# A SURVEY OF RECENT RESULTS

on

# TORSION FREE ABELIAN GROUPS

by

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## ABSTRACT

This thesis is a survey of some recent results concerning torsion free abelian groups, hereafter referred to as groups. The emphasis is on countable groups, particularly groups of finite rank.

Section 1 contains the introduction and some notation used throughout this thesis. We begin in section 2 by describing the general nature of the existing characterizations for countable groups and by describing why these characterizations do not provide satisfactory systems of invariants. We include here a brief description of a classification for groups of arbitrary power. Pathologies of groups are discussed in section 3. We briefly discuss rank one groups and completely decomposable groups and then present examples to show the vast number of indecomposable groups which exist and that a group may have two different decompositions into the direct sum of indecomposable groups. Quasi-isomorphism and the ring of quasi-endomorphisms of a group are introduced in section 4 and discussed briefly. We present the theorems which establish the importance of these notions; namely that (i) quasi-decompositions of certain groups are unique up to quasi-isomorphism and (ii) the quasi-decomposition theory of certain groups is equivalent to the decomposition theory of the quasi-endomorphism ring considered as a right module over itself. Included under "certain groups" are the groups of finite rank.

Section 5 is devoted to rank two groups. We outline the development of the quasi-isomorphism invariants for rank two groups, due to Beaumont and Pierce, and discuss some of their

applications. For example, conditions, in terms of the invariants, are given for quasi-isomorphic rank two groups to be isomorphic. Type sets are reviewed in section 6. We present both necessary and sufficient conditions for sets of types to be the type sets of rank two groups and of groups of arbitrary finite rank. We devote section 7 to a brief discussion of the notion and importance of quasi-essential groups. The ideas of irreducibility and the psuedo-socle are defined in section 8. We demonstrate how these ideas affect the structure of the quasi-endomorphism ring by showing how they can be used to compute the quasi-endomorphism ring of rank two groups.

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## INTRODUCTION

The problem of classifying countable abelian groups by a complete set of invariants has not yet been solved. Satisfactory complete systems of invariants have been found for some special classes of abelian groups such as the torsion groups and the rank one torsion free groups. However the mixed countable abelian groups have not yet been classified. Every mixed group can be regarded as an extension of a torsion group, namely its torsion part, by a torsion free group. Hence the classification of torsion free groups plays an important part in the classification of mixed groups. Certain systems of invariants do exist for the countable torsion free abelian groups. These systems provide schemes for the construction of such groups and the groups obtained are isomorphic iff the schemes are equivalent in some sense. The schemes are usually in terms of matrices and the equivalence problem for the schemes is unsolved. As a consequence, these invariants do not provide a satisfactory characterization. The purpose of this thesis is to survey some of the work that has been done in the study and classification of torsion free abelian groups and to present resumeés of the main results. As a result, very few of the theorems will be proved and none will be proved in detail. Complete details of proof can, of course, be found in the papers indicated. The papers reviewed here will, for the large part, be papers presented since the publication in 1958 of L. Fuchs book "Abelian Groups". The emphasis will be on torsion-free abelian groups of finite rank.

Throughout this thesis we will use the following notation:

$Z$: the domain of rational integers

$N$: $\{n \in Z \mid n \geq 0\} \cup \{\infty\}$

$R$: the field of rational numbers

$Z^{(p)}$: The domain of p-adic integers

$R^{(p)}$: the field of p-adic numbers

$R_p = \{\frac{a}{b} \in R \mid (b,p) = 1\}$ where $p$ is a rational prime

$R^n$ : an n-dimensional rational vector space

$\pi$ : the set of rational primes

$[\theta]$: the type containing the characteristic $\theta$

If $G, G_\alpha$ are groups and $x \in G$ then we will use

$\prod_\alpha G_\alpha$: the cartesian product of the $G_\alpha$ where $\alpha$ varies over some index set.

$h_p(x,G)$: the p-height of $x$ in $G$

$h(x,G)$ : the height of $x$ in $G$ ; i.e. $h(x,G)(p) = h_p(x,G)$

$t(x,G)$ : the type of $x$ in $G$ . Where there is no chance of confusion we will write $t(x)$ .

$T(G)$ : the type set of $G$ i.e. $T(G) = \{t(x,G) \mid x \in G\}$

$G_t = \{g \in G \mid t(g,G) \geq t\}$ where $t$ is a type

$r(G)$ : the rank of $G$ .

If $G$ is a torsion free abelian group we will use $V$ to denote the minimal divisible group containing $G$. $V$ can be considered as a rational vector space with dimension equal to $r(G)$. Hence $G$ is a full subgroup of $V$ (a group $A$ is a full subgroup of a torsion free abelian group $B$ if $B/A$ is a torsion group). All torsion free abelian groups with rank $\leq r(G)$ can be considered as subgroups of $V$. If $r(G) = n$, we will sometimes find it convenient to write $V = R^n$. Throughout this thesis we will use the word group, unless otherwise specified, to denote a torsion free abelian group and thus a full subgroup of a rational vector space with dimension equal to the rank of the group.

By a basis of a group we will mean a maximal linearly independent set of elements of the group. A basis of a group $G$ will also be a basis, in the usual sense, of the vector space $V$.

## 2. Isomorphism Invariants for Countable Groups.

As we mentioned in the introduction, certain systems of invariants do exist for the countable groups. These systems usually arise as follows. A basis of a countable group $G$ is selected and used to derive a scheme of invariants for the group. Then it is shown that such schemes can be used to construct all countable groups, in some cases of finite rank, and in other cases of both finite and infinite rank. However, different bases of a group give rise to different schemes and different schemes can be used to construct isomorphic groups. To rectify this, an equivalence is then defined on the schemes so that equivalent schemes will correspond to isomorphic groups. The equivalence classes of the schemes then form a complete system of invariants for the groups. Since the schemes are usually in terms in matrices, the systems of invariants consist of certain equivalence classes of matrices. It is here that the problem with these systems arises. As Fuchs [13] has shown, it is possible to determine which equivalence classes of matrices correspond to countable groups. However the problem of actually determining the equivalence classes has not been solved. Hence these various systems do not give satisfactory characterizations of the countable groups. These systems are valuable, though, in that they provide methods of describing the countable groups and thus deepen our knowledge of their structure. They have also provided methods for constructing new examples of indecomposable groups.

The first of these systems of invariants were provided by Kurosh [24], Derry [9], and Mal'cev [27]. Kurosh found invariants for primitive groups of finite rank and Derry and Mal'cev presented invariants for arbitrary groups of finite rank. All three used certain equivalence classes of infinite sequences of finite matrices of p-adic numbers to describe the groups. Kurosh's classification provided the first examples of indecomposable groups of arbitrary finite rank. The only previous examples of indecomposable groups were rank two groups found by Levi [26] and Pontryagin [28]. The results of Kurosh, Derry and Mal'cev have been generalized in Fuchs [13] to provide invariants, in terms of infinite matrices, for all countable groups of both finite and infinite rank. Szekeres [36] has given another classification for arbitrary countable groups in terms of certain p-adic and integral invariants. However this classification is not complete for Szekeres does not study the effect of a change in basis or determine conditions for equivalence of his schemes. Two other systems have been developed by Campbell [5] and Rotman [35], who use somewhat similar approaches. Campbell's invariants are for arbitrary countable groups whereas Rotman's are for finite rank groups only. Campbells approach is based on the fact that a group is determined once the divisibility properties of a basal subgroup (a subgroup generated by a basis of a group) are known. The schemes produced are certain systems of sequences of additive groups of suitable ordered sets of integers and are called D-systems. The equivalence classes of D-systems under a suitable equivalence relation are called D-types. Rotman, instead of considering basal subgroups, considers ordered bases of a group with finite rank. A generalized height function is defined and an equivalence relation is set up on these functions.

To illustrate we will now examine the invariants of Szekeres, Campbell, and Rotman. Szekeres was the first to give invariants for arbitrary countable groups. Campbell's and Rotman's invariants are the most recent ones presented. We will also examine a system of invariants developed by Erdös [12]. This system is not for the countable groups only but for groups of arbitrary power. Erdös uses torsion free factor groups of free abelian groups to classify groups of arbitrary power with infinite matrices. Hence it is appropriate to consider Erdös' system here.

Szekeres makes use of the notions of independence and dependence modulo $p^n$ and $p^\infty$ where $p \in \pi$. Let $g_1, \dots, g_k$ be an independent set of elements of a group $G$. Then we say that $g_1, \dots g_k$ are dependent modulo $p^n$ ($n$ is a positive integer) if $\sum_{i=1}^{k} a_i g_i \equiv 0 \pmod{p^n}$ has a solution with at least one $(a_i, p) = 1$ where all $a_j$ are integers. Otherwise $g_1, \dots g_k$ will be called independent modulo $p^n$.

Now suppose that $\alpha \in Z^{(p)}$ has the standard representation $\alpha = a_o + a_1 p + a_2 p^2 + \dots$, where $0 \leq a_i < p$ for all $i$. Then we will write $\alpha^{(n)} = a_o + a_1 p + \dots + a_{n-1} p^{n-1}$. If $\alpha_1, \dots, \alpha_k \in Z^{(p)}$, we will write $\sum_{i=1}^{k} \alpha_i g_i \equiv 0 \pmod{p^\infty}$ to indicate that $\sum_{i=1}^{k} \alpha_i^{(n)} g_i \equiv 0 \pmod{p^n}$ for every $n$. If there exist p-adic integers $\alpha_1, \dots, \alpha_k$, not all zero, such that $\sum_{i=1}^{k} \alpha_i g_i \equiv 0 \pmod{p^\infty}$, then we will say that $g_1, \dots, g_k$ are dependent modulo $p^\infty$. Otherwise we will say that they are independent modulo $p^\infty$. The construction of the invariants employs the following lemma.

Lemma If $g_1, \dots g_k$ are independent modulo $p$ and $g$ is independent of $g_1, \dots, g_k$ modulo $p^\infty$, then $g$ cannot be dependent on $g_1, \dots, g_k$ modulo $p^n$ for large enough $n$.

We now construct the invariants. Let $G$ be a countable group and $S = \{a_1, a_2, \dots\}$ be a basis of $G$. Choose some $p \in \pi$. Let $a_{i_1}$ be the first element of $S$ that is not independent modulo $p^\infty$ ($i_1$ need not exist). If $i_1$ exists then, by the lemma, there is a largest $h_1$ with $a_{i_1} \equiv 0 \pmod{p^{h_1}}$. We write $p^{h_1} b_1 = a_{i_1}$, $b_1 = p^{-h_1} a_{i_1}$ and construct inductively a sequence $b_1, b_2, \ldots$ of elements that are independent modulo $p^\infty$. $b_1$ has already been constructed. Let $a_{i_k}$ be the first element of $S$ that is linearly independent of $b_1, \ldots, b_{k-1}$ modulo $p^\infty$. Then there is a greatest $h_k$ and uniquely defined coefficients $0 \leq \alpha_{i_{k,\ell}} < p^{h_k}$ ($\ell = 1, \ldots, k-1$) with $a_{i_k} \equiv \sum_{\ell=1}^{k-1} b_\ell \pmod{p^{h_k}}$. We write $b_k = p^{-h_k}(a_{i_k} - \sum_{\ell=1}^{k-1} \alpha_{i_{h,\ell}} b_\ell)$.

The $b_k$'s are constructed until all the $a_i$'s are exhausted. The number, $s$, of $b_k$'s is finite or infinite and does not depend on the choice of $S$. We call $s = s(p)$ the rank of $G$ modulo $p^\infty$. For all $p$ we have $0 \leq s(p) \leq r = r(G)$.

Let $M(p)$ denote the set of indices $i_k = i_k(p)$ ($1 \leq k \leq s(p)$), and $N(p)$ denote the set of indices $j \notin M(p)$ ($i \leq j \leq r$). Then $N(p) = \emptyset$ if $s(p) = r$ and $i_k = k$ for every $k \leq r$. Suppose $N(p) \neq \emptyset$ and $j \in N(p)$. Writing $i_0 = 0$, $i_{s+1} = r + 1$, we then have $i_k < j < i_{k+1}$ for some $0 \leq k \leq s$. Now $a_j$ is independent of $b_1 \ldots b_k$ modulo $p^\infty$ and hence $\beta a_j \equiv \Sigma_{i=1}^k \beta_i b_i \pmod{p^\infty}$ where $\beta, \beta_1, \ldots, \beta_k \in Z^{(p)}$. Not all of the $\beta, \beta_1, \ldots, \beta_k$ are $\equiv 0 \pmod p$ and in particular $\beta \not\equiv 0 \pmod p$. Hence we have $a_j \equiv \sum_{i=1}^{k} \alpha_{ji} b_i \pmod{p^\infty}$ for some $\alpha_{ji}, \ldots, \alpha_{jk} \in Z^{(p)}$ and uniquely determined by $a_j$. Because of this last congruence there exists an infinite sequence, $b_{jn} (n=0,1,2,\ldots)$, of elements of $G$ with $b_{jo} = a_j$, $b_{jn} = p^{-n}(a_j - \sum_{i=1}^{k} \alpha_{ji}^{(n)} b_i)$.

