixed constant and λ as a variable parameter. When these substitutions are made in the above differential equations we obtain a set in y_i in which there are no terms independent of y_i or λ . If this set admits the periodic solutions

$$y_i = \psi_i(\varepsilon_0, \lambda; t)$$

having the period

$$NT(1 + \text{a power series in } \lambda)$$

N being an integer, then the solutions

$$x_i = \theta_i(\varepsilon_0; t) + \psi_i(\varepsilon_0, \lambda; t)$$

are said to be of the second genus. Since the ψ_i vanish with λ , the second genus orbits approach those of the

first genus as λ approaches zero.

§5. The Differential Equations.

In equations (9) make the substitutions

$$\begin{aligned}\varepsilon &= \varepsilon_0(1+\lambda), & \text{where } \varepsilon \text{ occurs explicitly,} \\ \rho &= \rho_0 + p \\ \mathcal{F} &= \mathcal{F}_0 + \mathcal{F} \\ (1+\delta) &= (1+\delta_0)(1+\gamma)\end{aligned}$$

where the zero subscript is attached to ρ , \mathcal{F} , δ , ε merely to indicate that they are the original values of these quantities as given by equation (10). (Note: there are no solutions for second genus orbits in the plane.)

We thus obtain the equations:

$$\begin{aligned}& p'' + (1+\delta_0)p \left[1 - 6\rho_0\varepsilon_0 + 18\rho_0^2\varepsilon_0^2 - 40\rho_0^3\varepsilon_0^3 \dots \right] \\&= (1+\delta_0) \left[\{ 3\rho_0^2\varepsilon_0\lambda - 6\rho_0^3\varepsilon_0^2(2\lambda+\lambda^2) + 10\rho_0^4\varepsilon_0^3(3\lambda+3\lambda^2+\lambda^3) \} \right. \\&\quad + p \{ 6\rho_0\varepsilon_0\lambda - 36\rho_0^2\varepsilon_0^2\lambda - 18\rho_0^2\varepsilon_0^2\lambda^2 + 120\rho_0^3\varepsilon_0^3\lambda + 120\rho_0^3\varepsilon_0^3\lambda^2 + \dots \} \\&\quad + p^2 \{ 3\varepsilon_0(1+\lambda) - 18\rho_0\varepsilon_0^2(1+\lambda)^2 + 60\rho_0^2\varepsilon_0^3(1+\lambda)^3 \} \\&\quad \left. + \mathcal{F}^2 \left\{ \frac{3}{2\mu}\varepsilon_0(1+\lambda) - \frac{6}{\mu}\rho_0\varepsilon_0^2(1+\lambda)^2 + \frac{15}{\mu}\rho_0^2\varepsilon_0^3(1+\lambda)^3 \right\} \dots \right] \\&+ \gamma(1+\delta_0) \left[\{ -\rho_0 + 3\rho_0^2\varepsilon_0(1+\lambda) - 6\rho_0^3\varepsilon_0^2(1+\lambda)^2 + 10\rho_0^4\varepsilon_0^3(1+\lambda)^3 \} \right. \\&\quad + p \{ -1 + 6\rho_0\varepsilon_0(1+\lambda) - 18\rho_0^2\varepsilon_0^2(1+\lambda)^2 + 40\rho_0^3\varepsilon_0^3(1+\lambda)^3 \} \\&\quad + p^2 \{ 3\varepsilon_0(1+\lambda) - 18\rho_0\varepsilon_0^2(1+\lambda)^2 + 60\rho_0^2\varepsilon_0^3(1+\lambda)^3 \} \\&\quad \left. + \mathcal{F}^2 \left\{ \frac{3}{2\mu}\varepsilon_0(1+\lambda) - \frac{6}{\mu}\rho_0\varepsilon_0^2(1+\lambda)^2 + \frac{15}{\mu}\rho_0^2\varepsilon_0^3(1+\lambda)^3 \right\} \dots \right] \quad (11) \\& \ddot{q} + \frac{1}{\mu}(1+\delta_0)q \left[1 - 3\rho_0\varepsilon + 6\rho_0^2\varepsilon^2 - 10\rho_0^3\varepsilon^3 \dots \right] \\&= \frac{1}{\mu}(1+\delta_0) \left[q \{ 3\rho_0\varepsilon_0\lambda - 6\rho_0^2\varepsilon_0^2(2\lambda+\lambda^2) + \dots \} \right. \\&\quad + p q \{ 3\varepsilon_0(1+\lambda) - 12\rho_0\varepsilon_0^2(1+\lambda)^2 + 30\rho_0^2\varepsilon_0^3(1+\lambda)^3 \} \dots \left. \right] \\&+ \frac{1}{\mu}\gamma(1+\delta_0) \left[q \{ -1 + 3\rho_0\varepsilon_0(1+\lambda) - 6\rho_0^2\varepsilon_0^2(1+\lambda)^2 + 10\rho_0^3\varepsilon_0^3(1+\lambda)^3 \} \right. \\&\quad \left. + p q \{ 3\varepsilon_0(1+\lambda) - 12\rho_0\varepsilon_0^2(1+\lambda)^2 + 30\rho_0^2\varepsilon_0^3(1+\lambda)^3 \} \dots \right]\end{aligned}$$

the terms independent of p and q having dropped out by reason of equations (9).

§6. The Equations of Variation and Their Solutions.

The equations of variation of (11) are obtained by equating to zero the left side:

$$\begin{aligned} p'' + p \left[1 + \varepsilon(-6 \cos \tau) + \varepsilon^2(3 + 6 \cos \tau + 12 \cos 2\tau) + \dots \right] \\ \ddot{q} + \frac{1}{\mu} q \left[1 + \varepsilon(-3 \cos \tau) + \varepsilon^2\left(\frac{3}{2} + 3 \cos \tau + \frac{9}{2} \cos 2\tau\right) + \dots \right] \end{aligned} \quad (12)$$

Since these two equations are independent, their solutions will be considered separately.

