MAXIMAL ABELIAN SUBALGEBRAS OF VON NEUMANN ALGEBRAS

by

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ABSTRACT

We are concerned with constructing examples of maximal abelian von Neumann subalgebras (MA subalgebras) in hyperfinite factors of type III. Our results will show that certain phenomena known to hold for the hyperfinite factor of type II_1 also hold for type III factors.

Let $\mathcal{M}$ and $\mathcal{N}$ be subalgebras of the factor $\mathfrak{a}$. We call $\mathcal{M}$ and $\mathcal{N}$ equivalent if $\mathcal{M}$ is the image of $\mathcal{N}$ by some automorphism of $\mathfrak{a}$. Let $N(\mathcal{M})$ denote the subalgebra of $\mathfrak{a}$ generated by all those unitary operators in $\mathfrak{a}$ which induce automorphisms of $\mathcal{M}$, and let $N^2(\mathcal{M})$, $N^3(\mathcal{M}), \ldots$ be defined in the obvious inductive fashion. Following J. Dixmier and S. Anastasio, we call a MA subalgebra $\mathcal{M}$ of $\mathfrak{a}$ singular if $N(\mathcal{M}) = \mathcal{M}$, regular if $N(\mathcal{M}) = \mathfrak{a}$, semi-regular if $N(\mathcal{M})$ is a factor distinct from $\mathfrak{a}$, and m-semi-regular $(m \geq 2)$ if $N(\mathcal{M}), \ldots N^{m-1}(\mathcal{M})$ are not factors but $N^m(\mathcal{M})$ is a factor.

The MA subalgebras of the hyperfinite II_1 factor $\mathfrak{B}$ have received much attention in the literature, in the papers of J. Dixmier, L. Pukánszky, Sister R. J. Tauer, and S. Anastasio. It is known that $\mathfrak{B}$ contains a MA subalgebra of each type. Further, $\mathfrak{B}$ contains pairwise inequivalent sequences of singular, semi-regular, 2-semi-regular, and 3-semi-regular MA subalgebras.

The only hitherto known example of a MA subalgebra in a type III factor is regular. In 1956 Pukánszky gave a general method for constructing MA subalgebras in a class of (probably non-hyperfinite) type III factors. Because of an error in a calculation, the types of these subalgebras is not known.

The main result of this thesis is the construction, in each of the uncountably many mutually non-isomorphic hyperfinite type III factors of R. Powers, of:

(i) a semi-regular MA subalgebra

(ii) two sequences of mutually inequivalent 2-semi-regular MA subalgebras

(iii) two sequences of mutually inequivalent 3-semi-regular MA subalgebras.

Let $\mathfrak{A}$ denote one of these type III factors and let $\mathfrak{B}$ denote the hyperfinite II_1 factor. Roughly speaking, whenever a non-singular MA subalgebra of $\mathfrak{B}$ is constructed by means of group operator algebras, our method will produce a MA subalgebra of $\mathfrak{A}$ of the same type.

H. Araki and J. Woods have shown that $\mathfrak{A} \otimes \mathfrak{B} \cong \mathfrak{A}$, and it is therefore only necessary to construct MA subalgebras of $\mathfrak{A} \otimes \mathfrak{B}$ of the desired type. We obtain MA subalgebras of $\mathfrak{A} \otimes \mathfrak{B}$ by tensoring a MA subalgebra in $\mathfrak{A}$ with one in $\mathfrak{B}$. In order to determine the type of such a MA subalgebra, we realize $\mathfrak{B}$ as a constructible algebra and then regard $\mathfrak{A} \otimes \mathfrak{B}$ as a constructible algebra; this allows us to consider operators in $\mathcal{A} \otimes \mathcal{B}$ as functions from a group into an abelian von Neumann algebra.

As a corollary to our calculations, we are able to construct mutually inequivalent sequences of 2-semi-regular and 3-semi-regular MA subalgebras of the hyperfinite II_1 factor which differ from those of Anastasio.

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1 REVIEW OF VON NEUMANN ALGEBRAS

In general, our notation and terminology is that of Dixmier's book [6].

A Hilbert space $\mathcal{H}$ is a non-zero vector space over the complex numbers $\mathbb{C}$ together with an inner product $x,y \to (x,y)$ such that $\mathcal{H}$ is complete with respect to the norm $x \to \|x\| = (x,x)^{\frac{1}{2}}$. By an operator on $\mathcal{H}$ we mean a bounded (equivalently: norm-continuous) linear transformation of $\mathcal{H}$ into $\mathcal{H}$. We use $\mathcal{L}(\mathcal{H})$ to denote the algebra of all operators on $\mathcal{H}$, $I_{\mathcal{H}}$ (or I, when $\mathcal{H}$ is understood) to denote the identity operator on $\mathcal{H}$, and $C_{\mathcal{H}}$ to denote the scalar multiples of $I_{\mathcal{H}}$. If $\mathcal{G} \subset \mathcal{H}$, $[\mathcal{G}]$ is the smallest closed linear subspace of $\mathcal{H}$ containing $\mathcal{G}$, and $\mathrm{pr}[\mathcal{G}]$ is the (orthogonal) projection onto this subspace. If $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$, $\mathfrak{A}'$ is the set of all those $B \in \mathcal{L}(\mathcal{H})$ such that $AB = BA$ for all $A \in \mathfrak{A}$; $\mathfrak{A}'$ is called the commutant of $\mathfrak{A}$. A von Neumann algebra (or ring of operators) on $\mathcal{H}$ is a $*$-algebra of operators on $\mathcal{H}$ satisfying $\mathfrak{A}'' = \mathfrak{A}$. If $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ is arbitrary, $\mathcal{R}(\mathfrak{A})$, the smallest von Neumann algebra on $\mathcal{H}$ containing $\mathfrak{A}$, is easily seen to be $(\mathfrak{A} \cup \mathfrak{A}^*)''$. This algebraic definition of a von Neumann algebra (which is used by Dixmier in his book [6]) is equivalent to the topological one originally employed by von Neumann: $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ is a von Neumann algebra if $\mathfrak{A}$ is a weakly closed $*$-algebra containing $I_{\mathcal{H}}$. The equivalence of these two definitions is a part of the following more general result, known as the Double Commutant Theorem (see [6; p.44], [7; p.885], or [14; §2]): if $\mathcal{A}$ is a $*$-algebra of operators on $\mathcal{H}$ which contains $I_{\mathcal{H}}$, then $\mathcal{R}(\mathcal{A}) = \mathcal{A}''$ is the closure of $\mathcal{A}$ in each of the four topologies: weak, strong, ultraweak, and ultrastrong - on $\mathcal{L}(\mathcal{H})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. An isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ is a linear and multiplicative map ϕ of $\mathcal{A}$ onto $\mathcal{B}$ which satisfies $\phi(A^*) = (\phi(A))^*$ for all $A \in \mathcal{A}$. If there is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, and we write $\mathcal{A} \cong \mathcal{B}$. It turns out that an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ is necessarily ultraweakly and ultrastrongly bicontinuous [6; p.57]. An isomorphism ϕ of $\mathcal{A}$ onto $\mathcal{B}$ is called spatial if there is a linear isometry $\mathcal{V}$ of $\mathcal{H}$ onto $\mathcal{K}$ such that $\phi(A) = \mathcal{V}A\mathcal{V}^{-1}$ for all $A \in \mathcal{A}$.

Let $\mathcal{A}$ be a von Neumann algebra on $\mathcal{H}$. A trace on $\mathcal{A}^+ = \{A \in \mathcal{A} : A \geq 0\}$ is a mapping $\omega : \mathcal{A}^+ \to [0,\infty) \cup \{\infty\}$ which satisfies the following:

(i) for all $S,T \in \mathcal{A}^+$, $\omega(S + T) = \omega(S) + \omega(T)$

(ii) for all $S \in \mathcal{A}^+$ and all $\lambda \geq 0$, $\omega(\lambda S) = \lambda\omega(S)$ (where the convention $0 \cdot \infty = 0$ is used)

(iii) for all $S \in \mathcal{A}^+$ and all unitary $U \in \mathcal{A}$, $\omega(USU^*) = \omega(S)$.

The trace ω on $\mathfrak{A}^+$ is called

(a) finite, if $\omega(I) < \infty$

(b) semi-finite, if, given $T \in \mathfrak{A}^+ - \{0\}$, there is an $S \in \mathfrak{A}^+$ with $0 < S \leq T$ and $\omega(S) < \infty$

(c) faithful, if $S \in \mathfrak{A}^+$ and $\omega(S) = 0$ imply $S = 0$

(d) normal, if, whenever $\mathfrak{F}$ is an upwardly-directed set in $\mathfrak{A}^+$ with least upper bound $T \in \mathfrak{A}^+$, then $\omega(T) = \sup\{\omega(S) : S \in \mathfrak{F}\}$.

A factor on $\mathfrak{H}$ is a von Neumann algebra $\mathfrak{A}$ on $\mathfrak{H}$ with $\mathfrak{A} \cap \mathfrak{A}' = \mathbb{C}_{\mathfrak{H}}$. It is the factors that have received the most attention in the literature. Their extreme non-commutativity actually makes them relatively easy to study; moreover, every von Neumann algebra looks locally like a factor, and in fact is built up from factors by means of the direct integral [17]. The comparison theorem ([6; p. 338] or [12; Theorem VI]) implies that if ω is a normal trace on $\mathfrak{A}^+$, where $\mathfrak{A}$ is a factor, then one of the following must be the case:

(i) $\omega(A) = 0$ for all $A \in \mathfrak{A}^+$

(ii) $\omega(A) = \infty$ for all $A \in \mathfrak{A}^+ - \{0\}$

(iii) ω is faithful and semi-finite.

Moreover, to within a positive multiple, there is at most one non-trivial normal trace on $\mathfrak{A}^+$. A factor $\mathfrak{A}$ such that there is no normal non-zero semi-finite trace on $\mathfrak{A}^+$ is said to be of type III. If a factor $\mathfrak{A}$ is not of type III there is a normal faithful semi-finite trace ω on $\mathcal{G}^+$ which, in some normalization, must satisfy one of:

(i) $\omega(\mathcal{G}^p) = \{0,1,\ldots,n\}$ for some integer $n \geq 1$

(ii) $\omega(\mathcal{G}^p) = \{0,1,\ldots,\infty\}$

(iii) $\omega(\mathcal{G}^p) = [0,1]$

(iv) $\omega(\mathcal{G}^p) = [0,\infty)\cup\{\infty\}$,

where $\mathcal{G}^p$ is the set of projections in $\mathcal{G}$. In case (i), $\mathcal{G}$ is said to be of type I_n; in this case $\mathcal{G}$ is isomorphic to the algebra of all $n\times n$ matrices with complex entries. In case (ii), $\mathcal{G}$ is said to be of type I_∞; there is a unique infinite cardinal α such that $\mathcal{G}$ is isomorphic to the algebra of all bounded linear operators on an α-dimensional Hilbert space. If (iii) holds, $\mathcal{G}$ is of type II_1, and if (iv) holds, of type II_∞. It is clear that the notion of a factor and its type are invariant under isomorphisms. Given that factors of each type exist on separable Hilbert spaces, the tensor product enables one to construct factors of each type on arbitrary infinite-dimensional Hilbert spaces.

Let $\mathcal{H}$ be separable infinite-dimensional Hilbert space. At present, three [two] non-isomorphic factors on $\mathcal{H}$ of type II_1 [II_∞] are known ([23; p. 3.85], [24]]). In this thesis, the only factor of type II_1 which is of interest is the hyperfinite one. In general, a factor $\mathcal{G}$ on $\mathcal{H}$ is called hyperfinite if it is generated by an increasing sequence $(\mathcal{G}_n)$ with each $\mathcal{G}_n$ a factor of type I_{2^n}. Murray and von Neumann showed that all hyperfinite factors of type II_1

on $\mathcal{H}$ are isomorphic [13; Theorem XIV] (see also [6; p.291]); hence one can speak of the hyperfinite II_1 factor on $\mathcal{H}$. Recently, Powers [19] announced the existence of an uncountable number of pairwise non-isomorphic hyperfinite factors of type III on $\mathcal{H}$ (for the proof, see [18]; in [2] Araki and Woods give a different proof of this result). It is these factors that we shall be primarily concerned with in this thesis. Two non-isomorphic non-hyperfinite factors of type III have been constructed on $\mathcal{H}$, one by Pukánszky [20] and one by Schwartz [25].

The remainder of this section discusses the three methods which we employ to obtain von Neumann algebras. These constructions - the group operator algebra, the constructible algebra, and the infinite tensor product - are all due to Murray and von Neumann.

Let G be a group with identity e. We use $\hat{G}$ to denote the Hilbert space with orthonormal basis $(\hat{g})_{g \in G}$; notice that $\hat{G}$ is separable whenever G is at most countable. For each $g \in G$ there is a unique unitary operator V_g on $\hat{G}$ satisfying

$$(1.1) \qquad V_g \hat{h} = (gh)^\wedge \qquad \text{for all } h \in G .$$

This defines a unitary representation $g \to V_g$ of G on $\hat{G}$. The group operator algebra over the group G is the von Neumann algebra $\mathcal{C}_G = \mathcal{R}(V_g : g \in G)$ on $\hat{G}$ (for a complete discussion of the group operator algebra, see either [6; pp. 301-303] or [13; §5.3]). Alternatively, $\mathcal{C}_G$ can be described as the set of all those operators T on $\hat{G}$ with $(T\hat{g}, \hat{h}) = (T\hat{e}, (hg^{-1})^\wedge)$ for all $g,h \in G$. The algebra $\mathcal{C}_G$ is a factor if and only if G has the infinite conjugate class property, i.e., whenever

$$\{hgh^{-1} : h \in G\} \text{ is infinite whenever } g \neq e; \tag{1.2}$$

in this case, $\mathcal{C}_G$ is necessarily of type II_1. If $\hat{G}$ is separable and if $\mathcal{C}_G$ is a factor, then $\mathcal{C}_G$ is hyperfinite whenever G is hyperfinite, i.e.,

$$G = \bigcup_{n=1}^{\infty} G_n \text{, where } G_1 \subset G_2 \subset \cdots \text{ and each } G_n \text{ is a finite subgroup of } G. \tag{1.3}$$

Before proceeding to the constructible algebra, we will briefly consider the tensor product of two Hilbert spaces. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with orthonormal basis $(\varphi_i)_{i\in I}$ and $(\psi_j)_{j\in J}$, respectively. Then $(\varphi_i \otimes \psi_j)_{(i,j)\in I\times J}$ is an orthonormal basis for $\mathcal{H}\otimes\mathcal{K}$. For each $j \in J$ we denote by ϕ_j the canonical embedding $x \to x \otimes \psi_j$ of $\mathcal{H}$ into $\mathcal{H}\otimes\mathcal{K}$. Given $A \in \mathcal{L}(\mathcal{H}\otimes\mathcal{K})$, the $\phi_j^* A \phi_k$ (which are operators on $\mathcal{H}$) are called the matrix elements of A relative to the orthonormal basis $(\psi_j)_{j\in J}$; an operator on $\mathcal{H}\otimes\mathcal{K}$ is completely determined by its matrix elements.

<u>Lemma 1.1</u> With the notation of the preceding paragraph:

(i) for each $j,k \in J$ and $A,B \in \mathcal{L}(\mathcal{H}\otimes\mathcal{K})$,

$$\phi_j^* AB \phi_k = \sum_{\ell\in J} (\phi_j^* A \phi_\ell)(\phi_\ell^* B \phi_k) ,$$

the sum converging in the strong topology on $\mathcal{L}(\mathcal{H})$

(ii) If $(A_\alpha)_{\alpha\in D}$ is a net in $\mathcal{L}(\mathcal{H}\otimes\mathcal{K})$ which converges weakly to an $A \in \mathcal{L}(\mathcal{H}\otimes\mathcal{K})$, then for each $j,k \in J$, $(\phi_j^* A_\alpha \phi_k)_{\alpha\in D}$ converges weakly to $\phi_j^* A \phi_k$.

<u>Proof</u>: Simple calculations (see [6; pp. 23-24] or [12; §2.4]).

Constructible algebras were first considered by Murray and von Neumann in [12] and [16], and further developed by Dixmier in [6; pp. 127-137]; our notation and terminology is taken from Bures [3]. The system $[\mathfrak{M},\mathcal{H},G,g \to U_g]$ is called a C-system if $\mathfrak{M}$ is a maximal abelian von Neumann algebra on the Hilbert space $\mathcal{H}$, if G is a group, and if $g \to U_g$ is a unitary representation of G on $\mathcal{H}$ with $U_g \mathfrak{M} U_g^* = \mathfrak{M}$ for all $g \in G$. Let $[\mathfrak{M},\mathcal{H},G,g \to U_g]$ be a C-system. Finite linear combinations of the operators $(M\otimes I_{\hat{G}})(U_g\otimes V_g)$, $M \in \mathfrak{M}$ and $g \in G$, form a $*$-algebra on $\mathcal{H}\otimes\hat{G}$ (V_g as in (1.1)); we use $\mathfrak{C}[\mathfrak{M},\mathcal{H},G,g \to U_g]$ to denote the von Neumann algebra on $\mathcal{H}\otimes\hat{G}$ generated by this $*$-algebra. If $A \in \mathfrak{C}[\mathfrak{M},\mathcal{H},G,g \to U_g]$, the matrix elements of A relative to the orthonormal basis $(\hat{g})_{g\in G}$ for $\hat{G}$ are such that for all $g,h \in G$, $\phi_g^* A \phi_h = \phi_{gh^{-1}}^* A \phi_e$ and

$\phi_g^* A \phi_e U_g^* \in \mathfrak{M}$. Setting $M_g = \phi_g^* A \phi_e U_g^*$, we obtain a family $(M_g)_{g\in G}$ in $\mathfrak{M}$ which completely determines A , and we write $A \sim [M_g : g \in G]$. Alternatively, the algebra $\mathfrak{G}[\mathfrak{M},\mathcal{H},G,g \to U_g]$ can be described as the set of all those $T \in \mathcal{L}(\mathcal{H}\otimes\hat{G})$ such that for some family $(M_g)_{g\in G}$ in $\mathfrak{M}$, $\phi_g^* T \phi_h = M_{gh^{-1}}U_{gh^{-1}}$ for all $g,h \in G$.

