PLANE COORDINATES

by

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PLANE COORDINATES

Chapter I

1. Introduction:-

The primary purpose of this thesis is to develop the ordinary relations of solid analytic geometry by the use of plane-coördinates. The significance of various equations of the Cartesian system with reference to this new system will also be discussed.

As far as possible, the treatment parallels the treatment of line-coördinates, as contained in the theses submitted by Valgardsson of Manitoba and Heaslip and James of British Columbia for the degree of Master of Arts.

2. Fundamental Definitions:-

We use the rectangular reference system, i.e. three mutually perpendicular planes intersecting in three mutually perpendicular straight lines $X'OX$, $Y'OY$, $Z'OZ$, which are called the X, Y, Z axes, respectively. The X axis is formed by the intersection of the ZX and XY planes; the Y axis by the intersection of the XY and YZ planes; and the Z axis by the intersection of the YZ and ZX planes. The point O, common to all three planes, is called the origin. The customary conventions with regard to sign are observed. For example, the directions $X'OX$, $Y'OY$, $Z'OZ$ are considered positive, and the directions XOX', YOY', ZOZ' are considered negative.

The coördinates of a plane are defined to be the reciprocals of its intercepts on the coördinate axes. Thus the plane ABC in figure (1) has coördinates (a, b, c), since

(1) $$OA = \frac{1}{a}, \quad OB = \frac{1}{b}, \quad OC = \frac{1}{c}$$

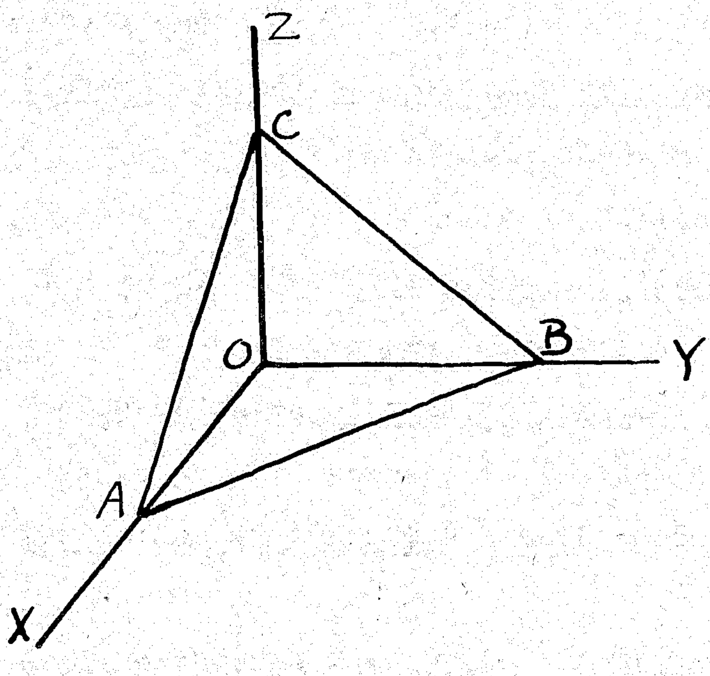

Fig. (1)

Conversely, any plane (a, b, c) makes intercepts $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ on the X, Y, Z axes, respectively.

In Cartesian coördinates the point (a, b, c) is such that its directed perpendicular distances from the YZ, ZX, XY planes are a, b, c, respectively. The plane

$$ax + by + cz - 1 = 0$$

has intercepts $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ on the coördinate axes.

3. Coördinates of Planes:-

Any plane whose intercepts on the coördinate axes are all finite and different from zero is seen to be represented uniquely by (a, b, c). The following is a summary of some special cases:

(i) Coördinate Planes. The XY plane is denoted by (a, b, ∞), where a and b are both finite.

(ii) A plane through a coördinate axis and cutting the other axes obliquely. Such a plane through the X axis has the coördinates (a, ∞, ∞), where a is finite.

(iii) The coördinates of a plane parallel to that given in (ii) are (o, b, c), where b and c are finite.

(iv) The coördinates of a plane parallel to that given in (i) are (o, o, c), where c is finite.

(v) The "plane at infinity" has the coördinates (o, o, o).

(vi) A plane through the origin and oblique to all three axes has the coördinates (∞, ∞, ∞).

It is to be noted that the coördinates in (ii) and (vi) do not represent one plane uniquely, and that the planes in (i) and (ii) do not possess unique coördinates.

4. Parallel Planes:-

Theorem: The necessary and sufficient conditions for the parallelism of two planes (x_1, y_1, z_1) and (x_2, y_2, z_2) are

(2) $$\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

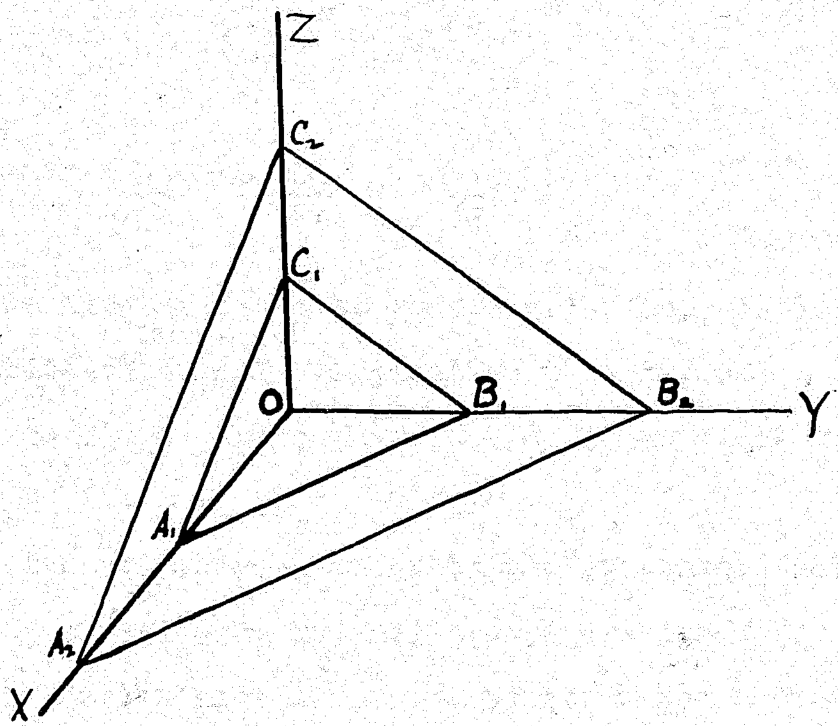

Fig. (2)

The conditions are necessary. For suppose that the planes $A_1 B_1 C_1$, i.e. (x_1, y_1, z_1), and $A_2B_2C_2$, i.e. (x_2, y_2, z_2), are parallel. Then they cut the coördinate planes in parallel lines,(1) that is, A_1B_1 and A_2B_2 are parallel. Hence

$$\frac{OA_2}{OA_1} = \frac{OB_2}{OB_1};$$

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(1) Wilson "Solid Geometry and Conic Sections", p. 12.

i.e.
$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

In the same way
$$\frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

Therefore
$$\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}.$$

The conditions are also sufficient. Suppose relations (2) hold. Then A_1B_1 is parallel to A_2B_2, and B_1C_1 is parallel to B_2C_2. Hence plane $A_1B_1C_1$ is parallel to plane $A_2B_2C_2$.(1)

This theorem is equivalent to the statement that the planes (a, b, c) and (ka, kb, kc) are parallel.

In the Cartesian system, two points whose coördinates satisfy equations (2) are collinear with the origin, and conversely. If two planes
$$A_1x + B_1y + C_1z - 1 = 0,$$
$$A_2x + B_2y + C_2z - 1 = 0$$
are parallel, then
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2},$$
and conversely.

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(1) Wilson, loc.cit., p. 13.

5. <u>Direction Cosines of a Line</u>:-[1]

Let l be any directed line in space, and let l' be the line through the origin with the same direction as l. Let α, β, γ be the angles between the X, Y, Z axes, respectively, and l'.[2]

By definition these are the angles which l makes with the axes. They are called the "direction angles" of the line l, and their cosines are called its "direction cosines". The direction cosines will be denoted by λ, μ, ν respectively.

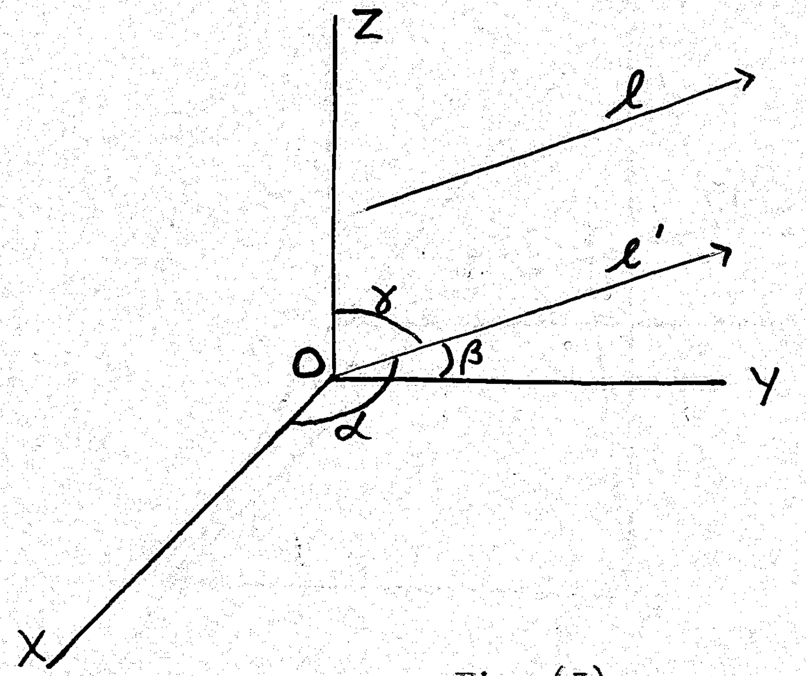

Fig. (3)

(1) As in Snyder and Sisam "Analytic Geometry of Space".

(2) See Snyder and Sisam, p. 3.

It is easily proved that the relation

$$\lambda^2 + \mu^2 + \nu^2 = 1$$

holds.[1]

6. Angle between Two Directed Lines:-[2]

Suppose that l_1 and l_2 are two directed lines with direction cosines λ_1, μ_1, ν_1, and λ_2, μ_2, ν_2, respectively. In solid geometry the angle between two directed lines is defined to be the angle between the two similarly directed lines through the origin.

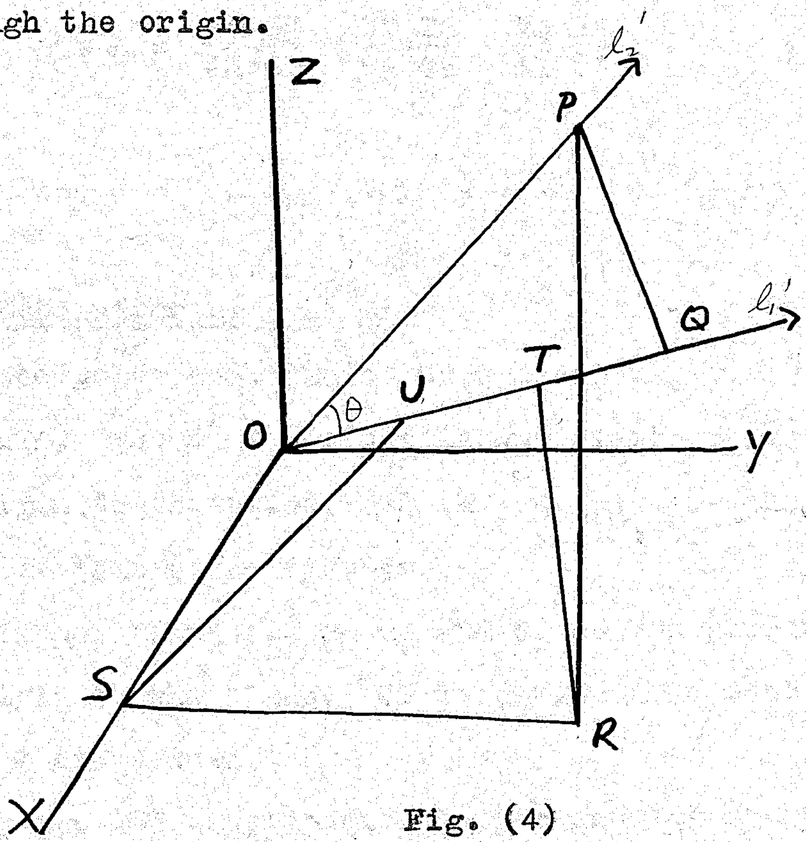

Fig. (4)

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(1) Snyder and Sisam, p. 6.

(2) As in Snyder and Sisam.

In figure (4), l_1' and l_2' are parallel to l_1 and l_2 respectively. If OP is any segment taken along the positive direction of l_2', PQ is perpendicular to l_1', and PR is perpendicular to the plane XOY at R. Perpendiculars RT, RS, SU are drawn to OQ, OS, OT, respectively, as shown in the diagram. The angle between l_1 and l_2 is the angle θ in the figure. Now

$$\cos\theta = \frac{OQ}{OP} = \frac{OU + UT + TQ}{OP};$$

therefore

$$\cos\theta = \frac{OU}{OS}\cdot\frac{OS}{OP} + \frac{UT}{SR}\cdot\frac{SR}{OP} + \frac{TQ}{PR}\cdot\frac{PR}{OP};$$

and hence

(3) $$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 .$$

7. <u>Polar Coördinates of a Plane:-</u>

Let the polar coördinates of a plane (x, y, z) be $(\rho, \alpha, \beta, \gamma)$, where ρ is the length of the perpendicular from the origin to the plane, and α, β, γ are the direction angles of this perpendicular.

ABC is any plane (x, y, z) and OP is the perpendicular from the origin to the plane. CP is produced to meet AB at Q and O and Q are joined.

