AN EXTENSION OF AIRY'S EQUATION

by

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ABSTRACT

We consider the differential equation $d^2u/dz^2 - z^n u = 0$ (z, u complex variables, n a positive integer), which is the simplest second order ordinary differential equation with a turning point of order n . The solutions which we study, herein called A_n functions, are generalizations of Airy functions.

Most of their properties are then deduced from those of related Bessel functions of order $\frac{1}{n+2}$, but in the discussion of the zeros in section 3, results are deduced directly from the differential equation.

It is easy to see that the A_n functions are special cases of functions studied by Turrittin [9]. The relation of the former to Bessel functions, however, enables us to use methods not available in [9] to obtain uniform asymptotic representations for large z .

We obtain new results on the distribution of the zeros which extend a property [6] of Airy functions, that is, of A_1 functions, to all positive integers n . A similar remark applies to bounds [8] for Airy functions and their reciprocals.

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AN EXTENSION OF AIRY'S EQUATION

INTRODUCTION

We shall define a class of functions, the A_n functions, which are solutions of the differential equation $d^2u/dz^2 - z^n u = 0$, (z, u complex variables, n a positive integer). This is the simplest example of a second order ordinary differential equation with a turning point of order n . The differential equation has no finite singularity and an irregular singularity of finite rank at infinity. The solutions to be considered are real on the real axis and analytic in the finite complex plane; one of them tends to zero as z approaches infinity in a sector containing the positive real axis. Their properties can be deduced directly from the differential equation, as is done in the discussion of the zeros in section 3, or from the properties of related Bessel functions of order $1/(n+2)$.

The A_n functions are essentially special cases of functions studied by Turrittin [9], who obtained uniform asymptotic representations for large z . We derive for A_n functions the analogues of his representations by using known properties of Bessel functions. In addition, we obtain new results on the distribution of the zeros. These theorems extend a property [6] of Airy functions, that is, of A_1 functions, to all positive integers n . The bounds we obtain in section 4 for A_n functions

in zero-free regions present, in explicit form, results implicit in Turrittin's work. The modifications made in regions containing zeros are new for values of $n \neq 1$. These results also extend known properties [8] of A_1 functions.

Although we shall not do this here, the properties of A_n functions obtained below may be used to obtain uniform asymptotic expansions of the solutions of a more general second order equation with a turning point of order n for large values of a (real or complex) parameter. In formulating a precise result the main problem would be to find the most general domain of validity.

1. The differential equation

Consider the second order differential equation

$$(1.1) \quad \frac{d^2 u}{dz^2} - z^n u = 0$$

where z and u are complex variables and n is an integer > -2 .

The point at infinity is an irregular singularity of rank $\frac{1}{2}(n+2)$ for n even, $\frac{1}{2}(n+3)$ for n odd, and the origin is a transition point of order n . All other points are ordinary points.

LEMMA 1: A fundamental set of solutions of (1.1) is given by

$$(1.2) \quad u_{\pm}(z) = z^{1/2} I_{\pm \nu}(\zeta), \quad \zeta = \frac{2}{n+2} z^{(n+2)/2}, \quad \nu = \frac{1}{n+2}$$

where $I_{\nu}(\zeta)$ is the modified Bessel function of the first kind, of order ν .

PROOF: In [10, p. 96(5)] replace ν by u , ζ by z , c by 1, then set $n = 2q - 2$ to obtain

$$u_{\pm}(z) = \left[\frac{2}{n+2} z^{(n+2)/2} \right]^{1/(n+2)} J_{\pm \frac{1}{n+2}} \left(i \frac{2}{n+2} z^{\frac{n+2}{2}} \right),$$

($n \neq -2$).

Ignoring a multiplicative constant, we have

$$u_{\pm} z = z^{1/2} I_{\pm \frac{1}{n+2}} \left(\frac{2}{n+2} z^{\frac{n+2}{2}} \right).$$

Alternatively, the transformation

$$(1.3) \quad z = \varphi(\zeta), \quad u = [\varphi'(\zeta)]^{1/2} y(\zeta)$$

sends the differential equation (1.1) into

$$(1.4) \quad \frac{d^2 y}{d\zeta^2} + \left[-\varphi'^2 \varphi^n + \frac{\varphi'^2}{\psi} \frac{d^2 \psi}{dz^2} \right] y = 0, \quad \psi = \varphi'^{1/2}.$$

Choosing φ so that

$$(1.5) \quad \varphi'^2 \varphi^n = 1, \quad \varphi = \left(\frac{n+2}{2} \zeta \right)^{\frac{2}{n+2}},$$

we reduce equation (1.4) to

$$\frac{d^2 y}{d\zeta^2} + \left[-1 + \frac{\frac{1}{4} - \left(\frac{1}{n+2} \right)^2}{\zeta^2} \right] y = 0.$$

This may be written in the form

$$\frac{d^2}{d\zeta^2} (\zeta^{-1/2} y) + \frac{1}{\zeta} \frac{d}{d\zeta} (\zeta^{-1/2} y) - \left[1 + \frac{1}{(n+2)^2 \zeta^2} \right] \zeta^{-1/2} y = 0$$

and thus possesses linearly independent solutions [10, p. 77]

$$y_{\pm}(\zeta) = \zeta^{1/2} I_{\pm \frac{1}{n+2}}(\zeta).$$

By (1.3) and (1.5) it now follows that (1.2) gives solutions of (1.1). This completes the proof of the lemma.

We shall consider, in what follows, only positive integral values of n , although (1.2) holds for $n \neq -2$. In the case $n = -2$ the differential equation (1.1) reduces to an Euler equation, with a fundamental set of solutions

$$u_{\pm}(z) = z^{(1 \pm \sqrt{5})/2}.$$

DEFINITION: The A_n functions of the first and second kind are given respectively by

$$(1.6) \quad A_n(z) = \frac{1}{n+2} z^{1/2} \left[I_{-\frac{1}{n+2}}(\zeta) - I_{\frac{1}{n+2}}(\zeta) \right] \\ = \frac{2 \sin(\frac{\pi}{n+2})}{\pi(n+2)} z^{1/2} K_{\frac{1}{n+2}}(\zeta)$$

$$B_n(z) = \left(\frac{z}{n+2} \right)^{1/2} \left[I_{-\frac{1}{n+2}}(\zeta) + I_{\frac{1}{n+2}}(\zeta) \right],$$

$$\zeta = \frac{2}{n+2} z^{(n+2)/2}.$$

In the special case $n = 1$, these reduce to the Airy functions of the first and second kind respectively.

LEMMA 2: $A_n(z)$ and $B_n(z)$ are entire functions, and are a fundamental set of solutions of (1.1).

PROOF: The modified Bessel function of the first kind has the expansion

$$(1.7) \quad I_\nu(\zeta) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\zeta\right)^{\nu+2k}}{k! \Gamma(\nu+k+1)}.$$

Appealing to (1.6) we have

$$(1.8) \quad A_n(z) = \alpha_n y_{n_1}(z) - \beta_n y_{n_2}(z)$$

where

$$(1.9) \quad \alpha_n^{-1} = (n+2)^{(n+1)/(n+2)} \Gamma\left(\frac{n+1}{n+2}\right), \\ \beta_n^{-1} = (n+2)^{(n+3)/(n+2)} \Gamma\left(\frac{n+3}{n+2}\right),$$

and

$$(1.10) \quad y_{n_1}(z) = \sum_{k=0}^{\infty} a_k z^{(n+2)k},$$

$$a_k^{-1} = (n+1)(n+2)(2n+3)(2n+4) \dots \{k(n+2)-1\} k(n+2), \quad k \geq 1, \quad a_0 = 1$$

$$y_{n_2}(z) = \sum_{k=0}^{\infty} b_k z^{(n+2)k+1},$$

$$b_k^{-1} = (n+2)(n+3) \dots \{k(n+2)\} \{k(n+2)+1\}, \quad k \geq 1, \quad b_0 = 1.$$

Similarly,

$$(1.11) \quad B_n(z) = (n+2)^{1/2} \alpha_n y_{n_1}(z) + (n+2)^{1/2} \beta_n y_{n_2}(z).$$

From (1.7) it is clear that $z^{1/2} I_{\frac{1}{n+2}}(\zeta)$, $z^{1/2} I_{-\frac{1}{n+2}}(\zeta)$ are entire

functions of z , so that $A_n(z)$ and $B_n(z)$ are also entire functions.