Consider first, equation (12,a). By the theory of linear differential equations with periodic coefficients,³⁾ we may get particular solutions to this equation by differentiating partially its generating solution (10,a) with respect to the arbitrary constants which are contained in the latter but which do not appear in the original differential equation. In this case, two such arbitrariness occur: t_0 and ε .

The two particular solutions are

$$\frac{\partial}{\partial t_0} (\varepsilon p) \quad \text{and} \quad \frac{\partial}{\partial \varepsilon} (\varepsilon p)$$

Now, since

$$p(\tau) \equiv p \left[\frac{\mu^2}{c^3} (1 + \delta)^{-\frac{1}{2}} (t - t_0) \right]$$

then

$$\frac{\partial}{\partial t_0} (\varepsilon p) = \frac{\partial}{\partial \tau} (\varepsilon p) \cdot \frac{\partial}{\partial t_0} \left[\frac{\mu^2}{c^3} (1 + \delta)^{-\frac{1}{2}} (t - t_0) \right] = (\text{Constant}) \times \frac{\partial}{\partial \tau} (\varepsilon p)$$

Also,

$$\frac{\partial}{\partial \varepsilon} (\varepsilon p) = -\sin \tau \cdot \varepsilon + (\sin \tau + \sin 2\tau) \cdot \varepsilon^2 - \left(\frac{13}{8} \sin \tau + 2 \sin 2\tau + \frac{9}{8} \sin 3\tau \right) \cdot \varepsilon^3 + \dots$$

We therefore obtain

$$\phi(\tau) = (\text{constant}) \cdot \left[-\sin \tau + (\sin \tau + \sin 2\tau) \cdot \epsilon - \left(\frac{13}{8} \sin \tau + 2 \sin 2\tau + \frac{9}{8} \sin 3\tau \right) \cdot \epsilon^2 + \dots \right]$$

This being a particular solution, we may disregard the constant factor, since $\phi(\tau)$ will be multiplied by an undetermined constant later, and take

$$\phi(\tau) = \left[-\sin \tau + (\sin \tau + \sin 2\tau) \cdot \epsilon - \left(\frac{13}{8} \sin \tau + 2 \sin 2\tau + \frac{9}{8} \sin 3\tau \right) \cdot \epsilon^2 + \dots \right] \quad (13.1)$$

as our particular solution corresponding to $\frac{\partial}{\partial t_0}(\epsilon \rho)$.

The other particular solution of (12,a) will consist of a periodic function plus τ times another periodic function. For, since ϵ enters into ρ both explicitly and implicitly (in τ by δ), we have

$$\frac{\partial}{\partial \epsilon}(\epsilon \rho) = \left[\frac{\partial}{\partial \epsilon}(\epsilon \rho) \right] + \frac{\partial}{\partial \tau}(\epsilon \rho) \cdot \frac{\partial \tau}{\partial \delta} \cdot \frac{\partial \delta}{\partial \epsilon}$$

where the brackets enclosing $\frac{\partial}{\partial \epsilon}(\epsilon \rho)$ denote explicit differentiation only.

On performing the indicated differentiations, this solution becomes:

$$\psi(\tau) + A \tau \cdot \phi(\tau) \quad (13.2)$$

where $\psi(\tau)$ is given by

$$\begin{aligned} \psi(\tau) = \left[\frac{\partial}{\partial \epsilon}(\epsilon \rho) \right] = & \cos \tau + \epsilon (3 - 2 \cos \tau - \cos 2\tau) \\ & + \epsilon^2 \left(-9 + \frac{39}{8} \cos \tau + 3 \cos 2\tau + \frac{9}{8} \cos 3\tau \right) \\ & + \dots \end{aligned} \quad (13.3)$$

and τ appears in the differentiation

$$\frac{\partial \tau}{\partial \delta} = \frac{\mu^2}{c^3} (t - t_0) \cdot -\frac{1}{2} (1 + \delta)^{-3/2} = (\text{constant}) \times \tau$$

In order that (13.1) and (13.2) may constitute a fundamental set of solutions ⁴⁾ of equation (12,a), that is, in order that the general solution of (12,a) shall be

$$K_1 \phi(\tau) + K_2 [\psi(\tau) + A\tau \cdot \phi(\tau)]$$

K_1 and K_2 being arbitrary constants, the fundamental determinant of the set must not be zero.

This determinant is:

$$D = \begin{vmatrix} \phi(\tau) & \psi(\tau) + A\tau \cdot \phi(\tau) \\ \dot{\phi}(\tau) & \dot{\psi}(\tau) + A\phi(\tau) + A\tau \cdot \dot{\phi}(\tau) \end{vmatrix}$$

Now it has been shown ⁵⁾ that the value of the determinant of such a fundamental set is constant. We can therefore compute the value of D most conveniently by setting τ equal to zero.

Thus

$$D = \begin{vmatrix} \phi(0) & \psi(0) \\ \dot{\phi}(0) & \dot{\psi}(0) + A\phi(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 + \dots \\ -1 + \dots & 0 \end{vmatrix} = 1 + \text{a power series in } \epsilon.$$

It therefore follows that the most general solution of equation (12,a) is

$$p = K_1 \phi(\tau) + K_2 [\psi(\tau) + A\tau \cdot \phi(\tau)] \quad (13.4)$$

ϕ and ψ being, of course, periodic (2π).

