OSCILLATION THEOREMS FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

Criteria will be obtained for a linear self-adjoint elliptic partial differential equation to be oscillatory or nonoscillatory in unbounded domains R of n-dimensional Euclidean space E^n . The criteria are of two main types: (i) those involving integrals of suitable majorants of the coefficients, and (ii) those involving limits of these majorants as the argument tends to infinity.

Our theorems constitute generalizations to partial differential equations of well-known criteria of Hille, Leighton, Potter, Moore, and Wintner for ordinary differential equations. In general, our method provides the means for extending in this manner any oscillation criterion for self-adjoint ordinary differential equations. Our results imply Glazman's theorems in the special case of the Schrodinger equation in E^n .

In the derivation of the oscillation criteria it is assumed that R is either quasiconical (i.e. contains an infinite cone) or limit-cylindrical (i.e. contains an infinite cylinder). In the derivation of the nonoscillation criteria no special assumptions regarding the shape of the domain are needed.

Examples illustrating the theory are given. In particular, it is shown that the limit criteria obtained in the second order case are the best possible of their kind.

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INTRODUCTION

Conditions on the coefficients of certain linear elliptic partial differential equations will be obtained which are sufficient for the equations to be oscillatory in unbounded domains of n-dimensional Euclidean space E^n. The criteria are of two main types: (i) those involving integrals of suitable majorants of the coefficients; and (ii) those involving limits of these majorants as the argument tends to infinity. Criteria of type (i) were obtained by Swanson [26] for second-order equations with one variable separable and the fundamental domain limit-cylindrical (i.e. containing an infinite cylinder); and conditions of type (ii) by Glazman [9] for the Schrödinger operator in the case that the domain is all of E^n.

Our theorems constitute generalizations to n dimensions (i.e. partial differential equations) of well-known one-dimensional results in the literature (cf. [9], [11], [16], [17], [19], [21], [28]). Our method provides the means for generalizing to n-dimensions (i.e. partial differential equations) any given one-dimensional oscillation criterion. The results we obtain serve to illustrate the power of the method.

There is an extensive literature on oscillation theorems for ordinary differential equations. A complete bibliography may be found in the forthcoming book of Swanson [27]. The corresponding theory for partial differential equations is not as well developed (see, however, [9], [10], [13], [14], [15], [26], [27]), largely because of the earlier lack of an n-dimensional analogue of Sturm's comparison theorem. In this work we use the recent Clark-Swanson result [4], together with a comparison theorem of Swanson [24] for eigenvalues, as the basic tools for deriving our oscillation criteria.

The definition of an oscillatory equation given below is closely related to the notion of conjugacy used by Kreith [14]. An equation oscillatory in our sense is often said to have the nodal property (cf. [26]). In general, this is stronger than the requirement that a solution exists with a zero in every neighbourhood of infinity. In fact, if a self-adjoint second-order linear elliptic equation is oscillatory in our sense, every solution of the equation has a zero in every neighbourhood of infinity. This will be seen to be a consequence of the Clark-Swanson separation theorem [4].

In Chapter I, equations of the second order will be considered on quasiconical domains (i.e. domains containing a cone). The criteria obtained are easily specialized to all of E^n, and will contain the corresponding results of Glazman [9] for the Schrödinger operator. Using the Clark-Swanson comparison theorem, we also derive nonoscillation theorems, but without making any special assumptions regarding the shape of the domain (in contrast to the oscillation theorems, where we assume that the domain is quasiconical).

Our results in this direction generalize well-known one-dimensional theorems of Hille [11], and Potter [21]. The last section of Chapter I will be devoted to examples illustrating the theory. In particular, we shall show that the limit criteria obtained are the best possible of their kind.

Equations of higher order will be studied in Chapter II. Our approach is an extension of that used by Kreith [13] and Swanson [26] for second-order equations. The fundamental domain will be limit-cylindrical (i.e. will contain an infinite cylinder). The results are fewer in this case, since there are not many one-dimensional theorems extant. Our theorems constitute generalizations of results of Glazman [9] for two-term ordinary differential equations of order $2m$ (m any positive integer), and Leighton-Nehari [17] for fourth-order equations. In the absence of a suitable comparison theorem, however, our methods will not yield nonoscillation theorems for higher order ($m \geq 2$) partial differential equations. The operator we consider has a relatively simple form, but our methods will work for more general self-adjoint operators of even order, since the variational principles we use are valid for general elliptic operators (cf. [1] and [18]). Although we know of no one-dimensional oscillation criteria for such general operators, we remark that such criteria could be generalized by using extensions of the Swanson comparison theorem [24, p.517].

CHAPTER I

SECOND ORDER SELF-ADJOINT EQUATIONS

1. Definitions and notations.

We shall obtain oscillation criteria of limit type and integral type for the linear elliptic partial differential equation

$$(1.1) \qquad Lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + bu = 0$$

in unbounded domains R in n-dimensional Euclidean space E^n. Our theorems are, for the most part, extensions of one-dimensional oscillation theorems of Kneser-Hille [11] (limit type), Leighton [16], Moore [19], Potter [21], and Wintner [28] (integral type). The remainder are the n-dimensional second order analogues of one-dimensional $2m$-th order oscillation theorems of Glazman [9].

Points in E^n are denoted by $x = (x_1, x_2, \ldots, x_n)$ and differentiation with respect to x_i is denoted by D_i, $i=1,2,\ldots,n$. The coefficients a_{ij} are supposed to be real and of class $C^1(\overline{R})$, and the matrix (a_{ij}) is positive definite in R. The coefficient b is assumed to be real and continuous on $\overline{R}$. The domain $D(L)$ of L is defined to be the set of all real-valued functions on $\overline{R}$ of class $C^2(\overline{R})$. The conditions on the coefficients, although not the weakest possible (see, for example, [4]), are the special case $m = 1$ of those we impose in the 2m-th order case treated in Chapter II.

<u>Definition</u>. A function u will be called a <u>solution</u> of $Lu = 0$ if $u \in D(L)$ and u satisfies (1.1) everywhere in R.

<u>Shape of the domain R</u>. We assume that R contains the origin and that R is large enough in the x_n direction to contain the cone $C_\alpha = \{x \in E^n : x_n \geq |x| \cos \alpha\}$ for some α, $0 < \alpha < \pi$, $|x|$ being the Euclidean distance $\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.

The boundary ∂R of R is supposed to have a piecewise continuous unit normal vector at each point.

We shall make use of the following notation:

$$R_r = R \cap \{x \in E^n : |x| > r\}; \quad S_r = \{x \in R \cup \partial R : |x| = r\}.$$

<u>Definition</u>. A bounded domain $N \subset R$ is said to be a <u>nodal domain</u> of a nontrivial solution u of (1.1) iff $u = 0$ on ∂N.

<u>Definition</u>. The differential equation (1.1) is said to be <u>oscillatory</u> in R iff there exists a nontrivial solution u_r of (1.1) with a nodal domain in R_r for all $r > 0$.

It follows from Clark and Swanson's n-dimensional analogue of Sturm's separation theorem [4] that <u>every</u> solution of an oscillatory differential equation vanishes at some point in R_r for all $r > 0$.

<u>Definition</u>. The differential equation (1.1) is said to be

non-oscillatory iff there exists $r > 0$ such that the solutions in R_r have no nodal domains.

2. Auxiliary results.

The basic tool for deriving our oscillation theorems will be a recent comparison theorem of C.A. Swanson [24]. In addition we shall need two well-known properties of eigenvalues.

Minimum principle. (Cf. [6, p.399]) Let Ω be a bounded domain in E^n. The function $u \in C^2(\Omega)$ which minimizes the functional

$$(2.1) \qquad J[u] = \int_\Omega \{ \sum_{i,j=1}^{n} a_{ij} D_i u D_j u - bu^2 \} \, dx$$

under the condition $\|u\| = 1$ is an eigenfunction corresponding to the smallest eigenvalue of the problem

$$(2.2) \qquad -Lu = \lambda u \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad \partial\Omega .$$

The norm $\|u\|$ is the usual L^2 norm: $\|u\| = [\int_\Omega |u|^2 dx]^{\frac{1}{2}}$.

Proof. In [6] it is shown that if the minimizing function exists and is of class C^2, then it is an eigenfunction corresponding to the smallest eigenvalue of the problem (2.2). Let the minimum value of $J[u]$ be λ_o. Then the results of [18, § 11] show that there exists a minimizing function u_o which is a weak solution of (2.2) in the following sense:

$$\langle u_o, (L-\lambda_o)\varphi \rangle = 0, \qquad \varphi \in C_o^\infty(\Omega),$$

$\langle\, ,\, \rangle$ being the usual $L^2(\Omega)$ inner product. On account of our conditions on the coefficients a_{ij} and b, the results of [1, §9] imply that u_o is in fact a classical solution of (2.2).

Monotonicity principle for eigenvalues. For $0 < t < \infty$ let $G(t)$ be a bounded domain in R. If $0 < t_1 < t_2 < \infty$ implies $G(t_1) \subset G(t_2)$, $G(t_1) \neq G(t_2)$, then the first eigenvalue $\lambda_o(t)$ of the problem

$$-Lu = \lambda u \quad \text{in} \quad G(t); \qquad u = 0 \quad \text{on} \quad \partial G(t)$$

is monotone decreasing in t. Moreover, if for some $r_o > 0$, $G(t) \subset \Omega(t)$, where $\Omega(t) = \{x \in E^n : r_o < |x| < r_o+t\}$, then $\lim\limits_{t\to 0+} \lambda_o(t) = +\infty$.

Proof. The monotonicity of $\lambda_o(t)$ may be established by adapting the proof given in [7, pp.400-401] for the Laplace operator.

The continuity of a_{ij} implies that the smallest eigenvalue $\Lambda_o(x)$ of the matrix $(a_{ij}(x))$ has non-zero infimum in $G(t)$, since $\overline{G}(t) \subset R$ and (a_{ij}) is positive-definite in R. In other words, the operator L is uniformly elliptic in $G(t)$, i.e. there exists a number $\mu_o(t) > 0$ such that $\sum\limits_{i,j=1}^{n} a_{ij}(x)z_i z_j \geq \mu_o(t)|z|^2$ for all $x \in G(t)$, $z \in E^n$.

Since the function b is uniformly continuous on $\overline{G}(t)$, there exists a number $k_o(t) > -\infty$ such that

$$(2.3) \qquad -\int_{G(t)} bu^2 dx \geq k_o(t)\int_{G(t)} u^2 dx$$

for all $u \in C^1(R)$. Let

$$J_t[u] = \int_{G(t)} \left\{ \sum_{i,j=1}^{n} a_{ij} D_i u D_j u - bu^2 \right\} dx .$$

Then (2.3) and the uniform ellipticity of L in $G(t)$ imply that

$$(2.4) \qquad J_t[u] \geq \mu_o(t)\int_{G(t)} \sum_{i=1}^{n} (D_i u)^2 dx + k_o(t)\int_{G(t)} u^2 dx .$$

But

$$\int_{G(t)} u^2 dx = \int_{G(t)} u^2 F(r,S) dr dS ,$$

where $F(r,S)$ is the Jacobian of the transformation from rectangular coordinates $(x_1,x_2,\ldots x_n)$ to hyperspherical polar coordinates $(r,\theta_1,\theta_2,\ldots,\theta_{n-1}) \equiv (r,S)$, defined by the relations

$$x_1 = r \prod_{i=1}^{n} \sin \theta_i , \qquad x_n = r \cos \theta_1 ,$$

$$x_i = r \cos \theta_{n-i+1} \prod_{j=1}^{n-1} \sin \theta_j , \qquad i=2,3,\ldots,n-1.$$

We now extend u continuously to all of the annulus $\Omega(t)$

by requiring it to be zero outside $G(t)$. In particular, u is zero on $|x| = r_o$, r_o being the inner radius of the annulus $\Omega(t)$. Hence

$$u(x) = \int_{r_o}^{r} v_t(t,\theta_1,\ldots,\theta_{n-1})dt ,$$

where

$$u(x) = v(r,\theta_1,\ldots,\theta_{n-1}), \quad v_t = \partial v/\partial t .$$

By Schwarz's inequality,

$$|u(x)|^2 \leq \left|\int_{r_o}^{r} dy\right| \left|\int_{r_o}^{r} \left(\frac{\partial v}{\partial y}\right)^2 dy\right| \leq t\int_{r_o}^{r_o+t} \left(\frac{\partial v}{\partial r}\right)^2 dr$$

whenever $x \in G(t) \subset \Omega(t)$. Integrating this inequality over $\Omega(t)$ we obtain

$$\int_{\Omega(t)} u^2 dx = \int_{\Omega(t)} u^2 F(r,S)drdS$$

$$\leq t^2 \int_{\Omega(t)} \left(\frac{\partial v}{\partial r}\right)^2 F(r,S)drdS$$

$$\leq t^2 \int_{\Omega(t)} \sum_{i=1}^{n} (D_i u)^2 dx ,$$

since

$$\sum_{i=1}^{n} (D_i u)^2 = |\text{grad } u|^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \sum_{i=1}^{n-1}\left(h_{i+1}\frac{\partial u}{\partial \theta_i}\right)^2 ,$$

where the h_i are certain (known) functions of r and S.