Furthermore,

$$(1.12) \quad \begin{aligned} A_n(0) &= \alpha_n, & A'_n(0) &= -\beta_n \\ B_n(0) &= (n+2)^{1/2} \alpha_n, & B'_n(0) &= (n+2)^{1/2} \beta_n. \end{aligned}$$

From Abel's formula [3, Ch. 3 (6.5)], if $u(z)$ and $v(z)$ are solutions of the differential equation (1.1), then the Wronskian $\mathcal{W}[u, v]$ of $u(z)$ and $v(z)$ is a constant.

It follows from (1.12) that

$$(1.13) \quad \mathcal{W}[A_n(z), B_n(z)] = \frac{2}{\pi} (n+2)^{-1/2} \sin\left(\frac{\pi}{n+2}\right),$$

upon use of the formula

$$\Gamma(p) \Gamma(1-p) = \operatorname{cosec}(p\pi), \quad 0 < p < 1.$$

Since the right hand side of (1.13) is non-zero, it follows that $A_n(z)$ and $B_n(z)$ are linearly independent. The lemma follows at once.

It should be noted that, if n is a positive integer, the functions

$$u_k(z) = A_n(w^k z), \quad w = \exp\left(\frac{2\pi i}{n+2}\right), \quad k = 0, \pm 1, \pm 2, \dots$$

are also solutions of (1.1).

PROOF: By (1.6), the function $z^{-1/2} u_k(z)$ is a linear combination of the functions

$$\begin{aligned} I_{\pm \frac{1}{n+2}} \left(\frac{2}{n+2} w^{k(n+2)/2} z^{(n+2)/2} \right) &= I_{\pm \frac{1}{n+2}} \left(\frac{2}{n+2} e^{k\pi i} z^{(n+2)/2} \right) \\ &= \exp \left(\pm \frac{k\pi i}{n+2} \right) \cdot I_{\pm \frac{1}{n+2}}(\zeta). \end{aligned}$$

We shall now show that

$$(1.14) \quad A_n(-z) = \frac{z^{1/2}}{n+2} \left[J_{-\frac{1}{n+2}}(\zeta) + J_{\frac{1}{n+2}}(\zeta) \right], \quad (n \text{ odd})$$

$$A_n(-z) = (n+2)^{-1/2} B_n(z), \quad (n \text{ even}).$$

PROOF: Let $\theta = \frac{2}{n+2} (e^{\pi i} z)^{(n+2)/2}$.

If n is odd, let $n = 2m + 1$. Then, by the continuation formulae for $I_\nu(\zeta)$, [Watson, p.75], we have, putting $\nu = \frac{1}{n+2}$,

$$\begin{aligned} I_\nu(\theta) &= I_\nu \left(\zeta e^{m\pi i} + \frac{3\pi i}{2} \right) = e^{m\nu\pi i} I_\nu \left(\zeta e^{\frac{3\pi i}{2}} \right) \\ &= e^{m\nu\pi i} e^{\frac{3\nu\pi i}{2}} J_\nu(\zeta), \quad \left(\frac{\pi}{2} < \arg \zeta e^{\frac{3\pi i}{2}} \leq \pi \right) \\ &= e^{\pi i/2} J_\nu(\zeta), \quad \left(-\frac{2\pi}{n+2} < \arg z \leq -\frac{\pi}{n+2} \right) \end{aligned}$$

upon using the relations [10, p. 77] between J_ν and I_ν .

Similarly,

$$I_{-\nu}(\theta) = e^{-\pi i/2} J_{-\nu}(\zeta), \quad -\frac{2\pi}{n+2} < \arg z \leq -\frac{\pi}{n+2}$$

Hence

$$\begin{aligned} A_n(-z) &= \frac{(e^{\pi i} z)^{1/2}}{n+2} [I_{-\nu}(\theta) - I_\nu(\theta)] = \frac{z^{1/2}}{n+2} [J_{-\nu}(\zeta) - e^{\pi i} J_\nu(\zeta)] \\ &= \frac{z^{1/2}}{n+2} [J_{-\nu}(\zeta) + J_\nu(\zeta)], \quad -\frac{2\pi}{n+2} < \arg z \leq -\frac{\pi}{n+2}. \end{aligned}$$

By replacing $e^{\pi i} z$ successively by $e^{3\pi i} z$, $e^{5\pi i} z$, ..., we can verify the result in sectors which together cover the z -plane.

If n is even, let $n = 2m$. Then

$$\begin{aligned} I_\nu(\theta) &= I_\nu(\zeta e^{(m+1)\pi i}) = e^{(m+1)\nu\pi i} I_\nu(\zeta) \\ &= e^{\pi i/2} I_\nu(\zeta). \end{aligned}$$

Similarly,

$$I_{-\nu}(\theta) = e^{-\pi i/2} I_{-\nu}(\zeta).$$

Hence

$$\begin{aligned} A_n(e^{\pi i} z) &= \frac{(e^{\pi i} z)^{1/2}}{n+2} [I_{-\nu}(\theta) - I_\nu(\theta)] \\ &= \frac{z^{1/2}}{n+2} [I_{-\nu}(\zeta) + I_\nu(\zeta)] \\ &= (n+2)^{-1/2} B_n(z). \end{aligned}$$

This establishes the result.

LEMMA 3: If k is any integer, then

$$(1.15) \quad A_n(w^k z) = \frac{z^{1/2}}{n+2} [I_{-\nu}(\zeta) - w^k I_{\nu}(\zeta)],$$

$$w = \exp \frac{2\pi i}{n+2}, \quad \nu = \frac{1}{n+2},$$

$$B_n(w^k z) = \left(\frac{z}{n+2}\right)^{1/2} [I_{-\nu}(\zeta) + w^k I_{\nu}(\zeta)].$$

PROOF: Let $\theta_k = \frac{2}{n+2} (w^k z)^{(n+2)/2}$. Then

$$\begin{aligned} I_{\nu}(\theta_k) &= I_{\nu}(\zeta e^{k\pi i}) = e^{k\nu\pi i} I_{\nu}(\zeta) \\ &= w^{k/2} I_{\nu}(\zeta). \end{aligned}$$

Similarly,

$$I_{-\nu}(\theta_k) = w^{-k/2} I_{-\nu}(\zeta).$$

Hence

$$\begin{aligned} A_n(w^k z) &= \frac{(w^k z)^{1/2}}{n+2} [I_{-\nu}(\theta_k) - I_{\nu}(\theta_k)] \\ &= \frac{z^{1/2}}{n+2} [I_{-\nu}(\zeta) - w^k I_{\nu}(\zeta)]. \end{aligned}$$

Similarly, we can prove the analogous formula for $B_n(w^k z)$.

Any three solutions of the differential equation (1.1) are linearly dependent, and in particular

$$(1.16) \quad (w^s - w^r)A_n(z) + (1-w^s)A_n(w^r z) - (1-w^r)A_n(w^s z) = 0$$

where r and s are integers. This is a consequence of the relations (1.15); for if we assume a linear relation between $A_n(z)$, $A_n(w^r z)$ and $A_n(w^s z)$ with undetermined coefficients, an appeal to the linear independence [10, pp. 77(3)] of $I_{\nu}(\zeta)$ and $I_{-\nu}(\zeta)$

yields (1.16).