<u>Lemma 1.2</u> Let $[\mathfrak{M},\mathcal{H},G,g \to U_g]$ be a C-system and let A and B be operators in $\mathfrak{G}[\mathfrak{M},\mathcal{H},G,g \to U_g]$ with $A \sim [M_g : g \in G]$ and $B \sim [N_g: g \in G]$. For all $g,h \in G$ and $M \in \mathfrak{M}$:

(i) $\phi_g^* AB \phi_e U_g^* = \sum_{k\in G} M_{gk^{-1}} U_{gk^{-1}} N_k U_{gk^{-1}}^*$, where the sum converges in the strong topology on $\mathfrak{M}$

(ii) $\phi_g^* A^* \phi_e U_g^* = U_g M_{g^{-1}}^* U_g^*$

(iii) $\phi_g^*(M\otimes I)(U_h\otimes V_h) \phi_e U_g^* = \delta_{g,h}M$

(iv) $\phi_g^* A(U_h\otimes V_h) \phi_e U_g^* = M_{gh^{-1}}$

(v) $\phi_g^* (U_h\otimes V_h) A \phi_e U_g^* = U_h M_{h^{-1}g} U_h^*$.

<u>Proof</u>. Simple calculations.

<u>Definition 1.3</u> The C-system $[\mathfrak{M},\mathcal{H},G,g \to U_g]$ is called:

(i) free, if $\mathfrak{M} \cap U_g\mathfrak{M} = \{0\}$ for all $g \in G - \{e\}$

(ii) ergodic, if $\mathfrak{M} \cap \{U_g : g \in G\}' = \mathbb{C}_{\mathcal{H}}$.

A von Neumann algebra is called constructible if it is spatially isomorphic to $\mathfrak{G}[\mathfrak{M},\mathcal{H},G,g \to U_g]$ for some free C-system $[\mathfrak{M},\mathcal{H},G,g \to U_g]$.

Proposition 1.4 ([3; §4] and [4; §7]). The C-system $[\mathcal{M},\mathcal{H},G,g \to U_g]$ is free if and only if, for each $g \in G - \{e\}$, there exists a family $(E_i)_{i \in I}$ of projections in $\mathcal{M}$ such that $\sum_{i \in I} E_i = I$ and $E_i U_g E_i U_g^* = 0$ for all $i \in I$.

Proposition 1.5 ([6]). Let $[\mathcal{M},\mathcal{H},G,g \to U_g]$ be a free C-system, and let $\mathcal{G} = \mathcal{G}[\mathcal{M},\mathcal{H},G,g \to U_g]$. Then $\mathcal{M} \otimes \mathbb{C}_{\hat{G}}$ is a maximal abelian subalgebra of $\mathcal{G}$, and $\mathcal{G}$ is a factor if and only if $[\mathcal{M},\mathcal{H},G,g \to U_g]$ is ergodic. If $\mathcal{G}$ is a factor, then:

(i) $\mathcal{G}$ is of type I if and only if $\mathcal{M}$ contains a minimal projection; if n is the cardinality of a maximal family of pairwise orthogonal minimal projections in $\mathcal{M}$, then $\mathcal{G}$ is of type I_n

(ii) $\mathcal{G}$ is finite (i.e., of type II_1 or I_n, $n < \infty$) if and only if there is a normal finite faithful trace ω on $\mathcal{M}^+$ with $\omega(U_g M U_g^*) = \omega(M)$ for all $g \in G$ and all $M \in \mathcal{M}^+$

(iii) $\mathcal{G}$ is of type III if and only if there does not exist a normal semi-finite faithful trace ω on $\mathcal{M}^+$ with $\omega(U_g M U_g^*) = \omega(M)$ for all $g \in G$ and all $M \in \mathcal{M}^+$.

Proposition 1.6 ([8], [13; Lemma 5.2.3]). Let $[\mathcal{M},\mathcal{H},G,g \to U_g]$ be a free C-system, and suppose that $\mathcal{G}[\mathcal{M},\mathcal{H},G,g \to U_g]$ is a factor of type II_1. If G is abelian, then $\mathcal{G}[\mathcal{M},\mathcal{H},G,g \to U_g]$ is hyperfinite.

In [15] a complete discussion of the infinite tensor product of von Neumann algebras can be found. Let I be an infinite indexing set, and let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces. A family $(f_i)_{i\in I}$ will be called a C_0-sequence if each $f_i \in \mathcal{H}_i$ and if $\sum_{i\in I} |\, 1 - \| f_i \| \,| < \infty$. Two C_0-sequences $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$ are called equivalent if $\sum_{i\in I} |\, 1 - (f_i, g_i)| < \infty$; this is an equivalence relation on the set of all C_0-sequences.

Let $(f_i^0)_{i\in I}$ be a fixed C_0-sequence, and let $\mathcal{E}_0$ denote the equivalence class determined by $(f_i^0)_{i\in I}$. For each $(f_i)_{i\in I} \in \mathcal{E}_0$, let $\underset{i\in I}{\otimes} f_i$ denote the map $(g_i)_{i\in I} \to \prod_{i\in I} (f_i, g_i)$ of $\mathcal{E}_0$ into $\mathbb{C}$. Defining finite linear combinations of the $\underset{i\in I}{\otimes} f_i$ in the obvious manner, we obtain a vector space V. The map $(f_i)_{i\in I} \to \underset{i\in I}{\otimes} f_i$ of $\mathcal{E}_0$ into V is clearly multi-linear. The form $(\underset{i\in I}{\otimes} f_i, \underset{i\in I}{\otimes} g_i) \to \prod_{i\in I} (f_i, g_i)$ extends to a sesqui-linear form on V which can be shown to be an inner product. We will refer to the completion of V relative to this inner product, which we denote by $\underset{i\in I}{\otimes} (\mathcal{H}_i, f_i^0)$, as the infinite tensor product of the $\mathcal{H}_i$ relative to $(f_i^0)_{i\in I}$; von Neumann, reserving the phrase infinite tensor (direct) product for a much larger Hilbert space, called this space the $\mathcal{E}_0$-adic incomplete direct product. Note that $\underset{i\in I}{\otimes} (\mathcal{H}_i, f_i^0)$ really depends on $\mathcal{E}_0$, and not on the particular $(f_i^0)_{i\in I} \in \mathcal{E}_0$ selected. The following result facilitates working with the infinite tensor product space.

Proposition 1.7 ([15; Lemma 4.1.4 and Theorem VII]). Let I be an infinite indexing set, let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces, and for each $i \in I$, let f_i^0 be a unit vector in $\mathcal{H}_i$.

(i) For each $i \in I$, choose an orthonormal basis $(f_i^j)_{j\in J_i}$ with $0 \in J_i$ for $\mathcal{H}_i$. Let J be the set of all those $j \in \prod_{i\in I} J_i$ with $j(i) = 0$ for all but finitely many $i \in I$, and for each $j \in J$, let $f_j = \otimes_{i\in I} f_i^{j(i)}$. Then $(f_j)_{j\in J}$ is an orthonormal basis for $\otimes_{i\in I}(\mathcal{H}_i, f_i^0)$.

(ii) Let $I = \bigcup_{k\in K} I_k$ be a disjoint union. Then there is a unique linear isometry (called the associativity isomorphism) of $\otimes_{i\in I}(\mathcal{H}_i, f_i^0)$ onto $\otimes_{k\in K}(\otimes_{i\in I_k}(\mathcal{H}_i, f_i^0), \otimes_{i\in I_k} f_i^0)$ which carries $\otimes_{i\in I} f_i$ into $\otimes_{k\in K}(\otimes_{i\in I_k} f_i)$ for each C_0-sequence $(f_i)_{i\in I}$ equivalent to $(f_i^0)_{i\in I}$.

Let $(\mathcal{H}_i)_{i\in I}$ and $(f_i^0)_{i\in I}$ be as in Proposition 1.7, and let $\mathcal{H} = \otimes_{i\in I}(\mathcal{H}_i, f_i^0)$. If $T \in \mathcal{L}(\mathcal{H}_{i_0})$, there is a unique

$\alpha_{i_o}(T) \in \mathcal{L}(\mathcal{H})$ which satisfies

$$[\alpha_{i_o}(T)][\underset{i\in I}{\otimes} f_i] = (\underset{i\in I-\{i_o\}}{\otimes} f_i) \otimes (T f_{i_o})$$

for each C_o-sequence $(f_i)_{i\in I}$ equivalent to $(f_i^o)_{i\in I}$. It is easily seen that α_{i_o} is a $*$-isomorphism; following the usual notation, we write $\bar{T}$ for $\alpha_{i_o}(T)$. If $\mathcal{G}_{i_o}$ is a von Neumann algebra on $\mathcal{H}_{i_o}$, then $\bar{\mathcal{G}}_{i_o} = \{\bar{T} : T \in \mathcal{G}_{i_o}\}$ is a von Neumann algebra on $\mathcal{H}$. If, for each $i \in I$, $\mathcal{G}_i$ is a von Neumann on $\mathcal{H}_i$, then $\underset{i\in I}{\otimes}(\mathcal{G}_i, f_i^o)$ denotes the von Neumann algebra $\mathcal{R}(\bar{\mathcal{G}}_i : i \in I)$ on $\mathcal{H}$; we call $\underset{i\in I}{\otimes}(\mathcal{G}_i, f_i^o)$ the infinite tensor product of the $\mathcal{G}_i$ relative to $(f_i^o)_{i\in I}$.

<u>Proposition 1.8</u> ([3; §3]). Let $(\mathcal{H}_i)_{i\in I}$, $(f_i^o)_{i\in I}$ and $(\mathcal{G}_i)_{i\in I}$ be as above, and let $\mathcal{H} = \underset{i\in I}{\otimes}(\mathcal{H}_i, f_i^o)$, $\mathcal{G} = \underset{i\in I}{\otimes}(\mathcal{G}_i, f_i^o)$. Then:

(i) $\mathcal{G}$ is maximal abelian on $\mathcal{H}$ if each $\mathcal{G}_i$ is maximal abelian on $\mathcal{H}_i$

(ii) $\mathcal{G}$ is a factor if and only if each $\mathcal{G}_i$ is a factor

(iii) $\mathcal{G} = \mathcal{L}(\mathcal{H})$ if each $\mathcal{G}_i = \mathcal{L}(\mathcal{H}_i)$.

Let I be an infinite indexing set. For each $i \in I$, let G^i be a group with identity e^i , let $\mathcal{G}^i = \mathcal{G}[\mathfrak{M}^i, \mathcal{H}^i, G^i, g \to U_g^i]$, where $[\mathfrak{M}^i, \mathcal{H}^i, G^i, g \to U_g^i]$ is a free C-system, and let f_i^o be a unit vector in $\mathcal{H}^i$. Set $\mathcal{H} = \underset{i\in I}{\otimes}(\mathcal{H}^i, f_i^o)$, $\mathcal{M} = \underset{i\in I}{\otimes}(\mathcal{M}^i, f_i^o)$, $\mathcal{G} = \underset{i\in I}{\otimes}(\mathcal{G}^i, f_i^o \otimes (e^i)^\wedge)$, and let G be the weak direct product of the G^i. For each $g = (g^i)_{i\in I} \in G$, let $U_g = \prod_{i\in I} \overline{U^i_{g^i}}$ (a finite product in which the factors commute). From Proposition 1.7 it follows that there is a linear isometry γ of

$$\underset{i\in I}{\otimes}(\mathcal{H}^i \otimes (G^i)^\wedge\ ,\ f_i^o \otimes (e^i)^\wedge)$$

onto $\mathcal{H} \otimes \hat{G}$ with

$$\gamma(\underset{i\in I}{\otimes}(f_i \otimes (g^i)^\wedge)) = (\underset{i\in I}{\otimes} f_i) \otimes ((g^i)_{i\in I})^\wedge$$

whenever $(f_i)_{i\in I}$ is a C_o-sequence equivalent to $(f_i^o)_{i\in I}$ and $(g^i)_{i\in I} \in G$.

<u>Proposition 1.9</u> ([3; Proposition 4.1] and Proposition 1.4). With the notation of the previous paragraph, $[\mathcal{M}, \mathcal{H}, G, g \to U_g]$ is a free C-system which is ergodic if and only if each $[\mathcal{M}^i, \mathcal{H}^i, G^i, g \to U^i_g]$ is ergodic. The map $A \to \gamma A \gamma^{-1}$ is an isomorphism of $\mathcal{G}$ onto $\mathcal{G}[\mathcal{M}, \mathcal{H}, G, g \to U_g]$.

2 MAXIMAL ABELIAN SUBALGEBRAS: DEFINITIONS AND SOME KNOWN RESULTS

Only separable Hilbert spaces will be considered in the remainder of this thesis.

The first part of this section consists of the basic definitions which, to some extent, serve to classify the maximal abelian (MA) subalgebras of a factor. Next, a summary of the known results concerning MA subalgebras of the hyperfinite II_1 factor is given. We conclude this section with a complete classification of the MA subalgebras of $\mathcal{L}(\mathcal{H})$; although this result was known to von Neumann, its proof does not seem to appear explicitly in the literature.

If $\mathfrak{M}$ and $\mathfrak{N}$ are subalgebras of a von Neumann algebra $\mathfrak{A}$, we say that $\mathfrak{M}$ and $\mathfrak{N}$ are equivalent in $\mathfrak{A}$ (or simply equivalent, if $\mathfrak{A}$ is understood) if there is an automorphism of $\mathfrak{A}$ which carries $\mathfrak{M}$ onto $\mathfrak{N}$. This defines an equivalence relation on the collection of all subalgebras of $\mathfrak{A}$. One problem in the structure theory of von Neumann algebras is to classify up to equivalence all of the subalgebras of a given von Neumann algebra, i.e., the determination of all equivalence classes of subalgebras. This problem is, of course, extremely difficult. The multiplicity theories of Halmos [10] and of Segal [27] give solutions to the classification up to equivalence of the abelian subalgebras of a factor of type I acting on a Hilbert space of arbitrary dimension. For factors of type II_1, the analogous problem has been examined and some results have been obtained by Bures [4].

Recall that a subalgebra $\mathfrak{M}$ of a von Neumann algebra $\mathfrak{a}$ is MA in $\mathfrak{a}$ if and only if $\mathfrak{M}' \cap \mathfrak{a} = \mathfrak{M}$.

<u>Definition 2.1</u> Let $\mathfrak{M}$ be a subalgebra of the von Neumann algebra $\mathfrak{a}$. For each integer $m \geq 0$, we inductively define subalgebras $N^m(\mathfrak{M})$ of $\mathfrak{a}$ by:

$$N^o(\mathfrak{M}) = \mathfrak{M}$$

$$N^m(\mathfrak{M}) = \mathcal{R}(U \in \mathfrak{a} : U \text{ unitary and } UN^{m-1}(\mathfrak{M})\, U^* = N^{m-1}(\mathfrak{M})\ \} \qquad m \geq 1 .$$

We will write $N(\mathfrak{M})$ instead of $N^1(\mathfrak{M})$, and we call this the normalizer of $\mathfrak{M}$ (in $\mathfrak{a}$).

Notice that $(N^m(\mathfrak{M}))_{m=0,1,\ldots}$ is an expanding sequence of subalgebras of $\mathfrak{a}$.

<u>Definition 2.2</u> If $\mathfrak{M}$ is a MA subalgebra of factor $\mathfrak{a}$, we call $\mathfrak{M}$:

(i) regular if $N(\mathfrak{M}) = \mathfrak{a}$

(ii) semi-regular, if $N(\mathfrak{M})$ is a factor distinct from $\mathfrak{a}$

(iii) singular, if $N(\mathfrak{M}) = \mathfrak{M}$

(iv) m-semi-regular ($m \geq 1$ and an integer), if $\mathfrak{M}, N(\mathfrak{M}), \ldots, N^{m-1}(\mathfrak{M})$ are not factors but $N^m(\mathfrak{M})$ is a factor.

<u>Definition 2.3</u> Let $\mathcal{M}$ be a MA subalgebra of a von Neumann algebra $\mathcal{A}$, and let $m \geq 1$ be an integer. We say that $\mathcal{M}$ has:

(i) proper length m, if $N^{m-1}(\mathcal{M}) \neq \mathcal{A}$ but $N^{m}(\mathcal{M}) = \mathcal{A}$

(ii) improper length m, if

$$N^{m-1}(\mathcal{M}) \subsetneqq N^{m}(\mathcal{M}) = N^{m+1}(\mathcal{M}) \subsetneqq \mathcal{A} \quad .$$

The definitions of regular, semi-regular and singular MA subalgebras were first given by Dixmier [5], while the notion of m-semi-regularity is due to Anastasio [1]. Definition 2.3 is a refinement of Tauer's length of a MA subalgebra [28].

It is easy to see that if $\mathcal{M}$ and $\mathcal{N}$ are equivalent subalgebras of a von Neumann algebra $\mathcal{A}$, then so are $N(\mathcal{M})$ and $N(\mathcal{N})$. Consequently, each of the properties of Definitions 2.2 and 2.3 is an invariant of the equivalence class determined by a MA subalgebra.

