SOME COMBINATORIAL PROPERTIES OF THE DIAGONAL SUMS OF DOUBLY STOCHASTIC MATRICES

by

EDWARD TZU-HSIA WANG

B.Sc., National Taiwan University, Taipei, Taiwan, 1965

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September, 1971

Supervisor: Professor R. Westwick

ABSTRACT

Let Ω_n denote the convex polyhedron of all $n\times n$ d.s. (doubly stochastic) matrices. The main purpose of this thesis is to study some combinatorial properties of the diagonal sums of matrices in Ω_n.

In Chapter I, we determine, for all d.s. matrices unequal to J_n; the maximum number of diagonals that can have a common diagonal sum. The key will be a Decomposition Theorem that enables us to characterize completely the structure of a d.s. matrix when this maximum number is attained, provided that the common sum is not one. When the common sum is one, the question is more difficult and remains open. Several applications of the Decomposition Theorem are also given.

In Chapter II, we concentrate on the diagonals with maximum diagonal sum h and the diagonals with minimum diagonal sum k. We obtain the best possible bounds for entries on these diagonals and for various kinds of functions of h and k. The key will be a Covering Theorem that enables us to analyze the cases when those bounds are attained. A conjecture is given.

In Chapter III, we study the properties of the h-function and the k-function, the functions that associate with each d.s. matrix its maximum and minimum diagonal sums respectively. In particular, we investigate the behavior of these functions on the Kronecker product of d.s. matrices. Furthermore, we show that the h-function is very similar to the rank function ρ in many respects. We also prove that for $A \varepsilon \Omega_n$, $h(A) \leq \rho(A)$ and $\mathrm{per}(A) \leq \{\frac{h(A)}{n}\}^{\frac{1}{2}}$ which improves a result by M. Marcus and H. Minc. A conjecture is given.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

I am deeply indebted to my supervisor, Dr. R. Westwick for rending geneous assistance and invaluable guidance throughout the research and preparation of this thesis.

My further thanks are due to Dr. B. N. Moyls, Dr. D. C. Murdoch, and Dr. A. Adler who carefully read the draft of this thesis and gave helpful suggestions and criticisms.

The financial support of the National Research Council of Canada and the University of British Columbia is gratefully acknowledged.

I dedicate this work to my wife for her constant encouragements during my research and for her excellent typing of this thesis.

INTRODUCTION

Let S_n denote the full symmetric group of degree n. If A is any $n \times n$ matrix and $\sigma \varepsilon S_n$, then the sequence of elements $a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}$ is called the diagonal of A corresponding to σ. Following the usual convention, we shall use σ to denote both the permutation σ and the diagonal corresponding to it. If σ is the identity permutation, then the diagonal corresponding to it is called the main diagonal of A. If for some $\sigma \varepsilon S_n$, $a_{i\sigma(i)} = 1$, $i = 1, 2, \ldots, n$, and $a_{ij} = 0$ otherwise, then A is called a permutation matrix. If σ is a diagonal of A, then the sum $\sum_{i=1}^{n} a_{i\sigma(i)}$ is called the diagonal sum of σ. Diagonal product is defined in a similar way.

An n-square real matrix $A = (a_{ij})$, where $0 \leq a_{ij} \leq 1$ for all $i,j = 1, 2, \ldots, n$ is called row stochastic if each row sum is one, column stochastic if each column sum is one, stochastic if it is either row stochastic or column stochastic and doubly stochastic if it is both row stochastic and column stochastic. The fundamental theorem in the theory of doubly stochastic matrices is the theorem of Birkhoff [1] which states that the set Ω_n of all n-square doubly stochastic matrices is a convex polyhedron with permutation matrices as vertices. (For more about this theorem, see [15, 18]).

Much of the study of doubly stochastic matrices is motivated by the well-known, long-standing and challenging van der Waerden conjecture [22] which states that for $A \in \Omega_n$, $\mathrm{per}(A) \geq n!/n^n$ with equality if and only if $A = J_n$, the doubly stochastic matrix with all entries equal to $1/n$. Here, per denotes the permanent function.

Desipte a great deal of effort, this conjecture is still unsolved though it is generally believed to be true [13]. The conjecture has been verified to be true for $n \leq 5$ [4, 5, 16], and many partial results, mainly due to Marcus, Minc and Newman have also been obtained [4, 5, 13, 16].

It has been pointed out by P. Erdös that the following two statements are consequences of the van der Waerden conjecture:

(1) There exists a diagonal σ of A such that $\prod_{i=1}^{n} a_{i\sigma(i)} \geq 1/n^n$.

(2) There exists a diagonal σ of A such that $\sum_{i=1}^{n} a_{i\sigma(i)} \geq 1$ and $a_{i\sigma(i)} > 0$ for $i = 1, 2, \ldots, n$.

In 1958, Marcus and Ree [18] verified (2) by proving that for $A \in \Omega_n$, there is a diagonal σ such that $\sum_{i=1}^{n} a_{i\sigma(i)} \geq \sum_{i,j=1}^{n} a_{ij}^2$ and $a_{i\sigma(i)} > 0$ for $i = 1, 2, \ldots, n$. Then in 1962, Marcus and Minc [10] verified (1) by proving that for $A \in \Omega_n$, $\max_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} \geq 1/n^n$ with equality if and only if $A = J_n$.

Erdös' observation and the proofs furnished by Marcus, Minc, and Ree suggested that local properties such as diagonal sum (as well as diagonal product) are closely related to the structure of doubly stochastic matrices. As an extreme case, notice that if any diagonal sum is n, then the doubly stochastic matrix must be a permutation matrix. A less trivial proposition, which will be used frequently in this work, also illustrates this point.

<u>Proposition</u>: If $A \in \Omega_n$, then

(i) $\max\limits_{\sigma \in S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \geq 1$ with equality if and only if $A = J_n$.

(ii) $\min\limits_{\sigma \in S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \leq 1$ with equality if and only if $A = J_n$.

Since for $A \in \Omega_n$, $\sum\limits_{i,j=1}^{n} a_{ij}^2 \geq 1$ with equality if and only if $a_{ij} = 1/n$ for all $i,j = 1, 2, \cdots, n$, i.e., if and only if $A = J_n$, (i) is an immediate consequence of Marcus and Ree's result. (ii) follows by applying (i) to the doubly stochastic matrix $B = (nJ_n - A)/(n - 1)$.

As we will see later, the above proposition can also be obtained independently as a corollary of one of our theorems (Theorem 1.3).

With this picture in mind, the main purpose of this work is to study some combinatorial properties of the diagonal sums of doubly stochastic matrices. The starting point will be the theorem of Marcus and Minc [10] which states that if $A \in \Omega_n$ has more than $(n-1)(n-1)!$ diagonals with a common non-zero diagonal product, then $A = J_n$.

In Chapter I, we first show that the above theorem still holds if we replace "non-zero diagonal product" by "diagonal sum". Then we pose this question, "For $A \varepsilon \Omega_n$, what is the maximum number of diagonals with sum greater than one and what is the maximum number of diagonals with sum less than one?" To answer this question, we obtain a decomposition theorem (Theorem 1.11) which shows that this common maximum number is again $(n-1)(n-1)!$, a somewhat surprising, if not significant, result. This theorem also enables us to characterize completely the structure of a doubly stochastic matrix when it has precisely $(n-1)(n-1)!$ diagonals with a common sum $\alpha \neq 1$. The decomposition theorem gives no information for $\alpha = 1$ in which case the question becomes more difficult and we are only able to obtain partial answer. Several consequences of this theorem are also derived.

In Chapter II, we confine ourselves to the diagonals with maximum diagonal sum h and the diagonals with minimum diagonal sum k. The key will be a covering theorem (Theorem 2.3) which enables us to obtain the best possible bounds for entries on these diagonals and for various kinds of functions of h and k, and to analyze the cases when these bounds are attained. When investigating the lower bound for $h + k$, a conjecture (Conjecture 2.21) naturally presents itself.

In Chapter III, we study the properties of the h-function and the k-function, the functions that associate with each doubly stochastic matrix its maximum and minimum diagonal sum respectively. In particular, we show that the h-function is convex and is very similar to the rank function ρ in many respects. We also investigate the behavior of these functions on the Kronecker product of doubly stochastic matrices. Furthermore, we prove that for $A \varepsilon \Omega_n$, $h(A) \leq \rho(A)$ and $\text{per}(A) \leq \{h(A)/n\}^{\frac{1}{2}}$ which improves a result of Marcus and Minc [11]. We also make the conjecture that the h-function defined on Ω_n obeys Sylvester's law for the rank function, i.e., $h(A) + h(B) - h(AB) \leq n$ for $A,B \varepsilon \Omega_n$.

On the whole, our study will be of a combinatorial nature. In particular, we will apply twice in this work (cf. Theorem 2.13 and Proposition 3.18) the well-known theorem of P. Hall [20] on systems of distinct representatives which states that the subsets $S_1, S_2, \cdots, S_n$ of a set S have a system of distinct representatives if and only if for each $k = 1, 2, \cdots, n$ and for all sequences ω of k terms such that $1 \leq \omega_1 < \omega_2 < \cdots < \omega_k \leq n$, it is true that $|S_{\omega_1} \cup S_{\omega_2} \cup \cdots \cup S_{\omega_k}| \geq k$.

CHAPTER I

DIADONAL SUMS OF d.s. MATRICES

In this chapter, we show that for d.s. matrices unequal to J_n, the maximum number of diagonals that can have a common sum is $(n-1)(n-1)!$. We then pose the questions, "For $A \varepsilon \Omega_n$, what is the maximum number of diagonals with sum greater than one and what is the maximum number of diagonals with sum less than one?" To answer these questions, we obtain a decomposition theorem which shows that the answers to both questions are again $(n-1)(n-1)!$. This theorem also enables us to characterize completely the structure of a d.s. matrix when there are precisely $(n-1)(n-1)!$ diagonals with a common sum $\alpha \neq 1$. For $\alpha = 1$, the question is more difficult and remains open.

In [10], Marcus and Minc proved the following results:

<u>Lemma 1.1</u>: Let A be an $n \times n$ matrix such that more than $(n-1)(n-1)!$ diagonals have a common non-zero product. Then $\rho(A) = 1$.

<u>Theorem 1.2</u>: Let $A \varepsilon \Omega_n$ be such that more than $(n-1)(n\ 1)!$ diagonals have a common non-zero product. Then $A = J_n$.

We begin this chapter by obtaining, using Lemma 1.1, an analogue of Theorem 1.2 concerning the diagonal sums of d.s. matrices.

<u>Theorem 1.3</u>: Let $A \varepsilon \Omega_n$ be such that more than $(n-1)(n-1)!$ diagonals have a common sum. Then $A = J_n$.

Proof: Let α denote the common diagonal sum. Define the matrix $B = (b_{ij})$ as: $b_{ij} = \exp(a_{ij})$. Then $b_{ij} > 0$ and $\ell n\, b_{ij} = a_{ij}$ for all $i,j = 1, 2, \cdots, n$. Hence $\ell n \prod_{i=1}^{n} b_{i\sigma(i)} = \sum_{i=1}^{n} a_{i\sigma(i)} = \alpha$ or $\prod_{i=1}^{n} b_{i\sigma(i)} = \exp(\alpha) \neq 0$ for more than $(n-1)(n-1)!$ diagonals σ. By Lemma 1.1, $\rho(B) = 1$ and hence there is one row of B, the 1st row say, such that every other row is a scalar multiple α_i of it, $i = 2, 3, \cdots, n$. Hence $\exp(a_{i1})/\exp(a_{11}) = \exp(a_{i2})/\exp(a_{12}) = \cdots = = \exp(a_{in})/\exp(a_{1n}) = \alpha_i > 0$ or $a_{i1} - a_{11} = a_{i2} - a_{12} = \cdots = = a_{in} - a_{1n} = \ell n\, (\alpha_i) = \beta_i$ for each $i = 2, 3, \cdots, n$. Therefore, $\sum_{j=1}^{n} a_{ij} = n\,\beta_i + \sum_{j=1}^{n} a_{1j}$. Since each row sum of A is one, we get $\beta_i = 0$ or $a_{i1} = a_{11}$, $a_{i2} = a_{12}, \cdots, a_{in} = a_{1n}$ for all $i = 2, 3, \cdots, n$. Since each column sum of A is also one, we get, for all $j = 1, 2, \cdots, n$, $n\, a_{1j} = 1$ or $a_{11} = a_{12} = \cdots = a_{1n} = 1/n$. Hence $A = J_n$.

Corollary 1.4: Let $A \in \Omega_n$. Then $\prod_{\sigma \in S_n} \sum_{i=1}^{n} a_{i\sigma(i)} \leq 1$, with equality iff $A = J_n$.

Proof: By the elementary arithmetic-geometric inequality, we have $\prod_{\sigma} \sum_{i=1}^{n} a_{i\sigma(i)} \leq \left\{\frac{1}{n!} \sum_{\sigma} \sum_{i=1}^{n} a_{i\sigma(i)}\right\}^{n!} = \left\{\frac{1}{n!}\, (n-1)! \sum_{i,j=1}^{n} a_{ij}\right\}^{n!} = = (n!/n!)^{n!} = 1$. We have equality iff $\sum_{i=1}^{n} a_{i\sigma(i)}$ are equal for all diagonals σ , and hence iff $A = J_n$.

The next corollary will be used frequently.

Corollary 1.5: Let $A \varepsilon \Omega_n$. Then

(i) $\max\limits_{\sigma\varepsilon S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \geq 1$ with equality iff $A = J_n$,

(ii) $\min\limits_{\sigma\varepsilon S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \leq 1$ with equality iff $A = J_n$.

Proof: (i) Since $n! \cdot \max\limits_{\sigma} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \geq \sum\limits_{\sigma} \sum\limits_{i=1}^{n} a_{i\sigma(i)} =$ $= (n-1)! \sum\limits_{i,j=1}^{n} a_{ij} = n!$, we get $\max\limits_{\sigma} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \geq 1$. Equality holds iff $\sum\limits_{i=1}^{n} a_{i\sigma(i)}$ are equal for all diagonals σ and hence iff $A = J_n$. The proof of (ii) is similar.

The next theorem is an analogue to a result of Sinkhorn and Knopp [21, corollary 4].

Theorem 1.6: Let $A,B \varepsilon \Omega_n$ be such that more than $(n-1)(n-1)!$ corresponding diagonals of A and B have equal sums. Then $A = B$.

Proof: Put $C = \frac{1}{n}(A - B + n J_n)$. Then $C \varepsilon \Omega_n$ and more than $(n-1)(n-1)!$ diagonals of C have the common sum one. Hence $C = J_n$ by Theorem 1.3. Therefore $A = B$.

Remark 1.7: (i) The number $(n-1)(n-1)!$ in Theorem 1.2 and 1.3 above is best possible in the sense that it is attainable for all n .

For consider the matrix

$$A = \begin{bmatrix} x & y & \cdots & y \\ y & z & \cdots & z \\ \vdots & \vdots & & \vdots \\ y & z & \cdots & z \end{bmatrix}$$

where $0 \leq x,y,z \leq 1$ are such that $x + (n-1)y = y + (n-1)z = 1$. Then clearly $A \in \Omega_n$. Since this particular matrix will be considered frequently in the sequel, we call it a (x,y,z)-matrix, and denote it by $A(x, y, z)$. If we choose x, y, and z such that $y^2z^{n-2} \neq xz^{n-1}$ (i.e., $y^2 \neq xz$), then $A \neq J_n$ and there are $(n-1)!$ diagonals passing through x with the common non-zero product xz^{n-1} and $(n-1)(n-1)!$ diagonals missing x with the common non-zero product y^2z^{n-2}. Similarly, if we choose x, y, and z such that $x + (n-1)z \neq 2y + (n-2)z$ (i.e., $2y \neq x + z$), then $A \neq J_n$ and A has precisely $(n-1)(n-1)!$ diagonals with the common sum $2y + (n-2)z$. In fact, we shall prove later on that if $A \in \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with a common sum $\alpha \neq 1$, then $A = A(x, y, z)$ for some suitable choices of x, y, and z.

(ii) Later on we shall show via a decomposition theorem that Theorem 1.3 and Theorem 1.6 are in fact equivalent.

(iii) In Theorem 1.6, the assumption "corresponding" is clearly indispensable for we can always permute the rows and columns of $A \in \Omega_n$ to get $B \neq A$. Obviously, A and B have all the diagonal sums equal (not corresponding diagonals). Furthermore, the number $(n-1)(n-1)!$ in Theorem 1.6 is attained for $n = 3$. For example, let $A = J_3$,

$B = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/6 & 1/2 & 1/3 \\ 1/3 & 1/6 & 1/2 \end{bmatrix}$. Then $A \neq B$, and there are precisely 4

corresponding diagonals with equal sums. However, we are unable to show that the number $(n-1)(n-1)!$ is attained for all n or for any $n \geq 4$. The next proposition shows that this question is equivalent to the existence of a d.s. matrix having precisely $(n-1)(n-1)!$ diagonals with the common sum one.