It can also be shown in an analogous manner that

$$(1.17) \quad (w^s - w^r)B_n(z) + (n+2)^{1/2}(1+w^r)A_n(w^s z) \\ - (n+2)^{1/2}(1+w^s)A_n(w^r z) = 0$$

where, as before, r and s are integers. In the special case for which $s = 1$, $r = -1$ and $n = 1$, the equation (1.17) reduces to the well-known relation [8, (4.5)], [6, p. 364] between Airy functions of the first and second kind.

2. Asymptotic behaviour as $z \rightarrow \infty$.

It is known that as $\zeta \rightarrow \infty$, [10, p. 202 (1), (2), (3); p. 199 (1), (3)]

$$(2.1) \quad K_\nu(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} e^{-\zeta} [1 + o(\frac{1}{\zeta})], \quad |\arg \zeta| < \frac{3\pi}{2},$$

$$(2.2) \quad I_\nu(\zeta) = \left(\frac{1}{2\pi\zeta}\right)^{1/2} \{e^\zeta [1 + o(\frac{1}{\zeta}) + e^{-\zeta + (\nu + \frac{1}{2})\pi i} [1 + o(\frac{1}{\zeta})]]\},$$

$$-\frac{\pi}{2} < \arg \zeta < \frac{3\pi}{2},$$

$$I_\nu(\zeta) = \left(\frac{1}{2\pi\zeta}\right)^{1/2} \{e^\zeta [1 + o(\frac{1}{\zeta})] + e^{-\zeta - (\nu + \frac{1}{2})\pi i} [1 + o(\frac{1}{\zeta})]\},$$

$$-\frac{3\pi}{2} < \arg \zeta < \frac{\pi}{2},$$

$$(2.3) \quad J_\nu(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{1/2} [\cos(\zeta - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + o(\zeta^{-1}) \sin(\zeta - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)],$$

$$|\arg \zeta| < \pi.$$

Each asymptotic form holds uniformly with respect to $\arg z$ in any closed sector within the stated open sector.

LEMMA 4: The following asymptotic forms are valid as $z \rightarrow \infty$:

$$(2.4) \quad A_n(z) = (n+2)^{-1/2} \pi^{-1/2} \sin(\frac{\pi}{n+2}) z^{-n/4} e^{-\zeta} [1 + o(\frac{1}{\zeta})],$$

$$|\arg z| < \frac{3\pi}{n+2}$$

$$(2.5) \quad A_n(-z) = 2(n+2)^{-1/2} \pi^{-1/2} \cos(\frac{\pi}{2n+4}) z^{-n/4} [\cos(\zeta - \frac{\pi}{4})$$

$$+ o(\zeta^{-1}) \sin(\zeta - \frac{\pi}{4})],$$

$$|\arg z| < \frac{2\pi}{n+2}, \quad n \text{ odd}$$

$$(2.6) \quad B_n(z) = \pi^{-1/2} z^{-n/4} e^{\zeta} [1 + O(\frac{1}{\zeta})], \quad |\arg z| < \frac{\pi}{n+2},$$

$$(2.7) \quad B_n(-z) = 2 \pi^{-1/2} \sin(\frac{\pi}{2n+4}) z^{-n/4} [\cos(\zeta + \frac{\pi}{4}) + O(\zeta^{-1}) e^{|\operatorname{Im} \zeta|}],$$

$$|\arg z| < \frac{2\pi}{n+2}, \quad n \text{ odd.}$$

Each asymptotic form holds uniformly with respect to $\arg z$ in any closed sector within the stated open sector.

PROOF: The form (2.4) follows from (1.6) by direct substitution of $\zeta = \frac{2}{n+2} z^{(n+2)/2}$ in the asymptotic form (2.1), whilst the form (2.6) follows from (1.6) and the forms (2.2) by noting that e^{ζ} dominates $e^{-\zeta}$ in the sector $|\arg \zeta| < \frac{\pi}{2}$.

To obtain (2.5) we use the relation (1.14) and the asymptotic form (2.3). Similarly we derive the form (2.7), provided we note that

$$\begin{aligned} |\sin(\zeta + v \frac{\pi}{2} - \frac{\pi}{4}) - \sin(\zeta - v \frac{\pi}{2} - \frac{\pi}{4})| &= |2 \cos(\zeta - \frac{\pi}{4}) \sin(\frac{v\pi}{2})| \\ &\leq 2 e^{|\operatorname{Im} \zeta|} \sin(v \frac{\pi}{2}) \end{aligned}$$

and

$$\begin{aligned} \cos(\zeta + \frac{v\pi}{2} - \frac{\pi}{4}) - \cos(\zeta - \frac{v\pi}{2} - \frac{\pi}{4}) &= -2 \sin(\zeta - \frac{\pi}{4}) \sin(\frac{v\pi}{2}) \\ &= 2 \cos(\zeta + \frac{\pi}{4}) \sin(\frac{v\pi}{2}). \end{aligned}$$

This completes the proof of the lemma.

The asymptotic behaviour in those sectors of the z -plane not covered by the forms (2.4) - (2.7) can be deduced from the forms (2.2) by making use of the relations (1.15). The resulting asymptotic forms will be valid in one of the sectors

$$\frac{(2k-1)\pi}{n+2} < \arg z < \frac{(2k+3)\pi}{n+2}$$

$$\frac{(2k-3)\pi}{n+2} < \arg z < \frac{(2k+1)\pi}{n+2}$$

where k is any integer. Each of the sectors has an angular width of $4\pi/(n+2)$, and the angle between rays corresponding to neighbouring $(n+2)$ -th roots of unity is $2\pi/(n+2)$; thus the whole z -plane can be covered by these sectors.

If n is even, we can deduce the asymptotic behaviour of $A_n(-z)$ from that of $B_n(z)$ by appealing to (1.14). A similar remark applies to $B_n(-z)$.

The complete asymptotic expansions, valid in the appropriate sectors, can be deduced from those corresponding to the forms (2.1)-(2.3).

3. Zeros of A_n functions.

THEOREM 1: The functions $A_n(z)$, $A'_n(z)$ have no real zeros for n even, but for n odd these functions have an infinite number of simple zeros on the negative real axis with no finite accumulation point. Their non-real zeros all lie in the region $\bigcup_k S_k$, where S_k is the double sector $2k\pi/(n+1) < |\arg z| < (2k+1)\pi/(n+2)$, and k runs over the integers in the range $0 < 2k < n+1$.

PROOF: Since (1.1) is a linear homogeneous equation, the zeros are simple with no finite accumulation point [3, p. 208], [1, p. 27].

If z is real and n even, $u''(z)$ and $u(z)$ have the same sign. Since $A_n(0) > 0$, $\lim_{z \rightarrow +\infty} A_n(z) = 0$, $\lim_{z \rightarrow -\infty} A_n(z) = +\infty$, we see that $A_n(z)$, $A'_n(z)$ can have no real zeros in this case.

If $z > 0$ and n is odd, a similar argument shows that there are no zeros in this case.

If $z < 0$ and n is odd, an easy application of the Sturm comparison theorem shows that any real solution of (1.1) has an infinite number of real zeros.

We now turn to the non-real zeros. Using the differential equation (1.1) we may verify that

$$\frac{d^2}{dz^2} A_n(az) - a^{n+2} z^n A_n(az) = 0$$

$$\frac{d^2}{dz^2} A_n(bz) - b^{n+2} z^n A_n(bz) = 0$$

where a and b are complex variables. Multiplication of the first

equation by $A_n(bz)$ and the second by $A_n(az)$, subtracting and integrating from c to 1 gives

$$(3.1) \quad (a^{n+2} - b^{n+2}) \int_c^1 z^n A_n(az) A_n(bz) dz \\ = [A_n(bz) \frac{d}{dz} A_n(az) - A_n(az) \frac{d}{dz} A_n(bz)]_c^1.$$

Multiplication of the first equation by $\frac{b^{n+1}}{a} A_n(bz)$ and the second by $\frac{a^{n+1}}{b} A_n(az)$ and subtracting gives

$$0 = \frac{d}{dz} \left[\frac{a^{n+1}}{b} A_n(az) \frac{d}{dz} A_n(bz) - \frac{b^{n+1}}{a} A_n(bz) \frac{d}{dz} A_n(az) \right] \\ - \left(\frac{a^{n+1}}{b} - \frac{b^{n+1}}{a} \right) \frac{d}{dz} A_n(az) \frac{d}{dz} A_n(bz)$$

which leads to

$$(3.2) \quad (a^{n+2} - b^{n+2}) \int_c^1 A'_n(az) A'_n(bz) dz \\ = [a^{n+1} A_n(az) A'_n(bz) - b^{n+1} A_n(bz) A'_n(az)]_c^1$$

where a prime denotes differentiation with respect to the argument of the A_n functions.