Thus we see that each basis of $G$ uniquely determines a system of invariants

$$\text{(I)} \qquad \begin{array}{ll} M(p) = [i_k(p)], \quad h_k(p) & (k = 1, \ldots, s(p)) \\ \alpha_{j\ell}(p) & (j > i_\ell(p)) \end{array}$$

for every $p \in \pi$. There systems are subject to the conditions

$$\text{(II)} \qquad \begin{array}{l} 0 \leq s(p) \leq r, \quad h_k(p) \geq 0 \\ 0 \leq \alpha_{ik,\ell}(p) < p^{h_k(p)} \quad (i_k \in M(p)) \\ \alpha_{j\ell}(p) \in Z^{(p)} \end{array}$$

Szekeres then demonstrates that his systems of invariants can be used to describe all countable groups by proving;

<u>Theorem 2.1</u> The set of elements $[b_k(p), b_{jn}(p)]$ generates the group $G$ and every element of $G$ can be expressed uniquely in the form

$$\sum_{i=1}^{r} x_i a_i + \sum_p \sum_{k=1}^{s(p)} y_k(p) b_k(p) + \sum_p \sum_{j\in N(p)} \sum_{n=1}^{\infty} Z_{jn}(p) b_{jn}(p)$$

where $0 \leq y_k(p) < p^{h_k(p)}$, $0 \leq Z_{jn}(p) < p$, $N_i \in Z$.

Theorem 2.2 If, for every $p \in \pi$, an arbitrary system (I) is given, satisfying conditions (II), then there is exactly one group $G$ belonging to this sytem of invariants (i.e. no two non-imorphic groups belong to this system).

These two theorems allow construction and characterization of all countable groups. However, as we mentioned above, since a basis change is not considered and no conditions for the equivalence of two systems are found, the characterization is not complete.

The classification of countable groups due to Campbell is a complete classification. In order to describe it we will need the following notions. Let $F$ denote the set of all row-finite matrices over $R$. All matrices considered will be in $F$. A square matrix with an inverse will be called regular. If $A = (a_{ij})$ is an integral matrix with $n$ columns and $g = (g_1, g_2, \ldots)$ is an ordered set of $n$ elements of a group $G$ then $Ag$ will denote the ordered set $(h_1, h_2, \ldots)$ where $h_i = \sum_j a_{ij} g_j$. By a vector we mean a matrix with one row and by a vector module we mean an additive group of vectors. If $A \in F$, we will denote by $(A)$ the vector module generated by the rows of $A$.

Now let $r$ be a finite or countably infinite cardinal.

$Q$ will denote the module consisting of all vectors with $r$ coordinates and $J$ will denote the submodule of $Q$ consisting of the integral vectors of $Q$. If $M$ is a submodule of $J$ and $P$ is an integral square matrix of order $r$ then the set of integral vectors $c$ with $cP \in M$ is a submodule of $J$ which we will denote by $M : P$ and call the quotient of $M$ by $P$. If $I$ is the unit matrix and $P = mI$ $(m \in Z)$, we will write $M : m$ for $M : mI$.

Suppose that $G$ is a countable group and that $r(G) = r$. Let $g = (g_1, g_2, \ldots)$ be an ordered basis of $G$ generating the basal subgroup $H$. For $m \in Z$, $m > 0$, let $f(m)$ denote the set of all integral vectors $c$ with $cg$ divisible in $G$ by $m$. $f(m)$ is a submodule of $J$. The function $f$ thus defined on the positive integers is called the divisibility function of $G$ with respect to $g$, for it completely describes in $G$ the divisibility properties of the elements of $H$.

<u>Theorem 2.3</u> The function $f$ completely determines the group $G$ to within isomorphism.

Furthermore $f(m)$ is the intersection of all $f(q)$, where $q$ ranges over the prime power factors of $m$. Hence $f$ is completely determined by its values at the prime powers and the condition $f(1) = J$. This gives for each $p \in \pi$ a sequence

$$f(p^o) = J \supseteq f(p^2) \supseteq \ldots \supseteq f(p^n) \supseteq \ldots$$

The system of these sequences, for all $p \in \pi$, is called the divisibility system of $G$ with respect to $g$. Groups with a common divisibility system are isomorphic. We also have that $f(p^{n+1}) : p = f(p^n)$ for $n = 0,1,2,\ldots$

Now let $A = (A_o, A_1, \ldots, A_n, \ldots)$ be an infinite sequence of submodules of $J$. We will call $A$ a Dp-sequence $(p \in \pi)$ if $A_o = J$ and $A_{n+1} : p = A_n$ for $n = 0,1,2,\ldots$. A system $[A(p)]$, containing precisely one Dp-sequence $A(p) - (A_o(p), A_1(p), A_2(p), \ldots)$ for each $p \in \pi$, will be called a D-system. It is clear that every divisibility system is a D-system. We also have

<u>Theorem 2.4</u> Every D-system is a divisibility system of a suitable group $G$.

A D-system which is a divisibility system of a group $G$ is said to belong to $G$. Thus we have that any given D-system belongs to a unique group.

The determination of the divisibility system for the group $G$ depended on the basis $g$. Thus distinct D-systems may belong to isomorphic groups. To complete the classification we now look at the effect of a change of basis of $G$ and determine conditions for two D-systems to be equivalent. An ordered basis of $G$ is expressible in the form $Pg$, where $P$ is an integral square matrix of order $r$. In fact, $Pg$ is a basis of $G$ iff $P$ is regular. And so, if $[A(p)]$ is the divisibility system of $G$ with respect to $g$, then the divisibility system of $G$ with respect to $Pg$, where $P$ is regular, is $[A(p) : P]$.

Let $g, g'$ be ordered bases of $G$ generating the basal subgroups $H, H'$ respectively. Then $H \cap H'$ is also a basal subgroup and we may select a basis $g''$ of $G$ in $H \cap H'$. For suitable regular matrices $P$ and $S$ we have $Pg = g'' = Sg'$. Suppose $[A(p)]$ and $[B(p)]$ are the divisibility systems of $G$ with respect to $g, g'$ respectively. Then for all $p \in \pi$ $A(p) : P = B(p) : S$. This relation provides the equivalence on the D-systems. We will call two D-systems $[A(p)]$, $[B(p)]$ associated if, for all $p \in \pi$ and for suitable regular integral matrices $P, S$, independant of $p$, $A(p) : P = B(p) : S$.

The relation of association between D-system is an equivalence. The resulting equivalence classes are called D-types. The set of D-systems belonging to a given group is a D-type. Every D-type corresponds to some group and two groups are isomorphic iff their D-types are the same. Hence the D-types provide a complete classification of the countable groups of rank $r$ where $r$ is a finite or countably infinite cardinal.

The final classification for countable groups that we will discuss here is one for groups of finite rank due to Rotman. Let $G$ and $G'$ be groups of finite rank $r$. Suppose $x_1, \ldots x_r$ is a basis of $G$ and $y_1, \ldots, y_r$ is a basis of $G'$ such that $h_p(\sum_{i=1}^{r} m_i x_i, G) = h_p(\sum_{i=1}^{r} m_i y_i, G')$ for all primes $p$ and all integers $m_i$. Define a mapping $f : G \to G'$ as follows. Let $f(x_i) = y_i$ $(i = 1, 2, \ldots, r)$. Suppose $0 \neq x \in G$. Then there exists integers $m, m_1, \ldots m_r$ with $mx = \sum m_i x_i$. We can assume that $m = p^k$ for some $p \in \pi$ and some $k \geq 0$. Thus $h_p(\Sigma m_i x_i) \geq k$ and there exists a unique $y \in G'$ such that $p^k y' = \Sigma m_i y_i$. Set $f(x) = y$. $f$ is then a well defined isomorphism. The result is:

Theorem 2.5 If $G$ and $G'$ are groups of finite rank $r$, then $G \cong G'$ iff there exist bases $x_1, \ldots x_r$ of $G$ and $y_1, \ldots, y_r$ of $G'$ with $h_p(\Sigma m_i x_i, G) = h_p(\Sigma m_i y_i, G')$ for all $p \in \pi$ and all integers $m_i$.

Let $Z^r$ denote $\prod_{i=1}^{r} Z$ and let $x_1, \ldots x_r$ be an ordered basis of $G$. We define a height function $f : \pi \times Z^r \to N$ by $f(p, m_1, \ldots, m_r) = h_p(\Sigma m_i x_i, G)$. Such functions describe the groups, as theorem 2.5 indicates. But the description is not complete as different bases will yield different functions. Hence suppose that $y_1, \ldots, y_r$ is another basis of $G$ and suppose further that $y_1, \ldots y_r$ induces a function $g : \pi \times Z^r \to N$. Now, there exists a rational non-singular $r \times r$ matrix $A = (a_{ij})$ with $y_i = \sum_j a_{ij} x_j$. Let $n$ be the product of the denominators of $a_{ij}$. Then $ng_i = \sum_j na_{ij} x_j$. The coefficients $na_{ij}$ are all integers and we have

$$g(p, nm_1, \ldots nm_r) = f(p, \Sigma m_i na_{i1}, \ldots, \Sigma m_i na_{ir}) = f(p, [m_1, \ldots, m_r]nA)$$

This relation is an equivalence relation on the functions and any two ordered bases of $G$ will determine the same equivalence class of functions. Hence the equivalence class in an invariant of $G$. Theorem 2.5 can now be restated as;

Theorem 2.6 Let $G$ and $G'$ be groups of finite rank $r$. Then $G \cong G'$ iff they have the same equivalence class of height functions.

Rotman then determines which equivalence classes of the functions $f : \pi \times Z^r \to N$ actually correspond to groups of rank $r$.

We conclude this section with a look at a classification for arbitrary groups due to Erodös. The approach here is through torsion free factor groups of free abelian groups. The classification requires some results on these groups. The main result needed is;

Lemma Let $F/H$ and $F'/H'$ be isomorphic torsion free factor groups of the free abelian groups $F$ and $F'$. Then there exists an isomorphism $\varphi$ of $F$ onto $F'$ with $H\varphi = H'$ iff $r(H) = r(H')$.

Here, as previously, matrices will be row finite matrices over $R$. All matrices will be square $m \times m$ matrices where $m$ is a cardinal. A matrix $A$ is called right regular if there exists a matrix $A'$ with $AA' = I$. We will call two $m \times m$ matrices $A$ and $B$ equivalent if there exist regular matrices $P$ and $Q$ with $PAQ = B$ and both $Q$ and $Q^{-1}$ are integral matrices.

Theorem 2.7 Let $m$ be any infinite cardinal. Then there exists a one-to-one correspondence between all groups of cardinality $\leq m$ (up to isomorphism) and all right regular $m \times m$ matrices (if we do not make distinction between equivalent matrices).

Proof: Let $F$ be a free group of rank $m$. Any group $G$ of cardinality $\leq m$ is isomorphic to a factor group of $F$ modulo a subgroup $H$ of rank $m$. Let each subgroup $H$ of $F$ correspond to the factor group $F/H$. Then by the lemma and the above remark this establishes a one-to-one correspondence between all groups of cardinality $\leq m$ (up to isomorphism) and all pure subgroups of rank $m$ of $F$ if we do not make distinction between subgroups of $F$ which can be mapped onto each other by automorphisms of $F$.

Let $F$ be a full subgroup of a rational vector space $V$. The correspondence $S \to S \cap F$ is one-to-one between subspaces $S$ of $V$ and the pure subgroups of $F$. We have that $r(S) = r(S \cap F)$ and that $S_1 \cap F$ can be mapped onto $S_2 \cap F$ by an automorphism of $F$ iff $V$ has an automorphism mapping $S_1$ onto $S_2$ and $F$ onto itself. Thus our problem is equivalent to the problem of classifying all subspaces of rank $m$ of $V$ under the group of automorphisms of $V$ which map $F$ onto itself. Now a subspace $S$ of $V$ has rank $m$ iff $V$ can be mapped onto $S$ by an endomorphism with a right inverse. If $\ell_1, \ell_2$ are endomorphisms of $V$ with right inverses then $V\ell_1 = V\ell_2$ iff there exists an automorphism $\varphi$ of $V$ with $\varphi\ell_1 = \ell_2$. An automorphism $\varphi$ of $V$ maps $V\ell_1$ onto $V\ell_2$ iff there is an automorphism $\varphi_1$ with $\varphi_1\ell_1\varphi = \ell_2$. The theorem follows from the representation of endomorphisms of $V$ by $m \times m$ matrices.

Now if $G$ is a group of cardinality $\leq m$, let $G$ be represented as $F/H$ where $F$ is a free group and $r(F) = r(H) = m$. Let $b_\lambda$ and $b'_\lambda$ $(\lambda \in \Lambda)$ be bases of $F$ and $H$ respectively. Then the matrix $A = (a_{\lambda\mu})$ defined by $b'_\lambda = \sum_{\mu \in \Lambda} a_{\lambda\mu} b_\mu$, $\lambda \in \Lambda$, $a_{\lambda\mu} \in R$, is the matrix corresponding to $G$. Here, as with the classifications of the countable groups, this classification does not provide a satisfactory system of invariants because of the problem of determining equivalent matrices.

## 3. Indecomposability and Direct Summation

In the previous section we mentioned all the classifications of the countable and arbitrary groups that are known. We now mention and discuss briefly one further classification, namely the well known classification of the rank one groups by their types. For a description of this characterization we refer to Fuchs [13]. This is the only satisfactory characterization for groups that exist. Not only does it completely describe the rank one groups, but it also completely describes the structure of completely decomposable groups (groups which are direct sums of rank one groups) for Baer [2] has proved —

Theorem 3.1 Let $G$ be a completely decomposable group and suppose $G = \Sigma_\lambda \oplus G_\lambda = \Sigma_n \oplus H_n$ where the $G_\lambda$ and $H_n$ are rank one groups. Then there exists a one-to-one correspondence between the summands $G_\lambda$ and the summands $H_n$ such that corresponding summands are isomorphic. Hence any decomposition of a completely decomposable group into a direct sum of rank one groups is essentially unique.