Since u is defined to be zero outside $G(t)$, we have

$$(2.5) \qquad \int_{G(t)} u^2 dx \leq t^2 \int_{G(t)} \sum_{i=1}^{n} (D_i u)^2 dx .$$

Combining this with inequality (2.4) we get

$$J_t[u] \geq [k_o(t) + \mu_o(t)/t^2] \int_{G(t)} u^2 dx .$$

According to Courant's minimum principle [6, p.399],

$$\lambda_o(t) = \inf\{J_t[u] : \|u\|_t = 1\} ,$$

where $\|u\|_t$ is the L^2 norm of u on $G(t)$. Thus

$$(2.6) \qquad \lambda_o(t) \geq k_o(t) + \mu_o(t)/t^2 .$$

Since $\mu_o(t)$ may be chosen to be $\inf\{\Lambda_o(x) : x \in G(t)\}$, it cannot decrease as the domain $G(t)$ shrinks, and therefore remains positive as $t \to 0+$.

Similarly, we may choose

$$k_o(t) = \inf\left\{-\int_{G(t)} bu^2 dx/\|u\|_t^2 : u \in C(G(t)\right\} ,$$

that is,

$$k_o(t) = \inf\left\{-\int_{\Omega(t)} bu^2 dx/\int_{\Omega(t)} u^2 dx : u \in C(G(t))\right\}$$

since u is zero outside $G(t)$. Now b is continuous in R and is therefore a continuous function of $|x|$. Thus b is a continuous function of t in every interval $0 \leq t \leq \delta$, and is thus bounded on $0 \leq t \leq \delta$ for all $\delta > 0$. Therefore $k_o(t)$ must remain bounded as $t \to 0+$. (This implies that we could have chosen $k_o(t)$ independent of t in (2.3)). In any event, the inequality (2.6) now implies that

$\lim_{t \to 0+} \lambda_o(t) = +\infty$, and the theorem is proved.

<u>Remark</u>. In the applications below, we actually need $\lambda_o(t)$ to tend <u>continuously</u> to $+\infty$. To prove that $\lambda_o(t)$ depends continuously on t, we proceed as follows:

$$\begin{aligned}
\lambda_o(t) &= \inf\{J_t[u] : \|u\|_t = 1\} \quad \text{(by Courant's principle)} \\
&= J_t[u_o] \qquad \text{(for some } u_o \text{ in the domain of } J_t) \\
&= \int_{G(t)} \left\{ \sum_{i,j=1}^{n} a_{ij} D_i u_o D_j u_o - b u_o^2 \right\} dx \\
&= \int_{\Omega(t)} \left\{ \sum_{i,j=1}^{n} a_{ij} D_i u_o D_j u_o - b u_o^2 \right\} dx ,
\end{aligned}$$

by defining u_o to be zero outside $G(t)$. This is possible because u_o is an eigenfunction of the problem

$$-Lu = \lambda u \quad \text{in} \quad G(t) \; ; \quad u = 0 \quad \text{on} \quad \partial G(t)$$

according to Courant's minimum principle. It is clear that the last integral above depends continuously on t, as may be seen by transforming to hyperspherical polar coordinates and considering the limits of the resulting multiple integral.

<u>Definition</u>. Let $\Lambda(x)$ denote the largest eigenvalue of the matrix $(a_{ij}(x))$, $x \in R$. A <u>majorant</u> of (a_{ij}) is a positive-valued function $f \in C^1(0,\infty)$ such that

$$f(r) \geq \max\{\Lambda(x) : x \in S_r\}, \qquad (0 < r < \infty).$$

The function g defined by

$$g(r) = \min\{b(x) : x \in S_r\}, \qquad (0 < r < \infty)$$

is called a majorant of $b(x)$.

Let A and B be functions in G defined by the equations $A(x) = f(|x|)$, $B(x) = g(|x|)$, respectively.

<u>The comparison equation</u>. We shall obtain oscillation theorems for equation (1.1) by comparing it with the separable equation

$$\sum_{i=1}^{n} D_i(AD_i v) + Bv = 0. \tag{2.7}$$

As before, let $r, \theta_1, \theta_2, \ldots, \theta_{n-1}$ denote hyperspherical polar coordinates. By writing (2.7) in terms of these coordinates, we find that (2.7) has solutions (in particular) of the form

$$v(x) = \rho(r)\varphi(\theta_1), \quad 0 \leq r < \infty, \quad 0 \leq \theta_1 \leq \alpha, \tag{2.8}$$

where ρ and φ satisfy the ordinary differential equations

$$\frac{d}{dr}[r^{n-1} f(r)\frac{d\rho}{dr}] + r^{n-1}[g(r) - \lambda_\alpha r^{-2} f(r)]\rho = 0, \tag{2.9}$$

$$\frac{d}{d\theta_1}[\sin^{n-2}\theta_1 \frac{d\varphi}{dr}] + \lambda_\alpha \varphi \sin^{n-2}\theta_1 = 0, \tag{2.10}$$

respectively. We choose λ_α to be the smallest number for which (2.10) has a nontrivial solution φ on $0 \le \theta_1 \le \alpha$ satisfying $\varphi(\alpha) = 0$. It is well known [5] that λ_α exists as the smallest eigenvalue of a singular Sturm-Liouville problem. We shall suppose, for definiteness, that the corresponding eigenfunction has been normalized by the condition $\varphi(0) = 1$.

3. Oscillation criteria of integral type.

The theorems in this section constitute extensions of the one-dimensional theorems of Leighton [16], Moore [19], Potter [21] and Wintner [28]. They may be specialized to all of E^n by taking $v = \rho$, $\varphi \equiv 1$, $\lambda_\alpha = 0$.

Theorem 1. Equation (1.1) is oscillatory in R if R contains a cone C_α $(\alpha > 0)$, and (a_{ij}), b have majorants f,g respectively such that

$$(3.1) \qquad \int_1^\infty \frac{dr}{r^{n-1}f(r)} = +\infty \text{ and } \int_1^\infty r^{n-1}[g(r)-\lambda_\alpha r^{-2}f(r)]dr = +\infty.$$

Proof. The hypotheses (3.1) imply that the ordinary differential equation (2.9) is oscillatory in $0 < r < \infty$ by the Leighton-Wintner theorem [16], [28]. Let $\rho(r)$ be a non-trivial solution of (2.9) with zeros at $r = \delta_1, \delta_2, \ldots, \delta_k, \ldots,$ where $\delta_k \uparrow \infty$. If φ is an eigenfunction of (2.10) with boundary condition $\varphi(\alpha) = 0$ corresponding to the eigenvalue λ_α, the function v defined by (2.8) is a solution of the comparison equation (2.7) with nodal domains in the form of truncated cones

$$C_{\alpha k} = \{x \in E^n : x_n > |x| \cos \alpha,\ \delta_k < |x| < \delta_{k+1}\}, \quad 0 < \alpha < \pi ,$$

$k=1,2,\ldots,$ with piecewise smooth boundaries. Thus v has a nodal domain $C_{\alpha k} \subset R_p$ for all $p > 0$; for if $p > 0$ is given, choose k large enough so that $\delta_k \geq p$, and clearly $x \in C_{\alpha k}$ implies that $|x| > \delta_k \geq p$ and $x \in C_\alpha \subset R$, so that $x \in R_p$. Since

$$\sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \leq \Lambda(x)|z|^2 \leq f(r)|z|^2 = A(x)|z|^2, \quad z \in E^n$$

and $b(x) \geq g(|x|) = B(x)$, equation (1.1) majorizes equation (2.7) in the following sense:

$$\int_{C_{\alpha k}} \left\{ \sum_{i,j=1}^{n} (A\delta_{ij} - a_{ij}) D_i u D_j u + (b-B)u^2 \right\} dx \geq 0 .$$

It then follows from a comparison theorem of C.A. Swanson [24] that the smallest eigenvalue of the problem

$$-Lw = \mu w \ \text{ in } \ C_{\alpha k}, \qquad w = 0 \ \text{ on } \ \partial C_{\alpha k}$$

satisfies $\mu \leq 0$. Let $M_{\alpha kt} \doteq \{x \in C_{\alpha k} : \delta_k < |x| < t\}$, $\delta_k < t \leq \delta_{k+1}$, and let $\mu_1(t)$ denote the smallest eigenvalue of the problem

$$-Lw = \mu(t)w \quad \text{in} \quad M_{\alpha kt}, \qquad w = 0 \quad \text{on} \quad \partial M_{\alpha kt}.$$

By the monotonicity principle for eigenvalues, $\mu_1(t)$ is monotone nonincreasing in $\delta_k < t \leq \delta_{k+1}$ and $\lim_{t \to \delta_k +} \mu_1(t) = +\infty$. Since $\mu_1(\delta_{k+1}) \leq 0$, there exists a number T in $(\delta_k, \delta_{k+1}]$ such that $\mu_1(T) = 0$. This means that $M_{\alpha kT}$ is a nodal domain of a nontrivial solution u_k of (1.1), and since $M_{\alpha kT} \subset C_{\alpha k} \subset R_p$ for arbitrary $p > 0$ provided k is sufficiently large, equation (1.1) is oscillatory in R and the theorem is proved.

It is convenient to state the integral conditions with the number 1 as the lower limit of integration, but the theorems remain valid if the integral conditions hold when 1 is replaced by a positive number. In fact, in the literature (Cf., e.g. [19], [21], [26], [27], [28]) the integral criteria are often stated in this manner. We shall have occasion to make use of this fact in the applications (Cf. §6, Example 4).

The first part of condition (3.1) requires the function f to grow quite slowly, and in fact does not hold for the Schrödinger operator $\Delta + b(x)$ in three dimensions, since in this case $f(r) \geq 1$, so that $\int_1^\infty \frac{dr}{r^{n-1} f(r)} < \infty$. In such cases the following extension of Moore's oscillation theorem [19] is valid:

<u>Theorem 2</u>. The equation (1.1) is oscillatory in R if R contains a cone $C_\alpha (\alpha > 0)$, and (a_{ij}), b have majorants

f, g, respectively, such that

$$(3.2) \qquad \int_1^\infty \frac{dr}{r^{n-1} f(r)} < \infty \quad \text{and} \quad \int_1^\infty r^{n-1}\, h_n^m(r)[g(r)-\lambda_\alpha r^{-2} f(r)]dr = \infty$$

for some number $m > 1$, where $h_n(r) = \int_r^\infty \frac{dt}{t^{n-1}\, f(t)}$.

Proof. According to Moore's oscillation theorem [19], the ordinary differential equation (2.9) is oscillatory in $0 < r < \infty$ on account of the hypotheses (3.2). The remainder of the proof follows that of Theorem 1 without change and will be omitted.

The criteria obtained in Theorems 1 and 2 may be sharpened slightly in the case that the largest eigenvalue $\Lambda(x)$ of (a_{ij}) is bounded in R.

Theorem 3. Let R contain the cone C_α for some $\alpha > 0$, and let $\Lambda(x)$ be bounded in R. Suppose (a_{ij}) and b have majorants f,g, respectively. Then equation (1.1) is oscillatory in R for $n = 2$ if

$$(3.3) \qquad \int_1^\infty r[g(r) - \lambda_\alpha r^{-2}]dr = +\infty ,$$

and for $n \geq 3$ if there exists a number $\delta > 0$ such that

$$(3.4) \qquad \int_1^\infty r^{1-\delta}[g(r) - \lambda_\alpha r^{-2} f(r)]dr = +\infty .$$

In the case $n = 1$, equation (1.1) is oscillatory if (3.4) holds with $\delta = 1$ (the Leighton-Wintner theorem).

Proof. If $\Lambda(x)$ is bounded in R, say $\Lambda(x) \leq \Lambda_1$, $x \in R$, we can choose $f(r) = \Lambda_1$, $0 \leq r < \infty$. Then, for $n = 2$, the conditions (3.1) are fulfilled and hence the first statement of the theorem follows from Theorem 1.