The study of MA subalgebras of the hyperfinite II_1 factor was initiated by Dixmier in his seminal paper [5]. Let G be a group, and consider the group operator algebra $\mathcal{C}_G$ on $\hat{G}$. If G_o is a subgroup of G, let $N(G_o)$ be the normalizer of G_o in G, and let $\mathcal{M}(G_o) = \mathcal{R}(V_g : g \in G_o) \subset \mathcal{C}_G$. Dixmier showed that, under certain conditions on G and G_o, $\mathcal{M}(G_o)$ is a MA subalgebra of $\mathcal{C}_G$ and $N(\mathcal{M}(G_o)) = \mathcal{M}(N(G_o))$. Using these results and choosing suitable groups G and subgroups G_o, he constructed examples of a regular, a semi-regular and a singular MA subalgebra of the hyperfinite II_1 factor.

The groups used by Dixmier in these constructions may be described as follows. Let F be a countably infinite field which is the increasing union of a sequence of finite subfields (in particular, we may take for F the algebraic completion of a finite field), and let K be the multiplicative group of non-zero elements of F. The set $K \times F$ becomes a group under the operation

$$(a,b)(c,d) = (ac, ad + b) \quad .$$

The group $K \times F$ is hyperfinite and has the infinite conjugate class property (see the proof of Theorem 4.1). The subgroup $K \times \{0\}$ of $K \times F$ is its own normalizer and $\mathcal{M}(K \times \{0\})$ is a singular MA subalgebra of $\mathcal{C}_{K \times F}$, while $\{1\} \times F$ is a normal subgroup and $\mathcal{M}(\{1\} \times F)$ is a regular MA subalgebra. It is a bit more difficult to obtain a semi-regular MA subalgebra. Let H be the group of all non-singular 2×2 matrices over F and let L be the normal subgroup of H consisting of all scalar multiples of the identity matrix. Let $G = H/L$, let H_0 and H_1 be the subgroups of H with typical elements

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & c \\ 0 & 1 \end{pmatrix} , \quad b \neq 0 \ ,$$

respectively, and let $G_0 = H_0/L$. Then the normalizer of

G_o in G is H_1/L , and $\mathcal{M}(G_o)$ is a semi-regular MA subalgebra of $\mathcal{C}_G$.

Let F and K be as above. Pukánszky has shown that for some subgroups K_o of K , $\mathcal{M}(K_o \times \{0\})$ is a singular MA subalgebra of $\mathcal{C}_{K_o \times F}$ [21]. By varying F and K_o appropriately, he constructed a sequence of pairwise inequivalent singular MA subalgebras of the hyperfinite II_1 factor. The mutual inequivalence of these subalgebras was established by means of the multiplicity theory of Segal.

Using group operator algebras over groups of matrices, Anastasio constructed infinite sequences of pairwise inequivalent 2-semi-regular and 3-semi-regular MA subalgebras of the hyperfinite II_1 factor [1]. The invariant of proper length was used to establish the mutual inequivalence of these subalgebras. In the proofs of Theorems 4.2 and 4.3 the groups used will be described.

Tauer's constructions of MA subalgebras of the hyperfinite II_1 factor are based on a different method. For each integer $p \geq 1$, let M_p denote the algebra of all $2^p \times 2^p$ matrices with complex entries. Embedding M_p in M_{p+1} in a suitable manner and using the normalized trace on each M_p , $M = \bigcup_{p=1}^{\infty} M_p$ becomes a pre-Hilbert space; let $\mathcal{H}$ denote its completion. We can regard M as a set of operators on $\mathcal{H}$ by letting each element of M act on $\mathcal{H}$ by left multiplication. The von Neumann algebra $\mathfrak{a}$ on $\mathcal{H}$ generated by M is the hyperfinite II_1 factor. Tauer constructs examples to show that:

(i) for each integer $m \geq 2$, $\mathfrak{a}$ contains m pairwise inequivalent semi-regular MA subalgebras of proper length m ([28], [29])

(ii) for each integer $m \geq 2$, $\mathfrak{a}$ contains an m-semi-regular MA subalgebra [30].

The remainder of this section is taken up with the classification of the MA subalgebras of $\mathcal{L}(\mathcal{H})$.

Lemma 2.4 Let $\mathcal{H}$ be a Hilbert space of dimension at least two, and let $\mathfrak{M}$ be a MA subalgebra of $\mathcal{L}(\mathcal{H})$ such that there is a family $(E_i)_{i\in I}$ of minimal projections in $\mathfrak{M}$ with $\sum_{i\in I} E_i = I$. Then $\mathfrak{M}$ is regular.

Proof. As $\mathfrak{M}$ is MA on $\mathcal{H}$, each E_i must be of rank one. Hence we can select an orthonormal basis $(\varphi_i)_{i\in I}$ for $\mathcal{H}$ such that $E_i\varphi_i = \varphi_i$ for each $i \in I$. In particular, I must contain at least two elements.

Suppose that an $A \in \mathcal{L}(\mathcal{H})$ commutes with each E_i . As each E_i is a minimal projection, a simple calculation shows that each $E_i A E_i \in \mathfrak{M}$. And as

$$A = \sum_{i\in I} E_i A = \sum_{i\in I} E_i A E_i$$

in the weak topology on $\mathcal{L}(\mathcal{H})$, $A \in \mathcal{M}$.

For distinct elements i and j of I , define unitary operators U_{ij} and V_{ij} on $\mathcal{H}$ by setting

$$U_{ij}\,\varphi_k = \begin{cases} \varphi_k & k \neq i,j \\ \varphi_j & k = i \\ \varphi_i & k = j \end{cases}$$

$$V_{ij}\,\varphi_k = \begin{cases} \varphi_k & k \neq i,j \\ \varphi_j & k = i \\ -\varphi_i & k = j \end{cases}$$

for all $k \in I$. Given an $A \in \mathcal{M}$, it is easy to verify that each $U_{ij}\,A(U_{ij})^*$ and each $V_{ij}\,A(V_{ij})^*$ commute with every E_k , and hence

$$U_{ij}\mathcal{M}(U_{ij})^* = V_{ij}\mathcal{M}(V_{ij})^* = \mathcal{M} .$$

Therefore

$$\mathcal{R}(U \in \mathcal{L}(\mathcal{H}) : \; U \text{ unitary and } U\mathcal{M}U^* = \mathcal{M}) \supset$$
$$\supset \mathcal{R}(U_{ij}, V_{ij} : i,j \in I \text{ and } i \neq j) ,$$

and so it suffices to show that if an $A \in \mathcal{L}(\mathcal{H})$ commutes with each U_{ij} and each V_{ij} , then $A \in \mathbf{C}_{\mathcal{H}}$.

Suppose that $A \in \mathcal{L}(\mathcal{H})$ commutes with each U_{ij} and each V_{ij} . For each $k \in I$ we can write

$A\varphi_k = \sum_{\ell \in I} \alpha_{\ell k} \varphi_\ell$, where the $\alpha_{\ell k}$ are complex numbers.

Fix $i, j \in I$ with $i \neq j$. Then

$$\sum_{k \in I} \alpha_{ki} \varphi_k = A\varphi_i = U_{ij} A\varphi_j = \sum_{k \in I} \alpha_{kj} U_{ij} \varphi_k$$

$$\sum_{k \in I} \alpha_{ki} \varphi_k = A\varphi_i = -V_{ij} A\varphi_j = -\sum_{k \in I} \alpha_{kj} V_{ij} \varphi_k \ .$$

On comparing coefficients in these two expansions, we see that $\alpha_{ii} = \alpha_{jj}$, $\alpha_{ij} = \alpha_{ji}$ and $\alpha_{ij} = -\alpha_{ji}$, and therefore $A \in \mathfrak{C}_{\mathcal{H}}$.

<u>Lemma 2.5</u> Let (X, Σ, μ) be a finite measure space, where Σ is a σ-algebra of subsets of X . For each $\varphi \in L^\infty(X, \Sigma, \mu)$, the relation

$$(M_\varphi f)(x) = \varphi(x)\, f(x) \qquad f \in L^2(X, \Sigma, \mu) \text{ and } x \in X$$

defines an $M_\varphi \in \mathcal{L}(L^2(X, \Sigma, \mu))$, and $\varphi \to M_\varphi$ is an isometric $*$-isomorphism of $L^\infty(X, \Sigma, \mu)$ onto a von Neumann algebra which is MA in $\mathcal{L}(L^2(X, \Sigma, \mu))$.

<u>Proof</u>. Easy calculations (see e.g. [6; pp. 117-118] or [11; pp. 6-14]).

<u>Lemma 2.6</u> Let $X = [0,1]$, let Σ be the Borel subsets of X , and let λ be Lebesgue measure on Σ . Let $\mathcal{H} = L^2(X, \Sigma, \lambda)$ and let $\mathfrak{M} = \{M_\varphi : \varphi \in L^\infty(X, \Sigma, \lambda)\}$. Then $\mathfrak{M}$ is a regular MA subalgebra of $\mathcal{L}(\mathcal{H})$.

<u>Proof</u>. By Lemma 2.5, $\mathfrak{M}$ is a MA subalgebra of $\mathcal{L}(\mathcal{H})$. Let $r \in (0,1)$ be a fixed irrational number, and let

$T : X \to X$ be addition by r modulo 1. It is clear that the map of $f \to f\circ T$ is a unitary transformation, say U, of $\mathcal{H}$. Moreover, $U\mathcal{M}U^* = \mathcal{M}$, for if $f \in \mathcal{H}$ and $\varphi \in L^\infty(X,\Sigma,\lambda)$ are arbitrary,

$$U\, M_\varphi\, U^* f = U\, M_\varphi(f\circ T^{-1}) = U(\varphi\cdot(f\circ T^{-1})) =$$

$$= (\varphi\circ T)\cdot f = M_{\varphi\circ T}\, f\ .$$

To show that $\mathcal{M}$ is regular, it will suffice to show that if an $A \in \mathcal{L}(\mathcal{H})$ commutes with U and with each unitary operator in $\mathcal{M}$, then $A \in \mathbb{C}_{\mathcal{H}}$.

For each $n \in \mathbb{Z}$, let $\varphi_n(x) = e^{2\pi i n x}$, $x \in X$; it is well-known that $(\varphi_n)_{n\in Z}$ is an orthonormal basis for $\mathcal{H}$. A simple calculation shows that $U\varphi_n = e^{2\pi i n r}\varphi_n$ for each integer n. Now suppose that an operator $A \in \mathcal{L}(\mathcal{H})$ commutes with U and with each unitary in $\mathcal{M}$. For each $n \in Z$ we can write $A\varphi_n = \sum_{m=-\infty}^{\infty} \alpha_{mn}\varphi_m$, where the α_{mn} are complex numbers. Then

$$\sum_{m=-\infty}^{\infty} \alpha_{mn}\varphi_m = A\varphi_n = e^{-2\pi i n r}\, U\, A\, \varphi_n$$

$$= e^{-2\pi i n r} \sum_{m=-\infty}^{\infty} \alpha_{mn}\, U\, \varphi_m$$

$$= \sum_{m=-\infty}^{\infty} \alpha_{mn}\, e^{2\pi i(m-n)r}\, \varphi_m\ .$$

As r is irrational, $e^{2\pi i(m-n)r} \neq 1$ unless $m = n$; comparing coefficients and using this remark, we see that $\alpha_{mn} = 0$ unless $m = n$. Consequently, there is a family $(\alpha_n)_{n\in Z}$ of complex numbers such that $A\varphi_n = \alpha_n\varphi_n$ for each

$n \in Z$. Now for each integer n, M_{φ_n} is a unitary operator in $\mathcal{M}$, and therefore

$$\alpha_o \varphi_o = A\varphi_o = M_{\varphi_n} A\varphi_{-n} = \alpha_{-n} M_{\varphi_n} \varphi_{-n} = \alpha_{-n} \varphi_o .$$

Thus $\alpha_o = \alpha_{\pm 1} = \alpha_{\pm 2} = \cdots$, and so $A \in \mathbb{C}_{\mathcal{H}}$.

<u>Lemma 2.7</u> Let $\mathcal{M}$ be a MA von Neumann algebra on $\mathcal{H}$ which possesses no minimal projections. Then $\mathcal{M}$ is regular.

<u>Proof</u>. As $\mathcal{H}$ is separable, there is a unit vector $x \in \mathcal{H}$ which is separating for $\mathcal{M}$, i.e., $M \in \mathcal{M}$ and $M x = 0$ imply $M = 0$ [27; Lemma 2.5]. A simple calculation [6; p.6] shows that x is cyclic for $\mathcal{M}' = \mathcal{M}$, i.e. $[\mathcal{M} x] = \mathcal{H}$. Applying now [27; Lemma 1.2], there is a compact Hausdorff space X, a regular measure μ on the σ-field Σ generated by the compact subsets of X with $\mu(X) = 1$, and a linear isometry of $\mathcal{H}$ onto $L^2(X,\Sigma,\mu)$ carrying $\mathcal{M}$ onto $\{M_\varphi : \varphi \in L^\infty(X,\Sigma,\mu)\}$.

As $\mathcal{M}$ does not possess minimal projections, the measure algebra of (X,Σ,μ) is non-atomic. Let (f_n) be an everywhere-dense sequence in $L^2(X,\Sigma,\mu)$, and for each n, let $E_n = \{x \in X : |f_n(x) - 1| \leq \frac{1}{2}\}$. Certainly each $E_n \in \Sigma$. Given $E \in \Sigma$ and $\epsilon > 0$, there is an integer n such that

$$\frac{\epsilon}{2} \geq \| f_n - \chi_E \|^2$$

$$\geq \int_{E-E_n} |f_n(x) - 1|^2 \, d\mu(x) + \int_{E_n-E} |f_n(x)|^2 \, d\mu(x)$$

$$\geq \tfrac{1}{2} \mu(E \Delta E_n) \quad ,$$

where Δ denotes symmetric difference. Hence the measure algebra of (X,Σ,μ) is separable. By a classification theorem of Halmos and von Neumann (cf. [9; p. 173]), there is an isomorphism Φ of the measure algebra of (X,Σ,μ) onto that of $([0,1],\mathcal{S}, \lambda)$, where $\mathcal{S}$ is the Borel subsets of $[0,1]$ and λ is Lebesgue measure on $\mathcal{S}$. In an obvious manner, we can regard Φ as a mapping from Σ into $\mathcal{S}$; as such, Φ is not onto $\mathcal{S}$, but every member of $\mathcal{S}$ is equivalent to a member of $\mathcal{S}$ in the range of Φ. It is routine to check that modulo the equivalence relation "equal almost everywhere", the map

$$\sum_{i=1}^{n} a_i \chi_{E_i} \to \sum_{i=1}^{n} a_i \chi_{\Phi(E_i)} \qquad a_i \in \mathbb{C} \text{ and } E_i \in \Sigma$$

is well-defined, linear, and is an L^2-isometry of the set of simple functions on (X,Σ,μ) onto the set of simple functions on $([0,1],\mathcal{S}, \lambda)$; hence the map extends to a linear isometry of $L^2(X,\Sigma,\mu)$ onto $L^2([0,1],\mathcal{S}, \lambda)$. It is readily seen that this isometry carries $\{M_\varphi : \varphi \in L^\infty(X,\Sigma,\mu)\}$ onto $\{M_\varphi : \varphi \in L^\infty([0,1],\mathcal{S},\lambda)\}$.

Therefore $\mathcal{M}$ acting on $\mathcal{H}$ is spatially isomorphic to $\{M_\varphi : \varphi \in L^\infty([0,1],\mathcal{S},\lambda)\}$ acting on $L^2([0,1],\mathcal{S}, \lambda)$.

As the latter is regular (Lemma 2.6), so is the former.

Remarks

(1) Our proof of Lemma 2.4 does not make use of the assumption that $\mathcal{H}$ is a separable Hilbert space.

(2) Segal has shown that Lemma 2.5 holds provided only that the measure space is semi-finite (in the sense that every set of infinite measure contains sets of arbitrarily large finite measure) and localizable (i.e. the measure algebra is complete as a partially ordered set) [26].

(3) Lemma 2.7 consists essentially in showing that a MA von Neumann algebra without minimal projections on a separable Hilbert space is spatially isomorphic to $\{M_\varphi : \varphi \in L^\infty([0,1], \mathcal{S}, \lambda)\}$. This is essentially due to von Neumann, and is well-known, although an explicit proof does not seem to appear in the literature. It can be deduced from the general Maharam classification theory of measure algebras (cf. [26; Corollary 5.1]). Our proof avoids this deep theorem, using instead a weaker classification theorem.

Let $\mathcal{M}$ be a MA subalgebra of $\mathcal{L}(\mathcal{H})$. If $\mathcal{M}$ satisfies the hypothesis of Lemma 2.4, set $c(\mathcal{M}) = 0$; otherwise, set $c(\mathcal{M}) = 1$. Let $n(\mathcal{M})$ be the maximal number of pairwise orthogonal minimal projections in $\mathcal{M}$ $(0 \leq n(\mathcal{M}) \leq \infty)$. The combination $c(\mathcal{M}) = 0$, $n(\mathcal{M}) = 0$ is impossible, while examples of all other combinations can be realized as $L^\infty(X,\Sigma,\mu)$ acting on $L^2(X,\Sigma,\mu)$ under pointwise multiplication for some finite measure space (X,Σ,μ).

<u>Theorem 2.8</u> Let $\mathcal{M}$ be a MA von Neumann algebra on $\mathcal{H}$. $\mathcal{M}$ is regular if $c(\mathcal{M}) = 0$ or if $c(\mathcal{M}) = 1$ and $n(\mathcal{M}) = 0$; for all other possible combinations, does not fall into any of the classes of Definition 2.2.