Unfortunately, the problem of determining those groups which are completely decomposable has not yet been solved. Some necessary and sufficient conditions for a group to be completely decomposable are known but either these conditions are applicable only to restricted classes of groups or the groups with the conditions have not been determined. For example, Rotman [35] has used his invariants for finite rank groups (see the previous section) to show that a group $G$ of finite rank is decomposable iff $G$ contains a basis $x_1, \ldots, x_r, y_1, \ldots y_s$ such that, for all $p \in \pi$ and all $m_i, m_j \in Z$, $h_p(\Sigma m_i x_i + \Sigma m_j y_j, G)$ $= \min\{h_p(\Sigma m_i x_i, G), h_p(\Sigma m_j y_j, G)\}$. This can obviously be used to derive a necessary and sufficient condition for the group $G$ to be completely decomposable. However the problem of determining the groups with such a basis has not been solved. More of these conditions, along with further information on completely decomposable groups, can be found in Baer [2], Kurosh [25], Fuchs [13], and Wang [38]. We conclude our brief discussion of rank one groups and completely decomposable groups with a review of some of the results on such groups that have been published recently.

Baer [2] has proved that if $G$ is a completely decomposable group and the rank one summands of $G$ all have the same type then every pure subgroup of $G$ is also completely decomposable. Prochazha [31] has generalized this result and proved that if $G = \sum_{\alpha < \tau} \oplus G_\alpha$ is a direct sum of rank one groups whose types are inversely well ordered in the natural partial order of types (i.e. $\alpha \leq \beta < \tau$ implies type $G_\beta \leq$ type $G_\alpha$) then any pure subgroup of $G$ is completely decomposable. Actually, in proving this result, Prochazka proves that any pure subgroup of $G$ is a direct summand of $G$. Hence the result is stronger than indicated for any direct summand of $G$ is completely decomposable. Kovas [23] has shown that if $G$ is a group and $H$ a subgroup of $G$ with $nG \subseteq H$ for some positive integer $n$ and if either $G$ or $H$ is a direct sum of rank one groups whose types are inversely well ordered in the natural partial order of types then $G \cong H$.

Homological methods and the concept of regular groups have been employed by Harrison [17] to help throw light on the problem of determining those groups which are completely decomposable and gain some insight into the number of groups which are not completely decomposable. Let $G$ be a group. For $p \in \pi$ let $f(p,G)$ denote the dimension of $G/pG$ as a vector space over the prime field of characteristic $p$. We will write $f(G) = \prod_p p^{f(p,G)}$. If $S$ is a pure subgroup of $G$, then $f(S) \cdot f(G/S) = f(G)$ (i.e. $f(p,S) + f(p,G/S) = f(p,G)$ for all $p$). We will also write $n(G) = \prod_p p^r$ where $r = r(G)$ If $H$ is any subgroup of $G$ $n(H) \cdot n(G/H) = n(G)$.

For $p \in \pi$ let $A_p$ be the subgroup of $R$ with denominators of powers of $p$. We will write $\ell(p,G) = r(\mathrm{Hon}(A_p,G))$ and $e(G) = \prod_p p^{e(p,G)}$. Then if $H$ is any subgroup of $G$ $e(G) \leq e(H) . e(G/H)$. These three functions are used to define regular groups. For any group $G$, $f(G) \cdot e(G) \leq n(G)$. We will call a group $G$ regular if $r(G)$ is finite and $f(G) \cdot e(G) = n(G)$. If $G$ is a regular group and $S$ is a pure subgroup of $G$, then both $S$ and $G/S$ are regular. All rank one groups and all finite direct sums of rank one groups are regular. Harrison's results are contained in two theorems which are;

1. A group $G$ which is divisible for all but a finite number of primes is a direct sum of a finite number of rank one groups iff it is regular. If $S$ is a pure subgroup of a group $G$ which is a finite direct sum of rank one groups all of which are divisible for all but a finite number of primes then both $S$ and $G/S$ are also direct sums of rank one groups.
2. Let $G$ and $H$ be groups of finite rank such that $G$ is divisible for all but a finite number of primes. Then the number of non-isomorphic groups $K$ which have a subgroup $H'$ isomorphic to $H$ and with factor group $K/H'$ isomorphic to $G$ is either one or the cardinality of the continuum depending on whether or not ${}^{n(G)}/f(G)$ is relatively prime to $f(H)$ (i.e. whether or not either $r(G) - f(p,G) = 0$ or $f(p,H) = 0$ for all $p \in \pi$).

These results show the immense number of groups which exist even for small ranks and that only relatively few of them are completely decomposable. For example the number of rank two groups formed by putting $A_q$ and $A_p$ (see above) together, where $p \neq q$; $p,q \in \pi$, is the power of the continuum but only one of them is completely decomposable.

Theorem 3.1 shows that direct sums of rank one groups present no irregularities for two different direct summations of rank one groups yield non-isomorphic groups. It is naturally to be hoped that the property of theorem 3.1 will carry over to direct sums of arbitrary indecomposable groups. Unfortunately, this does not happen. We will now demonstrate that, in general, direct summations behave in a very erratic manner. For instance, Corner [8], using one of his own results, [7], has constructed, for any positive integer $r$, an example of a countable, reduced group $G$ with the property that $\sum_m G \cong \sum_n G$ iff $m \equiv n \pmod{r}$. Here $m$ and $n$ are postive integers and $\sum_n G$ indicates the direct sum of $n$ copies of $G$. In particular, there then exists a group $G$ that is isomorphic to the direct sum of $r + 1$ copies of itself but not to the direct sum of $s$ copies of itself for all $1 < s < r + 1$.

The main problem in this connection is presented by direct sums of indecomposable groups, not all of which are rank one groups. It is possible for two different (in the sense of theorem 3.1) direct sums of indecomposable groups to be direct decompositions of the same group. This problem is further complicated by the fact that besides the rank one groups which are of course, indecomposable, vast numbers of indecomposable groups are known to exist. We will first discuss some examples of indecomposable groups and then some examples to indicate the failure of theorem 3.1 in the general sense.

Rotman [35] has used his invariants (see previous section) to construct examples of indecomposable groups of any finite rank. The pure subgroups of $Z^{(p)}$ yield examples of indecomposable groups of any rank up to and including $\aleph$ (the power of the continuum). de Groot and de Vries [16] have constructed examples of rank $m$ for $\aleph_0 \leq m \leq \aleph$. indecomposable groups of any rank $\leq 2^{\aleph}$ are presented in Fuchs [13]. The best possible result however is the example constructed by Fuchs [14]. We will look at this example in some detail. Fuchs proves

Theorem 3.2 For every infinite cardinal $m$ there exists a rigid system of $2^m$ groups of power $m$.

By a rigid system of groups we mean a set of groups, $G_\lambda (\lambda \in \Lambda)$, with the properties

(i) If $\lambda \neq n$ then $\mathrm{Hom}(G_\lambda, G_n) = 0$ for all $\lambda, n \in \lambda$.

(ii) For every endomorphism $\varphi$ of $G_\lambda$ there exists $r_\varphi \in R$ such that $g\varphi = r_\varphi g$ for all $g \in G_r$. Groups in any rigid system are indecomposable and pairwise non-isomorphic. We will use the following notation in the construction. If $t$ is a type and $G$ is a group, recall that we write

$G_t = \{g \in G \mid t(g,G) \geq t\}$. Let $t_0 = [(\infty,0,0,\ldots)]$ and $t_1 = [(0,\infty,0,0,\ldots)]$.

Lemma If for some cardinal $m$ we have a system of groups $G_\lambda(\lambda \in \Lambda)$ satisfying

$1_0$ $|G_\lambda| = m$

$2_0$ $|\Lambda| = 2^m$

$3_o$ $|G_\lambda / G_{\lambda_{t_o}}| = m$

$4_o$ for every $\lambda \in \Lambda$ elements $g_\lambda, \bar{g}_\lambda \in G_\lambda, \notin 2G_\lambda$, which are independant mod $G_{\lambda_{t_o}}$, can be selected such that

$G'_\lambda = \{G_\lambda, 2^{-\infty}g_\lambda, 2^{-\infty}\bar{g}_\lambda\}$ (i.e. the group generated by the direct sum of $G_\lambda$ and the elements of the form $2^{-n}g_\lambda$, $2^{-n}\bar{g}_\lambda$, $n=1,2,\ldots$) has the property: if $\varphi$ is a homomorphism of $G_\lambda$ into $G'_\mu$ then either $\varphi = 0$ or $\lambda = \mu$ and there exists $r \in R$ with $g\varphi = rg$ for all $g \in G_\lambda$.

Then, for any $n$ with $m < n \leq 2^m$ there exists a system of groups $H_a (a \in A)$ which also has properties $1_o - 4_o$

Note that any system satisfying properties $1_o - 4_o$ is a rigid system by property $4_o$. The construction of the $H_a$ proceeds as follows; let $\Lambda_a (a \in A)$ be a collection of subsets of $\Lambda$, each of power $n$, such that $\lambda_o \in \Lambda_a$ for a fixed $\lambda_o$ and every $a$ and $\Lambda_a \subseteq \Lambda_b$ implies $a = b$. There exists $2^n$ sets $\Lambda_a$ with these properties and hence we may assume $A$ has power $2^n$. For every $\Lambda_a$ let $H_a = \{\sum_{\lambda \in \lambda_a} G_\lambda, 2^{-\infty}(g_{\lambda_o} + g'_\lambda), \ldots\}$, $(\lambda' \neq \lambda_o, \lambda' \in \Lambda_a, g_\lambda \in G_\lambda)$.

As the lemma indicates the construction will yield a rigid system with additional properties to (i) and (ii). The next step in the construction is to construct a system of $2^{\aleph_1}$, groups of power $\aleph_1$. Let $G_\lambda (\lambda \in \Lambda)$ be rank one groups whose types are pairwise incomparable and incomparable with the types $t_o$ and $t_1$ . We may assume the index set has power $\aleph_1$ . We can form $2^{\aleph_1}$ subsets $\Lambda_a (a \epsilon A)$ of $\Lambda$ such that $|\Lambda_a| = \aleph_1$, a fixed $\lambda_o$ belongs to all $\Lambda_a$ and $\Lambda_a \subseteq \Lambda_b$ implies $a = b$ . Select $g_\lambda \epsilon G_\lambda$, $\notin 3G_\lambda$ and, for every $\Lambda_a$ , define

$$H_a = \{ \sum_{\lambda \epsilon \Lambda_a} G_\lambda, 3^{-\infty}(g_{\lambda_o} + g'_\lambda), \ldots \} \; (\lambda' \neq \lambda_o, \lambda' \epsilon \Lambda_a, g_\lambda \epsilon G_\lambda) .$$

$g_{\lambda_o} + g'_\lambda$ will have type $t_1$ in $H_a$ and the system $H_a (a \epsilon A)$ has properties $1_o - 4_o$

The final step is to construct a system of groups with properties $1_o - 4_o$ for a limit cardinal $n$ . Let $\aleph_1 = m_1, m_2, \ldots, m_\alpha, \ldots$ be the sequence of all cardinals greater than $\aleph_o$ and less than $n$ . Suppose that for each $m_\alpha$ there exists a system of groups $H_a^{(\alpha)}$ satisfying $1_o - 4_o$ . With the $G_\lambda$ as in the previous step, a system $K_x$ is constructed for $n$ , such that every $K_x$ arises as the union of a sequence $G_\lambda \subseteq H_a^{(1)} \subseteq H_{(b)}^{(2)} \subseteq \cdots \subseteq H_c^{(\alpha)} \subseteq \cdots$ . Different sequences may produce isomorphic $K_x$ . We consider only the non-isomorphic $K_x$ that result. It can then be proved that the $K_x$ have property $4_o$ and that there are at least $n$ non-isomorphic groups $K_x$ . Then applying the procedure of the lemma we obtain $2^n$ groups of power $n$ satisfying the conditions $1_o - 4_o$ .

This completes the construction and the proof of theorem 3.2. We remark that this result can be used to answer problems 20, 21 and 46 in Fuchs book [13].

The first example demonstrating that different direct sums of indecomposable groups could yield isomorphic groups was due to Jonsson [19] who discovered a rank 3 group that could be decomposed into a direct sum of indecomposable groups in two different ways. Jesmanowicz [18] presented another example in answer to a problem posed by Fuchs [13, problem 22]. Fuchs asked: If $r_i (i=1,2,3,4)$ are positive integers with $r_1 + r_2 = r_3 + r_4$ and $r_1 \neq r_3$, $r_1 \neq r_4$, does there exist indecomposable groups $G_i (i=1,2,3,4)$ with $r(G_i) = r_i (i=1,2,3,4)$ and $G_1 \oplus G_2 \cong G_3 \oplus G_4$. Jesmanowicz answered this problem affirmatively and also proved a similar result for three direct summands. The strongest example in this connection is due to Corner [6] who generalized Fuch's problem and answered it affirmatively.

<u>Theorem 3.3</u> Let $N,k$ be positive integers with $N \geq k$. Then there exists a group $G$ of rank $N$ such that for any partition,

$$N = r_1 + r_2 + \dots + r_k, \text{ of } N \text{ into } k$$

positive integers there exists indecomposable subgroups $G_1, \dots, G_k$ of $G$ satisfying

(i) $r(G_i) = r_i (i=1,\dots,k)$

(ii) $G = \sum_{i=1}^{k} \oplus G_i$

Proof: Let $n = N - k$. If $n = 0$ take $G$ to be free with $r(G) = k = N$. Hence we may assume $n \geq 1$. Let $p, p_1, \ldots, p_n, q_1, \ldots, q_n$ be distinct primes and let $u_1, \ldots, u_k, x_1, \ldots, x_n$ be a basis of $R^N$. Let $G$ be the group generated by $\{p^{-m}u_i, p_j^{-m}x_j, q_j^{-1}(u_1+x_j)\}$ $(1 \leq i \leq k, 1 \leq j \leq n, m \geq 0)$. Then $G$ is a subgroup of $R^N$ and $r(G) = N$.