The plane QOC is perpendicular to each of the planes XOY and ABC. Hence it is perpendicular to AB, their line of intersection. Therefore CQ and OQ are both perpendicular to AB. Since the triangles OQB and AOB are similar, it

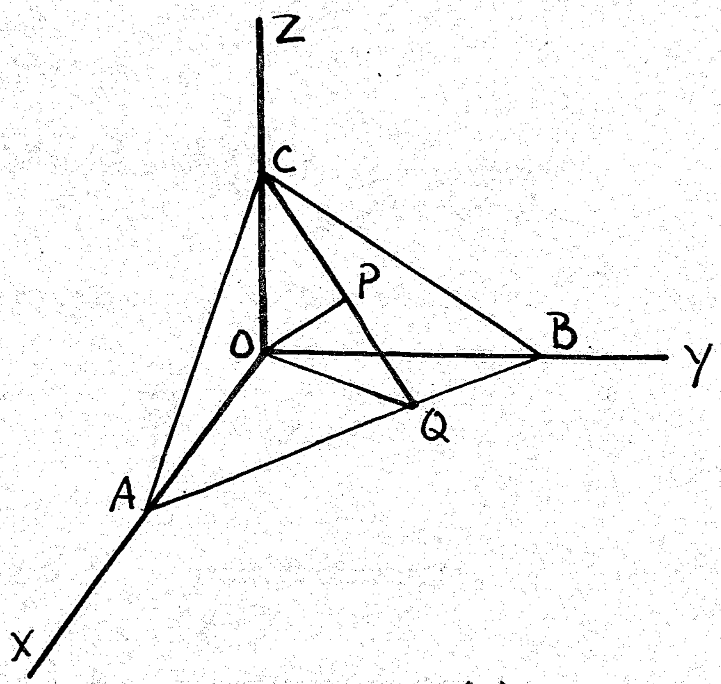

Fig. (5)

follows that

$$\frac{OQ}{OB} = \frac{OA}{AB} ;$$

so that

$$OQ = \frac{OA.OB}{AB} = \frac{\frac{1}{x}\cdot\frac{1}{y}}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}}} = \frac{1}{\sqrt{x^2 + y^2}}.$$

In the triangle QOC

$$CQ^2 = OC^2 + OQ^2 ;$$

therefore

$$CQ = \sqrt{\frac{1}{z^2} + \frac{1}{x^2 + y^2}} = \sqrt{\frac{x^2 + y^2 + z^2}{z^2(x^2 + y^2)}}.$$

Again, the triangles OPQ, COQ are similar; therefore

$$\frac{OP}{OQ} = \frac{OC}{CQ} ,$$

from which we obtain

$$\rho = OP = \frac{OC \cdot OQ}{CQ} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \text{ (1)}.$$

Since OP is perpendicular to the plane ABC

$$\cos\alpha = \frac{OP}{OA} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$

Similarly

$$\cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Therefore

$$\rho = \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

(4)

$$\cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

---

(1) The perpendicular from the origin to a plane is always considered positive.

The inverse transformations are

(5)
$$x = \frac{\cos\alpha}{\rho},$$
$$y = \frac{\cos\beta}{\rho},$$
$$z = \frac{\cos\gamma}{\rho}.$$

8. Rotation of Axes:-

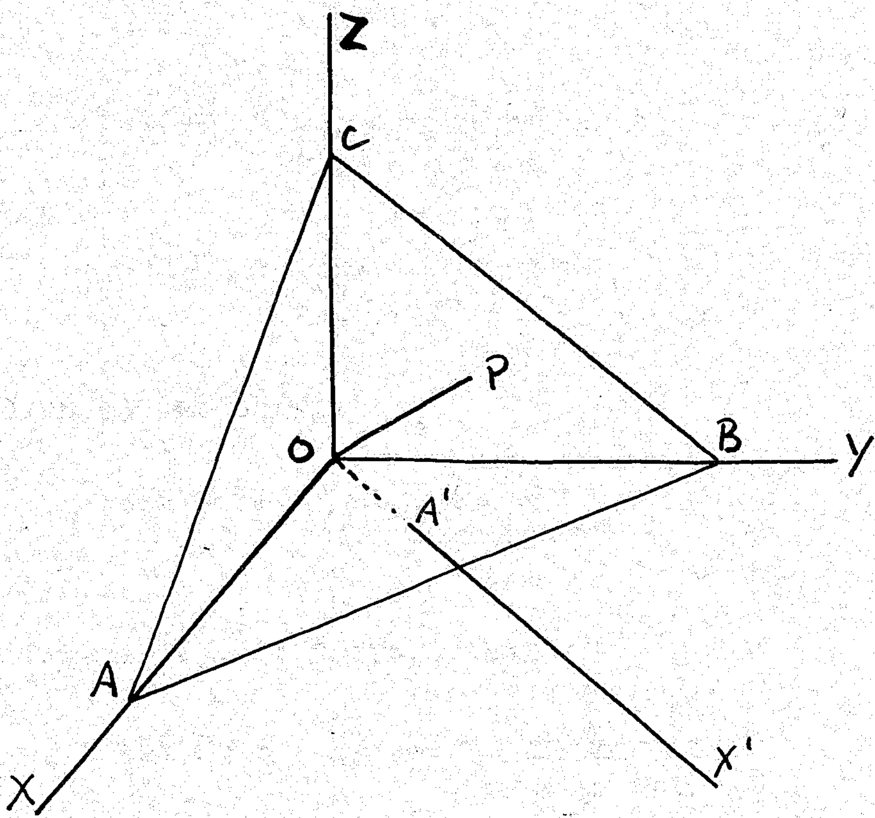

Fig. (6)

Let the original reference system be rotated about the origin to a new position so that the new X axis has direction cosines λ_1, μ_1, ν_1, the new Y axis has direction cosines λ_2, μ_2, ν_2, and the new Z axis has direction cosines λ_3, μ_3, ν_3, all with respect to the old axes. We shall denote the new axes by primed letters.

Suppose the X' axis cuts any plane (x, y, z) at A', as in figure (6). Denote the angle POA' by θ. By equations (4), the direction cosines of OP are

$$\cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

From equation (3) we obtain

$$\cos\theta = \frac{\lambda_1 x + \mu_1 y + \nu_1 z}{\sqrt{x^2 + y^2 + z^2}}.$$

But, from figure (6), it follows that

$$\cos\theta = \frac{OP}{OA'} = \frac{x'}{\sqrt{x^2 + y^2 + z^2}}.$$

By equating these two values for $\cos\theta$, we get

$$x' = \lambda_1 x + \mu_1 y + \nu_1 z.$$

similarly

(6)

$$y' = \lambda_2 x + \mu_2 y + \nu_2 z,$$

and

$$z' = \lambda_3 x + \mu_3 y + \nu_3 z.$$

The inverse transformations are

$$
(7) \qquad
\begin{aligned}
x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \\
y &= \mu_1 x' + \mu_2 y' + \mu_3 z', \\
z &= \nu_1 x' + \nu_2 y' + \nu_3 z'.
\end{aligned}
$$

We can express results (6) and (7) in tabulated form as follows:

(8)

	x'	y'	z'
x	λ_1	λ_2	λ_3
y	μ_1	μ_2	μ_3
z	ν_1	ν_2	ν_3

These relations are exactly the same as those obtained for Cartesian coördinates.

9. <u>Standard Form of the Equation of a Point</u>:-

The standard equation of a point will be that relation which involves the directed perpendicular distances from the three coördinate planes to the point. Let P be the point whose directed perpendicular distances from the YZ, ZX, and XY planes are r, s, and t respectively.

In figure (7), OR = r, RQ = s, QP = t. Rotate the axes so that the X' axis passes through P. Then $\frac{1}{OP}$ is the x' coördinate of all planes which pass through P. Therefore

$$x' = \frac{1}{\sqrt{r^2 + s^2 + t^2}}.$$

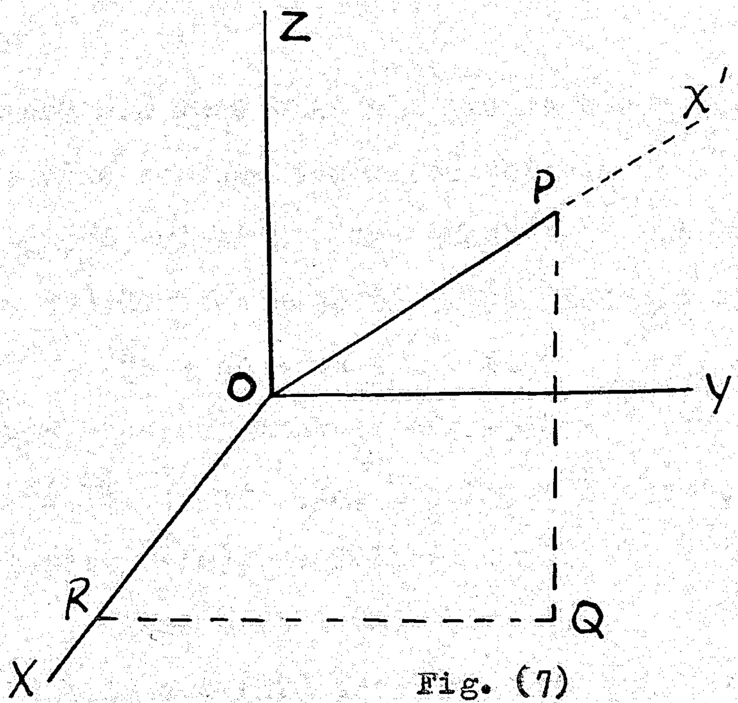

Fig. (7)

But, from (6), we have

$$x' = \lambda_1 x + \mu_1 y + \nu_1 z,$$

where

$$\lambda_1 = \frac{OR}{OP} = \frac{r}{\sqrt{r^2 + s^2 + t^2}},$$

(9) $$\mu_1 = \frac{RQ}{OP} = \frac{s}{\sqrt{r^2 + s^2 + t^2}},$$

$$\nu_1 = \frac{QP}{OP} = \frac{t}{\sqrt{r^2 + s^2 + t^2}}.$$

Therefore

$$\frac{rx + sy + tz}{\sqrt{r^2 + s^2 + t^2}} = \frac{1}{\sqrt{r^2 + s^2 + t^2}},$$

and hence

(10) $$rx + sy + tz - 1 = 0.$$

We must now show that all planes whose coördinates satisfy (10) pass through the given point.

Let (a, b, c) be a plane which does not pass through P, but whose coördinates satisfy (10). Then

(11) $$ra + sb + tc - 1 = 0.$$

From section 4, the coördinates of a plane through P and parallel to (a, b, c) are (ka, kb, kc). Since these coördinates must satisfy (10), it follows that

(12) $$k(ra + sb + tc) - 1 = 0.$$

The equations (11) and (12) are both true only if k = 1, in which case the plane (ka, kb, kc) is coincident with the plane (a, b, c). Therefore the plane (a, b, c) must pass through the point.

10. Equations of Points (Continued):-

The standard equation of a point P is given by (10). The direction cosines of OP are given in (9).

If we denote the length of OP by ρ, equation (10) may be written

(13) $$\lambda x + \mu y + \nu z - \frac{1}{\rho} = 0.$$

We shall call (13) the "directed" equation of the point.

If $(\rho, \alpha, \beta, \gamma)$ are the polar coördinates of a plane, whose intercept coördinates are (x, y, z), passing through the point

$$rx + sy + tz - 1 = 0$$

then

$$\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{rx}{\sqrt{x^2+y^2+z^2}} + \frac{sy}{\sqrt{x^2+y^2+z^2}} + \frac{tz}{\sqrt{x^2+y^2+z^2}}.$$

Therefore

(14) $$\rho = r\cos\alpha + s\cos\beta + t\cos\gamma$$

We shall call (14) the "polar" equation of the point.

The equation of the origin is

$$ox + oy + oz - 1 = 0.$$

The equation of the "point at infinity" is

$$\lambda x + \mu y + \nu z = 0.$$

The equation of a point on the X axis is

$$rx - 1 = 0,$$

and the equation of a point in the XY plane is

$$rx + sy - 1 = 0.$$

In Cartesian coördinates the plane

$$ox + oy + oz - 1 = 0$$

is known as the "plane at infinity".(1) The plane

$$\lambda x + \mu y + \nu z = 0$$

passes through the origin and λ, μ, ν are the direction cosines of the normal to the plane.

The plane

$$rx - 1 = 0$$

is parallel to the YZ plane.

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(1) Snyder and Sisam, p. 34.

11. Distance between Two Points:-

Let two points P_1 and P_2 be denoted by the equations

$$r_1 x + s_1 y + t_1 z - 1 = 0$$

and

$$r_2 x + s_2 y + t_2 z - 1 = 0.$$

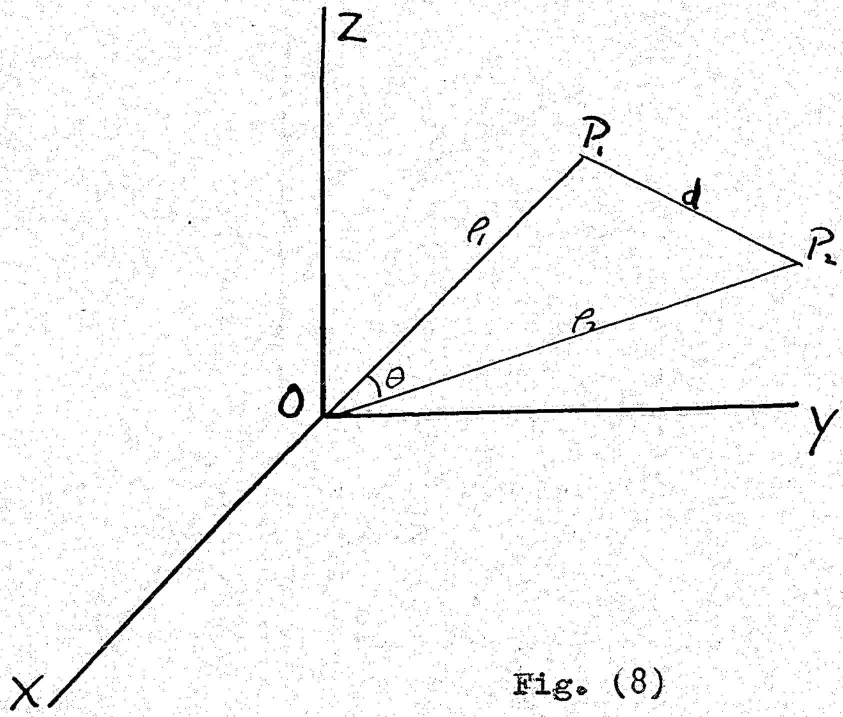

Fig. (8)

Let the lengths of P_1P_2, OP_1, OP_2 be d, ρ_1, ρ_2 respectively, and let angle P_1OP_2 be θ. We have