Proposition 1.8: The following two statements are equivalent.

(i) There exist $A, B \in \Omega_n$, $A \neq B$ such that precisely $(n-1)(n-1)!$ corresponding diagonals of A and B have equal sums.

(ii) There exists $C \in \Omega_n$ with precisely $(n-1)(n-1)!$ diagonals having sum one.

Proof: (i) => (ii) . Put $C = \frac{1}{n}(A - B + nJ_n)$. Then $C \in \Omega_n$ and, since $A \neq B$, $C \neq J_n$. Since C has at least $(n-1)(n-1)!$ diagonals with sum one, it has precisely $(n-1)(n-1)!$ diagonals with sum one by Theorem 1.3.

(ii) => (i) . Since C has precisely $(n-1)(n-1)!$ diagonals with sum one, $C \neq J_n$ by Theorem 1.3. Now, C and J_n have precisely $(n-1)(n-1)!$ corresponding diagonals with equal sums.

Since for a d.s. matrix A , each row sum and column sum is identically one, it is natural to ask the following questions, "How many diagonals of A can have sums greater than one, and how many diagonals of A can have sums smaller than one?" In the following, we shall obtain a decomposition theorem from which the answers to both questions follow immediately. Since this theorem is a combinatorial theorem concerning the positions in a matrix without reference to the actual entries, we find it convenient to extend the notion of diagonal as follows:

Definition 1.9: Let $\sigma \in S_n$. By a σ-diagonal or simply a diagonal we mean the set of ordered pairs $\{(i, \sigma(i)); i = 1, 2, \cdots, n\}$. Two diagonals σ and τ are said to be disjoint if for all $i = 1, 2, \cdots, n$, $\sigma(i) \neq \tau(i)$. A collection E of diagonals is a mutually disjoint collection if for all $\sigma, \tau \in E$, $\sigma \neq \tau$, σ and τ are disjoint.

Definition 1.10: A collection E of diagonals is said to cover a matrix A if each entry of A appears in at least one of the diagonals of E . If, in addition, E is a mutually disjoint collection, then we say the covering is exact. It is clear that a mutually disjoint collection E of diagonals covers A exactly iff $|E| = n$.

Theorem 1.11: (Decomposition Theorem for the set of all diagonals.)

Let D be the set of all diagonals of order n . Then there exists a decomposition of D into $(n-1)!$ mutually disjoint subsets each containing n mutually disjoint diagonals.

Proof: Let $\sigma_o \varepsilon S_n$ be any full cycle permutation, and let $G = <\sigma_o>$ be the cyclic subgroup of order n generated by σ_o. Consider the class of all (left) cosets σG of G, $\sigma \varepsilon S_n$. Since $|\sigma G| = |G| = n$ for all $\sigma \varepsilon S_n$ and $\sum_{\sigma} |\sigma G| = |S_n| = n!$, where the summation is taken over a complete set of coset representatives σ, it is clear that there are $(n-1)!$ such (disjoint) cosets. Furthermore, since σ_o is a full cycle, we have, for each $i = 1, 2, \cdots, n$, that $(\sigma g_1)(i) = (\sigma g_2)(i) \Rightarrow \sigma(g_1(i)) = \sigma(g_2(i)) \Rightarrow g_1(i) = g_2(i) \Rightarrow g_1 = g_2 \Rightarrow$ $\Rightarrow \sigma g_1 = \sigma g_2$ for all $g_1, g_2 \varepsilon G$, $\sigma \varepsilon S_n$. Therefore, the diagonals in each coset are mutually disjoint. This completes the proof.

As an immediate application, we have the following:

Theorem 1.12: Let m be an integer, $1 \leq m \leq n$. Let K_m be the smallest positive integer with the property that any collection of K_m diagonals would contain m disjoint ones. Then $K_m = (m-1)(n-1)! + 1$.

Proof: Let E be a collection of diagonals such that $|E| \geq (m-1)(n-1)! + 1$. Let $D = D_1 \cup D_2 \cup \cdots \cup D_{(n-1)!}$ be a decomposition of the set D of all diagonals as described in Theorem 1.11. By the pigeon-hole principle, $|E \cap D_i| \geq m$ for some i, $1 \leq i \leq (n-1)!$, for otherwise $|E| \leq (m-1)(n-1)!$, a contradiction. Since D_i is a mutually disjoint collection, $K_m \leq (m-1)(n-1)! + 1$ follows. On the other hand, let F be the collection of all diagonals passing through any one of a_{1i}, $i = 1, 2, \cdots, m-1$. Then $|F| = (m-1)(n-1)!$. Clearly, F does not contain m mutually disjoint diagonals for otherwise the pigeon-hole principle would imply that some two of these disjoint diagonals must pass through the smae a_{1i} for some i, which is obviously absurd. Hence $K_m \geq (m-1)(n-1)! + 1$. Therefore, $K_m = (m-1)(n-1)! + 1$.

<u>Corollary 1.13</u>: Let $A \in \Omega_n$. Let $\Delta_n(A)$ and $\delta_n(A)$ be the maximum number of diagonals of A with sums greater than one, and smaller than one, respectively. Let $\Delta_n = \max_{A\in\Omega_n} \Delta_n(A)$ and $\delta_n = \max_{A\in\Omega_n} \delta_n(A)$. Then $\Delta_n = \delta_n = (n-1)(n-1)!$. This bound is always attainable.

<u>Proof</u>: If $\Delta_n(A) \geq (n-1)(n-1)! + 1$ for some $A \in \Omega_n$, then by Theorem 1.12, we can select n mutually disjoint diagonals each having sum > 1. Since they cover A exactly, we get $\sum_{i,j=1}^{n} a_{ij} > n$, a contradiction. Hence $\Delta_n \leq (n-1)(n-1)!$. Similarly, $\delta_n \leq (n-1)(n-1)!$. To see that this bound is always attainable, consider the d.s. matrices $A(0, 1/(n-1), (n-2)/(n-1)^2)$ and $A(1, 0, 1/(n-1))$ respectively (cf. Remark 1.7.(i)). In both cases, there are precisely $(n-1)(n-1)!$ diagonals with sum $2y + (n-2)z$, which is equal to $[(n-1)^2 + 1]/(n-1)^2$ in the first case and $(n-2)/(n-1)$ in the second. This completes the proof.

In [10], it was shown that for any d.s. matrix A, $\max_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \geq 1/n^n$. The next corollary is now clear from the elementary arithmetic-geometric inequality and Corollary 1.13 above.

Corollary 1.14: Let $A \in \Omega_n$. Then at most $(n-1)(n-1)!$ diagonals can have diagonal product $> 1/n^n$.

We mention that Corollaries 1.13 and 1.14 are in fact true for the class of stochastic matrices. As another immediate application of the Decomposition Theorem, we show that Theorem 1.3 and Theorem 1.6 are indeed equivalent. (Remark 1.7.(ii).)

Theorem 1.15: The following two statements for d.s. matrices are equivalent.

(i) If $A \in \Omega_n$ has more than $(n-1)(n-1)!$ diagonals with a common sum, then $A = J_n$.

(ii) If $A, B \in \Omega_n$ have more than $(n-1)(n-1)!$ corresponding diagonals with equal sum, then $A = B$.

Proof: (i) => (ii) . This is just Theorem 1.6.

(ii) => (i) . Let $A \in \Omega_n$ be such that more than $(n-1)(n-1)!$ diagonals have the common sum α . Then by Corollary 1.13, it is impossible that $\alpha > 1$ or $\alpha < 1$. Hence $\alpha = 1$ and so A and J_n have more than $(n-1)(n-1)!$ corresponding diagonals with equal sum. Therefore $A = J_n$ by (ii) . This completes the proof.

For $A \in \Omega_n$, we have seen (Corollary 1.5) that

$\max\limits_{\sigma \in S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \geq 1$ and $\min\limits_{\sigma \in S_n} \sum\limits_{i=1}^{n} a_{i\sigma(i)} \leq 1$ with either (and hence both) equality iff $A = J_n$. Regarding Corollary 1.13, the following question naturally presents itself. "If $A \in \Omega_n$, $A \neq J_n$, what is the maximum number $\overline{\Delta}_n(A)$ of diagonals σ such that $\sum_{i=1}^{n} a_{i\sigma(i)} \geq 1$, and what is the maximum number $\overline{\delta}_n(A)$ of diagonals τ such that $\sum_{i=1}^{n} a_{i\tau(i)} \leq 1$?" Let $\overline{\Delta}_n = \max\limits_{A \in \Omega_n \sim \{J_n\}} \overline{\Delta}_n(A)$ and $\overline{\delta}_n = \max\limits_{A \in \Omega_n \sim \{J_n\}} \overline{\delta}_n(A)$.

It can be easily seen that for $n = 2$, $\overline{\Delta}_2 = \overline{\delta}_2 = 1$. In general, however, the question seems to be quite difficult. When $n = 3$, for example, the following matrices show that $\overline{\Delta}_3 = \overline{\delta}_3 = 5$.

$$A = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/6 & 1/2 & 1/3 \\ 1/3 & 1/6 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

In the next proposition, we show that $\overline{\Delta}_n = \overline{\delta}_n$ and obtain an upper bound for this value in general.

<u>Proposition 1.16</u>: $\overline{\Delta}_n = \overline{\delta}_n \leq (n-1)(n-1)! + \left[\frac{(n-1)(n-1)!}{n}\right]$ where $[\ \ ]$ denotes the greatest integral part function. This bound is attained for $n = 2$ and $n = 3$.

<u>Proof</u>: For any $A \in \Omega_n$, it is clear that $\frac{nJ_n - A}{n - 1} \in \Omega_n$ and that $A \neq J_n$ iff $\frac{nJ_n - A}{n - 1} \neq J_n$. Since $\sum_{i=1}^{n} a_{i\sigma(i)} \geq 1$ iff $\frac{n - \sum_{i=1}^{n} a_{i\sigma(i)}}{n - 1} \leq 1$, it follows that $\overline{\Delta}_n = \overline{\delta}_n$. Now, let $A \in \Omega_n$, $A \neq J_n$. Let $E = \left\{\sigma ; \sum_{i=1}^{n} a_{i\sigma(i)} = 1\right\}$. Since $A \neq J_n$, we have, by Theorem 1.3

that $|E| \leq (n-1)(n-1)!$. Let $D = D_1 \cup D_2 \cup \cdots \cup D_{(n-1)!}$ be a decomposition of the set D of all diagonals as stated in Theorem 1.11. Let m be the number of D_i's such that $D_i \subset E$. Then $m \leq [\frac{(n-1)(n-1)!}{n}]$. For each D_i such that $D_i \not\subset E$, there is at least one diagonal with sum < 1 . Hence there are at least $(n-1)! - [\frac{(n-1)(n-1)!}{n}]$ diagonals with sum < 1 . Therefore, $\overline{\Delta}_n \leq n! - \left\{(n-1)! - [\frac{(n-1)(n-1)!}{n}]\right\} = (n-1)(n-1)! + [\frac{(n-1)(n-1)!}{n}]$. For $n = 2$ and $n = 3$, we get $\overline{\Delta}_2 = \overline{\delta}_2 = 1$ and $\overline{\Delta}_3 = \overline{\delta}_3 = 5$ respectively.

<u>Remark 1.17</u>: We are unable to determine if the bound given in the above proposition is attainable in general. For $n = 4$, $(n-1)(n-1)! + [\frac{(n-1)(n-1)!}{n}] = 22$. The best we can get for $\overline{\Delta}_4(A)$ and $\overline{\delta}_4(A)$ so far is 20 as shown by the matrix below:

$$A = \begin{bmatrix} x & y & x & y \\ y & x & y & x \\ x & y & x & y \\ y & x & y & x \end{bmatrix}$$

where $x + y = 1/2$, $x > 1/4$, $y < 1/4$. Then $A \in \Omega_n$, $A \neq J_n$ and there are 16 diagonals with sum $2x + 2y = 1$, 4 diagonals with sum $4x > 1$ and 4 diagonals with sum $4y < 1$. Hence $\overline{\Delta}_4(A) = \overline{\delta}_4(A) = 20$.

In what follows, we will derive more consequences of the Decomposition Theorem.

<u>Proposition 1.18</u>: Let $A \in \Omega_n$. If $\Delta_n(A) = (n-1)(n-1)!$, then the remaining $(n-1)!$ diagonals must have sums strictly less than one and if $\delta_n(A) = (n-1)(n-1)!$, then the remaining $(n-1)!$ diagonals must have sums strictly greater than one. (For notations, cf. Corollary 1.13.)

<u>Proof</u>: Let $E = \left\{\sigma ; \sum_{i=1}^{n} a_{i\sigma(i)} > 1\right\}$. Then $|E| = (n-1)(n-1)!$. Let $D = D_1 \cup D_2 \cup \cdots \cup D_{(n-1)!}$ be a decomposition as stated in Theorem 1.11. If $D_i \subset E$ for any i, then since D_i covers A exactly, we get a contradiction. Hence for all $i = 1, 2, \cdots, (n-1)!$, $D_i \not\subset E$ or $|E \cap D_i| \leq n-1$. Furthermore, since the D_i's are disjoint, so are the $E \cap D_i$'s. Hence $(n-1)(n-1)! = |E| = |E \cap D| =$
$= |E \cap (\bigcup_{i=1}^{(n-1)!} D_i)| = |\bigcup_{i=1}^{(n-1)!} (E \cap D_i)| = \sum_{i=1}^{(n-1)!} |E \cap D_i| \leq (n-1)(n-1)!$. Thus $|E \cap D_i| = n-1$ for all $i = 1, 2, \cdots, (n-1)!$. Therefore, for each i, there is exactly one diagonal $\sigma_i \in D_i \sim E$. Since each D_i covers A exactly, these σ_i's must have sum strictly less than one. The other assertion follows by a parallel argument. This completes the proof.

We are now in a position to characterize all $n \times n$ d.s. matrices that have precisely $(n-1)(n-1)!$ diagonals with the common sum $\alpha \neq 1$.

<u>Lemma 1.19</u>: If $A \in \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with the common sum $\alpha \neq 1$, then the remaining $(n-1)!$ diagonals also have a common sum which is $\beta = n - (n-1)\alpha \neq \alpha$.

Proof: Let $E = \left\{\sigma ; \sum_{i=1}^{n} a_{i\sigma(i)} = \alpha\right\}$. Then $|E| = (n-1)(n-1)!$. Following the notations and arguments given in Proposition 1.18, we get $|E \cap D_j| = n-1$ for all $j = 1, 2, \cdots, (n-1)!$, and hence for each j, there is exactly one $\sigma_j \varepsilon D_j \sim E$. Since D_j covers A exactly, $\sum_{i=1}^{n} a_{i\sigma_j(i)} = n - (n-1)\alpha$ for all $j = 1, 2, \cdots, (n-1)!$. Since $\alpha \neq 1$, it is clear that $\beta \neq \alpha$.

Theorem 1.20: If $A \varepsilon \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with the common sum $\alpha \neq 1$, then there exist permutation matrices P and Q such that for some suitable choices of x, y and z, $PAQ = A(x, y, z)$. (cf. Remark 1.7.(i).)