<u>Proof.</u> Lemma 2.4 [Lemma 2.7] shows that $\mathcal{M}$ is regular if $c(\mathcal{M}) = 0$ [$c(\mathcal{M}) = 1$ and $n(\mathcal{M}) = 0$]. Now suppose that $c(\mathcal{M}) = 1$ and $n(\mathcal{M}) \geq 1$. Let $(E_i)_{i\in I}$ be a maximal family of pairwise orthogonal minimal projections in $\mathcal{M}$, and let

$$E = \sum_{i\in I} E_i \quad , \quad F = I - E .$$

Then both E and F are non-zero projections in $\mathcal{M}$. Notice that $\mathcal{M}_E[\mathcal{M}_F]$ is a MA von Neumann algebra on $E(\mathcal{H})$ [$F(\mathcal{H})$] satisfying the hypothesis of Lemma 2.4 [Lemma 2.7] and therefore

$$N(\mathcal{M}_E) = \mathcal{L}(E(\mathcal{H})) \ , \quad N(\mathcal{M}_F) = \mathcal{L}(F(\mathcal{H})) .$$

The canonical isomorphism of $\mathcal{H}$ onto $E(\mathcal{H}) \oplus F(\mathcal{H})$ induces an isomorphism of $\mathcal{M}$ onto $\mathcal{M}_E \oplus \mathcal{M}_F$ [6; p.22], and so it suffices to show that $\mathcal{M}_E \oplus \mathcal{M}_F$ is a semi-regular subalgebra of $\mathcal{L}(E(\mathcal{H}) \oplus F(\mathcal{H}))$.

Let U and V be unitary operators on $E(\mathcal{H})$ and $F(\mathcal{H})$, respectively, such that $U \mathcal{M}_E U^* = \mathcal{M}_E$ and $V \mathcal{M}_F V^* = \mathcal{M}_F$. Then $U \oplus V$ is a unitary operator on $E(\mathcal{H}) \oplus F(\mathcal{H})$ with

$$(U \oplus V)\ (\mathcal{M}_E \oplus \mathcal{M}_F)\ (U \oplus V)^* = \mathcal{M}_E \oplus \mathcal{M}_F \ .$$

Therefore $N(\mathfrak{M}_E) \oplus N(\mathfrak{M}_F) \subset N(\mathfrak{M}_E \oplus \mathfrak{M}_F)$. Conversely, suppose that W is a unitary operator on $E(\mathcal{H}) \oplus F(\mathcal{H})$ with $W(\mathfrak{M}_E \oplus \mathfrak{M}_F)W^* = \mathfrak{M}_E \oplus \mathfrak{M}_F$. As automorphisms of a von Neumann algebra map minimal projections into minimal projections,

$$W(E \oplus 0)W^* = E \oplus 0 \quad \text{and} \quad W(0 \oplus F)W^* = 0 \oplus F .$$

Therefore $W = U \oplus V$, where U and V are unitary operators on $E(\mathcal{H})$ and $F(\mathcal{H})$, respectively, such that $U \mathfrak{M}_E U^* = \mathfrak{M}_E$ and $V \mathfrak{M}_F V^* = \mathfrak{M}_F$. This shows that

$$\begin{aligned} N(\mathfrak{M}_E \oplus \mathfrak{M}_F) &= N(\mathfrak{M}_E) \oplus N(\mathfrak{M}_F) \\ &= \mathcal{L}(E(\mathcal{H})) \oplus \mathcal{L}(F(\mathcal{H})) , \end{aligned}$$

which is not a factor.

<u>Theorem 2.9</u> Two MA subalgebras $\mathfrak{M}$ and $\mathfrak{N}$ on $\mathcal{H}$ are equivalent in $\mathcal{L}(\mathcal{H})$ if and only if $c(\mathfrak{M}) = c(\mathfrak{N})$ and $n(\mathfrak{M}) = n(\mathfrak{N})$.

<u>Proof</u>. The proof of this theorem is contained in the proofs of the preceding results.

3 THE MAIN CONSTRUCTION

Throughout this section, p will denote a fixed point in $(0,\frac{1}{2})$ and G will denote a fixed countably infinite group with identity e.

We begin with a summary of this section. Our first task is to construct a type III factor $\mathcal{G}_p$ containing a regular MA subalgebra $\mathfrak{M}_p$, a type II_1 factor $\mathcal{B}(p,G)$, and, for each subgroup G_o of G, a subalgebra $\mathcal{N}(p,G,G_o)$ of $\mathcal{B}(p,G)$. For a subgroup G_o of G, we will use $\mathfrak{M}(G,G_o)$ to denote the subalgebra of the group operator algebra $\mathcal{L}_G$ generated by $\{V_g : g \in G_o\}$. Recall that $N(G_o)$ denotes the normalizer of a subgroup G_o of G. Our second task is to prove the following six theorems, which constitute the main results of this section:

<u>Theorem 3.1</u> Let G_o be a subgroup of G. Then $\mathfrak{M}_p \otimes \mathcal{N}(p,G,G_o)$ is a MA subalgebra of $\mathcal{G}_p \otimes \mathcal{B}(p,G)$ if and only if

(α) : G_o is abelian and $\{g_o\, g\, g_o^{-1} : g_o \in G_o\}$ is infinite whenever $g \in G - G_o$.

<u>Theorem 3.2</u> Suppose that G_o is a subgroup of G satisfying

(β) : given a finite subset F of G and a $g \in G$, there are infinitely many $g_o \in G_o$ such that:

(i) $h, k \in F$ and $h g_o k^{-1} = g_o$ imply $h = k$

(ii) if $g \notin N(G_o)$, then $g g_o g^{-1} \notin G_o$.

Then

$$N(\mathfrak{M}_p \otimes \mathfrak{N}(p,G,G_o)) = \mathfrak{a}_p \otimes \mathfrak{N}(p,G,N(G_o))$$

$$N(\mathfrak{a}_p \otimes \mathfrak{N}(p,G,G_o)) = \mathfrak{a}_p \otimes \mathfrak{N}(p,G,N(G_o)) .$$

Theorem 3.3 For a subgroup G_o of G, $\mathfrak{a}_p \otimes \mathfrak{N}(p,G,G_o)$ is a factor if and only if G_o has the infinite conjugate class property (see (1.2)).

Theorem 3.4 Let G_o be a subgroup of G. Then $\mathfrak{M}_p \otimes \mathfrak{M}(G,G_o)$ is a MA subalgebra of $\mathfrak{a}_p \otimes \mathcal{C}_G$ if and only if G_o satisfies condition (α) of Theorem 3.1.

Theorem 3.5 If G_o is a subgroup of G satisfying condition (β) of Theorem 3.2, then

$$N(\mathfrak{M}_p \otimes \mathfrak{M}(G,G_o)) = \mathfrak{a}_p \otimes \mathfrak{M}(G,N(G_o))$$

$$N(\mathfrak{a}_p \otimes \mathfrak{M}(G,G_o)) = \mathfrak{a}_p \otimes \mathfrak{M}(G,N(G_o)) .$$

Theorem 3.6 For a subgroup G_o of G, $\mathfrak{a}_p \otimes \mathfrak{M}(G,G_o)$ is a factor if and only if G_o has the infinite conjugate class property.

The algebra $\mathfrak{a}_p$ and its subalgebra $\mathfrak{M}_p$ are defined in the text preceding Lemma 3.10 while $\mathfrak{B}(p,G)$ and the $\mathfrak{N}(p,G,G_o)$ are defined after Lemma 3.13 and in Definition 3.15, respectively. The proofs of the six theorems are given at the end of this section.

Before proceeding to the actual constructions, we first establish a technical result.

Lemma 3.7 Let $[\mathcal{M}, \mathcal{H}, G, g \to U_g]$ be a C-system, and let $\mathcal{Q} = \mathcal{Q}[\mathcal{M}, \mathcal{H}, G, g \to U_g]$. Let $\mathcal{M}_o$ be a subalgebra of $\mathcal{M}$, let G_o be a subgroup of G , and suppose that $U_g \mathcal{M}_o U_g^* = \mathcal{M}_o$ for all $g \in G_o$. Then

$$\mathcal{R}(\ (M \otimes I_{\hat{G}})(U_g \otimes V_g)\ :\ M \in \mathcal{M}_o \text{ and } g \in G_o)$$

consists of all those operators $A \in \mathcal{Q}$ with $A \sim [M_g : g \in G]$ satisfying:

(i) $M_g \in \mathcal{M}_o$ whenever $g \in G_o$

(ii) $M_g = 0$ whenever $g \in G - G_o$.

Proof. Let

$$\mathcal{P}_o = \{ \sum_{g \in F} (M_g \otimes I_{\hat{G}})(U_g \otimes V_g)\ :\ \text{each } M_g \in \mathcal{M}_o \text{ and } F \subset G_o \text{ finite} \}$$

$$\mathcal{P}_1 = \mathcal{R}(\ (M \otimes I_{\hat{G}})(U_g \otimes V_g)\ :\ M \in \mathcal{M}_o \text{ and } g \in G_o)$$

$$\mathcal{P}_2 = \{A \in \mathcal{Q}\ :\ A \text{ satisfies (i) and (ii)} \}\ .$$

Observe that $\mathcal{P}_o \subset \mathcal{P}_2$ (Lemma 1.2) and that, by the continuity of matrix elements (Lemma 1.1), $\mathcal{P}_2$ is a von Neumann algebra. A simple calculation together with an application of the double commutant theorem shows that $\mathcal{P}_o$ is a weakly dense sub-*-algebra of $\mathcal{P}_1$, and therefore that $\mathcal{P}_1 \subset \mathcal{P}_2$. To show that $\mathcal{P}_1 = \mathcal{P}_2$, it will suffice to show that $\mathcal{P}_o' \subset \mathcal{P}_2'$.

Suppose that $T \in \mathcal{P}_o'$, i.e., T is an operator on $\mathcal{H} \otimes \hat{G}$ which commutes with each $(M \otimes I)(U_g \otimes V_g)$, $M \in \mathcal{M}_o$ and $g \in G_o$. For any $k \in G_o$, $g,h \in G$, and $M \in \mathcal{M}_o$,

$$\begin{aligned} M\, U_k \phi^*_{k^{-1}g} T\, \phi_h &= \phi^*_g (M \otimes I)(U_k \otimes V_k)\phi_{k^{-1}g}\, \phi^*_{k^{-1}g}\, T\, \phi_h \\ &= \phi^*_g (M \otimes I)(U_k \otimes V_k)\, T\, \phi_h \\ &= \phi^*_g\, T (M \otimes I)(U_k \otimes V_k)\, \phi_h \\ &= \phi^*_g\, T\, \phi_{kh}\, \phi^*_{kh} (M \otimes I)(U_k \otimes V_k)\, \phi_h \\ &= \phi^*_g\, T\, \phi_{kh}\, M\, U_k\, . \end{aligned}$$

Let $A \in \mathcal{P}_2$ with $A \sim [M_g : g \in G]$ be given. For all $g,h \in G$,

$$\begin{aligned} \phi^*_g\, A\, T\, \phi_h &= \sum_{k \in G} \phi^*_g\, A\, \phi_{k^{-1}g}\, \phi^*_{k^{-1}g}\, T\, \phi_h \\ &= \sum_{k \in G} \phi^*_k\, A\, \phi_e\, \phi^*_{k^{-1}g}\, T\, \phi_h \\ &= \sum_{k \in G_o} M_k\, U_k\, \phi^*_{k^{-1}g}\, T\, \phi_h \\ &= \sum_{k \in G_o} \phi^*_g\, T\, \phi_{kh}\, M_k\, U_k \\ &= \sum_{k \in G} \phi^*_g\, T\, \phi_{kh}\, \phi^*_{kh}\, A\, \phi_h \\ &= \phi^*_g\, T\, A\, \phi_h\, , \end{aligned}$$

where the sums all converge in the weak topology. As an operator is completely determined by its matrix elements, $T \in \mathcal{P}_2'$.

<u>Corollary 3.8</u> Let $[\mathcal{M},\mathcal{H},G,g \to U_g]$ be a C-system, let $\mathcal{G} = \mathcal{G}[\mathcal{M},\mathcal{H},G,g \to U_g]$, and let G_o be a subgroup of G . Then

$$\mathcal{R}(U_g \otimes V_g : g \in G_o)$$

consists of all those operators $A \in \mathcal{G}$ with $A \sim [M_g : g \in G]$ satisfying:

(i) $M_g \in \mathbb{C}_{\mathcal{H}}$ whenever $g \in G_o$

(ii) $M_g = 0$ whenever $g \in G - G_o$.

For each $g \in G$, let $\mathcal{H}^g$ be 2-dimensional Hilbert space with orthonormal basis $(\varphi_n^g)_{n \in Z_2}$. The vectors

$$\gamma_o^g = \sqrt{p}\ \varphi_o^g + \sqrt{1-p}\ \varphi_1^g$$

$$\gamma_1^g = \sqrt{1-p}\ \varphi_o^g - \sqrt{p}\ \varphi_1^g$$

form a second orthonormal basis for $\mathcal{H}^g$. Let

$$F_n^g = \mathrm{pr}\,[\gamma_n^g] \qquad n \in Z_2$$

$$\mathcal{M}^g = \{a\ F_o^g + b\ F_1^g : a,b \in \mathbb{C}\} .$$

Define a unitary representation $n \to U_n^g$ of Z_2 on $\mathcal{H}^g$ by setting $U_n^g \gamma_m^g = \gamma_{n+m}^g$ for all $n,m \in Z_2$. Then

$$(3.1) \qquad U_n^g F_m^g (U_n^g)^* = F_{n+m}^g \qquad n,m \in Z_2$$

$$U_n^g \mathcal{M}^g (U_n^g)^* = \mathcal{M}^g \qquad n \in Z_2 .$$

Lemma 3.9 For each $g \in G$, $[\mathcal{M}^g, \mathcal{H}^g, Z_2, n \to U_n^g]$ is a free and ergodic C-system.

Proof. If $A \in (\mathcal{M}^g)'$, then $A \gamma_n^g = F_n^g A \gamma_n^g$, $n \in Z_2$, which implies that $\mathcal{M}^g$ is MA on $\mathcal{H}^g$. Hence $[\mathcal{M}^g, \mathcal{H}^g, Z_2, n \to U_n^g]$ is a C-system.

Proposition 1.4, the projections F_0^g and F_1^g, and (3.1) imply that the C-system $[\mathcal{M}^g, \mathcal{H}^g, Z_2, n \to U_n^g]$ is free. To show ergodicity, suppose that $a F_0^g + b F_1^g$ $(a, b \in \mathbb{C})$ commutes with U_1^g. Then

$$a F_0^g + b F_1^g = U_1^g (a F_0^g + b F_1^g)(U_1^g)^*$$

$$= a F_1^g + b F_0^g ,$$

which implies that $a = b$.

As F_0^g and F_1^g are minimal projections in $\mathcal{M}^g$ with $F_0^g + F_1^g = I$, each $\mathcal{C}^g = \mathcal{C}[\mathcal{M}^g, \mathcal{H}^g, Z_2, n \to U_n^g]$ is a factor of type I_2 on the 4-dimensional Hilbert space $\mathcal{H}^g \otimes \hat{Z}_2$.

Let Δ be the set of all functions from G into Z_2 which have finite support. Under component-wise addition, Δ is an abelian group; we use 0 to denote the identity in Δ.

For each $g \in G$, let γ_g be the element of Δ defined by

$$\gamma_g(h) = \begin{cases} 1 & h = g \\ 0 & h \in G - \{g\} . \end{cases}$$

Given $\alpha, \beta \in \Delta$, we define elements $\alpha \wedge \beta$ and $\alpha \vee \beta$ of Δ by setting

$$(\alpha \wedge \beta)(g) = \min \{\alpha(g), \beta(g)\} \qquad g \in G$$

$$(\alpha \vee \beta)(g) = \max \{\alpha(g), \beta(g)\} \qquad g \in G$$

(we consider Z_2 to be ordered in the natural way, i.e., $0 \leq 1$). For $\alpha, \beta \in \Delta$, we will write $\alpha \leq \beta$ whenever $\alpha \wedge \beta = \alpha$.

Let $\mathcal{H} = \bigotimes_{g \in G} (\mathcal{H}^g, \varphi_o^g)$, $\mathcal{M} = \bigotimes_{g \in G} (\mathcal{M}^g, \varphi_o^g)$, and for each $\alpha \in \Delta$, let $\varphi_\alpha = \bigotimes_{g \in G} \varphi^g_{\alpha(g)}$ and let $U_\alpha = \prod_{g \in G} \overline{U^g_{\alpha(g)}}$ (a finite product in which the factors commute). From Propositions 1.7 and 1.9 we know that $(\varphi_\alpha)_{\alpha \in \Delta}$ is an orthonormal basis for $\mathcal{H}$, that $[\mathcal{M}, \mathcal{H}, \Delta, \alpha \to U_\alpha]$ is a free and ergodic C-system, and that $\mathcal{A}_p = \mathcal{A}[\mathcal{M}, \mathcal{H}, \Delta, \alpha \to U_\alpha]$ is spatially isomorphic to $\bigotimes_{g \in G} (\mathcal{A}^g, \varphi_o^g \otimes \hat{\delta})$. $\mathcal{A}_p$ is therefore a hyperfinite factor acting on a separable Hilbert space; moreover, it follows from [3; Prop. 5.5] that $\mathcal{A}_p$ is of type III. As the group G has served merely as an indexing set in this construction, $\mathcal{A}_p$ is actually independent of the particular choice of G. Let $\mathcal{M}_p = \mathcal{M} \otimes \mathbb{C}_{\hat{\Delta}}$.

<u>Lemma 3.10</u> $\mathcal{M}_p$ is a regular MA subalgebra of $\mathcal{G}_p$.