Suppose now that $a = re^{i\theta}$ is a non-real zero of $A_n(z)$ or $A'_n(z)$. Let b denote its conjugate $\bar{a}$, and $c = 0$. Then (3.1) and (3.2) become

$$(3.3) \quad \int_0^1 z^n A_n(az) A_n(\bar{a}z) dz = - \frac{a^{n+2} - \bar{a}^{n+2}}{a^{n+2} - \bar{a}^{n+2}} A_n(0) A'_n(0),$$

$$(3.4) \quad \int_0^1 A'_n(az) A'_n(\bar{a}z) dz = - \frac{a^{n+1} - \bar{a}^{n+1}}{a^{n+2} - \bar{a}^{n+2}} A_n(0) A'_n(0).$$

By the Schwarz reflection principle, the integrands become

$z^n |A_n(az)|^2$ and $|A'_n(az)|^2$ respectively, which are both non-negative for $0 \leq z \leq 1$. Moreover, $A_n(0)A'_n(0) < 0$. Suppose without loss of generality that θ is in the upper half-plane. Let T_k be the sector

$$(2k-1)\pi / (n+2) \leq \theta \leq 2k\pi / (n+1), \quad k=1,2,\dots,n/2 \text{ if } n \text{ is even} \\ =1,2,\dots,(n-1)/2 \text{ if } n \text{ is odd.}$$

Let T_0 be the sector $\pi(2k_{\max}+1)/(n+2) \leq \theta < \pi$; that is, T_0 is the sector $(n+1)\pi / (n+2) \leq \theta < \pi$ if n is even, and is the sector $n\pi / (n+2) \leq \theta < \pi$ if n is odd.

If $\theta \notin S_k$, $0 < 2k < n+1$, we shall derive a contradiction.

If $\theta \in T_0$, n even, then $(n+1)\pi \leq (n+2)\theta < (n+2)\pi$, and $\sin(n+2)\theta \leq 0$, which contradicts (3.3).

If $\theta \in T_0$, n odd, then $n\pi \leq \theta < (n+2)\theta$, and we have the following cases:

(a) $n\pi \leq (n+2)\theta < (n+1)\pi$, whence $\sin(n+2)\theta \leq 0$.

(b) $(n+1)\pi \leq (n+2)\theta < (n+2)\pi$, whence

$n\pi < (n+1)^2\pi / (n+2) < (n+1)\theta < (n+1)\pi$, and therefore $\sin(n+1)\theta < 0$.

This contradicts (3.4).

If $\theta \in T_k$, where

$$(3.5) \quad 0 < 2k < n+1,$$

then $(2k-1)\pi \leq (n+2)\theta \leq 2k\pi + 2k\pi / (n+1) = 2k\pi (n+2)/(n+1)$, and

we have the following cases:

(a) $(2k-1)\pi \leq (n+2)\theta < 2k\pi$, whence $\sin(n+2)\theta \leq 0$.

(b) $2k\pi \leq (n+2)\theta \leq 2k\pi (n+2)/(n+1)$, whence

$$2k\pi (n+1)/(n+2) \leq (n+1)\theta \leq 2k\pi.$$

But $2k(n+1)/(n+2) = 2k - 2k/(n+2) > 2k-1$, by hypothesis (3.5), and therefore $\sin(n+1)\theta \leq 0$.

If $0 < \theta < \pi/(n+2)$, let $c = \infty$. The integral in (3.1) exists, since $A_n(z)$ is an entire function with asymptotic representation (2.4). Then (3.1) gives

$$(a^{n+2} - \bar{a}^{n+2}) \int_{\infty}^1 z^n |A_n(az)|^2 dz = 0.$$

Since $z > 0$, and $|A_n(az)|$ is not identically zero for $1 \leq z < \infty$, this implies that $a^{n+2} - \bar{a}^{n+2} = 0$; that is, at least one of

$$a = \bar{a}, \bar{a}w, \dots, \bar{a}w^{n+1}$$

must hold. This implies that a is real or $\arg a - \arg \bar{a}$ is a multiple of $2\pi/(n+2)$. This contradicts the hypothesis $0 < \arg a < \pi/(n+2)$. Hence there are no zeros of $A_n(z)$ in this range.

Similarly, if a is a zero of $A'_n(z)$ with $0 < \arg a < \pi/(n+2)$, the integral in (3.2) exists for $c = \infty$, since

$$A'_n(z) = -M z^{(n+1)/2} K_{1-\nu}(\zeta), \quad M = 2\pi^{-1}(n+2)^{-1} \sin\left(\frac{\pi}{n+2}\right),$$

$\nu = \frac{1}{n+2}$, and $K_{1-\nu}(\zeta)$ has the asymptotic representation (2.1). Then (3.2) gives $\int_1^{\infty} |A'_n(az)|^2 dz = 0$, which is a contradiction.

Hence $A'_n(z)$ has no zeros in the sector $|\arg z| < \pi/(n+2)$. This completes the proof of the theorem.

It should be noted that $A_n(z)$ and $A'_n(z)$ do have zeros in S_k .

PROOF: Set $\zeta = Z e^{(k+\frac{1}{2})\pi i}$. Then

$$I_{\nu}(\zeta) = e^{(k+1)\nu\pi i} I_{\nu}(Z e^{-\frac{i\pi}{2}}) = e^{(k+1/2)\nu\pi i} J_{\nu}(Z),$$

$$(-\frac{\pi}{2} < \arg Z \leq \pi)$$

and

$$I_{-\nu}(\zeta) = e^{-(k+1/2)\nu\pi i} J_{-\nu}(Z).$$

Using the relation

$$J_{-\nu}(Z) = Y_{\nu}(Z) \sin \nu\pi + J_{\nu}(Z) \cos \nu\pi,$$

where $Y_{\nu}(Z)$ is Weber's Bessel function of the second kind of order ν , we see that

$$\begin{aligned} I_{-\nu}(\zeta) - I_{\nu}(\zeta) &= e^{-(k+1/2)\nu\pi i} [-Y_{\nu}(Z) \sin \nu\pi + J_{\nu}(Z) \cos \nu\pi] - e^{(k+1/2)\nu\pi i} J_{\nu}(Z) \\ &= e^{-(k+1/2)\nu\pi i} [-Y_{\nu}(Z) \sin \nu\pi + J_{\nu}(Z) \cos \nu\pi] - e^{(k+1/2)\nu\pi i} J_{\nu}(Z) \end{aligned}$$

$$(-\frac{\pi}{2} < \arg Z \leq \pi).$$

This can be written in the form

$$J_{\nu}(Z) \cos \alpha' - Y_{\nu}(Z) \sin \alpha'$$

provided that

$$\tan \alpha' = \frac{\sin \nu\pi}{\cos \nu\pi - e^{(2k+1)\nu\pi i}}.$$

It is well-known that $\tan \alpha'$ takes every value once and only once with two exceptions:

$$\tan \alpha' \neq \pm i.$$

These exceptional values can only occur if $\cos(2k+1)\nu\pi = \cos \nu\pi$ and $\sin(2k+1)\nu\pi = \mp \sin \nu\pi$; that is to say $(2k+1)\nu\pi = 2\ell\pi \mp \nu\pi$, ($\ell = 0, 1, 2, \dots$). But $0 < 2k < n+1$, and therefore these

exceptional values cannot arise. Hence α' is uniquely determined. We now use the fact that the function $J_\nu(Z)\cos \alpha' - Y_\nu(Z)\sin \alpha'$ has zeros with positive real part. This implies that $A_n(z)$ has zeros in the sectors (3.7). Similarly, since $A'_n(z) = -M z^{(n+1)/2} K_{1-\nu}(\zeta)$, $M = 2(n+2)^{-1} \pi^{-1} \sin(\frac{\pi}{n+2})$, we can show that $A'_n(z)$ also has zeros in the stated sectors.