Proof: Since by Lemma 1.19, A has only 2 distinct values α and β for diagonal sums, a theorem by J. Kapoor ([7], Theorem 2.15) implies that there exist permutation matrices P and Q and a positive integer k, $1 \leq k \leq n$ such that PAQ or $(PAQ)^t$ has the following form:

$$\begin{bmatrix} (\beta-\alpha)+\lambda_n+\delta_1 & \lambda_1+\delta_1 & \cdots & \lambda_{n-1}+\delta_1 \\ (\beta-\alpha)+\lambda_n+\delta_2 & \lambda_1+\delta_2 & \cdots & \lambda_{n-1}+\delta_2 \\ \vdots & \vdots & & \vdots \\ (\beta-\alpha)+\lambda_n+\delta_{k-1} & \lambda_1+\delta_{k-1} & \cdots & \lambda_{n-1}+\delta_{k-1} \\ \lambda_n+\delta_k & \lambda_1+\delta_k & \cdots & \lambda_{n-1}+\delta_k \\ \vdots & \vdots & & \vdots \\ \lambda_n+\delta_n & \lambda_1+\delta_n & \cdots & \lambda_{n-1}+\delta_n \end{bmatrix}.$$

Since $A \varepsilon \Omega_n$, $n\lambda_j + \sum_{i=1}^{n} \delta_i = 1$ for all $j = 1, 2, \cdots, n-1$. Hence

$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda$ for some λ . Similarly,

$$(\beta-\alpha) + (\sum_{i=1}^{n} \lambda_i) + n\,\delta_j = 1 \quad \text{for all} \quad j = 1, 2, \cdots, k-1 \,. \tag{1}$$

Hence $$\delta_1 = \delta_2 = \cdots = \delta_{k-1} = \delta \quad \text{for some} \quad \delta \,. \tag{2}$$

Also $$n\,\delta_j + \sum_{i=1}^{n} \lambda_i = 1 \quad \text{for all} \quad j = k, k+1, \cdots, n \,. \tag{3}$$

From (1), (2) and (3), $n\,\delta_j - n\delta - (\beta-\alpha) = 0$ or $\delta_j = \delta + \frac{1}{n}(\beta-\alpha) = \delta'$ for some δ' for all $j = k, k+1, \cdots, n$. Consequently, PAQ or $(PAQ)^t$ takes the form:

$$\begin{array}{l} \left[\begin{array}{cccc} (\beta-\alpha)+\lambda_n+\delta & \lambda+\delta & \cdots & \lambda+\delta \\ (\beta-\alpha)+\lambda_n+\delta & \lambda+\delta & \cdots & \lambda+\delta \\ \vdots & \vdots & & \vdots \\ (\beta-\alpha)+\lambda_n+\delta & \lambda+\delta & \cdots & \lambda+\delta \\ \lambda_n+\delta' & \lambda+\delta' & \cdots & \lambda+\delta' \\ \vdots & \vdots & & \vdots \\ \lambda_n+\delta' & \lambda+\delta' & \cdots & \lambda+\delta' \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \longrightarrow (k-1)\text{th row} \\ \\ \\ \\ \end{array}$$

For convenience, put $x = (\beta-\alpha) + \lambda_n + \delta$, $y = \lambda + \delta$, $y' = \lambda_n + \delta'$, and $z = \lambda + \delta'$. Then PAQ or $(PAQ)^t$ becomes

$$\left[\begin{array}{cccc} x & y & \cdots & y \\ \vdots & \vdots & & \vdots \\ x & y & \cdots & y \\ y' & z & \cdots & z \\ \vdots & \vdots & & \vdots \\ y' & z & \cdots & z \end{array}\right] \begin{array}{l} \\ \\ \longrightarrow (k-1)\text{th row} \\ \\ \\ \\ \end{array}$$

Clearly, this matrix has two distinct diagonal sums values: there are

$(k-1)(n-1)!$ diagonals passing through some x with sum $x + (k-2)y + (n-k+1)z$ and $(n-k+1)(n-1)!$ diagonals passing through some y' with sum $y' + (n-k)z + (k-1)y$. Since, by assumption, there are precisely $(n-1)(n-1)!$ diagonals with the same sum, we have either $k-1 = n-1$ or $n-k+1 = n-1$. Hence $k = n$ or $k = 2$. For $k = n$, PAQ or $(PAQ)^t$ becomes

$$\begin{bmatrix} x & y & y & \cdots & y \\ x & y & y & \cdots & y \\ \vdots & \vdots & \vdots & & \vdots \\ x & y & y & \cdots & y \\ y' & z & z & \cdots & z \end{bmatrix} .$$

Since $A \varepsilon \Omega_n$, we have $x = z$ and hence after permuting the rows, the matrix becomes

$$\begin{bmatrix} y' & x & \cdots & x \\ x & y & \cdots & y \\ \vdots & \vdots & & \vdots \\ x & y & \cdots & y \end{bmatrix} .$$

Since this matrix is symmetric, we can conclude that for some permutation matrices P' and Q' , $P'AQ' = A(y', x, y)$. For $k = 2$, $PAQ = (PAQ)^t$ becomes

$$\begin{bmatrix} x & y & \cdots & y \\ y' & z & \cdots & z \\ \vdots & \vdots & & \vdots \\ y' & z & \cdots & z \end{bmatrix} .$$

Since $A \varepsilon \Omega_n$, we have $y = y'$. Therefore, $PAQ = A(x, y, z)$.

As an application of the above theorem, we have:

Corollary 1.21: If $A \in \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with the common sum $\alpha \neq 1$, then $1 - \frac{1}{n-1} \leq \alpha \leq 1 + \frac{1}{(n-1)^2}$. Conversely, given any $\alpha \neq 1$, with $1 - \frac{1}{n-1} \leq \alpha \leq 1 + \frac{1}{(n-1)^2}$, there exists a matrix $A \in \Omega_n$ that has precisely $(n-1)(n-1)!$ diagonals with the common sum α. This matrix is unique up to permutation of rows and columns.

Proof: If $A \in \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with the common sum $\alpha \neq 1$, then by Theorem 1.20, there exist permutation matrices P and Q such that

$$PAQ = A(x, y, z) = \begin{bmatrix} x & y & \cdots & y \\ y & z & \cdots & z \\ \vdots & \vdots & & \vdots \\ y & z & \cdots & z \end{bmatrix}$$

for some x, y, and z. Since $x + (n-1)y = y + (n-1)z = 1$, we have $y = \frac{1-x}{n-1}$, $z = \frac{1-y}{n-1} = \frac{n+x-2}{(n-1)^2}$. Hence $\alpha = 2y + (n-2)z =$
$= \frac{1}{(n-1)^2}\{2(1-x)(n-1) + (n-2)(n+x-2)\} = \frac{1}{(n-1)^2}\{n^2-2n+2-nx\} = 1 + \frac{1-nx}{(n-1)^2}$.
Consequently, α attains its maximum $1 + \frac{1}{(n-1)^2}$ and its minimum $1 - \frac{1}{n-1}$ when $x = 0$ and $x = 1$ respectively. Conversely, let $\alpha \neq 1$ be given with $1 - \frac{1}{n-1} \leq \alpha \leq 1 + \frac{1}{(n-1)^2}$. A matrix $A \in \Omega_n$ that has precisely $(n-1)(n-1)!$ diagonals with a common sum α must be, in view of Theorem 1.20, equivalent to $A(x, y, z)$ for some x, y, and z via

permutation matrices. Hence

$$x + (n-1)y = 1 \quad , \tag{4}$$

$$y + (n-1)z = 1 \quad , \tag{5}$$

$$2y + (n-2)z = \alpha \quad . \tag{6}$$

From (5) and (6), we get $z = \frac{2 - \alpha}{n} > 0$ since $\alpha < 2$. Substitute this into (5), we get $y = \frac{1}{n}\{(2-n) + (n-1)\alpha\} \geq 0$ since $(n-1)\alpha \geq n-2$. Substitute this into (4), we get $x = 1 - (n-1) + \frac{2(n-1)^2 - \alpha(n-1)^2}{n} \geq$ $\geq 2 - n + \frac{(n-1)^2 - 1}{n} = 0$ since $\alpha(n-1)^2 \leq (n-1)^2 + 1$. Hence the matrix $A(x, y, z)$ where $x = (2-n) + (n-1)^2(\frac{2 - \alpha}{n})$, $y = \frac{1}{n}\{(2-n) + (n-1)\alpha\}$, and $z = \frac{2 - \alpha}{n}$ is d.s. and has precisely $(n-1)(n-1)!$ diagonals with the common sum α. This proves both the existence and uniqueness up to permutation of rows and columns.

<u>Remark 1.22</u>: (i) Corollary 1.21 above shows that the bound given in Theorem 1.3, though best possible in general, is not uniform in the sense that if the value $\alpha \neq 1$ of the common diagonal sum does not belong to the interval $[\frac{n-2}{n-1}, 1 + \frac{1}{(n-1)^2}]$, then the bound $(n-1)(n-1)!$ can not be attained. Consequently, a better (smaller) bound depending on the value of α should be expected. To get an explicit formula for such a bound, however, seems to be difficult. For instance, it is intuively clear (though a simple and rigorous proof using the König-Frobenius theorem [6, 8] can be given) that for $A \in \Omega_n$, the maximum number of zero diagonals is D_n - the number of derangements of n object. (cf. e.g. [20], p. 22). An explicit formula, if it exists, must then give the value D_n for $\alpha = 0$.

(ii) Theorem 1.20 and Corollary 1.21 above do not hold for $\alpha = 1$. In fact, the example given in Remark 1.7.(i) is not valid for $\alpha = 1$ since $2y + (n-2)z = 1$ together with $y + (n-1)z = 1 = x + (n-1)y$ implies $x = y = z$ or $A = J_n$. For $A = J_n$, clearly all diagonal sums are equal to one. Hence, excluding J_n, we can ask the question: "What is the maximum number of diagonals with sum one?" The bound $(n-1)(n-1)!$ still holds in general, but we are unable to construct examples to show that this bound is attained for all n. (cf. Proposition 1.8) For $n = 3$, the matrix B given in Remark 1.7.(iii) shows that this bound is attained. For $n = 4$, the best example we can get is the one given in Remark 1.17 which gives 16, rather than 18, diagonals with sum one.

We close this chapter by giving a proposition that answers partially the problem mentioned in Remark 1.22.(i) above.

Proposition 1.23: Let $A \in \Omega_n$ and let α be a given real number such that $1 < \alpha \leq n$. Then at most $(m-1)(n-1)!$ diagonals of A can have diagonal sums $\geq \alpha$, where $m = [\frac{n}{\alpha}] + 1$.

Proof: Suppose more than $(m-1)(n-1)!$ diagonals of A have sums $\geq \alpha$. Then by Theorem 1.12, we can select from them m mutually disjoint ones with total sum $\geq m\alpha = \alpha([\frac{n}{\alpha}] + 1) > \alpha(\frac{n}{\alpha}) = n$, a contradiction.

CHAPTER II

MAXIMUM AND MINIMUM DIAGONAL SUM OF d.s. MATRICES

The purpose of this chapter is to carry out a combinatorial investigation of the diagonals with maximum diagonal sum h and the diagonals with minimum diagonal sum k. First, we obtain the best possible upper and lower bounds for entries on these diagonals. Secondly, we obtain the best possible upper and lower bounds for various kinds of functions of h and k. The key will be a covering theorem that enables us to analyze the cases when these bounds are attained. When studying the lower bound for $h+k$, an interesting combinatorial question presents itself. Concerning this question, a conjecture and partial solutions are given. Finally, we obtain the bounds for h and k when the d.s. matrices under consideration have properties discussed in Chapter I. Throughout, we shall use $A \geq 0$ to mean that all the entries of the matrix A are non-negative, and we shall assume that $n \geq 2$, since the case for $n = 1$ is always a triviality.

<u>Definition 2.1</u>: Let $A \in \Omega_n$. A diagonal σ of A is called a maximum diagonal if its sum is a maximum among all diagonal sums of A, and a minimum diagonal if its sum is a minimum among all diagonal sums of A.

The first result gives the upper bound for entries on a minimum diagonal.

Proposition 2.2: Let $A \in \Omega_n$, $n \geq 3$ and let σ be a minimum diagonal. Then $\max_i a_{i\sigma(i)} \leq \frac{2}{n+1}$ with equality iff $a_{i\sigma(i)} = \frac{2}{n+1}$ for some i and $a_{j\sigma(j)} = 0$ for all $j \neq i$.

Proof: Since permuting the rows and columns of A does not affect the set of diagonal sums, we may assume, without lose of generality, that σ is the main diagonal, and that $\max_i a_{ii} = a_{11}$. By assumption, $a_{11} + a_{ii} \leq a_{1i} + a_{i1}$ for all $i = 1, 2, \cdots, n$. Summing over i , we get $(n+1)a_{11} + \sum_{i=2}^{n} a_{ii} \leq \sum_{i=1}^{n} (a_{1i} + a_{i1}) = 2$. Hence $a_{11} \leq \frac{2}{n+1}$ with equality iff $a_{11} = \frac{2}{n+1}$, $a_{ii} = 0$ and $a_{1i} + a_{i1} = \frac{2}{n+1}$ for all $i = 2, 3, \cdots, n$.

Remark 2.3:

(i) If $n = 2$, the upper bound is $1/2$ and is attained iff $A = J_2$.

(ii) For all $n \geq 3$, the upper bound is the best possible. Consider

$$A = \begin{bmatrix} \frac{2}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & 0 & \cdots & \frac{n}{(n+1)(n-2)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n+1} & \frac{n}{(n+1)(n-2)} & \cdots & 0 \end{bmatrix} \in \Omega_n$$

where $a_{11} = \frac{2}{n+1}$, $a_{1i} = a_{i1} = \frac{1}{n+1}$, $a_{ii} = 0$ for all $i = 2, 3, \ldots, n$, and $a_{ij} = \frac{n}{(n+1)(n-2)}$ otherwise. Clearly, the main diagonal is a minimum diagonal, and its maximum entry is $\frac{2}{n+1}$. Incidentally, this example also shows that in general the maximum entry of a d.s. matrix can lie on a minimum diagonal.

If we were to apply the argument used in the proof of Proposition 2.2 to a maximum diagonal σ, we should get the lower bound $\max_i a_{i\sigma(i)} \geq 1/n$ which is trivial since $\sum_{i=1}^{n} a_{i\sigma(i)} \geq 1$ by Corollary 1.5. However, the upper bound for $\max_i a_{i\sigma(i)}$ and the lower bound for $\min_i a_{i\sigma(i)}$ can be obtained in terms of the diagonal sum h and the bounds are non-trivial when $h \leq 2$. Since $h \geq 1$ by Corollary 1.5, we have $n\cdot\max_i a_{i\sigma(i)} \geq h \geq 2-h$ and $n\cdot\min_i a_{i\sigma(i)} - 1 \leq h-1 \leq (n-1)(h-1)$, i.e., $\max_i a_{i\sigma(i)} \geq \frac{2-h}{n}$ and $\min_i a_{i\sigma(i)} \leq \frac{(n-1)(h-1)+1}{n}$. However, the following stronger results hold.

Proposition 2.4: Let $A \in \Omega_n$ and let σ be a maximum diagonal with sum h. Then (i) $\min_i a_{i\sigma(i)} \geq \frac{2-h}{n}$.

(ii) $\max_i a_{i\sigma(i)} \leq \frac{(n-1)(h-1)+1}{n}$.

Proof: As before, assume that σ is the main diagonal.

(i) Suppose $a_{11} = \min_i a_{ii}$. By assumption, $a_{11} + a_{ii} \geq a_{1i} + a_{i1}$ for all $i = 1, 2, \cdots, n$. Summing over i, we get

$$n\, a_{11} + h \geq \sum_{i=1}^{n} (a_{1i} + a_{i1}) = 2, \text{ or } a_{11} \geq \frac{2-h}{n}.$$

(ii) Suppose $a_{11} = \max_i a_{ii}$. By assumption, for each fixed $i = 2, 3, \cdots, n$, we have $a_{ij} + a_{ji} \leq a_{ii} + a_{jj}$ for all

$j = 1, 2, \cdots, n$. Summing over all $j \neq 1, i$, we get

$$\sum_{j \neq 1, i} (a_{ij} + a_{ji}) \leq (n-2)a_{ii} + \sum_{j \neq 1, i} a_{jj} \quad \text{, or}$$

$$2 - a_{1i} - a_{i1} - 2a_{ii} \leq (n-2)a_{ii} + \sum_{j \neq 1, i} a_{jj} \; ; \quad \text{or}$$

$2 - a_{1i} - a_{i1} \leq (n-1)a_{ii} + \sum_{j \neq 1} a_{jj}$. Summing over all $i \neq 1$, we get $2(n-1) - \sum_{i=2}^{n} (a_{1i} + a_{i1}) \leq 2(n-1) \sum_{j \neq 1} a_{jj}$, or $2(n-1) - (2 - 2a_{11}) \leq 2(n-1)(h - a_{11})$. Simplifying, we get $a_{11} \leq \frac{1}{n}\{(n-1)(h-1) + 1\}$. This completes the proof.

If we consider a minimum diagonal σ instead, the same argument as above yields:

<u>Proposition 2.5</u>: Let $A \in \Omega_n$, and let σ be a minimum diagonal with sum k . Then (i) $\min_i a_{i\sigma(i)} \geq \dfrac{1 - (n-1)(1-k)}{n}$

(ii) $\max_i a_{i\sigma(i)} \leq \dfrac{2-k}{n}$.

Since 1 and 0 are the upper and lower bound, respectively, for any entry of a d.s. matrix, and since $\frac{2-h}{n} \geq 0$ iff $h \leq 2$ iff $\frac{(n-1)(h-1) + 1}{n} \leq 1$, Proposition 2.4 can be restated as follows:

<u>Proposition 2.6</u>: Let $A \in \Omega_n$, and let σ be a maximum diagonal with sum h . Then

$$\text{(i)} \quad \min_i a_{i\sigma(i)} \geq \begin{cases} \frac{2-h}{n} & \text{if } h \leq 2 \\ 0 & \text{if } h \geq 2 \end{cases}$$

$$\text{(ii)} \quad \max_i a_{i\sigma(i)} \leq \begin{cases} \frac{(n-1)(h-1)+1}{n} & \text{if } h \leq 2 \\ 1 & \text{if } h \geq 2 \end{cases} .$$

Similarly, since $\frac{1}{n}\{1 - (n-1)(1-k)\} \geq 0$ iff $k \geq \frac{n-2}{n-1}$, and $\frac{2-k}{n} \leq \frac{2}{n+1}$ iff $\frac{2}{n+1} \leq k$, Propositions 2.2 and 2.4 can be put together and restated in:

Proposition 2.7: Let $A \in \Omega_n$, and let σ be a minimum diagonal with sum k. Then

$$\text{(i)} \quad \min_i a_{i\sigma(i)} \geq \begin{cases} \frac{1-(n-1)(1-k)}{n} & \text{if } k \geq \frac{n-2}{n-1} \\ 0 & \text{if } k \leq \frac{n-2}{n-1} \end{cases}$$

$$\text{(ii)} \quad \max_i a_{i\sigma(i)} \leq \begin{cases} \frac{2-k}{n} & \text{if } k \geq \frac{2}{n+1} \\ \frac{2}{n+1} & \text{if } k \leq \frac{2}{n+1} \end{cases} .$$

Clearly, (ii) above is a refinement of Proposition 2.2. In the following examples, we discuss the case of equality for the bounds given in Propositions 2.6 and 2.7 for arbitrarily preassigned values of h and k.