For $n \geq 3$, the first part of (3.2) is fulfilled, and $h_n(r) = r^{2-n}/(n-2)\Lambda_1$. By hypothesis there exists $\delta > 0$ such that (3.4) holds. Let $m = 1 + \delta/(n-2)$. Then direct computation shows that condition (3.4) implies the second part of condition (3.2), and therefore the second statement of the theorem follows from Theorem 2.

It is clear that our method enables us to generalize to n dimensions any sufficient condition for a self-adjoint ordinary linear differential equation of the second order to be oscillatory. In what follows we shall therefore generalize only a representative number of the existing one-dimensional oscillation theorems.

Our next theorem generalizes Potter's refinement [21] of Leighton's theorem. We shall find it convenient to introduce the following notation. Let h be a positive C^2 function defined by

$$(3.5) \qquad [h(r)]^{-2} = g(r) - \Lambda_1[\lambda_\alpha + (n-1)(n-3)/4]r^{-2}, \quad 0 < r < \infty .$$

Let the functions H_1 and H_2 be defined by

$$H_1(r) = \frac{1}{h(r)} - \frac{[h'(r)]^2}{4h(r)} + \frac{h''(r)}{2} ,$$

$$H_2(r) = \frac{1}{h(r)} - \frac{[h'(r)]^2}{4h(r)} - \frac{h''(r)}{2} ,$$

where primes denote differentiation with respect to r.

Theorem 4. Let R contain the cone C_α for some $\alpha > 0$, and let $\Lambda(x)$ be bounded in R, say $\Lambda(x) \leq \Lambda_1$, $x \in R$. Then equation (1.1) is oscillatory in R if there exists a positive C^2 function h satisfying (3.5) for large r and either

$$(3.6) \qquad \int_1^\infty \frac{dt}{h(t)} = \int_1^\infty H_1(t)\, dt = +\infty$$

or

$$(3.7) \qquad \int_1^\infty H_2(t) dt = +\infty .$$

Proof. In equation (2.9) choose $f(r) = \Lambda_1$. The normal form of this equation, obtained by making the oscillation-preserving transformation $\rho = r^{(1-n)/2}\sigma$, is

$$(3.8) \qquad \Lambda_1 \frac{d^2\sigma}{dr^2} + \{g(r) - \Lambda_1[\lambda_\alpha + (n-1)(n-3)/4]r^{-2}\}\sigma = 0.$$

The hypothesis (3.6) (or (3.7)) implies that the equation (3.8) is oscillatory by the theorem of Potter mentioned in the remark above. Thus the equation (2.9) is also oscillatory if (3.6) or (3.7) holds. The remainder of the proof is similar to that of Theorem 1 and will be omitted.

Because of the positivity condition on h, it is clear that Theorem 4 is in some respects less general than Theorem 3. However, in section 6 we shall exhibit an example for which Theorem 4 gives information not obtainable in any obvious way from Theorem 3.

4. Conditions of limit type.

The first theorem in this section is a generalization of the classical Kneser-Hille theorem [11]. Our theorem also contains Glazman's generalization [9, Th.7, p.158] for the Schrödinger operator $-\nabla^2 - b(x)$, $x \in E^n$, and in fact provides a new proof of his result. It will also be seen that our condition is sharp.

Theorem 5. Suppose that R contains the cone C_α for some $\alpha > 0$, and that $\Lambda(x)$ is bounded in R_s for some $s > 0$, say $\Lambda(x) \leq \Lambda_1$, $x \in R_s$. Let b have majorant g. Then equation (1.1) is oscillatory in R if

$$(4.1) \qquad \liminf_{r\to\infty} r^2 g(r) > \Lambda_1[\lambda_\alpha + (n-2)^2/4].$$

Proof. The hypothesis (4.1) implies that there exist constants r_o and γ such that

$$r^2\, g(r) > \gamma > \Lambda_1[\lambda_\alpha + (n-2)^2/4]$$

for all $r > r_o$. We then compare (2.9) with the Euler equation

$$(4.2) \qquad \frac{d}{dr}\,[\Lambda_1 r^{n-1}\,\frac{d\rho}{dr}] + (\gamma - \lambda_\alpha\Lambda_1)r^{n-3}\,\rho = 0,$$

with solutions $\rho = r^\beta$, where

$$\beta^2 + (n-2)\beta + (\gamma/\Lambda_1) - \lambda_\alpha = 0. \tag{4.3}$$

Since $\gamma > \Lambda_1[\lambda_\alpha + (n-2)^2/4]$, the quadratic equation (4.3) has complex roots, and therefore equation (4.2) is oscillatory in (r_o, ∞). We may choose $f(r) = \Lambda_1$ and apply Sturm's comparison theorem [5, p.208] on (r_1, ∞), $(r_1 = \max\{r_o, s\})$ to deduce that equation (2.9) is oscillatory on account of the hypothesis

$$r^{n-1}[g(r) - \lambda_\alpha r^{-2} f(r)] > r^{n-3}(\gamma - \lambda_\alpha \Lambda_1), \quad r > \max\{r_o, s\}.$$

The remainder of the proof proceeds as in Theorem 1 and will be omitted.

Our next theorem is an extension of a well-known theorem of Hille [11, Th.5]. As Hille points out in the paper just cited, the conditions are a refinement of those in Theorem 5, since integration smooths out irregularities in the function g.

Theorem 6. Let R contain the cone C_α for some $\alpha > 0$, and let $\Lambda(x)$ be bounded in R, say $\Lambda(x) \leq \Lambda_1$, $x \in R$. Then equation (1.1) is oscillatory in R if

$r^2\, g(r) \geq \Lambda_1[\lambda_\alpha + (n-1)(n-3)/4]$ for large r and either

$$\liminf_{r\to\infty} r\int_r^\infty g(t)\, dt > \Lambda_1[\lambda_\alpha + (n-2)^2/4] \tag{4.4}$$

or

$$(4.5) \qquad \limsup_{r\to\infty} r\int_r^\infty g(t)\,dt > \Lambda_1[\lambda_\alpha + (n^2-4n+7)/4],$$

where $g(r) = \min\{b(x) : x \in S_r\}$.

Proof. In equation (2.9) choose $f(r) = \Lambda_1$. We recall that the normal form of this equation, obtained by making the oscillation-preserving transformation $\rho = r^{(1-n)/2}\sigma$, is

$$(3.8) \qquad \Lambda_1 \frac{d^2\sigma}{dr^2} + \{g(r) - \Lambda_1[\lambda_\alpha + (n-1)(n-3)/4]r^{-2}\}\sigma = 0.$$

The hypothesis (4.4) implies that the equation (3.8) is oscillatory in $0 < r < \infty$ by a theorem of Hille [11, Th.5], since

$$\liminf_{r\to\infty} r\int_r^\infty \{g(t)/\Lambda_1 - [\lambda_\alpha + (n-1)(n-3)/4]t^{-2}\}dt$$

$$= \Lambda_1^{-1} \liminf_{r\to\infty} r\int_r^\infty g(t)\,dt - [\lambda_\alpha + (n-1)(n-3)/4]$$

$$> \lambda_\alpha + (n-2)^2/4 - [\lambda_\alpha + (n-1)(n-3)/4] = \frac{1}{4}.$$

Thus equation (2.9) is also oscillatory in $0 < r < \infty$ if hypothesis (4.4) holds.

If the hypothesis (4.5) holds, then equation (3.8) is oscillatory in $0 < r < \infty$ by the theorem of Hille cited above, since

$$\limsup_{r\to\infty} r\int_r^{\infty} \{g(t)/\Lambda_1 - [\lambda_\alpha + (n-1)(n-3)/4]t^{-2}\}dt$$

$$= \Lambda_1^{-1} \limsup_{r\to\infty} r\int_r^{\infty} g(t)dt - [\lambda_\alpha + (n-1)(n-3)/4]$$

$$> [\lambda_\alpha + (n^2-4n+7)/4] - [\lambda_\alpha + (n-1)(n-3)/4] = 1.$$

Thus the equation (2.9) is also oscillatory in $0 < r < \infty$ if (4.5) holds. The remainder of the proof will be omitted, since it follows that of Theorem 1 without change.

In section 6 we shall show, by exhibiting a counter-example, that the inequality (4.1) is sharp. Our next theorem permits us to relax the condition (4.1) provided we impose an additional hypothesis. This gives the second order, n-dimensional analogue of a 2m-th order, one-dimensional result of Glazman [9, p.102]. Although this result is not the sharpest possible, it is useful in the applications of the theory, as we shall demonstrate in section 6.

<u>Theorem 7</u>. Let R contain the cone C_α for some $\alpha > 0$, and let $\Lambda(x)$ be bounded in R, say $\Lambda(x) \le \Lambda_1$, $x \in R$. Then equation (1.1) is oscillatory if $r^2 g(r) \ge \Lambda_1[\lambda_\alpha + (n-2)^2/4]$ for large r and

(4.6) $$\limsup_{r\to\infty} (\log r) \int_r^{\infty} t\{g(t) - \Lambda_1[\lambda_\alpha + (n-2)^2/4]t^{-2}\}dt = +\infty .$$

<u>Proof</u>. In equation (2.9) choose $f(r) = \Lambda_1$. We then reduce

(2.9) to normal form as in the proof of Theorem 6:

$$(3.8) \qquad \Lambda_1 \frac{d^2\sigma}{dr^2} + \{g(r) - \Lambda_1[\lambda_\alpha + (n-1)(n-3)/4]r^{-2}\}\sigma = 0,$$

where $\rho(r) = r^{(1-n)/2}\sigma$. By a theorem of Glazman [9, p.102], the equation (3.8) is oscillatory in $0 < r < \infty$, since the hypothesis (4.6) implies

$$\limsup_{r\to\infty} (\log r) \int_r^\infty t\{g(t)/\Lambda_1 - [\lambda_\alpha + (n-1)(n-3)/4]t^{-2} - 1/(4t^2)\}dt$$

$$= \Lambda_1^{-1} \limsup_{r\to\infty} (\log r) \int_r^\infty t\{g(t) - \Lambda_1[\lambda_\alpha + (n-2)^2/4]t^{-2}\}dt$$

$$= +\infty .$$

Thus the equation (2.9) is also oscillatory, since the transformation $\rho = r^{(1-n)/2}\sigma$ preserves oscillatory behaviour in $0 < r < \infty$. The proof of the theorem may now be completed as in Theorem 1. We therefore omit the details.

5. Nonoscillation theorems.

All of our nonoscillation theorems will be proved by contradiction, and this gives rise to an interesting feature: since nodal domains are not constructed, no special assumptions are needed regarding the shape of the domain R. It is not necessary for R to be quasiconical (as in §§3 and 4), quasicylindrical (as in [26]) or quasibounded [9]. There is no loss of generality in assuming that R contains the origin. Since R is connected and unbounded, S_r is nonvoid for each $r > 0$.

Theorem 8. Let L be uniformly elliptic in R_s for some $s > 0$, i.e. there exists a number $\Lambda_o > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \geq \Lambda_o |z|^2$ for all $x \in R_s$, $z \in E^n$. Let $g_o(r) = \max \{b(x) : x \in S_r\}$, $0 < r < \infty$. Then equation (1.1) is non-oscillatory in R if

$$\limsup_{r \to \infty} r^2 g_o(r) < (n-2)^2 \Lambda_o/4. \tag{5.1}$$

Proof. Suppose the conclusion of the theorem is false, i.e. that (1.1) is oscillatory in R. Then there exists a nontrivial solution u of (1.1) with a nodal domain $N_r \subset R_r$ for all $r > 0$. But the hypothesis (5.1) implies that there exist constants r_o, γ_o such that

$$r^2 g_o(r) < \gamma_o < (n-2)^2 \Lambda_o/4,$$

for all $r > r_o$. We compare (1.1) with the equation

$$\sum_{i=1}^{n} \Lambda_o D_i^2 v + \gamma_o |x|^{-2} v = 0, \qquad x \in R. \tag{5.2}$$

Because of the hypotheses

$$\sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \geq \Lambda_o |z|^2, \quad x \in R_s,\ z \in E^n,$$

$$\gamma_o r^{-2} > g_o(r) \geq b(x), \quad x \in S_r,\ r > r_o,$$

equation (5.2) majorizes equation (1.1) in the following sense:

$$\int_{N_r} \left\{ \sum_{i,j=1}^{n} (a_{ij}-\Lambda_o\delta_{ij}) D_i u D_j u + (\gamma_o |x|^{-2} - b)u^2 \right\} dx \geq 0,$$

$$(r > \max\{r_o, s\})$$

and therefore the Clark-Swanson n-dimensional analogue [4] of Sturm's comparison theorem implies that every solution of (5.2) has a zero in $N_r \cup \partial N_r$ for all $r > \max\{r_o, s\}$. But the solution $\rho = r^\alpha$ of the ordinary differential equation

$$(5.3) \qquad \Lambda_o \frac{d}{dr} [r^{n-1} \frac{d\rho}{dr}] + \gamma_o r^{n-3} \rho(r) = 0$$

is also a solution of (5.2) (equation (5.3) being the radial form of (5.2)), and α satisfies the quadratic equation

$$\alpha^2 + (n-2)\alpha + \gamma_o/\Lambda_o = 0.$$

This equation has real roots because $\gamma_o < (n-2)^2\Lambda_o/4$, and hence the solution $v = r^\alpha$ of (5.2) is non-zero in $N_r \cup \partial N_r$ for all $r > 0$. This contradiction establishes the theorem.