The above method cannot be used in the solution of equation (12,b) because of the absence of a generating solution,

or rather, because its generating solution is identically zero. This equation, a form of Hill's equation,⁶⁾ will be solved by making the standard transformation

$$q = e^{i\beta\tau} \cdot v \quad (14.1)$$

in which

$$\beta = \frac{1}{\sqrt{\mu}} + \beta_1 \varepsilon + \beta_2 \varepsilon^2 + \dots$$

$$v = v_0 + v_1 \varepsilon + v_2 \varepsilon^2 + \dots$$

The various β_i and v_i will be determined so as to fulfill the following two conditions:

(i) The periodicity condition

$$v_i(\tau) - v_i(\tau + 2\pi) = 0 \quad (14.2)$$

(ii) The initial-value condition:

$$v_0(0) = 1 ; v_i(0) = 0, \quad i = 1, 2, \dots$$

The substitution of (14.1) in equation (12,b) yields:

$$\ddot{v} + 2i\beta\dot{v} - \beta^2 v + \frac{1}{\mu} v \left[1 + \varepsilon(-3\cos\tau) + \varepsilon^2\left(\frac{3}{2} + 3\cos\tau + \frac{9}{2}\cos 2\tau\right) + \dots \right]$$

or

$$\begin{aligned} & (\ddot{v}_0 + \ddot{v}_1 \varepsilon + \ddot{v}_2 \varepsilon^2 + \dots) + 2i(\dot{v}_0 + \dot{v}_1 \varepsilon + \dot{v}_2 \varepsilon^2 + \dots) \left(\frac{1}{\sqrt{\mu}} + \beta_1 \varepsilon + \beta_2 \varepsilon^2 + \dots \right) \\ & - (v_0 + v_1 \varepsilon + v_2 \varepsilon^2 + \dots) \left[\frac{1}{\mu} - \frac{3}{\mu} \cos\tau \cdot \varepsilon \right. \\ & \quad \left. + \frac{1}{\mu} \left(\frac{3}{2} + 3\cos\tau + \frac{9}{2} \cos 2\tau \right) \cdot \varepsilon^2 + \dots \right] \equiv 0 \end{aligned}$$

As before, we equate to zero the coefficients of ε^j .

Each such equation will have for its complementary function the solution of

$$V_j \equiv \ddot{v}_j + 2i \cdot \frac{1}{\sqrt{\mu}} \dot{v}_j = 0$$

Coefficient of ε^0 : $\ddot{v}_0 + 2i \cdot \frac{1}{\sqrt{\mu}} \dot{v}_0 = 0$

whence

$$v_0 = C_1 + C_2 e^{-\frac{2}{\sqrt{\mu}} i \tau}$$

Applying conditions (14.2), we get

$$C_1 + C_2 = 1, \quad C_2 \left(1 - e^{-\frac{4\pi}{\sqrt{\mu}} i}\right) = 0$$

Since $\frac{1}{\sqrt{\mu}}$, in general, is not rational, we must have

$$C_2 = 0.$$

Therefore

$$v_0 = 1$$

Coefficient of ε .

$$\ddot{v}_1 + 2i \left(\frac{1}{\sqrt{\mu}} \dot{v}_1 + \beta_1 \dot{v}_0 \right) - \left(\frac{2}{\sqrt{\mu}} v_0 \beta_1 \right) + \left(-\frac{3}{\mu} \cos \tau \cdot v_0 \right) = 0$$

or

$$\ddot{v}_1 + \frac{2}{\sqrt{\mu}} i \cdot \dot{v}_1 = \frac{2}{\sqrt{\mu}} \beta_1 + \frac{3}{2\mu} (e^{i\tau} + e^{-i\tau})$$

Now in order to introduce no terms proportional to τ into our solution, it is evident that the constant term on the right side of this equation must vanish. That is,

$$\beta_1 = 0$$

Integration gives

$$v_1 = C_2 + C_3 e^{-\frac{2}{\sqrt{\mu}} i \tau} + \frac{3}{2\sqrt{\mu}} \left[\frac{1}{2-\sqrt{\mu}} e^{-i\tau} - \frac{1}{2+\sqrt{\mu}} e^{i\tau} \right]$$

and applying initial and periodic conditions we obtain

$$C_3 \left(1 - e^{-\frac{4\pi}{\sqrt{\mu}} i}\right) = 0$$

whence, $C_3 = 0$, and $C_2 = -\frac{3}{4-\mu}$.

Therefore

$$v_1 = \frac{3}{\mu-4} + \frac{3}{2\sqrt{\mu}} \left[\frac{1}{2-\sqrt{\mu}} e^{-i\tau} - \frac{1}{2+\sqrt{\mu}} e^{i\tau} \right]$$

Coefficient of ε^2 .

$$\ddot{v}_2 + 2i(\dot{v}_0 \beta_2 + \dot{v}_1 \beta_1 + \frac{1}{\sqrt{\mu}} \dot{v}_2) - \left(\frac{2}{\sqrt{\mu}} \beta_2 v_0 + v_0 \beta_1^2 + \frac{2}{\sqrt{\mu}} \beta_1 v_1 \right) + \frac{1}{\mu} v_0 \left(\frac{3}{2} + 3 \cos 2\tau + \frac{9}{2} \cos 2\tau \right) - \frac{3}{\mu} v_1 = 0$$

Substitution for β_1, v_0, v_1 yields

$$\ddot{v}_2 + \frac{2}{\sqrt{\mu}} i \dot{v}_2 = -\frac{1}{\mu} \left[\frac{3}{2} + \frac{3}{2} e^{i\tau} + \frac{3}{2} e^{-i\tau} + \frac{9}{4} e^{2i\tau} + \frac{9}{4} e^{-2i\tau} \right] - \frac{3}{\mu} \left[\frac{3}{\mu-4} + \frac{3}{2\sqrt{\mu}} \left(\frac{1}{2-\sqrt{\mu}} e^{-i\tau} - \frac{1}{2+\sqrt{\mu}} e^{i\tau} \right) \right]$$