Example 2.8: Let h be a given number such that $1 \leq h \leq 2$.

Consider the d.s. matrix $A = A(x, y, z)$ with $x = \frac{(n-1)(h-1) + 1}{n}$, $y = \frac{2-h}{n}$ and $z = \frac{n+h-2}{n(n-1)}$; i.e.,

$$A = \begin{bmatrix} \frac{(n-1)(h-1)+1}{n} & \frac{2-h}{n} & \cdots & \frac{2-h}{n} \\ \frac{2-h}{n} & \frac{n+h-2}{n(n-1)} & \cdots & \frac{n+h-2}{n(n-1)} \\ \vdots & \vdots & & \vdots \\ \frac{2-h}{n} & \frac{n+h-2}{n(n-1)} & \cdots & \frac{n+h-2}{n(n-1)} \end{bmatrix} .$$

Then $\sum_{i=1}^{n} a_{ii} = h$, and any diagonal missing a_{11} has sum $\frac{2(2-h)}{n} + \frac{(n-2)(n+h-2)}{n(n-1)} = \frac{n-h}{n-1} \leq 1 \leq h$. Hence the main diagonal is a maximum diagonal with sum h . Finally, $\frac{(n-1)(h-1) + 1}{n} =$ $= \frac{(n-1)^2(h-1) + (n-1)}{n(n-1)} \geq \frac{(h-1) + (n-1)}{n(n-1)} = \frac{n+h-2}{n(n-1)}$ implies that $\max_i a_{ii} = a_{11} = \frac{(n-1)(h-1) + 1}{n}$. If $h > 2$, then the permutation matrices are examples for which $\max_i a_{i\sigma(i)} = 1$. Hence the upper bound in Proposition 2.6 is always attainable.

Remark 2.9: If we consider the lower bound in Proposition 2.7 we have two cases: (i) $0 \leq k \leq \frac{n-2}{n-1}$. In this case, permutation matrices provide examples for which $\min_i a_{i\sigma(i)} = 0$. (ii) $\frac{n-2}{n-1} < k \leq 1$. In this case, Example 2.8 above with h replaced by k suffices since a change from $h \geq 1$ to $k \leq 1$ merely reverses all the inequalities therein and consequently the main diagonal becomes a minimum diagonal and $\min_i a_{ii} = a_{11} = \frac{1-(n-1)(1-k)}{n}$. Hence the lower bound in Proposition 2.7 is always attainable.

Example 2.10: Let $n \geq 3$ and let k be a given number such that $\frac{2}{n+1} \leq k \leq 1$. Consider the d.s. matrix

$$A = \begin{bmatrix} \frac{2-k}{n} & \frac{n+k-2}{n(n-1)} & \cdots & \frac{n+k-2}{n(n-1)} \\ \frac{n+k-2}{n(n-1)} & \frac{(n+1)k-2}{n(n-1)} & \cdots & \frac{(n^2-2n+4)-(n+2)k}{n(n-1)(n-2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{n+k-2}{n(n-1)} & \frac{(n^2-2h+4)-(n+2)k}{n(n-1)(n-2)} & \cdots & \frac{(n+1)k-2}{n(n-1)} \end{bmatrix},$$

where $a_{11} = \frac{2-k}{n}$, $a_{ii} = \frac{(n+1)k-2}{n(n-1)}$, $a_{1i} = a_{i1} = \frac{n+k-2}{n(n-1)}$ for all $i = 2, 3, \cdots, n$, and $a_{ij} = \frac{(n^2-2n+4)-(n+2)k}{n(n-1)(n-2)}$ otherwise. Since $(n+1)k - 2 \geq 0$, and since $n^2 - 3n + 2 = (n-1)(n-2) \geq 0$ implies that $n^2 - 2n + 4 \geq n + 2 \geq (n+2)k$, we have $A \geq 0$. Straightforward computations show that indeed $A \in \Omega_n$. Furthermore, $\sum_{i=1}^{n} a_{ii} =$ $= \frac{2-k}{n} + \frac{(n+1)k-2}{n} = k$, $\frac{2-k}{n} + \frac{(n+1)k-2}{n(n-1)} = \frac{2(n+k-2)}{n(n-1)}$, and $\frac{(n^2-2n+4)-(n+2)k}{n(n-1)(n-2)} - \frac{(n+1)k-2}{n(n-1)} = \frac{n(1-k)}{(n-1)(n-2)} \geq 0$. Hence the main diagonal is a minimum diagonal. Finally, since $\frac{2-k}{n} - \frac{(n+1)k-2}{n(n-1)} = \frac{2(1-k)}{n-1} \geq 0$, we have $\max_i a_{ii} = a_{11} = \frac{2-k}{n}$. The case for $n = 2$ is easy since $a_{11} = \frac{2-k}{2}$ implies that $a_{22} = \frac{2-k}{2}$ and $a_{12} = a_{21} = \frac{k}{2}$. If

$\frac{2-k}{2} \leq \frac{k}{2}$, then $k \geq 1$. Hence $k = 1$ or $A = J_2$. Therefore, the equality holds iff $k = 1$. Hence the upper bound in Proposition 2.7 is always attainable for all n , provided that $k \geq \frac{2}{n+1}$. If $k < \frac{2}{n+1}$, however, the fact that $\max_i a_{i\sigma(i)} \leq k < \frac{2}{n+1}$ clearly shows that the upper bound is not attainable.

The equality case for the lower bound in Proposition 2.6 is somewhat complicated. Before making a complete analysis, we need a lemma which gives a non-trivial lower bound for any entry on a diagonal with "comparatively large" diagonal sum α .

Lemma 2.11: Let $A \in \Omega_n$ and let σ be a diagonal with sum $\alpha \geq n-2$. Then for all $i = 1, 2, \cdots, n$, $a_{i\sigma(i)} \geq \frac{\alpha - n + 2}{2}$ with equality iff there exist permutation matrices P and Q such that PAQ takes the form:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & 1-a_2 & & \\ \vdots & & \ddots & 0 \\ & 0 & & \\ a_n & & & 1-a_n \end{bmatrix} ,$$

where $\sum_{i=1}^{n} a_i = 1$.

Proof: Permuting the rows and columns of A, if necessary, we may assume that σ is the main diagonal and a_{11} is the entry being considered. Let R_i and C_i denote the ith row sum and column sum respectively. Then $2n - 2 = \sum_{i=2}^{n} (R_i + C_i) = 2(\sum_{i=2}^{n} a_{ii}) +$

$$+ \sum_{i=2}^{n} (a_{1i} + a_{i1}) + 2 \sum_{\substack{i,j=2 \\ i \neq j}}^{n} a_{ij} = 2(\alpha - a_{11}) + (n - \alpha) + \beta \quad \text{where}$$

$\beta = \sum_{\substack{i,j=2 \\ i \neq j}}^{n} a_{ij} \geq 0$. Therefore $2\, a_{11} = \alpha - n + 2 + \beta$, and thus

$a_{11} \geq \frac{\alpha - n + 2}{2}$ with equality iff $\beta = 0$; i.e., iff $a_{ij} = 0$ for all $i \neq j$, $i,j = 2, 3, \cdots, n$. Since A is d.s., $a_{1i} = a_{i1}$ for all $i = 2, 3, \cdots, n$. Therefore, if we write a_i for a_{1i} , A takes the described form where $\sum_{i=1}^{n} a_i = 1$. This completes the proof.

Now we are in a position to analyze the equality case for the lower bound in Proposition 2.6. There are two cases to be considered:

Case I: $1 \leq h \leq 2$. In this case, Example 2.10 with k replaced by h suffices provided that $n \geq 4$ since then $(n^2 - 2n + 4) - (n + 2)h \geq (n^2 - 2n + 4) - 2(n + 2) = n^2 - 4n \geq 0$ and hence $A \geq 0$. Furthermore, a change from $k \leq 1$ to $h \geq 1$ merely reverses all the inequalities therein. Consequently, the main diagonal becomes a maximum diagonal and a_{11} becomes a minimum entry thereon. The case for $n = 2$ is again easy since $a_{11} = \frac{2 - h}{2}$ implies that $h = 1$ and hence equality is attained iff $h = 1$, i.e., iff $A = J_2$. The case for $n = 3$ calls for separate consideration. Since

$(n^2 - 2n + 4) - (n + 2)h = 7 - 5h$ for $n = 3$, Example 2.10 (with k replaced by h) still stands for $h \le \frac{7}{5}$. If $h > \frac{7}{5}$, however, we claim that the equality can not hold. Assume the contrary, and let $A \in \Omega_3$ be such that the main diagonal is a maximum diagonal with sum h, $\frac{7}{5} < h \le 2$, and that $a_{11} = \min_i a_{ii} = \frac{2-h}{3}$. Examining the proof of Proposition 2.4.(i), we see that $a_{11} + a_{22} = a_{12} + a_{21}$. Hence A takes the form:

$$A = \begin{bmatrix} \frac{2-h}{3} & x & \frac{1+h}{3} - x \\ \frac{2-h}{3} - x+y & y & x-2y+\frac{1+h}{3} \\ x-y+\frac{2h-1}{3} & 1-x-y & 2y+\frac{1-2h}{3} \end{bmatrix}.$$

But $\sum_{i=1}^{3} a_{ii} = h$ implies that $3y + 1 = h$ or $y = \frac{2h-1}{3}$. Substituting this into the matrix, we find that A takes the form:

$$A = \begin{bmatrix} \frac{2-h}{3} & x & \frac{1+h}{3} - x \\ \frac{1+h}{3} - x & \frac{2h-1}{3} & 1+x-h \\ x & \frac{4-2h}{3} - x & \frac{2h-1}{3} \end{bmatrix}.$$

Since $1 + x - h \ge 0$, we get $x \ge h - 1 > \frac{2}{5}$. On the other hand, $\frac{4 - 2h}{3} - x \ge 0 \Rightarrow x \le \frac{1}{3}(4 - 2h) < \frac{1}{3}(4 - 2\cdot\frac{7}{5}) = \frac{2}{5}$, a contradiction. In this case, however, Lemma 2.11 gives the lower bound $a_{11} \ge \frac{h-1}{2}$. Since $h > \frac{7}{5} \Rightarrow \frac{h-1}{2} > \frac{2-h}{3}$, this bound is a better one. (Note that since there is a "gap" between $\frac{h-1}{2}$ and $\frac{2-h}{3}$, the lower bound $\frac{2-h}{3}$ is not even approachable.) Furthermore, this new lower bound is attainable. Consider the d.s. matrix:

$$A = \begin{bmatrix} \frac{h-1}{2} & \frac{3-h}{4} & \frac{3-h}{4} \\ \frac{3-h}{4} & \frac{h+1}{4} & 0 \\ \frac{3-h}{4} & 0 & \frac{h+1}{4} \end{bmatrix} .$$

Then $\sum_{i=1}^{3} a_{ii} = h$, $\frac{h-1}{2} + \frac{h+1}{4} = \frac{3h-1}{4} > 2(\frac{3-h}{4})$ $(\because\ h > \frac{7}{5})$, and $\frac{h-1}{2} < \frac{h+1}{4}$ $(\because\ h \leq 2)$. Hence the main diagonal is a maximum diagonal on which the minimum entry is $\frac{h-1}{2}$.

Case II: $h > 2$. In this case, we claim that the lower bound 0 in Proposition 2.6.(i) is attainable iff $h \leq n-2$. (Hence in particular, it is not attainable for $n = 1, 2, 3, 4$.) Suppose $h > n-2$. Then by Lemma 2.11, $a_{i\sigma(i)} \geq \frac{h-n+2}{2} > 0$ for all entries on this maximum diagonal and hence the lower bound 0 is not attainable. For $h \leq n-2$, consider the d.s. matrix:

$$A = \begin{bmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{h}{n-1} & \cdots & \frac{n-h-2}{(n-1)(n-2)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n-1} & \frac{n-h-2}{(n-1)(n-2)} & \cdots & \frac{h}{n-1} \end{bmatrix}$$

where $a_{11} = 0$, $a_{1i} = a_{i1} = \frac{1}{n-1}$, $a_{ii} = \frac{h}{n-1}$ for all $i = 2, 3, \cdots, n$, and $a_{ij} = \frac{n-h-2}{(n-1)(n-2)}$ otherwise. Since $n-h-2 \geq 0$, $A \geq 0$ and straightforward computations show that $A \in \Omega_n$. Since $\frac{h}{n-1} > \frac{2}{n-1}$ and $\frac{h}{n-1} - \frac{n-h-2}{(n-1)(n-2)} = \frac{h(n-1) - (n-2)}{(n-1)(n-2)} > 0$, the main diagonal is a maximum diagonal with minimum entry 0 .

Now we can summarize and state the equality case for the lower bound given in Proposition 2.6 in the following:

<u>Proposition 2.12</u>: Let h be a given number such that $1 \leq h \leq n$.

(I) If $1 \leq h \leq 2$, then there exists a d.s. matrix with maximum diagonal sum h and such that the minimum entry on the diagonal is $\frac{2-h}{2}$ iff (i) $n = 2$ and $h = 1$ or (ii) $n = 3$ and $h < \frac{7}{5}$ or (iii) $n \geq 4$.

(II) If $2 < h \leq n$, then there exists a d.s. matrix with maximum diagonal sum h and such that the minimum entry on the diagonal is 0 iff $h \leq n-2$.

In what follows, we shall obtain the best possible upper and lower bounds for various kinds of functions of h and k and discuss the cases for equality. When the bounds are attained, examples will be constructed. The key will be Theorem 2.13 referred to as the "Covering Theorem", the proof of which depends on a well-known combinatorial theorem of P. Hall on SDR (systems of distinct representatives). A conjecture is given.

<u>Theorem 2.13</u>: (The Covering Theorem)

Let A be a given $n\times n$ matrix, and let $\sigma_1, \sigma_2, \cdots, \sigma_p$ be p given mutually disjoint diagonals of A , $1 \leq p \leq n$. Then one can select $n-p$ mutually disjoint diagonals $\sigma_{p+1}, \sigma_{p+2}, \cdots, \sigma_n$ such that $\{\sigma_i;\ i = 1, 2, \cdots, n\}$ cover A exactly. (Definition 1.10.)

<u>Proof</u>: If $p = n$, the given diagonals already cover A exactly and there is nothing to prove. Hence assume that $1 \leq p \leq n-1$. Since each diagonal of A corresponds to a permutation on the set $S = \{1, 2, \cdots, n\}$, the assertion is equivalent to saying that given p mutually disjoint permutations $\sigma_1, \sigma_2, \cdots, \sigma_p$ on S , $1 \leq p \leq n-1$, one can always exhibit another permutation σ on S such that $\sigma \cap \sigma_i = \phi$ for all $i = 1, 2, \cdots, p$. Let $S_1 = S \sim \bigcup_{i=1}^{p} \{\sigma_i(1)\}$, $S_2 = S \sim \bigcup_{i=1}^{p} \{\sigma_i(2)\}$, $\cdots$, $S_n = S \sim \bigcup_{i=1}^{p} \{\sigma_i(n)\}$. Let m be an integer such that $1 \leq m \leq n$. We claim that $|S_{\omega_1} \cup S_{\omega_2} \cup \cdots \cup S_{\omega_m}| \geq m$ for all sequences $\{\omega\}$ such that $1 \leq \omega_1 < \omega_2 < \cdots < \omega_m \leq n$. Since for all $i \neq j$, $i,j = 1, 2, \cdots, p$, $\sigma_i \cap \sigma_j = \phi$, we have $|S_j| = n-p$ for all $j = 1, 2, \cdots, n$, and hence (I) the number of elements, counting repetitions, in the set $S_{\omega_1} \cup S_{\omega_2} \cup \cdots \cup S_{\omega_m}$ is $m(n-p)$. On the other hand, each index from S appears exactly p times in $\bigcup_{i=1}^{n} \bigcup_{j=1}^{p} \{\sigma_j(i)\}$,hence appears exactly $n-p$ times in $\bigcup_{i=1}^{n} S_i$, and hence (II) each index from S appears at most $n-p$ times in

$S_{\omega_1} \cup S_{\omega_2} \cup \cdots \cup S_{\omega_m}$. The two statements (I) and (II) imply that $|S_{\omega_1} \cup S_{\omega_2} \cup \cdots \cup S_{\omega_m}| \geq m$. By a well-known theorem of P. Hall on SDR [20, p. 48], there exists an SDR for $S_1, S_2, \cdots, S_n$; i.e., there exist $d_i \in S_i$, $i = 1, 2, \cdots, n$ that form an SDR. Define the permutation σ on S by $\sigma(i) = d_i$ for all $i = 1, 2, \cdots, n$. It is clear from the construction of S_i that $\sigma \cap \sigma_j = \phi$ for all $j = 1, 2, \cdots, p$. This completes the proof.