<u>Proof</u>. That $\mathcal{M}_p$ is a MA subalgebra of $\mathcal{G}_p$ is part of Proposition 1.5. For all unitary operators $U \in \mathcal{M}$ and all $\alpha \in \Delta$,

$$(U \otimes I)\mathcal{M}_p(U \otimes I)^* = U\mathcal{M}U^* \otimes \mathbb{C}_{\hat{\Delta}} = \mathcal{M}_p \quad .$$

$$(U_\alpha \otimes V_\alpha)\mathcal{M}_p(U_\alpha \otimes V_\alpha)^* = U_\alpha \mathcal{M} U_\alpha^* \otimes \mathbb{C}_{\hat{\Delta}} = \mathcal{M}_p \quad .$$

As

$$\mathcal{G}_p = \mathcal{R}(U \otimes I \ , \ U_\alpha \otimes V_\alpha : \alpha \in \Delta \ , \quad U \text{ a unitary} \in \mathcal{M}),$$

$\mathcal{M}_p$ is regular.

For each $g \in G$, let

$$P_{\gamma_g} = \left(\frac{1-p}{p}\right)^{\frac{1}{2}} \overline{F_o^g} - \left(\frac{p}{1-p}\right)^{\frac{1}{2}} \overline{F_1^g} \quad ,$$

and for each $\alpha \in \Delta$, let

$$P_\alpha = \begin{cases} I_{\mathcal{H}} & \alpha = 0 \\ \prod_{\alpha(g)=1} P_{\gamma_g} & \text{otherwise} \end{cases}$$

(a finite product in which the factors commute). Notice that

$$(3.2) \qquad P_\alpha P_\beta = P_{\alpha\wedge\beta+\alpha\vee\beta} P^2_{\alpha\wedge\beta} \quad \text{for all } \alpha,\beta \in \Delta \quad .$$

Each P_α is a self-adjoint operator on $\mathcal{H}$ satisfying

$P_\alpha \varphi_o = \varphi_\alpha$. As $\mathcal{M}$ is MA on $\mathcal{H}$, φ_o is both cyclic and separating for $\mathcal{M}$.

Let $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}$ and let $\tilde{\mathcal{M}} = \mathcal{M} \otimes \mathcal{M}$. Then $\tilde{\mathcal{M}}$ is a MA von Neumann algebra on $\tilde{\mathcal{H}}$, $(\varphi_\alpha \otimes \varphi_\beta)_{\alpha,\beta \in \Delta}$ is an orthonormal basis for $\tilde{\mathcal{H}}$, and $\varphi_o \otimes \varphi_o$ is both cyclic and separating for $\tilde{\mathcal{M}}$.

Lemma 3.11

(i) $\mathcal{J} = \{ \sum_{\alpha \leq \bar{\alpha}} c_\alpha P_\alpha : c_\alpha \in \mathbb{C}$ and $\bar{\alpha} \in \Delta \}$ is a strongly dense sub-*-algebra of $\mathcal{M}$ containing $I_{\mathcal{H}}$.

(ii) $\tilde{\mathcal{J}} = \{ \sum_{\alpha,\beta \leq \bar{\alpha}} c_{\alpha,\beta} P_\alpha \otimes P_\beta : c_{\alpha,\beta} \in \mathbb{C}$ and $\bar{\alpha} \in \Delta \}$ is a strongly dense sub-*-algebra of $\tilde{\mathcal{M}}$ containing $I_{\tilde{\mathcal{H}}}$.

Proof. It is clear that $I_{\mathcal{H}} \in \mathcal{J}$ and that $\mathcal{J}$ is a linear space closed under the *-operator. For each $g \in G$, $\overline{F_o^g} \in \mathcal{J}$, for

$$P_{\gamma_g} = \left(\frac{1-p}{p}\right)^{\frac{1}{2}} \overline{F_o^g} - \left(\frac{p}{1-p}\right)^{\frac{1}{2}} (P_o - \overline{F_o^g})$$

$$\overline{F_o^g} = \sqrt{p(1-p)} \left(P_{\gamma_g} + \left(\frac{p}{1-p}\right)^{\frac{1}{2}} P_o\right) \quad . \tag{3.3}$$

To show that $\mathcal{J}$ is closed under multiplication, let $\alpha,\beta \in \Delta$ be given; from (3.2) and the observation that

$$(\alpha\wedge\beta + \alpha\vee\beta) \wedge (\alpha\wedge\beta) = 0 \quad ,$$

it is sufficient to show that for each $\gamma \in \Delta$,

$$P_\gamma^2 = \sum_{\alpha\leq\gamma} c_\alpha P_\alpha$$

for some $c_\alpha \in \mathbb{C}$. If $\gamma = 0$ this is obvious, and if $\gamma \in \Delta - \{0\}$,

$$\begin{aligned} P_\gamma^2 &= \prod_{\gamma(g)=1} P_{\gamma_g}^2 \\ &= \prod_{\gamma(g)=1} \left[\frac{1-p}{p} \overline{F_o^g} + \frac{p}{1-p} (P_o - \overline{F_o^g}) \right] \\ &= \prod_{\gamma(g)=1} \left[\frac{1-2p}{\sqrt{p(1-p)}} P_{\gamma_g} + P_o \right], \end{aligned}$$

which is of the required form (in this calculation we used (3.3)).

This shows that $\mathcal{S}$ is a sub-*-algebra of $\mathfrak{M}$ which contains I and generates $\mathfrak{M}$. By the double commutant theorem, this proves (i).

The proof of (ii) is similar.

<u>Lemma 3.12</u> If $S = \sum_{\alpha\leq\bar\alpha} c_\alpha P_\alpha$, $T = \sum_{\alpha\leq\bar\alpha} d_\alpha P_\alpha \in \mathcal{S}$ are such that $\alpha \leq \bar\alpha$, $c_\alpha \neq 0$ and $d_\alpha \neq 0$ imply $\alpha = 0$, then

$$(ST\varphi_o,\varphi_o) = (S\varphi_o,\varphi_o)(T\varphi_o,\varphi_o) .$$

<u>Proof</u>. A simple calculation:

$$\begin{aligned} (ST\varphi_o,\varphi_o) &= \sum_{\alpha,\beta\leq\bar\alpha} c_\beta d_\alpha (P_\alpha\varphi_o, P_\beta\varphi_o) \\ &= \sum_{\alpha\leq\bar\alpha} c_\alpha d_\alpha \end{aligned}$$

$$= c_o d_o$$

$$= (S\varphi_o, \varphi_o)(T\varphi_o, \varphi_o) \quad .$$

For each $\alpha \in \Delta$ and $g \in G$ we define an element $g\alpha$ of Δ by setting

$$(g\alpha)(h) = \alpha(g^{-1}h) \qquad h \in G.$$

Notice that $(gh)\alpha = g(h\alpha)$ for all $g,h \in G$ and $\alpha \in \Delta$. For each $g \in G$, the relation

$$U_g \varphi_\alpha = \varphi_{g\alpha} \qquad \alpha \in \Delta$$

defines a unique unitary operator U_g on $\mathcal{H}$, and the map $g \to U_g$ is a unitary representation of G on $\mathcal{H}$; moreover,

$$U_g \overline{F_n^h} U_g^* = \overline{F_n^{gh}} \qquad g,h \in G \ , \ \alpha \in \Delta$$

$$U_g \mathfrak{M} U_g^* = \mathfrak{M} \qquad g \in G \ .$$

Lemma 3.13

(i) If an $M \in \mathfrak{M}$ satisfies $U_g M U_g^* = M$ for all $g \in G_o$, where G_o is an infinite subgroup of G , then $M \in \mathfrak{C}_{\mathcal{H}}$.

(ii) $[\mathfrak{M}, \mathcal{H}, G, g \to U_g]$ is a free and ergodic C-system.

Proof. (i) For such an M and for all $g \in G_o$,

$$M\varphi_o = U_g M U_g^* \varphi_o = U_g M \varphi_o \ .$$

If $M\varphi_o = \sum_{\alpha\in\Delta} c_\alpha \varphi_\alpha$, then

$$\sum_{\alpha\in\Delta} c_\alpha \varphi_\alpha = U_g \sum_{\alpha\in\Delta} c_\alpha \varphi_\alpha = \sum_{\alpha\in\Delta} c_\alpha \varphi_{g\alpha} = \sum_{\alpha\in\Delta} c_{g^{-1}\alpha} \varphi_\alpha ,$$

and thus $c_\alpha = c_{g^{-1}\alpha}$ for all $\alpha \in \Delta$. As $\alpha \in \Delta - \{0\}$ implies that $\{g\alpha : g \in G_o\}$ is infinite, we must have $c_\alpha = 0$ unless $\alpha = 0$, and therefore $M\varphi_o = c_o \varphi_o$. As φ_o is separating for $\mathcal{M}$, $M = c_o I$.

(ii) From the preceding we know that $[\mathcal{M},\mathcal{H},G,g \to U_g]$ is an ergodic C-system. If the system is not free, there is an $M \in \mathcal{M} - \{0\}$ and a $g \in G - \{e\}$ such that $U_g M \in \mathcal{M}$. Let $\epsilon > 0$ be fixed but arbitrary; as φ_o is separating for $\mathcal{M}$, $\delta = \dfrac{\epsilon \| M\varphi_o \|}{1 + \epsilon} > 0$, and $\delta = \epsilon(\| M\varphi_o\| - \delta)$. By Lemma 3.11, there is an $S = \sum_{\alpha\leq\bar{\alpha}} c_\alpha P_\alpha \in \mathcal{S}$ such that $\| (S - M)\varphi_o\| \leq \delta$. Hence $\| S\varphi_o\| \geq \| M\varphi_o\| - \delta > 0$, and consequently

$$\| (S - M)\varphi_o\| \leq \delta = \epsilon(\| M\varphi_o\| - \delta) \leq \epsilon \| S\varphi_o\| .$$

As $\bar{\alpha}$ has finite support, we can find an $h \in G - \{e\}$ with $\bar{\alpha}(h) = \bar{\alpha}(g^{-1}h) = 0$. Now

$$S S^* = \sum_{\alpha,\beta\leq\bar{\alpha}} c_\alpha \overline{c_\beta} P_\alpha P_\beta = \sum_{\alpha\leq\bar{\alpha}} d_\alpha P_\alpha$$

for some $d_\alpha \in \mathbb{C}$. Applying Lemma 3.12 and (3.3) ,

$$\| \overline{F_o^h}\, U_g\, S\varphi_o - U_g S\, \overline{F_o^h}\, \varphi_o \|^2 =$$

$$= \| \overline{F_o^{g^{-1}h}}\, S\varphi_o - S\, \overline{F_o^h}\, \varphi_o \|^2$$

$$= (\overline{F_o^{g^{-1}h}}\, S\, S^*\, \varphi_o, \varphi_o) + (\overline{F_o^h}\, S\, S^*\, \varphi_o, \varphi_o) -$$

$$- 2(\overline{F_o^h}\, \overline{F_o^{g^{-1}h}}\, S\, S^*\, \varphi_o, \varphi_o)$$

$$= 2(p - p^2)\, \| S\varphi_o \|^2 .$$

On the other hand,

$$\| \overline{F_o^h}\, U_g\, S\varphi_o - U_g\, S\, \overline{F_o^h}\, \varphi_o \| \leq$$

$$\leq \| \overline{F_o^h}\, U_g\, S\, \varphi_o - \overline{F_o^h}\, U_g\, M\, \varphi_o \| +$$

$$+ \| U_g\, M\, \overline{F_o^h}\, \varphi_o - U_g\, S\, \overline{F_o^h}\, \varphi_o \|$$

$$\leq \| S\varphi_o - M\varphi_o \| + \| M\varphi_o - S\varphi_o \|$$

$$\leq 2\, \epsilon\, \| S\varphi_o \| .$$

Combining these two calculations yields

$$4\, \epsilon^2 \| S\varphi_o \|^2 \geq 2(p - p^2) \| S\varphi_o \|^2$$

$$2\, \epsilon^2 \geq p - p^2 .$$

As $p - p^2 > 0$, this contradicts the arbitrariness of ϵ, and the system $[\mathcal{M}, H, G\ g \to U_g]$ is therefore free.

Let $\mathcal{B}(p,G) = \mathcal{C}[\mathcal{M}, H, G, g \to U_g]$.

Lemma 3.14 $\mathcal{B}(p,G)$ is a factor of type II_1 acting on a separable Hilbert space. $\mathcal{B}(p,G)$ is hyperfinite whenever G is either hyperfinite or abelian.

Proof. We use Proposition 1.5. That $\mathcal{B}(p,G)$ is a factor follows from Lemma 3.13. As $M \to (M\varphi_o,\varphi_o)$ is a finite normal faithful trace on $\mathfrak{M}^+$ satisfying

$$(U_g\, M\, U_g^* \varphi_o,\varphi_o) = (M\varphi_o,\varphi_o) \quad \text{for all } M \in \mathfrak{M}^+ \text{ and } g \in G\,,$$

$\mathcal{B}(p,G)$ is finite. And as $\mathcal{C}_p$ is of type III, $\mathfrak{M}$ cannot contain any minimal projections, which implies that $\mathcal{B}(p,G)$ is not of type I. Therefore $\mathcal{B}(p,G)$ is a factor of type II_1.

If G is abelian, then $\mathcal{B}(p,G)$ is hyperfinite, by Proposition 1.6. Suppose now that G is hyperfinite, say $G = \bigcup_{n=1}^{\infty} G_n$, where $G_1 \subset G_2 \subset \cdots$, and each G_n is a finite subgroup of G. For each n, let

$$\mathcal{B}_n = \mathcal{R}((M \otimes I_{\hat{G}})(U_g \otimes V_g) : \; g \in G_n \text{ and } M \in \overline{\mathfrak{M}^h} \text{ for some } h \in G_n) \;;$$

each $\mathcal{B}_n$ is finite-dimensional as a vector space, and moreover,

$$\mathcal{B}(p,G) = \mathcal{R}(\mathcal{B}_n : \; n = 1,2,\ldots) \;.$$

Using [13; Theorem XII] (or [6; p. 299]), we conclude that $\mathcal{B}(p,G)$ is hyperfinite.

Let $\mathcal{G} = \Delta \times G$, the group-theoretic direct product, and for $a = (\alpha, g)$, let $\tilde{U}_a = U_\alpha \otimes U_g$. There is a unique linear isometry γ of $\mathcal{H} \otimes \hat{\Delta} \otimes \mathcal{H} \otimes \hat{G}$ onto $\tilde{\mathcal{H}} \otimes \hat{\mathcal{G}}$ with

$$\gamma(\varphi_\alpha \otimes \hat{\gamma} \otimes \varphi_\beta \otimes \hat{g}) = \varphi_\alpha \otimes \varphi_\beta \otimes (\gamma, g)^\wedge \quad \text{for all } \alpha,\beta,\gamma \in \Delta \text{ and } g \in G .$$

It is straightforward to prove (cf. Proposition 1.9) that $[\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \mathcal{G}, a \to \tilde{U}_a]$ is a free and ergodic C-system, and that if $\tilde{\mathcal{G}} = \mathcal{G}[\tilde{\mathcal{M}}, \tilde{\mathcal{H}}, \mathcal{G}, a \to \tilde{U}_a]$, then $A \to \gamma A \gamma^{-1}$ is an isomorphism of $\mathcal{G}_p \otimes \mathcal{B}(p,G)$ onto $\tilde{\mathcal{G}}$. Notice that for $M,N \in \mathcal{M}$ and $a = (\alpha, g) \in \mathcal{G}$,

$$\gamma((M \otimes I_{\hat{\Delta}})(U_\alpha \otimes V_\alpha) \otimes (N \otimes I_{\hat{G}})(U_g \otimes V_g))\gamma^{-1} =$$

(3.4)
$$= ((M \otimes N) \otimes I_{\hat{\mathcal{G}}})(\tilde{U}_a \otimes V_a) \quad .$$

<u>Definition 3.15</u> For each subgroup G_o of G, define a subalgebra $\mathcal{N}(p,G,G_o)$ of $\mathcal{B}(p,G)$ and subalgebras $\tilde{\mathcal{P}}(G_o)$ and $\tilde{\mathcal{J}}(G_o)$ of $\tilde{\mathcal{G}}$ as follows:

$$\mathcal{N}(p,G,G_o) = \mathcal{R}(U_g \otimes V_g : g \in G_o)$$

$$\tilde{\mathcal{P}}(G_o) = \gamma \mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)\gamma^{-1}$$

$$\tilde{\mathcal{J}}(G_o) = \gamma \mathcal{G}_p \otimes \mathcal{N}(p,G,G_o)\gamma^{-1} .$$

Notice that these subalgebras are all proper.