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\theta,$$

and hence, from (3) and (9), it follows that

$$d^2 = (r_1^2 + s_1^2 + t_1^2) + (r_2^2 + s_2^2 + t_2^2) - 2(r_1 r_2 + s_1 s_2 + t_1 t_2),$$

so that

(15) $$d = \sqrt{(r_2 - r_1)^2 + (s_2 - s_1)^2 + (t_2 - t_1)^2}.$$

12. Division of a Segment in a Given Ratio:-

Let the segment be defined by the two points given in section 11, and let the given ratio of division be h : k. Suppose that P, the division point, has the equation

$$rx + sy + tz - 1 = 0.$$

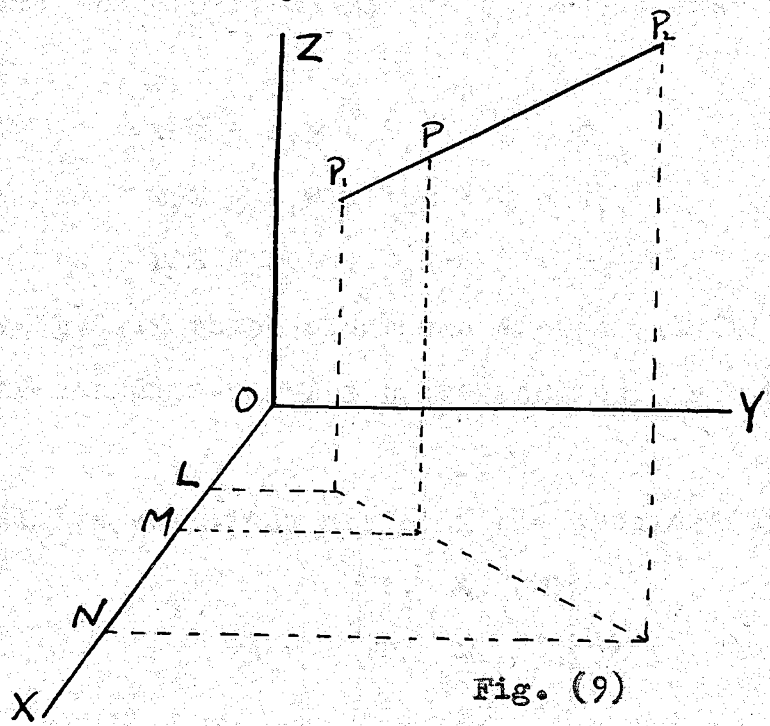

Fig. (9)

If λ, μ, ν are the direction cosines of the line P_1P_2, then

$$P_1P \cdot \lambda = LM = r - r_1,$$
$$PP_2 \cdot \lambda = MN = r_2 - r.$$

Hence

$$\frac{h}{k} = \frac{P_1P}{PP_2} = \frac{P_1P \cdot \lambda}{PP_2 \cdot \lambda} = \frac{r - r_1}{r_2 - r}.$$

On solving for r we obtain

$$r = \frac{kr_1 + hr_2}{h + k}.$$

(16) Similarly
$$s = \frac{ks_1 + hs_2}{h + k},$$
$$t = \frac{kt_1 + ht_2}{h + k}.$$

13. Plane through Three Points:-

Let the equations of the three distinct points P_1, P_2, P_3 be

$$r_1x + s_1y + t_1z - 1 = 0,$$
$$r_2x + s_2y + t_2z - 1 = 0,$$
$$r_3x + s_3y + t_3z - 1 = 0,$$

respectively. If these equations are solved for x, y, z, we obtain the coördinates of a plane passing through the three points.

Finite solutions are possible provided that

$$\Delta = \begin{vmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{vmatrix} \neq 0.$$

If $\Delta = 0$, then each element of any one row is a linear combination of the corresponding elements of the other two rows. Suppose that

$$r_3 = k_1r_1 + k_2r_2,$$
$$s_3 = k_1s_1 + k_2s_2,$$
$$t_3 = k_1t_1 + k_2t_2.$$

Let us consider the point P whose equation is

$$rx + sy + tz - 1 = 0,$$

where

$$r = \frac{k_1 r_1 + k_2 r_2}{k_1 + k_2},$$

$$s = \frac{k_1 s_1 + k_2 s_2}{k_1 + k_2},$$

$$t = \frac{k_1 t_1 + k_2 t_2}{k_1 + k_2}.$$

From (16) we see that the point P is collinear with P_1 and P_2. Therefore any plane through P_1 and P_2 must pass through P.

From (9) we see that the vectors OP and OP_3 are one and the same straight line. Therefore the origin, P, and P_3 are collinear. Hence, any plane passing through P and P_3 must pass through the origin, and one, at least, of x, y, z must be infinite.

In Cartesian coördinates three planes determine a point except when one plane is parallel to the line of intersection of the other two. The condition for this exception is $\Delta = 0$.

14. <u>The Expression</u> $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$:-

Let θ be the angle between the perpendiculars from the origin to two planes (x_1, y_1, z_1) and (x_2, y_2, z_2) and let d be the distance between the feet of these perpendiculars. Then

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\theta ,$$

where ρ_1 and ρ_2 are the lengths of the polar normals as given in (4) and $\cos\theta$ is determined by the relation

$$(17) \qquad \cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\cdot\sqrt{x_2^2 + y_2^2 + z_2^2}};$$

that is

$$d^2 = \frac{1}{x_1^2 + y_1^2 + z_1^2} + \frac{1}{x_2^2 + y_2^2 + z_2^2} - 2\frac{(x_1x_2 + y_1y_2 + z_1z_2)}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}$$

or

$$d^2 = \frac{x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2(x_1x_2 + y_1y_2 + z_1z_2)}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)},$$

which reduces to

$$(18) \qquad d = \rho_1\rho_2\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

15. Distance between Parallel Planes:-

The distance between the parallel planes (x, y, z) and (kx, ky, kz) is equal to the distance between the feet of their polar normals. From equation (18) we obtain

$$(19) \qquad d = \frac{k - 1}{k\sqrt{x^2 + y^2 + z^2}}.$$

16. Distance to a Point from a Plane:-

Let the point be defined by the equation

$$rx + sy + tz - 1 = 0$$

and the plane by the coördinates (x_1, y_1, z_1). Through the point draw a plane with coördinates, say, (kx_1, ky_1, kz_1), parallel to the given plane. Then the distance to the point from the plane is equal to the distance between these two planes. Since the new plane passes through the given point, we have

$$k(rx_1 + sy_1 + tz_1) - 1 = 0;$$

that is

$$k = \frac{1}{rx_1 + sy_1 + tz_1}.$$

On substituting this value for k in (19), we obtain

(20) $$d = -\frac{rx_1 + sy_1 + tz_1 - 1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}.$$

<u>Theorem:</u> Two points P_1, P_2 whose equations are

$$r_1x + s_1y + t_1z - 1 = 0,$$

$$r_2x + s_2y + t_2z - 1 = 0,$$

respectively, are on the same side or on opposite sides of the plane (x_1, y_1, z_1), according as its coördinates give the first members of the equations of the points like or unlike signs. For, let the point of intersection of the line P_1P_2 and the plane be P whose equation is

$$rx + sy + tz - 1 = 0,$$

where

$$r = m_1r_1 + m_2r_2,$$

$$s = m_1s_1 + m_2s_2,$$

$$t = m_1t_1 + m_2t_2,$$

and

$$m_1 + m_2 = 1 \text{ (Section 12).}$$

Therefore

$$(m_1 r_1 + m_2 r_2)x_1 + (m_1 s_1 + m_2 s_2)y_1 + (m_1 t_1 + m_2 t_2)z_1 - 1 = 0;$$

that is,

$$m_1(r_1 x_1 + s_1 y_1 + t_1 z_1 - 1) + m_2(r_2 x_1 + s_2 y_1 + t_2 z_1 - 1) = 0.$$

If $r_1 x_1 + s_1 y_1 + t_1 z_1 - 1$ and $r_2 x_1 + s_2 y_1 + t_2 z_1 - 1$ have unlike signs, then m_1 and m_2 have the same sign, and the point P lies between P_1 and P_2. If $r_1 x_1 + s_1 y_1 + t_1 z_1 - 1$ and $r_2 x_1 + s_2 y_1 + t_2 z_1 - 1$ have the same sign, then the numbers m_1, m_2 have opposite signs, hence the point P is not between P_1 and P_2.

A point whose equation is

$$rx + sy + tz - 1 = 0$$

will be considered to be on the positive or negative side of the plane (x_1, y_1, z_1) according as the expression

$$rx_1 + sy_1 + tz_1 - 1$$

is positive or negative respectively.

From (20) and the theorem just proved we can say that the distance to a point from a plane is positive or negative according as the point and the origin are on the same side or on opposite sides of the plane.

17. Angles between Line and Plane; Plane and Plane:-

The angle between a line and a plane is the complement of the angle between the line and the polar normal to the plane. If λ, μ, ν are the direction cosines of a line which makes an angle θ with the plane (x, y, z), then from

(3) and (4) we get

(21) $$\sin\theta = \frac{\lambda x + \mu y + \nu z}{\sqrt{x^2 + y^2 + z^2}}.$$

The angle between two planes is equal to the angle between their polar normals and is given by (17).

18. Two-Point Equations of a Line:-

Two distinct points will determine a straight line since the totality of planes, which pass through the two points simultaneously, define a line. Hence the simultaneous equations

(22) $$r_1 x + s_1 y + t_1 z - 1 = 0,$$
$$r_2 x + s_2 y + t_2 z - 1 = 0,$$

give the equations of the line. We shall refer to (22) as the "Two-Point" equations of a line.

19. Equations of Lines (Continued):-

The most general equations of a line are given by (22). The following is a summary of special cases:[1]

(i) A coördinate axis. The X axis has the equations

$$rx - 1 = 0,$$
$$ox + oy + oz - 1 = 0.$$

---

(1) It is understood that r, s, and t are not zero in the following work.

(ii) <u>A line parallel to (i) and passing through the Y axis</u> has the equations

$$x = 0,$$
$$sy-1 = 0.$$

(iii) <u>A line parallel to (i) and cutting the YZ plane</u> has the equations

$$x = 0,$$
$$sy+tz-1 = 0.$$

(iv) <u>A line through the origin and lying in a coördinate plane.</u> Such a line in the XY plane has the equations

$$rx+sy-1 = 0,$$
$$ox+oy+oz-1 = 0.$$

(v) <u>A line through the origin oblique to all three axes</u> has the equations

$$rx+sy+tz-1 = 0,$$
$$ox+oy+oz-1 = 0.$$

(vi) <u>A line through the X and Y axes but not through the origin</u> has the equations

$$rx-1 = 0,$$
$$sy-1 = 0.$$

(vii) <u>A line through the X axis and parallel to the YZ plane</u> has the equations

$$rx-1 = 0,$$
$$rx+sy+tz-1 = 0.$$

20. <u>Two-Plane Form of the Equations of a Line</u>:-

Let the line be defined by the planes (x_1, y_1, z_1) and (x_2, y_2, z_2). If the line passes through the origin then one or more of the coördinates of each plane will be infinite. If it does not pass through the origin, all the members of at least one set of coördinates will be finite.

Suppose the points

$$r_1 x + s_1 y + t_1 z - 1 = 0,$$
$$r_2 x + s_2 y + t_2 z - 1 = 0,$$

lie on the line. The point in which the line cuts the XY plane can be found by eliminating z from the two equations, and the point where it cuts the YZ plane can be found by eliminating x. Let these two points be denoted by the equations

$$r_3 x + s_3 y - 1 = 0,$$

(23) $$s_4 y + t_4 z - 1 = 0, \quad \text{respectively}$$

Then

$$r_3 x + s_3 y - 1 = 0,$$
$$r_3 x_1 + s_3 y_1 - 1 = 0,$$
$$r_3 x_2 + s_3 y_2 - 1 = 0.$$

If these equations in r , s are to be consistent we must have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

whence

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

In the same way, from the second of equations (23) we obtain

$$\frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

Therefore

(24) $$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

Equations (24) are called the "Two-Plane" equations of a straight line. Obviously these have no meaning if one of the denominators is zero. Suppose $x_1 - x_2$ is zero. Then x must be equal to x_1 and instead of (24) we write

$$x = x_1,$$

$$\frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

In Cartesian coördinates (24) give the "two-point" equations of a straight line.

21. Direction Cosines of a Line:-

If the line is defined by the two points whose equations are (22), the direction cosines are found to be

$$\lambda = \frac{r_2 - r_1}{\sqrt{(r_2 - r_1)^2 + (s_2 - s_1)^2 + (t_2 - t_1)^2}},$$

(25)
$$\mu = \frac{s_2 - s_1}{\sqrt{(r_2 - r_1)^2 + (s_2 - s_1)^2 + (t_2 - t_1)^2}},$$

$$\nu = \frac{t_2 - t_1}{\sqrt{(r_2 - r_1)^2 + (s_2 - s_1)^2 + (t_2 - t_1)^2}}.$$

Suppose the line is defined by (24). Equate the first two fractions. Then

$$(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2).$$

This equation is reducible to the form

(26)
$$\frac{y_2 - y_1}{x_1 y_2 - x_2 y_1} x + \frac{x_1 - x_2}{x_1 y_2 - x_2 y_1} y - 1 = 0,$$

which is the equation of a point on the line. In the same way the equations

(27)
$$\frac{z_2 - z_1}{y_1 z_2 - y_2 z_1} y + \frac{y_1 - y_2}{y_1 z_2 - y_2 z_1} z - 1 = 0,$$

and

(28)
$$\frac{z_2 - z_1}{x_1 z_2 - x_2 z_1} x \quad \frac{x_1 - x_2}{x_1 z_2 - x_2 z_1} z - 1 = 0,$$

represent points on the line. We can therefore select two of these points and find the direction cosines of the line joining them by means of (25).

If the denominator $x_1 y_2 - x_2 y_1$ has the value zero, i.e., if

$$\frac{x_1}{x_2} = \frac{y_1}{y_2},$$

from section 3 we know that $x_1 z_2 - x_2 z_1$ and $y_1 z_2 - y_2 z_1$ cannot also be zero. In this case we can use the two points whose equations are (27) and (28).

22. <u>Plane Parallel to a Line</u>:-

<u>Theorem</u>: The plane

(29) $(k_1 x_1 + k_2 x_2,\ k_1 y_1 + k_2 y_2,\ k_1 z_1 + k_2 z_2)$

is parallel to the line determined by the planes (x_1, y_1, z_1) and (x_2, y_2, z_2).

If θ is the angle between the line and plane, from equation (21) we obtain

$$\sin\theta = \frac{\lambda x + \mu y + \nu z}{\sqrt{x^2 + y^2 + z^2}}.$$

Let (27) and (28) be the equations of the line. Then, from (25) we have

$$\lambda = \frac{\dfrac{z_2 - z_1}{x_1 z_2 - x_2 z_1}}{\sqrt{\left(\dfrac{z_2 - z_1}{x_1 z_2 - x_2 z_1}\right)^2 + \left(\dfrac{z_1 - z_2}{y_1 z_2 - y_2 z_1}\right)^2 + \left(\dfrac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \dfrac{y_1 - y_2}{y_1 z_2 - y_2 z_1}\right)^2}},$$

(30)
$$\mu = \frac{\dfrac{z_1 - z_2}{y_1 z_2 - y_2 z_1}}{\sqrt{\left(\dfrac{z_2 - z_1}{x_1 z_2 - x_2 z_1}\right)^2 + \left(\dfrac{z_1 - z_2}{y_1 z_2 - y_2 z_1}\right)^2 + \left(\dfrac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \dfrac{y_1 - y_2}{y_1 z_2 - y_2 z_1}\right)^2}},$$

$$\nu = \frac{\dfrac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \dfrac{y_1 - y_2}{y_1 z_2 - y_2 z_1}}{\sqrt{\left(\dfrac{z_2 - z_1}{x_1 z_2 - x_2 z_1}\right)^2 + \left(\dfrac{z_1 - z_2}{y_1 z_2 - y_2 z_1}\right)^2 + \left(\dfrac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \dfrac{y_1 - y_2}{y_1 z_2 - y_2 z_1}\right)^2}} .$$

The substitution of (30) in the expression for $\sin\theta$ gives us

$$\sin\theta = \frac{P + Q + R}{S.T},$$

where

$$P = \frac{z_2 - z_1}{x_1 z_2 - x_2 z_1} \left(k_1 x_1 + k_2 x_2\right),$$

$$Q = \frac{z_1 - z_2}{y_1 z_2 - y_2 z_1} \left(k_1 y_1 + k_2 y_2\right.,$$

$$R = \left(\frac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \frac{y_1 - y_2}{y_1 z_2 - y_2 z_1}\right)(k_1 z_1 + k_2 z_2),$$

$$S = \sqrt{(k_1 x_1 + k_2 x_2)^2 + (k_1 y_1 + k_2 y_2)^2 + (k_1 z_1 + k_2 z_2)^2},$$

$$T = \sqrt{\left(\frac{z_2 - z_1}{x_1 z_2 - x_2 z_1}\right)^2 + \left(\frac{z_1 - z_2}{y_1 z_2 - y_2 z_1}\right)^2 + \left(\frac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \frac{y_1 - y_2}{y_1 z_2 - y_2 z_1}\right)^2}.$$

The numerator reduces to zero and hence

$$\sin\theta = 0,$$

and the plane is parallel to the line.