To ensure that v_2 shall have no non-periodic terms, we cause the constant terms on the right side of this last equation to vanish. That is,

$$\frac{2}{\sqrt{\mu}} \beta_2 - \frac{3}{2\mu} - \frac{9}{\mu(\mu-4)} = 0$$

whence

$$\beta_2 = \frac{3\sqrt{\mu}(\mu+2)}{4\mu(\mu-4)}$$

Then

$$v_2 = C_3 + C_4 e^{-\frac{2}{\sqrt{\mu}} i\tau} + \frac{3(\sqrt{\mu}+3)(\sqrt{\mu}-1)}{2\mu(2+\sqrt{\mu})^2} e^{i\tau} + \frac{3(\sqrt{\mu}-3)(\sqrt{\mu}+1)}{2\mu(2-\sqrt{\mu})^2} e^{-i\tau} + \frac{9\sqrt{\mu}}{16\mu(1+\sqrt{\mu})} e^{2i\tau} - \frac{9\sqrt{\mu}}{16\mu(1-\sqrt{\mu})} e^{-2i\tau}$$

As before, it can be shown that $C_4 = 0$, and that C_3 has the value

$$C_3 = \frac{33\mu^3 - 264\mu^2 + 264\mu + 288}{8\mu(1-\mu)(4-\mu)^2}$$

In similar fashion we can find as many more β_i 's and v_j 's

as will determine β and v to any desired degree of approximation.

Having now obtained the particular solution q of (14.1), the other particular solution is obtained by replacing i by $-i$ in v and $e^{i\beta\tau}$, that is, the conjugate of q is also a particular solution ⁷⁾.

Thus, the general solution of equation (12,b) is

$$q = K_3 e^{i\beta\tau} v^{(1)} + K_4 e^{-i\beta\tau} v^{(2)}$$

where

$$\begin{aligned} \beta &= \frac{1}{\sqrt{\mu}} + 0.\varepsilon + \frac{3\sqrt{\mu}(\mu+2)}{4\mu(\mu-4)} \varepsilon^2 + \dots \\ v^{(1)} &= 1 + \frac{3}{\mu-4} \left\{ (1-\cos\tau) + \frac{2}{\sqrt{\mu}} \cdot i \sin\tau \right\} \cdot \varepsilon \\ &\quad + \left\{ C_3 + \frac{3(\mu^2-7\mu-12)}{\mu(4-\mu)^2} \cos\tau - \frac{9}{8(1-\mu)} \cos 2\tau \right. \\ &\quad \left. + \frac{6\sqrt{\mu}(10-\mu)}{\mu(4-\mu)^2} \cdot i \sin\tau + \frac{9\sqrt{\mu}}{8\mu(1-\mu)} \cdot i \sin 2\tau \right\} \cdot \varepsilon^2 \\ &\quad + \dots \end{aligned}$$

and $v^{(2)}$ differs from $v^{(1)}$ only in the sign of i .

If the signs of both τ and i are changed in (14.4), q does not change. Therefore, because of the parity of $\cos \tau$ and $\sin \tau$, it must be evident that the coefficients of the cosine terms in $v^{(1)}$ are always real and those of the sine terms always purely imaginary.

The fundamental determinant of this set of solutions will be computed for later use. It is :

$$\Delta = \begin{vmatrix} e^{i\beta\tau} v^{(1)} & e^{-i\beta\tau} v^{(2)} \\ e^{i\beta\tau} (i\beta v^{(1)} + \dot{v}^{(1)}) & e^{-i\beta\tau} (-i\beta v^{(2)} + \dot{v}^{(2)}) \end{vmatrix} \begin{vmatrix} v^{(1)} & v^{(2)} \\ (i\beta v^{(1)} + \dot{v}^{(1)}) & (-i\beta v^{(2)} + \dot{v}^{(2)}) \end{vmatrix}$$

Δ being constant in value, its evaluation will be most easily effected by setting $\tau = 0$.

Now $v^{(1)}(0) = v^{(2)}(0) = 1$, by equation (14.2), ii, and $\dot{v}^{(1)}(0) = -\dot{v}^{(2)}(0) = \frac{2}{\sqrt{\mu}} \cdot i (\epsilon + \epsilon^3 + \epsilon^5 + \dots)$ by equation (14.4).

Hence

$$\Delta = -\frac{2}{\sqrt{\mu}} \cdot i (1 + \text{a power series in } \epsilon) \quad (14.5)$$

This completes the solutions of the equations of variation (12). These solutions are the complementary functions of equations (11) whose particular integrals are next to be determined as power series in λ .

§7. Notation

As the algebraic expressions become too unwieldy and so obscure the methods of attaining certain results in the subsequent constructions, we shall employ the "foundation-letter" notation of Dr. Buchanan⁸⁾.

The notation consists of symbols of the form

$$C(,) \quad , \quad S(,)$$

and represents power series in λ having for their coefficients sums of cosines or sines respectively. Of the two parentheses

in the superscripts, the first has two entries: an integer 0, 1, 2, ..., followed by the letter e or o. The integer designates both the lowest power of ξ in the series and the parity in ξ , while the letter e or o denotes that the arguments of the cosines or sines are even or odd multiples of τ respectively. The second parenthesis contains an integer which denotes the amount by which the highest multiple of τ in the arguments of the trigonometric terms occurring in the coefficient of any power of ξ exceeds that power.