As an immediate consequence, we get:

<u>Corollary 2.14</u>: Let $A \in \Omega_n$. Then

(i) $h + (n-1)k \leq n$ with equality iff for any set of n diagonals $\sigma_1, \sigma_2, \cdots, \sigma_n$ that cover A exactly where σ_1 is a maximum diagonal, we have $\sum_{i=1}^{n} a_{i\sigma_j(i)} = k$ for all $j = 2, 3, \cdots, n$.

(ii) $(n-1)h + k \geq n$ with equality iff for any set of n diagonals $\tau_1, \tau_2, \cdots, \tau_n$ that cover A exactly where τ_1 is a minimum diagonal, we have $\sum_{i=1}^{n} a_{i\tau_j(i)} = h$, for all $j = 2, 3, \cdots, n$.

<u>Proof</u>: (i) Let σ_1 be any maximum diagonal. By the Covering Theorem, let $\sigma_2, \sigma_3, \cdots, \sigma_n$ be $n-1$ diagonals such that $\{\sigma_i ; i = 1, 2, \cdots, n\}$ cover A exactly. Then $h + (n-1)k \leq$ $\leq h + \sum_{j=2}^{n} \sum_{i=1}^{n} a_{i\sigma_j(i)} = n$, with equality iff $\sum_{i=1}^{n} a_{i\sigma_j(i)} = k$ for all $j = 2, 3, \cdots, n$. The proof of (ii) is similar.

Remark 2.15:

(i) The equalities in the above corollary are always attainable. For example, if A is a permutation matrix or $A = J_n$, then $h + (n-1)k = n$, and if $A = J_n$ or $A = \frac{1}{n-1}(nJ_n - I)$, then $(n-1)h + k = n$.

(ii) The above corollary is, in fact, true for any $n\times n$ matrix B with $\sum_{i,j=1}^{n} b_{ij} \leq n$, e.g., substochastic matrices.

Proposition 2.16: Let $A \in \Omega_n$. Then $h+k \leq n$ with equality iff $n = 2$ or A is a permutation matrix.

Proof: For $n = 2$, $h+k = 2$ always holds. We assume that $n \geq 3$. By Corollary 2.14, $h+k \leq h+(n-1)k \leq n$. If $h+k = n$, then $h+k = h+(n-1)k = n$. Hence $k = (n-1)k$ and so $k = 0$. Therefore $h = n$ and A is a permutation matrix. The converse is obvious.

Proposition 2.17: Let $A \in \Omega_n$. Then $h+k \geq \frac{n}{n-1}$ with equality iff $n = 2$ or there exist permutation matrices P and Q such that

$$PAQ = \frac{1}{n-1}(nJ_n - I) = \begin{bmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 \end{bmatrix}.$$

Proof: For $n = 2$, $h+k = 2$ always holds. We assume that $n \geq 3$. By Corollary 2.14, $(n-1)(h+k) \geq (n-1)h + k \geq n$. Hence $h+k \geq \frac{n}{n-1}$. If $h+k = \frac{n}{n-1}$, then $(n-1)k = k$, from which $k = 0$. Hence $h = \frac{n}{n-1}$. Furthermore, let τ be any minimum (zero) diagonal and σ_2 be any diagonal disjoint from τ . Then the Covering Theorem implies the existence of $n-2$ diagonals $\sigma_3, \sigma_4, \cdots, \sigma_n$ such that $\{\tau, \sigma_i;\ i = 2, 3, \cdots, n\}$ cover A exactly. Since $(n-1)h + k = n$, we get from Corollary 2.14.(ii) that $\sum_{i=1}^{n} a_{i\sigma_j(i)} = h = \frac{n}{n-1}$ for all $j = 2, 3, \cdots, n$. In other words, if $h+k = \frac{n}{n-1}$, then there is a zero diagonal τ such that any diagonal disjoint from τ is a maximum diagonal with sum $h = \frac{n}{n-1}$. We claim that this implies the existence of permutation matrices P and Q such that $PAQ = \frac{1}{n-1}(nJ_n - I)$; i.e., A has a zero diagonal τ and all entries off τ are equal to $\frac{1}{n-1}$. The proof of this which turns out to be an interesting combinatorial argument must be split into several steps and will follow from the subsidiary results contained in Lemma 2.18, Theorem 2.19, and Theorem 2.20.

Lemma 2.18: If an $n \times n$ matrix A has the property that there is a certain diagonal τ such that every diagonal disjoint from τ has a constant sum, then every 2×2 submatrix that does not contain any entry from τ must have both diagonal sums equal.

Proof: Without loss of generality, we may assume that τ is the main diagonal. For $n = 2$ or $n = 3$, there is no 2×2 submatrix that does not contain any entry from the main diagonal. We assume that $n \geq 4$.

Consider any 2×2 submatrix $A[i,j \mid i',j']$ that does not contain any entry from the main diagonal. Interchanging the 1st row with the ith row, the 2nd row with the i'th row, the 1st column with the jth column, and the 2nd column with the j'th column, we can bring $A[i,j \mid i',j']$ to the upper left corner. On the 1st and 2nd row, there is an entry from the original main diagonal at the $(1, k)$ and $(2, \ell)$ positions, say, $k \geq 2$, $\ell \geq 2$. Interchanging the 3rd column with the kth column, the 4th column with the ℓth column, we can bring a_{1k} and $a_{2\ell}$ into the submatrix $A[1,2 \mid 3,4]$. Similarly, we can bring the two main diagonal entries on the 1st and 2nd column into the submatrix $A[3,4 \mid 1,2]$. Consequently, the matrix takes the form:

$$\left[\begin{array}{cc|cc|c}
a_{ij} & a_{ij'} & * & & \\
a_{i'j} & a_{i'j'} & & * & \\
\hline
* & & \alpha & & \\
 & * & & \beta & \\
\hline
 & & & & \\
 & & & & B \\
 & & & &
\end{array}\right],$$

where a $*$ denotes an entry form the original main diagonal of A . Now, in the $(n-4)\times(n-4)$ submatrix $B = A(1,2,3,4 \mid 1,2,3,4)$, we can chose any diagonal σ such that σ and τ (to be precise, τ restricted to B) are disjoint. Consider the diagonals $\sigma \cup \{\alpha, \beta, a_{ij}, a_{i'j'}\}$ and

$\sigma \cup \{\alpha, \beta, a_{ij'}, a_{i'j}\}$. By assumption, they have the same sum, and hence $a_{ij} + a_{i'j'} = a_{ij'} + a_{i'j}$.

<u>Theorem 2.19</u>: If an $n\times n$ matrix A has the property that there is a certain diagonal τ such that every 2×2 submatrix that does not contain any entry from τ has both diagonal sums equal, then for some permutation matrices P and Q, PAQ takes the form:

$$B = PAQ = \begin{bmatrix} * & \beta_2 & \beta_3 & \cdots & \beta_n \\ \alpha_2+\beta_1 & * & \alpha_2+\beta_3 & \cdots & \alpha_2+\beta_n \\ \alpha_3+\beta_1 & \alpha_3+\beta_2 & * & \cdots & \alpha_3+\beta_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n+\beta_1 & \alpha_n+\beta_2 & \alpha_n+\beta_3 & \cdots & * \end{bmatrix} .$$

Here, $b_{ij} = \alpha_i + \beta_j$ for all $i \neq j$, $i,j = 1, 2, \cdots, n$ and for convenience $\alpha_1 = 0$, and a $*$ denotes an entry from τ .

<u>Proof</u>: We prove this by induction on n . For $n = 2$ or $n = 3$, there is no 2×2 submatrix satisfying the assumed condition and hence the assertion is "vacuously true". We start with the case $n = 4$. Without loss of generality, we may assume that τ is the main diagonal. If we put $\alpha_1 = 0$, $\alpha_2 = a_{24} - a_{14}$, $\alpha_3 = a_{31} - a_{21} + a_{24} - a_{14}$, $\alpha_4 = a_{41} - a_{21} + a_{24} - a_{14}$, $\beta_1 = a_{21} - a_{24} + a_{14}$, $\beta_2 = a_{12}$, $\beta_3 = a_{13}$ and $\beta_4 = a_{14}$, then from the assumption, one can verify easily that $a_{ij} = \alpha_i + \beta_j$ for all $i \neq j$. Now assume that the assertion is true for $n \geq 4$. Let A be an $(n-1)\times(n-1)$ matrix with the described

property. As usual, assume that τ is the main diagonal. Since the $n\times n$ submatrix $A(n+1 \mid n+1)$ also has the described property, the induction hypothesis implies that A takes the form:

$$A = \left[\begin{array}{cccccc|c} * & \beta_2 & \beta_3 & \cdots\cdots & & \beta_n & a_{1\ n+1} \\ \alpha_2+\beta_1 & * & \alpha_2+\beta_3 & \cdots\cdots & & \alpha_2+\beta_n & a_{2\ n+1} \\ \alpha_3+\beta_1 & \alpha_3+\beta_2 & * & \cdots\cdots & & \alpha_3+\beta_n & a_{3\ n+1} \\ \cdot & \cdot & \cdot & & & \vdots & \vdots \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & \alpha_{n-2}+\beta_{n-1} & \alpha_{n-2}+\beta_n & a_{n-2\ n+1} \\ \cdot & \cdot & \cdot & & * & \alpha_{n-1}+\beta_n & a_{n-1\ n+1} \\ \alpha_n+\beta_1 & \alpha_n+\beta_2 & \alpha_n+\beta_3 & \cdots & \alpha_n+\beta_{n-1} & * & a_{n\ n+1} \\ \hline a_{n+1\ 1} & a_{n+1\ 2} & a_{n+1\ 3} & \cdots & a_{n+1\ n-1} & a_{n+1\ n} & * \end{array}\right].$$

Let $\beta_{n+1} = a_{1\ n+1}$. For each $a_{i\ n+1}$, $i = 2, 3, \cdots, n-1$, consider the 2×2 submatrix $A[1,i \mid n,n+1]$. By assumption, $\beta_n + a_{i\ n+1} = a_{1\ n+1} + \alpha_i + \beta_n$, and hence $a_{i\ n+1} = \alpha_i + a_{1\ n+1} = \alpha_i + \beta_{n+1}$. For $a_{n\ n+1}$, consider the 2×2 submatrix $A[n-2,n \mid n-1,n+1]$. By assumption, $\alpha_{n-2} + \beta_{n-1} + a_{n\ n+1} = a_{n-2\ n+1} + \alpha_n + \beta_{n-1}$, and hence $a_{n\ n+1} = a_{n-2\ n+1} + \alpha_n - \alpha_{n-2} = \alpha_{n-2} + \beta_{n+1} + \alpha_n - \alpha_{n-2} = \alpha_n + \beta_{n+1}$. Similarly, define $\alpha_{n+1} = a_{n+1\ 1} - \beta_1$. For each $a_{n+1,j}$, $j = 2, 3, \cdots, n-1$, consider the 2×2 submatrix $A[n,n+1 \mid 1,j]$. By assumption, $\alpha_n + \beta_1 + a_{n+1\ j} = \alpha_n + \beta_j + a_{n+1\ 1}$, and hence $a_{n+1\ j} = \beta_j + a_{n+1\ 1} - \beta_1 = \alpha_{n+1} + \beta_j$. Finally, for $a_{n+1\ n}$, consider the 2×2 submatrix

$A[n-2, n+1 \mid n-1, n]$. By assumption, $\alpha_{n-2} + \beta_{n-1} + a_{n+1\ n} =$ $= \alpha_{n-2} + \beta_2 + a_{n+1\ n-1}$, and hence $a_{n+1\ n} = a_{n+1\ n-1} + \beta_n - \beta_{n-1} =$ $= \alpha_{n+1} + \beta_n$. Therefore $a_{ij} = \alpha_i + \beta_j$ for all $i \neq j$, $i, j = 1, 2, \cdots, n+1$ if we set $\alpha_1 = 0$ for notational convenience. This completes the proof of Theorem 2.19.

<u>Theorem 2.20</u>: If $A \varepsilon \Omega_n$ has a zero diagonal τ such that every diagonal disjoint from τ has a constant sum, then all entries off τ are equal to $\frac{1}{n-1}$.

<u>Proof</u>: For $n = 2$ and $n = 3$, this is clear. We assume $n \geq 4$. As usual, assume that τ is the main diagonal. By Lemma 2.18 and Theorem 2.19, A takes the form:

$$A = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \cdots & \beta_n \\ \alpha_2+\beta_1 & 0 & \alpha_2+\beta_3 & \cdots & \alpha_2+\beta_n \\ \alpha_3+\beta_1 & \alpha_3+\beta_2 & 0 & \cdots & \alpha_3+\beta_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n+\beta_1 & \alpha_n+\beta_2 & \alpha_n+\beta_3 & \cdots & 0 \end{bmatrix} .$$

Clearly, any diagonal disjoint from τ has the constant sum $\sum_{i=1}^{n} (\alpha_i + \beta_i)$ where we set $\alpha_1 = 0$ as before. Let $\alpha = \sum_{i=1}^{n} \alpha_i$, $\beta = \sum_{i=1}^{n} \beta_i$. Since by the Covering Theorem, we can always select $n-1$ diagonals such that together with τ , they cover A exactly, we have $\alpha + \beta = \frac{n}{n-1}$ or

$$(n-1)\alpha + (n-1)\beta = n \quad . \tag{1}$$

Since the 1st row sum and the 1st column sum are one, we have

$$\beta - \beta_1 = 1 \tag{2}$$

and

$$\alpha + (n-1)\beta_1 = 1 \quad . \tag{3}$$

From (2) and (3), we get

$$\alpha + (n-1)\beta = n \quad . \tag{4}$$

From (1) and (4), we get $(n-2)\alpha = 0$ from which $\alpha = 0$. Hence $\beta = \frac{n}{n-1}$ and from (2), $\beta_1 = \beta - 1 = \frac{1}{n-1}$. Since the ith row sum and the ith column sum are one, $i = 2, 3, \cdots, n$, we get $(n-1)\alpha_i + \beta - \beta_i = 1$ and $\alpha - \alpha_i + (n-1)\beta_i = 1$. Hence

$$(n-1)\alpha_i - \beta_i = 1 - \beta = \frac{-1}{n-1} \tag{5}$$

and

$$\alpha_i - (n-1)\beta_i = \alpha - 1 = -1 \quad . \tag{6}$$

From (5) and (6), we get $[(n-1)^2 - 1]\alpha_i = 0$ or $n(n-2)\alpha_i = 0$, from which $\alpha_i = 0$ for all $i = 2, 3, \cdots, n$. Therefore, from (5), $\beta_i = \frac{1}{n-1}$ for all $i = 2, 3, \cdots, n$. This completes the proof.

The proof of Proposition 2.17 is thus also completed.

Theorem 2.20 has a generalized from which we shall state as a conjecture since we have not been able to prove it completely.

Conjecture 2.21: Let $A \in \Omega_n$, and let $\tau_1, \tau_2, \cdots, \tau_m$ be m mutually disjoint zero diagonals of A , $1 \leq m \leq n-1$. If every diagonal disjoint from each τ_i , $i = 1, 2, \cdots, m$ has a constant sum (this constant sum is $\frac{n}{n-m}$ by the Covering Theorem), then all entries off the m zero diagonals are equal to $\frac{1}{n-m}$.

We remark that this conjecture is clearly true for $m = n-1$ since A is then a permutation matrix. Also, Theorem 2.20 is the special case where $m = 1$. Furthermore, the arguments used in the proof of Lemma 2.18 and Theorem 2.19 do not yield an answer for Conjecture 2.21 since for arbitrary n and m, there may not be any 2×2 submatrix that does not contain any entry from the given m zero diagonals. The next proposition shows that Conjecture 2.21 is true for $m = n - 2$ also.

<u>Proposition 2.22:</u> Let $A \in \Omega_n$ and let $\tau_1, \tau_2, \cdots, \tau_{n-2}$ be $n-2$ mutually disjoint zero diagonals of A. If every diagonal disjoint from each τ_i, $i = 1, 2, \cdots, n-2$, has a constant sum, then all entries off the $n-2$ zero diagonals are equal to $1/2$.