Conversely, if the plane (x_3, y_3, z_3) is parallel to the line of intersection of (x_1, y_1, z_1) and (x_2, y_2, z_2), its coördinates must be of the form (29). We have

$$\sin\theta = \frac{\lambda x + \mu y + \nu z}{\sqrt{x^2 + y^2 + z^2}} = 0,$$

and therefore

$$\frac{z_2 - z_1}{x_1 z_2 - x_2 z_1} x_3 + \frac{z_1 - z_2}{y_1 z_2 - y_2 z_1} y_3$$

$$+ \left(\frac{x_1 - x_2}{x_1 z_2 - x_2 z_1} - \frac{y_1 - y_2}{y_1 z_2 - y_2 z_1} \right) z_3 = 0.$$

This equation reduces to

(31) $$(z_2 - z_1) \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

$(z_2 - z_1)$ cannot always be zero, since we do not have to restrict the line in this manner. Therefore we must have the relation

(32) $$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

satisfied under all conditions. If (32) holds, then x_3, y_3, z_3 must be a linear combination of the corresponding elements of the other two rows, and hence must be of the form (29).

In Cartesian coördinates a point (29) is co-planar with the points (x_1, y_1, z_1), (x_2, y_2, z_2) and the origin.

23. <u>Pencil of Planes</u>:-

Suppose the plane (29) passes through the line of intersection of the planes (x_1, y_1, z_1) and (x_2, y_2, z_2). Then it passes through all points on the line and its coördinates must satisfy the equation of any point on the line. Let a point on the line be defined by the equation

$$rx + sy + tz - 1 = 0.$$

We must have

$$rx_1 + sy_1 + tz_1 - 1 = 0,$$

(33)
$$rx_2 + sy_2 + tz_2 - 1 = 0,$$

$$r(k_1 x_1 + k_2 x_2) + s(k_1 y_1 + k_2 y_2) + t(k_1 z_1 + k_2 z_2) - 1 = 0,$$

that is

(34) $$k_1(rx_1 + sy_1 + tz_1) + k_2(rx_2 + sy_2 + tz_2) - 1 = 0.$$

Equations (33) and (34) hold simultaneously only if

(35) $$k_1 + k_2 = 1.$$

This relation is the necessary and sufficient condition that a plane, whose coördinates are given by (29), will pass through the line of intersection of the planes (x_1, y_1, z_1) and (x_2, y_2, z_2).

In (29), if we let

$$k_1 = \frac{k}{h + k},$$

$$k_2 = \frac{h}{h + k},$$

we have the system of planes whose coördinates are givem by

$$x = \frac{kx_1 + hx_2}{h + k},$$

(36)
$$y = \frac{ky_1 + hy_2}{h + k},$$

$$z = \frac{kz_1 + hz_2}{h + k},$$

which is a pencil of planes, since relation (35) still holds.

In Cartesian coördinates all points (36) are collinear, and divide the segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $h : k$.

24. <u>Three-Plane Equation of a Point</u>:-

Let (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) be the coördinates of three planes such that no plane is parallel to the line of intersection of the other two.

The conditions that these three planes pass through the point,

$$rx + sy + tz - 1 = 0,$$

are

$$rx_1 + sy_1 + tz_1 - 1 = 0,$$
$$rx_2 + sy_2 + tz_2 - 1 = 0,$$
$$rx_3 + sy_3 + tz_3 - 1 = 0.$$

The condition that r, s, t exist so as to satisfy these four simultaneous equations is that

(37)
$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

This is the required equation, since it is of the first degree in x, y, z, and is obviously satisfied by the coördinates of the three planes.

If

$$\omega = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0,$$

the point is finite. If $\omega = 0$, (37) gives an equation of the form

$$rx + sy + tz = 0,$$

which has already been defined as a point at infinity. If $\omega = 0$, the elements of any one row of ω must be a linear combination of the corresponding elements of the other two rows, and hence the plane must be parallel to the line of intersection of the other two.

25. <u>Translation of Axes</u>:-

Suppose the origin is translated to the point

$$rx + sy + tz - 1 = 0,$$

without any rotation of axes. Let any plane be represented by the polar coördinates $(\rho, \alpha, \beta, \gamma)$ and $(\rho', \alpha', \beta', \gamma')$ with respect to the original and new systems, respectively. Then

$$\alpha' = \alpha,$$
$$\beta' = \beta,$$
$$\gamma' = \gamma.$$

From (20) we have

$$\rho' = \frac{-(rx + sy + tz - 1)}{x + y + z}.$$

Therefore

(38)
$$\rho' = -(rx + sy + tz - 1)\rho,$$
$$\alpha' = \alpha,$$
$$\beta' = \beta,$$
$$\gamma' = \gamma.$$

and hence

(39)
$$x' = \frac{-x}{rx + sy + tz - 1},$$
$$y' = \frac{-y}{rx + sy + tz - 1},$$
$$z' = \frac{-z}{rx + sy + tz - 1}.$$

The inverse transformations are

(40)
$$x = \frac{x'}{rx' + sy' + tz' + 1},$$
$$y = \frac{y'}{rx' + sy' + tz' + 1},$$
$$z = \frac{z'}{rx' + sy' + tz' + 1}.$$

26. <u>The Degree of an Equation is Unchanged by Transformations</u>:- (1)

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(1) Tanner and Allen "Analytic Geometry", p. 127.
Wentworth "Analytic Geometry", p. 109.

Let the degree of the equation be n. A general term would be

(41) $$Ax^p y^q z^m,$$

where p, q, m are not negative and

$$p + q + m \leq n.$$

If we rotate axes by equations (7), in place of (41) we obtain

$$A(\lambda_1 x' + \lambda_2 y' + \lambda_3 z')^p(\mu_1 x' + \mu_2 y' + \mu_3 z')^q(\nu_1 x' + \nu_2 y' + \nu_3 z')^m$$

Since each term in each bracket is of the first degree, we cannot obtain terms of degree higher than n.

If we translate axes according to equations (40), (41) becomes

$$\frac{A\, x'^p\, y'^q\, z'^m}{(rx' + sy' + tz')^{p + q + m}}$$

If every term in the new equation be multiplied by

$$(rx' + sy' + tz' + 1)^n,$$

the term (41) finally becomes

(42) $$A\, x'^p\, y'^q\, z'^m(rx' + sy' + tz' + 1)^{n - (p + q + m)}$$

Any term in (42) cannot be of degree higher than n. Hence the degree of an equation is not raised by translation or rotation of axes.

Suppose the degree were lowered by a transformation of coördinates. Then, by applying the inverse transformation, we should be raising the degree of the equation. This has been proved impossible. Therefore the degree is unchanged by rotation and translation.

CHAPTER II

The General Second Degree Equation

The most general second degree equation in x, y, z is

$$(1) \qquad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxz + 2ux + 2vy + 2wz + d = 0,$$

where at least one of a, b, c, f, g, h is different from zero. We shall show that (1) always represents a conicoid in the planar system of coördinates.

1. Equation of the Tangent Point:-

The line of intersection of the planes (x_1, y_1, z_1) and (x_2, y_2, z_2) is given (Section 20, Chap. I) by the equations

$$(2) \qquad \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} = p.$$

The coördinates of any plane through (2) are

$$(3) \qquad \begin{aligned} x &= x_1 + p(x_1 - x_2), \\ y &= y_1 + p(y_1 - y_2), \\ z &= z_1 + p(z_1 - z_2). \end{aligned}$$

If a plane (3) touches the surface (1), its coördinates must satisfy equation (1). Substituting (3) in (1) we obtain a quadratic equation in p, which shows that, in general, through any line two planes can be drawn to touch the surface (1).

Suppose that one of these is the plane (x_1, y_1, z_1). It follows that one root of the quadratic in p must be zero, and hence the constant term must be zero. We therefore have

$$(4) \quad ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0.$$

Suppose (3) determines one plane only. In this case the plane is the tangent plane (x_1, y_1, z_1), and both roots of the quadratic are zero. Both the constant term and the coefficient of p must be zero, so that

$$(5) \quad \begin{aligned} &ax_1(x_1 - x_2) + by_1(y_1 - y_2) + cz_1(z_1 - z_2) + f\{y_1(z_1 - z_2) \\ &+ z_1(y_1 - y_2)\} + g\{z_1(x_1 - x_2) + x_1(z_1 - z_2)\} \\ &+ h\{x_1(y_1 - y_2) + y_1(x_1 - x_2)\} + u(x_1 - x_2) \\ &+ v(y_1 - y_2) + w(z_1 - z_2) = 0. \end{aligned}$$

It follows from (2) that

$$(x_1 - x_2) : (y_1 - y_2) : (z_1 - z_2) = (x - x_1) : (y - y_1) : (z - z_1),$$

and from (5) we get

$$(6) \quad \begin{aligned} &axx_1 + byy_1 + czz_1 + f(y_1z + z_1y) + g(z_1x + x_1z) \\ &+ h(x_1y + y_1x) + ux + vy + wz = ax_1^2 + by_1^2 + cz_1^2 \\ &+ 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + ux_1 + vy_1 + wz_1. \end{aligned}$$

As a consequence of (4) the right number of (6) is equal to

$$-(ux_1 + vy_1 + wz_1 + d).$$

Therefore (6) reduces to

(7) $axx_1 + byy_1 + czz_1 + f(y_1z + z_1y) + g(z_1x + x_1z)$

$+ h(x_1y + y_1x) + u(x + x_1) + v(y + y_1) + w(z + z_1)$

$+ d = 0.$

Formula (7) is the equation of the point of tangency of the plane (x_1, y_1, z_1) to the surface (1).

2. <u>Condition that a Point Lies on the Surface</u>:-[1]

Let the equation of the point on the surface be

(8) $$rx + sy + tz - 1 = 0.$$

Comparing equations (7) and (8) we have

$$\frac{ax_1 + hy_1 + gz_1 + u}{r} = \frac{hx_1 + by_1 + fz_1 + v}{s}$$

$$= \frac{gx_1 + fy_1 + cz_1 + w}{t} = \frac{-(ux_1 + vy_1 + wz_1 + d)}{1}.$$

Put each fraction equal to $-\lambda$. Then

$$ax_1 + hy_1 + gz_1 + u + \lambda r = 0,$$
$$hx_1 + by_1 + fz_1 + v + \lambda s = 0,$$
$$gx_1 + fy_1 + cz_1 + w + \lambda t = 0,$$
$$ux_1 + vy_1 + wz_1 + d - \lambda = 0.$$

We also have

$$rx_1 + sy_1 + tz_1 - 1 = 0.$$

Eliminating x_1, y_1, z_1, λ from the above equations, we obtain the required condition, namely

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(1) C. Smith "Solid Geometry", p. 41.

$$\begin{vmatrix} a & h & g & u & r \\ h & b & f & v & s \\ g & f & c & w & t \\ u & v & w & d & -1 \\ r & s & t & -1 & 0 \end{vmatrix} = 0,$$

which is the same as

(9) $Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs + 2Wt + D=0,$

where A, B, C, etc., are the co-factors of a, b, c, etc., respectively, in the determinant

$$\delta \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}.$$

The relation (9) is a condition that the point (8) lies on the surface (1).[1]

Incidentally, (9) represents a conicoid in the Cartesian system. Hence, for a point to lie on the surface (1), it must lie on a conicoid; that is, (1) represents a conicoid in the planar system of coördinates.

A proof that (1) represents a conicoid will be given in section 3, where no reference is made, as above, to Cartesian coördinates.

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(1) For a similar discussion see Snyder and Sisam, pp. 130, 131.

3. Locus of Middle Points of a System of Parallel Chords:-

Let the equation of the surface be (1), and let (8) be the equation of any point on this conicoid; r, s, t must satisfy (9). Let

(10) $$\ell x + my + nz - 1 = 0$$

be the equation of a point on a line whose direction cosines are λ, μ, ν. The point (8) will lie on this line and be distant p from (10) if

$$r - \ell = p\lambda,$$
$$s - m = p\mu,$$
$$t - n = p\nu,$$

that is, if

(11) $$r = \ell + p\lambda,$$
$$s = m + p\mu,$$
$$t = m + p\nu.$$

If we substitute (11) in (9) we obtain a quadratic equation in p, which shows that any given line cuts the surface in two points. It follows that all straight lines in a plane cut the surface in two points, and therefore all plane sections of the surface are conic sections. This is the definition of a conicoid.

We have

(12) $$p^2(A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu) + 2p(A\ell\lambda + Bm\mu + Cn\nu + Fn\mu + Fm\nu + G\ell\nu + Gn\lambda + H\ell\mu + Hm\lambda + U\lambda + V\mu + W\nu) + (A\ell^2 + Bm^2 + Cn^2 + 2Fmn + 2Gn\ell + 2H\ell m + 2U\ell + 2Vm + 2Wn + D) = 0,$$

where A, B, C, ... , have the same values as in section 2. If (10) is the equation of the middle point of the line, the values of p obtained from (12) must be equal numerically but opposite in sign. The condition for this is that the coefficient of p equals zero. Hence

$$(13)\quad l(A\lambda + H\mu + G\nu) + m(H\lambda + B\mu + F\nu) + n(G\lambda + F\mu + C\nu) + U\lambda + V\mu + W\nu = 0.$$

Therefore the plane whose polar coördinates are given by

$$(14)\quad \begin{aligned} \rho &= \frac{P}{\sqrt{Q^2 + R^2 + S^2}}, \\ \cos\alpha &= \frac{Q}{\sqrt{Q^2 + R^2 + S^2}}, \\ \cos\beta &= \frac{R}{\sqrt{Q^2 + R^2 + S^2}}, \\ \cos\gamma &= \frac{S}{\sqrt{Q^2 + R^2 + S^2}}, \end{aligned}$$

where

$$\begin{aligned} P &= U\lambda + V\mu + W\nu, \\ Q &= A\lambda + H\mu + G\nu, \\ R &= H\lambda + B\mu + F\nu, \\ S &= G\lambda + F\mu + C\nu, \end{aligned}$$

passes through the point (10).[1] But (14) represents a

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(1) C.f. equation (14), Chapter I.

fixed plane when λ, μ, ν are fixed. Therefore the mid-points of all parallel chords whose direction cosines are λ, μ, ν lie in the plane (14).