Now let $r_1 + \ldots + r_k = N$ be a partition of $N$ into $k$ positive integers. Set $s_o = 0$ and $s_i = r_1 + \ldots + r_i - i$ $(i=1,\ldots,k)$. If we write $q_1 \ldots q_n = q_j Q_j$ then $(Q_1, \ldots, Q_n) = 1$ and there exists $t_1, \ldots, t_n \in Z$ such that $\sum_{i=1}^{n} t_i Q_i = 1$. Let

$$a_i = \sum_{s_{i-1} < j \leq s_i} t_j Q_j .$$

Then $\sum_{i=1}^{k} a_i = 1$ and $a_i \equiv 1 \pmod{q_j}$ if $s_{i-1} < j < s_i$; $a_i \equiv 0 \pmod{q_j}$ otherwise. We now define $b_1, \ldots, b_k \in R^N$ by:

$$\begin{aligned}
u_1 &= a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_kb_k \\
u_2 &= -b_1 + b_2 \\
u_3 &= -b_1 \qquad + b_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
u_k &= -b_1 \qquad\qquad\qquad + b_k .
\end{aligned}$$

These equations can be solved to give $b_1, \ldots, b_k$ as integral linear combinations of $u_1, \ldots, u_k$.

For $i = 1,\ldots,k$, let $G_i$ be the group generated by

$\{p^{-m}b_i,\ p_j^{-m}x_j,\ q_j^{-1}(b_i+x_j)\}\ (s_{i-1} < j \leq s_i,\ m \geq 0)$ .

These $G_i$ are the required indecomposable groups.

Corner also proves a similar result for the countable rank care, namely;

Theorem 3.4 There exists a group $G$ of countable rank such that for any sequence $r_1, r_2, \ldots$ of positive integers, infinitely many of which are greater than one, there exists indecomposable subgroups $G_i$ of $G$ satisfying

(i) $r(G_i) = r_i\ (i=1,2,\ldots)$

(ii) $G = \sum_i \oplus G_i$ .

4. Quasi-isomorphism and Quasi-endomorphisms.

The examples of section 3 indicate that a large number of groups exist which do not have unique decompositions into a sum of indecomposable groups. This is because the condition of uniqueness up to isomorphism is too strong. If we replace isomorphism by the weaker condition of quasi-isomorphism and indecomposability by the corresponding notion of strong indecomposability, we retain the essential uniqueness of decompositions of groups of finite rank. In this section we define and discuss the notion of quasi-isomorphism and state a theorem analogous to theorem 3.1 . We will also define and establish the importance of the notions of quasi-endomorphisms and the ring of quasi-endomorphisms of a group.

The definition of quasi-isomorphism is due to Jonsson [20] and was originally given for groups in general.

<u>Definition 4.1</u> Let $G$ and $H$ be groups.

(i) $G$ and $H$ are said to be quasi-isomorphic iff each is isomorphic to a subgroup of the other. We will write $G \stackrel{\bullet}{\cong} H$

(ii) $G$ is said to be quasi-contained in $H$ iff $G \stackrel{\bullet}{\cong} G \cap H$. We will write $G \stackrel{\bullet}{\subseteq} H$.

(iii) $G$ and $H$ are said to be quasi-equal iff $G \stackrel{\bullet}{\subseteq} H$ and $H \stackrel{\bullet}{\subseteq} G$. We will write $G \doteq H$.

(iv) $G$ is said to be quasi-decomposable iff there exist non-zero independent groups $G_1$ and $G_2$, such that $G \doteq G_1 \oplus G_2$.

(v) $G$ is said to be strongly indecomposable iff it is not quasi-decomposable.

Other formulations of these notions, better suited to the case of torsion free-abelian groups are as follows. Let $G$ and $H$ be torsion free abelian groups. Then

(i)' $G \stackrel{\bullet}{\cong} H$ if there exists subgroups $G' \subseteq G$, $H' \subseteq H$ and positive integers $m,n$ such that $G' \cong H'$, $nG \subseteq G'$, and $mH \subseteq H'$.

(ii)' $G \stackrel{\bullet}{\subseteq} H$ if, for some positive integer $n$, $nG \subseteq H$.

If $G$ and $H$ are torsion free abelian groups of finite rank then

(i)" $G \stackrel{\bullet}{\cong} H$ if there exists subgroups $G' \subseteq G$, $H' \subseteq H$ such that $G' \cong H'$ and $G'$ and $H'$ have finite index in $G$ and $H$ respectively. Also $G \stackrel{\bullet}{\cong} H$ if there exists a subgroup $H' \subseteq H$ such that $H'$ has finite index in $H$ and $G \cong H'$.

(i) - (v) are Jonsson's original definitions. As defined by Jonsson $\doteq$ is an equivalence relation on the class of all groups. $\dot{\subseteq}$ in general is not transitive over the class of all subgroups of a group G and hence $\dot{=}$ need not be an equivalence relation. (i)$'$ is from Beaumont and Pierce [3]. (ii)$'$ is from Ried [33] and (i)$''$ from Beaumont and Pierce [4]. (i)$''$ can be seen by applying Theorem 2.4 of Jonsson [20]. Both $\doteq$ and $\dot{=}$ are equivalence relations on the class of all torsion free abelian groups.

The definitions of quasi-isomorphism was, as we stated above, introduced by Jonsson to provide the following theorem, analogous to theorem 3.1.

<u>Theorem 4.2</u> Let $G_1, G_2, \ldots, G_m$, $H_1, H_2, \ldots, H_n$ be strongly indecomposable groups of finite rank such that $\sum_{i=1}^{m} \oplus G_i \doteq \sum_{j=1}^{n} \oplus H_j$.

Then $m = n$ and there exists a permutation $\varphi$ of $\{1,2,\ldots,m\}$ such that $G_i \doteq H_{\varphi(i)}$ $(i=1,\ldots,m)$.

Actually a stronger result than this is possible if we make use of the ring of quasi-endomorphisms of a group. We will give this presently.

As a result of theorem 4.2, quasi-isomorphism has come to play an important role in the study of groups. Some properties of groups that are invariant under quasi-isomorphism have been found and in the case of rank two groups a complete set of quasi-isomorphism invariants have been found by Beaumont and Pierce [4]. We will now review some results on quasi-isomorphism. The invari- for the rank two groups will be discussed in another section.

Beaumont and Pierce [3] have proved that if $G \overset{\bullet}{\cong} H$ and $G' \overset{\bullet}{\cong} H'$ then $G \oplus G' \overset{\bullet}{\cong} H \oplus H'$. It has also been proved here that if $G$ and $H$ are groups then the following conditions are equivalent

(i) $G \overset{\bullet}{\cong} H$

(ii) there exists subgroups $G' \subseteq G$, $H' \subseteq H$ and a positive integer $n$ such that $G \cong H'$, $H \cong G'$, $nG \subseteq G'$ and $nH \subseteq H'$.

(iii) There exists a subgroup $G' \subseteq G$ and a positive integer $n$ such that $G' \cong H$ and $nG \subseteq G'$.

If $G$ and $H$ are quasi-isomorphic groups then $r(G) = r(H)$ and $T(G) = T(H)$ ([4]). Hence rank and type sets are quasi-isomorphic invariants.

Koehler [22] has proved some properties of quasi-isomorphic groups of finite rank.

If $G$ and $H$ are groups of finite rank then we can consider them as both being subgroups of $R^n$ for some $n$. Then the following conditions are equivalent.

(i) $G \overset{\bullet}{\cong} H$

(ii) there exists a subgroup $H'$ of $H$ and a monomorphism $\varphi$ from $H'$ to $G$ such that $G \overset{\bullet}{\subseteq} \varphi(H')$ and $H \overset{\bullet}{\subseteq} H'$.

(iii) there exists a monomorphism $\varphi$ from $H$ to $G$ such that $G \overset{\bullet}{\subseteq} \varphi(H) \subseteq G$.

(iv) there exists a subgroup $G'$ of $G$ such that $H \cong G' \overset{\bullet}{=} G$.

(v) there exists non-singular linear transformations $\lambda_1, \lambda_2$ of $R^n$ such that $\lambda_1(G) \subseteq H$ and $\lambda_2(H) \subseteq G$.

Another result proved by Koehler uses the notation: $\overline{G}$ denotes the minimal divisible group containing $G$. Now let $G$ and $H$ be quasi-isomorphic subgroups of $R^n$. Then

(i) $G_t \doteq H_t$ for all types $t$.

(ii) there exists a non-singular linear transformation $L$ of $R^n$ such that $L(\overline{H}_t) = \overline{G}_t$ for all types $t$.

(iii) If $G \doteq H$ then $G_t = H_t$ ; $\overline{G}_t = \overline{H}_t$ for all types $t$.

Note that $\overline{G}_t$ , $\overline{H}_t$ are subspaces of $R^n$.

We conclude our discussion of quasi-isomorphism with an alternate definition proposed by Walker [37]. Walker considers the quotient category $\mathcal{G}/\mathcal{B}$ where $\mathcal{G}$ is the category of all abelian groups and $\mathcal{B}$ is the class of all bounded abelian groups. two abelian groups $G$ and $H$ are quasi-isomorphic if there exists isomorphic subgroups $G'$ and $H'$ of $G$ and $H$ respectively with $G/G'$ and $H/H' \in \mathcal{B}$. If $G$ and $H$ are torsion free groups then this is equivalent to each being isomorphic to a subgroup of the other with bounded quotients. Hence two torsion free groups are quasi-isomorphic iff they are isomorphic in $\mathcal{G}/\mathcal{B}$. Furthermore quasi-decomposition, quasi-endomorphisms (see definition 4.4) etc. become decompositions, endomorphisms etc. in the quotient category $\mathcal{G}/\mathcal{B}$. Hence quasi-decomposition theory of torsion free groups in $\mathcal{G}$ is equivalent to decomposition theory of torsion free groups in $\mathcal{G}/\mathcal{B}$. Since quasi-isomorphism and quasi-decomposition theory are of principal value only in the study of torsion free groups, Walker submits that the proper definition of quasi-isomorphism should be:

<u>Definition 4.3</u> Two groups $G$ and $H$ are said to be quasi-isomorphic iff they are isomorphic in $\mathcal{G}/\mathcal{B}$ .

This definition makes available for application the homological algebra of $\mathcal{G}/\mathcal{B}$ and category theory in general.

We now discuss quasi-endomorphisms.

<u>Definition 4.4</u> Let $G$ be a full subgroup of a rational vector space $V$ . Let $L(V)$ denote the ring of linear transformations of $V$ . We define $E(G) = \{\lambda \in L(V) \mid G\lambda \mathrel{\dot{\subseteq}} G\}$ . If $\lambda \in E(G)$ we call $\lambda$ a quasi-endomorphism of $G$ . If $\lambda \in L(V)$ is non-singular and $\lambda, \lambda^{-1} \in E(G)$ then we call $\lambda$ a quasi-automorphism of $G$ . Hence $\lambda$ is a quasi-automorphism of $G$ iff $G\lambda \doteq G$ . $E(G)$ is the rational algebra generated by the endomorphisms of $G$ in $L(V)$ and, in particular, is a ring and the quasi-automorphisms are its units.

$E(G)$ plays an important part in the quasi-decomposition theory of $G$ . This has been demonstrated in two papers by Reid [33], [34] . We will establish the importance of $E(G)$ by reviewing the pertinent results from these papers. It is obvious that $G \doteq G_1 \oplus G_2$ where $G_1 = Ve \cap G \doteq Ge$ and $G_2 = V(1-e) \cap G \doteq G(1-e)$ where $e$ is an idempotent of $E(G)$ . Conversely if $G \doteq G_1 \oplus G_2$ then there exists a unique idempotent $e$ with $G_1 \doteq Ge$ and $G_2 \doteq G(1-e)$ . Hence we have that $G$ is strongly indecomposable iff $E(G)$ contains no proper $(\neq 0,1)$ idempotents.

Now let $G \doteq \sum_{i=1}^{n} \oplus G_i$ be a finite quasi-decomposition of $G$. Set $H = \sum_{i=1}^{n} \oplus G_i$. Then $G \doteq H$ and $E(G) = E(H)$. Hence the projections $\ell_i (i=1,\cdots,n)$ defined by the decomposition of $H$ belong to $E(G)$. Furthermore they are mutually orthogonal idempotents and their sum is the identity of $E(G)$. Also $G\ell_i \doteq G_i (i=1,\cdots,n)$ and $G \doteq \sum_{i=1}^{n} \oplus G\ell_i$. Thus any finite quasi-decomposition of $G$ is quasi-equal to one of the form $G \doteq \sum_{i=1}^{n} \oplus G\ell_i$ where $\ell_i$ are mutually orthogonal idempotents whose sum is the identity of $E(G)$. A quasi-decomposition of this form is said to be normalized. Next, if $\ell_1,\cdots,\ell_n$ are mutually orthogonal non-zero idempotents whose sum is the identity of $E(G)$ then $E(G)$ has a decomposition $E(G) = \sum_{i=1}^{n} \oplus \ell_i E(G)$ into a direct sum of right ideals.

<u>Theorem 4.5</u> The correspondence $G \doteq \sum_{i=1}^{n} + G\ell_i \to E(G) = \sum_{i=1}^{n} + \ell_i E(G)$ is one-to-one between normalized finite quasi-decompositions of $G$ and finite decompositions of the $E(G)$-module $E(G)$. Also $E(G\ell_i) \cong \ell_i E(G) \ell_i$ $(i=1,\cdots,n)$ and $G\ell_i$ is strongly indecomposable iff $\ell_i E(G)$ is an indecomposable $E(G)$-module. If $e$ and $f$ are any idempotents of $E(G)$ then $Ge \doteq Gf$ iff $eE(G)$ and $fE(G)$ are isomorphic $E(G)$-modules.