<u>Proof:</u> It is clear that in each row and each column, there are exactly two entries off every τ_i, $i = 1, 2, \cdots, n-2$. For convenience, we call any such entry a "star". By the Covering Theorem, the constant sum in the hypothesis must be $\frac{n}{2}$. Permuting the rows and columns, we can assume that there are stars at the (1, 1)th, (1, 2)th, and the (2, 2)th positions. If the other star on the 2nd row is at the (2, 1)th position, then we get a direct sum. Otherwise, by permuting the columns, we can assume that it is at the (2, 3)th position, and by permuting the rows, we can assume that the other star on the 3rd column is at the (3, 3)th position. Repeat this process. Eventually, we can write A as a direct sum, $A = \oplus \sum_{i=1}^{r} A_i$, where $A_i \in \Omega_{n_i}$, $\sum_{i=1}^{r} n_i = n$, each $n_i \geq 2$; and each A_i as two diagonals consisting of stars only - the main diagonal and the diagonal corresponding to the permutation $(1,2,\cdots,n_i)$.

Since each A_i is d.s., we can write x_i for all entries on the main diagonal and $1-x_i$ for all entries on the other diagonal. For each $i = 1, 2, \cdots, r$, we have, by assumption, that

$n_1x_1 + n_2x_2 + \cdots + n_ix_i + \cdots + n_rx_r = \frac{n}{2}$ and

$n_1x_1 + n_2x_2 + \cdots + n_i(1-x_i) + \cdots + n_rx_r = \frac{n}{2}$. Hence $x_i = 1 - x_i$ or $x_i = \frac{1}{2}$. This completes the proof.

In view of Theorem 2.20 and Proposition 2.22, Conjecture 2.21 is true for $n = 2, 3, 4$.

Since $h+k \leq n$ by Proposition 2.16, we get $hk \leq (\frac{h+k}{2})^2 = \frac{n^2}{4}$. Hence there exists a smallest constant $\beta(n)$ such that $hk \leq \frac{1}{4}\beta(n)\cdot n^2$. We claim that $\beta(n) = \frac{1}{n-1}$.

<u>Proposition 2.23</u>: Let $A \in \Omega_n$. Then $hk \leq \frac{n^2}{4(n-1)}$ with equality iff $h = \frac{n}{2}$ and any diagonal disjoint from a maximum diagonal is a minimum diagonal with sum $k = \frac{n}{2(n-1)}$.

<u>Proof</u>: Since $n^2 - 4nh + 4h^2 = (n-2h)^2 \geq 0$, we get

$$h(n-h) = hn - h^2 \leq \frac{n^2}{4}. \tag{7}$$

Also, from Corollary 2.14.(i), we get

$$(n-1)k \leq n-h. \tag{8}$$

From (7) and (8), we get $(n-1)hk \leq \frac{n^2}{4}$ or $hk \leq \frac{n^2}{4(n-1)}$. If equality holds, it must hold in both (7) and (8). Hence $h = \frac{n}{2}$ and the equality case in Corollary 2.14.(i) implies that any diagonal from a maximum diagonal is a minimum diagonal with sum $\frac{n}{2(n-1)}$. The converse is obvious. This completes the proof.

The next examples show that the above bound is always attainable and is not unique for $n \geq 3$.

<u>Example 2.24</u>:

(i) Let A be the d.s. matrix:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2(n-1)} & \cdots & \frac{1}{2(n-1)} \\ \frac{1}{2(n-1)} & \frac{1}{2} & \cdots & \frac{1}{2(n-1)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2(n-1)} & \frac{1}{2(n-1)} & \cdots & \frac{1}{2} \end{bmatrix}$$

where $a_{ii} = \frac{1}{2}$ for all $i = 1, 2, \cdots, n$ and $a_{ij} = \frac{1}{2(n-1)}$ otherwise. Clearly, $h = \sum_{i=1}^{n} a_{ii} = \frac{n}{2}$ and any diagonal disjoint from the main diagonal is a minimum diagonal with sum $k = \frac{1}{2(n-1)}$.

(ii) For $n \geq 3$, let A be the d.s. matrix:

$$A = \begin{bmatrix} \frac{n+1}{2n} & \frac{1}{2n} & \cdots & \frac{1}{2n} \\ \frac{1}{2n} & \frac{n^2-n-1}{2n(n-1)} & \cdots & \frac{n^2-2n+2}{2n(n-1)(n-2)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2n} & \frac{n^2-2n+2}{2n(n-1)(n-2)} & \cdots & \frac{n^2-n-1}{2n(n-1)} \end{bmatrix},$$

where $a_{11} = \frac{n+1}{2n}$, $a_{1i} = a_{i1} = \frac{1}{2n}$, $a_{ii} = \frac{n^2-n-1}{2n(n-1)}$ for all $i = 2, 3, \cdots, n$, and $a_{ij} = \frac{n^2-2n+2}{2n(n-1)(n-2)}$ otherwise. It is readily verified that A is d.s. and $\frac{1}{2n} < \frac{n^2-2n+2}{2n(n-1)(n-2)} < \frac{n^2-n-1}{2n(n-1)} < \frac{n+1}{2n}$. Hence $h = \sum_{i=1}^{n} a_{ii} = \frac{n+1}{2n} + \frac{n^2-n-1}{2n} = \frac{n}{2}$ and any diagonal disjoint from the main diagonal is a minimum diagonal with sum $k = \frac{2}{2n} + \frac{n -2n+2}{2n(n-1)} = \frac{n}{2(n-1)}$.

For any $A \in \Omega_n$, since $h \geq 1$ and $k \leq 1$ with either equality iff $A = J_n$ (Corollary 1.5), we have $(h-1)(1-k) \geq 0$ or $h+k-hk \geq 1$ with equality iff $A = J_n$. The next proposition gives the upper bound for the corresponding quantity $h+k+hk$.

<u>Proposition 2.25</u>: Let $A \in \Omega_n$. Then $h+k+hk \leq n + \frac{1}{n-1}$ with equality iff $h = n-1$ and any diagonal disjoint from a maximum diagonal is a minimum diagonal with sum $k = \frac{1}{n-1}$.

<u>Proof</u>: Let $B = \frac{1}{2}(A + J_n)$. Then $B \in \Omega_n$ and the maximum diagonals and minimum diagonals of A and B correspond to each other. An application of Proposition 2.23 to B gives $(\frac{h+1}{2})(\frac{k+1}{2}) \leq \frac{n^2}{4(n-1)}$, or $h+k+hk \leq \frac{n^2}{n-1} - 1 = n + \frac{1}{n-1}$. If equality holds, then by Proposition 2.23, $\frac{h+1}{2} = \frac{n}{2}$ or $h = n-1$ and any diagonal disjoint from a maximum diagonal is a minimum diagonal with sum k such that $\frac{k+1}{2} = \frac{n}{2(n-1)}$ or $k = \frac{1}{n-1}$. The converse is obvious. This completes the proof.

We close this chapter by obtaining bounds for h, k and $h+k$ when the d.s. matrices under consideration have properties discussed in Chapter I.

Proposition 2.26: If $A \in \Omega_n$ has precisely $(n-1)(n-1)!$ diagonals with common sum $\alpha \neq 1$, then $h \leq 2$, $k \geq \frac{n-2}{n-1}$ and $2 - \frac{n-2}{(n-1)^2} \leq h+k \leq 2 + \frac{n-2}{n-1}$. These bounds are always attainable.

Proof: By Theorem 1.20, there exist permutation matrices P and Q such that PAQ is a (x, y, z)-matrix. Examining the proof of Corollary 1.21 shows that $\alpha = 1 + \frac{1-nx}{(n-1)^2}$. Hence $\beta = x + (n-1)z = n - (n-1)\alpha = 1 - \frac{1-nx}{n-1}$ and $\alpha + \beta = 2 - \frac{(1-nx)(n-2)}{(n-1)^2}$ where β denotes the only other diagonal sum (Lemma 1.19). Clearly, α attains its maximum $1 + \frac{1}{(n-1)^2}$ when $x = 0$ and its minimum $\frac{n-2}{n-1}$ when $x = 1$; and β attains its maximum 2 when $x = 1$ and its minimum $\frac{n-2}{n-1}$ when $x = 0$. Finally, $\alpha + \beta$ attains its maximum $2 + \frac{n-2}{n-1}$ when $x = 1$ and its minimum $2 - \frac{n-2}{(n-1)^2}$ when $x = 0$. Therefore, $h \leq 2$, $k \geq \frac{n-2}{n-1}$ and $2 - \frac{n-2}{(n-1)^2} \leq h+k \leq 2 + \frac{n-2}{n-1}$. It is clear from the proof that these bounds are always attainable.

Proposition 2.27: If $A \in \Omega_n$ $(n \geq 3)$ has precisely $(n-1)(n-1)!$ diagonals with sum (i) greater than one, then $h + k < 2$,

(ii) smaller than one, than $h + k > 2$.

This bound is not attainable but best possible in the sense that $h+k$ can become arbitrarily close to 2 in either case.

<u>Proof</u>: Let $E = \left\{\sigma;\ \sum_{i=1}^{n} a_{i\sigma(i)} > 1\right\}$. Then $|E| = (n-1)(n-1)!$.

Let $D = \bigcup_{j=1}^{(n-1)!} D_j$ be a decomposition of the set D of all diagonals as described in Theorem 1.11. Then we have seen (cf. Proposition 1.18) that $|E \cap D_j| = n-1$ for all $j = 1, 2, \cdots, (n-1)!$. In case (i), let σ_o be a maximum diagonal with sum h where $\sigma_o \in D_1$, say. Let $h, \alpha_2, \alpha_3, \cdots, \alpha_{n-1}, \beta$ be the diagonal sum of each diagonal in D_1 where $\beta < 1$ and $\alpha_i > 1$, $i = 2, 3, \cdots, n-1$. Since D_1 covers A exactly, $n = h + \beta + \sum_{i=2}^{n-1} \alpha_i > h + \beta + n-2$ and hence $h+k \leq h+\beta < 2$.

Similarly, in case (ii), let τ_o be a minimum diagonal with sum k where $\tau_o \in D_1$, say. Let $k, \beta_2, \beta_3, \cdots, \beta_{n-1}, \alpha$ be the diagonal sum of each diagonal in D_1 where $\alpha > 1$ and $\beta_i < 1$, $i = 2, 3, \cdots, n-1$. Since D_1 covers A exactly, $n = k + \alpha + \sum_{i=2}^{n-1} \beta_i < k + \alpha + n-2$ and hence $h+k \geq \alpha+k > 2$. Furthermore, it is clear from the proof that this bound is not attainable. Finally, to see that this bound is approachable, consider a d.s. (x, y, z)-matrix A with $x \neq \frac{1}{n}$ (so that $A \neq J_n$). As seen before, A has only 2 distinct diagonal sums α and β with $\alpha + \beta = 2 - \frac{(1-nx)(n-2)}{(n-1)^2}$. (cf. Proposition 2.26.) It is clear that $\alpha + \beta \to 2^{-o}$ as $x \to \frac{1}{n}^{-o}$, and $\alpha + \beta \to 2^{+o}$ as $x \to \frac{1}{n}^{+o}$.

CHAPTER III

PROPERTIES OF THE h-FUNCTION AND THE k-FUNCTION OF d.s. MATRICES

The main prupose of this chapter is to study the properties of the h-function and the k-function, the functions that associate with each d.s. matrix its maximum and minimum diagonal sums, respectively. In particular, we investigate the behaviour of these functions on the Kronecker product of d.s. matrices. We will show that the h-function is very similar to the rank function ρ in many respects. Furthermore, we shall prove that, for $A \in \Omega_n$, $h(A) \leq \rho(A)$ and $\operatorname{per}(A) \leq \left\{\frac{h(A)}{n}\right\}^{\frac{1}{2}}$, which improves a result of Marcus and Minc [11]. A conjecture is given.

We mention first that the functions h and k are, in fact, defined for any $n \times n$ matrices. It is when restricted to d.s. matrices that they have interesting properties.

We shall denote the set of all $n \times n$ matrices with non-negative entries by H_n.

Lemma 3.1: Let A, B be in H_n. Then (i) $h(A+B) \leq h(A) + h(B)$ with equality iff A and B have a corresponding maximum diagonal. (ii) $k(A+B) \geq k(A) + k(B)$ with equality iff A and B have a corresponding minimum diagonal.

Proof: (i) Let σ, τ and μ be any maximum diagonals of A, B and $A+B$ respectively. Then $h(A+B) = \sum_{i=1}^{n} (a_{i\mu(i)} + b_{i\mu(i)}) =$

$= \sum_{i=1}^{n} a_{i\mu(i)} + \sum_{i=1}^{n} b_{i\mu(i)} \leq \sum_{i=1}^{n} a_{i\sigma(i)} + \sum_{i=1}^{n} b_{i\tau(i)} = h(A) + h(B)$. If equality holds, then $\sum_{i=1}^{n} a_{i\mu(i)} = \sum_{i=1}^{n} a_{i\sigma(i)}$ and $\sum_{i=1}^{n} b_{i\mu(i)} = \sum_{i=1}^{n} b_{i\tau(i)}$. Hence μ is a maximum diagonal for both A and B . The converse is obvious. The proof of (ii) is similar.

The above lemma can be generalized immediately to:

Corollary 3.2: Let $A_i \in H_n$, where $i = 1, 2, \cdots, m$. Then

(i) $h(\sum_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} h(A_i)$ with equality iff all A_i's have a corresponding maximum diagonal.

(ii) $k(\sum_{i=1}^{m} A_i) \geq \sum_{i=1}^{m} k(A_i)$ with equality iff all A_i's have a corresponding minimum diagonal.

Proof: By Lemma 3.1 and induction on m .

Corollary 3.3: The function h is convex on H_n , and the function k is concave on H_n .

Proof: Let A,B be in H_n and $\lambda \in [0, 1]$. Then

$$h\{\lambda A + (1-\lambda)B\} \leq h(\lambda A) + h((1-\lambda)B) = \lambda h(A) + (1-\lambda)h(B) \quad \text{and}$$

$$k\{\lambda A + (1-\lambda)B\} \geq k(\lambda A) + k((1-\lambda)B) = \lambda k(A) + (1-\lambda)k(B) .$$

Theorem 3.4: Let A,B be in Ω_n . Then

(i) $h(AB) \leq \min \{h(A), h(B)\}$

(ii) $k(AB) \geq \max \{k(A), k(B)\}$.

Proof: (i) By the well-known theorem of Birkhoff [1], every d.s. matrix can be expressed as a convex combination of permutation matrices. Hence $B = \sum_{i=1}^{m} \lambda_i P_i$, where $\lambda_i \geq 0$ such that $\sum_{i=1}^{m} \lambda_i = 1$, and P_i is a permutation matrix, $i = 1, 2, \cdots, m$. Since $h(AP_i) = h(A)$ for all i , Corollary 3.3 gives that $h(AB) = h(\sum_{i=1}^{m} \lambda_i AP_i) \leq \sum_{i=1}^{m} \lambda_i h(AP_i) =$ $= (\sum_{i=1}^{m} \lambda_i)(h(A)) = h(A)$. Similarly, $h(AB) \leq h(B)$. Therefore, $h(AB) \leq \min\{h(A), h(B)\}$. The proof of (ii) is similar.

The next two corollaries are immediate consequences of Theorem 3.4.

Corollary 3.5: Let A,B be in Ω_n . Then AB is a permutation matrix iff both A and B are permutation matrices.

Proof: If AB is a permutation matrix, then $h(AB) = n$, and hence $h(A) \geq n$ and $h(B) \geq n$ by Theorem 3.4. Therefore $h(A) = h(B) = n$, and A and B are both permutation matrices. The converse is obvious.

Corollary 3.6: Let $A \in \Omega_n$. Then for all m , (i) $h(A^m) \leq h(A)$; and (ii) $k(A^m) \geq k(A)$.

Proof: Since $A \in \Omega_n$ implies that $A^m \in \Omega_n$ for all $m = 1, 2, \cdots$, it suffices to put $B = A^{m-1}$ in Theorem 3.4.

Remark 3.7: To determine when equality holds in Theorem 3.4 seems to be quite difficult. For example, $h(AB) = h(A)$ will hold if B is a permutation matrix or if $PAQ = J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_r}$, where $\sum_{i=1}^{r} n_i = n$

and P and Q are permutation matrices. The equality can hold, however, when both A and B are fully indecomposable. For example,

let $A = B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

Hence $h(AB) = \frac{3}{2} = h(A) = h(B)$. We also observe another condition for equality as follows. Let $\beta(B)$ denote the minimum number of permutation matrices necessary to represent B as a convex combination [15]. Then, in view of Corollary 3.2 and the proof of Theorem 3.4, A must have at least $\beta(B)$ maximum diagonals in order that $h(AB) = h(B)$ can possibly hold. This condition, however, is rather weak since $\beta(B)$ is small compared with $n!$, the number of diagonals of a matrix.

Corollary 3.8: Let A,B be in Ω_n . Then

(i) $h(AB) \leq h(A)h(B)$ with equality iff $A = B = J_n$.

(ii) $k(AB) \geq k(A)k(B)$ with equality iff $A = B = J_n$.

Proof: (i) For any d.s. matrix X , we have, by Corollary 1.5, that $h(X) \geq 1$ with equality iff $X = J_n$. Hence Theorem 3.4 implies that $h(AB) \leq \min\{h(A), h(B)\} \leq h(A)h(B)$ with equality only if $h(A) = h(B) = 1$ or $A = B = J_n$. The converse is obvious. The proof of (ii) is similar.