A plane which passes through the mid-points of a system of parallel chords of a conicoid is known as a diametral plane. If a diametral plane is perpendicular to the chords it bisects, it is called a principal plane.

4. <u>The Principal Plane</u>:-

If the plane (14) is perpendicular to the chords whose direction cosines are λ, μ, ν, the direction cosines of its polar normal must be λ, μ, ν. Therefore

$$\frac{A\lambda + H\mu + G\nu}{\lambda} = \frac{H\lambda + B\mu + F\nu}{\mu} = \frac{G\lambda + F\mu + C\nu}{\nu}$$

Put ξ for the common value of each of these fractions; then

$$\begin{aligned} (A-\xi)\lambda + H\mu + G\nu &= 0, \\ H\lambda + (B-\xi)\mu + F\nu &= 0, \qquad (16) \\ G\lambda + F\mu + (C-\xi)\nu &= 0. \end{aligned}$$

Eliminating λ, μ, ν we get

$$\begin{vmatrix} A-\xi & H & G \\ H & B-\xi & F \\ G & F & C-\xi \end{vmatrix} = 0,$$

which, when expanded, becomes the cubic

$$\xi^3 - \mathcal{I}\xi^2 + \mathcal{J}\xi - \mathcal{D} = 0, \qquad (17)$$

where

$$\mathcal{I} = A + B + C, \quad \mathcal{J} = AB + BC + CA - F^2 - G^2 - H^2,$$

and $$\mathcal{D} = ABC + 2\,FGH - AF^2 - BG^2 - CH^2.$$

When ξ is determined, any two of the three relations (16) will give the corresponding values of λ, μ, ν. Since one root of a cubic equation is always real, it follows that there is always at least one principal plane.

5. The Roots of (17):-[1]

Let ξ_1, be any root of (17) and let λ_0, μ_0, ν_0 (not all zero) be values of λ, μ, ν that satisfy (16) when $\xi = \xi_1$. If ξ_1 is a complex number, λ_0, μ_0, ν_0 may be complex. Let

$$\lambda_0 = \lambda_1 + i\lambda_2,$$
$$\mu_0 = \mu_1 + i\mu_2,$$
$$\nu_0 = \nu_1 + i\nu_2,$$

where $i = \sqrt{-1}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ are real.

Substitute ξ_1 and these values of λ_0, μ_0, ν_0 for ξ, λ, μ, ν in (16), multiply the resulting equations by $\lambda_1 - i\lambda_2, \mu_1 - i\mu_2, \nu_1 - i\nu_2$, respectively, and add. The result is

$$(\lambda_1^2 + \lambda_2^2 + \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2)\xi_1 = (\lambda_1^2 + \lambda_2^2)A$$
$$+ (\mu_1^2 + \mu_2^2)B + (\nu_1^2 + \nu_2^2)C + 2(\mu_1\nu_1 + \mu_2\nu_2)F$$
$$+ 2(\nu_1\lambda_1 + \nu_2\lambda_2)G + 2(\lambda_1\mu_1 + \lambda_2\mu_2)H.$$

The coefficient of ξ_1 is real and different from zero, and

(1) Snyder and Sisam, p. 79.

the right member of the equation is also real. Hence ξ_1 is real. Since ξ_1 is any root of (17), all the roots of (17) are real.

The conditions that all the roots of (17) are zero are

$$
(18)\qquad
\begin{aligned}
ABC + 2FGH - AF^2 - BG^2 - CH^2 &= 0,\\
AB + BC + CA - F^2 - G^2 - H^2 &= 0,\\
A + B + C &= 0.
\end{aligned}
$$

Square (18, 3), i.e. the third equation of (18), and subtract twice (18, 2) from it. The result is

$$A^2 + B^2 + C^2 + 2F^2 + 2G^2 + 2H^2 = 0.$$

Since A, B, C, etc., are assumed to be real, it follows that

$$(19)\qquad A = B = C = F = G = H = 0.$$

If (19) is true, (9) reduces to

$$2Ur + 2Vs + 2Wt + D = 0.$$

But this is the condition that the plane whose polar coördinates are

$$
(20)\qquad
\begin{aligned}
\rho &= \frac{D}{2\sqrt{U^2 + V^2 + W^2}},\\
\cos\alpha &= \frac{-U}{\sqrt{U^2 + V^2 + W^2}},\\
\cos\beta &= \frac{-V}{\sqrt{U^2 + V^2 + W^2}},\\
\cos\gamma &= \frac{-W}{\sqrt{U^2 + V^2 + W^2}},
\end{aligned}
$$

passes through the point whose equation is

$$rx + sy + tz - 1 = 0.$$

Therefore the fixed plane (20) will pass through all the points on the conicoid, and hence the conicoid reduces to a plane. This degenerate case is obtained by letting all the roots of the cubic be zero. Henceforth we shall assume that at least one root of the cubic is different from zero.

6. Elimination of the yz, zx, z terms:-

Since at least one of the principal planes is not at infinity, we can translate and rotate the system of reference so that the new XY plane is a principal plane of the surface.

Let the equation of the conicoid referred to the new axes be (1). Since the surface is symmetrical with respect to the XY plane, the two parts into which the XY plane divides the surface must be exactly alike. If there is a tangent plane (x_1, y_1, z_1) at a point on one side of the XY plane, there must be a corresponding tangent plane $(x_1, y_1, -z_1)$ at a point on the other side. Substituting each set of coördinates in (1), we obtain

$$ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

and

$$ax_1^2 + by_1^2 + cz_1^2 - 2fy_1z_1 - 2gz_1x_1 + 2hx_1y_1 + 2ux_1 + 2vy_1 - 2wz_1 + d = 0.$$

Since these relations are true for all tangent planes, it follows that

$$f = g = w = 0.$$

These results may be derived in a second way as follows. Consider the three points

$$\text{(21)} \qquad \begin{aligned} r_1 x + s_1 y + t_1 z - 1 &= 0, \\ r_2 x + s_2 y + t_2 z - 1 &= 0, \\ r_3 x + s_3 y + t_3 z - 1 &= 0, \end{aligned}$$

on the surface and on one side of the XY plane. Let these points be considered as distinct. Later we shall require that they approach coincidence. On the other side of the XY plane we must have the corresponding points

$$\text{(22)} \qquad \begin{aligned} r_1 x + s_1 y - t_1 z - 1 &= 0, \\ r_2 x + s_2 y - t_2 z - 1 &= 0, \\ r_3 x + s_3 y - t_3 z - 1 &= 0. \end{aligned}$$

The coördinates (x_1, y_1, z_1) of the plane through the three points (21) are given (Section 13, Chap. I) by

$$x_1 = \frac{\begin{vmatrix} 1 & s_1 & t_1 \\ 1 & s_2 & t_2 \\ 1 & s_3 & t_3 \end{vmatrix}}{\begin{vmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{vmatrix}} = \frac{\delta_1}{\Delta_1},$$

$$y_1 = \frac{\begin{vmatrix} r_1 & 1 & t_1 \\ r_2 & 1 & t_2 \\ r_3 & 1 & t_3 \end{vmatrix}}{\Delta_1} = \frac{\delta_2}{\Delta_1},$$

$$z_1 = \frac{\begin{vmatrix} r_1 & s_1 & 1 \\ r_2 & s_2 & 1 \\ r_3 & s_3 & 1 \end{vmatrix}}{\Delta_1} = \frac{\delta_3}{\Delta_1}.$$

and the coördinates of the plane (x_2, y_2, z_2) through the three points (22) are given by

$$x_2 = \frac{\begin{vmatrix} 1 & s_1 & -t_1 \\ 1 & s_2 & -t_2 \\ 1 & s_3 & -t_3 \end{vmatrix}}{\begin{vmatrix} r_1 & s_1 & -t_1 \\ r_2 & s_2 & -t_2 \\ r_3 & s_3 & -t_3 \end{vmatrix}} = \frac{-\delta_1}{-\Delta_1} = \frac{\delta_1}{\Delta_1},$$

$$y_2 = \frac{\begin{vmatrix} r_1 & 1 & -t_1 \\ r_2 & 1 & -t_2 \\ r_3 & 1 & -t_3 \end{vmatrix}}{-\Delta_1} = \frac{-\delta_2}{-\Delta_1} = \frac{\delta_2}{\Delta_1},$$

$$z_2 = \frac{\begin{vmatrix} r_1 & s_1 & 1 \\ r_2 & s_2 & 1 \\ r_3 & s_3 & 1 \end{vmatrix}}{-\Delta_1} = \frac{\delta_3}{-\Delta_1} = -\frac{\delta_3}{\Delta_1}.$$

Therefore

$$x_2 = x_1,$$

(23) $$y_2 = y_1,$$

$$z_2 = -z_1.$$

In the case where $\Delta_1 = 0$, results similar to (23) can be obtained by using polar coördinates.

Let the points (21) approach coincidence; then the points (22) will do likewise. At all steps in this process relation (23) holds for the coördinates of the planes through the respective sets of points. In the limit, i.e. where tangency occurs, the relation must still be true. Therefore, for every tangent plane (x_1, y_1, z_1) at a point on one side of the XY plane there must be a corresponding tangent plane $(x_1, y_1, -z_1)$ at a point on the other side.

If $f = g = w = 0$, equation (1) becomes

(24) $ax^2 + by^2 + cz^2 + 2hxy + 2ux + 2vy + d = 0.$

7. Reduction when $d \neq 0$:-

If we translate the origin to the point whose equation is

$$-\frac{u}{d}x - \frac{v}{d}y - 1 = 0,$$

(24) becomes

$$\left(a - \frac{u^2}{d}\right)x^2 + \left(b - \frac{v^2}{d}\right)y^2 + cz^2 + 2\left(h - \frac{uv}{d}\right)xy + d = 0.$$

The term in xy can be eliminated by rotating the X, Y axes through an angle θ determined by

$$\tan 2\theta = \frac{2\left(h - \frac{uv}{d}\right)}{\left(a - \frac{u}{d}\right) - \left(b - \frac{v}{d}\right)},$$

according to the rotation formulae

$$x = x'\cos\theta - y'\sin\theta,$$

$$y = x'\sin\theta + y'\cos\theta,$$

$$z = z'.$$

Dropping primes, we get an equation of the form

$$a_1 x^2 + b_1 y^2 + c_1 z^2 + d = 0.$$

Since $d \neq 0$, we can divide by $-d$ and the resulting equation has the form

(25) $$a_0 x^2 + b_0 y^2 + c_0 z^2 = 1.$$

Hence for $d \neq 0$, under all conditions we can reduce equation (1) to the form (25).

8. Reduction when $d = 0$:-

The equation to be considered is

$$ax^2 + by^2 + cz^2 + 2hxy + 2ux + 2vy = 0.$$

(i) If $u = v = 0$, by rotating the X, Y axes through an angle θ given by

$$\tan 2\theta = \frac{2h}{a - b},$$

we eliminate the xy term. The resulting equation has the form

(26) $$a_0 x^2 + b_0 y^2 + c_0 z^2 = 0.$$

(ii) If v is not zero we eliminate the y term by rotating the X, Y axes according to the transformations

$$x = \frac{ux' - vy'}{\sqrt{u^2 + v^2}},$$

$$y = \frac{vx' + uy'}{\sqrt{u^2 + v^2}}$$

$$z = z',$$

and we obtain an equation of the form

$$a_1 x^2 + b_1 y^2 + c_1 z^2 + 2h_1 xy + 2u_1 x = 0.$$

If $u_1 = 0$ we have case (i). If $u_1 \neq 0$, by translating the origin to the point whose equation is

$$-\frac{a_1}{2u_1}x - \frac{h_1}{u_1}y - 1 = 0,$$

we obtain

(27) $$b_1 y^2 + c_1 z^2 + 2u_1 x = 0.$$

Therefore equation (1) can be reduced to one of the forms (25), (26), or (27).

9. <u>Center of Conicoid</u>:-

Consider equation (25), namely

$$ax^2 + by^2 + cz^2 = 1.$$

The center lies on the plane midway between parallel tangents to the surface. If (x, y, z) is tangent to the surface, (-x, -y, -z) is also tangent. Therefore the origin is the center of this type of conicoid.

Consider equation (26), namely

$$ax^2 + by^2 + cz^2 = 0.$$

As before, the origin is the center.

Suppose the conicoid reduces to

$$by^2 + cz^2 + 2ux = 0.$$

Let the parallel planes (x_1, y_1, z_1) and (kx_1, ky_1, kz_1) touch this surface; that is

$$by_1^2 + cz_1^2 + 2ux_1 = 0,$$

$$bk^2y_1^2 + ck^2z_1^2 + 2ukx_1 = 0.$$

If $u = 0$ this conicoid is a degenerate of (26). If $u \neq 0$ then $k = 1$, or else the parallel tangent planes are all at infinity. Therefore the surface has no finite center.

10. Polar Plane:-

We shall show that the points of contact of all tangent planes through a given point to a conicoid lie on a plane. This plane is called the polar plane of the point with respect to the conicoid. Conversely, the point is called the polar point of the plane with respect to the conicoid.

(i) Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 = 1.$$

The equation of the tangent point of the plane (x_1, y_1, z_1) is given (Section 1, Chap. 2) by

(28) $$axx_1 + byy_1 + czz_1 - 1 = 0.$$

Suppose the plane (x_1, y_1, z_1) passes through the point

(29) $$rx + sy + tz - 1 = 0;$$

then

$$rx_1 + sy_1 + tz_1 - 1 = 0.$$

The point (28) lies on the plane $\left(\frac{r}{a}, \frac{s}{b}, \frac{t}{c}\right)$, since its coördinates satisfy the equation. Hence the points of tangency all lie on the plane

(30) $$\left(\frac{r}{a}, \frac{s}{b}, \frac{t}{c}\right),$$

which must therefore be the polar plane of the point (29) with respect to the conicoid.

(ii) Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 = 0.$$

The equation of the tangent point is

(31) $$axx_1 + byy_1 + czz_1 = 0.$$

The point (31) lies on the plane (o, o, o), and therefore all tangent points are at infinity. This type of conicoid will be discussed later.

(iii) Let the equation of the conicoid be

$$by^2 + cz^2 + 2ux = 0.$$

The equation of the tangent point is

(32) $$byy_1 + czz_1 + u(x + x_1) = 0.$$

If the plane (x_1, y_1, z_1) also passes through the point whose equation is (29), the point (32) lies on the plane

(33) $$\left(\frac{-1}{r}, \frac{us}{br}, \frac{ut}{cr}\right),$$

since its coördinates satisfy (32). Therefore (33) is the polar plane of the point (29) with respect to this conicoid.