In the following work we shall use the modified notation obtained by deleting the letters e and o in the first parenthesis since in our case the arguments of the sines and cosines are neither exclusively even nor exclusively odd multiples of τ .

We may have occasion also to adjoin a subscript to a given foundation letter, in which case we understand that we are considering a particular series of that type.

An Example.

$$\begin{aligned} C^{(1)(0)} &= \xi (C_0^{(1)} + C_1^{(1)} \cos \tau) + \xi^2 (C_0^{(2)} + C_1^{(2)} \cos \tau + C_2^{(2)} \cos 2\tau) + \\ &\quad \dots + \xi^j \cdot \sum_{K=0}^j C_K^{(j)} \cos K\tau + \dots \\ &= \sum_{j=1}^{\infty} \sum_{K=0}^j C_K^{(j)} \cos K\tau \cdot \xi^j. \end{aligned}$$

For future reference, we may note that in this notation,

$$\phi(\tau) = S^{(0)(1)}, \quad \psi(\tau) = C^{(0)(1)}.$$

§8. The Period of The Orbits of the Second Genus.

For the first genus orbits, the period is 2π in τ which is $2\pi \cdot \frac{c^3}{\mu^2} (1+\delta)^{\frac{1}{2}} = T_1$ in t .

The period of the second genus orbits is to be, by definition,

$$N \cdot T_1 (1 + \text{a power series in } \lambda)$$

N being integral. Now the period of the solution q is $\frac{2\pi}{\beta}$.

Therefore in order that our final solutions be periodic (2π) in τ , we must have the period

$$\begin{aligned} T_2 &= \frac{2\pi}{\beta} = n \cdot 2\pi && \text{in } \tau \\ &= n \cdot \frac{c^3}{\mu^2} (1+\delta)^{\frac{1}{2}} \cdot 2\pi && \text{in } t \\ &= n \cdot \frac{c^3}{\mu^2} (1+\epsilon^2 (\dots) \dots) \cdot 2\pi = n \cdot \frac{c^3}{\mu^2} \{ 1 + \epsilon_0^2 (1+\lambda)^2 + \dots \} \cdot 2\pi \\ &= N \cdot T_1 (1 + \text{a power series in } \lambda) \end{aligned}$$

as required by definition. All values of ϵ for which this does not hold are excluded.

§9. The Solutions of Equation (11).

After substituting for the value of ρ_0 (equation (10)) in equation (11), we arrive at the following:

$$\begin{aligned} \ddot{p} + \rho [P] &= (1+\delta) \left[\{ \lambda P_{01} + \lambda^2 P_{02} + \dots \} + \rho \{ \lambda P_{11} + \lambda^2 P_{12} + \dots \} \right. \\ &\quad \left. + \rho^2 \{ P_{20} + \lambda P_{21} + \dots \} + \rho^2 \{ Q_{20} + \lambda Q_{21} + \dots \} \right] \\ &\quad + \gamma (1+\delta) \left[\{ P_{00} + \lambda P_{01} + \dots \} + \rho \{ P_{100} + \lambda P_{110} + \dots \} \right. \\ &\quad \left. + \rho^2 \{ P_{20} + \lambda P_{21} + \dots \} + \rho^2 \{ Q_{20} + \lambda Q_{21} + \dots \} \right] \quad (15) \\ \ddot{q} + \frac{1}{\mu} q [Q] &= \frac{1}{\mu} (1+\delta) \left[q \{ \lambda Q_{01} + \lambda^2 Q_{02} + \dots \} + \rho q \{ Q_{100} + \lambda Q_{110} + \dots \} \right] \\ &\quad + \frac{1}{\mu} \gamma (1+\delta) \left[q \{ Q_{00} + \lambda Q_{01} + \dots \} + \rho q \{ Q_{100} + \dots \} \right] \end{aligned}$$

where

$$P = 1 + \varepsilon(-6 \cos \tau) + \varepsilon^2(3 + 6 \cos 2\tau + 12 \cos 2\tau) + \dots$$

$$Q = 1 + \varepsilon(-3 \cos \tau) + \varepsilon^2\left(\frac{3}{2} + 3 \cos \tau + \frac{9}{2} \cos 2\tau\right) + \dots$$

$$P_{00} = -\cos \tau + \varepsilon(\cos \tau + 2 \cos 2\tau) + \varepsilon^2(\quad) + \dots$$

$$P_{01} = \varepsilon\left(\frac{3}{2} + \frac{3}{2} \cos 2\tau\right) + \varepsilon^2(\quad) + \dots$$

$$\varepsilon^2(-3 \cos \tau + \cos 3\tau) + \dots$$

$$P_{02} =$$

$$P_{11} = \varepsilon(6 \cos \tau) + \varepsilon^2(-9 - 6 \cos 2\tau - 21 \cos 2\tau) + \dots$$

$$P_{20} = \varepsilon(3) + \varepsilon^2(-18 \cos \tau) + \dots$$

$$P_{100} = -1 + \varepsilon(6 \cos \tau) + \varepsilon^2\left(\frac{17}{2} - 6 \cos \tau - \frac{7}{2} \cos 2\tau\right) + \dots$$

$$Q_{20} = \varepsilon\left(\frac{3}{2\mu}\right) + \varepsilon^2\left(\frac{3}{2\mu} \cdot 4 \cos \tau\right) + \dots$$

$$Q_{11} = \varepsilon(3 \cos \tau) + \varepsilon^2\left(-\frac{3}{2} - 3 \cos \tau - \frac{15}{2} \cos 2\tau\right) + \dots$$

$$Q_{100} = -1 + \varepsilon(3 \cos 2\tau) + \varepsilon^2\left(\frac{3}{2} - 3 \cos \tau - \frac{9}{2} \cos 2\tau\right) + \dots$$

$$Q_{110} = \varepsilon(3) + \varepsilon^2(-12 \cos \tau) + \dots$$

In order to integrate equations (15) as a power series in λ , we put

$$p = \sum_{j=1}^{\infty} p_j \lambda^j, \quad q = \sum_{j=1}^{\infty} q_j \lambda^j, \quad \gamma = \sum_{j=1}^{\infty} \gamma_j \lambda^j.$$

On equating coefficients of like powers of λ in the resulting equation we obtain a series of differential equations in p_j and q_j having for their complementary functions the solutions (13.4) and (14.4) respectively. It will be shown that the γ_j and the constants of integration will be so determined that p_j and q_j will have the period T_2 .

