SOME GENERALIZATIONS OF NILPOTENCE IN RING THEORY

by

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ABSTRACT

The study of certain series of groups has greatly aided the development and understanding of group theory. Normal series and central series are particularly important. This paper attempts to define analogous concepts in the theory of rings and to study what interrelationships exist between them.

Baer and Freidman have already studied chain ideals, the ring theory equivalent of accessible subgroups. Also, Kegel has studied weakly nilpotent rings, the ring theory equivalent of groups possessing upper central series. Some of the more important results of these authors are given in the first three sections of this paper.

Power nilpotent rings, the ring theory equivalent of groups possessing lower central series, are defined in section 4. The class of power nilpotent rings is not homomorphically closed. However, it does possess many of the other properties that the class of weakly nilpotent rings has.

In section 5 meta* ideal and U*-ring are defined in terms of descending chains of subrings of the given ring. Not every power nilpotent ring is a U*-ring. This is contrary to the result for semigroups. It is also shown that an intersection of meta* ideals is always a meta* ideal. It follows that not every meta* ideal is a meta ideal since the intersection of meta ideals is not always a meta ideal.

Section 6 is concerned with rings in which only certain kinds of multiplicative decomposition take place. The rings studied here are called prime products rings and it is proved that all weakly nilpotent and power nilpotent rings are prime products rings. A result given in the section on U-rings suggests that all U-rings may be prime products rings. The class of prime products rings is very large but does not include any rings with a non-zero idempotent.

The last section studies ring types which are defined analogously to group types. The study of which ring types actually occur is nearly completed here. Finally, it is shown that every weakly nilpotent ring has a ring type similar to that of some ring which is power nilpotent. This suggests (but does not prove) the conjecture that all weakly nilpotent rings are power nilpotent.

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INTRODUCTION

The understanding of the structure of rings has been greatly advanced by the study and results of radical theory. Radical theory has focused attention on the "radical-free" part of rings and great attempts have been made to give an explicit description of rings which are semi-simple with respect to various radical properties. While these efforts have been moderately successful, very little serious work has been done on radical rings. This paper attempts to explore some classes of rings that contain most of the radical rings for the commonly studied radical properties. While the results given here do not give complete information on radical rings, they do give some indications of the nature of such rings. There is every reason to believe that further research in this area will provide even more information.

Much of the inspiration behind the ring theory concepts introduced here comes from the study of generalizations of nilpotence in group theory. It is surprising how many parallel results are obtained in the two theories. An indication of the equivalent group theory concept is given at the beginning of each appropriate section in this paper.

NOTATION

The following symbols and notations are used in this paper to mean exclusively the following things. They are not defined later on when they appear in the text.

C is any index set.

N is the set of natural numbers $\{1,2,3, \ldots\}$.

Z is the set of all integers.

ω is the first non-finite ordinal number.

the integer (p,q) is the least common multiple of the integers p and q.

the set (a,b) is the open interval of the real number line with endpoints a and b.

$[g,h]$ is the set of all integers between g and h including both g and h.

$[x]$ is the largest integer $\leq x$.

$I(S)$ is the ideal of a ring R generated by the subset S of R.

$\langle A\rangle$ is the subring of a ring R generated by the subset A of R.

$\oplus$ is used to denote the direct sum of groups, rings, etc.

iff means "if and only if".

Ξ is used to denote the end of the proof of a theorem or the end of an example following a remark.

1. I-CHAINS

There is a close enough relationship between a ring and its ideals to make it possible to infer some properties of a ring from the nature of its ideals. This suggests that in some cases it might even be useful to know something about the ideals of the ideals of a ring or, more generally, about certain subrings which are related to the ideals of a ring. A promising class of such subrings is the set of meta-ideals of a ring. Meta ideals were originally defined by Baer (1).

DEFINITION: *An* <u>*I-chain*</u> *of a ring* R *is a chain of subrings of* R, $I_1 \subseteq I_2 \subseteq \dots \subseteq I_\beta = R$, *where* I_α *is an ideal of* $I_{\alpha+1}$ *for every* α *and if* α *is a limit ordinal,* $I_\alpha = \bigcup_{\gamma<\alpha} I_\gamma$.

DEFINITION: *A subring of* S *of a ring* R *is a* <u>*meta ideal*</u> *of* R *if there exists an I-chain in* R *which begins with* S.

DEFINITION: *A subring* S *of a ring* R *is a* <u>*chain ideal*</u> *of* R *if there exists a finite I-chain in* R *which begins with* S.

DEFINITION: *The* <u>*index*</u> *of a chain ideal* S *is the smallest natural number* j *such that there exists an I-chain in* R *which begins with* S *and reaches* R *after* j *steps, i.e.* $R = I_{j+1}$.

I-chains are principally tools used to analyse the ring structure associated with meta ideals. Meta ideals appear to have been first studied by R. Baer (1). Later and apparently independently, Freidman (2), (3), (4) studied them in connection with rings in which every subring is a meta ideal. Although the results below are interesting in themselves, they are mainly introduced to aid in proving more complicated results which will appear later in the paper.

<u>Theorem 1</u>. (Freidman) S is a chain ideal of index n in R implies that $RS^n + S^nR \subseteq S$.

PROOF:

Let $S \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_{n+1} = R$ be an I-chain. Since I_p is an ideal in I_{p+1} and $S \subseteq I_p$ for all p in [2,n], $RS^n = (I_{n+1}S)S^{n-1} \subseteq (I_nS)S^{n-2} \subseteq \ldots \subseteq I_2S \subseteq S$. Similarily, $S^nR \subseteq S$. ∎

<u>Theorem 2</u>. (Baer) If S is a chain ideal of index n in a ring R and if I(S) is the ideal in R generated by S, then $[I(S)]^{3^n} \subseteq S \subseteq I(S)$.

PROOF:

It is easy to see that $I(S)^3 \subseteq RSR$. Let $S \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_{n+1} = R$ be an I-chain. For every integer p in [2,n],

$(I_p S I_p)^3 = (I_p S I_p{}^2) S (I_p{}^2 S I_p) \subseteq I_{p-1} S I_{p-1}$. Hence it can be seen that $[I(S)]^{3^n} \subseteq I_2 S I_2 \subseteq S$. Ξ

<u>Theorem 3</u>. (Baer) If S is a subring of R, a ring, and I is an ideal of R such that $I^p \subseteq S \subseteq I$, then S is a chain ideal of R of index $m \leq p$.

PROOF:

The following is an I-chain in R: $S \subseteq S+I^{p-1} \subseteq S+I^{p-2} \subseteq \dots \subseteq S+I = I \subseteq R$. This chain has p steps and is an I-chain beginning with S. Hence the index of S in R is some integer $m \leq p$. Ξ

<u>Theorem 4</u>. If S is a chain ideal of index n in a ring R, then S^k is a chain ideal of index $m \leq n$ in R where k is any natural number.

PROOF:

Suppose that the following is an I-chain in R: $S \subseteq I_2 \subseteq \dots \subseteq I_{n+1} = R$. Then $S^k \subseteq I_2 \subseteq \dots \subseteq I_{n+1} = R$ is also an I-chain in R. Hence S^k has index $m \leq n$ where k is any natural number. Ξ

<u>Theorem 5</u>. A nilpotent chain ideal is always contained in a nilpotent ideal.

PROOF:

Suppose that S is a chain ideal and that $S^k = 0$. By theorem 2, $[I(S)]^{3^n} \subseteq S$ where n is the index of S in the ring R. Hence $[I(S)]^{k\cdot 3^n} \subseteq S^k = 0$ and I(S) is therefore nilpotent itself. ∎

Chain ideals have considerably different properties from the more general meta ideals as a comparison between theorem 5 and the following remark indicates.

Remark A. (Baer) A nilpotent meta ideal need not be contained in a nilpotent ideal.

EXAMPLE:

Let $V = \bigoplus_{i\varepsilon N} V_i$, where each V_i is the one dimensional vector space generated by the vector v_i over the field of integers modulo 2. Let R_n denote the ring of all linear transformations on V with the property that $f(V_i) \subseteq V_{i+1} \oplus V_{i+2} \oplus \ldots \oplus V_{3n}$ if $i < 3n$ and $f(V_i) = 0$ if $i \geq 3n$ for all $f\varepsilon R_n$. Let $f_n \varepsilon R_n$ be defined by: $f_n(v_i) = v_{i+1}$ if i is even and $i < 3n$; otherwise $f_n(v_i) = 0$. Then $f_n^2 = 0$ and $S_n = \{0, f_n\}$ is a subring of R_n. Hence $S_n^2 = 0$. Let I_n be the ideal of R_n generated by S_n. There exists f in R_n such that $f(v_{2i-1}) = v_{2i}$ for $i = 1,2,\ldots, [3n/2]$. Given an integer $t > 1$ such that $2i+t < 3n$, there exists $g\varepsilon R_n$ such that $g(v_{2i+1}) = v_{2i+t}$ and

$g(v_k) = 0$ if $k \neq 2i+1$. The functions f and g are defined so that $g \cdot f_n \cdot f(v_{2i-1}) = v_{2i+t}$ and $g \cdot f_n \cdot f(v_k) = 0$ if $k \neq 2i-1$. It is also true that $g \cdot f_n \cdot (v_{2i}) = v_{2i+t}$ and $g \cdot f_n(v_k) = 0$ if $k \neq 2i$. Since I_n must contain all elements of the form $g \cdot f_n \cdot f$ and $g \cdot f_n$, I_n contains all elements f of R which have the property that $f(V_i) \subseteq V_{i+3} \oplus V_{i+4} \oplus \cdots \oplus V_{3n}$. Hence $R_n{}^3 \subseteq I_n$ and since $R_n{}^{3n-1} \neq 0$, $I_n{}^{n-1} \neq 0$. Now let R be the discrete direct sum of the rings R_n, let S be the discrete direct sum of the rings S_n, and let I be the discrete direct sum of the rings I_n where n ranges over the natural numbers. Then S is a subring of R and $S^2 = 0$, while I is an ideal of R which is not nilpotent, since I^n is the discrete direct sum of the rings I_j and $I_j{}^n \neq 0$ if $j > n$. The ideal generated by S in R is I and therefore S is not contained in a nilpotent ideal. Each R_n is a nilpotent ring and therefore is weakly nilpotent (a ring is weakly nilpotent if every non-zero homomorphic image of the ring contains a two-sided annihilator different from 0). It is proved below that all sums of weakly nilpotent rings are weakly nilpotent and that all subrings of weakly nilpotent rings are meta ideals. Ξ

It is a fact that most of the unresolved problems in the theory of meta ideals apply to locally nilpotent rings. The following theorem is particularly interesting since it suggests why the study of meta ideals might be closely related to the studies of generalized types of nilpotence.