<u>Lemma 3.16</u> Let G_o be a subgroup of G. The subalgebra $\mathcal{N}(p,G,G_o)$ is MA in $\mathcal{B}(p,G)$ if and only if G_o satisfies

$$(\alpha) : G_o \text{ is abelian and } \{g_o\, g\, g_o^{-1} : g_o \in G_o\} \text{ is infinite whenever } g \in G - G_o .$$

<u>Proof</u>. Suppose that the subgroup G_o satisfies condition (α). Then $\mathcal{N}(p,G,G_o)$ is an abelian algebra, and to show that it is MA in $\mathcal{B}(p,G)$, we must verify that

$$\mathcal{B}(p,G) \cap (\mathcal{N}(p,G,G_o))' \subset \mathcal{N}(p,G,G_o) .$$

Let $B \in \mathcal{B}(p,G) \cap (\mathcal{N}(p,G,G_o))'$ with $B \sim [M_g : g \in G]$ be given. From Lemma 1.2 we have that for all $g \in G$ and $h \in G_o$,

$$(3.5) \qquad \phi_{gh}^{*}\, B(U_h \otimes V_h)\, \phi_e\, U_{gh}^{*} = M_g$$

$$(3.6) \qquad \phi_{gh}^{*}\, (U_h \otimes V_h)\, B\, \phi_e\, U_{gh}^{*} = U_h\, M_{h^{-1}gh}\, U_h^{*}$$

$$(3.7) \qquad (\phi_e^{*}\, B\, B^{*}\, \phi_e \varphi_o, \varphi_o) = (\sum_{k \in G} M_k\, M_k^{*} \varphi_o, \varphi_o) = \sum_{k \in G} \| M_k \varphi_o \|^{2} ,$$

where the expressions (3.5) and (3.6) are equal. If $g \in G_o$, then $M_g = U_h\, M_g\, U_h^{*}$ for all $h \in G_o$, which, by Lemma 3.13, implies that $M_g \in \mathbb{C}_{\mathcal{H}}$. If $g \in G - G_o$, then for all $h \in G_o$,

$$\| M \varphi_o \| = \| U_h\, M_{h^{-1}gh}\, U_h^{*}\, \varphi_o \| = \| M_{h^{-1}gh} \varphi_o \| ;$$

by (3.7) and condition (α), this means that $M_g \varphi_o = 0$, and consequently that $M_g = 0$. Corollary 3.8 now implies that $B \in \mathcal{N}(p,G,G_o)$.

Conversely, suppose that $\mathcal{N}(p,G,G_o)$ is a MA subalgebra of $\mathcal{B}(p,G)$. If condition (α) fails, then, as $\mathcal{N}(p,G,G_o)$ abelian implies G_o abelian, there is a $g \in G - G_o$ such that

$$F = \{g_o g\, g_o^{-1} : g_o \in G_o\}$$

is finite. Let

$$B = \sum_{h \in F} U_h \otimes V_h \quad ;$$

then $B \in \mathcal{B}(p,G)$ and, as $g \in F$, $B \notin \mathcal{N}(p,G,G_o)$. For any $h \in G_o$ and $k \in G$,

$$\phi_k^* B(U_h \otimes V_h)\, \phi_e U_k^* =$$

$$= \phi_{kh^{-1}}^* B\, \phi_e U_{kh^{-1}}^*$$

$$= \begin{cases} I & kh^{-1} \in F \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi_k^* (U_h \otimes V_h) B\, \phi_e U_k^* =$$

$$= U_h\, \phi_{h^{-1}k}^* B\, \phi_e U_{h^{-1}k} U_h^*$$

$$= \begin{cases} I & h^{-1}k \in F \\ 0 & \text{otherwise}, \end{cases}$$

where we used Lemma 1.2. As $h^{-1}k \in F$ is equivalent to

$kh^{-1} \in F$ $(h \in G_o$, $k \in G)$, B commutes with $U_h \otimes V_h$ for all $h \in G_o$, i.e., $B \in (\mathcal{N}(p,G,G_o))'$. This implies that $\mathcal{N}(p,G,G_o)$ is not MA in $\mathcal{B}(p,G)$, which is a contradiction. Hence condition (α) must hold.

Lemma 3.17 Let G_o be a subgroup of G. An $\tilde{A} \in \tilde{\mathcal{G}}$ with $\tilde{A} \sim [\tilde{M}_a : a \in \mathcal{G}]$ is an element of (i) $\tilde{\mathcal{G}}(G_o)$ if and only if

$$\begin{aligned} &\tilde{M}_a \in \mathcal{M} \otimes \mathbb{C}_{\mathcal{H}} && a \in \{0\} \times G_o \\ &\tilde{M}_a = 0 && \text{otherwise}, \end{aligned}$$

and (ii) $\tilde{\mathcal{J}}(G_o)$ if and only if

$$\begin{aligned} &\tilde{M}_a \in \mathcal{M} \otimes \mathbb{C}_{\mathcal{H}} && a \in \Delta \times G_o \\ &\tilde{M}_a = 0 && \text{otherwise}. \end{aligned}$$

Proof. Using [6; p. 57] and (3.4),

$$\begin{aligned} \tilde{\mathcal{G}}(G_o) &= \gamma\, [\mathcal{M} \otimes \mathbb{C}_{\hat{\Delta}} \otimes \mathcal{N}(p,G,G_o)]\gamma^{-1} \\ &= \gamma\, \mathcal{R}(M \otimes I_{\hat{\Delta}} \otimes U_g \otimes V_g : M \in \mathcal{M},\ g \in G_o)\gamma^{-1} \\ &= \mathcal{R}((M \otimes I_{\mathcal{H}} \otimes I_{\hat{\mathcal{G}}})(U_{(o,g)} \otimes V_{(o,g)}) : M \in \mathcal{M}, \\ &\qquad g \in G_o) \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{J}}(G_o) &= \gamma\, \mathcal{G}_p \otimes \mathcal{N}(p,G,G_o)\gamma^{-1} \\ &= \gamma\, \mathcal{R}((M \otimes I_{\hat{\Delta}})(U_\alpha \otimes V_\alpha) \otimes (U_g \otimes V_g) : M \in \mathcal{M}, \\ &\qquad \alpha \in \Delta,\ g \in G_o)\gamma^{-1} \end{aligned}$$

$$= \mathcal{R}((M \otimes I_{\mathcal{H}} \otimes I_{\hat{\mathcal{G}}})(U_{(\alpha,g)} \otimes V_{(\alpha,g)}) : M \in \mathcal{M},$$

$$\alpha \in \Delta , g \in G_o) .$$

The desired conclusions now follow from Lemma 3.7.

Lemma 3.18 Let $\mathcal{N}$ be an abelian von Neumann algebra on the Hilbert space $\mathcal{K}$, and let x be a non-zero vector in $\mathcal{K}$. Let $(M_i)_{i\in I}$ and $(N_i)_{i\in I}$ be two families in $\mathcal{N}$ such that $M = \sum_{i\in I} M_i N_i$ and $N = \sum_{i\in I} N_i N_i^*$ exist in $\mathcal{N}$ in the strong topology, and suppose that $N \leq I$. Then

$$\| M x\|^2 \leq \sum_{i\in I} \| M_i x\|^2 .$$

Proof. As $\mathcal{N}$ is a uniformly closed commutative B^*-algebra with identity, the Gelfand-Naimark representation theorem (see e.g. [7; p. 876]) gives an isometric $*$-isomorphism $A \to f_A$ of $\mathcal{N}$ onto $C(X)$, X some compact Hausdorff space. Let F be an arbitrary finite subset of I, and set $M_F = \sum_{i\in F} M_i N_i$ and $N_F = \sum_{i\in F} N_i N_i^*$. As $I \geq N \geq N_F$, $1 \geq f_{N_F} \geq \sum_{i\in F} |f_{N_i}|^2$, and consequently

$$|f_{M_F}| \leq \sum_{i\in F} |f_{M_i}| \, |f_{N_i}|$$

$$|f_{M_F}|^2 \leq \Big(\sum_{i\in F} |f_{M_i}|^2\Big) \Big(\sum_{i\in F} |f_{N_i}|^2\Big) \leq \sum_{i\in F} |f_{M_i}|^2 .$$

Passing back to $\mathcal{N}$, $|M_F|^2 \leq \sum_{i\in F} |M_i|^2$, and therefore

$$\| M_F x\|^2 = (M_F M_F^* x,x) \leq \sum_{i\in F} (M_i M_i^* x,x) = \sum_{i\in F} \| M_i x\|^2 .$$

Taking the supremum over all finite $F \subset I$, we are done.

In order to simplify the notation, let $\imath$ denote the identity in $\mathcal{G}$ and for each $a \in \mathcal{G}$, let $\tilde{T}_a = \tilde{U}_a \otimes V_a$.

Lemma 3.19 (cf. [20; Lemma 15]). Let a unitary operator $\tilde{U} \in \tilde{\mathfrak{a}}$ with $\tilde{U} \sim [\tilde{M}_a : a \in \mathcal{G}]$ and an $\epsilon > 0$ be given. Then there is a finite subset $\mathfrak{F}_0$ of $\mathcal{G}$ such that for any finite subset $\mathfrak{F}$ of $\mathcal{G}$ containing $\mathfrak{F}_0$, there is a family $(\tilde{N}_b)_{b\in\mathfrak{F}}$ of elements of $\tilde{\mathcal{J}}$ such that:

(3.8) (i) $\| \tilde{M}_b\varphi_0\otimes\varphi_0 - \tilde{N}_b\varphi_0\otimes\varphi_0\| \leq \frac{\epsilon}{2}$ for all $b \in \mathfrak{F}$

(ii) if $\tilde{V} = \sum_{b\in\mathfrak{F}} (\tilde{N}_b\otimes I_{\hat{\mathcal{G}}})(\tilde{U}_b\otimes V_b)$, then for all $c,d \in \{0\} \times G$,

$$\| \phi_c^* [\tilde{U} \tilde{T}_d \tilde{U}^* - \tilde{V} \tilde{T}_d \tilde{V}^*] \phi_\imath \tilde{U}_c^* \varphi_0 \otimes \varphi_0\| \leq \frac{\epsilon^2}{4} . \tag{3.9}$$

Proof. Fix c and d in $\{0\} \times G$. By Lemma 1.2,

$$I_{\tilde{\mathcal{H}}} = \phi_\imath^* \tilde{U} \tilde{U}^* \phi_\imath = \sum_{a\in\mathcal{G}} \tilde{M}_a \tilde{M}_a^* , \tag{3.10}$$

where the sum converges strongly. Hence there is a finite subset $\mathfrak{F}_0$ of $\mathcal{G}$ such that

$$(\sum_{a\notin\mathfrak{F}_o} \tilde{M}_a \tilde{M}_a^* \varphi_o \otimes \varphi_o , \varphi_o \otimes \varphi_o) = \sum_{a\notin\mathfrak{F}_o} \|\tilde{M}_a \varphi_o \otimes \varphi_o\|^2 \leq \frac{\epsilon^4}{256} .$$

Fix a finite subset $\mathfrak{F}$ of $\mathcal{G}$ containing $\mathfrak{F}_o$ and let

$$\tilde{W} = \sum_{b\in\mathfrak{F}} (\tilde{M}_b \otimes I)(\tilde{U}_b \otimes V_b) .$$

Again using Lemma 1.2, we find that

$$\phi_c^* \tilde{U} \tilde{T}_d \tilde{U}_l^* \phi_l \tilde{U}_c^* =$$

$$= \sum_{a\in\mathcal{G}} [\phi^*_{ca^{-1}} \tilde{U} \tilde{T}_d \phi_l \tilde{U}^*_{ca^{-1}}] \tilde{U}_{ca^{-1}} [\tilde{U}_a \tilde{M}^*_{a^{-1}} \tilde{U}^*_a] \tilde{U}^*_{ca^{-1}}$$

$$= \sum_{a\in\mathcal{G}} \tilde{M}_{ca^{-1}d^{-1}} \tilde{U}_c \tilde{M}^*_{a^{-1}} \tilde{U}^*_c$$

$$= \sum_{a\in\mathcal{G}} \tilde{M}_a \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c , \tag{3.11}$$

and similarly

$$\phi_c^* \tilde{W} \tilde{T}_d \tilde{W}^* \phi_l \tilde{U}_c^* = \sum_{a\in\mathfrak{F}'} \tilde{M}_a \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c , \tag{3.12}$$

where the sum in (3.11) converges strongly and $\mathfrak{F}' = \mathfrak{F} \cap c\,\mathfrak{F}\, d^{-1}$ (we use the convention that the empty sum is zero). By means of (3.10) and Lemma 3.18, we obtain

$$\| \phi_c^* [\tilde{U} \tilde{T}_d \tilde{U}^* - \tilde{W} \tilde{T}_d \tilde{W}^*] \phi_l \tilde{U}_c^* \varphi_o \otimes \varphi_o \| =$$

$$= \| \sum_{a\notin\mathfrak{F}'} \tilde{M}_a \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c \varphi_o \otimes \varphi_o \|$$

$$\leq \| \sum_{a \in \mathfrak{F}-\mathfrak{F}'} \tilde{M}_a \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c \varphi_o \otimes \varphi_o \| +$$

$$+ \| \sum_{a \notin \mathfrak{F}} \tilde{M}_a \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c \varphi_o \otimes \varphi_o \|$$

$$\leq \Big(\sum_{a \in \mathfrak{F}-\mathfrak{F}'} \| \tilde{U}_c \tilde{M}^*_{c^{-1}ad} \tilde{U}^*_c \varphi_o \otimes \varphi_o \|^2 \Big)^{\frac{1}{2}} +$$

$$+ \Big(\sum_{a \notin \mathfrak{F}} \| \tilde{M}_a \varphi_o \otimes \varphi_o \|^2 \Big)^{\frac{1}{2}}$$

$$\leq \Big(\sum_{a \in \mathfrak{F}-\mathfrak{F}'} \| \tilde{M}^*_{c^{-1}ad} \varphi_o \otimes \varphi_o \|^2 \Big)^{\frac{1}{2}} + \frac{\epsilon^2}{16}$$

$$\leq \Big(\sum_{a \notin \mathfrak{F}} \| \tilde{M}^*_a \varphi_o \otimes \varphi_o \|^2 \Big)^{\frac{1}{2}} + \frac{\epsilon^2}{16}$$

$$\leq \frac{\epsilon^2}{8} .$$

It follows from (3.10) that each $\tilde{M}_a$ is in the unit ball of $\tilde{\mathfrak{M}}$. Hence, by Lemma 3.11 and the Kaplansky density theorem [6; p. 46], there is a family $(\tilde{N}_b)_{b \in \mathfrak{F}}$ of elements in the unit ball of $\tilde{\mathcal{J}}$ such that

$$\| \tilde{M}_b \varphi_o \otimes \varphi_o - \tilde{N}_b \varphi_o \otimes \varphi_o \| \leq \min \left(\frac{\epsilon}{2}, \frac{\epsilon^2}{16n}\right) \quad \text{for all } b \in \mathfrak{F} ,$$

where n is the number of elements in $\mathfrak{F}$. In particular, (3.8) is satisfied. Letting $\tilde{V} = \sum_{b \in \mathfrak{F}} (\tilde{N}_b \otimes I)(\tilde{U}_b \otimes V_b)$, we have that (cf. (3.12))

$$\| \phi_c^*[\tilde{W}\ \tilde{T}_d\ \tilde{W}^* - \tilde{V}\ \tilde{T}_d\ \tilde{V}^*]\ \phi_c\ \tilde{U}_c^*\ \varphi_o \otimes \varphi_o \| =$$

$$= \| \sum_{a\in\mathfrak{F}'} (\tilde{M}_a\ \tilde{U}_c\ \tilde{M}^*_{c^{-1}ad}\ \tilde{U}_c^* - \tilde{N}_a\ \tilde{U}_c\ \tilde{N}^*_{c^{-1}ad}\ \tilde{U}_c^*)\varphi_o \otimes \varphi_o \|$$

$$\leq \| \sum_{a\in\mathfrak{F}'} \tilde{M}_a\ \tilde{U}_c\ (\tilde{M}^*_{c^{-1}ad} - \tilde{N}^*_{c^{-1}ad})\ \tilde{U}_c^*\ \varphi_o \otimes \varphi_o \| +$$

$$+ \| \sum_{a\in\mathfrak{F}'} (\tilde{M}_a - \tilde{N}_a)\ \tilde{U}_c\ \tilde{N}^*_{c^{-1}ad}\ \tilde{U}_c^*\ \varphi_o \otimes \varphi_o \|$$

$$\leq \sum_{a\in\mathfrak{F}'} \| \tilde{M}_a\ \tilde{U}_c\ (\tilde{M}^*_{c^{-1}ad} - \tilde{N}^*_{c^{-1}ad})\ \tilde{U}_c^*\ \varphi_o \otimes \varphi_o \| +$$

$$+ \sum_{a\in\mathfrak{F}'} \| \tilde{U}_c\ \tilde{N}^*_{c^{-1}ad}\ \tilde{U}_c^*(\tilde{M}_a - \tilde{N}_a)\ \varphi_o \otimes \varphi_o \|$$

$$\leq \sum_{a\in\mathfrak{F}'} \| (\tilde{M}^*_{c^{-1}ad} - \tilde{N}^*_{c^{-1}ad})\ \varphi_o \otimes \varphi_o \| +$$

$$+ \sum_{a\in\mathfrak{F}'} \|(\tilde{M}_a - \tilde{N}_a)\ \varphi_o \otimes \varphi_o \|$$

$$\leq \frac{\epsilon^2}{8} .$$

Combining the last two inequalities by means of the triangle inequality gives the estimate (3.9).

<u>Lemma 3.20</u> (cf. [20; Lemma 17]). Suppose G_o is a subgroup of G satisfying

(β) : given a finite subset F of G and a $g \in G$, there are infinitely many $g_o \in G_o$ such that:

(i) $h,k \in F$ and $h\ g_o\ k^{-1} = g_o$ imply $h = k$

(ii) if $g \notin N(G_o)$, then $g\ g_o\ g^{-1} \notin G_o$.

Then $N(\tilde{\mathcal{P}}(G_o)) = N(\tilde{\mathcal{S}}(G_o)) = \tilde{\mathcal{S}}(N(G_o))$.

<u>Proof</u>. It is easy to see that

$$\tilde{\mathcal{S}}(N(G_o)) \subset N(\tilde{\mathcal{P}}(G_o)) \ , \quad \tilde{\mathcal{S}}(N(G_o)) \subset N(\tilde{\mathcal{S}}(G_o)) \ .$$

Conversely, suppose that we are given a unitary operator $\tilde{U} \in \tilde{\mathfrak{a}}$ satisfying one of

$$\tilde{U}\,\tilde{\mathcal{P}}\,(G_o)\tilde{U}^* = \tilde{\mathcal{P}}(G_o) \tag{3.13}$$

$$\tilde{U}\,\tilde{\mathcal{S}}\,(G_o)\tilde{U}^* = \tilde{\mathcal{S}}(G_o) \ . \tag{3.14}$$

We will be done if we can show that $\tilde{U} \in \tilde{\mathcal{S}}\,(N(G_o))$.