REMARKS.

By using the methods of Theorem 3 below, we can find the number of zeros of $A_n(z)$, $A'_n(z)$ in a certain closed subregion of S_k . If $n = 1$, $A_n(z)$ is the Airy function of the first kind, the zeros of which are all real and negative. The assertion in Theorem 1 concerning non-real zeros is thus vacuously true in this case.

THEOREM 2: The functions $B_n(z)$, $B'_n(z)$ have no real zeros if n is even, but for n odd these functions have an infinite number of single zeros on the negative real axis, with no finite accumulation point. Their non-real zeros all lie in the region $\bigcup S'_k$, where S'_k is the double sector $\frac{(2k-1)\pi}{n+2} < |\arg z| < \frac{(2k-1)\pi}{n+1}$, and k runs through the integers in the range $0 < 2k < n+3$.

PROOF: If z is real, arguments similar to those in Theorem 1 apply here.

Suppose that $a = re^{i\theta}$ is a non-real zero of $B_n(z)$ or $B'_n(z)$.

Then the relations (3.3) and (3.4) remain valid when $F_1(z)$ is replaced by $F_2(z)$. In this case, however, $B_n(0) B'_n(0) > 0$.

We suppose without loss of generality that θ is in the upper half-plane, since we may use the reflection principle. Let T'_k be the sector $\frac{(2k-1)\pi}{n+1} \leq \theta \leq \frac{(2k+1)\pi}{n+2}$, $k = 1, 2, \dots, (n+2)/2$ if n is even, $k = 1, 2, \dots, (n+1)/2$ if n is odd. Let T'_0 be the sector $(2k_{\max} - 1)\pi/(n+1) \leq \theta \leq \pi$; that is T'_0 is the ray $\theta = \pi$ if n is even, and is the sector $n\pi/(n+1) \leq \theta \leq \pi$ if n is odd.

If $\theta \notin S'_k$, $0 < 2k < n+3$, we shall derive a contradiction.

If $\theta \in T'_0$, n even, this reduces to the real case which has been treated earlier.

If $\theta \in T'_0$, n odd, then $n(n+2)\pi/(n+1) \leq (n+2)\theta \leq (n+2)\pi$ and we have the following cases:

(a) $(n+1)\pi \leq (n+2)\theta \leq (n+2)\pi$, whence $\sin(n+2)\theta \geq 0$.

(b) $n(n+2)\pi/(n+1) \leq (n+2)\theta < (n+1)\pi$, so that $n\pi \leq (n+1)\theta < (n+1)^2\pi/(n+2)$.

But $(n+1)^2/(n+2) < (n+1)^2/(n+1) = n+1$

and $n(n+2)/(n+1) > n(n+1)/(n+1) = n$.

Hence $\sin(n+1)\theta \leq 0$ and $\sin(n+2)\theta < 0$.

If $\theta \in T'_k$, where

$$(3.6) \quad 0 < 2k < n+3,$$

then $(2k-1)\pi(n+2)/(n+1) \leq (n+2)\theta \leq (2k+1)\pi$.

But
$$(2k-1)(n+2)/(n+1) = 2k-1+(2k-1)/(n+1)$$

$$= 2k+(2k-n-2)/(n+1) \leq 2k, \text{ by hypothesis (3.6).}$$

Hence we have the following cases:

(a) $2k\pi \leq (n+2)\theta \leq (2k+1)\pi$, whence $\sin(n+2)\theta \geq 0$.

(b) $(2k-1)\pi (n+2)/(n+1) \leq (n+2)\theta < 2k\pi$,

whence $\sin(n+2)\theta < 0$

and $(2k-1)\pi \leq (n+1)\theta < 2k\pi (n+1)/(n+2) < 2k\pi$,

so that $\sin(n+1)\theta \leq 0$.

In the case $0 < \theta \leq \pi/(n+2)$, we have

$$0 < (n+1)\theta < (n+1)\pi/(n+2) < \pi$$

and $0 < (n+2)\theta < \pi$, so that

$$\sin(n+1)\theta > 0 \quad \text{and} \quad \sin(n+2)\theta > 0.$$

Thus if $\theta \notin S'_k$ for any k , $0 < 2k < n+3$, either $\sin(n+2)\theta \geq 0$ or $\sin(n+1)\theta$ and $\sin(n+2)\theta$ have the same sign. This contradicts the analogue of (3.3) or (3.4), and completes the proof of the theorem.

Figures 1 and 2 below illustrate the case $n = 4$.

Figure 1.

Figure 2.

Notation. Let OA, OC, and OE be the rays $\arg z = (2k-2)\pi/(n+2)$, $\arg z = (2k-1)\pi/(n+2)$, $\arg z = 2k\pi/(n+2)$ respectively. Let AB and DE be arcs of the circle $|z| = R$, with R arbitrarily large, and B, D the points $R \exp[(2k-1)\pi/(n+2) \mp \delta]i$, δ being an arbitrary number in the range $0 < \delta < \pi/(n+2)$. Let the curve BCD have the equation

$$\operatorname{Im} \zeta = \text{constant} = (-)^{k+1} 2^{(n+2)-1} R^{(n+2)/2} \cos(n/2 + 1)\delta.$$

Figure 3 illustrates the case $n = 2$, $k = 1$.

Let $\Delta_{PQ} \arg f(z)$ denote the increase in $\arg f(z)$ as z traverses the arc PQ.

Let $x_0 = |z|$ on OA.

$n = 2, k = 1$

Figure 3.

THEOREM 3: If j is a sufficiently large integer, the closed path OABCDEO (defined above) has inside it j zeros of $B_n(z)$ if

$$R = \left[\left(\frac{n}{2} + 1 \right) \left(j - \frac{1}{4} \right) \pi \sec \left(\frac{n}{2} + 1 \right) \delta \right]^{2/(n+2)} \text{ and } j \text{ zeros of } B'_n(z) \\ \text{if } R = \left[\left(\frac{n}{2} + 1 \right) \left(j - \frac{1}{4} \right) \pi \sec \left(\frac{n}{2} + 1 \right) \delta \right]^{2/(n+2)} .$$

PROOF: We examine the changes in $\arg B_n(z)$ and $\arg B'_n(z)$ as z traverses the closed path OABCDEO. Along OA we use the relations (1.15) with k replaced by $k-1$ to obtain

$$B_n \{ x_0 e^{(2k-2)\pi i/(n+2)} \} = (n+2)^{-\frac{1}{2}} x_0^{-\frac{1}{2}} [I_{-\nu}(\xi_0) + \omega^{k-1} I_{\nu}(\xi_0)],$$

where $\xi_0 = \frac{2}{n+2} x_0^{(n+2)/2}$, $\nu = 1/(n+2)$.