11. Rectilinear Generators:-

Let the equation of the surface be

(34) $$a^2x^2 + b^2y^2 - c^2z^2 = 1,$$

which may be written in the form

$$(ax + cz)(ax - cz) = (1 + by)(1 - by),$$

or

(35) $$\frac{ax + cz}{1 + by} = \frac{1 - by}{ax - cz} = \eta, \text{ say.}$$

Then

$$ax + cz = \eta(1 + by),$$

(36) $$(ax - cz)\eta = 1 - by .$$

For every value of η, these equations define a line. Every point lying on the surface (34) must satisfy the relation (Section 2, Chap. 2).

(37) $$\frac{r^2}{a^2} + \frac{s^2}{b^2} - \frac{t^2}{c^2} = 1.$$

If the point whose equation is

$$r_1 x + s_1 y + t_1 z - 1 = 0,$$

lies on the line (36), it follows (Section 12, Chap. 1) that

$$r_1 = m_1 \frac{a}{\eta} + m_2 a \eta,$$

(38) $$s_1 = -m_1 b + m_1 b,$$

$$t_1 = m_1 \frac{c}{\eta} - m c \eta,$$

$$m_1 + m_2 = 0.$$

Relation (37) holds when we replace r, s, t by r_1, s_1, t_1 respectively. Therefore any point on the line (36) also lies on the surface (34), and (36) is a system of rectilinear generators of (34).

Equation (34) may be written

(39) $$\frac{ax + cz}{1 - by} = \frac{1 + by}{ax - cz} = \rho\text{, say.}$$

Then

$$ax + cz = \rho(1 - by),$$

(40) $$(ax - cz)\rho = 1 + by,$$

which is a second system of rectilinear generators of (34).

We can find in a similar manner the equations of the generating lines of the surface

(41) $$b^2y^2 - c^2z^2 = 2ux.$$

The equations of the generators of one system are

$$by - cz = 2\sigma x,$$

$$by + cz = \frac{u}{\sigma};$$

and of the other system

$$by + cz = 2\tau x,$$

$$by - cz = \frac{u}{\tau}.$$

12. Invariants:-

Let the equation of the surface be

$$ax^2 + by^2 + cz^2 = 1.$$

If the axes are rotated to new positions according to equations (8) of Chapter I, the resulting equation is of the form

$$a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy = 1,$$

where

$$a_1 = a\lambda_1^2 + b\mu_1^2 + c\nu_1^2,$$

$$b_1 = a\lambda_2^2 + b\mu_2^2 + c\nu_2^2,$$

$$c_1 = a\lambda_3^2 + b\mu_3^2 + c\nu_3^2,$$
$$f_1 = a\lambda_2\lambda_3 + b\mu_2\mu_3 + c\nu_2\nu_3,$$
$$g_1 = a\lambda_3\lambda_1 + b\mu_3\mu_1 + c\nu_3\nu_1,$$
$$h_1 = a\lambda_1\lambda_2 + b\mu_1\mu_2 + c\nu_1\nu_2,$$

Making use of the relation

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}^2 = 1,$$

we obtain

$$D \equiv \begin{vmatrix} a_1 & h_1 & g_1 \\ h_1 & b_1 & f_1 \\ g_1 & f_1 & c_1 \end{vmatrix} = abc = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}.$$

Therefore D is unchanged by rotation.

In the same way it can be shown that

$$I \equiv a + b + c,$$
$$J \equiv bc + ca + ab - f^2 - g^2 - h^2$$

are unchanged by rotation.

It can readily be shown that these expressions are not invariant under translation.

The condition that a point

$$rx + sy + tz - 1 = 0$$

lies on the general conicoid is (Section 1, Chap. 2)

(42) $\quad Ar^2 + Bs^2 + Ct^2 + \ldots\ldots\ldots\ldots = 0,$

where A, B, C, ... are the co-factors of a, b, c, ... in the determinant

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

Let the axes be rotated to new positions according to the formulae (8) of Chapter I, namely

$$x = \lambda_1 x' + \lambda_2 y' + \lambda_3 z',$$
$$y = \mu_1 x' + \mu_2 y' + \mu_3 z',$$
$$z = \nu_1 x' + \nu_2 y' + \nu_3 z'.$$

The point whose equation referred to the old system is

$$rx + sy + tz - 1 = 0,$$

becomes

$$(r\lambda_1 + s\mu_1 + t\nu_1)x' + (r\lambda_2 + s\mu_2 + t\nu_2)y' + (r\lambda_3 + s\mu_3 + t\nu_3)z' - 1 = 0,$$

in the new system; that is

$$r' = r\lambda_1 + s\mu_1 + t\nu_1,$$
$$s' = r\lambda_2 + s\mu_2 + t\nu_2,$$
$$t' = r\lambda_3 + s\mu_3 + t\nu_3.$$

The inverse relations are readily found to be

$$r = r'\lambda_1 + s'\lambda_2 + t'\lambda_3,$$
(43) $$s = r'\mu_1 + s'\mu_2 + t'\mu_3,$$
$$t = r'\nu_1 + s'\nu_2 + t'\nu_3.$$

The degree of equation (42) will be unaltered, as proved in Chapter I, Section 26, by the substitutions (43). If, by a change of rectangular axes through the same origin,

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs$$

becomes changed into

$$A'r^2 + B's^2 + C't^2 + 2F'st + 2G'tr + 2H'rs;$$

then, since $r^2 + s^2 + t^2$ is unaltered by this change of axes,

(44)
$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs - \xi(r^2 + s^2 + t^2)$$

will be transformed into

(45)
$$A'r^2 + B's^2 + C't^2 + 2F'st + 2G'tr + 2H'rs - \xi(r^2 + s^2 + t^2).$$

The expressions (44) and (45) will therefore be the product of linear factors for the same values of ξ.

The condition that (44) is the product of linear factors is

$$\begin{vmatrix} A-\xi & H & G \\ H & B-\xi & F \\ G & F & C-\xi \end{vmatrix} = 0,$$

that is

$$\xi^3 - \xi^2(A + B + C) + \xi(BC + CA + AB - F^2 - G^2 - H^2)$$
$$- (ABC + 2FGH - AF^2 - BG^2 - CH^2) = 0.$$

The condition that (45) is the product of linear factors is similarly

$$\xi^3 - \xi^2(A' + B' + C') + \xi(B'C' + C'A' + A'B' - F'^2 - G'^2 - H'^2)$$
$$-(A'B'C' + 2F'G'H' - A'F'^2 - B'G'^2 - C'H'^2) = 0.$$

Since the roots of the above cubic equations in ξ are the same, the coefficients must be equal.

Hence

$$\mathcal{I} = A + B + C,$$

$$\mathcal{J} = BC + CA + AB - F^2 - G^2 - H^2,$$

$$\mathcal{D} = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix},$$

are unaltered by rotation.

Translation of axes to the point whose equation is

$$\alpha x + \beta y + \gamma z - 1 = 0$$

can be accomplished (Section 25, Chap. I) by means of the formulae

$$x = \frac{x'}{\alpha x' + \beta y' + \gamma z' + 1},$$

$$y = \frac{y'}{\alpha x' + \beta y' + \gamma z' + 1},$$

$$z = \frac{z'}{\alpha x' + \beta y' + \gamma z' + 1}.$$

The point, whose equation referred to the old axes is

$$rx + sy + tz - 1 = 0,$$

has the equation

$$rx' + sy' + tz' - (\alpha x' + \beta y' + \gamma z' + 1) = 0$$

referred to the new axes; that is

$$r' = r - \alpha,$$

$$s' = s - \beta,$$

$$t' = t - \gamma;$$

therefore

$$r = r' + \alpha \,,$$
(46) $$s = s' + \beta \,,$$
$$t = t' + \gamma \,.$$

The substitution of (46) in (42) does not change any of the coefficients of the second degree terms. Therefore $\mathcal{I}$, $\mathcal{J}$, $\mathcal{D}$ are unaltered by translation of axes. Thus $\mathcal{I}$, $\mathcal{J}$, $\mathcal{D}$ are unaltered by translation or rotation, and are therefore invariants.

The proof that Δ is invariant is similar to that given for $\mathcal{D}$. The condition that a point lies on a conicoid is

(47) $$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs + 2Wt + D = 0.$$

Let this equation be transformed by a rotation into

$$A'r^2 + B's^2 + C't^2 + 2F'st + 2G'tr + 2H'rs + 2U'r + 2V's + 2W't + D' = 0.$$

This rotation transforms the expression

(48) $$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs + 2Wt + D - k(r^2 + s^2 + t^2 + 1)$$

into

(49) $$A'r^2 + B's^2 + C't^2 + 2F'st + 2G'tr + 2H'rs + 2U'r + 2V's + 2W't + D' - k(r^2 + s^2 + t^2 + 1).$$

The discriminants of (48) and (49) are, respectively

$$\begin{vmatrix} A-k & H & G & U \\ H & B-k & F & V \\ G & F & C-k & W \\ U & V & W & D-k \end{vmatrix} \quad \text{and}$$

$$\begin{vmatrix} A'-k & H' & G' & U' \\ H' & B'-k & F' & V' \\ G' & F' & C'-k & W' \\ U' & V' & W' & D'-k \end{vmatrix}$$

The expressions (48) and (49) are factorable into linear expressions for the same values of k. The condition that each is factorable is that its discriminant equals zero. Hence, since the coefficient of k^4 in each case is unity, the constant terms of these discriminants must be equal; that is $\Delta = \Delta'$. Hence, Δ is invariant under rotation.

In order to prove that Δ is invariant under translation, let the axes be translated to the point whose equation is

$$\alpha x + \beta y + \gamma z - 1 = 0.$$

The condition that the point lies on the conicoid becomes

$$\begin{aligned} &Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2(A\alpha + H\beta + G\gamma + U)r \\ (50) \quad &+ 2(H\alpha + B\beta + F\gamma + V)s + 2(G\alpha + F\beta + C\gamma + W)t \\ &+ D' = 0, \end{aligned}$$

where D' is the left member of (47) when r, s, t are replaced by α, β, γ. The discriminant of (50) is

$$\begin{vmatrix} A & H & G & A\alpha+H\beta+G\gamma+U \\ H & B & F & H\alpha+B\beta+F\gamma+U \\ G & F & C & G\alpha+F\beta+C\gamma+W \\ A\alpha+H\beta+G\gamma+U, & H\alpha+B\beta+F\gamma+V, & G\alpha+F\beta+C\gamma+W, & D' \end{vmatrix}$$

Multiply the first column by α, the second by β, the third by γ, and subtract their sum from the last column. In the resulting determinant, multiply the first row by α, the second by β, the third by γ, and subtract their sum from the last row. The resulting determinant is Δ. Hence $\Delta' = \Delta$, so that Δ is invariant under both translation and rotation.

CHAPTER III

Classification of Surfaces

1. Review of Previous Work:-

In Chapter II we have seen that the condition that a point whose equation is

$$(1) \qquad rx + sy + tz - 1 = 0$$

lies on the surface whose equation is given by

$$(2) \qquad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

is

$$(3) \qquad Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs + 2Wt + D = 0,$$

where A, B, C, ... , are the co-factors of a, b, c, ... , in the determinant

$$\delta \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

For brevity we shall refer to (3) as the "point-condition" equation. We have also seen that $\mathcal{I} \equiv (A + B + C)$,

$\mathcal{J} \equiv (AB + BC + CA - F^2 - G^2 - H^2)$,

$$\mathcal{D} \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}, \quad \text{and} \quad \Delta \equiv \begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{vmatrix},$$

are invariant under translation and rotation.

2. The Sphere:-

The sphere is defined to be the locus of a point which moves so as to remain at a constant distance from a fixed point. This distance is known as the radius and the fixed point is the center of the sphere. Let the equation of the center be

$$\alpha x + \beta y + \gamma z - 1 = 0,$$

and let the radius be R; then we have

$$\sqrt{(r-\alpha)^2 + (s-\beta)^2 + (t-\gamma)^2} = R,$$

or

$$(r-\alpha)^2 + (s-\beta)^2 + (t-\gamma)^2 = R^2.$$

Therefore the general point-condition equation of a sphere is

$$Ar^2 + As^2 + At^2 + 2Ur + 2Vs + 2Wt + D = 0,$$

where A is different from zero. Conversely, any point $rx + sy + tz = 1$, where r, s, t satisfy the condition equation, lies on a sphere.

The point-condition equation of a sphere whose center is the origin, is seen to be

$$r^2 + s^2 + t^2 = R^2.$$

The sphere may also be defined as the envelope of planes which move so as always to remain at a constant distance from a fixed point. Thus

$$\frac{\sqrt{x^2 + y^2 + z^2}}{\alpha x + \beta y + \gamma z - 1} = \frac{1}{R};$$

that is

(4) $$R^2(x^2 + y^2 + z^2) = (\alpha x + \beta y + \gamma z - 1)^2.$$

The equation of a sphere, center at the origin, is seen to be

$$R^2(x^2 + y^2 + z^2) = 1,$$

or

(5) $$ax^2 + ay^2 + az^2 + d = 0.$$

(If a and d have the same sign the sphere is imaginary.)

The point-condition equation of the sphere (5) is

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs$$
$$+ 2Wt + D = 0,$$

where $A = a^2d$, $B = a^2d$, $C = a^2d$, $D = a^3$, and

$$F = G = H = U = V = W = 0.$$

Therefore

$$\mathcal{I} = 3a^2d,$$
$$\mathcal{J} = 3a^4d^2,$$
$$\mathcal{D} = a^6d^3,$$
$$\Delta = a^9d^3.$$

3. <u>The Ellipsoid</u>:-

Consider the surface whose equation is

(6) $$a^2x^2 + b^2y^2 + c^2z^2 = 1.$$

The point-condition equation of this surface is found to be

(7) $$b^2c^2r^2 + c^2a^2s^2 + a^2b^2t^2 = a^2b^2c^2.$$

For a, b, c are all different from zero, and a, b, c in descending order of magnitude, we have

$$\frac{r^2}{a^2} + \frac{s^2}{a^2} + \frac{t^2}{a^2} \not> 1,$$

and

$$\frac{r^2}{c^2} + \frac{s^2}{c^2} + \frac{t^2}{c^2} \not< 1.$$

Hence a point on the surface can not be at a distance from the origin greater than a nor less than c. The surface is therefore limited in every direction; and, since all plane sections of a conicoid are conics, it follows that all plane sections of (6) are ellipses. This is the usual definition of an ellipsoid.

The surface is clearly symmetrical with respect to the three coördinate planes, the three coördinate axes, and the origin. The points in which it cuts the axes are found by letting $s = t = 0$, $t = r = 0$, $r = s = 0$, respectively, in equation (7). These points are determined by the relations

$$r = \pm a, \quad \pm ax - 1 = 0,$$
$$s = \pm b, \quad \pm by - 1 = 0,$$
$$t = \pm c, \quad \pm cz - 1 = 0,$$

respectively.