Let $\tilde{U} \sim [\tilde{M}_a : a \in \mathcal{G}]$, and for each $a \in \mathcal{G}$, let

$$\tilde{M}_a\, \varphi_o \otimes \varphi_o = \sum_{\alpha,\beta\in\Delta} \theta(a;\ \alpha,\beta)\ \varphi_\alpha \otimes \varphi_\beta \ ,$$

where the $\theta(a;\ \alpha,\beta)$ are complex numbers. Suppose we knew that

$$\theta(a;\ \alpha,\beta) = 0 \qquad \text{whenever} \quad \beta \neq 0 \tag{3.15}$$

$$\theta(a;\ \alpha,0) = 0 \qquad \text{whenever} \quad a \notin \Delta \times N(G_o) \ . \tag{3.16}$$

If $a \notin \Delta \times N(G_o)$, then $\tilde{M}_a\, \varphi_o \otimes \varphi_o = 0$; as $\varphi_o \otimes \varphi_o$ is separating for $\tilde{\mathcal{M}}$, $\tilde{M}_a = 0$. And if $a \in \Delta \times N(G_o)$, then for all $\alpha,\beta \in \Delta$,

$$\begin{aligned} M_a\, \varphi_\alpha \otimes \varphi_\beta &= (P_\alpha \otimes P_\beta)\, \tilde{M}_a\, \varphi_o \otimes \varphi_o \\ &= P_\alpha \otimes P_\beta \sum_{\gamma\in\Delta} \theta(a;\ \gamma,0)\ \varphi_\gamma \otimes \varphi_o \end{aligned}$$

$$= \Big(\sum_{\gamma\in\Delta} \theta(a;\ \gamma,0)\ P_\alpha\, \varphi_\gamma \Big) \otimes \varphi_\beta \ ,$$

and therefore $\tilde{M}_a \in \mathcal{M} \otimes \mathbb{C}_{\mathcal{H}}$. Lemma 3.17 now implies that $\tilde{U} \in \mathfrak{J}(N(G_o))$. Hence it is sufficient to show that (3.15) and (3.16) hold.

Fix an $(\alpha,g) \in \mathcal{G}$ and an $(\alpha_1,\alpha_2) \in \Delta \times \Delta$, and let $\epsilon > 0$ be given. Applying Lemma 3.19 to $\tilde{U}$ and ϵ , we get a finite subset $\mathfrak{F}_o$ of $\mathcal{G}$ and, with $\mathfrak{F} = \mathfrak{F}_o \cup \{(\alpha,g)\}$, a family $(\tilde{N}_b)_{b\in\mathfrak{F}}$ of elements of $\tilde{\mathfrak{J}}$ satisfying (3.8) and (3.9). By the finiteness of $\mathfrak{F}$ and the definition of $\tilde{\mathfrak{J}}$ (Lemma 3.11), there are complex numbers $\sigma(b;\ \beta,\gamma)$ and an $\bar{\alpha} \in \Delta$ such that

$$(3.17) \qquad \tilde{N}_b = \sum_{\beta,\gamma\leq\bar{\alpha}} \sigma(b;\ \beta,\gamma)\ P_\beta \otimes P_\gamma \qquad \text{for all} \quad b \in \mathfrak{F}\ ;$$

without loss of generality, we may assume that $\alpha_1,\alpha_2 \leq \bar{\alpha}$. From (3.8)

$$(3.18) \qquad \frac{\epsilon}{2} \geq \| \tilde{M}_b\ \varphi_o \otimes \varphi_o - \tilde{N}_b\ \varphi_o \otimes \varphi_o \|$$

$$\geq |\theta(b;\ \beta,\gamma) - \sigma(b;\ \beta,\gamma)|$$

for all $b \in \mathfrak{F}$ and all $\beta,\gamma \leq \bar{\alpha}$. Let

$$F = \{h \in G : (\beta,h) \in \mathfrak{F} \ \text{ for some } \ \beta \in \Delta\},$$

a finite subset of G containing g . Applying condition (β) to the set $g^{-1}F$ and the element g and using the fact that $\bar{\alpha}$ has finite support, we can find a $g_o \in G_o$ such that

$$(3.19) \qquad \bar{\alpha} \wedge g\, g_o\, g^{-1}\, \bar{\alpha} = 0$$

(3.20) $\qquad h,k \in g^{-1} F$ and $h\, g_o\, k^{-1} = g_o$ imply $h = k$

(3.21) $\qquad$ if $g \notin N(G_o)$, then $g\, g_o\, g^{-1} \notin G_o$.

In order to simplify the notation, let $h = g\, g_o\, g^{-1}$, let $c = (0, g_o)$, and let $d = (0, g\, g_o\, g^{-1})$. Let $\tilde{V}$ be as in (3.9) of Lemma 3.19 , let $\mathfrak{F}' = \mathfrak{F} \cap d\, \mathfrak{F}\, c^{-1}$, and let

$$S = \phi_d^* \tilde{V} \tilde{T}_c \tilde{V}^* \phi_\iota \tilde{U}_d \ .$$

Notice that $(\alpha, g) \in \mathfrak{F}'$, also that $\mathfrak{F}'$ is not empty. Now (cf. (3.12))

$$S = \sum_{a \in \mathfrak{F}'} \tilde{N}_a \tilde{U}_d \tilde{N}^*_{d^{-1}ac} \tilde{U}_d^*$$

$$= \sum_{\substack{a,b \in \mathfrak{F} \\ b = d^{-1}ac}} \tilde{N}_a \tilde{U}_d \tilde{N}_b^* \tilde{U}_d^* \ .$$

If $a = (\beta, k)$ and $b = (\gamma, \ell)$ are elements of $\mathfrak{F}$, the relation $d^{-1}ac = b$ implies that $(\beta, g\, g_o^{-1}\, g^{-1}\, k\, g_o) = (\gamma, \ell)$. Hence $\beta = \gamma$ and $(g^{-1}k)\, g_o\, (g^{-1}\, \ell)^{-1} = g_o$; as $k, \ell \in F$, (3.20) may be applied, giving $k = \ell$. Therefore $a = b$, and the double sum reduces to a single sum. On substituting (3.17) into this sum we get

$$S = \sum_{b \in \mathfrak{F}'} \tilde{N}_b \tilde{U}_d \tilde{N}_b^* \tilde{U}_d^*$$

$$= \sum_{\substack{b \in \mathfrak{F}' \\ \beta,\gamma,\delta,\eta \leq \bar{\alpha}}} \sigma(b; \beta,\gamma) \overline{\sigma(b; \delta,\eta)}\, (P_\beta \otimes P_\gamma) \cdot$$

$$\cdot\, \tilde{U}_d (P_\delta \otimes P_\eta)\, \tilde{U}_d^*$$

$$= \sum_{\substack{b\in\mathfrak{F}' \\ \beta,\gamma,\delta,\eta\leq\bar{\alpha}}} \sigma(b;\beta,\gamma)\overline{\sigma(b;\delta,\eta)}(P_\beta\otimes P_\gamma)(P_\delta\otimes U_h P_\eta U_h^*)$$

$$= \sum_{\substack{b\in\mathfrak{F}' \\ \beta,\gamma,\delta,\eta\leq\bar{\alpha}}} \sigma(b;\beta,\gamma)\overline{\sigma(b;\delta,\eta)}\ P_\beta P_\delta \otimes P_\gamma P_{h\eta} \ .$$

From (3.19) and the assumption that $\alpha_2 \leq \bar{\alpha}$, it follows that $\gamma,\eta \leq \bar{\alpha}$ and $\alpha_2 + h\alpha_2 = \gamma + h\eta$ together imply that $\gamma = \eta = \alpha_2$. Hence, for all $\beta,\gamma,\delta,\eta \leq \bar{\alpha}$,

$$(P_\beta P_\delta \otimes P_\gamma P_{h\eta}\ \varphi_o \otimes \varphi_o\ ,\ I\otimes P_{\alpha_2+h\alpha_2}\ \varphi_o\otimes\varphi_o) =$$

$$= (P_\beta P_\delta\ \varphi_o,\varphi_o)(P_{\gamma+h\eta}\ \varphi_o\ ,\ P_{\alpha_2+h\alpha_2}\ \varphi_o)\ .$$

$$= \begin{cases} 1 & \beta=\delta \text{ and } \gamma=\eta=\alpha_2 \\ 0 & \text{otherwise,}\end{cases}$$

and therefore

$$(S\varphi_o\otimes\varphi_o\ ,\ I\otimes P_{\alpha_2+h\alpha_2}\ \varphi_o\otimes\varphi_o) =$$

$$= \sum_{\substack{b\in\mathfrak{F}' \\ \beta\leq\bar{\alpha}}} |\sigma(b;\ \beta,\alpha_2)|^2$$

$$\geq |\sigma((\alpha,g);\ \alpha_1,\alpha_2)|^2\ .$$

To show that (3.16) holds, suppose that $(\alpha,g) \notin \Delta \times N(G_o)$ and that $\alpha_2 = 0$. Let

$$T = \phi_d^*\ \tilde{U}\ \tilde{T}_c\ \tilde{U}^*\ \phi_2\ \tilde{U}_d^*\ .$$

As $\tilde{U}$ satisfies one of (3.13), (3.14), $T = 0$ (Lemma 3.17 and (3.21)). The inequality (3.9) now gives

$$\begin{aligned}\frac{\epsilon^2}{4} &\geq \| T\varphi_o \otimes \varphi_o - S\varphi_o \otimes \varphi_o \| \\ &\geq |(S\varphi_o \otimes \varphi_o , \varphi_o \otimes \varphi_o)| \\ &\geq |\sigma((\alpha,g);\ \alpha_1,0)|^2 \quad .\end{aligned}$$

Combining this estimate with (3.18), we get

$$\begin{aligned}|\theta((\alpha,g);\ \alpha_1,0)| &\leq |\theta((\alpha,g);\ \alpha_1,0) - \sigma((\alpha,g);\ \alpha_1,0)| + \\ &\quad + |\sigma((\alpha,g);\ \alpha_1,0)| \\ &\leq \epsilon \quad .\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that $\theta((\alpha,g);\ \alpha_1,0) = 0$, and therefore that (3.16) holds.

To show that (3.15) holds, suppose that $\alpha_2 \neq 0$. If $g \notin N(G_o)$, then, as before, $T = 0$; and if $g \in N(G_o)$, then $T \in \mathcal{M} \otimes \mathbb{C}_{\mathcal{H}}$ for similar reasons. In any case,

$$(T\varphi_o \otimes \varphi_o , \varphi_o \otimes \varphi_{\alpha_2+h\alpha_2}) = 0 ,$$

and therefore (using (3.9))

$$\begin{aligned}\frac{\epsilon^2}{4} &\geq \| T\varphi_o \otimes \varphi_o - S\varphi_o \otimes \varphi_o \| \\ &\geq |(T\varphi_o \otimes \varphi_o - S\varphi_o \otimes \varphi_o , \varphi_o \otimes \varphi_{\alpha_2+h\alpha_2})| \quad .\end{aligned}$$

$$= |(S\varphi_o \otimes \varphi \ , \ I \otimes P_{\alpha_2 + h\alpha_2} \varphi_o \otimes \varphi_o)|$$

$$\geq |\sigma((\alpha,g);\ \alpha_1,\alpha_2)|^2 \ .$$

As before, this implies that $\theta(\alpha,g);\ \alpha_1,\alpha_2) = 0$.

This completes the proof of Lemma 3.20.

Lemma 3.21 There is an isomorphism Φ_o of $\mathcal{C}_G$ onto $\mathcal{N}(p,G,G_o)$ such that

$$\Phi_o(V_g) = U_g \otimes V_g \qquad \text{for all } g \in G$$

$$\Phi_o(\mathcal{M}(G,G_o)) = \mathcal{N}(p,G,G_o) \qquad \text{for all subgroups } G_o \text{ of } G \ .$$

Proof. Let η be the unique unitary operator on $\mathcal{H} \otimes \hat{G}$ with $\eta(\varphi_\alpha \otimes \hat{g}) = \varphi_{g\alpha} \otimes \hat{g}$ for all $\alpha \in \Delta$, $g \in G$. For any $T \in \mathcal{C}_G$, let $\Phi_o(T) = \eta(I_{\mathcal{H}} \otimes T)\eta^{-1}$. It is trivial that Φ_o is a normal $*$-isomorphism of $\mathcal{C}_G$ into $\mathcal{L}(\mathcal{H} \otimes \hat{G})$ with $\Phi_o(I) = I$. If $g \in G$, then for all $\alpha \in \Delta$, $h \in G$,

$$\begin{aligned} \Phi_o(V_g)\ \varphi_\alpha \otimes \hat{h} &= \eta(I \otimes V_g)\ \eta^{-1}\ \varphi_\alpha \otimes \hat{h} \\ &= \eta(\varphi_{h^{-1}\alpha} \otimes (gh)^\wedge) \\ &= \varphi_{g\alpha} \otimes (gh)^\wedge \\ &= (U_g \otimes V_g)(\varphi_\alpha \otimes \hat{h}) \ , \end{aligned}$$

and therefore $\Phi_o(V_g) = U_g \otimes V_g$. Using [6; p.57], we have that for any subgroup G_o of G ,

$$\Phi_o(\mathcal{M}(G,G_o)) = \Phi_o(\mathcal{R}(V_g : g \in G_o))$$

$$= \mathcal{R}(\Phi_o(V_g) : g \in G_o)$$

$$= \mathcal{N}(p,G,G_o) ;$$

in particular, this implies that

$$\Phi_o(\mathcal{C}_G) = \Phi_o(\mathcal{M}(G,G)) = \mathcal{N}(p,G,G) .$$

Lemma 3.22 There is a $*$-isomorphism Φ of $\mathcal{A}_p \otimes \mathcal{C}_G$ into $\mathcal{A}_p \otimes \mathcal{B}(p,G)$ such that for any subgroup G_o of G,

$$\Phi(\mathcal{M}_p \otimes \mathcal{M}(G,G_o)) = \mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)$$

$$\Phi(\mathcal{A}_p \otimes \mathcal{M}(G,G_o)) = \mathcal{A}_p \otimes \mathcal{N}(p,G,G_o) .$$

Proof. The result follows easily from Lemma 3.21 and [6; pp. 57 and 60].

Lemma 3.23 For any subgroup G_o of G, $\mathcal{C}_{G_o} \cong \mathcal{M}(G,G_o)$.

Proof. As $G_o \subset G$, we may consider $\hat{G}_o$ to be a subspace of $\hat{G}$. If $T \in \mathcal{M}(G,G_o)$ and $g \in G - G_o$, then (cf. Lemma 3.21) $\Phi_o(T) \in \mathcal{N}(p,G,G_o)$, and therefore, using Corollary 3.8,

$$0 = (\phi_g^* \eta(I \otimes T) \eta^{-1} \phi_e \varphi_o , \varphi_o)$$

$$= (\eta(I \otimes T) \eta^{-1} \varphi_o \otimes \hat{e} , \varphi_o \otimes \hat{g})$$

$$= (I \otimes T \varphi_o \otimes \hat{e} , \varphi_o \otimes \hat{g})$$

$$= (T \hat{e} , \hat{g}) .$$

Now $\hat{G}_o$ is invariant under $\mathcal{M}(G,G_o)$, for if $T \in \mathcal{M}(G,G_o)$ and $g \in G_o$, then by the above calculation,

$$T \hat{g} = \sum_{h \in G} (T \hat{g}, \hat{h})\hat{h} = \sum_{h \in G} (T \hat{e} , (hg^{-1})^\wedge)\hat{h} =$$

$$= \sum_{k \in G_o} (T \hat{e}, \hat{k}) \, (kg)^\wedge .$$

Hence the restriction $\Phi'(T)$ of a $T \in \mathcal{M}(G,G_o)$ to $\hat{G}_o$ is an operator on $\hat{G}_o$. It is easy to verify that Φ' is a normal *-isomprphism of $\mathcal{M}(G,G_o)$ into $\mathcal{C}_{G_o}$.
Using [6; p. 57],

$$\begin{aligned} \Phi'(\mathcal{M}(G,G_o)) &= \Phi'(\mathcal{R}(V_g : g \in G_o)) \\ &= \mathcal{R}(\Phi'(V_g) : g \in G_o) \\ &= \mathcal{C}_{G_o} . \end{aligned}$$

Proof of Theorem 3.1. As $\mathcal{M}_p$ is MA in $\mathcal{Q}_p$, a result of Saitô and Tomiyama [22] implies that $\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)$ is MA in $\mathcal{Q}_p \otimes \mathcal{B}(p,G)$ if and only if $\mathcal{N}(p,G,G_o)$ is MA in $\mathcal{B}(p,G)$. But by Lemma 3.16, this is the case if any only if condition (α) holds.