Appealing to the asymptotic form (2.2) we deduce

$$\Delta_{OA} \arg B_n(z) = \arg(1 + \omega^{k-1}) + o(1)$$

and so

$$(3.7) \quad \Delta_{OA} \arg B_n(z) = \frac{(k-1)\pi}{n+2} + o(1).$$

Along AB and DE we put $z = x e^{2k\pi i/(n+2)}$, $\xi = \frac{2}{n+2} x^{(n+2)/2}$,

and use the relation (1.15) to derive

$$F_2(z) = (n+2)^{-\frac{1}{2}} x^{\frac{1}{2}} [I_{-\nu}(\xi) + \omega^k I_{\nu}(\xi)].$$

From the asymptotic forms (2.2) it now follows that

$$(3.8) \quad \Delta_{AB} \arg B_n(z) = -\frac{n}{4} \left(\frac{\pi}{n+2} - \delta \right) - \operatorname{Im} \xi \Big|_B + o(1)$$

$$\Delta_{DE} \arg B_n(z) = -\frac{n}{4} \left(\frac{\pi}{n+2} - \delta \right) - \operatorname{Im} \xi \Big|_D + o(1),$$

upon noting that $e^{-\xi}$ is dominant in the sector $-\frac{3\pi}{2} < \arg \xi < -\frac{\pi}{2}$ and e^{ξ} is dominant in the sector $-\frac{\pi}{2} < \arg \xi < \frac{\pi}{2}$. Moreover, $x^{1/2} \xi^{-1/2} = 2^{-1/2} (n+2)^{-1/2} x^{-n/4}$.

From (1.15) and (2.2) we deduce

$$(3.9) \quad \Delta_{EO} \arg B_n(z) = -\frac{k\pi}{n+2} + o(1).$$

On BCD we put $z = x_1 e^{(2k-1)\pi i/(n+2)}$, $\xi_1 = \frac{2}{n+2} x_1^{(n+2)/2}$.

Then

$$\begin{aligned} I_{\nu}(\zeta) &= I_{\nu}(\xi_1 e^{(2k-1)\pi i/2}) = e^{k\nu\pi i} I_{\nu}(\xi_1 e^{-\pi i/2}) \\ &= e^{(k-\frac{1}{2})\nu\pi i} J_{\nu}(\xi_1), \quad (-\frac{\pi}{2} < \arg \xi_1 \leq \pi). \end{aligned}$$

Similarly,

$$I_{-\nu}(\zeta) = e^{-(k-\frac{1}{2})\nu\pi i} J_{-\nu}(\xi_1), \quad (-\frac{\pi}{2} < \arg \xi_1 \leq \pi).$$

Therefore

$$\begin{aligned} (3.10) \quad B_n(z) &= (n+2)^{-\frac{1}{2}} x_1^{\frac{1}{2}} e^{(k-\frac{1}{2})\nu\pi i} [e^{-(k-\frac{1}{2})\nu\pi i} [e^{-(k-\frac{1}{2})\nu\pi i} J_{-\nu}(\xi_1) \\ &\quad + e^{(k-\frac{1}{2})\nu\pi i} J_{\nu}(\xi_1)], \\ &\quad -\frac{\pi}{2} < \arg \xi_1 \leq \pi. \end{aligned}$$

Hence the asymptotic form (2.3) yields

(3.11)

$$\begin{aligned} \Delta_{BCD} \arg B_n(z) &= -\frac{n}{4}(2\delta) + \Delta_{BCD} \arg [e^{-(k-\frac{1}{2})\nu\pi i} \cos(\xi_1 + \frac{\nu\pi}{2} - \frac{\pi}{4}) \\ &\quad + e^{(k-\frac{1}{2})\nu\pi i} \cos(\xi_1 - \frac{\nu\pi}{2} - \frac{\pi}{4})] + o(1). \end{aligned}$$

We now put $\xi = \alpha + i\beta$, (α, β real). Then, noting that $x_1 = x e^{\pi i/(n+2)}$ and therefore $\xi_1 = \xi e^{\pi i/2}$, we see that $\xi_1 = i\alpha - \beta$.

Hence

$$\begin{aligned} & \cos(\xi_1 + \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) \cdot e^{-(k-\frac{1}{2})\sqrt{\pi}i} + \cos(\xi_1 - \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) \cdot e^{(k-\frac{1}{2})\sqrt{\pi}i} \\ &= \cos(\beta + \frac{\pi}{4}) [e^{\alpha} \cos \frac{k\pi}{n+2} + e^{-\alpha} \cos \frac{(k-1)\pi}{n+2}] + i \sin(\beta + \frac{\pi}{4}) [e^{\alpha} \cos \frac{k\pi}{n+2} \\ & \quad - e^{-\alpha} \cos \frac{(k-1)\pi}{n+2}], \end{aligned}$$

after some simplification.

We now choose R so that $\beta + \frac{\pi}{4}$ is a multiple of π , say

$$(*) \quad \beta + \frac{\pi}{4} = -j\pi, \quad (j \text{ a positive integer}).$$

This is always possible since

$$(3.12) \quad \beta = \operatorname{Im} \xi = (-)^k \operatorname{Im} \zeta = -\frac{2}{n+2} R^{(n+2)/2} \cos(\frac{n}{2} + 1)\delta$$

along BCD, so that β is large and negative. With this choice of R , the expression in brackets on the right side of (3.11) is real and one-signed along BCD, so that

$$\Delta_{BCD} \arg B_n(z) = -\frac{n\delta}{2} + o(1).$$

Combining this with (3.7), (3.8) and (3.9), we find, upon use of (*),

$$\Delta_{OABCDEO} \arg B_n(z) = -2\beta - \frac{\pi}{2} + o(1) = 2j\pi + o(1).$$

Now let

$$C_v(\xi) = (n+2)^{-1/2} \omega^{k/2} [I_{-v}(\xi) + \omega^k I_v(\xi)].$$

Then

$$B_n(z) = B_n(\omega^k x) = x^{1/2} C_v(\xi).$$

Hence

$$\begin{aligned}\omega^k B'_n(z) &= \frac{1}{2} x^{-1/2} C_v(\xi) + x^{1/2} C'_v(\xi) \cdot x^{n/2} \\ &= \frac{1}{2} x^{-1/2} C_v(\xi) + x^{(n+1)/2} [C_{v-1}(\xi) - v\xi^{-1} C_v(\xi)] \\ &= x^{(n+1)/2} C_{v-1}(\xi)\end{aligned}$$

which leads to

$$(3.13) \quad B'_n(z) = (n+2)^{-1/2} \omega^{-k/2} x^{(n+1)/2} [I_{-v+1}(\xi) + \omega^k I_{v-1}(\xi)].$$

Noting that $\xi^{-1/2} x^{(n+1)/2} = 2^{-1/2} (n+2)^{1/2} x^{n/4}$, we deduce from (2.2) that

$$\Delta_{AB} \arg B'_n(z) = \frac{n}{4} \left(\frac{\pi}{n+2} - \delta \right) - \operatorname{Im} \xi \Big|_B + o(1)$$

$$\Delta_{DE} \arg B'_n(z) = \frac{n}{4} \left(\frac{\pi}{n+2} - \delta \right) - \operatorname{Im} \xi \Big|_D + o(1)$$

$$\Delta_{EO} \arg B'_n(z) = -\arg \omega^{-k} (1 + \omega^k) + o(1) = \frac{k\pi}{n+2} + o(1).$$

Replacing k by $k-1$ in (3.13) leads to

$$\Delta_{OA} \arg B'_n(z) = \arg \omega^{-k+1} (1 + \omega^{k-1}) + o(1) = -\frac{(k-1)\pi}{n+2} + o(1).$$

Using the recurrence relations

$$\xi_1 J'_v(\xi_1) + v J_v(\xi_1) = \xi_1 J_{v-1}(\xi_1)$$

$$\xi_1 J'_{-v}(\xi_1) + v J_{-v}(\xi_1) = -\xi_1 J_{-v+1}(\xi_1)$$

we can deduce from (3.10) that

(3.14)

$$\begin{aligned}B'_n(z) &= (n+2)^{-1/2} e^{-(k-1/2)v\pi i} x_1^{(n+1)/2} [e^{(k-1/2)v\pi i} J_{v-1}(\xi_1) \\ &\quad - e^{-(k-1/2)v\pi i} J_{-v+1}(\xi_1)], \\ &\quad \left(-\frac{\pi}{2} < \arg \xi_1 \leq \pi \right).\end{aligned}$$