Consider the system of tangent planes through the point

(8) $$mz - 1 = 0,$$

on the Z axis. The coördinates of all planes through this point and touching the surface are $(x, y, \frac{1}{m})$ where

$$a^2x^2 + b^2y^2 + \frac{c^2}{m^2} = 1.$$

The polar plane of the point (8) is $(0, 0, \frac{m}{c^2})$. Translate the origin to the point

$$\frac{c^2}{m}z - 1 = 0;$$

the new XY plane will be the polar of the point (8). The equation of (8) becomes

$$\frac{m^2 - c^2}{m}z - 1 = 0;$$

that is, the coördinates of all planes through (8) will be $(x, y, \frac{m}{m^2 - c^2})$. Let these planes touch the surface whose new equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = \left(\frac{c^2}{m}z + 1\right)^2;$$

so that

(9) $$a^2x^2 + b^2y^2 = \frac{m^2}{m^2 - c^2} = \frac{1}{1 - \frac{c^2}{m^2}}$$

Therefore we have an ellipse.[1] For $m > c$ the ellipse is real, and for $m < c$ it is imaginary. The ratio of the semi-axes remains constant, namely $a : b$. The major semi-axis is equal to $a\sqrt{1 - \frac{c^2}{m^2}}$, which is seen to be zero for $m = c$

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(1) Valgardsson "Line Coördinates".

and equal to a for m infinitely large. As m becomes indefinitely large the polar plane $(0, 0, \frac{m}{c})$ approaches coincidence with the XY plane.

In the same way we could show that the section of the surface made by the YZ plane is an ellipse of semi axes b and c and that the section made by the ZX plane is am ellipse of semi-axes c, a. We call a, b, c the "semi-axes" of the ellipsoid. If a = b, the sections parallel to the XY plane are circles and the surface is a surface of revolution. If a = b = c we have a sphere.

For the ellipsoid

$$\mathcal{I} = -(a^2b^2 + b^2c^2 + c^2a^2)$$
$$\mathcal{J} = (a^2b^2c^2)(a^2 + b^2 + c^2)$$
$$\mathcal{D} = -a^4b^4c^4$$
$$\Delta = -a^6b^6c^6.$$

If c = 0, (6) becomes

$$a^2x^2 + b^2y^2 = 1,$$

and the point-condition equation (7) becomes

$$a^2b^2t^2 = 0.$$

If a, b are different from zero, then t = 0. Hence for c = 0, the surface must lie wholly in the XY plane. In this case

$$\mathcal{I} = -a^2b^2,$$
$$\mathcal{J} = 0,$$
$$\mathcal{D} = 0,$$
$$\Delta = 0.$$

If $b = c = 0$, (6) becomes

$$a^2x^2 = 1.$$

Hence the surface has degenerated into the two points

$$ax \pm 1 = 0.$$

Let $a = a_1\lambda$, $b = b_1\lambda$, $c = c_1\lambda$. Equation (6) then becomes

$$a_1^2x^2 + b_1^2y^2 + c_1^2z^2 = \frac{1}{\lambda^2} \tag{10}$$

Let λ increase indefinitely but let a_1, b_1, c_1 remain fixed. In the limit we have

$$a_1^2x^2 + b_1^2y^2 + c_1^2z^2 = 0.$$

Hence this equation is the limiting case of an ellipsoid as the semi-axes a, b, c become infinitely large. It is to be noticed that translation does not affect the latter equation. The only plane which is tangent to the surface is the plane (0, 0, 0).

The point-condition equation of (10) is

$$\frac{b_1^2c_1^2}{\lambda^2}r^2 + \frac{c_1^2a_1^2}{\lambda^2}s^2 + \frac{a_1^2b_1^2}{\lambda^2}t^2 = a_1^2b_1^2c_1^2 .$$

In the limit, when λ becomes infinitely large, this equation becomes

$$0r^2 + 0s^2 + 0t^2 = a_1^2 b_1^2 c_1^2 ,$$

which can be satisfied only by points at infinity.

In this case $\mathcal{I} = \mathcal{J} = \mathcal{D} = \Delta = 0$.

4. The Hyperboloid of One Sheet:-

Consider the surface whose equation is

(11) $$a^2x^2 + b^2y^2 - c^2z^2 = 1.$$

The point-condition equation of this surface is found to be

(12) $$b^2c^2r^2 + c^2a^2s^2 - a^2b^2t^2 = a^2b^2c^2.$$

Let a, b, c be all different from zero. The surface is clearly symmetrical with respect to the coördinate planes, coördinate axes, and the origin. By the same method as employed in Section 3, we can show that the plane sections of the surface parallel to the XY plane are ellipses whose axes have minimum values in the XY plane section, and increase indefinitely as the section is moved further away from the XY plane. Thus

$$a^2x^2 + b^2y^2 = \frac{m^2}{m^2 + c^2}$$

is the equation of the ellipse when the plane passes through the point

$$-\frac{c}{m}z - 1 = 0.$$

The semi-axes are in the ratio a : b and the semi-major axis has the value $a\sqrt{\dfrac{m^2 + c^2}{m^2}}$, which becomes infinitely large as m approaches zero.

In the same way we find that sections parallel to the YZ plane are hyperbolas. In particular, if we consider the section made by the plane $\left(\dfrac{m}{a^2}, 0, 0\right)$, we obtain the equation

$$b^2y^2 - c^2z^2 = \frac{m^2}{m^2 - a^2}$$

This curve is well-defined except for m = a, and this is seen to be the case where the plane is at a distance from the YZ plane equal to the semi-axis a of the ellipse which is formed by the intersection of the surface by the XY plane. We can discuss this case easier with reference to the point-condition equation which is

$$c^2a^2s^2 - a^2b^2t^2 = a^2b^2c^2 - b^2c^2r^2.$$

When r = a we have

$$\frac{s^2}{b^2} = \frac{t^2}{c^2};$$

that is

$$\frac{s}{t} = \pm\frac{b}{c}.$$

The system of points whose equations are

(13) $$\frac{b}{c}\,ty + tz - 1 = 0$$

and

(14) $$-\frac{b}{c}\,ty + tz - 1 = 0$$

can be shown to define two lines. For the direction cosines of the line joining (13) to the origin (Section 21, Chap. I) are

$$\cos\alpha = 0,$$

$$\cos\beta = \frac{b}{\sqrt{b^2 + c^2}},$$

$$\cos\gamma = \frac{c}{\sqrt{b^2 + c^2}},$$

which are constant. In the same way we can show that (14) defines a line. Therefore when m = a, we have a pair of straight lines through the origin.

For this surface

$$\mathcal{J} = b^2c^2 + c^2a^2 - a^2b^2,$$

$$\mathcal{J} = a^2b^2c^2(c^2 - a^2 - b^2),$$

$$\mathcal{D} = -a^4b^4c^4,$$

$$\Delta = a^6b^6c^6.$$

Suppose c = 0. This case has already been discussed under the ellipsoid.

If b = 0 and a and c are different from zero, the equation becomes

(14) $$a^2x^2 - c^2z^2 = 1$$

(which is Valgardsson's hyperbola in line coördinates). When b = 0

$$\mathcal{J} = a^2c^2,$$

$$\mathcal{J} = \mathcal{D} = \Delta = 0.$$

If a = b = 0, the surface is imaginary. If b = c = 0 we have the case

$$a^2x^2 = 1,$$

which represents a pair of points, as we have already seen.

If we let $a = a_1\lambda$, $b = b_1\lambda$, $c = c_1\lambda$, then it follows that

$$a_1^2x^2 + b_1^2y^2 - c_1^2z^2 = \frac{1}{\lambda^2}$$

The section of this surface made by a plane parallel to the XY plane has the equation

$$a_1^2x^2 + b_1^2y^2 = \frac{k^2}{\lambda^2},$$

where k depends only on the position of the cutting plane. This is an ellipse whose semi-axes are $\frac{\lambda a_1}{k}$ and $\frac{\lambda b_1}{k}$, both of which become infinite as λ becomes infinite.

In the same way we can show that the major axes of the hyperbolic sections parallel to the other coördinate planes become infinite as λ becomes infinite.

In the limit we have

$$a_1^2x^2 + b_1^2y^2 - c_1^2z^2 = 0.$$

For this last equation

$$\mathcal{I} = \mathcal{J} = \mathcal{D} = \Delta = 0.$$

5. The Hyperboloid of Two Sheets:-

Consider the equation

(15) $$a^2x^2 - b^2y^2 - c^2z^2 = 1.$$

The point-condition equation for (15) is

(16) $$b^2c^2r^2 - c^2a^2s^2 - a^2b^2t^2 = a^2b^2c^2.$$

This surface is symmetrical with respect to the coördinate planes, coördinate axes, and the origin. As before we can find the sections made by planes parallel to the coördinate planes. The sections parallel to the XY and ZX planes are found to be hyperbolas, and the sections by planes parallel to the YZ plane are ellipses. Suppose the plane parallel to the YZ plane passes through the point

$$rx - 1 = 0.$$

It is readily seen that the ellipses are imaginary unless

$$r^2 \geqq a^2.$$

If $r = a$, the ellipses degenerate into points on the X axis.

For this surface

$$\mathcal{I} = (c^2a^2 + a^2b^2 - b^2c^2)$$

$$\mathcal{J} = a^2b^2c^2(a^2 - b^2 - c^2)$$

$$\mathcal{D} = - a^4b^4c^4$$

$$\triangle = - a^6b^6c^6.$$

When b or c is zero, cases are obtained which have been discussed already. Let us consider the case when the semi-axes become infinite; suppose the equation is

$$a^2x^2 - b^2y^2 - c^2z^2 = \frac{1}{\lambda^2}$$

Then there is no part of the surface between the planes parallel to the YZ plane and passing through the points

$$\pm a\lambda x - 1 = 0.$$

If λ approaches infinity the distance between these points becomes infinite. In the limit we have the hyperboloid of two sheets at infinity. We have $\mathcal{J} = \mathcal{j} = \mathcal{D} = \Delta = 0.$

6. <u>The Paraboloid</u>:-

Consider the surface defined by the equation

(17) $$b^2y^2 + c^2z^2 + 2ux = 0.$$

The point-condition equation of (17) is

(18) $$c^2u^2s^2 + b^2u^2t^2 + 2b^2c^2ur = 0.$$

If b, c, u are all different from zero, we may write, instead of (18),

$$\frac{s^2}{b^2} + \frac{t^2}{c^2} + \frac{2r}{u} = 0.$$

The surface (17) is symmetrical with respect to the XY and ZX planes and the X axis. The polar of the point

(19) $$mx - 1 = 0$$

is (Section 10, Chap. II) the plane $\left(-\frac{1}{m}, 0, 0\right)$. Translate the origin to the point

$$-mx - 1 = 0.$$

Then the polar plane will be the new XY plane. Equations (19) and (17), referred to the new axes, are respectively

$$2mx - 1 = 0,$$

$$b^2y^2 + c^2z^2 - 2umx^2 + 2ux = 0.$$

Let all the tangent planes pass through the point (19); that is $x = \frac{1}{2m}$. Therefore we have

$$b^2y^2 + c^2z^2 = -\frac{u}{2m}.$$

Hence plane sections parallel to the YZ plane are ellipses of semi-axes $b\sqrt{-\frac{2m}{u}}$ and $c\sqrt{-\frac{2m}{u}}$. This ellipse degenerates to a point when $m = 0$; that is, the YZ plane touches the surface at the origin. The ellipse increases in size as the cutting plane is moved further from the origin. It is to be noted that m and u must be opposite in sign for real ellipses. If u is positive the surface lies wholly on the positive side of the YZ plane.

Consider any plane parallel to the XZ plane, (0, m, 0), say. Translate the origin to the point

$$\frac{1}{m}y - 1 = 0;$$

that is the new XZ plane is this plane. Equation (18) becomes (Chapter II)

$$\frac{\left(S + \frac{1}{m}\right)^2}{b^2} + \frac{t^2}{c^2} + \frac{2r}{u} = 0.$$

For any point in the new XZ plane $S = 0$. Therefore the point-condition equation of the plane section by the new XZ plane becomes

(20) $$\frac{t^2}{c^2} + \frac{2r}{u} + \frac{1}{m^2b^2} = 0.$$

It can easily be shown the line-condition equation for a parabola has the same form as (20).[1] Therefore the

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(1) This can be done by a method similar to that employed in Chapter II, Section 2. See Snyder and Sisam, p. 91.

section by this plane is a parabola. In the same way we can show that sections parallel to the XY plane yield parabolas.

We call the surface whose equation is (17) an elliptic paraboloid, because the sections parallel to one coördinate plane are ellipses and the sections parallel to the other two coördinate planes are parabolas.

In the same way we can investigate the surface whose equation is

(21) $$b^2y^2 - c^2z^2 + 2ux = 0.$$

Sections parallel to the YZ plane yield hyperbolas and sections parallel to the other two coördinate planes yield parabolas. Therefore (21) represents an hyperbolic paraboloid.

For (17)
$$\mathcal{I} = -u^2(b^2 + c^2),$$
$$\mathcal{J} = b^2c^2u^4,$$
$$\mathcal{D} = 0,$$
$$\Delta = b^6c^6u^6.$$

For (21)
$$\mathcal{I} = -u^2(b^2 - c^2),$$
$$\mathcal{J} = -b^2c^2u^4,$$
$$\mathcal{D} = 0,$$
$$\Delta = -b^6c^6u^6.$$

If $u = 0$, we have

$$b^2y^2 + c^2z^2 = 0$$

or

$$b^2y^2 - c^2z^2 = 0.$$

The first is a special case of the infinite ellipsoid and the second represents a pair of infinitely distant points.[1]

For these two cases $\mathcal{I} = \mathcal{J} = \mathcal{D} = \Delta = 0$.

When $c = 0$, we have

$$b^2y^2 + 2ux = 0.$$

This is a parabola in the XY plane.[2] In this case $\mathcal{I} = -b^2u^2$, $\mathcal{J} = \mathcal{D} = \Delta = 0$. The point-condition equation reduces to

$$b^2u^2t^2 = 0;$$

that is, $t = 0$, and the points all lie in the XY plane.

7. <u>Invariants for the Various Equations</u>:-[3]

Equation	Δ	$\mathcal{D}$	$\mathcal{J}$	$\mathcal{I}$
$a^2x^2 + b^2y^2 + c^2z^2 = 1$	−	−	+	−
$a^2x^2 + b^2y^2 - c^2z^2 = 1$	+	−	?	?
$a^2x^2 - b^2y^2 - c^2z^2 = 1$	−	−	?	?
$b^2y^2 + c^2z^2 + 2ux = 0$	−	0	+	−
$b^2y^2 - c^2z^2 + 2ux = 0$	+	0	−	?

(1) Valgardsson "Line Coördinates", Ch. III.

(2) Valgardsson, Ch. II, Sect. 4.

(3) It is understood that all coefficients appearing in the following table are different from zero.