We proceed then as indicated.

Coefficient of λ .

$$\ddot{p}_1 + p_1 [P] = (1+\delta) [P_{01} + \gamma, P_{00}] = P''$$

$$\ddot{q}_1 + \frac{1}{\mu} q_1 [Q] = 0 \quad (16)$$

Knowing the complementary function for the first of these equations, we may solve it completely by the method of variation of parameters. Thus,

$$\begin{aligned} D \dot{k}_1^{(1)} &= \begin{vmatrix} 0 & \psi + A\tau \cdot \phi \\ P'' & \dot{\psi} + A\tau \dot{\phi} + A\phi \end{vmatrix} \\ &= -(1+\delta) (\psi + A\tau \phi) [P_{01} + \gamma, P_{00}] \end{aligned}$$

$$D \dot{k}_2^{(1)} = \begin{vmatrix} \phi & 0 \\ \dot{\phi} & P'' \end{vmatrix} = (1+\delta) \cdot \phi [P_{01} + \gamma, P_{00}] ,$$

where the arbitraries have been assigned the superscripts 1 in order to associate them with p_1 and q_1 . In general $k_1^{(j)}$ and $k_2^{(j)}$ will be associated with the solutions of p_j and q_j .

Here D is the fundamental determinant of § 6 and has the value

$$D = \text{constant} = 1 + \text{a power series in } \epsilon .$$

Employing the foundation-letter notation, these equations may be written:

$$\frac{D}{1+\delta} \dot{k}_1^{(1)} = -[C_3^{(1)(2)} + \gamma_1 C_4^{(0)(2)} + A\tau(S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)})]$$

$$\frac{D}{1+\delta} \dot{k}_2^{(1)} = [S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}],$$

whence, upon integration,

$$\begin{aligned} \frac{D}{1+\delta} k_1^{(1)} = & -[d_1^{(1)}\tau + S_5^{(1)(2)} + \gamma_1(d_2^{(1)}\tau + S_6^{(0)(2)})] \\ & -A[\tau \int (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau - \iint (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau d\tau] \end{aligned}$$

$$\frac{D}{1+\delta} k_2^{(1)} = \int (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau,$$

the d's representing power series in ϵ with constant coefficients.

Substitution of these expressions in the complementary function yields

$$\begin{aligned} \frac{D}{1+\delta} p_1 = & -\phi \left\{ (d_1^{(1)} + \gamma_1 d_2^{(1)}) \cdot \tau + S_5^{(1)(2)} + \gamma_1 S_6^{(0)(2)} \right\} \\ & + A \left\{ \tau \int (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau - \iint (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau d\tau \right\} \\ & + A\tau \cdot \phi \left[\int (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau \right. \\ & \left. + \psi \left[\int (S_1^{(1)(2)} + \gamma_1 S_2^{(0)(2)}) d\tau \right] \right] \end{aligned}$$

On reducing, and noting that the terms in $A\tau$ cancel off,

we have

$$\frac{D}{1+\delta} p_1 = \phi(d_1^{(1)} + \gamma_1 d_2^{(1)}) \cdot \tau + C^{(1)(2)} + \gamma_1 C^{(0)(2)},$$

The complete solution at this stage will now be

$$p_1 = K_1^{(1)} \phi + K_2^{(1)} (\psi + A \tau \phi) + \phi (d_1^{(1)} + \gamma, d_2^{(1)}) \cdot \tau + C^{(1)(3)} + \gamma, C^{(1)(3)} \quad (17)$$

In order that p_1 be periodic, terms containing τ as a factor must be eliminated. That is,

$$A K_2^{(1)} + d_1^{(1)} + \gamma, d_2^{(1)} = 0 \quad (18)$$

Therefore the periodic solutions for p_1 and q_1 are

$$p_1 = K_1^{(1)} \phi + K_2^{(1)} \psi + C^{(1)(3)} + \gamma, C^{(1)(3)}$$

$$q_1 = K_3^{(1)} e^{i\beta\tau} v^{(1)} + K_4^{(1)} e^{-i\beta\tau} v^{(2)}$$

The arbitrary constants $K_j^{(1)}$ may be evaluated by the use of the initial conditions

$$\dot{p}_j(0) = 0, \quad q_j(0) = 0 \quad (19)$$

Applying these conditions to the above equations it is found that

$$K_1^{(1)} = 0; \quad K_3^{(1)} = -K_4^{(1)}$$

The periodic solutions thus become

$$p_1 = K_2^{(1)} \psi + C^{(1)(3)} + \gamma, C^{(1)(3)} \quad (20)$$

$$q_1 = K_3^{(1)} e^{i\beta\tau} v^{(1)} - K_3^{(1)} e^{-i\beta\tau} v^{(2)}$$

In equation (18) we have a linear relation connecting $K_2^{(1)}$ and γ . Another such relationship will be found in the process of integrating p_2 and q_2 which will uniquely determine these two constants and hence further simplify (20).