Now

$$(3.15) \quad \begin{aligned} \cos[\xi_1 - (\nu-1) \frac{\pi}{2} - \frac{\pi}{4}] &= -\sin(\xi_1 - \frac{\nu\pi}{2} - \frac{\pi}{4}) \\ \cos[\xi_1 - (-\nu+1) \frac{\pi}{2} - \frac{\pi}{4}] &= \sin(\xi_1 + \frac{\nu\pi}{2} - \frac{\pi}{4}) \end{aligned}$$

and

$$(3.16) \quad \begin{aligned} &-e^{(k-\frac{1}{2})\nu\pi i} \sin(\xi_1 - \frac{\nu\pi}{2} - \frac{\pi}{4}) - e^{-(k-\frac{1}{2})\nu\pi i} \sin(\xi_1 + \frac{\nu\pi}{2} - \frac{\pi}{4}) \\ &= -\sin(\beta + \frac{\pi}{4}) [e^{\alpha} \cos \frac{k\pi}{n+2} + e^{-\alpha} \cos \frac{(k-1)\pi}{n+2}] \\ &\quad -i \cos(\beta + \frac{\pi}{4}) [e^{\alpha} \cos \frac{k\pi}{n+2} - e^{-\alpha} \cos \frac{(k-1)\pi}{n+2}] \end{aligned}$$

after some simplification.

If R is chosen so that $\beta + \frac{\pi}{4}$ is an odd multiple of $\frac{\pi}{2}$, say,

$$\beta + \frac{\pi}{4} = (-2j + 1) \frac{\pi}{2}, \quad (j \text{ a positive integer})$$

then the right side of (3.16) is real and one-signed on BCD.

It then follows from (2.3), (3.14), (3.15) and (3.16) that

$$\Delta_{BCD} \arg B'_n(z) = \frac{n\delta}{2} + o(1)$$

for the above choice of R . Hence

$$\Delta_{OABCDEO} \arg B'_n(z) = -2\beta + \frac{\pi}{2} + o(1) = 2j\pi + o(1).$$

This completes the proof of the theorem.

Remark: Theorems 2 and 3 include as a special case ($n=1$) the distribution of the zeros [6] of the Airy function of the second kind. Theorem 3 shows that the A_n functions of the second kind do have zeros in S'_k .

If n is even, the fraction V_n of the plane containing

zeros of $F_1(z)$ or $F'_1(z)$ is $\frac{n^2}{4}(n+1)(n+2) = \frac{1}{4} - \frac{(3n+2)}{4(n+1)(n+2)}$,

so that

$$\frac{1}{12} \leq v_n < \frac{1}{4}.$$

If n is odd, $v_n = \frac{n-1}{4(n+2)}$, so that

$$0 \leq v_n < \frac{1}{4}.$$

Similarly, if v'_n is the fraction of the plane containing zeros of $F_2(z)$ or $F'_2(z)$, then

$$v'_n = \frac{(n+2)}{4(n+1)} \quad \text{for } n \text{ even}$$

$$v'_n = \frac{(n+1)}{4(n+2)} \quad \text{for } n \text{ odd}$$

Hence

$$\frac{1}{4} < v'_n \leq \frac{1}{3} \quad \text{for } n \text{ even}$$

$$\frac{1}{6} \leq v'_n < \frac{1}{4} \quad \text{for } n \text{ odd.}$$

4. Bounds for A_n functions.

LEMMA 5: The following functions are bounded (for finite n):

$$(4.1) \quad (1 + |z|^{\frac{n}{4}})A_n(z)e^{\zeta}, \quad (1 + |z|^{\frac{n}{4}})^{-1}A'_n(z)e^{\zeta}, \quad |\arg z| \leq \frac{3\pi}{n+2} - \varepsilon$$

($\varepsilon > 0$)

$$(4.2) \quad (1 + |z|^{\frac{n}{4}})B_n(z)e^{-\zeta}, \quad (1 + |z|^{\frac{n}{4}})^{-1}B'_n(z)e^{-\zeta}, \quad |\arg z| \leq \frac{\pi}{n+2} - \varepsilon$$

$$(4.3) \quad (1 + |z|^{\frac{n}{4}})A_n(z)\exp(-)^{k+1}\zeta, \quad (1 + |z|^{\frac{n}{4}})^{-1}A'_n(z)\exp(-)^{k+1}\zeta,$$

$$\frac{(2k-1)\pi}{n+2} + \varepsilon \leq |\arg z| \leq \frac{(2k+1)\pi}{n+2} - \varepsilon,$$

$$(k = 1, 2, \dots), \quad (0 < \varepsilon < \frac{\pi}{n+2})$$

$$(4.4) \quad (1 + |z|^{\frac{n}{4}})B_n(z)\exp(-)^{k+1}\zeta, \quad (1 + |z|^{\frac{n}{4}})^{-1}\exp(-)^{k+1}\zeta,$$

$$(k=1, 2, \dots), \quad \frac{(2k-1)\pi}{n+2} + \varepsilon \leq \arg z \leq \frac{(2k+1)\pi}{n+2} - \varepsilon.$$

PROOF: This is trivial for the functions $(1+|z|^{\frac{n}{4}})A_n(z)e^{\zeta}$ and $(1+|z|^{\frac{n}{4}})B_n(z)e^{-\zeta}$, being a direct consequence of the asymptotic forms (2.4) and (2.6). To prove the corresponding result for the functions involving derivatives, we note that

$$A_n(z) = M z^{\frac{1}{2}} K_{\nu}(\zeta), \quad M = 2(n+2)^{-1} \pi^{-1} \sin\left(\frac{\pi}{n+2}\right)$$

Hence

$$\begin{aligned} A'_n(z) &= M \left[\frac{1}{2} z^{-\frac{1}{2}} K_{\nu}(\zeta) + z^{\frac{1}{2}} K'_{\nu}(\zeta) \cdot z^{\frac{n}{2}} \right] \\ &= M \left[\frac{1}{2} z^{-\frac{1}{2}} K_{\nu}(\zeta) + z^{(n+1)/2} \{ \nu \zeta^{-1} K_{\nu}(\zeta) - K_{\nu-1}(\zeta) \} \right] \\ &= -M z^{(n+1)/2} K_{\nu-1}(\zeta). \end{aligned}$$

Use of the asymptotic form (2.1) yields the desired result for $A'_n(z)$. Moreover, by definition,

$$B_n(z) = z^{\frac{1}{2}} C_\nu(\zeta), \quad C_\nu(\zeta) = (n+2)^{-\frac{1}{2}} [I_{-\nu}(\zeta) + I_\nu(\zeta)].$$

Hence

$$\begin{aligned} B'_n(z) &= \frac{1}{2} z^{-\frac{1}{2}} C_\nu(\zeta) + z^{\frac{1}{2}} C'_\nu(\zeta) \cdot z^{\frac{n}{2}} \\ &= \frac{1}{2} z^{-\frac{1}{2}} C_\nu(\zeta) + \frac{1}{2} z^{(n+1)/2} [C_{\nu-1}(\zeta) - \nu \zeta^{-1} C_\nu(\zeta)] \\ &= z^{(n+1)/2} C_{\nu-1}(\zeta), \end{aligned}$$

upon use of the recurrence relations

$$\begin{aligned} \zeta I'_\nu(\zeta) + \nu I_\nu(\zeta) &= \zeta I_{\nu-1}(\zeta) \\ \zeta I'_{-\nu}(\zeta) + \nu I_{-\nu}(\zeta) &= \zeta I_{-\nu+1}(\zeta). \end{aligned}$$

It now follows from the asymptotic form (2.2) that the function $(1+|z|^{n/4})^{-1} B'_n(z) e^{-\zeta}$ is bounded in the sector $|\arg z| < \frac{\pi}{n+2}$.