<u>Theorem 6.</u> (Baer) Every idempotent meta ideal is an ideal.

PROOF:

Let S be an idempotent meta ideal. Suppose that the following is an I-chain: $S \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_\beta = R$. Let α be the largest ordinal number such that S is an ideal of I_α. Since I_α is an ideal of $I_{\alpha+1}$ and $S \subseteq I$, $SI_{\alpha+1} \subseteq I_\alpha$ and therefore it follows that $SI_{\alpha+1} = S^2I_{\alpha+1} = S(SI_{\alpha+1}) \subseteq SI_\alpha \subseteq S$. Similarly, $I_{\alpha+1}S \subseteq S$. This shows that S is an ideal of $I_{\alpha+1}$ and hence $I_\alpha = R$ and S is an ideal of R. ∎

It is easy to show that the intersection of a finite number of meta ideals is a meta ideal. It is also true that a meta ideal of a meta ideal of R is itself a meta ideal of R. However, unlike the result for ideals, the intersection of an infinite number of meta ideals need not be a meta ideal.

<u>Remark B</u>. The intersection of chain ideals need not be a meta ideal.

EXAMPLE:

Let S be the semigroup generated by the set $\{x_n : n \in N\}$ with the following defining relations:

(1) S is commutative

(2) $x_1{}^2 = 0$

(3) $x_n{}^n = x_1$ for all $n \in N$.

Let R be the algebra over the field of integers modulo 2 with basis S. Let T be the subring $\langle \{x_p{}^k : p \text{ is odd and } k \leq p\} \rangle + x_1 R$. The following is an I-chain in R beginning with $T + T^pR$: $T + T^pR \subseteq T + T^{p-1}R \subseteq \ldots \subseteq T + TR \subseteq R$. Hence $T + T^pR$ is a chain ideal of R. However, $T = \bigcap_{p \in N} (T + T^pR)$, which is an intersection of chain ideals. But if $y \in R \sim T$, then y is a sum of terms of the form $x_{n_1}^{m_1} \ldots x_{n_k}^{m_k}$ where $m_i < n_i$ for all i in $[1,k]$, and for at least one such term some n_i is even. Let h be an odd natural number different from all the subscripts which appear in the sum of terms which equals y. Then $yx_h \notin T$ since one of the terms still has an even subscript. So y is not in the idealizer of T. Due to the arbitrary nature of y, T is its own idealizer and hence is not a meta ideal. Ξ

The following theorem has far reaching implications. It also raises a problem which is still open. Namely, if S is a meta ideal of R, does R necessarily have a proper ideal I which <u>contains</u> S? Many additional results could be proved if this stronger version of the theorem were true.

<u>Theorem 7</u>. (Levic) If a ring R has a proper, non-zero meta ideal S, then R is not simple.

PROOF:

Let $S \subsetneqq I_2 \subsetneqq \ldots \subsetneqq I_\beta = R$ be an I-chain. If β is not a limit ordinal, then $I_{\beta-1}$ is an ideal of R which is proper and non-zero. Therefore suppose β is a limit ordinal. Select any non-zero element $x \varepsilon R$. If $xR + Rx = 0$, then S, the subring of R generated by x, annihilates R on both sides. However $x \varepsilon I_\alpha$ for some $\alpha < \beta$ and hence $S \subset I_\alpha \neq R$. Therefore S is a proper, non-zero ideal of R. If $RxR = 0$ and $xR + Rx \neq 0$, then $R^2 \neq 0$ and hence $Rx \neq R$ and $xR \neq R$. It follows that either Rx or xR is a proper, non-zero ideal of R. Hence from now on it will be assumed that $RxR \neq 0$. Let $K = \{\sum_{i=1}^{n} a_i x b_i = n \varepsilon N$ and $a_i, b_i \varepsilon R$ for every i in $[1,n]\}$. If $K = R$, then $x \varepsilon K$ and therefore $x = \sum_{i=1}^{m} c_i x d_i$ for some elements $c_i, d_i \varepsilon R$. Let γ be the smallest ordinal number such that $x, c_i, d_i \varepsilon I_\gamma$ for all i in $[1,m]$. Then I_γ is a proper meta ideal of R since γ cannot be a limit ordinal. Now it can be shown that $K \subset I_\gamma$. To do this it is sufficient to show that $\sum_{i=1}^{n} a_i x b_i \varepsilon I_\gamma$ for all $a_i, b_i \varepsilon R$. If $a_i, b_i \varepsilon I_\gamma$ for all i in $[1,n]$, then $\sum_{i=1}^{n} a_i x b_i \varepsilon I_\gamma$.

Suppose that α is an ordinal number $\geq \gamma$ and suppose that all the elements in the set $\{\sum_{i=1}^{n} a_i x b_i : n\varepsilon N$ and $a_i, b_i \varepsilon I_\alpha$ for all i in $[1,n]\}$ lie in I_γ. Then if $n\varepsilon N$ and $a_i, b_i \varepsilon I_{\alpha+1}$ for all i in $[1,n]$, $\sum_{i=1}^{n} a_i x b_i$ lies in I_γ since $\sum_{i=1}^{n} a_i x b_i$ $= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i c_j) x (d_j b_i)$ and since all the elements $a_i c_j$ and $d_j b_i$ lie in I_α. Hence by transfinite induction (the step at limit ordinal numbers is obvious) the ideal K lies in I_γ. This negates the possibility that K = R and hence R is not simple. Ξ

2. J-CHAINS

Weakly nilpotent rings have been studied before, especially by Kegel (8), (9). Theorem 8 below gives an equivalent definition of weakly nilpotent rings based on a chain condition. Actually, J-chains are the ring theory equivalent of upper central series in group theory. Moreover, weakly nilpotent rings are the ring theory equivalent of ZA groups (see Kurosh (1) for definition). An important relationship between weakly nilpotent rings and meta ideals is proved in section 3 on U-rings.

DEFINITION: *A ring R is <u>weakly nilpotent</u> if every non-zero homomorphic image of R contains a two-sided annihilator different from 0.*

From this definition it is easy to see that all homomorphic images of weakly nilpotent rings are weakly nilpotent rings.

DEFINITION: *The <u>J-chain</u> of a ring R is the chain of ideals of R,* $J_1 \subset J_2 \subset \ldots \subset J_\beta = J_{\beta+1}$, *where* J_1 *is the ideal consisting of all the two-sided annihilators in R,* $J_{\alpha+1}$ *is the largest ideal of R with the property that* $J_{\alpha+1}R + RJ_{\alpha+1} \subseteq J_\alpha$, *and if* α *is a limit ordinal, then* $J_\alpha = \bigcup_{\gamma<\alpha} J_\gamma$.

DEFINITION: *The J-chain of a ring R <u>terminates</u> (or <u>ends</u>) at J_β if $J_\beta = J_{\beta+1}$.*

DEFINITION: *A ring R has a <u>trivial J-chain</u> if its J-chain consists of 0.*

<u>Theorem 8</u>. (Kegel) A ring R is weakly nilpotent iff R's J-chain terminates at $J_\beta = R$.

PROOF:

Suppose that R is weakly nilpotent. Suppose also that J_β occurs in R's J-chain and that $J_\beta \neq R$. Then R/J_β, a homomorphic image of R, must contain an annihilator of R/J_β different from 0. However the set of all annihilators of R/J_β forms an ideal, K*, of R/J_β. Moreover, K* is isomorphic to K/J_β where K is an ideal of R. But K satisfies the relation $KR + RK \subset J_\beta$ which shows that $J_\beta \neq J_{\beta+1}$. Therefore R's J-chain does not terminate until it reaches R.

Suppose R's J-chain terminates at $J_\beta = R$. Let R/K be a non-zero homomorphic image of R and let γ be the maximal ordinal number such that $J_\gamma \subseteq K$. Then there exists an x in $J_{\gamma+1} \subset K$ which must satisfy the relation: $xR + Rx \subseteq J_\gamma \subseteq K$. Hence x^*, the image of x under the homomorphism $R \to R/K$, has the property that $x^*(R/K) + (R/K)x^* = 0$. Hence R/K does contain a non-zero annihilator. This shows that R is weakly nilpotent. Ξ

<u>Theorem 9</u>. A weakly nilpotent ring with ACC on two-sided ideals is nilpotent.

PROOF:

Consider R's J-chain: $0 \subset J_1 \subset \ldots \subset J_\beta = R$. By ACC this chain must terminate at $J_p = R$ for some natural number p. Since $J_n{}^2 \subseteq J_nR + RJ_n \subseteq J_{n-1}$, it follows that $J_n{}^{n+1} = 0$ for every natural number n and therefore $R^{p+1} = 0$. Ξ

<u>Theorem 10</u>. A subring S of a weakly nilpotent ring R is weakly nilpotent.