Coefficient of λ^2 . We obtain the set:

$$\begin{aligned} \ddot{p}_2 + p_2 [P] &= P^{(2)} \\ \ddot{q}_2 + \frac{1}{\mu} p_2 [Q] &= Q^{(2)}, \end{aligned} \quad (21)$$

in which

$$P^{(2)} = (1+\delta) [P_{02} + p_1 P_{11} + p_1^2 P_{20} + q_1^2 Q_{20} + \gamma_1 P_{01} + \gamma_2 P_{00} + \gamma_1 p_1 P_{100}]$$

$$Q^{(2)} = \frac{1}{\mu} (1+\delta) [q_1 Q_{11} + p_1 q_1 Q_{100} + \gamma_1 q_1 Q_{100}]$$

Upon varying the parameters, there result

$$\begin{aligned} D \dot{k}_1^{(2)} &= -(\psi + A\tau\phi) \cdot P^{(2)} \\ D \dot{k}_2^{(2)} &= \phi P^{(2)} \\ \Delta \dot{k}_3^{(2)} &= -e^{-i\beta\tau} v^{(2)} Q^{(2)} \\ \Delta k_4^{(2)} &= e^{i\beta\tau} v^{(1)} Q^{(2)}, \end{aligned} \quad (22)$$

D and Δ being the determinants given in §6.

Consider the last two of these equations. We shall investigate those parts of $Q^{(2)}$ which contain $K_2^{(1)}$ and γ_1 , with the view to obtaining a supplementary relation to (18).

$$\text{Now } v^{(1)} = C_1^{(0)(0)} + i S_1^{(1)(0)}, \quad v^{(2)} = C_1^{(0)(0)} - i S_1^{(1)(0)}$$

$$\begin{aligned} \text{and } Q^{(2)} &= \frac{1}{\mu} (1+\delta) [q_1 Q_{11} + \{K_2^{(1)} \psi + C^{(1)(3)} + \gamma_1 C^{(0)(3)}\} \{K_3^{(1)} e^{i\beta\tau} v^{(1)} - K_3^{(1)} e^{-i\beta\tau} v^{(2)}\} Q_{100} \\ &\quad + \gamma_1 \{K_3^{(1)} e^{i\beta\tau} v^{(1)} - K_3^{(1)} e^{-i\beta\tau} v^{(2)}\} Q_{100}] \end{aligned}$$

The terms of $Q^{(2)}$ in which we are interested are those

containing $e^{i\beta\tau}$ and $e^{-i\beta\tau}$, and are shown in the following table

Coefficient of $\frac{1}{\mu}(1+\delta) e^{i\beta\tau}$	Coefficient of $\frac{1}{\mu}(1+\delta) e^{-i\beta\tau}$
$K_3^{(1)} w^{(1)} [K_2^{(1)} \psi + C^{(1)(3)} + \gamma_1 C^{(0)(3)}] Q_{110}$ $+ K_3^{(1)} w^{(1)} \gamma_1 Q_{100}$ $= K_3^{(1)} [K_2^{(1)} (C^{(1)(0)} + i S^{(2)(0)})$ $+ \gamma_1 (C^{(0)(2)} + i S^{(3)(2)})$ $+ C^{(2)(2)} + i S^{(3)(2)})]$	$-K_3^{(1)} w^{(2)} [K_2^{(1)} \psi + C^{(1)(3)} + \gamma_1 C^{(0)(3)}] Q_{110}$ $- K_3^{(1)} w^{(2)} \gamma_1 Q_{100}$ <i>Conjugate</i>

Our two equations may now be written

$$\Delta \dot{k}_3^{(2)} = -e^{-i\beta\tau} Q_1^{(2)} - \frac{1}{\mu}(1+\delta) w^{(2)} K_3^{(1)} [K_2^{(1)} (C^{(1)(0)} + i S^{(2)(0)})$$

$$+ \gamma_1 (C^{(0)(2)} + i S^{(3)(2)}) + (C^{(2)(2)} + i S^{(3)(2)})]$$

$$\Delta \dot{k}_4^{(2)} = e^{i\beta\tau} Q_2^{(2)} - \frac{1}{\mu}(1+\delta) w^{(1)} K_3^{(1)} [\text{Conjugate}]$$

$Q_1^{(2)}$ and $Q_2^{(2)}$ being those parts of $Q^{(2)}$ not included in the above table.

Removal of constant terms gives

$$K_3^{(1)} [K_2^{(1)} d_0^{(2)} + \gamma_1 d_2^{(2)} + d_1^{(2)}] = 0$$

or

$$K_2^{(1)} d_0^{(2)} + \gamma_1 d_2^{(2)} + d_1^{(2)} = 0$$

since $K_3^{(1)} \neq 0$, otherwise we would have the case for analytical continuation of the first genus orbits. Evidently we must exclude all values of ε for which the determinant

$$\begin{vmatrix} d_0^{(2)} & d_2^{(2)} \\ A & d_2^{(1)} \end{vmatrix} \quad (24)$$

of equations (18) and (23) is equal to zero.