Let A, B be in Ω_n. Since $h(A) \geq 1$ and $h(AB) \leq h(B)$, it follows that $1 \leq h(A) + h(B) - h(AB)$ with equality iff $A = B = J_n$. The problem of determining the corresponding best possible upper bound for $h(A) + h(B) - h(AB)$ seems to be a difficult one. Clearly, $2n$ is an upper bound. The following conjecture which seems plausible is exactly the analogue of Sylvester's law for rank of two matrices:

$$\rho(A) + \rho(B) - \rho(AB) \leq n \qquad [14, \text{p. } 28].$$

<u>Conjecture 3.9</u>: Let A, B be in Ω_n. Then $h(A) + h(B) - h(AB) \leq n$.

<u>Remark 3.10</u>: With little computation, Conjecture 3.9 can be verified to be true for $n = 2$. However, even for $n = 3$, the manipulations get too involved to give any answer. The difficulty of this problem can perhaps be seen from the fact that the equality will not be attained uniquely. It is clear that $h(A) + h(B) - h(AB) = n$ if either A or B is a permutation matrix. This is, however, not the only case when equality can hold; e.g., consider the 4×4 d.s. matrices:

$$A = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & J_2 \end{array}\right], \qquad B = \left[\begin{array}{c|c} 0 & J_2 \\ \hline I_2 & 0 \end{array}\right].$$

Then $h(A) = h(B) = 3$, and

$$AB = \left[\begin{array}{c|c} 0 & J_2 \\ \hline J_2 & 0 \end{array}\right].$$

Hence $h(AB) = 2$ and $h(A) + h(B) - h(AB) = 4 = n$.

For arbitrary A,B in H_n , it is easy to see that $h(A) + h(B) - h(A + B)$ and $k(A + B) - k(A) - k(B)$ have no upper bound though zero is the lower bound in both cases by Lemma 3.1. If, however, A and B are d.s. matrices, then it is readily seen that the upper bounds exist as indicated in the next proposition.

Proposition 3.11: Let A,B be in Ω_n . Then

(i) $h(A) + h(B) - h(A + B) \leq n$ with equality iff A and B are permutation matrices corresponding to disjoint permutations.

(ii) $k(A + B) - k(A) - k(B) \leq 2$ with equality iff both A and B have a zero diagonal and $a_{ij} + b_{ij} = \frac{2}{n}$ for all $i,j = 1, 2, \cdots, n$.

Proof:

(i) Since $h(A) \leq n$, $h(B) \leq h(A + B)$, it is clear that $h(A) + h(B) - h(A + B) \leq n$. If equality holds, then $n = h(A) = h(B) = h(A + B)$ and hence A and B are permutation matrices corresponding to disjoint permutations. The converse is obvious.

(ii) Since $\frac{1}{2}(A + B) \in \Omega_n$, we have, $\frac{1}{2} k(A + B) = k(\frac{A+B}{2}) \leq 1$ or $k(A + B) \leq 2$, and hence $k(A + B) - k(A) - k(B) \leq 2$. If equality holds, then $k(A) = k(B) = 0$ and $k(\frac{A+B}{2}) = 1$. Hence both A and B have a zero diagonal and $\frac{A+B}{2} = J_n$, which implies that $a_{ij} + b_{ij} = \frac{2}{n}$ for all $i,j = 1, 2, \cdots, n$. The converse is obvious. This completes the proof.

Next, we study the behaviour of the h-function and the k-function on the Kronecker product (or direct product) [9] of d.s. matrices. The fact that $A, B \varepsilon \Omega_n$ implies that $A \times B \varepsilon \Omega_{n^2}$ makes the consideration of $h(A \times B)$ and $k(A \times B)$ quite natural.

Lemma 3.12: For any n-square matrices A and B,

(i) $h(A \times B) = h(B \times A)$ and $k(A \times B) = k(B \times A)$.

(ii) $h(PAQ \times P_1BQ_1) = h(A \times B)$ and $k(PAQ \times P_1BQ_1) = k(A \times B)$ for any permutation matrices P, Q, P_1, and Q_1.

Proof:

(i) Since it is known [2] that there exists a permutation matrix P such that $P^t(A \times B)P = B \times A$, the assertions are clear.

(ii) Since the Kronecker product satisfies the property that $AC \times BD = (A \times B)(C \times D)$, we have $(P \times P_1)(A \times B)(Q \times Q_1) = (PAQ) \times (P_1BQ_1)$; i.e., permuting the rows and columns of A and B only permutes the rows and columns of $A \times B$. Hence the assertions follows. This completes the proof.

We remark that the above lemma can be of practical use sometimes when one of the matrices $A \times B$ or $B \times A$ has a form easily handled (e.g. Proposition 3.18 below).

The next proposition presents a strong contrast between the Kronecker product and the ordinary product of d.s. matrices (cf. Corollary 3.8).

<u>Proposition 3.13</u>: Let A,B be in Ω_n . Then (i) $h(A \times B) \geq h(A)h(B)$ and (ii) $k(A \times B) \leq k(A)k(B)$.

<u>Proof</u>: (i) Let σ be a maximum diagonal of A with sum $h(A)$. In $A \times B$, consider the blocks at the $(i, \sigma(i))$th position, where $i = 1, 2, \cdots, n$. In each of these blocks, there is a diagonal (of that block) with sum $a_{i\sigma(i)} \cdot h(B)$, $i = 1, 2, \cdots, n$. The union of these diagonals clearly forms a diagonal for $A \times B$ with sum $h(A) \cdot h(B)$. Hence $h(A \times B) \geq h(A)h(B)$. The proof of (ii) is similar.

As an immediate consequence, we have:

<u>Corollary 3.14</u>: Let A,B be in Ω_n . Then $A \times B = J_{n^2}$ iff $A = B = J_n$.

<u>Proof</u>: If $A \times B = J_{n^2}$, then $h(A \times B) = 1$ and hence $h(A)h(B) \leq 1$ by Proposition 3.13. But $h(A) \geq 1$, and $h(B) \geq 1$ by Corollary 1.5. Hence $h(A) = h(B) = 1$ or $A = B = J_n$. The sufficiency is clear.

<u>Remark 3.15</u>: The case for equality in Proposition 3.13 is of some interest. It can be verified directly that for $n = 2$, equality always holds in both cases. In general, we are unable to determine the conditions for equality. For $n \geq 3$, however, there exist matrices such that the inequality is strict as shown by the next example.

Example 3.16: Let

$$A = B = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \varepsilon\ \Omega_3 \ .$$

Then $h(A) = h(B) = \frac{5}{4}$ and $h(A)h(B) = \frac{25}{16}$. Now,

$$A \times B = \frac{1}{16}\left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & \underline{4} & 4 & 0 & 4 & 4 \\ 0 & 0 & 0 & \underline{4} & 2 & 2 & 4 & 2 & 2 \\ 0 & 0 & 0 & 4 & 2 & 2 & \underline{4} & 2 & 2 \\ 0 & \underline{4} & 4 & 0 & 2 & 2 & 0 & 2 & 2 \\ \underline{4} & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\ 4 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & \underline{1} \\ 0 & 4 & \underline{4} & 0 & 2 & 2 & 0 & 2 & 2 \\ 4 & 2 & 2 & 2 & 1 & \underline{1} & 2 & 1 & 1 \\ 4 & 2 & 2 & 2 & 1 & 1 & 2 & \underline{1} & 1 \end{array}\right] \varepsilon\ \Omega_9 \ .$$

If we consider the diagonal of $A \times B$ consisting of the underlined entries, we see that $h(A \times B) \geq \frac{1}{16}(24 + 3) = \frac{27}{16} > h(A)h(B)$.

The next two propositions show that we have equality in both cases in Proposition 3.13 if one of A and B is some special d.s. matrix.

Proposition 3.17: Let $A \varepsilon \Omega_n$ and let P be any permutation matrix. Then (i) $h(A \times P) = h(P \times A) = h(P)h(A) = nh(A)$.

(ii) $k(A \times P) = k(P \times A) = k(P)k(A) = 0$.

Proof: In view of Lemma 3.12.(i), it suffices to consider $P \times A$. Let σ be the permutation corresponding to P . Then $P \times A$ has a copy of A in the $(i, \sigma(i))$th block, where $i = 1, 2, \cdots, n$, and all the other blocks are 0 . Hence $h(P \times A) = nh(A)$ and $k(P \times A) = 0$.

Proposition 3.18: Let $A \in \Omega_n$. Then

$$\text{(i)} \quad h(A \times J_n) = h(J_n \times A) = h(A) \ .$$

$$\text{(ii)} \quad k(A \times J_n) = k(J_n \times A) = k(A) \ .$$

Proof: (i) In view of Lemma 3.12.(i) and Proposition 3.13, it suffices to show that $h(J_n \times A) \leq h(A)$. We have

$$J_n \times A = \frac{1}{n}\begin{bmatrix} A & A & \cdots & A \\ A & A & \cdots & A \\ \vdots & \vdots & & \vdots \\ A & A & \cdots & A \end{bmatrix} \ .$$

Let σ be any diagonal of $J_n \times A$. Consider the following sets of indices (mod n) taken from the set $\{1, 2, \cdots, n\}$:

$$T_1 = \{\sigma(kn + 1) \ , \ (\text{mod } n) \ ; \ k = 0, 1, \cdots, n-1\} \ ,$$

$$T_2 = \{\sigma(kn + 2) \ , \ (\text{mod } n) \ ; \ k = 0, 1, \cdots, n-1\} \ ,$$

$$\vdots$$

$$T_n = \{\sigma(kn + n) \ , \ (\text{mod } n) \ ; \ k = 0, 1, \cdots, n-1\} \ .$$

In other words, T_i consists of the 2nd indices (mod n) of all entries of σ which lie on the ith row of each block, where $i = 1, 2, \cdots, n$. Note that the elements in each T_i are not necessarily distinct. We claim that there is an SDR for the sets $T_1, T_2, \cdots, T_n$. Let m be an integer such that $1 \leq m \leq n$. Let ω be an increasing sequence of m terms, $1 \leq \omega_1 < \omega_2 < \cdots < \omega_m \leq n$. We claim that $|T_{\omega_1} \cup T_{\omega_2} \cup \cdots \cup T_{\omega_m}| \geq m$. Since each T_{ω_i} , where

$i = 1, 2, \cdots, m$, has n elements counting repetition, $T_{\omega_1} \cup T_{\omega_2} \cup \cdots \cup T_{\omega_m}$ has mn elements counting repetition. Furthermore, it is clear from the definition of diagonal that each index (mod n) occurs precisely n times in $\bigcup_{j=1}^{n} T_j$ and hence occurs at most n times in $\bigcup_{i=1}^{m} T_{\omega_i}$. Therefore, $|T_{\omega_1} \cup T_{\omega_2} \cup \cdots \cup T_{\omega_m}| \geq m$. Now, the theorem of P. Hall [20, p. 48] implies the existence of an SDR for the sets $T_1, T_2, \cdots, T_n$; $\sigma(k_i n + i) \varepsilon T_i$ say, where $i = 1, 2, \cdots, n$, $0 \leq k_i \leq n-1$. Consider the set $S = \{a_{k_i n+i, \sigma(k_i n+i)}; i = 1, 2, \cdots, n\}$. We identify the rows of $J_n \times A$ that are the same rows of A ,and the columns of $J_n \times A$ that are the same columns of A . Since $k_i n+i \neq k_j n+j \pmod n$, and $\sigma(k_i n + i) \neq \sigma(k_j n + j) \pmod n$ for $i \neq j$, $i,j = 1, 2, \cdots, n$, the elements of S lie on distinct rows and columns even under the identification. Hence they constitute a diagonal for A . Now we permute the $(k_i n + i)$th row with the ith row, where $i = 1, 2, \cdots, n$. Since these two rows are the same row under the identification, this permutation will not affect the block structure of $J_n \times A$. Furthermore, since $\{\sigma(k_i n + i), \pmod n; i = 1, 2, \cdots, n\} = \{1, 2, \cdots, n\}$, there exists, for each i , a unique j (depending on i) such that $\sigma(k_j n + j) = i \pmod n$, $i = 1, 2, \cdots, n$. Hence permuting the ith column with the $\sigma(k_j n + j)$th (mod n) column, where $i = 1, 2, \cdots, n$, will not affect the block structure of $J_n \times A$. In this manner, we can bring the above found diagonal into the $(1, 1)$th block of $J_n \times A$. Since there are no entries of σ left on the first

n rows and columns, we can consider the (n^2-n)-square matrix H obtained by deleting the first n rows and columns of $J_n \times A$, and the remaining $n^2 - n$ entries of σ will form a diagonal for H. Now, we can repeat the above process and eventually bring the entries of σ into the (1,1)th, (2, 2)th, $\cdots$, and (n, n)th block such that each block has n entries from σ that form a diagonal for that block (namely, A), and such that the block structure of $J_n \times A$ remains unchanged. Hence $h(J_n \times A) \leq \frac{1}{n}\{nh(A)\} = h(A)$. The proof of (ii) is similar.

The Birkhoff theorem can be used to yield an upper bound for $h(A \times B)$ in terms of $h(A)$ and $h(B)$.

Proposition 3.19: Let A, B be in Ω_n. Then $h(A \times B) \leq$ $\leq \min\{nh(A), nh(B)\}$ with equality if either A or B is a permutation matrix.

Proof: By Birkhoff's theorem, let $B = \sum_{i=1}^{m} \lambda_i P_i$, where $\lambda_i \geq 0$ such that $\sum_{i=1}^{m} \lambda_i = 1$, and each P_i is a permutation matrix, $i = 1, 2, \cdots, m$. Since the Kronecker product is distributive over the summation [9, p. 82], we get, by Corollary 3.2.(i) and Proposition 3.17.(i), that $h(A \times B) = h(A \times \sum_{i=1}^{m} \lambda_i P_i) = h(\sum_{i=1}^{m} \lambda_i (A \times P_i)) \leq \sum_{i=1}^{m} \lambda_i h(A \times P_i) =$ $= nh(A) \sum_{i=1}^{m} \lambda_i = nh(A)$. Similarly, $h(A \times B) \leq nh(B)$. The assertion for equality is obvious from Proposition 3.17.(i).

In the rest of this chapter, we shall study the relation between $h(A)$ and $per(A)$, the permanent function of A, where $A \in \Omega_n$.

First, however, in order to increase understanding of the function $h(A)$, we list the similarities between $h(A)$ and the rank function $\rho(A)$.

(1) If $A \neq 0$, then $1 \leq \rho(A) \leq n$.

(1)' If $A \in \Omega_n$, then $1 \leq h(A) \leq n$. (Corollary 1.5)

(2) $\rho(A) = \rho(A^t)$.

(2)' $h(A) = h(A^t)$. (Trivial)

(3) $\rho(A + B) \leq \rho(A) + \rho(B)$.

(3)' $h(A + B) \leq h(A) + h(B)$ for A,B in D_n. (Lemma 3.1.(i))

(4) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.

(4)' $h(AB) \leq \min\{h(A), h(B)\}$ for A,B in Ω_n. (Theorem 3.4.(i))

(5) If $A = \oplus \sum_{i=1}^{m} A_i$, then $\rho(A) = \sum_{i=1}^{m} \rho(A)$.

(5)' If $A = \oplus \sum_{i=1}^{m} A_i$, then $h(A) = \sum_{i=1}^{m} h(A_i)$. (Trivial)

(6) If P and Q are non-singular, then $\rho(A) = \rho(PAQ)$.

(6)' If P and Q are permutation matrices, then $h(A) = h(PAQ)$. (Trivial)

The similarities would be even more striking if Conjecture 3.9 and its generalization (Conjecture 3.20 below) are true since they are exactly the analogue of Sylvester's law and Frobenius' inequality for the rank function, respectively [14, pp. 27-28].

<u>Conjecture 3.20</u>: Let $A,B,$ and C be in Ω_n. Then $h(AB) + h(BC) \leq h(B) + h(ABC)$. (Conjecture 3.9 is a special case of this for $B = I$.)

In fact, it was the above listed similarities together with some observations and experiments that tempted us to make the Conjectures 3.9 and 3.20.

In [11], Marcus and Minc proved the following theorem.

<u>Theorem 3.21</u>: If $A \in \Omega_n$, then $\mathrm{per}(A) \leq \left\{\frac{\rho(A)}{n}\right\}^{\frac{1}{2}}$ and in addition, if A is normal, then $\mathrm{per}(A) \leq \frac{\rho(A)}{n}$ with equality iff A is a permutation matrix or $n = 2$ and $A = J_2$.

Our main result concerning this will be to show that this theorem still holds if we replace $\rho(A)$ by $h(A)$ and that the new upper bound is an improvement on the old one. But first of all, a simple inequality between $\mathrm{per}(A)$ and $h(A)$:

<u>Proposition 3.22</u>: If $A \in \Omega_n$, then $\mathrm{per}(A) \leq \frac{n!}{n^n}\left\{h(A)\right\}^n$ with, equality iff $A = J_n$.