Theorem 4.5 shows that the quasi-decomposition theory of $G$ is equivalent to the decomposition theory of $E(G)$ as a (right) module over itself. As a result of this theorem we have the following generalization of theorem 4.2:

<u>Theorem 4.6</u> Let $E(G)$ have descending chain condition on right ideals. Then any quasi-decomposition of $G$ has only finitely many summands. If $\sum_{i=1}^{m} \oplus G_i \doteq G \doteq \sum_{j=1}^{n} \oplus H_j$, where all $G_i, H_j$ are strongly indecomposable, then $m = n$ and for some permutation $\varphi$ of $\{1,\ldots,m\}$, $G_i \doteq H_{\varphi(i)}$ $(i=1,\ldots,m)$.

If $G$ has finite rank then $E(G)$ is a finite dimensional rational algebra and so has descending chain condition on right ideals. There also exists groups of infinite rank with $E(G)$ satisfying the descending chain condition. Hence theorem 4.6 is a stronger result than theorem 4.2.

A ring $E$ with radical $N$ is said to be completely primary if $E/N$ is a division ring. It is said to be semi-primary if $E/N$ has descending chain condition on right ideals. Now suppose that $E(G)$ has descending chain condition on right ideals. Then the following are true, as Reid [34] has proven.

(i) If $\lambda \in E(G)$ then there exists a quasi-decomposition $G \doteq G_1 \oplus G_2$ such that $\lambda$ induces a quasi-automorphism on $G_1$ and a nilpotent quasi-endomorphism on $G_2$.

(ii) $G$ is strongly indecomposable iff $E(G)$ is completely primary.

It is also true that if $E(G)$ is semi-primary with nil radical, then $G$ has a quasi-decomposition into a finite number of strongly indecomposable summands whose quasi-endomorphism rings are completely primary. Any two such quasi-decompositions of $G$ are equivalent in the sense of theorem 4.6 .

We conclude this section by remarking that $E(G)$ is a quasi-isomorphism invariant. In fact if $G \doteq H$ then $E(G) = E(H)$ and if $G \stackrel{\bullet}{\cong} H$ then $E(G) \cong E(H)$ .

## 5. Rank Two Groups

Considerable effort has been devoted to the study of rank two groups. The reason for this is two fold; to help develop a complete picture of rank two groups; and, since rank two groups are relatively easy to work with and exhibit much of the pathology of groups of higher rank, to provide a basis for conjectures concerning groups of arbitrary finite rank. Beaumont and Pierce [4] have classified the rank two groups up to quasi-isomorphism and used their invariants to determine conditions for rank two groups to be quasi-decomposable and for quasi-isomorphic rank two groups to be isomorphic. They have also employed their invariants in determing $E(G)$ for rank two groups and in determing both necessary and sufficient conditions for a set of types to be the type set of a rank two group. Reid [34] has provided another approach to determing $E(G)$ for rank two groups and Dubois has devoted two papers, [10] and [11] , to determing type sets of rank two groups. This section will be devoted to reviewing and discussing some of the results of Beaumont and Pierce. The others will be presented in the following sections.

We start by outlining the development of the quasi-isomorphism invariants for rank two group of Beaumont and Pierce. We will employ the following notation. $(G;x_1,x_2)$ will be used to denote a full subgroup, $G$, of a two-dimensional rational vector space, $V$, with $x_1,x_2$ forming a basis of $G$. If $x \in V$ then $h_p^G(x)$ will denote $\sup\{k \mid ({}^c/p^k)x \in G, (c,p) = 1, c \in Z\}$. If $x \in G$ then $h_p^G(x) = h_p(x,G)$. $h_p$ will denote the ordinary logarithmic $p$-adic valuation on $R$ and $R^{(p)}$.

Given $(G;x_1,x_2)$, if $H_1$ and $H_2$ are the pure subgroups of $G$ generated by $x_1$ and $x_2$ respectively, then $G/(H_1 \oplus H_2)$ is either $0$ or a torsion group. Hence we can make the following definition.

<u>Definition 5.1</u> Let $\Sigma$ be the characteristic (i.e. a function from $\pi$ to $N$) satisfying $G/(H_1 \oplus H_2) \cong \sum_{p\in\pi} Z(p^{\Sigma(p)})$. We will write $(G;x_1,x_2) \to \Sigma$.

Now define $A = \{(\alpha,\beta) \mid \alpha,\beta \in \prod_{p\in\pi} Z^{(p)}\}$ and let $\theta$ be a characteristic. A pair $(\alpha,\beta) \in A$ is said to be $\theta$-equivalent to $(\alpha',\beta') \in A$ if

(i) $h_p(\alpha(p)) = h_p(\alpha'(p))$, $h_p(\beta(p)) = h_p(\beta'(p))$ for all $p$.

(ii) $h_p(\alpha(p)\beta'(p) - \alpha'(p)\beta(p)) \geq \theta(p) + h_p(\alpha(p)) + h_p(\beta(p))$ for all $p$.

We will write $(\alpha,\beta) \sim_\theta (\alpha',\beta')$.

$\sim_\theta$ is an equivalence relation on $A$. Furthermore if $(\alpha,p) \in A$, $(\alpha',\beta') \in A$ and if for all $p \in \pi(\alpha(p),\beta(p))$ and $(\alpha'(p),\beta'(p))$ satisfy the following conditions with $(\alpha_1,\beta_1)$ a pair of p-adic numbers.

(a) $h_p(\alpha_1) = h_p^G(x_2) = u$ and $h_p(\beta_1) = h_p^G(x_1) = v$

(b) If $0 < k \leq \Sigma(p)$ and if $m,n$ are integers with $h_p(m) = u$, $h_p(n) = v$ then $p^{-(k+u+v)}(mx_1+nx_2) \in G$ iff $h_p(m\beta_1-n\alpha_1) \geq k+u+v$,

then $(\alpha,\beta) \sim_\Sigma (\alpha',\beta')$. This result leads to

<u>Definition 5.2</u> Let $(G;x_1,x_2) \to \Sigma$ and let $(\alpha,\beta) \in A$ with $(\alpha(p),\beta(p))$ satisfying conditions (a) and (b) above for all $p$. We define $X = [(\alpha,\beta)]$ to be the $\Sigma$-equivalence class in A containing $(\alpha,\beta)$. We call $(\Sigma,X)$ the pair of invariants determined by $(G;x_1,x_2)$ and write $(G;x_1,x_2) \to (\Sigma,X)$.

Now assume that $\Sigma(p) \neq 0$ and that $\alpha(p) = 0$. Then $h_p(\alpha(p)) = \infty$ and so also $h_p^G(x_2) = \infty$. But this implies that $\Sigma(p) = 0$. Hence $\alpha(p) \neq 0$. Also $\beta(p) \neq 0$. Thus we have if $\Sigma(p) \neq 0$ then both $\alpha(p)$ and $\beta(p)$ are non-zero and so for all $(\alpha',\beta') \in X$ if $\Sigma(p) \neq 0$ then $\alpha'(p) \neq 0$ and $\beta'(p) \neq 0$. it is then possible to prove

<u>Theorem 5.3</u>

(i) Let $x_1,x_2$ be an independent pair in $V$ and $(\Sigma,X)$ a pair consisting of a characteristic $\Sigma$ and a $\Sigma$-equivalence class $X$ such that for all $p \in \pi$ and $(\alpha,\beta) \in X$, if $\Sigma(p) \neq 0$ then $\alpha(p) \neq 0$ and $\beta(p) \neq 0$. Then there exists a full subgroup $G$ of $V$, such that $x_1, x_2 \in G$ and $(G; x_1, x_2) \to (\Sigma, X)$.

(ii) Let $(G; x_1, x_2) \to (\Sigma, X)$ and $(H; y_1, y_2) \to (\overline{\Sigma}, \overline{X})$. Then the

non-singular linear transformation $\varphi$ of $V$ taking $x_1$ to $y_1$ and $x_2$ to $y_2$ satisfies $\varphi(G) = H$ iff $\Sigma = \overline{\Sigma}$ and $X = \overline{X}$.

From now on a pair $(\Sigma, X)$ will be a pair such as in theorem 5.3 (i). As a result of this theorem, the correspondence between the full subgroups $(G; x_1, x_2)$ of $V$ and the pairs $(\Sigma, X)$ is one-to-one, for an independent pair $x_1, x_2$ in $V$.

The next step in the development is to determine conditions, in terms of the invariants, for groups to be quasi-isomorphic. If $\varphi_1$ and $\varphi_2$ are functions from $\pi$ to $Z \cup \{\infty\}$ we define $\varphi_1 < \varphi_2$ if $\varphi_1(p) \leq \varphi_2(p)$ for almost all $p$, including all $p$ with $\varphi_1(p) = \infty$. $\varphi_1 \sim \varphi_2$ if $\varphi_1 < \varphi_2$ and $\varphi_2 < \varphi_1$.

Now for $\alpha \in \prod_{p \in \pi} R^{(p)}$ we define a function $K(\alpha)$ from $\pi$ to $Z \cup \{\infty\}$ by $K(\alpha)(p) = h_p(\alpha(p))$. We use this function to define an equivalence on the set of invaraints $(\Sigma, X)$.

<u>Definition 5.4</u> Let $(G; x_1 x_2) \to (\Sigma, X)$ and $(H; y_1, y_2) \to (\overline{\Sigma}, \overline{X})$ with $(\alpha,\beta) \in X$, $(\overline{\alpha},\overline{\beta}) \in \overline{X}$. We define $(\Sigma, X) \sim (\overline{\Sigma}, \overline{X})$ if $\Sigma \sim \overline{\Sigma}$, $K(\alpha) \sim K(\overline{\alpha})$, $K(\beta) \sim K(\overline{\beta})$ and $\Sigma + K(\alpha) + K(\beta) < K(\overline{\alpha}\beta - \alpha\overline{\beta})$.

If $G$ and $H$ are full subgroups of $V$ and $x_1, x_2$ is an independent pair in $G \cap H$ and if $(G;x_1,x_2) \to (\Sigma,X)$, $(H;x_1,x_2) \to (\overline{\Sigma},\overline{X})$, then $G \doteq H$ iff $(\Sigma,X) \sim (\overline{\Sigma},\overline{X})$. Making use of this result and theorem 5.3 we have

Theorem 5.5 Let $G$ and $H$ be full subgroups of $V$, $x_1,x_2$ an independent pair in $G$ and let $(G;x_1,x_2) \to (\Sigma,X)$. Then $G \doteq H$ iff there is an independent pair $y_1,y_2$ in $H$ with $(H;y_1,y_2) \to (\overline{\Sigma},\overline{X})$ such that $(\Sigma,X) \approx (\overline{\Sigma},\overline{X})$.

If we have $(G;x_1,x_2) \to (\Sigma,X)$ then the pair $(\Sigma,X)$ depends upon the basis $x_1,x_2$ of $G$. The final step in the development of the invariants is to determine the effect on $(\Sigma,X)$ of a change in basis; i.e. if $(G;x_1,x_2) \to (\Sigma,X)$ and $y_1,y_2$ is another independent pair in $G$ to determine $(\overline{\Sigma},\overline{X})$ where $(G;y_1,y_2) \to (\overline{\Sigma},\overline{X})$. Aong with this goes the problem of strengthening theorem 5.5 by determining strictly in terms of the invariants, a condition for two groups to be quasi-isomorphic regardless of the basis chosen. A condition for them to be isomorphic has not been found.

Let $x_1,x_2$, $y_1,y_2$ be independent pairs in $G$. Then there exists $r_1,r_2$, $s_1,s_2 \in R$ such that $y_i = r_i x_1 + s_i x_2$ $(i=1,2)$. Suppose $(G;x_1,x_2) \to (\Sigma,\overline{X})$ with $(\alpha,\beta) \in X$, $(\overline{\alpha},\overline{\beta}) \in \overline{X}$. It is possible to determine $\overline{\Sigma}(p)$, $\overline{\alpha}(p)$, $\overline{\beta}(p)$ for almost all primes $p$. Let $u(p) = h_p^G(y_2)$, $v(p) = h_p^G(y_1)$. Then we define

$$\tilde{\Sigma}(p) = 0 \text{ if } h_p(s_1\alpha(p) - r_1\beta(p)) + h_p(s_2\alpha(p) - r_2\beta(p))$$
$$\geq \Sigma(p) + h_p(\alpha(p)) + h_p(\beta(p))$$
$$= \Sigma(p) + h_p(\alpha(p)) + h_p\beta(p)) - (u(p) + v(p)) \text{ otherwise}$$

$$\tilde{\alpha}(p) = 0 \text{ if } u(p) = \infty$$
$$= p^{u(p)} \text{ if } u(p) < \infty \text{ and } \tilde{\Sigma}(p) = 0$$
$$= -s_2\alpha(p) + r_2\beta(p) \text{ if } \tilde{\Sigma}(p) > 0$$
$$\tilde{\beta}(p) = 0 \text{ if } v(p) = \infty$$
$$= p^{v(p)} \text{ if } v(p) < \infty \text{ and } \tilde{\Sigma}(p) = 0 .$$
$$= s_1\alpha(p) - r_1\beta(p) \text{ if } \tilde{\Sigma}(p) > 0 .$$

Then, if $\tilde{X} = [(\tilde{\alpha},\tilde{\beta})]$ is the $\tilde{\Sigma}$ equivalence class containing $(\tilde{\alpha},\tilde{\beta})$, $(\overline{\Sigma},\overline{X}) \sim (\tilde{\Sigma},\tilde{X})$ . The final result, which demonstrates that the pairs $(\Sigma,X)$ can be used to provide a complete quasi-isomorphism classification for the rank two groups, is;

<u>Theorem 5.6</u> Let $(G;x_1,x_2) \to (\Sigma,X)$ and $(H;y_1,y_2) \to (\overline{\Sigma},\overline{X})$ .
Then $G \doteq H$ iff

(i) $\Sigma + K(\alpha) + K(\beta) \sim \overline{\Sigma} + K(\overline{\alpha}) + K(\overline{\beta})$

(ii) $K(\alpha) \cap K(\beta) \sim K(\overline{\alpha}) \cap K(\overline{\beta})$ .