We have, therefore,

$$K_2^{(1)} = d_3^{(1)} \quad \text{and} \quad \gamma_1 = d_4^{(1)}$$

which gives for equations (20),

$$p_1 = d_3^{(1)} \psi + C^{(0)(2)}$$

$$q_1 = K_3^{(1)} e^{i\beta\tau} v^{(1)} - K_3^{(1)} e^{-i\beta\tau} v^{(2)}$$

When these values have been substituted in $P^{(2)}$ and $Q^{(2)}$, there result:

$$P^{(2)} = [e^{2i\beta\tau} (C^{(0)(0)} + i S^{(1)(0)}) + e^{-2i\beta\tau} (\text{Conjugate}) + C^{(0)(6)} + \gamma_2 C^{(0)(2)}]$$

$$Q^{(2)} = [e^{i\beta\tau} (C^{(0)(3)} + i S^{(1)(3)}) + e^{-i\beta\tau} (\text{Conjugate})]$$

The coefficient of γ_2 here is the same as that of γ_1 in the coefficient of λ . The significance of this fact will be apparent in the later development.

We shall now complete the integrations of the last pair of equations (22). They are of the form:

$$\Delta \dot{k}_3^{(2)} = - \left[C^{(0)(3)} + i S^{(1)(4)} + e^{-2i\beta\tau} (\text{Conjugate}) \right]$$

$$\Delta \dot{k}_4^{(2)} = \left[e^{2i\beta\tau} (C^{(0)(3)} + i S^{(1)(4)}) + (\text{Conjugate}) \right]$$

On integrating,

$$k_3^{(2)} = C_{11}^{(1)(4)} - i S_{11}^{(0)(3)} + e^{-2i\beta\tau} (C_{22}^{(0)(4)} - i S_{22}^{(0)(4)})$$

$$- k_4^{(2)} = e^{2i\beta\tau} (C_{22}^{(0)(4)} + i S_{22}^{(0)(4)}) + C_{11}^{(0)(3)} + i S_{11}^{(1)(4)} \quad (25)$$

Substituting these results in the complementary function we arrive at the particular integral

$$g_2 = e^{i\beta\tau} (C^{(0)(4)} + i S^{(0)(4)}) + e^{-i\beta\tau} (\text{Conjugate})$$

Thus the complete solution for q_2 is:

$$g_2 = K_3^{(2)} e^{i\beta\tau} v^{(1)} + K_4^{(2)} e^{-i\beta\tau} v^{(2)} + e^{i\beta\tau} (C^{(0)(4)} + i S^{(0)(4)}) + e^{-i\beta\tau} (\text{Conjugate}). \quad (26)$$

Consider now the first pair of equations (22). They will yield, on solution, a relation involving γ_2 . We have,

$$D \dot{k}_1^{(2)} = -\psi \left[e^{2i\beta\tau} (C^{(0)(0)} + i S^{(1)(0)}) + e^{-2i\beta\tau} (\text{Conjugate}) \right]$$

$$- \psi \left[C^{(0)(6)} + \gamma_2 C^{(0)(2)} \right] - A\tau \phi [P^{(2)}]$$

$$D \dot{k}_2^{(2)} = \phi [P^{(2)}].$$

Integration gives

$$D k_1^{(2)} = \int P_1^{(2)} d\tau - A\tau \int \phi [P^{(2)}] d\tau + (d_1^{(3)} + \gamma_2 d_2^{(1)}) \cdot \tau$$

$$D k_2^{(2)} = \int \phi [P^{(2)}] d\tau.$$

The particular integral is, therefore,

$$p_2 = \phi \left[(d_1^{(3)} + \gamma_2 d_2^{(1)}) \cdot \tau - A\tau \int \phi[P^{(2)}] d\tau + \int P_1^{(2)} d\tau \right] \\ + (\psi + A\tau\phi) \left[\int \phi[P^{(2)}] d\tau \right],$$

or

$$p_2 = \phi \left[(d_1^{(3)} + \gamma_2 d_2^{(1)}) \cdot \tau + \int P_1^{(2)} d\tau \right] + \psi \int \phi[P^{(2)}] d\tau,$$

and we get for the general solution:

$$p_2 = K_1^{(2)} \phi + K_2^{(2)} (\psi + A\tau\phi) + \phi (d_1^{(3)} + \gamma_2 d_2^{(1)}) \cdot \tau \\ + \phi \int P_1^{(2)} d\tau + \psi \int \phi[P^{(2)}] d\tau. \quad (27)$$

The condition for periodicity will give the relation

$$A \cdot K_2^{(2)} + \gamma_2 d_2^{(1)} + d_1^{(3)} = 0 \quad (28)$$

A second similar relation which, with (28), will uniquely determine $K_2^{(2)}$ and γ_2 , will arise in the next integration in the same manner as did equation (23). Moreover, the coefficients of $K_2^{(2)}$ and γ_2 will be the same as those of $K_2^{(1)}$ and γ_1 in (23). Therefore the functional determinant of these two equations will be the determinant (24).

Thus we have

$$K_2^{(2)} = d_3^{(2)} \quad \text{and} \quad \gamma_2 = d_4^{(2)}.$$

Completing the integrations in (27), we finally arrive at the general solutions for p_2 and q_2 :

$$p_2 = K_1^{(2)} \phi + e^{2i\beta\tau} (C^{(1)(4)} + iS^{(1)(4)}) + e^{-2i\beta\tau} (\text{Conjugate}) \\ + C^{(2)(7)},$$

$$\begin{aligned} \mathcal{G}_2 = & K_3^{(2)} e^{i\beta z} \mathcal{W}^{(1)} + K_4^{(2)} e^{-i\beta z} \mathcal{W}^{(2)} \\ & + e^{i\beta z} (C^{(0)(4)} + i S^{(0)(4)}) + e^{-i\beta z} (\text{Conjugate}) . \end{aligned} \quad (29)$$

On imposing initial conditions in equations (29), the integration constants may be evaluated.

Proceeding similarly, we can continue the process of determining the succeeding p_j and q_j , the two unknowns $K_2^{(j)}$ and γ_j being determined by two relations similar to those of equations (18) and (23). Moreover, the functional determinant will be (24).

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