Equation	Δ	$\mathcal{D}$	$\mathcal{J}$	$\mathcal{I}$
$ax^2 + by^2 + cz^2 = 0$	0	0	0	0
$ax^2 + by^2 = 0$	0	0	0	0
$a^2x^2 + b^2y^2 = 1$	0	0	0	$-$
$a^2x^2 - b^2y^2 = 1$	0	0	0	$+$
$b^2y^2 + 2ux = 0$	0	0	0	$\pm$
$a^2x^2 = 1$	0	0	0	0

Since these are all the possible equations, we can say that when $\Delta \neq 0$, $\mathcal{D} \neq 0$ we have an ellipsoid or an hyperboloid of one or two sheets. If $\Delta \neq 0$, $\mathcal{D} = 0$ we must have either an elliptic or hyperbolic paraboloid. If $\Delta = \mathcal{D} = \mathcal{J} = 0$, and $\mathcal{I} \neq 0$ we have a plane curve, which can be an ellipse, parabola, or hyperbola. If $\Delta = \mathcal{D} = \mathcal{J} = \mathcal{I} = 0$, the equation represents two points, or else may be satisfied only by points at infinity.

The original equation represents two points when it has two linear factors in x, y, z, for which a necessary condition is that the discriminant δ vanish.

CHAPTER IV

Reduction of the General Equation

1. General Statement:-

In this chapter we shall consider the reduction of the general equation when $\Delta \neq 0$, that is, when the equation represents an ellipsoid, hyperboloid, or paraboloid.

2. Reduction of the Point-Condition Equation:-

Let the equation

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + 2Ur + 2Vs$$

(1) $$+ 2Wt + D = 0$$

be the point-condition equation of the surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy$$

(2) $$+ 2wz + d = 0.$$

We have seen (Section 4, Chap. II) that there is at least one principal plane. Take this plane for the XY plane in a new system of coördinates. The degree of (1) will be unaltered by the transformation.

By supposition the XY plane bisects all chords parallel to the Z axis; therefore if

$$r_1 x + s_1 y + t_1 z - 1 = 0$$

be any point on the surface, the point

$$r_1 x + s_1 y - t_1 z - 1 = 0$$

will also be on the surface. From this we see that in the transformed equation

$$F = G = W = 0.$$

The reduced equation therefore is

$$Ar^2 + Bs^2 + Ct^2 + 2Hrs + 2Ur + 2Vs + D = 0.$$

Now rotate the X, Y axes through an angle θ given by the relation

$$\tan 2\theta = \frac{2H}{A-B},$$

according to the transformations (43) of Chapter II, namely

$$r = r'\cos\theta + s'\sin\theta,$$

$$s = -r'\sin\theta + s'\cos\theta,$$

$$t = t'.$$

Dropping primes, we get an equation of the form

(3) $$Ar^2 + Bs^2 + Ct^2 + 2Ur + 2Vs + D = 0.$$

(i) Let A, B, C be all finite and different from zero. We can then write equation (3) in the form

$$A\left(r + \frac{U}{A}\right)^2 + B\left(s + \frac{V}{B}\right)^2 + Ct^2 = \frac{U^2}{A} + \frac{V^2}{B} - D \equiv D'.$$

Hence, by changing the origin to the point

$$\frac{U}{A}x + \frac{V}{B}y - 1 = 0$$

by means of formulae (46) of Chapter II, we obtain

$$Ar^2 + Bs^2 + Ct^2 = D'.$$

If D' be not zero we have

$$\frac{r^2}{\frac{D'}{A}} + \frac{s^2}{\frac{D'}{B}} + \frac{t^2}{\frac{D'}{C}} = 1,$$

which we can write in the form

(4) $$\frac{r^2}{a^2} + \frac{s^2}{b^2} + \frac{t^2}{c^2} = 1,$$

or

(5) $$\frac{r^2}{a^2} + \frac{s^2}{b^2} - \frac{t^2}{c^2} = 1,$$

or

(6) $$\frac{r^2}{a^2} - \frac{s^2}{b^2} - \frac{t^2}{c^2} = 1,$$

according as $\frac{D'}{A}$, $\frac{D'}{B}$, $\frac{D'}{C}$ are all positive, two positive and one negative, or one positive and two negative, respectively. (If all three are negative the surface is clearly imaginary.)

If D' be zero, we have

(7) $$Ar^2 + Bs^2 + Ct^2 = 0.$$

(ii) Let A, any one of the coefficients, be zero. Write the equation in the form

$$2Ur + B\left(s + \frac{V}{B}\right) + Ct^2 + D - \frac{V^2}{B} = 0.$$

If U be not zero, by changing the origin to the point

$$Qx + \frac{U}{B}y - 1 = 0,$$

where

$$Q = \frac{1}{2U}\left(D - \frac{V^2}{B}\right),$$

we can reduce the equation to

(8) $$Bs^2 + Ct^2 + 2Ur = 0.$$

If $U = 0$, we have the form

(9) $$Bs^2 + Ct^2 + D' = 0,$$

or, if $D' = 0$, the form

(10) $$Bs^2 + Ct^2 = 0.$$

(iii) Let B, C, two of the three coefficients, be zero. We then have

$$A\left(r + \frac{U}{A}\right)^2 + 2Vs + D' - \frac{U^2}{A} = 0.$$

If we translate the origin to the point

$$\frac{U}{A}x + \frac{1}{2V}\left(D' - \frac{U^2}{A}\right)y - 1 = 0,$$

the equation reduces to the form

(11) $$r^2 = 2ks.$$

If, however, $V = 0$, the equation is equivalent to

(12) $$r^2 = k'.$$

3. <u>To Find the Equations of the Center of a Conicoid</u>:-

If the origin is the center of the surface, it is the middle point of all chords passing through it; if

$$r_1x + s_1y + t_1z - 1 = 0$$

be any point on the surface, the point

$$-r_1x - s_1y - t_1z - 1 = 0$$

will also be on the surface.

Hence we have

$$Ar_1^2 + Bs_1^2 + Ct_1^2 + 2Fs_1t_1 + 2Gt_1r_1 + 2Hr_1s_1 + 2Ur_1 + 2Vs_1 + 2Wt_1 + D = 0,$$

and

$$Ar_1^2 + Bs_1^2 + Ct_1^2 + 2Fs_1t_1 + 2Gt_1r_1 + 2Hr_1s_1 - 2Ur_1 - 2Vs_1 - 2Wt_1 + D = 0;$$

therefore

$$Ur_1 + Vs_1 + Wt_1 = 0.$$

Since this relation holds for all points on the surface, we must have U, V, W all zero. Hence, when the origin is the center of a conicoid, the coefficients of r, s, t are all zero.

Let

$$\alpha x + \beta y + \gamma z - 1 = 0$$

be the equation of the center of the surface; then if we take the center for origin, the coefficients of r, s, t in the transformed equation will all be zero. The transformed equation will be (Section 46, Chap. II)

$$A(r + \alpha)^2 + B(s + \beta)^2 + C(t + \gamma)^2 + 2F(s + \beta)(t + \gamma) + 2G(t + \gamma)(r + \alpha) + 2H(r + \alpha)(s + \beta) + 2U(r + \alpha) + 2V(s + \beta) + 2W(t + \gamma) + D = 0.$$

Hence the equations giving the center are

$$\begin{aligned} A\alpha + H\beta + G\gamma + U &= 0, \\ H\alpha + B\beta + F\gamma + V &= 0, \qquad (13) \\ G\alpha + F\beta + C\gamma + W &= 0. \end{aligned}$$

Therefore

$$\frac{\alpha}{\begin{vmatrix} H & G & U \\ B & F & V \\ F & C & W \end{vmatrix}} = \frac{-\beta}{\begin{vmatrix} A & G & U \\ H & F & V \\ G & C & W \end{vmatrix}} = \frac{\gamma}{\begin{vmatrix} A & H & U \\ H & B & V \\ G & F & W \end{vmatrix}} = \frac{-1}{\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}}. \qquad (13a)$$

The point-condition equation of the conicoid when the center is at the origin is

(14) $Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + D' = 0,$

where D' is obtained from (3) by putting $r = \alpha$, $s = \beta$, $t = \gamma$.

Multiply equations (13) in order by α, β, γ, and subtract the sum from D'; then we have

(15) $$D' = U\alpha + V\beta + W\gamma + D.$$

From (13) and (15) we have

$$\begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D-D' \end{vmatrix} = 0;$$

therefore

(16) $$D'\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{vmatrix};$$

which may be written

(17) $$D'\mathscr{D} = \Delta.$$

It is seen that the equation of the center is given by

(18) $$\mathscr{U}x + \mathscr{V}y + \mathscr{W}z - \mathscr{D} = 0,$$

where $\mathscr{U}$, $\mathscr{V}$, etc., are the co-factors of U, V, etc., in Δ.

4. The Discriminating Cubic:-

We have seen (Section 2) that by a proper choice of rectangular axes

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs$$

can always be reduced to the form

$$\alpha r^2 + \beta s^2 + \gamma t^2;$$

and this reduction can be effected without changing the origin, for the terms of second degree are not altered by transforming to any parallel axes.

Now $r^2 + s^2 + t^2$ is unaltered by a change of rectangular axes through the same origin. Hence, when the axes are so changed that

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs$$

becomes

$$\alpha r^2 + \beta s^2 + \gamma t^2,$$

(19) $$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs - \xi(r^2 + s^2 + t^2)$$

will become

(20) $$\alpha r^2 + \beta s^2 + \gamma t^2 - \xi(r^2 + s^2 + t^2).$$

Both these expressions will therefore be the product of linear factors for the same values of ξ. The condition that (19) is the product of linear factors is

(21) $$\begin{vmatrix} A-\xi & H & G \\ H & B-\xi & F \\ G & F & C-\xi \end{vmatrix} = 0.$$

But (20) is the product of linear factors when ξ is equal to α, β, or γ. Hence α, β, γ are the three roots of (21). The equation when expanded is

$$\xi^3 - \xi^2(A + B + C) + \xi(AB + BC + CA - F^2 - G^2 - H^2)$$
$$- (ABC + 2FGH - AF^2 - BG^2 - CH^2) = 0,$$

or

(22) $\xi^3 - \mathcal{I}\xi^2 + \mathcal{J}\xi - \mathcal{D} = 0.$

This equation is called the "discriminating cubic".

5. Discussion for $\mathcal{D} \neq 0$:-

From equation (18) we see that there is a definite center at a finite distance, unless $\mathcal{D} = 0$. If $\mathcal{D} = 0$ and one of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ is different from zero (i.e. $\Delta \neq 0$) there is a definite center at an infinite distance.

If $\mathcal{D}$ be not zero, change to parallel axes through the center, and the equation becomes

$$Ar^2 + Bs^2 + Ct^2 + 2Fst + 2Gtr + 2Hrs + D' = 0,$$

where D' is found as in Section 2. Now, keeping the origin fixed, change the axes in such a manner that the equation is reduced to the form

$$\alpha r^2 + \beta s^2 + \gamma t^2 + D' = 0.$$

Then, by Section 3, α, β, γ will be the three roots of the discriminating cubic.

Since $\mathcal{D} D' = \Delta$, the last equation may be written in the form

$$\mathcal{D}\alpha r^2 + \mathcal{D}\beta s^2 + \mathcal{D}\gamma t^2 + \Delta = 0.$$

If the three quantities $\frac{\mathcal{D}\alpha}{\Delta}$, $\frac{\mathcal{D}\beta}{\Delta}$, $\frac{\mathcal{D}\gamma}{\Delta}$ are all negative, the surface is an ellipsoid; if two of them are negative, the surface is an hyperboloid of one sheet; if one is negative, the surface is an hyperboloid of two sheets; and if they are all positive, the surface is an imaginary ellipsoid.

We have shown in Chapter II that the general equation can be reduced to one of the three forms

$$ax^2 + by^2 + cz^2 - 1 = 0, \tag{23}$$

$$ax^2 + by^2 + cz^2 = 0, \tag{24}$$

$$by^2 + cz^2 + 2ux = 0. \tag{25}$$

We see from Section 7 of Chapter III that $\mathcal{D} \neq 0$ always requires $\Delta \neq 0$, which is true only for (23).

6. Discussion of the Case $\mathcal{D} = 0$:-

When $\mathcal{D} = 0$, one root of the discriminating cubic must be zero. From Section 4, Chapter II, we see that one principal plane must be the plane (0, 0, 0). If $\mathcal{J} \neq 0$, we must have two finite principal planes, and therefore the center is at infinity and must lie on the line of intersection of the two finite principal planes.

If $\mathcal{D} = 0$ and $\Delta \neq 0$, equation (18) shows that the center is at infinity. Since one root of the discriminating cubic is zero, the equation can easily be solved; let the roots be 0, α, β. Find the direction cosines of the principal axis by means of equations (16), Chapter II, and take the X axis parallel to the principal axis. The equation will then become

$$\alpha s^2 + \beta t^2 + 2U'r + 2V's + 2W't + D = 0,$$

or, by a change of origin,

$$\alpha s^2 + \beta t^2 + 2U'r = 0.$$

Hence we have the surface, which, expressed in plane coördinates, is

$$ay^2 + bz^2 + 2ux = 0,$$ (1)

since $\Delta \neq 0$.

7. Summary:-

Let us investigate the general equation of a conicoid. If $\Delta \neq 0$ and $d \neq 0$, it follows that $\mathcal{D} \neq 0$ and we have an ellipsoid or hyperboloid. If Δ is positive we have the hyperboloid of one sheet. If Δ is negative we discover the nature of the surface by solving the discriminating cubic; three roots with the same sign denote an ellipsoid and roots which differ in sign denote an hyperboloid of two sheets.

If $\Delta \neq 0$ but $d = 0$, it follows that $\mathcal{D} = 0$.(2) This gives us an elliptic or hyperbolic paraboloid according as Δ is negative or positive, respectively.

The plane curves are found to be those surfaces for which all the invariants except $\mathcal{J}$ vanish. If $d = 0$ the plane curve is a parabola. If $d \neq 0$ the plane curve is an

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(1) Snyder and Sisam, p. 130.

(2) Section 7, Chapter III.

ellipse or hyperbola according as $\mathcal{J}$ is negative or positive, respectively.

A pair of points is given when $\mathcal{I} = \mathcal{J} = \mathcal{D} = \Delta = 0$ provided that the equation is factorable.

Otherwise the equation represents an infinite conicoid or an infinite conic.

BIBLIOGRAPHY

1. Lambert "Analytic Geometry",

   The Macmillan Co., 1904.

2. Smith "Solid Geometry",

   Macmillan and Co., 1889.

3. Snyder and Sisam "Analytic Geometry of Space",

   Henry Holt and Co., 1914.

4. Tanner and Allen "Brief Course in Analytic Geometry",

   American Book Co., 1911.

5. Valgardsson "Line Coördinates",

   M.A. Thesis at University of Manitoba.

6. Wilson "Solid Geometry and Conic Sections",

   Macmillan and Co., 1898.