PROOF:

Let R have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_\beta = R$. And let S have the J-chain H: $0 \subset H_1 \subset H_2 \subset \ldots \subset H_\gamma = H_{\gamma+1}$. If $x \varepsilon J_1$, then $xR + Rx = 0$ and hence $xS + Sx = 0$. It follows that $S \cap J_1 \subseteq H_1$. Suppose that $S \cap J_\alpha \subseteq H_\alpha$. Then if $x \varepsilon S \cap J_{\alpha+1}$, $xR + Rx \subseteq J_\alpha$ and hence $xS + Sx \subseteq S \cap J_\alpha \subseteq H_\alpha$. It follows that $S \cap J_{\alpha+1} \subseteq H_{\alpha+1}$. By transfinite induction (the step at limit ordinals is obvious) it follows that $S \cap J_\beta \subseteq H_\beta$, i.e. $H_\beta \supseteq S \cap R = S$. Hence S is weakly nilpotent. Ξ

<u>Theorem 11</u>. A complete direct sum of weakly nilpotent rings is weakly nilpotent.

PROOF:

Suppose $R = \bigoplus_{\gamma \varepsilon C} A_\gamma$ where each A_γ is weakly nilpotent.

Let R have the J-chain J: $0 \subset J_1 \subset J_2 \subset \cdots \subset J_\beta = J_{\beta+1}$.

Let A_γ have the J-chain H_γ: $0 \subset (H_\gamma)_1 \subset (H_\gamma)_2 \subset \cdots \subset (H_\gamma)_{\beta_\gamma} = A_\gamma$.

Note that $J_\alpha = \bigoplus_{\gamma \varepsilon C} (H_\gamma)_\alpha$ since $A_\eta A_\lambda \neq 0$ implies that $\eta = \lambda$.

Hence if $\delta = \max\{\beta_\gamma : \gamma \varepsilon C\}$, $J_\delta = \bigoplus_{\gamma \varepsilon C} A_\gamma = R$ and R is weakly nilpotent. Ξ

Corollary. A discrete direct sum of weakly nilpotent rings is weakly nilpotent and a subdirect sum of weakly nilpotent rings is weakly nilpotent.

PROOF:

This follows from theorems 10 and 11 and from the fact that such sums can be represented as subrings of a complete direct sum of weakly nilpotent rings. Ξ

Of all of the extensions of nilpotence studied in this paper weakly nilpotent rings have the greatest number of pleasant properties. In addition to the general properties proved above, the following results are also useful.

Theorem 12. (Kegel) If I is a weakly nilpotent ideal of a ring R and $R^p \subseteq I$, then R is weakly nilpotent.

PROOF:

It is sufficient to prove the case when $p = 2$. Since I is weakly nilpotent it has a non-zero ideal J such that $JI + IJ = 0$. Consider the ideal RJR of R. It annihilates R: $(RJR)R + R(RJR) \subseteq RJI + IJR = 0$. Suppose that $RJR = 0$. Then either $JR \neq 0$ or $RJ \neq 0$ or $JR + RJ = 0$. In the first case JR is a non-zero annihilator of R; in the second case RJ is a non-zero annihilator of R; in the last case J is a non-zero annihilator of R. Hence R has a non-zero annihilator. Now let $f\colon R \to R/K$ be a homomorphic image of R where K is a proper ideal of R. If $I \subseteq K$, then R/K is nilpotent and therefore has a non-zero annihilator. Otherwise the image $f(I)$ of I under the homomorphism f has a non-zero annihilator J^* which is an ideal of R/K. Again $(R/K)J^*(R/K)$ is an annihilator of R/K since $(R/K)^2 \subseteq f(I)$ and J^* annihilates $f(I)$. Suppose $(R/K)J^*(R/K) = 0$. As above, then either $J^*(R/K) \neq 0$ or $(R/K)J^* \neq 0$ or J^* annihilates R/K. In any event R/K has a non-zero annihilator and since this is true for every non-zero homomorphic image of R, R is weakly nilpotent. Ξ

In Kegel's terminology (9) the following theorem says that being weakly nilpotent is a left conservative property. It is easy to see that being weakly nilpotent is also a right conservative property.

Theorem 13. (Kegel) If L is a weakly nilpotent left ideal of R, then LR is a weakly nilpotent ideal.

PROOF:

Let LR have the J-chain H: $0 \subset H_1 \subset H_2 \subset \ldots \subset H_\gamma = H_{\gamma+1}$. Note that it can be proved easily by transfinite induction that every H_α is an ideal of R as well as of LR. Now suppose LR is not weakly nilpotent. Then $H_\gamma \neq LR$. Let f be the homomorphism of R given by f: $R \to R/H_\gamma = R^*$. Let $L^* = f(L)$ and note that $L^*R^* = f(L)f(R) = f(LR) = LR/H_\gamma$. Hence L^*R^* has a trivial J-chain and L^* is a weakly nilpotent left ideal of R^*. Let L^* have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_\beta = L^*$. Since L^*R^* cannot be nilpotent, $L^*J_\beta R^*$ cannot be 0 for otherwise $(L^*R^*)^2 \subseteq L^*L^*R^* = L^*J_\beta R^* = 0$. Let η be the smallest ordinal number such that $L^*J_\eta R^* \neq 0$. Then η cannot be a limit ordinal since if $L^*J_\alpha R^* = 0$ for all ordinal numbers $\alpha < \eta$ and η is a limit ordinal, then $L^*J_\eta R^* = 0$. Note that $L^*J_\eta R^*$ is an ideal of L^*R^* and $L^*J_\eta R^*(L^*R^*) \subseteq L^*J_{\eta-1}R^* = 0$ and $(L^*R^*)L^*J_\eta R^* \subseteq L^*J_{\eta-1}R^* = 0$. Hence $L^*J_\eta R^*$ is a non-zero two-sided annihilator of L^*R^* which contradicts the fact that L^*R^* must have a trivial J-chain. Ξ

A left J-chain may be defined in the same way as a J-chain exchanging only the requirement that $J_{\alpha+1}$ annihilate the ring R on both sides modulo J_α with the requirement that $J_{\alpha+1}$

annihilate R on the left modulo J_α (let J_0 be the 0 subring). If a ring R has a non-trivial left J-chain, does R have a non-trivial J-chain? The answer is no, even in the case when R has both a non-trivial left J-chain and a non-trivial right J-chain, as a study of the ring given in the example below will verify. Whether all possible examples can be written as a direct sum of two rings as in the given example below is an intriguing question.

<u>Remark C</u>. It is possible for a ring R to have a trivial J-chain and also to have non-zero ideals I and J such that $IR = RJ = 0$.

EXAMPLE:

Let A be the ring of all $\omega x \omega$ square matrices with integer entries and the restrictions that all of the entries on the main diagonal and to the left of the main diagonal are 0, and all but a finite number of the entries are 0. Let B be the anti-automorphic copy of A under the identity mapping and consider the ring $R = A \oplus B$. Let I be the one-sided ideal of R consisting of all elements of R of the form $(y,0)$ where y is a $\omega x \omega$ square matrix with every row except the first filled with zeros. Let J be the one-sided ideal of R consisting of all elements of R of the form $(0,z)$ where z is a $\omega x \omega$ square matrix with every row except the first row filled with zeros.

Then $RI = JR = 0$. However, R has a trivial J-chain since if (u,v) is an arbitrary element from R and $(u,v)R = (u,v)(A \oplus B) = uA \oplus vB = 0$, then u must be 0. Similarly, if $R(u,v) = 0$, then v must be 0. So $(0,0)$ is the only two-sided annihilator of R. Moreover, IR and RJ are two-sided ideals of R and $R(IR) = (RJ)R = 0$. Since IR and RJ are non-zero ideals, they satisfy the conditions in the statement of the remark. Ξ

The ring A in the example above is the union of the ideals $I(n)$ of A where $I(n)$ consists of those matrices in A where all the non-zero entries occur in the first n rows and the first n columns. Since $I(n)$ is nilpotent and therefore weakly nilpotent, A is the union of weakly nilpotent ideals. However A has a trivial J-chain. The following result shows that a union of a special kind of weakly nilpotent ideals is weakly nilpotent.

DEFINITION: *The <u>J(R)-chain</u> of an ideal $K \subset R$, a ring, is the chain:* $0 \subset J_1 \subset \dots \subset J_\beta = J_{\beta+1}$ *where* $J_1 = \{x \in K: xR + Rx = 0\}$, $J_{\alpha+1} = \{x \in K: xR + Rx \subseteq J_\alpha\}$, *and if* α *is a limit ordinal,* $J_\alpha = \bigcup_{\gamma < \alpha} J_\gamma$.

DEFINITION: *An ideal K of a ring R is <u>nilpotently embedded</u> in R if K's J(R)-chain ends at K.*

<u>Theorem 14.</u> (Kegel) If R is the union of ideals nilpotently embedded in itself, then R is weakly nilpotent.

PROOF:

Suppose $R = \bigcup_{\alpha \in C} I_\alpha$ where each I_α is nilpotently embedded in R. Let I_α have the J(R)-chain $(J_\alpha)_n$: $0 \subset (J_\alpha)_1 \subset \dots \subset (J_\alpha)_{\beta_\alpha} = I_\alpha$ for each α in A. Then R has the J-chain J: $0 \subset J_1 \subset \dots \subset J_\gamma = R$ where $J_\delta \supseteq \bigcup_{\alpha \in C} (J_\alpha)_\delta$ for every ordinal number δ and hence if $\gamma = \max\{\beta_\alpha : \alpha \in C\}$, J_γ must be R. Ξ

<u>Theorem 15</u>. If R is a non-zero weakly nilpotent ring, then $R^2 \neq R$.