<u>Proof</u>: This is immediate from the definition of $\mathrm{per}(A)$ and the arithmetic - geometric mean inequality.

<u>Theorem 3.23</u>: If $A \in \Omega_n$, then $\mathrm{per}(A) \leq \left\{\frac{h(A)}{n}\right\}^{\frac{1}{2}}$ with equality iff A is a permutation matrix. If A is also normal, then $\mathrm{per}(A) \leq \frac{h(A)}{n}$ with equality iff A is a permutation matrix or $n = 2$ and $A = J_2$.

<u>Proof</u>: Assume first that A is normal. Then Marcus and Minc proved [11, Theorem 1] that $\mathrm{per}(A) \leq \frac{1}{n}\sum_{i=1}^{n}|\lambda_i|^n$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of A. Since it is well known that each

eigenvalue of a d.s. matrix does not exceed one in modulus, we have $|\lambda_i|^n \leq |\lambda_i|^2$ for all i, and hence $\operatorname{per}(A) \leq \frac{1}{n}\sum_{i=1}^{n}|\lambda_i|^2$. Since A is normal, the classical Schur's inequality gives $\sum_{i=1}^{n}|\lambda_i|^2 = \sum_{i,j=1}^{n}|a_{ij}|^2 = \sum_{i,j=1}^{n} a_{ij}^2$ and hence $\operatorname{per}(A) \leq \frac{1}{n}\sum_{i,j=1}^{n} a_{ij}^2$. Furthermore, by a result of Marcus and Ree [18], there exists a diagonal σ such that $\sum_{i,j=1}^{n} a_{ij}^2 \leq \sum_{i=1}^{n} a_{i\sigma(i)}$ and consequently, $\operatorname{per}(A) \leq \frac{1}{n}\sum_{i=1}^{n} a_{i\sigma(i)} \leq \frac{h(A)}{n}$. If equality holds, then, for all $i = 1, 2, \cdots, n$, $|\lambda_i|^n = |\lambda_i|^2$. If $n \geq 3$, this implies that $|\lambda_i| = 1$ for all $i = 1, 2, \cdots, n$. Hence by a result of Mirsky and Perfect [19, Theorem 5], we conclude that A is a permutation matrix. If $n = 2$, then

$$A = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} .$$

Hence $\operatorname{per}(A) = \frac{h(A)}{n}$ would imply that $x^2 + (1-x)^2 = x$ or $1-x$ depending on whether $x \geq \frac{1}{2}$ or $x \leq \frac{1}{2}$. If $x^2 + (1-x)^2 = x$, we get $x = 1$ or $x = \frac{1}{2}$. If $x^2 + (1-x)^2 = 1-x$, we get $x = 0$ or $x = \frac{1}{2}$. Hence $A = J_2$ or A is a permutation matrix. The converse is obvious. For general A, we apply the inequality $(\operatorname{per}(AB))^2 \leq \operatorname{per}(AA^t)\operatorname{per}(B^tB)$ of Marcus and Newman [17, Theorem 5] to get, by putting $B = I_n$, $(\operatorname{per}(A))^2 \leq \operatorname{per}(AA^t)$. Since $AA^t \in \Omega_n$ is normal, the above result and Theorem 3.4.(i) together imply that $(\operatorname{per}(A))^2 \leq \frac{h(AA^t)}{n} \leq \frac{h(A)}{n}$ or $\operatorname{per}(A) \leq \left\{\frac{h(A)}{n}\right\}^{\frac{1}{2}}$. If equality holds, it must also hold in the inequality $(\operatorname{per}(AB))^2 \leq \operatorname{per}(AA^t)\operatorname{per}(B^tB)$. Equality implies that either

(i) a row of A or a column of B consists of zeros or (ii) $A^t = BDP$ where D is a diagonal matrix and P is a permutation matrix. In our case, since $B = I_n$ and since (i) is impossible, we get $A^t = DP$ or $D = A^tP^t \in \Omega_n$ and hence $D = I_n$. Therefore $A^t = P$ or $A = P^t$ is a permutation matrix. The converse is again obvious.

As an immediate consequence, we obtain the following well known upper bound for per(A) [16, Lemma 1].

<u>Corollary 3.24</u>: If $A \in \Omega_n$, then $\text{per}(A) \leq 1$ with equality iff A is a permutation matrix.

<u>Proof</u>: This follows from Theorem 3.23 since $h(A) \leq n$, with equality iff A is a permutation matrix.

<u>Theorem 3.25</u>: If $A \in \Omega_n$, then $h(A) \leq \rho(A)$.

<u>Proof</u>: Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the non-zero eigenvalues of A. Then $\rho(A) \geq r$. Since A is d.s., $|\lambda_i| \leq 1$ for all $i = 1, 2, \cdots, r$. Hence $\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_r = |\lambda_1 + \lambda_2 + \cdots + \lambda_r| \leq$ $\leq |\lambda_1| + |\lambda_2| + \cdots + |\lambda_r| \leq r \leq \rho(A)$. Now choose permutation matrices P and Q such that $B = PAQ$ has the main diagonal as one of its maximum diagonals. Then, since $B \in \Omega_n$, we have $h(A) = h(B) =$ $= \text{trace}(B) \leq \rho(B) = \rho(A)$.

In view of Theorem 3.25 and the fact that $\rho(A) = n$ for all non-singular $n \times n$ matrices whence Theorem 3.21 yields the trivial bound $\operatorname{per}(A) \leq 1$, we see that the bounds obtained in Theorem 3.23 are indeed improvements on those in Theorem 3.21.

Concerning Theorem 3.23, we do not know whether the square root can be removed in general. The following propositions answer this question partially.

Proposition 3.26: Let $A \in \Omega_n$. If for some permutation matrices P and Q, $PAQ = \oplus \sum_{i=1}^{r} A_i$, where $r > 1$, $A_i \in \Omega_{n_i}$, $i = 1, 2, \cdots, r$, and $\sum_{i=1}^{r} n_i = n$, and if $\operatorname{per}(A_i) \leq \frac{h_i}{n_i}$ for all $i = 1, 2, \cdots, r$, where $h_i = h(A_i)$, then $\operatorname{per}(A) \leq \frac{h}{n}$ with equality iff A is a permutation matrix.

Proof: Since $\dfrac{\sum_{i=1}^{r} n_i}{\prod_{i=1}^{r} n_i} = \sum_{i=1}^{r} \dfrac{1}{n_1 \cdots \hat{n}_i \cdots n_r} \leq \sum_{i=1}^{r} \dfrac{1}{h_1 \cdots \hat{h}_i \cdots h_r} = \dfrac{\sum_{i=1}^{r} h_i}{\prod_{i=1}^{r} h_i}$, where $\hat{}$ denotes the deletion of that factor, we have

$$\operatorname{per}(A) = \operatorname{per}(PAQ) = \prod_{i=1}^{r} \operatorname{per}(A_i) \leq \prod_{i=1}^{r} \frac{h_i}{n_i} \leq \frac{\sum_{i=1}^{r} h_i}{\sum_{i=1}^{r} n_i} = \frac{h}{n}.$$

If equality holds, then for all $i = 1, 2, \cdots, r$, $n_1 \cdots \hat{n}_i \cdots n_r = h_1 \cdots \hat{h}_i \cdots h_r$. Hence $\prod_{i=1}^{r} n_i = \prod_{i=1}^{r} h_i$. Since $h_i \leq n_i$, we get $h_i = n_i$ for all $i = 1, 2, \cdots, r$. Hence each A_i is a permutation matrix and therefore so is A. The converse is obvious.

Proposition 3.27: Let $A \in \Omega_n$. If there is a normal matrix $B \in \Omega_n$ such that $\text{per}(A) \leq \text{per}(B)$ and $h(B) \leq h(A)$, then $\text{per}(A) \leq \frac{h(A)}{n}$ with equality iff A is a permutation matrix or $n = 2$ and $A = J_2$.

Proof: Since $\text{per}(A) \leq \text{per}(B) \leq \frac{h(B)}{n} \leq \frac{h(A)}{n}$ by Theorem 3.23, the inequality is clear. If equality holds, then either (i) B is a permutation matrix or (ii) $n = 2$, and $B = J_2$. In case (i), $h(B) = n$ implies that $h(A) = n$ and hence A is a permutation matrix. In case (ii), $\text{per}(B) = \frac{1}{2}$ and hence $\text{per}(A) \leq \frac{1}{2}$. Since the van de Waerden conjecture is true for $n = 2$, $\text{per}(A) = \frac{1}{2}$, and therefore $A = J_2$. The converse is obvious.

It was once conjectured that for $A \in \Omega_n$, $\text{per}(A) \geq \max\{\text{per}(AA^t), \text{per}(A^tA)\}$ (e.g. [12], Conjecture 2). Newman has given examples to show that this is false in general. It turns out that if $A \in \Omega_n$ is such that the above inequality is reversed, then the square root can be removed from Theorem 3.23.

Corollary 3.28: If $A \in \Omega_n$ satisfies $\text{per}(A) \leq \max\{\text{per}(AA^t), \text{per}(A^tA)\}$, then $\text{per}(A) \leq \frac{h(A)}{n}$, with equality iff A is a permutation matrix or $n = 2$ and $A = J_2$.

Proof: We may assume that $\text{per}(AA^t) \geq \text{per}(A^tA)$. Since AA^t is normal, and since $h(AA^t) \leq h(A)$ by Theorem 3.4.(i), the assertion follows from Proposition 3.27.

Corollary 3.29: If $A \in \Omega_n$ satisfies $\mathrm{per}(A) \leq \mathrm{per}(\frac{A + A^t}{2})$, then $\mathrm{per}(A) \leq \frac{h(A)}{n}$, with equality iff A is a symmetric permutation matrix or $n = 2$ and $A = J_2$.

Proof: Since $\frac{A + A^t}{2}$ is symmetric and hence normal, and since $h(\frac{A + A^t}{2}) = \frac{1}{2} h(A + A^t) \leq \frac{1}{2} \{h(A) + h(A^t)\} = h(A)$ by Lemma 3.1.(i), the assertion except "symmetric" follows from Proposition 3.27. But if $\frac{A + A^t}{2}$ is a permutation matrix, then $a_{i\sigma(i)} + a_{\sigma(i)\,i} = 2$ for some σ, where $i = 1, 2, \cdots, n$. Hence A is a symmetric permutation matrix. The converse is obvious.

We close this chapter by giving some upper bounds for $\mathrm{per}(A \times B)$ in terms of $h(A)$ and $h(B)$, where A and B are in Ω_n. In [2], the upper bound for $\mathrm{per}(A \times B)$ was studied by Brualdi.

Proposition 3.30: Let A,B be in Ω_n. Then $\mathrm{per}(A \times B) \leq$

$\leq \min \left\{ (\frac{h(A)}{n})^{\frac{1}{2}}, (\frac{h(B)}{n})^{\frac{1}{2}} \right\}$. If A and B are also normal, then $\mathrm{per}(A \times B) \leq \min \left\{ \frac{h(A)}{n}, \frac{h(B)}{n} \right\}$ with equality iff A and B are both permutation matrices.

Proof: Since $A \times B \in \Omega_{n^2}$, Theorem 3.23 and Proposition 3.19 imply that $\mathrm{per}(A \times B) \leq \left\{ \frac{h(A \times B)}{n^2} \right\}^{\frac{1}{2}} \leq \left\{ \frac{1}{n^2} \min[nh(A), nh(B)] \right\}^{\frac{1}{2}} =$

$= \min \left\{ [\frac{h(A)}{n}]^{\frac{1}{2}}, [\frac{h(B)}{n}]^{\frac{1}{2}} \right\}$. If A and B are also normal, then so is $A \times B$, and hence, by the same argument, we get

$\mathrm{per}(A \times B) \leq \frac{h(A \times B)}{n^2} \leq \min\left\{\frac{h(A)}{n}, \frac{h(B)}{n}\right\}$. If equality holds, then since $n^2 \neq 2$, $A \times B$ must be a permutation matrix. Hence A and B are both permutation matrices. The converse is obvious.

Remark 3.31: In [2], Brualdi proved that for $A \in H_n$ and $B \in H_m$, $\mathrm{per}(A \times B) \leq K_{mn}(\mathrm{per}\ A)^n(\mathrm{per}\ B)^m$ where K_{mn} is a certain constant depending on m and n . He conjectured that $K_{mn} = \frac{(mn)!}{(m!)^n(n!)^m}$. If we restrict A and B to be d.s. matrices and put $n = m$, then it is natural to compare the bound given by this conjecture and that given in Proposition 3.30. We remark that in general, they are not comparable. For example, if A and B are permutation matrices, then Proposition 3.30 yields $\mathrm{per}(A \times B) \leq 1$ while Brualdi's conjecture yields $\mathrm{per}(A \times B) \leq \frac{(n^2)!}{(n!)^{2n}}$, and it is known that $\frac{(n^2)!}{(n!)^{2n}} \geq 1$ with equality iff $n = 1$ [2, §3.7]. On the other hand, if $A = B = J_n$, then Proposition 3.30 yields $\mathrm{per}(A \times B) \leq \frac{1}{n}$ while Brualdi's conjecture yields

$$\mathrm{per}(A \times B) \leq \frac{(n^2)!}{(n!)^{2n}} \left(\frac{n!}{n^n} \right)^n \left(\frac{n!}{n^n} \right)^n = \frac{(n^2)!}{(n^n)^{2n}} = \frac{(n^2)!}{(n^2)^{n^2}} .$$

Since

$$\mathrm{per}(A \times B) = \mathrm{per}(J_n \times J_n) = \mathrm{per}(J_{n^2}) = \frac{(n^2)!}{(n^2)^{n^2}} ,$$

clearly $\frac{(n^2)!}{(n^2)^{n^2}} \leq \frac{1}{n}$.

BIBLIOGRAPHY

[1] G. Birkhoff, Tres observaciones sobre el algebra lineal, Univ. Nac. de Tucumán, Rev. Ser. A 5 (1946), 147-151.

[2] R. A. Brualdi, Permanent of the Direct Product of Matrices, Pacific J. of Math., Vol. 16, No. 3 (1966), 471-482.

[3] R. A. Brualdi and H. W. Wielandt, A Spectral Characterization of Stochastic Matrices, Linear Algebra and Its Application, Vol. 1, No. 1 (1968), 65-71.

[4] P. J. Eberlein, Remarks on the van der Waerden Conjecture, II, Linear Algebra and Its Application, 2 (1969), 311-320.

[5] P. J. Eberlein and G. S. Mudholkar, Some Remarks on the van der Waerden Conjecture, J. Comb. Theory (1968), 386-396.

[6] G. Frobenius, Über Matrizen aus nicht negativen Elementen, S.-B. Kgl. Preuss Akad. Wiss. (1912), 456-477.

[7] J. Kapoor, Matrices which, Under Row Permutations, Give Specified Values of Certain Matrix Functions, University of British Columbia Ph.D. thesis, June 1970.

[8] D. König, Theorie der endlichen und unendlichen Graphen, Chelsea, New York, 1950.

[9] C. C. MacDuffee, The Theory of Matrices, rev. ed. New York, Chelsea 1946.

[10] M. Marcus and H. Minc, Some Results on Doubly Stochastic Matrices, Proc. of Amer. Math. Soc. 13, No. 4 (1962), 571-579.

[11] ________, Inequalities for General Matrix Functions, Bull. of Amer. Math. Soc. 70 (1964), 308-313.

[12] ________, Permanent, Amer. Math. Monthly 72 (1965), 577-591.

[13] ________, On a conjecture of B. L. van der Waerden, Proc. Comb. Phil. Soc. 63 (1967), 305-309.

[14] ________, A Survey of Matrix Theory and Matrix Inequalities, Prindle, Weber and Schmidt, Boston 1969.

[15] M. Marcus, H. Minc, and B. Moyls, Some Results on Non-Negative Matrices, J. of Research of the National Bureau of Standards, Vol. 65B, No. 3, July-Sept. 1961, 205-209.

[16] M. Marcus and M. Newman, On the inimum of the Permanent of a Doubly Stochastic Matrix, Duke Math. J., Vol. 26, No. 1 (1959), 61-72.

[17] M. Marcus and M. Newman, Inequalities for the permanent function, Ann. of Math. (2) 75 (1962), 47-62.

[18] M. Marcus and R. Ree, Diagonals of Doubly Stochastic Matrices, Quart. J. of Math. Oxford 10, (1959), 296-302.

[19] H. Perfect and L. Mirsky, Spectral Properties of Doubly Stochastic Matrices, Monat. fur Math. 69 (1965), 35-57.

[20] H. J. Ryser, Combinatorial Mathematics, Carus Monograph 14, Wiley, New York, 1963.

[21] R. Sinkhorn and P. Knopp, Problems Involving Diagonal Products in Non-Negative Matrices, Trans. of Amer. Math. Soc. 136 (1969), 67-75.

[22] van der Waerden, B. L., Aufgabe 45, Jber. Deutsch. Math. Verein., 35 (1926), 117.