In section 6 we shall exhibit a counterexample to show that the constant $(n-2)^2\Lambda_o/4$ in condition (5.1) cannot be improved, i.e. there exist oscillatory equations for which $\limsup_{r\to\infty} r^2 g_o(r) = (n-2)^2\Lambda_o/4$.

The theorem just proved, together with Theorem 5 above, contains Glazman's generalization [9, Th.7, p.158] of Hille's results [11]. In the cases where $n \neq 1$, both the result and the proof are new.

Theorem 9. Let L be uniformly elliptic in R_s for some $s > 0$, Λ_o being the ellipticity constant. Let g_1 be the function defined by

$$g_1(t) = g_o(t) - \Lambda_o(n-1)(n-3)/4t^2 ,$$

with $g_o(r) = \max\{b(x) : x \in S_r\}$, $0 < r < \infty$. Then equation (1.1) is nonoscillatory in R if

$$\limsup_{r\to\infty} r\int_r^\infty g_1^+(t)\,dt < \Lambda_o/4, \tag{5.4}$$

where $$g_1^+(r) = \max\{g_1(r), 0\}, \quad 0 < r < \infty.$$

Proof. Suppose the conclusion of the theorem is false, i.e. that (1.1) is oscillatory in R. Then there exists a nontrivial solution u of (1.1) with a nodal domain $N_r \subset R_r$ for all $r > 0$.

We now compare (1.1) with the equation

$$\sum_{i=1}^{n} \Lambda_o D_i^2 v + g_o(|x|)v = 0, \qquad x \in R. \tag{5.5}$$

Because of the hypotheses

$$\sum_{i,j=1}^{n} a_{ij}(x)\, z_i z_j \geq \Lambda_o|z|^2, \qquad x \in R_s, \qquad z \in E^n,$$

$$g_o(r) \geq b(x), \qquad x \in S_r, \qquad 0 < r < \infty,$$

equation (5.5) majorizes equation (1.1):

$$\int_{N_r} \left\{ \sum_{i,j=1}^{n} (a_{ij} - \Lambda_o \delta_{ij}) D_i u D_j u + (g_o - b) u^2 \right\} dx \geq 0, \quad (r > s)$$

and therefore the theorem of Clark and Swanson cited above [4] implies that <u>every</u> solution of (5.5) has a zero in $N_r \cup \partial N_r$ for all $r > s$. But we shall show that there exists a solution of (5.5) which has no zeros in R_p for some $p > 0$ and therefore has no zeros in R_r for all $r > p$. To see this, we note that the solutions of the ordinary differential equation

$$(5.6) \qquad \Lambda_o \frac{d}{dr} [r^{n-1} \frac{d\rho}{dr}] + r^{n-1} g_o(r) \rho = 0$$

are also (radial) solutions of (5.5). The normal form of (5.6), obtained by making the oscillation-preserving transformation $\rho = r^{(1-n)/2} w$, is

$$(5.7) \qquad \Lambda_o \frac{d^2 w}{dr^2} + [g_o(r) - (n-1)(n-3)\Lambda_o/4r^2] w = 0.$$

Since $g_1^+(r)$ is nonnegative, a well-known theorem of Hille [11, Th.7 Cor.1] implies that the equation

$$\frac{d^2 y}{dr^2} + g_1^+(r) y/\Lambda_o = 0$$

is nonoscillatory on account of hypothesis (5.4). Moreover,

$$g_1^+(r)/\Lambda_o \geq g_1(r)/\Lambda_o = g_o(r)/\Lambda_o - (n-1)(n-3)/4r^2,$$

so that Sturm's comparison theorem [5, p.208] implies that equation (5.7) is nonoscillatory. Thus there exists a solution $w = w(r)$ of (5.7) which has no zeros in (p,∞) for some $p > 0$. Hence there exists a solution $v = r^{(1-n)/2}\, w$ of (5.5) which has no zeros in R_p for some $p > 0$. We have therefore arrived at a contradiction, and the theorem is proved.

Remark. If $n = 1$, $b(x) \geq 0$ and $a_{11}(x) = 1$, this theorem reduces to the classical theorem of Hille cited in the proof.

It is possible to regard Theorem 9 as the special case $p = 0$ of the following n-dimensional analogue of a theorem of Hille [11, p.250, Th.12]:

Theorem 9A. The equation (1.1) is nonoscillatory in R if there exists a positive integer p such that

$$\int_r^\infty g_1^+(t)dt \leq \frac{1}{4}\,\Lambda_o \int_r^\infty S_p(t)dt$$

for sufficiently large r, where

$$S_p(r) = \sum_{k=0}^{p} [L_k(r)]^{-2}, \quad L_p(x) = L_{p-1}(x)\log_p x, \quad p=1,2,3,\ldots$$

with

$$L_o(x) = x, \quad \log_2 x = \log\log x, \quad \log_p x = \log\log_{p-1} x.$$

Proof. The proof is similar to that of Theorem 9. We appeal to Hille's Theorem 12 instead of the Corollary 1 to his

Theorem 7 at the appropriate places. We omit the details.

Our next theorem is the n-dimensional analogue of a 2m-th order one-dimensional oscillation theorem of Glazman [9, p.99, Th.10]. As will be noted below, our result contains a well-known criterion for non-oscillation first proved by Hille [11].

<u>Theorem 10</u>. Equation (1.1) is non-oscillatory in R if the inequality

$$\int_{M_\delta} r g_1^+(r)\,dr < \infty$$

holds for some $\delta > 0$, where

$$M_\delta = \{r : r^2 g_o(r)/\Lambda_o \geq \frac{(n-2)^2}{4} - \delta\},$$

and g_o, g_1^+, Λ_o have the meanings assigned in Theorem 9.

<u>Proof</u>. Since we shall use the argument of Theorem 9, it suffices to show that (5.7) is nonoscillatory. The conditions of [9, p.99, Th.10] are satisfied, since

$$r^2[g_o(r)/\Lambda_o - (n-1)(n-3)/4r^2] \geq \frac{1}{4} - \delta$$

for all $r \in M_\delta$. Thus (5.7) is nonoscillatory. The remaining details are as in Theorem 9.

<u>Corollary 1</u>. (Cf. [11, p.237, Th.2]). If b is bounded on $0 < x < \infty$, the ordinary differential equation $(a(x)y')' + b(x)y = 0$ is non-oscillatory if $\int_1^\infty x\, b^+(x)\, dx < \infty$,

where $b^+(x)$ is the positive part of $b(x)$, provided $a(x)$ is bounded below on $0 < x < \infty$.

Proof. The conditions of Theorem 10 are fulfilled, in particular, if for some $m \geq 1$,

$$\int_0^\infty r^{2m-1} g_1^+(r)\, dr < \infty . \tag{5.8}$$

If we set $m = 1$, $n = 1$ we obtain $\int_0^\infty x\, b^+(x)\, dx < \infty$, and this is equivalent to the hypothesis of the corollary, since b is bounded on $(0, \infty)$.

Since $g_1(r) = g_0(r)$ when $n = 1$ or $n = 3$, Theorem 10 gives a simple criterion for non-oscillation in E^3:

Corollary 2. The equation (1.1) is non-oscillatory in E^3 if L is uniformly elliptic and

$$\int_0^\infty r\, g_0^+(r)\, dr < \infty ,$$

where $g_0^+(r) = \max\{g_0(r), 0\}$ and $g_0(r) = \max\{b(x) : |x| = r\}$.

Proof. Set $m = 1$, $n = 3$ in (5.8). A similar result is true for general n:

Corollary 3. Let $r^2\, g_0(r) \geq \Lambda_0(n-1)(n-3)$ for large r. Then the equation (1.1) is non-oscillatory in E^n if

$$\int_0^\infty r[g_0(r) - \Lambda_0(n-1)(n-3)/4r^2]dr < \infty .$$

Proof. Set $m = 1$ in (5.8).

Remark. Each of the corollaries to Theorem 10 might have been deduced from the corresponding one-dimensional theorem of Hille [11, p.237, Th.2] by using the method of Theorem 9. It should be noted that we have improved Hille's result even in the one-dimensional case, since the use of the Clark-Swanson comparison theorem enables us to remove the requirement that the coefficient $b(x)$ by eventually positive, on account of the inequality

$$g_0(r) \geq b(x), \quad x \in S_r, \quad 0 < r < \infty ,$$

and the fact that the hypotheses imply the non-oscillation of equation (5.7).

Our next theorem, a generalization of a remark of Potter [21, p.468], is useful in the applications of the theory. In fact, we shall use it to show that the estimates in section 4 are the best possible of their kind. Following Potter, we introduce the following notation. Let η be a positive C^2 function defined by

$$(5.9) \qquad [\eta(r)]^{-2} = \Lambda_0^{-1} g_1(r) - (n-1)(n-3)/4r^2, \quad 0 < r < \infty .$$

Let the functions G_1 and G_2 be defined by

$$G_1(r) = \frac{1}{\eta(r)} - \frac{[\eta'(r)]^2}{4\eta(r)} + \frac{\eta''(r)}{2} ,$$

$$G_2(r) = \frac{1}{\eta(r)} - \frac{[\eta'(r)]^2}{4\eta(r)} - \frac{\eta''(r)}{2} ,$$

where primes denote differentiation with respect to r.

Theorem 11. Let L be uniformly elliptic in R_s for some $s > 0$, Λ_0 being the ellipticity constant. Let $g_1(r)$ be the maximum of the positive part of $b(x)$ on S_r, $0 < r < \infty$. Then equation (1.1) is non-oscillatory in R if there exists a positive C^2 function η satisfying (5.9) for large r and either

(5.10) $$G_1(r) \leq 0$$

or

(5.11) $$G_2(r) \leq 0$$

holds for large r.

Proof. We use the argument of Theorem 9, except that we appeal to the remark of Potter mentioned above and the Sturm comparison theorem [5, p.208] to show that the equation (5.7) is non-oscillatory. To see this, we note that the following ordinary differential equations are oscillatory or nonoscillatory together:

(5.12) $$w'' + [\eta(r)]^{-2} w = 0,$$

(5.13) $$[\eta^2(r)v']' + v = 0,$$

$$(5.14) \qquad [\eta(r)z']' + G_1(r)z = 0;$$

$$(5.15) \qquad [\eta(r)y']' + G_2(r)y = 0,$$

for the derivative of a solution of (5.12) is a solution of (5.13), equation (5.14) is obtained from (5.12) by the substitution $z = [\eta(r)]^{-\frac{1}{2}} w$, and (5.15) is obtained from (5.13) by the substitution $y = [\eta(r)]^{\frac{1}{2}} v$. If the hypothesis (5.10) (or (5.11)) holds, the equation (5.14) (or (5.15)) is non-oscillatory by the Sturm comparison theorem [5]. Thus the equation (5.12) is non-oscillatory, and therefore (5.7) is also non-oscillatory. The remaining details of the proof of this theorem are similar to those of Theorem 9 and will be omitted.

6. Sharpness of the results.

In this section we shall give examples to illustrate the theory. In particular, we shall show that the limit conditions of Kneser-Hille type in sections 4 and 5 are the best possible of their kind. Our first example shows that the estimates (4.1) and (4.4) are sharp for each positive integer n, i.e. there exists a non-oscillatory equation for which equality holds in (4.1) and (4.4).