PROOF:

Suppose that R is weakly nilpotent and has the J-chain J: $0 \subset J_1 \subset \dots \subset J_\beta = R$. Suppose also that $R^2 = R$. Then $J_2 = \{x \in R: xR + Rx \subseteq J_1\} = \{x \in R: (xR + Rx)R + R(xR + Rx) = 0\}$. That is, $J_2 = \{x \in R: xR^2 + xRx + R^2x = 0\}$. A similar computation shows that $J_3 = \{x \in R: xR^3 + RxR^2 + R^2xR + R^3x = 0\}$. However, since $R^2 = R$, $J_2 = J_3$ and hence J_2 must equal R. But then $R^3 = 0$, R is nilpotent, and therefore $R^2 \neq R$. This is a contradiction. Ξ

3. U-RINGS

U-rings have been studied extensively by Freidman (2) (3) (4) (5) (6). In (2) Freidman proves that the nil radical of a U-ring is equal to the Jacobson radical, the Brown-McCoy radical, and the Levitzky radical. It follows that every nil U-ring is locally nilpotent. Since Freidman has characterized the radical-free part of U-rings, the remaining problem is to characterize locally nilpotent U-rings. It has been conjectured that all locally nilpotent U-rings are weakly nilpotent. Freidman (6) states as a corollory the result that every locally nilpotent U_2-ring is weakly nilpotent. Unfortunately, Freidman does not prove this corollory and it is not obviously true. However, it is proved below that the result is true for U_2-rings which satisfy certain additional conditions. Some general results on U-rings are given first.

DEFINITION: *A ring R is a <u>U-ring</u> if each subring S of R is a meta ideal of R.*

<u>Theorem 16</u>. (Freidman) Every weakly nilpotent ring R is a U-ring.

PROOF:

Let R have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_\beta = R$. Let S be any proper subring of R. Let γ be the minimal ordinal number such that $J_\gamma \not\subset S$. Note that γ cannot be a limit ordinal. There exists x in $J_\gamma \sim S$ which satisfies the relation: $xS + Sx \subseteq J_\gamma R + RJ_\gamma \subseteq J_{\gamma-1} \subseteq S$. Hence S is not its own idealizer in R. Now suppose R is not a U-ring. Then R has a subring T which is not a meta ideal. This means that any I-chain in R starting at T must end at some proper subring S of R which is its own idealizer. This is a contradiction. Ξ

Theorem 17. (Freidman) Every homomorphic image of R, a U-ring, is a U-ring. Also every subring of R is a U-ring.

PROOF:

Let R/K be a homomorphic image of R where K is an ideal of R. Let S* be any subring of R/K. Then S* is isomorphic to S/K for some subring S of R. There exists an I-chain in R beginning with S: $S \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq R$. The following is an I-chain in R/K beginning with S*: $S^* \subseteq I_2/K \subseteq I_3/K \subseteq \ldots \subseteq R/K$. Hence S* is a meta ideal of R/K. It follows that R/K is a U-ring since S* is arbitrary.

Let T be a subring of R and S any subring of T. There exists an I-chain in R beginning with S: $S \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq R$.

The following is an I-chain in T: $S \subseteq T \cap I_2 \subseteq T \cap I_3 \subseteq \ldots \subseteq T \cap R = T$. Hence S is a meta ideal of T and T is a U-ring. ∎

The following theorem narrows the search for a U-ring which is not weakly nilpotent.

Theorem 18. There exists a locally nilpotent U-ring which is not weakly nilpotent iff there exists a locally nilpotent U-ring with a trivial J-chain.

PROOF:

Suppose R is a locally nilpotent U-ring which is not weakly nilpotent. Let R have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_\beta = J_{\beta+1}$. Then R/J_β is a homomorphic image of R and therefore a locally nilpotent U-ring. However R/J_β has a trivial J-chain.

The converse is clear. ∎

DEFINITION: *Let* $S = \{x_s\colon s \varepsilon (0,1)$ *and* s *is a rational number*$\}$. *Define multiplication in* S *by the rule:* $x_s x_t = x_{s+t}$ *if* $s + t < 1$; *otherwise* $x_s x_t = 0$. *Let* p *be any prime number. The* *__Zassenhaus Example modulo p__* *is the algebra over the field of integers modulo* p *with basis* S. *More generally, any algebra with basis* S *will be called a* *__Zassenhaus Example__*.

The class of locally nilpotent rings is not a very good upper bound on the class of nil U-rings. Although a Zassenhaus Example is not a U-ring, it is both locally nilpotent and a Baer Lower Radical ring. The following theorem places a different upper bound on U-rings which excludes this example and many others like it. It also suggests that all nil U-rings may have multiplicably indecomposable elements. In the section of this paper on prime products rings it is proved that all weakly nilpotent rings are generated by the multiplicably indecomposable elements they contain. It seems quite possible that this result may hold for nil U-rings as well.

Theorem 19. Suppose a ring R has a sequence of elements, $\{x_i : i \varepsilon N\}$, such that $x_i^{n_i} = x_{i-1}$ where $n_i \geq 2$ for all $i \varepsilon N$ and $x_1 \neq 0$ while $x_0 = 0$. Then R is not a U-ring.

PROOF:

It is sufficient to show that a subring of R is not a U-ring. Let S be the subring of R generated by $\{x_i : i \varepsilon N\}$. Then S is commutative since any two elements in the sequence $\{x_i : i \varepsilon N\}$ are powers of an element in the sequence and therefore commute. It would be pleasant if the elements in S had subscripts with the property that when two elements were multiplied together the result would be either 0 or an element with subscript equal to the sum of the subscripts of the two multipliers. If x_k is renamed $y_{(1/\prod_{i=1}^{k} n_i)}$, then

$$[y_{(1/\prod_{i=1}^{k} n_i)}]^{n_k} = y_{(1/\prod_{i=1}^{k-1} n_i)} = y_{(n_k/\prod_{i=1}^{k} n_i)}.$$ Consequently

$(x_k)^p$ can be renamed $y_{(p/\prod_{i=1}^{k} n_i)}$ for every $p\varepsilon\mathbb{N}$. Since any two elements of the form y_s and y_t in S are both powers of some x_i in the sequence generating S, $y_s y_t = y_{(s+t)}$ (which may be 0 if $s + t > (n_1-1/n_1)$. This gives a subscript structure very much like that found in a Zassenhaus Example. Every s such that $y_s \varepsilon S$ is a rational number which lies in the interval (0,1). Exactly which rational numbers s appear as y-subscripts for elements in S depends on the sequence of integers $\{n_i : i\varepsilon\mathbb{N}\}$.

<u>Lemma 19A</u>. Suppose y_{s_1} and $y_{s_2} \varepsilon S$ and $s_1 < s_2 < 1$ and L_i is the characteristic of y_{s_i} for i = 1, 2. Then L_2 divides L_1 .

PROOF:

Note that $L_1 y_{s_1} = 0$ implies that $(L_1 y_{s_1}) y_{(s_2-s_1)} = L_1 y_{s_2} = 0$. Since $L_2 y_{s_2} = 0$, L_3, the greatest common divisor of L_1 and L_2, must be a solution of the equation $Xy_{s_2} = 0$. Since L_2 is the smallest positive integral solution of this equation, L_2 must be L_3 and therefore L_2 does divide L_1 .

DEFINITION: *A* <u>*point*</u> *in S will be an element of the form* y_t.

If the additive characteristic of every or all but one non-zero element in the ring S is 0, define G = 0. Otherwise let $G^* = \min\{\text{characteristic } (y_s): y_s \varepsilon S$ and characteristic $(y_s) > 1\}$. Let $y_{s_0} \varepsilon S$ be any element with characteristic G^*. Either

(1) y_{s_0} is the only point in S which has characteristic G^*

or

(2) there exists a maximum open interval $(a_1,a_2) \subseteq (0,1)$ such that $t\varepsilon(a_1,a_2)$ implies that y_t has characteristic G^*.

In case (1) let $G = \min\{\text{characteristic } (y_s): y_s \varepsilon S$ and characteristic $(y_s) > G^*\}$ and let y_{s_1} be a point in S which has characteristic G. Then every point y_t where $s_1 < t < s_0$ must have characteristic G by lemma 19A. Hence there exists a maximum open $(a_1,a_2) \subseteq (0,1)$ such that $t\varepsilon(a_1,a_2)$ implies that y_t has characteristic G. In case (2) let $G = G^*$.

Note also that if G = 0, then there is a maximum open interval $(a_1,a_2) \subseteq (0,1)$ such that $t\varepsilon(a_1,a_2)$ implies that y_t has characteristic 0.

DEFINITION: *G is called the* ***primary characteristic*** *of S; (a_1, a_2) is called the* ***primary interval*** *of S.*

DEFINITION: *A* ***formal additive relationship*** *in S is an equation of the form $\sum_{i=1}^{h} L_i y_{s_i} = 0$ where $s_i = s_j$ implies that $i = j$, $L_i \varepsilon Z$, and $L_i y_{s_i} \neq 0$ for every i in $[1,h]$.*

Lemma 19B. There exists no formal additive relationships in S in which every term has subscripts which lie in the primary interval (a_1, a_2).