Proof of Theorem 3.2. As $A \to \gamma A \gamma^{-1}$ is a normal *-isomorphism of $\mathcal{Q}_p \otimes \mathcal{B}(p,G)$ onto $\tilde{\mathcal{Q}}$ (cf. the text preceding Definition 3.15), $\gamma N(\mathcal{n}) \gamma^{-1} = N(\gamma \mathcal{n} \gamma^{-1})$ for any subalgebra $\mathcal{n}$ of $\mathcal{Q}_p \otimes \mathcal{B}(p,G)$., for

$$\gamma N(\mathcal{N})\gamma^{-1} = \gamma\, \mathcal{R}(U : U \in \mathcal{A}_p \otimes \mathcal{B}(p,G) \text{ and unitary, and } U\mathcal{N}U^* = \mathcal{N})\gamma^{-1}$$

$$= \mathcal{R}(\gamma U \gamma^{-1} : U \in \mathcal{A}_p \otimes \mathcal{B}(p,G) \text{ and unitary, and } U\mathcal{N}U^* = \mathcal{N})$$

$$= \mathcal{R}(\tilde{U} : \tilde{U} \in \tilde{\mathcal{A}} \text{ and unitary, and } \tilde{U}\gamma\mathcal{N}\gamma^{-1}\tilde{U}^* = \gamma\mathcal{N}\gamma^{-1})$$

$$= N(\gamma\,\mathcal{N}\,\gamma^{-1}) .$$

In particular, using Definition 3.15 and Lemma 3.20,

$$N(\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)) = \gamma^{-1} N(\gamma\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)\gamma^{-1})\gamma$$

$$= \gamma^{-1} N(\tilde{\mathcal{P}}(G_o))\gamma$$

$$= \gamma^{-1} \mathcal{Z}(N(G_o))\gamma$$

$$= \mathcal{A}_p \otimes \mathcal{N}(p,G, N(G_o)) ,$$

and similarly,

$$N(\mathcal{A}_p \otimes \mathcal{N}(p,G,G_o)) = \mathcal{A}_p \otimes \mathcal{N}(p,G, N(G_o)) .$$

<u>Proof of Theorem 3.3</u> As $\mathcal{A}_p$ is a factor, $\mathcal{A}_p \otimes \mathcal{N}(p,G,G_o)$ is a factor if and only if $\mathcal{N}(p,G,G_o)$ is a factor [6; p. 30]. As the property of being a factor is preserved by isomorphisms, Lemmas 3.21 and 3.23 imply that $\mathcal{N}(p,G,G_o)$ is a factor if and only if $\mathcal{C}_{G_o}$ is a factor.

But the group operator algebra $\mathcal{C}_{G_o}$ is a factor if and only if G_o has the infinite conjugate class property.

Proof of Theorem 3.4 (cf. [5; Lemma 1]). As in the proof of Theorem 3.1, $\mathcal{M}_p \otimes \mathcal{M}(G,G_o)$ is MA in $\mathcal{A}_p \otimes \mathcal{C}_G$ if and only if $\mathcal{M}(G,G_o)$ is MA in $\mathcal{C}_G$. If (α) holds, then, by Lemmas 3.16 and 3.21, $\mathcal{N}(p,G,G_o) = \Phi_o(\mathcal{M}(G,G_o))$ is MA in $\mathcal{N}(p,G,G) = \Phi_o(\mathcal{C}_G)$, and so $\mathcal{M}(G,G_o)$ is MA in $\mathcal{C}_G$. Conversely, if $\mathcal{M}(G,G_o)$ is MA in $\mathcal{C}_G$, a calculation similar to that in the proof of the "only if" part of Lemma 3.16 shows that condition (α) must be satisfied.

Proof of Theorem 3.5 First of all, it is clear that

$$\mathcal{A}_p \otimes \mathcal{M}(G, N(G_o)) \subset N(\mathcal{M}_p \otimes \mathcal{M}(G,G_o))$$

$$\mathcal{A}_p \otimes \mathcal{M}(G, N(G_o)) \subset N(\mathcal{A}_p \otimes \mathcal{M}(G,G_o)) \quad .$$

To show that the opposite inclusions hold, let U be a unitary operator in $\mathcal{A}_p \otimes \mathcal{C}_G$ satisfying one of

$$U \mathcal{M}_p \otimes \mathcal{M}(G,G_o)\, U^* = \mathcal{M}_p \otimes \mathcal{M}(G,G_o)$$

$$U \mathcal{A}_p \otimes \mathcal{M}(G,G_o)\, U^* = \mathcal{A}_p \otimes \mathcal{M}(G,G_o) \quad .$$

By Lemma 3.22, $\Phi(U)$ is a unitary operator in $\mathcal{A}_p \otimes \mathcal{B}(p,G)$ such that either

$$\Phi(U)\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)\Phi(U)^* = \Phi(U \mathcal{M}_p \otimes \mathcal{M}(G,G_o)\, U^*)$$

$$= \mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)$$

or $$\Phi(U)\, \mathfrak{a}_p \otimes \mathcal{N}(p,G,G_o)\Phi(U)^* = \Phi(U\, \mathfrak{a}_p \otimes \mathcal{M}(G,G_o)U^*)$$

$$= \mathfrak{a}_p \otimes \mathcal{N}(p,G,G_o) \quad .$$

By Lemma 3.20, $\Phi(U) \in \mathfrak{a}_p \otimes \mathcal{N}(p,G, N(G_o))$, and therefore $U \in \mathfrak{a}_p \otimes \mathcal{M}(G,N(G_o)))$.

<u>Proof of Theorem 3.6</u>. Similar to the proof of Theorem 3.3.

4 EXAMPLES OF MAXIMAL ABELIAN SUBALGEBRAS

We begin by stating in four theorems the main results of this thesis. After a brief discussion of the constructions of the previous section, we turn to the proofs of the theorems.

<u>Theorem 4.1</u> Each of the type III factors $\mathfrak{a}_p$, $0 < p < \frac{1}{2}$, contains a semi-regular MA subalgebra.

<u>Theorem 4.2</u> For each integer $m \geq 2$ and each $p \in (0,\frac{1}{2})$, $\mathfrak{a}_p$ contains two 2-semi-regular MA subalgebras, one of improper length m and one of proper length m.

<u>Theorem 4.3</u> For each integer $m \geq 3$ and each $p \in (0,\frac{1}{2})$, $\mathfrak{a}_p$ contains two 3-semi-regular MA subalgebras, one of improper length m and one of proper length m.

<u>Theorem 4.4</u> For each integer $m \geq 2$, the hyperfinite II_1 factor contains

(i) a 2-semi-regular MA subalgebra of improper length m

(ii) a 3-semi-regular MA subalgebra of improper length $m+1$.

The factors $\mathfrak{a}_p$, $p \in (0,\frac{1}{2})$, were first studied by Pukánszky, who obtained them by a measure-theoretic construction [20]. In this paper Pukánszky also constructs, for each $p \in (0,\frac{1}{2})$ and each countably infinite group G, a type III factor $\mathcal{C}(p,G)$ and, for each subgroup G_o of G, a subalgebra $\mathcal{P}(p,G,G_o)$ of $\mathcal{C}(p,G)$. That $\mathcal{P}(p,G,G_o)$ is MA in $\mathcal{C}(p,G)$ whenever G_o satisfies condition (α) of Theorem 3.1 is not difficult to show. It is reasonable to conjecture that

$$N(\mathcal{P}(p,G,G_o)) = \mathcal{P}(p,G, N(G_o))$$

under condition (β) of Theorem 3.2; however, Pukánszky's proof of this statement is not valid. Our algebra $\mathcal{C}_p \otimes \mathcal{B}(p,G)$ is obtained by modifying the construction of Pukánszky's $\mathcal{C}(p,G)$.

Powers has shown that if $0 < p < q < \frac{1}{2}$, then $\mathcal{C}_p$ and $\mathcal{C}_q$ are non-isomorphic; unfortunately, his proof depends heavily on C^*-algebra techniques ([18], [19]). Araki and Woods have given a proof of this result which uses only methods of von Neumann algebras [2]; in addition, they show that

$$(4.1) \qquad \mathcal{C}_p \otimes \mathcal{B} \cong \mathcal{C}_p \qquad \text{for each } p \in (0,\tfrac{1}{2}),$$

where $\mathcal{B}$ is the hyperfinite II_1 factor.

Proof of Theorem 4.1 Recall the conditions (α) and (β) of Theorems 3.1 and 3.2, respectively. We first show that it will suffice to construct a countably infinite hyperfinite group G with the infinite conjugate class property and a normal subgroup G_o of G satisfying conditions (α) and (β). For then, by Theorems 3.1, 3.2, and 3.3, $\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)$ is a MA subalgebra of $\mathcal{C}_p \otimes \mathcal{B}(p,G)$ with normalizer $\mathcal{G}_p \otimes \mathcal{N}(p,G,G)$, a factor distinct from $\mathcal{G}_p \otimes \mathcal{B}(p,G)$. Applying the isomorphism (4.1) and Lemma 3.14, we are done.

We now turn to the construction of such a G and G_o . Let F be a countably infinite field which is the increasing union of a sequence of finite subfields (in particular, we may take for the F the algebraic completion of a finite field). The set

$$G = \{(\alpha,\beta) : \alpha,\beta \in F \text{ and } \alpha \neq 0\}$$

becomes a group under the operation

$$(\alpha,\beta)(\gamma,\delta) = (\alpha\,\gamma,\ \alpha\,\delta + \beta) .$$

It is easy to see that G is countably infinite and hyperfinite. To verify that G has the infinite conjugate class property, let a $(\alpha,\beta) \in G - \{(1,0)\}$ be given. For all $(\gamma,\delta) \in G$,

$$\begin{aligned}(\gamma,\delta)(\alpha,\beta)(\gamma,\delta)^{-1} &= (\gamma\,\alpha,\ \gamma\,\beta + \delta)(\gamma^{-1},\ -\gamma^{-1}\,\delta) \\ &= (\alpha,\ -\alpha\,\delta + \gamma\,\beta + \delta) .\end{aligned}$$

If $\alpha = 1$, then $\beta \neq 0$, and so $-\alpha\,\delta + \gamma\,\beta + \delta = \gamma\,\beta$ runs through infinitely many elements as γ runs through $F - \{0\}$; and if $\alpha \neq 1$, $-\alpha\,\delta + \gamma\,\beta + \delta$ runs through infinitely many elements as δ runs through F .

It is easy to verify that

$$G_o = \{(1,\beta) : \beta \in F\}$$

is a normal subgroup of G. The subgroup G_o has property (α), for if $(\alpha,\beta) \in G - G_o$, then $\alpha \neq 1$, and so

$$(1,\gamma)(\alpha,\beta)(1,\gamma)^{-1} = (\alpha,\beta + \gamma)(1,-\gamma)$$

$$= (\alpha, -\alpha\gamma + \beta + \gamma)$$

runs through infinitely many elements as γ runs through F. Finally, we show that G_o has property (β). Let $g_1,\ldots,g_n \in G$ be given, with, say,

$$g_i = (\alpha_i,\beta_i) \qquad i = 1,\ldots,n .$$

Let

$$H = \{(1 - \alpha_i)^{-1}(\beta_i - \beta_j) : \alpha_i \neq 1 \text{ and } 1 \leq i,j \leq n\} ,$$

a finite subset of F. If $g_o = (1,\beta_o)$ for some $\beta_o \in F - H$ and if $g_i\, g_o\, g_j^{-1} = g_o$, then

$$(1,\beta_o) = (\alpha_i,\beta_i)(1,\beta_o)(\alpha_j,\beta_j)^{-1}$$

$$= (\alpha_i,\alpha_i\beta_o + \beta_i)(\alpha_j^{-1}, -\alpha_j^{-1}\beta_j)$$

$$= (\alpha_i\alpha_j^{-1} , -\alpha_i\alpha_j^{-1}\beta_j + \alpha_i\beta_o + \beta_i) .$$

Hence $\alpha_i = \alpha_j$, and so $\beta_o = -\beta_j + \alpha_i\beta_o + \beta_i$. If $\alpha_i \neq 1$, then $\beta_o = (1 - \alpha_i)^{-1}(\beta_i - \beta_j)$, a contradiction; therefore $\alpha_i = 1$, and thus $\beta_i = \beta_j$, i.e., $g_i = g_j$.

<u>Proof of Theorem 4.2</u> Fix a $p \in (0,\frac{1}{2})$ and an integer $m \geq 2$. Suppose that we had a countably infinite hyperfinite group G with the infinite conjugate class property and a

subgroup G_o of G such that

(4.2) (i) G_o satisfies condition (α) of Theorem 3.1

(4.3) (ii) $G_o \subsetneqq N(G_o) \subsetneqq \cdots \subsetneqq N^m(G_o) = G$, and each $N^k(G_o)$, $0 \leq k \leq m-1$, satisfies condition (β) of Theorem 3.2

(4.4) (iii) $N(G_o)$ does not have the infinite conjugate class property while $N^2(G_o)$ does.

Then, from Section 3, $\mathfrak{M}_p \otimes \mathfrak{N}(p,G,G_o)$ is a 2-semi-regular MA subalgebra of $\mathfrak{a}_p \otimes \mathfrak{B}(p,G)$ of improper length m and $\mathfrak{M}_p \otimes \mathfrak{M}(G,G_o)$ is a 2-semi-regular MA subalgebra of $\mathfrak{a}_p \otimes \mathcal{C}_G$ of proper length m. As $\mathfrak{B}(p,G)$ and $\mathcal{C}_G$ are both hyperfinite II_1 factors, two applications of (4.1) completes the proof of the theorem. Hence it suffices to construct such a group G and subgroup G_o.

Again, let F be a countably infinite field which is the increasing union of a sequence of finite subfields. Let G be the group of all $(m+2) \times (m+2)$ matrices (g_{ij}) over F with

(4.5) $$g_{11} \neq 0$$

(4.6) $$g_{ii} = 1 \qquad i = 2,\ldots,m+2$$

(4.7) $$g_{ij} = 0 \qquad i > j,$$

and let G_o be the subgroup of G consisting of all those matrices (g_{ij}) in G with

$$g_{11} = 1$$

$$g_{12} = g_{23}$$

$$g_{2j} = 0 \qquad j = 4,\ldots,m+2$$

$$g_{ij} = 0 \qquad 3 \leq i < j \ .$$

The group G is clearly countably infinite and hyperfinite. Anastasio has shown that G has the infinite conjugate class property and that the subgroup G_o satisfies (i), (ii), and (iii) [1].

<u>Proof of Theorem 4.3</u> The proof is similar to that of Theorem 4.2. Let the field F be as before, and let a $p \in (0,\frac{1}{2})$ and an integer $m \geq 3$ be fixed. Let G be the group of all $(m+2) \times (m+2)$ matrices (g_{ij}) over F satisfying (4.5), (4.6), and (4.7), and let G_o be the subgroup of G consisting of all those matrices (g_{ij}) in G with

$$g_{11} = 1$$

$$g_{12} = g_{23} = g_{34}$$

$$g_{13} = g_{24}$$

$$g_{2j} = g_{3j} = 0 \qquad j = 5,\ldots,m+2$$

$$g_{ij} = 0 \qquad 4 \leq i < j \ .$$

Then G is a countably infinite hyperfinite group with the infinite conjugate class property (see [1]); moreover,

(i) G_o satisfies condition (α) of Theorem 3.1

(ii) $G_o \subsetneqq N(G_o) \subsetneqq \cdots \subsetneqq N^m(G_o) = G$, and each $N^k(G_o)$, $0 \leq k \leq m-1$, satisfies condition (β) of Theorem 3.2

(iii) $N(G_o)$ and $N^2(G_o)$ do not have the infinite conjugate class property while $N^3(G_o)$ does.

As before, this is sufficient to establish our theorem.

Before proceeding to the proof of Theorem 4.4, we must first prove

Lemma 4.5 Let p be a point in $(0,\frac{1}{2})$, let G be a countably infinite group, and let G_o be a subgroup of G . If G_o satisfies condition (β) of Theorem 3.2, then

$$N(\mathcal{N}(p,G,G_o)) = \mathcal{N}(p,G,N(G_o)) .$$

Proof. That $\mathcal{N}(p,G,N(G_o)) \subset N(\mathcal{N}(p,G,G_o))$ is trivial. For the converse, let a unitary operator U in $\mathcal{B}(p,G)$ with

$$U\,\mathcal{N}(p,G,G_o)\,U^* = \mathcal{N}(p,G,G_o)$$

be given. Then $I \otimes U$ is a unitary operator in $\mathcal{C}_p \otimes \mathcal{B}(p,G)$ such that

$$(I \otimes U)\mathcal{M}_p \otimes \mathcal{N}(p,G,G_o)\,(I \otimes U)^* = \mathcal{M}_p \otimes \mathcal{N}(p,G,G_o) .$$

According to Theorem 3.2, this forces $I \otimes U \in \mathcal{C}_p \otimes \mathcal{N}(p,G,N(G_o))$, and therefore $U \in \mathcal{N}(p,G,N(G_o))$.

<u>Proof of Theorem 4.4</u>. Let an integer $m \geq 2$ and a point p in $(0,\frac{1}{2})$ be fixed. Let the field F, the group G of $(m+2) \times (m+2)$ matrices over F and its subgroup G_o be as in the proof of Theorem 4.2. Then $\mathcal{B}(p,G)$ is the hyperfinite II_1 factor (Lemma 3.14) and $\mathcal{N}(p,G,G_o)$ is a MA subalgebra of $\mathcal{B}(p,G)$ (Lemma 3.16 and (4.2)). By Lemma 4.5 and (4.3), $\mathcal{N}(p,G,G_o)$ has improper length m. By Lemma 3.21, Lemma 3.23, and (4.3),

$$N(\mathcal{N}(p,G,G_o)) = \mathcal{N}(p,G,N(G_o)) \cong \mathcal{C}_{N(G_o)}$$

$$N^2(\mathcal{N}(p,G,G_o)) = \mathcal{N}(p,G,N^2(G_o)) \cong \mathcal{C}_{N^2(G_o)} \quad .$$

As the notion of a factor is an invariant under isomorphisms, (4.4) shows that $\mathcal{N}(p,G,G_o)$ is 2-semi-regular.

This proves (i). The proof of (ii) is similar, the groups and subgroups from the proof of Theorem 4.3 being employed.

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