Example 1. For each positive integer n there is a non-oscillatory equation for which

$$\liminf_{r\to\infty} r^2 g(r) = \Lambda_1[\lambda_\alpha + (n-2)^2/4]$$

and

$$\liminf_{r\to\infty} r\int_r^\infty g(t)\,dt = \Lambda_1[\lambda_\alpha + (n-2)^2/4].$$

Construction. Let L be the Schrödinger operator $\nabla^2 + b(x)$, $x \in E^n$, with

(6.1) $$b(x) = (n-2)^2/(4r^2), \qquad |x| = r.$$

Then $\Lambda_1 = 1$, $\lambda_\alpha = 0$. Simple computations show that $\eta(r) = 2r$, $G_1(r) \equiv 0$. Thus equation (1.1) is non-oscillatory by Theorem 11.

On the other hand, since b is given by (6.1), we have

$$\liminf_{r\to\infty} r^2 g(r) = (n-2)^2/4 = \Lambda_1[\lambda_\alpha + (n-2)^2/4].$$

Also,

$$\liminf_{r\to\infty} r\int_r^\infty g(t)dt = \frac{(n-2)^2}{4} \liminf_{r\to\infty} r\int_r^\infty t^{-2}\,dt$$

$$= (n-2)^2/4 = \Lambda_1[\lambda_\alpha+(n-2)^2/4].$$

Our next example shows that the estimates (5.1) and (5.4) are sharp for every positive integer n. In the special case $n = 1$ this reduces to the example given by Hille [11].

Example 2. For each positive integer n there is an oscillatory equation for which $\limsup_{r\to\infty} r^2 g_0(r) = (n-2)^2\Lambda_0/4$.

Construction. Let L be the Schrödinger operator $\nabla^2 + b(x)$, $x \in E^n$, and take

$$(6.2) \qquad b(x) = (n-2)^2/4r^2 + \gamma_1/(r \log r)^2,$$

where $|x| = r$ and γ_1 is a number satisfying $\gamma_1 > \frac{1}{4}$.

Then it is clear that

$$\limsup_{r\to\infty} r^2 g_o(r) = (n-2)^2/4 = (n-2)^2\Lambda_o/4,$$

since $\Lambda_o = 1$ in this case. On the other hand, our equation has particular solutions (viz. solutions depending on r alone) of the form $u = r^{(2-n)/2} (\log r)^s$, where s is a root of the quadratic equation $s^2 - s + \gamma_1 = 0$. Since $\gamma_1 > \frac{1}{4}$, this quadratic equation has complex roots, and thus the radial form of (1.1) is oscillatory. The argument of Theorem 1 therefore shows that equation (1.1) is oscillatory.

It is also clear that if the function b is defined by (6.2), we have

$$\limsup_{r\to\infty} r\int_r^\infty g_1^+(t)\,dt$$

$$= \limsup_{r\to\infty} r\int_r^\infty [1/4t^2 + \gamma_1/(t \log t)^2]dt$$

$$= \frac{1}{4} \qquad = \Lambda_o/4.$$

However, we have shown that (1.1) is oscillatory in this case.

Remark. Examples 1 and 2 have shown, in particular, that for the Schrödinger operator in E^n the constant $(n-2)^2/4$

is critical for each positive integer n, i.e. there exist both oscillatory and non-oscillatory equations for which equality holds in the estimates (4.1), (4.4), (5.1) and (5.4).

We shall now give an example to illustrate the effect of relaxing one or both of the conditions (3.1) in the n-dimensional form of the Leighton-Wintner theorem.

<u>Example 3</u>. There exists a nonoscillatory equation for which

$$\int_1^\infty \frac{dr}{r^{n-1}\ f(r)} < \infty \quad \text{and} \quad \int_1^\infty r^{n-1}\ [g(r) - \lambda_\alpha r^{-2} f(r)]dr = \infty .$$

There also exists a nonoscillatory equation for which

$$\int_1^\infty \frac{dr}{r^{n-1}\ f(r)} = \infty \quad \text{and} \quad \int_1^\infty r^{n-1}[g(r)-\lambda_\alpha r^{-2} f(r)dr] < \infty .$$

<u>Construction</u>. Let L be the Schrödinger operator $\nabla^2 + b(x)$, $x \in E^n$, and take

(6.3) $$b(x) = e^{-r} + (n-2)^2/(4r^2), \qquad |x| = r.$$

Then we may take $f(r) = 1$ (since $a_{ij} = \delta_{ij}$) and $\lambda_\alpha = 0$ (since $R = E^n$).

<u>If $n \geq 3$</u>, then $\int_1^\infty \frac{dr}{r^{n-1}\ f(r)} < \infty$ and

$$\int_1^\infty r^{n-1}[g(r) - \lambda_\alpha r^{-2}\ f(r)]dr = \int_1^\infty r^{n-1}[e^{-r} + (n-2)^2/(4r^2)]dr = \infty .$$

On the other hand, a straightforward computation shows that (in the notation of Theorem 11) $G_1(r) < 0$ for large r, so that equation (1.1) is nonoscillatory by Theorem 11.

<u>If $n \leq 2$</u>, then $\int_1^\infty \frac{dr}{r^{n-1}\ f(r)} = \infty$ and

$$\int_1^\infty r^{n-1}[g(r) - \lambda_\alpha r^{-2} f(r)]dr = \int_1^\infty r^{n-1}[e^{-r} + (n-2)^2/(4r^2)]dr < \infty .$$

However, since $[\eta(r)]^{-2} = e^{-r} + 1/(4r^2)$; a routine computation shows that $G_1(r) < 0$ for large r, and therefore equation (1.1) is nonoscillatory if b is given by (6.3) and $a_{ij} = \delta_{ij}$.

<u>Remark</u>. This example also throws some light on the oscillatory behaviour of the one-dimensional equation $[a(x)u']' + b(x)u = 0$. In particular, the equation

$$(x^s u')' + x^s[e^{-x} + (s-1)^2/(4x^2)]u = 0,$$

(s a nonnegative integer), is nonoscillatory on $0 < x < \infty$.

We continue our inspection of the hypotheses of the theorems in this chapter with the following

<u>Remark</u>. Example 1 provides us with a nonoscillatory equation for which $\int_1^\infty \frac{dt}{h(t)} = \infty$ (Potter's condition). This illustrates the effect of relaxing one of the conditions (3.6). In fact, if b is given by (6.1), then $h(r) = 2r$, so that $\int_1^\infty \frac{dt}{h(t)} = \infty$.

However, $H_1(r) \equiv 0$, so that $\int_1^\infty H_1(t)\ dt < \infty$.

In the remark following Theorem 4 (the n-dimensional form of a result of Potter [21]) we noted that the requirement that the function h be positive was in general more restrictive than the corresponding conditions in Theorem 3 (the n-dimensional form of the Leighton-Moore-Wintner Theorem). However, our next example shows that there are equations for which Theorem 4 gives information not immediately obtainable from Theorem 3.

Example 4. The differential equation

$$\nabla^2 u + [(n-2)^2/(4r^2) + 1/(4r^2 \log r)]u = 0 \tag{6.4}$$

is oscillatory on E^n, even though condition (3.4) does not hold.

To see this, we note that (with the notation of Theorem 4)

$$[h(r)]^{-2} = 1/(4r^2) + 1/(4r^2 \log r).$$

A routine computation shows that $H_2(r) = O(r^{-1})$ for large r, and therefore $\int_s^\infty H_2(r)dr = \infty$ $(s > 1)$. Thus equation (6.4) is oscillatory on account of Theorem 4 and the remark following Theorem 1 regarding the limits of integration. Except in the case $n = 2$, this cannot be concluded in an obvious way from Theorem 3, since (for $s > 1$)

$$\int_s^\infty r^{1-\delta}\, [g(r) - \lambda_\alpha r^{-2} f(r)]dr = \int_s^\infty r^{1-\delta}[\frac{1}{4r^2} + \frac{1}{4r^2 \log r}]dr < \infty$$

for each $\delta > 0$.

Equation (6.4) also satisfies the conditions of Theorem 7. In fact, in this example $\Lambda_1 = 1$, $\lambda_\alpha = 0$, so that

$$g(t) - \Lambda_1[\lambda_\alpha + (n-2)^2/4]t^{-2} = 1/(4t^2 \log t).$$

Thus

$$\int_r^\infty t\{g(t) - \Lambda_1[\lambda_\alpha + (n-2)^2/4]t^{-2}\}dt = \int_r^\infty \frac{dt}{4t \log t} = \infty .$$

This shows that we might have used Theorem 7 instead of Theorem 4 to show that equation (6.4) is oscillatory.

CHAPTER II

EQUATIONS OF ARBITRARY EVEN ORDER

7. Preliminaries.

The definition of an oscillatory equation given here reduces to Glazman's in the case of ordinary equations of order 2m (m = 1, 2, ...)(cf. [9, p. 40]). The criteria obtained below for 2m-th order partial differential equations are direct generalizations of corresponding results of Glazman [9] for the ordinary differential equation

$$(-1)^m \frac{d^{2m}u}{dx^{2m}} - b(x)u = 0 \ , \ 0 \leq x \leq \infty \ .$$

We also obtain for fourth-order partial differential equations criteria extending those obtained by Leighton and Nehari [17] for the ordinary differential equation

$$\frac{d^2}{dx^2}\left[a(x) \frac{d^2u}{dx^2}\right] - b(x)u = 0 \ , \quad a(x) > 0 \ .$$

Most of our theorems are proved by appealing to their one-dimensional forms. An exception is Theorem 20, which is proved by investigating directly the solutions of the comparison equation.

8. Definitions and notation.

We shall consider the linear elliptic differential operator L defined by

$$(8.1) \quad Lu = (-1)^m \sum_{i,j=1}^{n} D_i^m(a_{ij} D_j^m u) - bu\,, \quad a_{ij} = a_{ji}\,,$$

on unbounded domains R in n-dimensional Euclidean space E^n. We shall use the notations of Chapter I, except where otherwise indicated. The coefficients a_{ij} are assumed to be real and of class C^m in $R \cup \partial R$ and the matrix (a_{ij}) is positive definite in R. The coefficient b is assumed to be real and continuous on $R \cup \partial R$. The domain $D(L)$ of L is defined to be the set of all real-valued functions on $R \cup \partial R$ of class $C^{2m}(\overline{R})$.

<u>Definition</u>. A <u>solution</u> of the equation $Lu = 0$ is a function $u \in D(L)$ which satisfies the equation everywhere in R.

We assume that R contains a cylinder of the form

$$G \times \{x_n : 0 \leq x_n < \infty\}\,,$$

where G is a bounded $(n-1)$-dimensional domain. The following notation will be used:

$$R_r = R \cap \{x \in E^n : |x| > r\} .$$

Definition. A bounded domain $N \subset R$ is said to be a nodal domain of a nontrivial solution u of $Lu = 0$ iff u and its partial derivatives of order $\leq m-1$ vanish on ∂N .

Definition. The differential equation $Lu = 0$ is said to be oscillatory in R iff there exists a nontrivial solution u_r of $Lu = 0$ with a nodal domain in R_r for all $r > 0$.

Definition. A function u is said to be oscillatory in R iff u has a zero in R_r for all $r > 0$.

We shall also use the standard notation

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} u , \quad \alpha = (\alpha_1, \ldots, \alpha_n), \ |\alpha| = \sum_{i=1}^n \alpha_i .$$

Our oscillation criteria in §10 will be proved under the assumption that the largest eigenvalue $\Lambda(x)$ of (a_{ij}) is bounded: For some $s > 0$ there exists Λ_1 such that $\Lambda(x) \leq \Lambda_1$, for all $x \in R_s$.

Let μ be the smallest eigenvalue of the problem

$$(8.2)\quad \begin{cases} (-1)^{m-1} \sum_{i=1}^{n} D_i^{\,m}(\Lambda_1 D_i^{\,m} \varphi) = \mu\varphi & \text{in } G \\ D^{\alpha}\varphi = 0 \quad \text{on } \partial G\ , & |\alpha| = 0, 1, \ldots, m-1\ . \end{cases}$$

Let the function $g \in C(0,\infty)$ be such that $g(x_n) \leq b(x)$ on each bounded subdomain of R. (For example, on a bounded domain $G_o \subset R$ we might set

$$g(t) = \min \{b(x) : x \in G_o \text{ and } x_n = t\} \ .)$$

9. Auxiliary results.

As in Chapter I, we shall make use of a monotonicity principle for eigenvalues, which we shall deduce from a form of Poincaré's inequality.

Definition. We shall say that a domain Ω has bounded width $\leq d$ iff there is a line ℓ such that each line parallel to ℓ intersects Ω in a set whose diameter is no greater than d.