(iii) there exist $r_1,r_2$, $s_1,s_2 \in R$ such that $r_1s_2 - r_2s_1 \neq 0$ and
$$K(\overline{\alpha}(s_1\alpha-r_1\beta) + \overline{\beta}(s_2\alpha-r_2\beta)) > \overline{\Sigma} + K(\overline{\alpha}) + K(\overline{\beta}) .$$

We note that the development of these invariants follows the general lines of the developments described in section 2. A basis is chosen and the invariants are developed. The effect of a change in basis is determined and an equivalence is defined on the system of invariants. Here, as in section 2, the problem of determining the equivalence classes prevents the invariants from being a really satisfactory classification. However the invaraints of Beaumont and Pierce are amenable to computation and thus do have several useful applications as we mentioned previously.

One of these applications is that the invariants provide necessary and sufficient conditions for a rank two group to be directly decomposable and for a rank two group to be quasi-decomposable. Two sufficient conditions for a rank two group to be strongly indecomposable are also provided. Clearly a rank two group $G$ is decomposable into the direct sum of two rank one groups iff $G$ contains a basis $x_1, x_2$ such that $(G; x_1, x_2) \to (\Sigma, X)$ where $\Sigma = 0$. Hence a rank two group $G$ is quasi-decomposable iff $G$ contains a basis $x_1, x_2$ such that $(G; x_1, x_2) \to (\Sigma, X)$ where $\Sigma \sim 0$. This may be restated as: If $(G; x_1, x_2) \to (\Sigma, X)$, $(\alpha, \beta) \in X$ then $G$ is quasi-decomposable iff there exist $r_1, r_2, s_1, s_2 \in R$ with $r_1 s_2 - r_2 s_1 \neq 0$ and $K(s_1\alpha - r_1\beta) + K(s_2\alpha - r_2\beta) > \Sigma + K(\alpha) + K(\beta$

Application of the above statement results in the two sufficient conditions for a rank two group to be strongly indecomposable. Let $(G; x_1, x_2) \to (\Sigma, X)$, $(\alpha, \beta) \in X$. Then $G$ is strongly indecomposable if either for some $p$, $\Sigma(p) = \infty$

and $\beta(p)/\alpha(p)$ is irrational or three distinct primes, $p_1, p_2, p_3$, exist with $\Sigma(p_1) = \Sigma(p_2) = \Sigma(p_3) = \infty$ and $\beta(p_1)/\alpha(p_1)$, $\beta(p_2)/\alpha(p_2)$, $\beta(p_3)/\alpha(p_3)$ all distinct. There results are from Beaumont and Pierce [4].

It is appropriate to mention here that Beaumont and Pierce's invariants are not the only ones that are amenable to computation. Mal'cev's invariants have been employed by Prochazha [30] to find conditions for rank two groups to be directly decomposable. We will not go into his results here but will just say that he has discovered criteria for direct decomposability of a rank two group. He has also found two sufficient conditions for a rank two group to be indecomposable, which are analogous to the conditions for strong indecomposability given above.

The ring of quasi-endomorphisms $E(G)$ of a group $G$ is a quasi-isomorphism invariant and hence it is desirable to determine $E(G)$. Beaumont and Pierce have used their invariants to do this for rank two groups. Let the type number of group $G$ be the cardinality of $T(G)$. Roughly their approach is as follows: If $G$ is a rank two group then the type number of $G$ is either one, two, or greater than two. In each case the invariants must satisfy one of two or three mutually exclusive, exhaustive conditions. $E(G)$ is then determined in each case. The result is that $E(G)$ must be (isomorphic to) one of $R$; $R \dot{+} R$ (ring direct sum): a quadratic field over $R$; the ring of all $2 \times 2$ triangular matrices in $R$; the ring of all

$2 \times 2$ triangular matrices in $R$ with equal diagonal elements; the ring of all $2 \times 2$ matrices in $R$. Examples are presented in each case. Another method of determining $E(G)$ for rank two groups, due to Reid [34], will be presented in section 8. Reid's end results are exactly the same as those of Beaumont and Pierce.

The question of when quasi-isomorphism implies isomorphism is of obvious importance. In the rank two case Beaumont and Pierce's invariants provide an answer. Use is made of a result by Baer [2], namely; if $G \cong G_1 \oplus G_2$ where $G_1$ and $G_2$ are rank one groups of comparable types then any finite extension of $G$ is isomorphic to $G$ (i.e. any group quasi-isomorphic to $G$ is isomorphic to $G$). Application of this result and the above mentioned determination of $E(G)$ leads to the following result.

<u>Theorem 5.7</u> Let $(G;x_1,x_2) \to (\Sigma,X)$, $(\alpha,\beta) \in X$. Then $G$ has the property that $G \doteq H$ implies $G \cong H$ iff either $\Sigma(p) + h_p(\alpha(p)) + h_p(\beta(p)) = \infty$ for all primes $p$ or $G$ is quasi-isomorphic to a direct sum of two rank one groups of comparable types.

While we are speaking of conditions for quasi-isomorphic groups to be isomorphic we should leave rank two groups for a moment and mention a conjecture of Beaumont and Pierce concerning groups of arbitrary finite rank which goes as follows: If $G$ is a group of rank $n$ containing a free subgroup $F$ of rank $n$

such that for each $p$, the divisible part of the p-primary component of $G/F$ has rank at least $n - 1$, and if $H \doteq G$ then $H \cong G$. That this is in fact true has been proved by Prochazha [32]. His proof makes use of Mal'cev's invariants. In another paper [29], Prochazha has proved that we may replace the condition that for each $p$, the divisible part of the p-primary component of $G/F$ has rank at least $n - 1$ by the condition that for each $p$, the p-rank of $G/F$ is at least $n-1$.

Discussion of the type sets of rank two groups will be presented in the next section along with some results on type sets of groups of arbitrary finite rank. Reids computation of $E(G)$ for rank two groups will be described in the final section to demonstrate how the notions of irreducable groups and the psuedo-socle affect the structure of $E(G)$.

6. Type Sets The type set of a group is a quasi-isomorphism invariant. As a result it would be helpful to have a necessary and sufficient condition for a set of types to be the type set of a group of finite rank. However no such condition is yet known. Some necessary conditions and some sufficient conditions are known. Beaumont and Pierce have used their invariants to determine conditions for the rank two case. Dubois [10], [11] has applied analytic number theory to the same case. The case of arbitrary finite rank has been examined by Koehler [22] from a lattice theoretical point of view.

When we say that $T(G)$ is a quasi-isomorphism invariant we mean that if $G \doteq H$ then $T(G) = T(H)$. The converse, however, is not true in general. We demonstrate this by presenting a theorem and an example from Beaumont and Pierce. The theorem determines $T(G)$ in terms of the invariants $(\Sigma, X)$. For $\beta \in \prod_p R^{(p)}$ we define a characteristic $\Delta(\beta)$ by $\Delta(\beta)(p) = 0$ if $\beta(p) \neq 0$ and $\Delta(\beta)(p) = \infty$ if $\beta(p) = 0$. If $(G; x_1, x_2) \to (\Sigma, X)$, $(\alpha, \beta) \in X$, we define $\sigma \in \prod_p R^{(p)}$ by $\sigma(p) = \beta(p)/\alpha(p)$ if $h_p(\beta(p)) = h_p(\alpha(p)) < \infty$, $\sigma(p) = 0$ otherwise,

<u>Theorem 6.1</u> Let $(G; x_1, x_2) \to (\Sigma, X)$, $(\alpha, \beta) \in X$. Then

$$T(G) = \{[\Sigma \cap (K(\sigma - s) + \Delta(\beta - s\alpha)) + (K(\alpha) \cap K(\beta))] \mid s \in R,\ s \neq 0\} \cup \{[K(\alpha)], [K(\beta)]\}.$$

<u>Example 6.2</u> Choose a fixed prime $p$ and let $\sigma_p$ be an irrational element of $R^{(p)}$ with $h_p(\sigma_p) = 0$. Now define:

$\alpha(q) = 1$ for all $q \in \pi$; $\beta(q) = 1$ if $q \neq p$, $\beta(p) = \sigma_p$;

$\Sigma(q) = 0$ if $q \neq p$, $\Sigma(p) = \infty$.

Then there exists a group $G$ with a basis $x_1, x_2$ such that $(G; x_1, x_2) \to (\Sigma, X)$, $(\alpha, \beta) \in X$. Theorem 6.1 can be used to show that $T(G)$ consists of the zero type alone. Furthermore it is possible to show that $G$ is strongly indecomposable. Let $H$ be a rank two free group. Then $T(H) = T(G)$ but $G \ncong H$.

Before proceeding further with the discussion of type sets we define to notion of a quotient divisible (q.d.) group. Such groups are important in the study of torsion free rings (c.f. Beaumont and Pierce [3]). We define them here because many of the results and examples involve them.

<u>Definition 6.3</u> A group G is said to be a q.d. group if G contains a full subgroup F, with F free, such that G/F is the direct sum of a divisible group and a group of bounded order. We note that if G has finite rank then the group of bounded order is finite.

The following properties of q.d. groups are proved in Beaumont and Pierce [3]. If G is q.d. and $G \doteq H$ then H is q.d. Thus the property of being a q.d. group is a quasi-isomorphic invariant. If G is q.d. then G has a free subgroup F such that G/F is divisible. We will also need the notion of a non-nil type. A characteristic is said to be non-nil if it is almost everywhere 0 or $\infty$. A type is said to be non-nil if it contains a non-nil characteristic. If t is a non-nil type then there exists a unique $\theta \in t$ such that $\theta(p)=0$ or $\infty$ for all p.

Beaumont and Pierce [4] have used their invariants for rank two groups to characterize the rank two q.d. groups. The result is that if $(G;x_1,x_2) \to (\Sigma,X)$, $(\alpha,\beta) \in X$ then G is a q.d. group iff $\Sigma + K(\alpha) + K(\beta)$ and $K(\alpha) \cap K(\beta)$ are non-nil: (as a consequence the group G of example 6.2 is q.d.)

In our discussion we will also use the following definitions and notions. For any group G, the type number of G is the cardinality of T(G) and $\overline{G}$ denotes the minimal divisible group containing G.

Let $C(G) = T(G) \cup \{$all finite intersections of members of $T(G)\}$. Then we will write $P(G) = \{G_t \mid t \in C(G)\}$ and $Q(G) = \{\overline{G}_t \mid t \in C(G)\}$. We note that ([13]) $G_t$ is a pure subgroup of $G$ for all types $t$ and that if $G$ has finite rank then $P(G)$ is countable. If there is no possibility of confusion we will sometimes write $G_k$ to denote $G_{t_k}$.

A group $G$ is said to be completely anisotropic if no two independent elements of $G$ have the same type.

The remaining definitions and notions are from Dubois [10] and [11]. Let $C$ denote the set of all coprime ordered pairs of integers $(a,b)$ with $0 \leq a$. Well order $C$ by the relation: if $\max\{a,|b|\} \leq \max\{c,|d|\}$ then $(a,b)$ precedes $(c,d)$. With this well order we will call $C$ the standard list.

Suppose that $t_o$ is a type, $T$ a set of types, $S : t_1, t_2, \ldots$ a sequence of types and $G$ is a group. $T$ is said to be a $t_o$-set if for $t', t'' \in T$, $t' \neq t''$, $t' \cap t'' = t_o$. $S$ is said to be a $t_o$-sequence if, for $0 < i < j$, $t_i \cap t_j = t_o$. $S$ is said to be a type sequence of $G$ if, for $x$ and $y$ independent in $G$, $C$ can be indexed so that $t(a_n x + b_n y, G) = t_n$ for all $n$. If $S$ is a type sequence of a rank two group $G$ then $T(G) = \{t_i \mid i = 1,2,\ldots\}$.

Next, let $S : t_1, t_2, \ldots$ be a $t_o$-sequence with $\theta_i \in t_i$ $(i=0,1,2,\ldots)$. We write $D(i,j,p)$ to represent the proposition $\theta_o(p) < \theta_i(p) < \theta_j(p) = \infty$. If $\theta_i$ is a term of

$S$ and $S'$ is a subsequence of $S$ then $\theta_i$ is said to be a snarl of $S'$ if there exists a subsequence $S''$ of $S'$, say $S'' : \theta_{n_1}, \theta_{n_2}, \ldots,$ such that for every $k$ there is a prime $p_k$ with $D(i,n_k,p_k)$. $\theta_j$ is said to be a snarl if it is a snarl of some subsequence. A subsequence with no snarls is said to be free.

Finally, let $T$ be a $t_o$-set with $\theta_o \in t_o$. A type $[\theta] \in T$ is a snarl of the subset $T' \subseteq T$ if $T'$ contains an infinite subset $T''$ such that for every $[\theta'']$ in $T''$ there is a $p \in \pi$ with $\theta_o(p) < \theta(p) < \theta''(p) = \infty$. A subset $T'$ of $T$ is said to be free if it has no snarls.