PROOF:

Let h be the fewest positive number of terms that a formal additive relationship has, when every term has subscripts in (a_1, a_2). Suppose $\sum_{i=1}^{h} L_i y_{s_i} = 0$ is a formal additive relationship where $s_i \varepsilon (a_1, a_2)$ for every i in [1,h]. Let $s_m = \max\{s_1, \ldots, s_h\}$ and $s_\ell = \min\{s_1, \ldots, s_h\}$. Given any $u > 0$ there exists a rational number $s < u$ such that $y_s \varepsilon S$. Due to this fact there exists $y_t \varepsilon S$ such that $t + s_\ell < a_2 < t + s_m$. Since $L_m y_{(s_m+t)} = 0$, $\sum_{i=1}^{h} L_i y_{(s_i+t)} = \left(\sum_{i=1}^{h} L_i y_{s_i}\right) y_t = 0$ can be rewritten as a formal additive relationship in (a_1, a_2) with fewer than h terms. This is a contradiction.

Lemma 19C. There exists no formal additive relationships in S in which any term has the form Hy_t where G does not divide H and $t < g/2$ where g is the length of the primary interval, (a_1, a_2).

PROOF:

Suppose $Hy_t + \sum_{j=1}^{m} L_j y_{s_j} = 0$ is a formal additive relationship where G does not divide H and $t < g/2$. Suppose also that $s_1 < \ldots < s_h < t < s_{h+1} < \ldots < s_m$. There exists $y_u \varepsilon S$ such that $a_1 + g/2 < t + u < a_2$. Then $(Hy_t + \sum_{j=1}^{m} L_j y_{s_j})y_u = 0$

is an additive relationship in which every term lies in (a_1, a_2) but not every term is 0 since $Hy_{(t+u)} \neq 0$. Consequently this can be rewritten as a formal additive relationship in the primary interval which contradicts lemma 19B.

DEFINITION: *A point $y_s \varepsilon S$ is an* <u>*M-endpoint*</u> *if $M \cdot y_s \neq 0$ but $My_t = 0$ for every $t > s$ where M is an integer.*

DEFINITION: *If y_s is an M-endpoint for some integer M and L is the smallest positive integer such that y_s is an L-endpoint, then L is the* <u>*near characteristic*</u> *of y_s.*

Lemma 19D. Every dense subset of an open interval $(b_1, b_2) \subseteq (0,1)$ contains points s such that y_s is not an M-endpoint for any $M \varepsilon Z$ or there is no point y_s in S.

PROOF:

If the M-endpoints in S are ordered according to their near characteristics then no two M-endpoints have the same near characteristics and as the near characteristics of the M-endpoints increase towards infinity, the y-subscripts decrease towards 0. Since the positive integers have only one limit point (plus infinity), the y-subscripts of the M-endpoints in S have at most one limit point. But every dense subset of the interval $(b_1, b_2) \subseteq (0,1)$ has infinitely many limit points. Hence some of the points in the dense subset of (b_1, b_2) either are not the y-subscripts of any M-endpoints in S or are not the y-subscripts of any points in S at all.

The proof of theorem 19 will now be finished.

Let $E = \{y_{1/k} \varepsilon S: \; k \varepsilon N\}$ and let $P(S) = \{$primes p: p divides k for some $k \varepsilon N$ such that $y_{1/k} \varepsilon E\}$.

Case (1): Suppose $P(S)$ is an infinite set. Then choose $p_0 \varepsilon P(S)$ and let $T = \{ \sum_{i=1}^{h} L_i y_{\ell_i/k_i} + \sum M_j y_{s_j} + \sum_{w=1}^{v} H_w y_{t_w} \varepsilon S:$

$L_i \varepsilon Z$, $(\ell_i, k_i) = 1$, and $(p_0, k_i) = 1$ for all i in $[1,h]$;

$M_j \varepsilon Z$, and y_{s_j} is an M_j-endpoint for all j in $[1,m]$;

$H_w \varepsilon Z$, and either $t_w \geq g/2$ or G divides H_w for every w in $[1,v]\}$.

Note that the set $\{\ell/k: k,\ell\varepsilon N$ and p_o divides $k\}$ is dense in $(0,g/2)$. From the proof of lemma 19D there exists some $y_t \varepsilon S$ such that $t\varepsilon(0,g/2)$, $t = \ell/k$ where p_o divides k, and y_t is not an M-endpoint for any integer M. By lemma 19C there exists no formal additive relationships involving elements of the form $H_w y_{t_w}$ where $t_w < g/2$ and G does not divide H_w. Hence $y_t \varepsilon S \sim T$ and therefore $T \neq S$. Note that the product of an M-endpoint with any other element in S is 0 and that $(H_w y_{t_w})\cdot(Ly_u) = LH_w y_{t_w+u}$ where either G divides LH_w or $t_w + u > g/2$ for every w in $[1,v]$. If p_o divides neither k_1 nor k_2, then p_o does not divide k_1k_2. Consequently, $(L_1 y_{\ell_1/k_1})(L_2 y_{\ell_2/k_2}) = L_1L_2 y_{(\ell_1 k_2 + \ell_2 k_1)/k_1k_2}$ lies in T if $L_i y_{\ell_i/k_i} \varepsilon T$ for $i = 1, 2$. Hence T is a subring of S since it is closed under addition and multiplication. If $Ly_{\ell/k}\varepsilon S\sim T$, and $(\ell,k) = 1$, then p_o divides k, $\ell/k < g/2$, L does not divide G, and there exists $t > \ell/k$ such that $Ly_t \neq 0$. Since P(S) is an infinite set there exists $y_{1/k_1}\varepsilon T$ such that $1/k_1 + \ell/k < \min\{g/2,t\}$. Consequently, $(Ly_{\ell/k})(y_{1/k_1}) = Ly_{(\ell k_1+k)/kk_1}$ is not 0 and is not in T since p_o divides kk_1, $(p_o, \ell k_1+k) = 1$ and by lemma 19C this element cannot be expressed as a sum of terms which lie in T. Hence $Ly_{\ell/k}$ is not in the idealizer of T and T is its own idealizer in S due to the arbitrary nature of this element.

Case (2): Suppose $P(S)$ is a finite set. Then choose $p_1 \varepsilon P(S)$ such that p_1 divides an infinite number of terms in the sequence $\{n_i : i \varepsilon N\}$. Note that every power of p_1 divides some k such that $y_{1/k} \varepsilon E$. Let $Q = \{\sum_{i=1}^{h} L_i y_{\ell_i/k_i} \varepsilon S : L_i \varepsilon Z,$ $(\ell_i, k_i) = 1$, and $k_i = p_1^{\ n}$ for some $n \varepsilon N$ for all i in $[1,h]\}$. Let q be a prime such that $q \not\varepsilon P(S)$ and let

$$Q^* = \{\sum_{i=1}^{h} L_i y_{q\ell_i/k_i} + \sum_{j=1}^{m} M_j y_{s_j} + \sum_{w=1}^{v} H_{y_w} y_{t_w} \varepsilon Q:$$

$L_i y_{\ell_i/k_i} \varepsilon Q$ for all i in $[1,h]$;

$M_j \varepsilon Z$, and y_{s_j} is an M_j-endpoint for all j in $[1,m]$;

$H_w \varepsilon Z$, and either $t_w \geq g/2$ or G divides H_w for all w in $[1,v]\}$.

Note that the set $\{\ell/p_1^{\ n} : \ell, n \varepsilon N$ and $(\ell, p_1 q) = 1\}$ is dense in $(0,g/2)$. From the proof of lemma 19D it follows that there exists a point $y_t \varepsilon S$ such that $t \varepsilon (0,g/2)$, y_t is not an M-endpoint for any integer M, and $t = \ell/p_1^{\ n}$, where $(\ell, p_1 q) = 1$. By lemma 19C there exists no formal additive relationships involving elements of the form $H_w y_{t_w}$ where $t_w < g/2$ and G does not divide H_w. Hence $y_t \varepsilon Q \sim Q^*$ and therefore $Q \neq Q^*$. Now, note that if

$L_1 y_{q\ell_1/k_1}$ and $L_2 y_{q\ell_2/k_2}$ are elements in Q^*, their product, $L_1 L_2 y_{q(\ell_1 k_2 + \ell_2 k_1)/k_1 k_2}$ is an element in Q^*. Since the statements found in case (1) on M_j-endpoints and elements of the form $H_w y_{t_w}$ where either $t_w \geq g/2$ or G divides H_w apply in this case also, Q^* is a subring of Q. If $L_{y_{\ell/k}} \varepsilon Q \sim Q^*$ and $(\ell,k) = 1$, then $(q,\ell) = 1$, G does not divide L, $\ell/k < g/2$, and there exists a rational number $t > \ell/k$ such that $L_{y_t} \neq 0$. Note that $\min\{t, g/2\} < (\ell/k + q/p_1{}^n)$ for some natural number n and there exists a point $y_{1/k_1} \varepsilon E$ such that $p_1{}^n$ divides k_1. Consequently, $(Ly_{\ell/k})(y_{q/p_1{}^n}) = Ly_{(\ell p_1{}^n + qk)/kp_1{}^n}$ which is not 0 and does not lie in Q^* since $(q, \ell p_1{}^n + qk) = 1$ and by lemma 19C this element cannot be expressed as a sum of terms which lie in Q^*. Hence $ly_{\ell/k}$ is not in the idealizer of Q^* and Q^* is its own idealizer in Q due to the arbitrary nature of this element. Ξ

DEFINITION: *A ring R is a* <u>*U_1-ring*</u> *if every subring of R is a chain ideal.*

It appears that very few results have been obtained which define the boundaries of the class of U_1-rings. The example given below shows that not every weakly nilpotent ring is a U_1-ring.

Remark D. Not every U-ring is a U_1-ring.