For example, the truncated cone $M_{\alpha k t}$ of section 3 has bounded width $< t \sec \alpha$, and the (open) cylinder $G(t)$ of section 10 has bounded width $\leq t$.

Lemma (Poincaré's inequality). If a domain Ω has bounded width $\leq d$, then

$$(9.1) \qquad |\varphi|_{j,\Omega} \leq \gamma\, d^{m-j}\, |\varphi|_{m,\Omega}$$

for all $\varphi \in C_o^m(\Omega)$, $0 \leq j \leq m-1$, where γ is a constant depending only on m and n , and

$$|\varphi|_{m,\Omega} = [\int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2\, dx]^{1/2} \quad .$$

Proof. This is given on pp. 73-75 of [1].

We shall state our next result for the simplest boundary value problem, but the method clearly works for general boundary conditions of the kind given in [24]. On account of the form of the Poincaré inequality here cited, the result (a monotonicity principle for eigenvalues) will be obtained for a more general operator than (8.1).

Let the linear elliptic differential operator M be defined by

$$(9.2) \qquad Mu \equiv (-1)^m \sum_{|p|=|q|=m} D^p(A_{pq}\, D^q u) - Bu \ ,$$

where $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ are multi-indices with integral nonnegative components. As usual, $|p| = \sum_{i=1}^{n} p_i$. For each $z \in E^n$ we write $z^p = \prod_{i=1}^{n} z_i^{p_i}$.

The coefficients $A_{pq}(x)$ are supposed to be of class $C^m(\overline{R})$

and symmetric in the indices. Following Browder [3], we call M elliptic if the following two conditions are fulfilled:

(a) The form $\sum_{|p|=|q|=m} A_{pq}(x)\, z^{p+q}$ is positive definite at each point $x \in R$.

(b) For each bounded domain G with $\overline{G} \subset R$ there exists a number $\mu_o(G) > 0$ such that

$$\int_G \sum_{|p|=|q|=m} A_{pq}\, D^p u\, D^q u\, dx \geq \mu_o(G) \int_G \sum_{|p|=m} (D^p u)^2\, dx$$

for all $u \in C^m(R)$.

We note that condition (a) is the usual ellipticity condition.

If the operator M is of the form (8.1), then condition (b) is redundant, and in fact is a consequence of condition (a). To see this, we note that if M has the form (8.1), then

$$A_{pq} = \begin{cases} a_{ij} & \text{if } p = (me_i) \text{ and } q = (me_j)\,, \\ 0 & \text{otherwise,} \end{cases}$$

where (me_i) is the vector in E^n with m in the i-th place and zeros elsewhere. If condition (a) holds, then there exists a number $\mu_o(G) > 0$ such that

$$\inf_{x\in G} \inf_{|z|=1} \sum_{i,j=1}^{n} a_{ij}(x)\, z_i^m\, z_j^m = \mu_o(G) ,$$

since $\overline{G}$ is compact and the coefficients a_{ij} are continuous. Hence

$$\sum_{i,j=1}^{n} a_{ij}\, z_i^m\, z_j^m \geq \mu_o(G)|z|^{2m} , \quad z \in E^n \quad .$$

This implies

$$\sum_{i,j=1}^{n} a_{ij}\, z_i^m\, z_j^m \geq \mu_o(G) \sum_{i=1}^{n} z_i^{2m} , \quad z \in E^n .$$

This may be written in the form

$$(*) \qquad \sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \geq \mu_o(G) \sum_{i=1}^{n} \xi_i^2 ,$$

where ξ is the vector $(\pm z_i^m)$, the signs being the same as those of the corresponding components in the vector z . Every vector $\xi \in E^n$ may be written in the above form; therefore condition (*) implies

$$\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \geq \mu_o(G) \sum_{i=1}^{n} \xi_i^2 , \qquad \xi \in E^n ,$$

and this implies condition (b).

<u>Monotonicity Principle</u>. For $0 < t < \infty$ let G_t be a domain contained within a domain Ω of bounded width $\leq t$. If $0 < r < s < \infty$ implies $G_r \subset G_s$, $G_r \neq G_s$, then the first eigenvalue $\lambda_0(t)$ of the problem

$$Mu = \lambda u \text{ in } G_t \; ; \quad D^\alpha u = 0 \text{ on } \partial G_t \; , \; |\alpha| = 0, 1, .., m-1$$

is monotone decreasing in t, and $\lim\limits_{t \to 0+} \lambda_0(t) = +\infty$.

<u>Proof</u>. For the first part we may adapt the argument in [7, pp. 400-401]. For the second part we note that since B is uniformly continuous on $\overline{G}$, there exists a constant $k_0 > -\infty$ such that $-\int_G Bu^2 dx \geq k_0 \int_G u^2 dx$ for all $u \in C^m(R)$. Let the Euler-Jacobi functional corresponding to M be defined by

$$J_G[u] = \int_G \{ \sum_{|p|=|q|=m} A_{pq} D^p u \, d^q u - Bu^2 \} \, dx \; .$$

Then condition (b) implies that

$$(9.3) \quad J_G[u] \geq \mu_0(G) \int_G \sum_{|p|=m} (D^p u)^2 dx + k_0 \int_G u^2 dx \; .$$

Extend u continuously to all of Ω by setting $u \equiv 0$

outside G . Apply Poincaré's inequality with $j = 0$ in (9.1) to obtain

$$\int_\Omega u^2 dx \leq \gamma^2 t^{2m} [\int_\Omega \sum_{|p|=m} |D^p u|^2 \, dx] \ .$$

Hence

$$\int_G u^2 dx \leq \gamma^2 t^{2m} [\int_G \sum_{|p|=m} (D^p u)^2 \, dx] \ .$$

Combining this with inequality (9.3) we get

$$J_G[u] \geq \left(k_o + u_o(G)/\gamma^2 t^{2m}\right) \int_G u^2 \, dx \ .$$

The remainder of the proof is as in section 2. (We note that $\gamma > 0$, since $\|u\|_G \neq 0$.)

10. Oscillation criteria.

In this section we obtain oscillation theorems under the hypothesis that the largest eigenvalue of (a_{ij}) is bounded. Our theorems generalize results of Glazman [9] for one dimension to the n-dimensional case. They may be specialized to all of E^n by taking $\varphi \equiv 1$, $\mu = 0$.

<u>THEOREM 10</u>. The differential equation

$$(10.1) \qquad Lu \equiv (-1)^m \sum_{i,j=1}^{n} D_i^{\,m} (a_{ij}\, D_j^{\,m} u) - bu = 0$$

is oscillatory in R if for some $s > 0$ there exists a number $\Lambda_1 > 0$ such that $\Lambda(x) \leq \Lambda_1$, for all $x \in R_s$, and if

$$(10.2) \qquad \int_0^\infty [g(t) + \mu]\, dt = +\infty ,$$

where μ and g are defined above in section 8.

<u>Proof</u>. We compare (8.1) with the separable equation

$$(10.3) \qquad (-1)^m \sum_{i=1}^{n} D_i^{\,m}(\Lambda_1\, D_i^{\,m} v) - b^* v = 0 ,$$

where $b^*(x) = g(x_n)$. The hypothesis (10.2) implies that the ordinary differential equation

$$(10.4) \qquad (-1)^m \Lambda_1\, D_n^{\,2m} w - [g(x_n) + \mu] w = 0$$

is oscillatory on account of Glazman's generalization [9, p. 104, Th. 13] of the theorem of Leighton [16] and

Wintner [28]. Let $r > 0$ be given. Then there exists a solution w of (10.4) with zeros of order m at $x_n = \delta_1, \delta_2$, where $\delta_2 > \delta_1 \geq \max\{r,s\}$. If φ is an eigenfunction of (8.2) corresponding to the eigenvalue μ, then the function v defined by $v(x) = w(x_n)\varphi(\overline{x})$, where $\overline{x} = (x_1, x_2, \ldots, x_{n-1})$, is a solution of (10.3) by direct calculation, with a nodal domain

$$G_1 = G \times \{x_n : \delta_1 < x_n < \delta_2\} .$$

In fact, if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\overline{\alpha}, \alpha_n)$, then

$$D^\alpha v = D_n^{\alpha_n} D^{\overline{\alpha}} v = D_n^{\alpha_n} w(x_n) D^{\overline{\alpha}} \varphi(\overline{x}) .$$

Hence $D^\alpha v = 0$ on ∂G_1 for $|\alpha| = 0, 1, \ldots, m-1$, since δ_1 and δ_2 are m-fold zeros of $w(x_n)$ and φ has nodal domain G. Thus v has a nodal domain $N_r \subset R_r$ for all $r > 0$. In fact, $x \in G_1$ implies $|x| \geq |x_n| > \delta_1 \geq r$, hence $x \in R_r$. Thus (10.3) is oscillatory. We now apply a theorem of Swanson [24, Th. 4]. The inequality

$$\int_{G_1} \{ \sum_{i,j=1}^{n} (\Lambda_1 \delta_{ij} - a_{ij}) D_i^m v \, D_j^m v + (b - b^*) v^2 \} \, dx \geq 0$$

holds whenever $G_1 \subset R_s$, on account of the hypotheses

$$\sum_{i,j=1}^{n} a_{ij}(x)z_i z_j \leq \Lambda_1 \sum_{i=1}^{n} z_i^2 \ , \ x \in R_s \ , \ z \in E^n$$

$$b^*(x) = g(x_n) \leq b(x) \ , \ x \in G_1 \qquad .$$

Hence the eigenvalue problem

$$Lu = \lambda u \quad \text{in} \quad G_1 \ ; \quad D^\alpha u = 0 \quad \text{on} \quad \partial G_1 \ , \ |\alpha| = 0, 1, \ldots, m-1$$

has at least one (in particular, the smallest) eigenvalue less than or equal to zero. Let

$$G(t) = G \times \{x_n : \delta_1 < x_n < t\} \ , \ \delta_1 < t \leq \delta_2 \ ,$$

and let $\lambda_o(t)$ denote the smallest eigenvalue of the problem

$$Lu = \lambda(t)u \quad \text{in} \quad G(t) \ ; \ D^\alpha u = 0 \quad \text{on} \quad \partial G(t) \ , \ |\alpha| = 0,1,\ldots,m-1.$$

By the monotonicity principle in section 9, $\lambda_o(t)$ is monotone nonincreasing in $\delta_1 < t \leq \delta_2$ and $\lim_{t \to \delta_1 +} \lambda_o(t) = +\infty$. Since $\lambda_o(\delta_2) \leq 0$, there exists a number T in $(\delta_1, \delta_2]$ such that $\lambda_o(T) = 0$. This means that $G(T)$ is a nodal domain of a nontrivial solution of (10.1), and since $G(T) \subset G_1 \subset R_r$ for arbitrary $r > 0$, equation (10.1) is oscillatory in R. This completes the proof of the theorem.

This theorem contains Glazman's generalization [9, p. 104] of the Leighton-Wintner theorem. To see this, set $n = 1$ and $a_{11}(x) = 1$, and recall that for $R = E^n$ we may take $\varphi \equiv 1$, $\mu = 0$.

Our next theorem extends to n dimensions Glazman's generalization [9, p. 100] of a result of Hille [11, Th. 5]. Glazman's result is the special case $R = E^n$, $n = 1$, $a_{11}(x) = 1$ of our theorem.

THEOREM 11. Let $\Lambda(x)$ be bounded, i.e. for some $s > 0$ there exists $\Lambda_1 > 0$ such that $\Lambda(x) \leq \Lambda_1$, $x \in R_s$. If $g(x_n) + \mu \geq 0$ for large x_n and

$$\limsup_{r \to \infty} r^{2m-1} \int_r^\infty [g(t)+\mu]dt > \Lambda_1 A_m^2 ,$$

where

$$A_m^{-1} = \frac{\sqrt{2m-1}}{(m-1)!} \sum_{k=1}^{m} \frac{(-1)^{k-1}\binom{m-1}{k-1}}{2m-k} ,$$

then the equation (10.1) is oscillatory.

Proof. As in the proof of Theorem 10, we compare (10.1) with (10.3), quoting Glazman's generalization [9, p. 100] of a theorem of Hille [11, Th. 5] to show that (10.4) is oscillatory. The remainder of the proof follows that of Theorem 10 without change.