The last of our preperatory notions is the construction, due to Dubois [11], of groups $R(S,x)$ which are useful in type considerations. $S$ is an independent set of reals in the open interval $(0,1)$ and $x$ is a function from $S$ into $\prod_p Z^{(p)}$ such that, for $s \in S$, $x(s)$ is a function on $\pi$ whose value at $p$ is a p-adic integer. We write $x(s)(p) = x(s,p)$. We define $R(S,x)$ to be the set of all finite rational combinations $\Sigma\, r_s s$ $(r_s \in R)$ such that for every $p$, $\Sigma\, r_s x(s,p) \in Z^{(p)}$. Then $R(S,x)$ is a group with rank $|S|$ such that $h_p(\Sigma\, r_s s,\ R(S,x)) = h_p(\Sigma\, r_s x(s,p))$ and for every $p$ the correspondence $\Sigma\, r_s s \to \Sigma\, r_s x(s,p)$ is a p-height preserving homomorphism with kernel equal to the set of all members of $R(S,x)$ with infinite p-height and a p-pure image in $Z^{(p)}$.

In the case where $S = \{x,y\}$, we denote the functions by $u$ and $v$ and the group by $R(x,y;u,v)$. Then $R(x,y;u,v)$ is the group of all rational combinations $ax + by$ where for all $p$ $au(p) + bv(p) \in Z^{(p)}$.

We now present necessary conditions on a type set to be the type set of a group $G$. Beaumont and Pierce [4] have used their invariants to examine the rank two case. Let $G$ be a rank two group and suppose that $x_1, x_2$ is a basis of $G$ and $(G;x_1,x_2) \to (\Sigma, X)$, $(\alpha,\beta) \in X$. Then $t(x_1,G) \cap t(x_2,G) = [K(\alpha) \cap K(\beta)]$.

<u>Theorem 6.4</u> If $G$ is a rank two group there is a unique type $t_o$ such that if $t_1, t_2 \in T(G)$, $t_1 \neq t_2$, then $t_1 \cap t_2 = t_o$. If $x$ and $y$ are non-zero elements of $G$ with $t(x) = t(y) \neq t_o$ then $x$ and $y$ are dependant. If $T(G)$ is finite then $t_o \in T(G)$.

If $r(G) = 2$ and $G$ has finite type number then $G$ is not completely anisotropic. Thus by theorem 6.4 if $t_o \notin T(G)$ then $G$ is completely anisotropic.

Note that theorem 6.4 can be restated as: if $r(G) = 2$ then $T(G)$ is a $t_o$-set. Dubois [10] has also obtained this result along with: every type sequence of $G$ is a $t_o$-sequence. Actually he has provided a stronger necessary condition. Employing some ideas from analytic number theory he proves

<u>Theorem 6.5</u> If $G$ is a rank two group then the type sequence of $G$ has an infinite free subsequence. If $G$ is completely

anisotropic then $T(G)$ contains an infinite free subset.

An example will demonstrate that the necessary condition of this theorem is stronger than that of the preceding one. The same example also provides a strong negative answer to a question posed by Beaumont and Pierce, namely: Given a countably infinite set $T$ of distinct types such that $T$ is a $t_o$-set does there exist a rank two group $G$ with $T(G) = T$ ?

Example 6.6 [10] Suppose the infinite sets $P_1, P_2, \ldots$ partition $\pi$. We define $\theta_1(p) = 1$ if $p \in P_1$, $\theta_1(p) = 0$ elsewhere and, for $n = 2,3,\ldots$, $\theta_n(p) = \infty$ if $p$ is the $n^{th}$ member of $P_i$ for some $i < n$, $\theta_n(p) = 1$ if $p \in P_n$, $\theta_n(p) = 0$ otherwise. Then $T = \{[\theta_1], [\theta_2], \ldots\}$ is a $t_o$-set where $t_o$ is the zero type and, by theorem 6.5 if $T'$ is any subset of $T$, then $T'$ is not the type set of any completely anisotropic rank two group.

Dubois has also applied the notion of groups of the form $R(x,y;u,v)$ to find a necessary and sufficient condition for a $t_o$-sequence to be the type sequence of a group. In [11] an example is constructed to prove that every type sequence of a rank two group is a type sequence of some rank two group $R(x,y;u,v)$. Examination of type sequences of arbitrary groups $R(x,y;u,v)$ will yield;

Theorem 6.7 A $t_o$-sequence is a type sequence of a rank two group iff the zero sequence obtained by subtracting $t_o$ from every term is likewise a type sequence of a rank two group.

Koehler [22] has obtained necessary conditions on type sets of groups of finite rank. His approach is to show that the type set of a finite rank group has certain lattices of types and of pure subgroups associated with it. His first result is to prove that, if $r(G) = n$, then $C(G)$ forms a lattice of length at most $n$ in which lattice meet is type intersection and $C(G)$ has a minimum type $t_o$ where $t_o = t(x_1) \cap \cdots \cap t(x_n)$ for any basis $x_1, x_2, \ldots, x_n$ of $G$.

The set of all types under the relation $\leq$ and the operations $\cap$ and $\cup$ forms a distributive lattice in which meet and join are $\cap$ and $\cup$ respectively. If $G$ has finite rank then the above remark tells us that $C(G)$ is also a lattice. However it need not be a sublattice of the lattice of all types (example 6.15).

Note that this remark is a generalization of theorem 6.4. This remark also provides an affirmative answer to another question posed by Beaumont and Pierce [4]: If $r(G) = n$ is the intersection of the types of elements of a given basis the same for all bases? Furthermore, suppose $G$ is a finite rank q.d. group and that $t_o$ is the minimum type in $C(G)$. Then ([11] or [22]) $t_o$ is non-nil. In this connection we also have ([11]) that the type set of rank two group $G$ is the type set of some q.d. group iff $t_o$, the minimal type in $C(G)$, is non-nil.

Theorem 6.8 If $G$ is a group of finite rank $n$, then $\overline{G} = R^n$ and

(i) $P(G)$ forms a lattice of pure subgroups of $G$, $Q(G)$ forms a lattice of subspaces of $R^n$ and as lattices, $P(G)$ is isomorphic to $Q(G)$ and both are dually isomorphic to $C(G)$ .

(ii) In the lattices $P(G)$ and $Q(G)$ if $\wedge$ and $\vee$ denote lattice meet and join respectively then, for

$$G_i, G_j \in P(G) \quad G_i \wedge G_j = G_i \cap G_j \; ; \; \overline{G}_i \wedge \overline{G}_j = \overline{G}_i \cap \overline{G}_j$$

$$G_i \vee G_j \supseteq G_i + G_j \; ; \; \overline{G}_i \vee \overline{G}_j \supseteq \overline{G}_i + \overline{G}_j$$

This theorem, due to Koehler , can be used to prove that if $T(G)$ is finite then $T(G) = C(G)$ and there are $r(G_t)$ independent elements of type $t$ in $G$ for every $t \in T(G)$. For an example of a group of finite rank and infinite type set $T(G)$ such that $T(G) \neq C(G)$ see Beaumont and Pierce [4, pg. 29] .

This concludes our review of necessary conditions on a type set to be the type set of a group of finite rank. We now turn to the problem of finding sufficient conditions. As with the necessary conditions we first turn our attention to the rank two case. In the case of finite type sets we have a complete answer.

Theorem 6.9 Let $T = \{t_o, t_1, \ldots, t_n\}$ be a finite $t_o$-set. Then their exists a rank two group $G$ with $T(G) = T$ .

Both Beaumont and Pierce [4] and Dubois [10] have constructed examples to prove this result. As a result of theorem 6.4 and 6.9 we have

Corollary 6.10 A finite type set $T$ is the type set of rank two group iff $T$ is a $t_o$-set containing $t_o$ .

Suppose $T$ is a finite $t_o$-set containing $t_o$ and $t_o$ is non-nil. Then does there exists a q.d. rank two group $G$ with $T(G) = T$ ? This question was posed by Beaumont and Pierce [4]. If $(G;x_1,x_2) \to (\Sigma,X)$, $(\alpha,\beta) \in X$ then $G$ is q.d. iff $\Sigma + K(\beta) + K(\alpha)$ and $K(\alpha) \cap K(\beta)$ are non-nil. In the question as posed $t_o = [K(\alpha) \cap K(\beta)]$ is given as non-nil and so to answer the question, a group $G$ must be found with $T(G) = T$ such that $G$ contains a basis $x_1,x_2$ with $(G;x_1,x_2) \to (\Sigma,X)$ where $\Sigma$ is non-nil and $\Sigma(p) = \infty$ for all but a finite number of primes $p$ such that $0 < h_p(\alpha(p)) < \infty$ and $0 < h_p(\beta(p)) < \infty$ whenever $(\alpha,\beta) \in X$. That such a group does exist has been proved by Koehler [21]. This question has also been answered by Dubois [10] who has construced an example to prove that: A free $t_o$-sequence, with $t_o$ non-nil, is the type sequence of a q.d. rank two group.

In the case of countably infinite type sets Beaumont and Pierce were only able to achieve partial results. They have constructed an example of a completely anisotropic rank two group $G$ with type set $T(G) \subseteq \{[\theta_o],[\theta_1],[\theta_2],\ldots\}$ where $\{\theta_o,\theta_1,\theta_2,\ldots\}$ is an infinite set of inequivalent characteristics with, for $i \neq j$, $\theta_i \cap \theta_j = \theta_o$ and used it to prove: If $T$ is an infinite $t_o$-set containing $t_o$ and each type in $T$ is finite then there is a completely anisotropic rank two group $G$ such that $T(G) \subseteq T$.

Koehler [21] has developed a new method of constructing rank two groups with infinite type sets which enables him to prove some partial results similar to those above, namely: If $T = \{t_o, t_1, t_2, \dots\}$ is a $t_o$-set then there exists a $t_o$-set $T = \{t_o, t_1', t_2', \dots\}$ with $t_i \leq t_i'$ if $i \geq 1$ and a rank two group $G$ with $T' \subseteq T(G)$. If $T = \{t_o, t_1, t_2, \dots\}$ is a $t_o$-set such that for at most finitely many $i$ $\theta_i(p) = \infty$ occurs where $\theta_i \in t_i$ then there exists a rank two group $G$ with $T \subseteq T(G)$. In some cases Koehler's construction yields good results. For instance, if we define $\theta_o(p) = \theta$ for all $p$; $\theta_1(p) = 1$ for all $p$; $\theta_k(p_{k-1}) = \infty$, $\theta_k(p) = 0$ for all other $p$, then $T = \{[\theta_o], [\theta_1], [\theta_2], \dots\}$ is a $[\theta_o]$ set and the rank two group $G$ resulting from the construction has type set $T(G) = T$.

Dubois has obtained some precise results in determining sufficient conditions for a type set to be the type set of a rank two group. In [10] an example is given which proves;

<u>Theorem 6.11</u>

(i) If $S$ is a free $t_o$-sequence then $S$ is a type sequence of some rank two group.

(ii) If $T$ is an infinite $t_o$-set whose members are infinite only where $t_o$ is infinite then $T$ is the type set of a rank two group.

We now present an example due to Dubois which demonstrates that the sufficient condition of theorem 6.11 is not necessary. This example will also provide a negative answer to another question posed by Beaumont and Pierce [4] : If $G$ is a group with infinite type set does there exists a completely anisotropic group $H$ with $T(H) = T(G)$ ? We define for $n = 0,1,2,\ldots$ : $\theta_{2n}(p) = 0$ for all $p$ ; $\theta_1(p) = 1$ for all $p$ ; for $n \geq 1$ , $\theta_{2n+1}(p) = 0$ for $p \neq p_n$, $\theta_{2n+1}(p_n) = \infty$ . Then $S : [\theta_1], [\theta_2], \ldots$ is a $[\theta_o]$-sequence and, if $T = \{[\theta_i] \mid i = 1,2,\ldots\}$ , $[\theta_1]$ is a snarl of every infinite subset of $T$ . Hence $T$ is not the type set of a completely anisotropic rank two group. Also neither $S$ nor $T$ is free. Now let $c_1 = (1,0)$, $c_2 = (0,1)$ . For $n \geq 1$ let $c_{2n+1} = (1,-p_n)$ and $c_{2n+2}$ equal to the first pair in the standard list not previously selected. We write $c_n = (a_n,b_n)$ . Choose independent real numbers $x$ and $y$ and let $G = \{ax+by \mid a,b\in R,\ ap + b\in Z^{(p)}$ for all $p\}$ . Then $G$ is a rank two group. Note that if $x,y \in (0,1)$ then $G$ is a group of the form $R(x,y;u,v)$ where $u(p) = p$ and $v(p) = 1$ for all $p$ . We also have, for all $k = 1,2,\ldots$ $[\theta_K] = t(a_k x+b_k y, G)$ . Hence $S$ is the type sequence of $G$ and $T = T(G)$ .

In [11] , Dubois strengthens the sufficient condition of theorem 6.11. A characteristic is said to be very large if it is infinite at infinitely many primes.

<u>Theorem 6.12</u> If a $t_o$-sequence, $S$, has an infinite free

subsequence and if the set of all snarls and very large elements of $S$ is free then $S$ is a type sequence of a rank two group.

The group constructed to prove theorem 6.12 is a group of the form $R(x,y;u,v)$. Dubois has constructed another group $R(x,y;u,v)$ which demonstrates that the condition that the set of all very large elements be free is not necessary.