EXAMPLE:

Let S be the semigroup consisting of the set of elements $\{0\} \cup \{x_n$: n has less than k prime factors where k is the smallest prime dividing n and n is square-free$\}$. Let multiplication in S be defined by the rule: $x_n x_m = x_{nm}$ if nm has less than k prime factors where k is the smallest prime dividing nm and nm is square-free; otherwise $x_n x_m = 0$. Let the primes be ordered according to size in the usual way. Let R be the algebra over the integers modulo 2 with basis $S \sim \{0\}$. Let Q be the subring of R generated by the set of elements $\{x_p$: p is a prime with an even index in the ordering of the primes$\}$. Any I-chain in R beginning with Q has an infinite number of steps since if p is the 2n+1-st prime in the ordering of the primes, then x_p does not occur in any subring in the I-chain until after I_{2n}. Hence R is not a U_1-ring. Let R have the J-chain J and note that x_n occurs in J_{p-k} if n has k prime factors and the smallest prime factor of n is p. Hence $\bigcup_{n \varepsilon N} J_n = R$ and R is weakly nilpotent and therefore R is a U-ring. Ξ

DEFINITION: *A ring R is a* <u>*U_2-ring*</u> *if every subring of R is a chain of index $n \leq M$, for some integer M.*

DEFINITION: *M is an* <u>*index bound*</u> *for a U_2-ring if every chain ideal of the ring has index $n \leq M$.*

The following theorems from Freidman (6) together with the example for remark F give a fairly good picture of what nil U_2-rings are like.

<u>Theorem 20</u>. (Freidman) Every nilpotent ring is a U_2-ring.

PROOF:

Suppose $R^p = 0$. Then $0 = R^p \subseteq S \subseteq R$ for every subring S of R. By theorem 3, S is a chain ideal of index $n \leq p$. Hence R has index bound p. ∎

It is also true that a homomorphic image of a U_2-ring of index bound M is a U_2-ring of index bound $n \leq M$ and that a subring of a U_2-ring of index M is a U_2-ring of index bound $n \leq M$. The proof of these statements is the same as that given for the corresponding statements for U-rings in theorem 17. However, unlike the result for weakly nilpotent rings, a direct sum of U-rings need not be a U-ring.

<u>Remark E.</u> (Freidman) A direct sum of U_2-rings need not be a U-ring.

EXAMPLE:

Let $R = Z/(2) + Z/(2)$. Let e be the identity of the first ring $Z/(2)$ and let f be the identity of the second ring in the summand. Then $e + f$ generates a subring proper in R which is not an ideal of R. Ξ

The following series of lemmas lead directly to the proof of the main theorem on U_2-rings which are locally nilpotent.

<u>Lemma 21A</u>. (Freidman) If R is a locally nilpotent U_2-ring and has index bound M and the additive group structure in R is either torsion free or every non-zero element has additive order p where p is a prime, then $x^{2M+1} = 0$ for every $x \varepsilon R$.

PROOF:

Let x be any element in R and observe that $\langle x\rangle \cdot \langle x^2\rangle^M \subseteq \langle x^2\rangle$ follows from theorem 1 and the fact that a subring of a U_2-ring is again a U_2-ring. Hence $x^{2M+1} = \sum_{i=1}^{j} c_i x^{2i}$ where c_i is an integer and $x^{2j} \neq 0$ while $x^{2j+2} = 0$. Suppose $x^{2j+1} \neq 0$. Then let $k = 2j+1$. Otherwise let $k = 2j$. The equations obtained by multiplying both sides of the equation: $x^{2M+1} = \sum_{i=1}^{j} c_i x^{2i}$ by x^{k-2n} for $n = 1, 2, \ldots, M$ show successively that $c_n = 0$

for $n = 1, 2, \ldots, M$. The equation finally obtained has the form: $x^{2M+1} = x^{2M+1}y$ where $y \in R$. If $x^{2M+1} \neq 0$, then the only element y which can satisfy such an equation in a locally nilpotent ring is 0 itself. Hence $x^{2M+1} = 0$ which is the desired result.

Lemma 21B. (Freidman) If R is a locally nilpotent U_2-ring with index bound M such that every non-zero x in R has additive order p where p is a prime and if S is a subring of R which is generated by exactly H elements, then $S^t = 0$ where $t = H(2M+1)3^M - H + 1$.

PROOF:

Let S be the ring generated by the elements $x_1, x_2, \ldots, x_H$ taken from R. According to lemma 21A, $(x_i)^{2M+1} = 0$ for every i in $[1,H]$. By theorem 2, $I(\langle x_i \rangle)^{3^M} \subseteq \langle x_i \rangle \subseteq I(\langle x_i \rangle)$. Hence $I(\langle x_i \rangle)^{3^M(2M+1)} = 0$ for all i in $[1,H]$. Due to the facts that $S \subseteq I(\langle x_1 \rangle) + I(\langle x_2 \rangle) + \ldots + I(\langle x_H \rangle)$ and that $[I(\langle x_1 \rangle) + I(\langle x_2 \rangle) + \ldots + I(\langle x_H \rangle)]^t = 0$, $S^t = 0$ where t is the number given in the statement of the lemma.

Lemma 21C. (Freidman) If R is a locally nilpotent U_2-ring with index bound M such that every $x \in R \sim \{0\}$ has additive order p where p is a prime and S is any subring of R generated by exactly H elements, then $|S| \leq p^u$ where $u+1 = H(H^{H(2M+1)3^M - H+1} - 1)/(H-1)$.

PROOF:

Let S be the subring of R generated by the elements $x_1, x_2, \ldots, x_H$. Then $S^t = 0$ and consequently the semigroup S^* generated by $x_1, \ldots, x_H$ contains no more than $H + H^2 + H^3 + \ldots + H^{t-1} = H(H^{t-1}-1)/(H-1)$ elements. Let $u + 1 = H(H^{H(2M+1)3^M-H+1}-1)/(H-1)$. The ring S therefore consists of sums of the form $\sum_{i=1}^{u} L_i y_i$ where each L_i is a positive integer $<p$ and each y_i is a non-zero member of the semigroup S^* and $y_i = y_j$ implies that $i = j$. However the number of such sums is no more than p^u.

<u>Lemma 21D</u>. (Freidman) If R is a locally nilpotent U_2-ring with index bound M such that every $x \in R \sim \{0\}$ has additive order p where p is a prime, then R is nilpotent.

PROOF:

Select by choice any $M+p^v$ elements from the ring R where $v+1 = M(M^{M(2M+1)3^M-M+1)}-1)/(M-1)$. Let these elements be denoted by $x_1, x_2, \ldots, x_M$; $y_1, y_2, \ldots, y_{p^v}$. Let S be the subring of R generated by the elements $x_1, x_2, \ldots, x_M$. Let $a_o = x_1 x_2 \ldots x_M$. Define a_i recursively by $a_i = a_{i-1} y_i$ for all i in $[1, p^v]$. Then it follows from theorem 1 that each $a_i \in S$ and therefore two of the p^v+1 elements $a_o, a_1, \ldots, a_{p^v}$ must be equal. Suppose $a_i = a_j$ where $i < j$. Then $a_i = a_i z$ where z is an element of R. Since R is a locally nilpotent ring a_i must be 0 and hence $a_{p^v} = 0$. $R^{M+p^v} = 0$ since every element in this subring of R can be written in the form of a_{p^v} by a proper choice of the elements $x_1, \ldots, x_M; y_1, \ldots, y_{p^v}$.

Theorem 21. *(Freidman) If R is a locally nilpotent U_2-ring and F(R), the periodic part of R, has characteristic $q > 0$, then R is nilpotent.

PROOF:

First consider F(R), the periodic part of R. Since F(R) has characteristic $q > 0$, F(R) may be represented as a direct sum of a finite number of rings whose additive group structure is a primary p group for distinct primes p. To show that F(R) is nilpotent it is sufficient to show that each of these subrings is nilpotent since a finite direct sum of nilpotent rings is nilpotent. Suppose $S \subset R$ is a ring which has a primary p group as its additive group structure. Then S is a U_2-ring and there exists n N such that $p^n S = 0$, but $p^{n-1} S \neq 0$. Then by lemma 21D $p^{n-1} S$ is a nilpotent ring. The ring $p^{n-2} S$ is also nilpotent since $p(p^{n-2} S / p^{n-1} S) = 0$ and therefore the factor ring $(p^{n-2} S / p^{n-1} S)$ is a nilpotent ring. Continuing to argue in this fashion it is easy to see after a finite number of steps that S itself is a nilpotent ring.

Now consider $R/F(R)$ which has a torsion free group as its additive group structure. According to lemma 21A there exists $w > 0$ such that $x^w = 0$ for all x in $R/F(R)$. M. Nagata proves in (13) that a torsion free ring with the property that $x^w = 0$ for all x in the ring must be nilpotent. Hence R is nilpotent since both $R/F(R)$ and $F(R)$ are nilpotent. Ξ

The following remark and example also appear in Freidman (6). The fact that every subring in the given example is an ideal of the ring makes it all the more startling.

Remark F. A U_2-ring need not be nilpotent if the periodic part of the ring has characteristic 0.