THEOREM 12. If the inequality

$$(10.5) \qquad g(x_n)+\mu \geq \Lambda_1 \alpha_m^2 x_n^{-2m} \quad (\alpha_m = (2m-1)!!/2^m)$$

holds for sufficiently large x_n and

$$(10.6) \qquad \limsup_{r \to \infty} (\log r)\int_r^\infty t^{2m-1}|g(t)+\mu-\Lambda_1\alpha_m{}^2 t^{-2m}|\, dt = \infty ,$$

then equation (10.1) is oscillatory in R.

Proof. The hypotheses imply that the ordinary differential equation (10.4) is oscillatory on account of a theorem of Glazman [9, p. 102]. The remainder of the proof is similar to that of Theorem 10 and will be omitted.

Corollary. Let the largest eigenvalue $\Lambda(x)$ of (a_{ij}) be bounded in R_s for some $s > 0$: $\Lambda(x) \leq \Lambda_1$, for all $x \in R_s$ and some $\Lambda_1 > 0$. Then (10.1) is oscillatory in R if for sufficiently large x_n and some $\delta > 0$

the inequality

$$x_n^{2m} [g(x_n)+\mu] > \Lambda_1(\alpha_m^{2}+\delta)$$

is satisfied.

Proof. The first hypothesis (10.3) of Theorem 12 is clearly fulfilled. Moreover, since $g(t) + \mu - \Lambda_1\alpha_m^{2}t^{-2m} > \Lambda_1\delta t^{-2m}$ for large t, the second hypothesis (10.6) of Theorem 12 is also satisfied.

Remark. This result generalizes the classical Kneser-Hille theorem [11] in four directions: (i) to equations of arbitrary even order, (ii) to equations with variable leading coefficients, (iii) to n dimensions, (iv) to equations not defined on all of E^n (i.e. on limit-cylindrical domains).

It is possible to prove this corollary by comparing (10.1) with (10.3), citing Glazman's generalization [9, Th. 9, p. 96] of the Kneser-Hille theorem [11] to show that the ordinary differential equation (10.4) is oscillatory.

It should be noted that the result of Glazman just cited [9, Th. 9, p. 96] is the special case $R = E^n$,

$n = 1$, $a_{11}(x) = 1$ of our corollary.

11. Equations with one variable separable.

In this section we shall consider the equation

$$(11.1)\quad D_n^{2m}u + \sum_{i,j=1}^{n-1} D_i^{m}[a_{ij}(\bar{x})D_j^{m}u] - (-1)^m b(x_n)u = 0 ,$$

where $\bar{x} = (x_1, x_2, \ldots, x_{n-1})$. Following Swanson [26] we let μ^* be the smallest eigenvalue of the problem

$$(11.2)\quad \begin{cases} (-1)^{m-1} \sum_{i,j=1}^{n-1} D_i^{m}[a_{ij}(\bar{x})D_j^{m}\varphi] = \mu^*\varphi & \text{in } G \\ D^{\bar{\alpha}}\varphi = 0 \quad \text{on } \partial G , & |\alpha| = 0, 1, \ldots, m-1 , \end{cases}$$

where G is as in section 8.

Each of the theorems of the preceding section has an analogue in this case, but <u>without the assumption that $\Lambda(x)$ is bounded</u>. As an example we state and prove the following analogue of Theorem 10:

<u>THEOREM 10A</u>. The differential equation

$$(11.3)\quad (-1)^m\{D_n^{2m}u + \sum_{i,j=1}^{n-1} D_i^{m}[a_{ij}(\bar{x})D_j^{m}u]\} - b(x_n)u = 0$$

is oscillatory in R if

$$(11.4) \qquad \int_0^\infty [b(t)+\mu^*]\,dt = +\infty \quad .$$

Proof. The hypothesis (11.4) implies that the ordinary differential equation

$$(11.5) \quad (-1)^m D_n^{2m} v - [b(x_n)+\mu^*]\, v = 0$$

is oscillatory on account of Glazman's generalization [9, p. 104] of the theorem of Leighton [16] and Wintner [28]. Let $r > 0$ be given. Then there exists a solution v of (11.5) with zeros of order m at $x_n = \delta_1, \delta_2$, where $\delta_2 > \delta_1 \geq r$. If φ is an eigenfunction of (11.2) corresponding to eigenvalue μ^*, then the function u defined by $u(x) = v(x_n)\varphi(\overline{x})$ is a solution of (11.3) by direct calculation, with a nodal domain $G_1 = \{x : \overline{x} \in G , \delta_1 < x_n < \delta_2\}$. Thus there exists u with a nodal domain $G_1 \subset R_r$ for arbitrary $r > 0$, since $x \in G_1$ implies $|x| \geq |x_n| > \delta_1 \geq r$, so that $x \in R_r$. Hence equation (11.3) is oscillatory.

The analogues of Theorems 11 and 12 for equation (11.3) are proved similarly. We state them without proof.

THEOREM 11A. If $b(x_n) + \mu^* \geq 0$ for large x_n and

$$\limsup_{r \to \infty} r^{2m-1}\int_r^{\infty} [b(t)+\mu^*]\, dt > A_m^2 ,$$

then the equation (11.3) is oscillatory in R .

THEOREM 12A. The equation (11.3) is oscillatory in R if the inequality $b(x_n) + \mu^* \geq \alpha_m^2 x_n^{-2m}$ holds for large x_n and

$$\limsup_{r \to \infty} (\log r)\int_r^{\infty} t^{2m-1}|b(t)+\mu^*-\alpha_m^2 t^{-2m}|\, dt = \infty .$$

Remark. Swanson [23] has obtained oscillation criteria for the second order separable equation

$$D_n[a(x_n)D_n u] + \sum_{i,j=1}^{n-1} D_i[a_{ij}(\overline{x})D_j u] + b(x_n)u = 0 .$$

Lack of suitable one-dimensional oscillation theorems has restricted us to equations with $a(x_n) \equiv 1$ in the general even order case.

12. Fourth-order equations on limit-cylindrical domains.

In this section we shall derive oscillation criteria for the equation

$$(12.1)\quad D_n^{\,2}[a(x_n)D_n^{\,2}u]+\sum_{i,j=1}^{n-1} D_i^{\,2}[a_{ij}(\overline{x})D_j^{\,2}u] - b(x_n)u = 0 ,$$

where the coefficients a , a_{ij} , b satisfy the conditions in section 8. We also suppose that $a(x_n)$ and $b(x_n)+\mu^*$ are positive for large x_n . Our theorems constitute extensions of well-known results of Leighton and Nehari [17].

<u>THEOREM 13</u>. Let α be an arbitrary real constant, and let $a(x_n) > 0$, $b(x_n)+\mu^* > 0$ for sufficiently large x_n . Then equation (12.1) is oscillatory in R if

$$(12.2)\quad \begin{cases} \limsup\limits_{t \to \infty} t^{-2-\alpha}a(t) < 1 \\[2ex] \liminf\limits_{t \to \infty} t^{2-\alpha}[b(t)+\mu^*] > \dfrac{(1-\alpha^2)^2}{16} \end{cases} ,$$

where μ^* is given by (11.2).

<u>Proof</u>. The hypotheses (12.2) imply that the ordinary differential equation

$$(12.3)\quad D_n^{\,2}[a(x_n)D_n^{\,2}v] - [b(x_n) + \mu^*]\, v = 0$$

has an oscillatory solution, i.e. has a solution with infinitely many zeros, on account of a theorem of Leighton and Nehari [17, Th. 6.2]. Let $r > 0$ be given. By another theorem in the paper just cited [17, Th. 3.6] there exists a solution v of (12.3) with double zeros at $x_n = \delta_1, \delta_2$, where $\delta_1 > \delta_2 \geq r$. The remainder of the proof follows that of Theorem 10A without change and will be omitted.

This result extends to n-dimensions part of the one-dimensional theorem of Leighton and Nehari [17, Th. 6.2] just cited.

In the case that $a(x_n) \equiv 1$, another part of [17, Theorem 6.2] shows that the conclusion of Theorem 13 holds if the hypotheses (12.2) are replaced by

$$\liminf_{t \to \infty} t^4[b(t) + \mu^*] > 9/16 .$$

<u>THEOREM 14</u>. The equation (12.1) is oscillatory in R if there exists $\alpha > 0$ such that

$$(12.4) \quad \int_\alpha^\infty \frac{dt}{a(t)} = \infty \quad \text{and} \quad \int_\alpha^\infty t^2[b(t) + \mu^*]\, dt = \infty .$$

<u>Proof</u>. The hypotheses (12.4) imply that the equation (12.3) has an oscillatory solution, on account of a theorem

of Leighton and Nehari [17, Th. 6.11]. The remaining details of the proof follow familiar lines and will be omitted.

In the case $a(x_n) \equiv 1$, the above result takes the following form:

<u>THEOREM 15</u>. The equation

$$D_n^4 u + \sum_{i,j=1}^{n-1} D_i^2 [a_{ij}(\bar{x}) D_j^2 u] - b(x_n) u = 0$$

is oscillatory in R if there exists $\alpha > 0$ such that

$$\int_\alpha^\infty t^r [b(t) + \mu^*] \, dt = \infty$$

holds for some $r < 3$.

<u>Proof</u>. Since we shall use the argument of Theorem 13, we need only show that (12.3) has an oscillatory solution. This fact is a consequence of a result of Leighton and Nehari [17, Cor. 6.10]. We omit the remaining details of the proof.

Our next two theorems give oscillation criteria for a special case of (12.1), namely

$$(12.5)\quad D_n^{\,2}[a(x_n)D_n^{\,2}u]+\sum_{i,j=1}^{n-1} D_i^{\,2}[a_{ij}(\bar{x})D_j^{\,2}u]+c(x_n)u = 0 ,$$

where the coefficient c is real and continuous on $\overline{R}$, and the following inequalities hold for large x_n :

$$(12.6)\qquad a(x_n) > 0 , \qquad c(x_n) - \mu^* > 0 .$$

<u>THEOREM 16</u>. Let s be an arbitrary real constant, and suppose that the inequalities (12.6) hold for sufficiently large x_n . Then equation (12.5) has a solution oscillatory in R if

$$\limsup_{t \to \infty} t^{-2-s}\, a(t) \leq 1$$

and

$$\liminf_{t \to \infty} t^{2-s}\, [c(t) - \mu^*] > s^2/4 ,$$

where μ^* is given by (11.2). In the case $a(x_n) \equiv 1$, the conclusion remains valid if (12.6) holds and

$$\liminf_{t \to \infty} t^4 [c(t) - \mu^*] > 1 .$$

Proof. The hypotheses imply that the ordinary differential equation

$$(12.7)\quad D_n^2[a(x_n)D_n^2 v] + [c(x_n) - \mu^*]\, v = 0$$

has an oscillatory solution, on account of a result of Leighton and Nehari [17, Th. 11.1]. Let $r > 0$ be given. Then there exists a solution v of (12.7) with a zero in R_r. If φ is an eigenfunction of (11.2) corresponding to the eigenvalue μ^*, then the function u defined by $u(x) = v(x_n)\varphi(\bar{x})$ is a solution of (12.5) by direct computation, with a zero in R_r. Since r is arbitrary, this implies that u is oscillatory in R, and the theorem is proved.

THEOREM 17. Let (12.6) hold for sufficiently large x_n and suppose $a(x_n) \equiv 1$. Then equation (12.5) has a solution oscillatory in R if there exists $\alpha > 0$ such that

$$\int_\alpha^\infty t^2[c(t) - \mu^*] = \infty \quad .$$

Proof. The proof is similar to that of Theorem 16, and appeals to a criterion of Leighton and Nehari [17, Th. 11.4] to show that (12.7) has an oscillatory solution. We omit the details.

13. Fourth order equations on all of E^n .

The equation to be considered is the special case $m = 2$ of (10.1), namely

$$(13.1) \quad Lu \equiv \sum_{i,j=1}^{n} D_i^2(a_{ij}\, D_j^2 u) - bu = 0 \ .$$

The general conditions on L are as in section 8, except that the domain R will be all of E^n . The nodal domains of the comparison equation will be annuli of the form $\{x : r_1 < |x| < r_2\}$, hence we need a slight extension of the monotonicity principle proved in Chapter I. We shall say that an annulus of the form $\{x : r_1 < |x| < r_1+t\}$ *has thickness* t . Even though this annulus has bounded width, the latter does not approach zero as $t \to 0+$, so that the form of the monotonicity principle in §9 is inapplicable here.