The various sufficiency theorems of Dubois, especially theorem 6.7 and the remark immediately preceeding it, suggest the following formulation of the problem of determining necessary and sufficient conditions in the rank two case. Let $\theta : \theta_1, \theta_2, \ldots$ be a sequence of characteristics with the corresponding sequence of types, $S$, a zero-sequence. Such a sequence $\theta$ is said to be solvable if there exists an indexing of the elements $(a,b)$ of $C$ so that for every $i$ there exists an $m$ such that for all $p \in \pi$ and $n > m$, if $\theta_n(p) = \infty$ then $h_p(a_i b_n - a_n b_i) = \theta_i(p)$. Then, with $S$ and $\theta$ as above

<u>Theorem 6.13</u> $S$ is a type sequence of a rank two group iff $\theta$ is a solvable sequence.

Koehler has found a sufficient condition for a finite type set to be the type set of a finite rank group. Let $t_\infty$ be the type greater than all types.

<u>Theorem 6.14</u> Let $T = \{t_\infty, t_o, t_1, \ldots, t_N\}$ be a set of distinct types forming a lattice under $\wedge$ and $\vee$ where $t_i \wedge t_j = t_i \cap t_j$ and $t_i \vee t_j$ is the l.u.b. in $T$. Let $L = \{0, \overline{G}_o, \overline{G}_1, \ldots \overline{G}_N\}$ be

a lattice of subspaces of $R^n = \overline{G}_o$ under $\wedge$ and $v$ where $\overline{G}_i \wedge \overline{G}_j = \overline{G}_i \cap \overline{G}_j$ and $v$ is the l.u.b. in $L$. Suppose that, as lattices, $T$ is dually isomorphic to $L$. Then there exists a rank $n$ group $G$ with $T(G) = T$ and $Q(G) = L$.

The results of theorems 6.8 and 6.14 indicate that the problem of finding all finite type sets which are type sets of a group of finite rank $n$ is equivalent to the problem of finding all possible finite lattices, under the operations $\wedge$ and $v$, of subspaces of $R^n$. This latter problem is as yet unsolved.

<u>Example 6.15</u> We define $\theta_o(p) = 0$ for all $p$; $\theta_1(2) = \infty$, $\theta_1(p) = 0$ otherwise; $\theta_2(3) = \infty$, $\theta_2(p) = 0$ otherwise; $\theta_3(2) = \theta_3(3) = \theta_3(5) = \infty$, $\theta_3(p) = 0$ otherwise. If we set $t_i = [\theta_i]$ then by theorem 6.14, since the dual of the lattice of types is realizable in $R^3$, there exists a rank 3 group $G$ with $T(G) = C(G) = \{t_\infty, t_o, t_1, t_2, t_3\}$. Hence $C(G)$ forms a lattice in which $t_1 \vee t_2 = t_3 > t_1 \cup t_2$.
Hence $C(G)$ is not a sublattice of the lattice of types.

We conclude our discussion of type sets with the following result from Dubois [11].

<u>Theorem 6.16</u> If $T$ is a set of types with the property that $t', t'' \in T$ implies $t' \cap T'' \in T$ then $T$ is the type set of a group $R(S,x)$ of rank $|T|$.

7. <u>Quasi Essential Groups</u> We devote this section to a brief discussion on the notion, due to Koehler [22], of quasi-essential groups. As we will see these groups can be used to suggest a possible approach to the problem of finding quasi-isomorphic invariants for groups of finite rank with finite type sets. The definition follows the construction of the group of theorem 6.14 and hence provides some idea of the method employed there.

<u>Definition 7.1</u> Let $G$ be a group and let $B = \{x_1, \ldots x_n\}$ be any finite set of independent elements of $G$. Let $F_B$ denote the free subgroup of $G$ generated by $B$. An element $x \in G$ is said to be $B$-reduced if $x \in F_B$ and $h_p(x, F_B) = 0$ for all $p \in \pi$.

<u>Definition 7.2</u> A group $G$ of finite rank is said to be an essential group if it has for a set of generators, the set

$$\{p^{-s_k(p)} y_i^k \mid p \in \pi,\ 0 \leq s_k(p) < \theta_k(p)+1;\ k=0,1,\ldots,N;\ i=1,2,\ldots,n_k\}$$

where

(i) $\theta_0, \theta_1, \ldots, \theta_N$ are characteristics with $t_i = [\theta_i]$ such that if $t_i \leq t_j$ then $\theta_i \leq \theta_j$ and if $t_i \cap t_j = t_k$ then $\theta_i \cap \theta_j = \theta_k$, $\theta \leq i,j,k \leq N$.

(ii) $n_k = r(G_{t_k})(=r(G_k))$ ; $k = 0,1,\ldots,N$.

(iii) $B_0 = \{y_1^0, y_2^0, \ldots, y_{n_0}^0\}$ is a basis of $\overline{G}$ such that $y_i^0 \notin \overline{G}_k$, $1 \leq k \leq N$, $1 \leq i \leq n_0$.

(iv) for each $k = 1,2,\ldots,N$, $\{y_1^k, y_2^k, \ldots, y_{n_k}^k\}$ is a basis of $\overline{G}_k$ such that $y_i^k$ is $B_o$-reduced and $y_i^k \notin \overline{G}_j$ if $\overline{G}_j \subseteq \overline{G}_k$ .

A group H is said to be a quasi-essential (q.e) group if it is quasi-isomorphic to some essential group G .

Suppose that $y_\lambda \in R^n$ and that $\theta_\lambda$ are characteristics $(\lambda \in \Lambda)$ . By the notation $G = \{(y_\lambda, \theta_\lambda) \mid \lambda \in \Lambda\}$ we will mean the group G generated by the set $\{p^{-s_\lambda(p)} y_\lambda \mid p \in \pi \ ; \ 0 \leq s_\lambda(p) \leq \theta_\lambda(p) + 1 \ ; \ \lambda \in \Lambda\}$ . Hence if G is the group of definition 7.2 then $G = \{(y_i^k, \theta_k)\}$. Furthermore $T(G) = \{t_\infty, t_o, t_1, \ldots, t_N\}$ and $Q(G) = \{0, \overline{G}_o, \overline{G}_1, \ldots, \overline{G}_N\}$ .

<u>Definition 7.3</u> Let $G'$ be an essential (q.e.) subgroup of a group G . $G'$ is said to be a maximal essential (q.e) subgroup if $G' \dot{\subseteq} H \subseteq G$ where H is an essential (q.e) subgroup of G then $G' \doteq H$ .

The principal result on these groups, for our purposes, is the following:

<u>Theorem 7.4</u> Let G be a finite rank group with finite type set. Then

(i) G has a maximal essential subgroup $G'$ , unique up to quasi-equality, with $T(G) = T(G')$ and $Q(G) = Q(G')$ .

(ii) If $x \in G$ there is a maximal essential subgroup $G'$ of G with $x \in G'$ .

(iii) G is q.e. iff $G/G'$ is a finite group for every maximal essential subgroup $G'$ of G .

(iv) If $G'$ is a maximal essential subgroup of G then $G/G'$ is a torsion group.

Proof: We will prove (i) and (ii) . Let $r(G) = n$ and $T(G) = \{t_\infty, t_o, t_1, \ldots, t_N\}$ with $t_o$ the minimal type in $T(G)$. There is an independent set $\{x, y_2^o, \ldots, y_n^o\}$ with $t(y_i^o) = t_o$ . If $t(x) = t_o$ let $y_1^o = x$ . Otherwise let $S$ be the pure subgroup of $G$ generated by $\{x, y_2^o\}$ . Now $t(y,S) = t(y,G)$ for all $y \in S$ and $t(y_2^o, S) = t_o$ . For some $m \in Z$ $t(x+my_2^o, S) = t_o = t(x+my_2^o, G)$ . Let $y_1^o = x + my_2^o$ and $B_o = \{y_1^o, y_2^o, \ldots, y_n^o\}$ . Then $x$ is $B_o$-reduced.

For each $t_k \in T(G)$ , $t_k \neq t_o$ we can find $n_k = r(G_k)$ independent $B_o$-reduced elements of type $t_k$ in $G$, $y_1^k, y_2^k, \ldots, y_{n_k}^k$ . Define $\theta_k = \bigcap_j h(y_j^k, G)$ for $k = 0,1,\ldots,N$ . It is possible to find characteristics $\theta_o', \theta_1', \ldots, \theta_N'$ such that, for $0 \leq i,j,j \leq N$ $\theta_i' \leq \theta_i$ ; $\theta_i' \sim \theta_i$ ; if $t_i \leq t_j$ then $\theta_i' \leq \theta_j'$ ; if $t_i \cap t_j = t_k$ then $\theta_i' \cap \theta_j' = \theta_k'$ .

Let $G' = \{(y_i^k, \theta_k') \mid k = 0,1,\ldots,N ; \ i = 1,2,\ldots,n_k\}$ . $G'$ is an essential subgroup of $G$ and $T(G') = T(G)$ , $Q(G') = Q(G)$ . Furthermore $G'$ is maximal essential, contains $x$ and is unique up to quasi-equality.

The results of this theorem suggest that the problem we mentioned in the opening paragraph of this section could possibly be solved by examining the groups of the form $G/G'$ where $G$ has finite rank, $G'$ is a maximal essential subgroup of $G$ .

(i) $G$ is irreducible.

(ii) $G \doteq \sum_{i=1}^{n} \oplus G_i$ where each $G_i$ is strongly indecomposable, irreducible, and $G_i \doteq G_j$ for all $i,j$.

(iii) $E(G) = A_n$ where $A$ is a division algebra, $n$ is the number of strongly indecomposable summands in a quasi-decomposition of $G$ and $n[A:R] = r(G)$.

An irreducible group of finite rank is strongly indecomposable iff $E(G)$ is a division ring.

An irreducible group of prime rank is either strongly indecomposable or a direct sum of isomorphic rank one groups.

If $E(G)$ has descending chain condition on right ideals, then $P$ is non-zero. Also if $G$ is strongly indecomposable and $N$ denotes the radical of $E(G)$ then as groups $r(H) = r(\overline{H}) = [E(G)/N:R]$, where $H$ is a minimal non-zero pure fully invariant subgroup of $G$.

If $E(G)$ has descending chain condition on right ideals then $E(G)$ is semi-simple iff $G = P$.

If $G$ has finite rank and $G \neq P$ then for $x \in G$, $x \notin S$ there exists an endomorphism $\lambda$ of $G$ with $0 \neq x\lambda \in S$.

Now suppose that $G$ is a rank two group. Then $G$ must satisfy one of three mutually exclusive exhaustive conditions; namely; $G$ is irreducible, $G$ is not irreducible but $G = P$, or $G \neq P$. Suppose $G$ is irreducible. If $G$ is strongly indecomposable then $E(G)$ is a division ring $A$. Furthermore $[A:R] = 2$. If $B$ is the center of $A$ then $[A:B] = x^2$ where $x$ is a positive integer. Now $[A:R] = [A:B][B:R]$ and so $[A:B] = 1$. Hence $A$ is equal to its center and is thus a quadratic field over $R$. If $G$ is not strongly indecomposable then $G = G_1 \oplus G_2$ where $G_1$ and $G_2$ are isomorphic rank one groups. In this case $E(G) = A_2$ where $A$ is a division algebra and $E(G_i) = A$ $(i=1,2)$. Since $G_i$ has rank one, $A = R$, and $E(G) = R_2$, the ring of all $2 \times 2$ matrices in $R$.

Next suppose that $G$ is not irreducible and $G = P$. If $G$ is strongly indecomposable, then $E(G)$ is a division algebra. Also $G$ contains a minimal non zero pure fully invariant subgroup of rank one. Hence $[E(G):R] = 1$, i.e. $E(G) = R$. If $G$ is quasi-decomposable then $E(G)$ is a semi-simple ring that is not a division ring. If $E(G)$ were simple then $G$ would be irreducible. Hence $E(G) = E_1 \dotplus E_2$ (ring direct sum) where the $E_i$ are simple ideals with central idempotent generators. Corresponding to this decomposition $E(G) = E_1 + E_2$ we have a quasi-decomposition $G \doteq G_1 \oplus G_2$ where the $G_i$ are rank one groups of incomparable types. Also $E(G_i) \cong E_i$ and therefore $E_i = R$ $(i=1,2)$. Thus $E(G) = R \dotplus R$.

Finally suppose that $G \neq P$. Assume $G$ is strongly indecomposable. Choose $0 \neq x_1 \in P$, $x_2 \in G$, $x_2 \notin P$. Then $\{x_1, x_2\}$ is a basis of $V$, which is an $E(G)$-module in a natural way. $\overline{P}$ is an $E(G)$-submodule of $V$ and so the matrix representation of $E(G)$ given by $\{x_1, x_2\}$ consists of triangular matrices. $N$, the radical of $E(G)$, is non-zero and is a rational algebra. Hence $N = \{\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \mid r \in R\}$. We also have that $E(G)/N \cong R$. Let $F$ denote the subalgebra of $E(G)$ generated by the unit matrix. Then $N \cap F = 0$. Under the natural map $F$ goes onto $E(G)/N$ and so $E(G) = N + F$. Hence $E(G)$ is the ring of triangular matrices in $R$ with equal diagonal elements. If $G$ is quasi-decomposable then $G \doteq P \oplus P'$ for some $P'$. Furthermore the type of $P'$ is less than the type of $P$. As before a triangular representation of $E(G)$ can be obtained. The radical $N$ is one-dimensional over $R$. Hence $E(G)/N$ is a two-dimensional semi-simple rational algebra that is not a division algebra. This implies $E(G)/N \cong R \dotplus R$ and hence $E(G)$ is the ring of all $2 \times 2$ triangular matrices in $R$.

We note in conclusion that these are the same end results at which Beaumont and Pierce [4] arrived in their computation of $E(G)$ for rank two groups.

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