EXAMPLE:

Let $\{x_p: p \text{ is a prime number}\}$ be a set of distinct elements and for each prime number p define $A_p = \{nx_p: 0 \leq n < p^p\}$. Make A_p a ring by defining addition and multiplication by the rules: $nx_p + mx_p = [(m+n) \bmod p^p]x_p$; $nx_p \cdot mx_p = [(nmp) \bmod p^p]x_p$. Hence $(A_p)^{p+1} = 0$, i.e. A_p is nilpotent. Let $R = \bigoplus_{p \text{ prime}} A_p$. Then $y \varepsilon R$ implies that

$$y = \sum_{i=1}^{n} m_{p_i} x_{p_i} \text{ where } p_i \text{ is a prime and } 0 < m_{p_i} < p_i^{\,p_i},\ m_{p_i} \varepsilon N$$

for all i in $[1,n]$. The subring S_y, generated in R by y, is really an ideal of R since $y \cdot x = [(p_j m_{p_j}) \bmod p^j] x_p$ if $p = p_j$ where j lies in $[1,n]$; otherwise $y \cdot x_p = 0$. Since the set of primes, $\{p_i : i = 1,2,\ldots,n\}$, are all distinct, $y \cdot x_p \varepsilon S_y$. Hence every subring of R is an ideal of R and therefore R is a U_2-ring. However R is not nilpotent since $(x_p)^p \neq 0$ for every prime number p. Ξ

DEFINITION: *A ring is* <u>*periodic*</u> *if the additive order of every non-zero element of R is a natural number.*

DEFINITION: *A ring is* <u>*primary*</u> *if the additive order of every non-zero element of the ring is a power of some prime number p.*

DEFINITION: *An element $x \varepsilon R$, a primary ring, has* <u>*infinite height*</u> *if the equation $mz = x$ has a solution z for every $m \varepsilon N$.*

<u>Lemma 22A.</u> If R is a primary ring and $x \varepsilon R$ is an element of infinite height in the additive group of R, then $xR + Rx = 0$.

PROOF:

Suppose that $x \varepsilon R$ is an element of infinite height in the additive group of R. Let p^m be the additive order of x. Then there exists a sequence of elements in R, $\{x_n : n \varepsilon N\}$, such that $x = px_1 = p^2 x_2 = \ldots = p^n x_n$ for every $n \varepsilon N$. If y is an element in R, then there is a non-negative integer k such that $p^k y = 0$. However $xy = (p^k x_k) y = x_k (p^k y) = x \cdot 0 = 0$. It follows that $xR = 0$. For similar reasons $Rx = 0$.

DEFINITION: *The <u>basic subring</u> of a primary ring R is the subring B which has for its additive group structure the basic subgroup of R's additive group. (See Kurosh (10) for a definition of basic subgroup).*

It follows from lemma 22A that a basic subring is always an ideal of the ring, since R/B has a complete group for its additive group structure. Hence every element other than 0 in R/B has infinite height and therefore $(R/B)^2 = 0$. It follows that $R^2 \subseteq B$.

<u>Theorem 22</u>. Let R be a locally nilpotent U_2-ring and let F(R) be the periodic part of R. Let $F(R) = \bigoplus_{i\varepsilon N} F_{p_i}$ where F_{p_i} is a p_i-primary ring and $\{p_i: i\varepsilon N\}$ is the sequence of primes in their usual ordering. Let A_i be the ideal of F_{p_i} consisting of all elements of infinite height in F_{p_i}. Then R is weakly nilpotent if for all $i\varepsilon N$ the basic subring of F_{p_i}/A_i is a direct sum of rings of finite positive characteristic.

PROOF:

Let F_{p_i} be the set of all elements in F(R) which have additive order any power of p_i. Then $F(R) = \bigoplus_{i\varepsilon N} F_{p_i}$. Hence F(R) is weakly nilpotent if each F_{p_i} is weakly nilpotent.

Let $A_i = \{x \varepsilon F_{p_i} : x$ has infinite height in the additive group of $F_{p_i}\}$. The group $F_{p_i}/A_i = F_i$ is an Ulm factor (see Kurosh (10) for definition) of the group F_{p_i} and hence the ring F_i is a primary ring which contains no elements of infinite height. Let B_i be the basic subgroup of F_i. Since $F_i^2 \subseteq B_i$, it follows from theorem 12 that F_i is weakly nilpotent if B_i is. By assumption, the ring B_i is a direct sum of rings of finite positive characteristic. It follows from theorem 21 that B_i is weakly nilpotent. Therefore F_{p_i} is weakly nilpotent since F_i is weakly nilpotent and since A_i is a subring of the first term of F_{p_i}'s J-chain. By theorem 21, R/F(R) is nilpotent, and therefore R is weakly nilpotent since F(R) is weakly nilpotent. Ξ

4. K-CHAINS

K-chains are the ring theory equivalent of lower central series in group theory. Power nilpotent rings are the ring theory equivalent of ZD groups. By comparing the theorems in this section with those in the section on J-chains, it can be seen that power nilpotent rings have many of the same properties that weakly nilpotent rings do. The statement for weakly nilpotent rings corresponding to theorem 28 below is also true. For many classes of rings J-chains and K-chains have useful relations to one another. Some of these relations are presented in section 7 below on ring types.

DEFINITION: *The* <u>*K-chain*</u> *of a ring R is the following descending chain of ideals of R:* $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = K_{\beta+1}$ *where* $R = K_0$, $K_{\alpha+1} = RK_\alpha + K_\alpha R$ *for every ordinal number* α, *and if* α *is a limit ordinal,* $K_\alpha = \bigcap_{\gamma<\alpha} K_\gamma$.

<u>Remark G</u>. Since for every natural number n, $K_n = R^{n+1}$, it is true that $K_\alpha R = RK_\alpha$ whenever α is finite. However, $K_\alpha R \neq RK_\alpha$ in the general case.

EXAMPLE:

Let R be the set of all 2x2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where a is of the form 2m/(2n+1) where m and n are integers and b is any rational number. Then R is a ring with the usual matrix addition and multiplication. Let R have the K-chain K: $R \supset K_1 \supset K_2 \ldots \supset K_\beta = K_{\beta+1}$. Then $K_\omega = \bigcap_{n\in N} R^n$ = the set of all 2x2 matrices where the only non-zero entry occurs in the first row, second column and any rational number may occur there. A simple calculation shows that $K_\omega R = 0$ while $RK_\omega = K_\omega$. ∎

DEFINITION: *A ring R is* *<u>power nilpotent</u>* *if R's K-chain ends at 0.*

The following theorem provides an alternative definition of power nilpotent rings.

<u>Theorem 23</u>. A ring R is power nilpotent iff $IR + RI \neq I$ for every non-zero ideal I of R.

PROOF:

Suppose R is power nilpotent and I is a non-zero ideal of R. Let R have the K-chain K: $R \supset K_1 \supset \ldots \supset K_\beta = 0$. Since $I \not\subset K_\beta = 0$, $I \not\subset K_\alpha$ for some smallest ordinal α with this property. The number α cannot be a limit ordinal. Hence $I \subseteq K_{\alpha-1}$ and therefore $IR + RI \subseteq K_{\alpha-1} R + RK_{\alpha-1} \subseteq K_\alpha$. It follows that $I \neq IR + RI$.

Now suppose that R's K-chain ends at $K_\beta \neq 0$. Then K_β is a non-zero ideal of R for which the relation $K_\beta R + RK_\beta = K_\beta$ holds. Hence if R is not power nilpotent, then there is some ideal I of R for which $IR + RI = I$. ∎

Theorem 24. A power nilpotent ring R with DCC on two-sided ideals is nilpotent.

PROOF:

Consider R's K-chain: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = 0$. By DCC this chain must end after a finite number of steps at $K_p = K_{p+1}$. Since $K_p = R^{p+1}$, $R^{p+1} = 0$ if R is power nilpotent. Ξ

Remark H. A power nilpotent ring with ACC on two-sided ideals need not contain any nilpotent elements different from 0.

EXAMPLE:

Let $R = 2Z$, the ring of even integers. Then if R has the K-chain K, $K_\omega = \bigcap_{n\varepsilon N} R^n = 0$. This is true since every even integer is the product of at most a bounded finite number of even integers. Hence R is a power nilpotent ring and has ACC on two-sided ideals. However R contains no non-zero nilpotent elements since no even integer other than 0 is nilpotent. Ξ

Theorem 25. A subring S of a power nilpotent ring R is power nilpotent.

PROOF:

Let S be a subring of R and let R have the K-chain K: $R \supset K_1 \supset \ldots \supset K_\beta = 0$. Suppose that S has the K-chain H: $S \supset H_1 \supset \ldots \supset H_\gamma = H_{\gamma+1}$. Then $S \subseteq R$. Suppose that $H_\alpha \subsetneq K_\alpha$.

Then $H_{\alpha+1} = H_\alpha S + SH_\alpha$, while $K_{\alpha+1} = K_\alpha R + RK_\alpha$. Therefore $H_{\alpha+1} \subseteq K_{\alpha+1}$. By transfinite induction (the step at limit ordinals is obvious) $H_\alpha \subseteq K_\alpha$ for every ordinal number α. Hence $H_\beta \subseteq K_\beta = 0$. Therefore $H_\gamma = 0$ and $\gamma \leq \beta$. This shows that S is power nilpotent. Ξ

<u>Theorem 26</u>. A complete direct sum of power nilpotent rings is power nilpotent.

PROOF:

Suppose $R = \bigoplus_{\gamma \varepsilon C} A_\gamma$ where each A_γ is power nilpotent.

Let R have the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\delta = K_{\delta+1}$. Let A_γ have the K-chain K_γ: $A_\gamma \supset (K_\gamma)_1 \supset (K_\gamma)_2 \supset \ldots \supset (K_\gamma)_{\beta_\gamma} = 0$. Note that for every ordinal number α, $K_\alpha = \bigoplus_{\gamma \varepsilon C} (K_\gamma)_\alpha$, since $A_{\gamma_1} \cdot A_{\gamma_2} \neq 0$ implies that $\gamma_1 = \gamma_2$. There exists an ordinal number β which is greater than all of the ordinal numbers $\{\beta_\gamma, \gamma \varepsilon C\}$. Hence $K_\beta = \bigoplus_{\gamma \varepsilon C} (K_\gamma)_\beta = \bigoplus_{\gamma \varepsilon C} 0_\gamma = 0$ where 0_γ represents the 0-subring of the ring A_γ. Hence R is power nilpotent. Ξ

<u>Corollary</u>. A discrete direct sum of power nilpotent rings is power nilpotent and a subdirect sum of power nilpotent rings is power nilpotent.