Lemma (*Poincaré's inequality for annuli*). If an annulus Ω has thickness t , then

$$(13.2) \qquad |u|_{o,\Omega} \leq t^2 \, |u|_{2,\Omega}$$

for all $u \in C^2(\Omega)$, where

$$|u|_{k,\Omega} = [\int_\Omega \sum_{|\alpha|=k} (D^\alpha u)^2 \, dx]^{1/2} \quad .$$

Proof. In the course of proving the monotonicity principle for eigenvalues in the second order case for annular domains, we showed (cf. (2.5)) that

(13.3) $$|u|^2_{0,\Omega} \leq t^2 |u|^2_{1,\Omega} \quad .$$

Applying this inequality to the first partial derivatives $D_i u$, we obtain

$$|D_i u|^2_{0,\Omega} \leq t^2 \, |D_i u|^2_{1,\Omega}$$

$$= t^2 \int_\Omega \sum_{|\alpha|=1} (D^\alpha D_i u)^2 \, dx \text{ (by definition)}$$

$$= t^2 \int_\Omega \sum_{j=1}^n (D_j D_i u)^2 \, dx \ , \ i=1,2,\ldots,n \ .$$

Hence

$$|u|^2_{1,\Omega} = \int_\Omega \sum_{i=1}^n (D_i u)^2 \, dx = \sum_{i=1}^n |D_i u|^2_{0,\Omega}$$

$$\leq t^2 \int_\Omega \sum_{i,j=1}^n (D_i D_j u)^2 \, dx = t^2 \int_\Omega \sum_{|\alpha|=2} (D^\alpha u)^2 \, dx$$

$$= t^2 |u|^2_{2,\Omega} \quad .$$

Combining this with (13.3) we obtain

$$|u|^2_{0,\Omega} \leq t^4 |u|^2_{2,\Omega} \quad ,$$

from which (13.2) follows immediately.

We now state the required form of the monotonicity principle for eigenvalues. In view of the form of Poincaré's inequality proved here, we state the result for the more general operator

$$(13.4) \qquad Mu \equiv \sum_{|p|=|q|=2} D^p \, (A_{pq} \, D^q u) - bu \; .$$

The principle will then be true for the operator L , since L is a special case of M (cf. section 9).

We note that the operator in (13.4) is the special case $m = 2$ of (9.2). We shall accordingly suppose that conditions (a) and (b) of section 9 are satisfied. As noted in section 9, however, when we apply the monotonicity principle for the operator L , its ellipticity alone (i.e. condition (a)) is enough to guarantee the truth of the principle, since condition (a) implies condition (b) in this case.

Monotonicity Principle (Annular Domains). Let $\Omega(t)$ be an annulus of thickness t . Then the first eigenvalue

$\lambda_o(t)$ of the problem

$$Mu = \lambda u \quad \text{in} \quad \Omega(t) \ ; \quad D^{\alpha}u = 0 \quad \text{on} \quad \partial\Omega(t), \ |\alpha| = 0, 1$$

is monotone nonincreasing (for $t > 0$) and $\lim_{t \to 0+} \lambda_o(t) = +\infty$.

Proof. The proof is similar to that of the corresponding result in section 9 and will be omitted.

14. Oscillation theorems.

The main result of this section is a theorem of the Kneser-Hille type for equation (13.1). It contains the corresponding result of Leighton and Nehari [17] for the fourth order ordinary differential equation $u^{iv} - bu = 0$ and extends the analogous theorem of Glazman [9] for an operator with harmonic leading term to one with (in particular) biharmonic leading term.

First we need a few technical lemmas of an elementary character. We shall compare equation (13.1) with the separable equation

$$\Lambda_1 \Delta^2 v - Bv = 0 \quad , \tag{14.1}$$

where Λ_1 is an upper bound on the largest eigenvalue $\Lambda(x)$

of the matrix $(a_{ij}(x))$; i.e. there exists a number Λ_1 such that $\Lambda(x) \leq \Lambda_1$. The continuous function B is such that there exists a function g_o satisfying

$$B(x) = g_o(|x|) \leq b(x) \quad , \quad x \in E^n .$$

<u>Notation</u>. Let $F(s,n)$ be the polynomial of degree four in s defined by

$$F(s,n) = s(s-2)(s+n-2)(s+n-4) \quad .$$

As in Chapter I, we introduce spherical polar coordinates r , θ_1 , θ_2 , ..., θ_{n-1} . By writing (14.1) in terms of these coordinates, we find that (14.1) has solutions (in particular) of the form

$$v(x) = \rho(r) \ , \ 0 \leq r < \infty \qquad ,$$

where ρ satisfies the ordinary differential equation

$$(14.2) \qquad \Lambda_1 \, \Delta^2\rho - g_o(r)\rho \;=\; 0 \quad .$$

<u>Proposition 18</u>. The polynomial $F(s,n)$ has a relative maximum at $s = 2-n/2$.

Proof. If $s = 2-n/2$ is a zero of $F(s,n)$, then it must be a repeated zero, since $F(s,n)$ is symmetrical about $s = 2-n/2$. Moreover, since $F(s,n)$ is a polynomial of degree four with positive leading coefficient and at least two distinct real zeros, a consideration of the shape of its graph shows that the repeated zero at $s = 2-n/2$ is a maximum point.

If $s = 2-n/2$ is not a zero of $F(s,n)$, we use logarithmic differentiation to show that

$$(14.3) \qquad \frac{F'(s,n)}{F(s,n)} = \frac{1}{s} + \frac{1}{s-2} + \frac{1}{s+n-2} + \frac{1}{s+n-4} ,$$

where the prime denotes differentiation with respect to s, and the formula (14.3) holds except at zeros of $F(s,n)$. Thus

$$F'(2-n/2,n) = 0 .$$

Differentiation of (14.3) yields

$$F''(s,n) = -F(s,n)\left[\frac{1}{s^2} + \frac{1}{(s-2)^2} + \frac{1}{(s+n-2)^2} + \frac{1}{(s+n-4)^2}\right]$$

$$+ F'(s,n)\left[\frac{1}{s} + \frac{1}{s-2} + \frac{1}{s+n-2} + \frac{1}{s+n-4}\right] .$$

It follows that when $F'(s,n) = 0$ (in particular, when

$s = 2-n/2)$, $F''(s,n)$ has sign opposite to that of $F(s,n)$. But

$$F(2-n/2,n) = (2-n/2)^2(n/2)^2 > 0 ,$$

since by hypothesis $F(2-n/2,n) \neq 0$. Hence $F''(2-n/2,n)<0$ and the proposition is proved.

Proposition 19. If the inequality

$$(14.4) \qquad \omega > \Lambda_1 n^2(n - 4)^2/16$$

holds, then the equation

$$(14.5) \qquad \Lambda_1 F(s,n) - \omega = 0$$

has at least one pair of complex roots.

Proof. By Proposition 18, the polynomial $F(s,n)$ has a nonnegative relative maximum at $s = 2-n/2$. Hence the polynomial $F(s,n) - \omega / \Lambda_1$ has a local maximum at $s = 2-n/2$. Condition (14.4) implies that $F(2-n/2,n) - \omega/\Lambda_1 = n^2(n-4)^2/16 - \omega/\Lambda_1 < 0$. Thus the relative maximum of the polynomial $F(s,n) - \omega/\Lambda_1$ at $s = 2-n/2$ is negative. Hence equation (14.5) has at least one pair of complex roots. We are now in a position to state and prove the main result of this section.

THEOREM 20. Suppose that the largest eigenvalue $\Lambda(x)$ of $(a_{ij}(x))$ is bounded in E^n (or at least outside some hypersphere), say $\Lambda(x) \leq \Lambda_1$. Then equation (13.1) is oscillatory in E^n if

$$(14.6) \qquad \liminf_{r \to \infty} r^4 g(r) > \Lambda_1 \, n^2(n-4)^2/16 \quad ,$$

where

$$g(r) = \min \{b(x) : |x| = r\} \quad .$$

Proof. The hypothesis (14.6) implies that these exist constants r_o and ω such that

$$r^4 \, g(r) > \omega > \Lambda_1 \, n^2(n-4)^2/16$$

for all $r > r_o$. We then compare (13.1) with the equation

$$(14.7) \qquad \Lambda_1 \, \Delta^2 v - \omega r^{-4} v = 0 \quad ,$$

which is the special case $B(x) = \omega|x|^{-4}$ of (14.1). The radial form of equation (14.7) is of Euler type and thus (14.7) has solutions (in particular) of the form $v(x) = |x|^s$, where s satisfies (14.5). This is easily seen by noting that

$$\Delta r^s = s(s+n-2)r^{s-2}$$

and

$$\Delta^2 r^s = s(s+n-2)\Delta r^{s-2}$$

$$= s(s+n-2)(s-2)(s+n-4)r^{s-4} .$$

Since $\omega > \Lambda_1 \, n^2(n-4)^2/16$, hypothesis (14.4) of Proposition 19 is fulfilled, and therefore equation (14.5) has at least pair of one complex roots. This implies that there exists an oscillatory solution of the radial form of (14.7), i.e. a solution with infinitely many zeros. Let $\alpha > 0$ be given. Then a theorem of Leighton and Nehari [17, Th. 3.6] implies that there exists a solution ρ of the ordinary differential equation

$$\Lambda_1 \, \Delta^2 \rho - \omega r^{-4} \rho = 0 \tag{14.8}$$

with double zeros at $r = \delta_1, \delta_2$, where $\delta_2 > \delta_1 > \max\{r_o, \alpha\}$, since $\Delta^2\rho(r) = (r^{n-1}\rho'')'' + [(1-n)r^{n-3}\rho']'$, so that (14.8) may be transformed into the form considered in [17,Th. 3.6]. (Note that $(1-n)r^{n-3} \leq 0$ for all positive integers n , and see the remark following [17, Th. 12.1]). It follows that the function v defined by $v(x) = \rho(x)$ is a solution of (14.7) with a nodal domain

$$N = \{x : \delta_1 < |x| < \delta_2\} .$$

In fact,

$$D_i v = \frac{d\rho}{dr} \frac{\partial r}{\partial x_i}$$

and the right side is zero on $r = \delta_1$, $r = \delta_2$. Thus for any $\alpha > 0$ there exists a solution v with a nodal domain in the region $\{x : |x| > \alpha\}$, since $x \in N$ implies $|x| > \delta_1 > \alpha$. Hence (14.7) is oscillatory.

We now apply a theorem [24, Th. 4] of C. A. Swanson. Because of the hypotheses

$$\sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \leq \Lambda(x)|z|^2 \leq \Lambda_1 |z|^2 , \; x \text{ suff. large}, \; z \in E^n ,$$

$$\omega r^{-4} < g(r) \leq b(x) , \; |x| > r_o ,$$

we have

$$\int_N \{ \sum_{i,j=1}^{n} (\Lambda_1 \delta_{ij} - a_{ij}) D_i^2 u \, D_j^2 u + (b - \omega |x|^{-4}) u^2 \} \, dx > 0 .$$

Hence the eigenvalue problem

$$Lu = \lambda u \quad \text{in} \quad N \; ; \quad u = D_i u = 0 \quad \text{on} \quad \partial N \; , \; i = 1, 2, \ldots, n$$

has at least one eigenvalue (in particular, the smallest) less than zero. Let

$$N(t) = \{x : \delta_1 < |x| < t\} \; , \quad \delta_1 < t \leq \delta_2 \; ,$$

and let $\lambda_o(t)$ denote the smallest eigenvalue of the problem

$$Lu = \lambda u \quad \text{in} \quad N(t) \; ; \; u = D_i u = 0 \quad \text{on} \quad \partial N(t), \; i = 1, 2, \ldots, n.$$

By the monotonicity principle of section 13, $\lambda_o(t)$ is monotone nonincreasing in $\delta_1 < t \leq \delta_2$ and $\lim_{t \to \delta_1 +} \lambda_o(t) = +\infty$. Since $\lambda_o(\delta_2) \leq 0$, there exists a number T in $(\delta_1, \delta_2]$ such that $\lambda_o(T) = 0$. Thus $N(T)$ is a nodal domain of a nontrivial solution of (13.1). Moreover, $N(T) \subset N \subset \{x : |x| > \alpha\}$ and α is arbitrary, therefore equation (13.1) is oscillatory in E^n.

<u>Corollary</u> (Leighton and Nehari [17, part of Th. 6.2]). The ordinary differential equation

$$u^{iv} - b(x)u = 0$$

is oscillatory if

$$\liminf_{x \to \infty} x^4 b(x) > 9/16 . \tag{14.9}$$

<u>Proof</u>. This corollary is the special case $a_{ij} = \delta_{ij}$, $n = 1$ of Theorem 20.

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