To see that the functions (4.3) and (4.4) are bounded in the stated sectors, we use the relations

$$\begin{aligned} A_n(z) &= (n+2)^{-1} x_2^{\frac{1}{2}} [I_{-\nu}(\xi_2) - \omega^{k+1} I_\nu(\xi_2)], \quad z = x_2 \omega^{k+1} \\ &\quad \xi_2 = \frac{2}{n+2} x_2^{(n+2)/2} \\ B_n(z) &= (n+2)^{-\frac{1}{2}} x_2^{\frac{1}{2}} [I_{-\nu}(\xi_2) + \omega^{k+1} I_\nu(\xi_2)]. \end{aligned}$$

As noted in section 3, $e^{-\xi_2}$ is dominant in the sector $-\frac{3\pi}{2} < \arg \xi_2 < -\frac{\pi}{2}$. This implies, upon use of the asymptotic forms (2.2), that the functions $(1+|z|^{n/4}) A_n(z) e^{\xi_2}$, $(1+|z|^{n/4})^{-1} A'_n(z) e^{\xi_2}$, $(1+|z|^{n/4}) B_n(z) e^{\xi_2}$, $(1+|z|^{n/4})^{-1} B'_n(z) e^{\xi_2}$

are bounded in this sector. Hence the functions (4.3), (4.4) are bounded in the stated sectors, since $\xi_2 = (-)^{k+1} \zeta$. This completes the proof of the lemma.

This result extends known properties [8] of Airy functions.

LEMMA 6: The reciprocals of the functions (4.1), (4.2) are bounded in the sectors $|\arg z| \leq \frac{2\pi}{n+1} - \varepsilon$, $|\arg z| \leq \frac{\pi}{n+1} - \varepsilon$ respectively where $0 < \varepsilon < \frac{\pi}{n+2}$. If k is any integer in the range $0 < 2k < n+1$, the reciprocals of the functions (4.3) are bounded in the sectors $\frac{(2k-1)\pi}{n+2} + \varepsilon \leq |\arg z| \leq \frac{2k\pi}{n+1} - \varepsilon'$, where $0 < \varepsilon, \varepsilon' < \frac{\pi}{n+2}$; for all other admissible values of k , they are bounded in the same sector as (4.3). The reciprocals of the functions (4.4) are bounded in the same sector as (4.4) provided $2k \geq n+3$; for all other admissible values of k the reciprocals of (4.4) are bounded in the sectors $\frac{(2k-1)\pi}{n+1} + \varepsilon \leq |\arg z| \leq \frac{(2k+1)\pi}{n+2} - \varepsilon'$.

PROOF: It was shown in section 3 that the zeros of $A_n(z)$, $A'_n(z)$ lie on the negative real axis (when there are real zeros) or in the region $\bigcup_k S_k$, where S_k is the double sector $\frac{2k\pi}{n+1} < |\arg z| < \frac{(2k+1)\pi}{n+2}$ and k takes integer values in the range $0 < 2k < n+1$. It was also shown there that the zeros of $B_n(z)$, $B'_n(z)$ lie on the negative axis (if there are real zeros) or in the region $\bigcup_k S'_k$, where S'_k is the double sector $\frac{(2k+1)\pi}{n+2} < |\arg z| < \frac{(2k-1)\pi}{n+1}$ and k takes integer values in the range $0 < 2k < n+3$. The lemma now follows from these results and the preceding lemma.

Bounds for sectors containing the rays $|\arg z| = \frac{(2k-1)\pi}{n+2}$,
 $(k = 1, 2, \dots)$ are given below.

LEMMA 7: The following functions and their reciprocals are bounded in the stated sectors:

$$(4.5) \quad \begin{aligned} & (1 + |z|^{\frac{n}{4}})[|A_n(z)|^2 + |B_n(z)|^2]^{\frac{1}{2}} \exp(-)^k \zeta, \\ & (1 + |z|^{\frac{n}{4}})^{-1}[|A'_n(z)|^2 + |B'_n(z)|^2]^{\frac{1}{2}} \exp(-)^k \zeta \\ & \frac{(2k-1)\pi}{n+2} - \varepsilon \leq |\arg z| \leq \frac{(2k-1)\pi}{n+2}, \quad (k=1, 2, \dots) \\ & (0 < \varepsilon < \frac{\pi}{n+2}). \end{aligned}$$

PROOF: Let $z = x_1 \exp[\frac{(2k-1)\pi i}{n+2}]$, $\xi_1 = \frac{2}{n+2} x_1^{\frac{n+2}{2}}$.

This substitution leads to the relations

$$(4.6) \quad I_{\vee}(\zeta) = e^{(k-\frac{1}{2})\sqrt{\pi}i} J_{\vee}(\xi_1), \quad I_{-\vee}(\zeta) = e^{-(k-\frac{1}{2})\sqrt{\pi}i} J_{-\vee}(\xi_1),$$

$$(-\frac{\pi}{2} < \arg \xi_1 \leq \pi).$$

Since $\xi_1 = \frac{2}{n+2} z^{\frac{n+2}{2}} e^{-\frac{(2k-1)\pi i}{2}}$, these relations hold for the

sector $\frac{(2k-2)\pi}{n+2} < \arg z \leq \frac{(2k+1)\pi}{n+2}$. Now let

$$F(z) = \cos(\xi_1 + \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) e^{-(k-\frac{1}{2})\sqrt{\pi}i} - \cos(\xi_1 - \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) e^{(k-\frac{1}{2})\sqrt{\pi}i}$$

$$G(z) = \cos(\xi_1 + \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) e^{-(k-\frac{1}{2})\sqrt{\pi}i} + \cos(\xi_1 - \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}) e^{(k-\frac{1}{2})\sqrt{\pi}i}.$$

Direct computation shows that

$$\begin{aligned}
 F(z) = & \sin \frac{\pi}{4} \cdot \left[\sin \frac{k\pi}{n+2} \exp(-)^k \zeta - \sin \frac{(k-1)\pi}{n+2} \exp(-)^{k+1} \zeta \right] \\
 (4.7) \quad & -i \cos \frac{\pi}{4} \left[\sin \frac{k\pi}{n+2} \exp(-)^k \zeta + \sin \frac{(k-1)\pi}{n+2} \exp(-)^{k+1} \zeta \right]
 \end{aligned}$$

$$\begin{aligned}
 G(z) = & \cos \frac{\pi}{4} \left[\cos \frac{k\pi}{n+2} \exp(-)^k \zeta + \cos \frac{(k-1)\pi}{n+2} \exp(-)^{k+1} \zeta \right] \\
 & +i \sin \frac{\pi}{4} \left[\cos \frac{k\pi}{n+2} \exp(-)^k \zeta - \cos \frac{(k-1)\pi}{n+2} \exp(-)^{k+1} \zeta \right].
 \end{aligned}$$

When $\arg z = \frac{(2k-1)\pi}{n+2}$, $\arg \zeta = \frac{(2k-1)\pi}{2}$, so that $e^{\pm \zeta} = e^{\pm i \operatorname{Im} \zeta}$

and therefore the functions $(1+|z|^{n/4})[|A_n(z)|^2 + |B_n(z)|^2]^{1/2} e^{\pm \zeta}$ are bounded on the rays $\arg z = \frac{(2k-1)\pi}{n+2}$, $(k=1, 2, \dots)$, upon use of (4.6), (4.7) and the asymptotic form (2.3). Recalling the notation of section 3, we note that $\exp(-)^{k+1} \zeta (=e^{-\xi})$ is dominant in the sector $\frac{(2k-3)\pi}{n+2} < \arg z < \frac{(2k-1)\pi}{n+2}$. Hence the functions (4.5) are bounded in the stated sectors.

Moreover, $A_n(z)$ and $B_n(z)$ have no common zeros, their Wronskian being non-zero, so that the reciprocals of the functions (4.5) are bounded in the same sectors as the functions themselves. This completes the proof of the lemma.

COROLLARY: The functions (4.5) and their reciprocals are bounded in the zero-containing regions.

PROOF: Choose ε so that the domain of definition of (4.5) includes the zero-containing regions. This is always possible, since the S_k , S'_k are restricted by (3.5) and (3.6).

We have therefore obtained bounds for A_n functions on the entire z -plane.

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