PROOF:

This follows from theorems 25 and 26 and the fact that such sums can be represented as subrings of a complete direct sum of power nilpotent rings. Ξ

Theorem 27. If I, a power nilpotent ring, is an ideal of a ring R and $R^p \subset I$ for some natural number p, then R is power nilpotent.

PROOF:

Let I have the K-chain H: $I \supset H_1 \supset H_2 \supset \ldots \supset H_\gamma = 0$. Let R have the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = K_{\beta+1}$. Then $K_{2p} \subseteq H_1$ since $H_1 = I^2$, $K_{2p} = R^{2p+1}$, and $R^{2p} \subseteq I^2$: Suppose that n is a natural number and that α is either 0 or a limit ordinal and that $K_{\alpha+2np} \subseteq H_{\alpha+n}$. Then if R^0 means that R does not appear, $K_{\alpha+2np+2p} = \sum_{s=0}^{2p} R^s K_{\alpha+2np} R^{2p-s}$. It follows that $K_{\alpha+2(n+1)p} \subseteq R^p K_{\alpha+2np} + K_{\alpha+2np} R^p \subseteq IH_{\alpha+n} + H_{\alpha+n} I \subset H_{\alpha+n+1}$. By using transfinite induction (the step at limit ordinals presents no problem) it can be concluded that $K_{\alpha+2np} \subseteq H_{\alpha+n}$ for all natural numbers n and all cases when α is 0 or a limit ordinal. Hence if $\gamma = \alpha+n$ where α is 0 or a limit ordinal and n is a natural number, then $K_{\alpha+2np} \subseteq H_\gamma = 0$ and this shows that R is power nilpotent. Ξ

It is also easy to see that power nilpotent rings contain no non-zero idempotents. More generally, every idempotent of a ring R is an element of every term of R's K-chain. For if e is an idempotent in the ring R and e lies in K_α, the α-th term of R's K-chain, then $e = e \cdot e$ and hence e lies in $K_{\alpha+1}$. Transfinite induction completes the proof.

DEFINITION: *If I is an ideal of the ring R, then the* <u>*K(R)-chain*</u> *of subrings of I is the following:* $I \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = K_{\beta+1}$ *where* $K_1 = IR + RI$, $K_{\alpha+1} = K_\alpha R + RK_\alpha$ *for every ordinal number* α, *and if* α *is a limit ordinal, then* $K_\alpha = \bigcap_{\gamma<\alpha} K_\gamma$.

DEFINITION: *An ideal I of a ring R is* <u>*power nilpotently embedded*</u> *in R if I's K(R)-chain ends at 0.*

<u>Theorem 28</u>. If a ring R has a homomorphic image T which is power nilpotent and if the kernel I of the homomorphism is power nilpotently embedded in R, then R is power nilpotent.

PROOF:

Let f be the homomorphism from R onto T, a power nilpotent ring.

Let R have the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = K_{\beta+1}$. Let T have the K-chain H: $T \supset H_1 \supset H_2 \supset \ldots \supset H_\gamma = 0$. Since

$f(R) \subseteq T$, $f(K_0) \subseteq H_0$. Suppose $f(K_\alpha) \subseteq H_\alpha$. Then $f(K_{\alpha+1}) = f(K_\alpha)f(R) + f(R)f(K_\alpha) \subseteq H_\alpha T + TH_\alpha \subseteq H_{\alpha+1}$. By transfinite induction (the step at limit ordinals is due to a set theoretic property of functions) $f(K_\alpha) \subseteq H_\alpha$ for all ordinal numbers α. Hence $f(K_\gamma) \subseteq H_\gamma = 0$. It follows that $K_\gamma \subseteq I$. Moreover $K'_{\gamma+\eta}$ is contained in the η-th term in I's K(R)-chain. Hence if I's K(R)-chain reaches 0 on the τ-th term, $K_{\gamma+\tau} = 0$. Hence R is power nilpotent if I is power nilpotently embedded in R. Ξ

The next example shows that the conditions used in the theorem are necessary to obtain the general result.

<u>Remark I</u>. The condition in theorem 28 that the kernel I of the homomorphism be power nilpotently embedded in the ring R rather than merely a power nilpotent ring itself is strictly necessary.

EXAMPLE:

Let S be the semigroup consisting of the set $\{0\} \cup \{A \subset N:$ A is any non-empty subset of N, the set of natural numbers$\}$. Define multiplication in S be the rule: $A \cdot B = A \cup B$ if $A \cap B = \emptyset$ and either A or B is a finite set; otherwise $A \cdot B = 0$. Let R be the algebra over the integers mod p with basis S. Then R has the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\omega$. Actually

$K_n = \langle \{cA\colon c \in Z,\ A \in S$, and A has at least $n+1$ elements$\} \rangle$ and $K_\omega = \langle \{cA\colon c \in Z,\ A \in S$, and A has an infinite number of elements$\}\rangle$. R is not power nilpotent since $K_{\omega+1} = K_\omega$. However $K_\omega^{\ 2} = 0$ and hence K_ω is nilpotent as well as power nilpotent. Moreover R/K_ω is isomorphic to T, the integral algebra over the subsemigroup of S consisting of the set $\{0\} \cup \{A \subset N\colon\ A \in S$ and A has a finite number of elements$\}$. If K^* is the K-chain for R/K_ω, then $K^*_n \cong K_n \cap T$ for every $n \in N$. Hence $K^*_\omega \cong K_\omega \cap T = 0$ and R/K_ω is power nilpotent. Ξ

The next example shows that unlike the case for weakly nilpotent rings, the very desireable property of being homomorphically closed does not hold for the class of power nilpotent rings.

Remark J. A homomorphic image of a power nilpotent ring need not be a power nilpotent ring.

EXAMPLE:

Let S be the free semigroup generated by the set $\{x_n\colon n \in N\}$. Let R be the algebra over the field of integers modulo 2 with basis S. Then R is a power nilpotent ring since $\bigcap_{n \in N} R^n = 0$. Define a function F from the generators of R into Q, the Zassenhaus Example modulo 2, by the rule: $F(x_n) = y_{1/n}$ for all $n \in N$. Note that $F(x_1) = 0$. Since the function F is defined on the generators of R, F can be extended to all of R so that it has ring homomorphic properties. Namely, define

$F(x_{K_1} \cdot \ldots \cdot x_{K_p} + x_{\ell_1} \cdot \ldots \cdot x_{\ell_q}) = F(x_{K_1}) \cdot \ldots \cdot F(x_{K_p}) + F(x_{\ell_1}) \cdot \ldots \cdot F(x_{\ell_q}) = y_{1/K_1} \cdot \ldots \cdot y_{1/K_p} + y_{1/\ell_1} \cdot \ldots \cdot y_{1/\ell_q}$

(= 0 if $\sum_{i=1}^{p} 1/K_i = \sum_{j=1}^{q} 1/\ell_j$). F is a homomorphism since both Q and R are algebras over the field of integers modulo 2. Moreover, F(R) = Q since the set $\{y_{1/k}: k \in N\}$ generates the ring Q. However, $Q^2 = Q$ since every element in Q has a square root in Q. Therefore Q is not power nilpotent. Ξ

Actually, every free ring is a power nilpotent ring. Since every ring is a homomorphic image of a free ring it is easy to see that the class of power nilpotent rings cannot be a subset of the radical class of any proper radical property.

5. U*-RINGS

U*-rings are defined below. It follows from their definition that all U_1-rings and U_2-rings are also U*-rings. This means that some results on U_2-rings carry over to U*-rings. For example, the direct sum of two U*-rings need not be a U*-ring. U*-rings do not have very many pleasant properties although they seem to be a considerably larger class of rings than the class of U-rings. The results below provide only an introduction to these rings.

DEFINITION: *A <u>D-chain</u> of a ring R is a chain of subrings of R,* $R = D_1 \supseteq D_2 \supseteq \ldots \supseteq D_\beta$ *where* $D_{\alpha+1}$ *is an ideal of D for every ordinal number* α, *and if* α *is a limit ordinal,* $D_\alpha = \bigcap_{\gamma<\alpha} D_\gamma$.

DEFINITION: *A subring S of a ring R is a <u>meta* ideal</u> of R if there exists a D-chain in R which ends at* $D_\beta = S$.

The definition of a meta* ideal makes it clear that chain ideals are meta* ideals and that meta* ideals of meta* ideals of a ring R are meta* ideals of R. Whether every meta ideal is also a meta* ideal is equivalent to the stronger version of theorem 7 mentioned above which has not been decided. The following result (compared with remark B) makes it clear that not every meta* ideal is a meta ideal.

Theorem 29. An intersection of meta* ideals is always a meta* ideal.

PROOF:

Suppose L and M are meta* ideals of R. Suppose the following are D-chains in R: $R \supseteq D_2 \supseteq D_3 \supseteq \dots \supseteq D_\beta = L$, $R \supseteq E_2 \supseteq E_3 \supseteq \dots \supseteq E_\gamma = M$. Then the following is also a D-chain in R: $R \supseteq D_2 \supseteq \dots \supseteq D_\beta = L \supseteq L \cap E_2 \supseteq \dots \supseteq L \cap E_\gamma = L \cap M$. Hence $L \cap M$ is a meta* ideal of R. If $\{M_\gamma : \gamma \varepsilon C\}$ is a set of meta* ideals of R, and C is a subset of the ordinal numbers, then there is a D-chain in R which passes through M_1, $M_1 \cap M_2$, ..., $\bigcap_{\gamma \varepsilon C} M_\gamma$ in that order of sucession. (There may be other subrings of R between these intersections as there are in the case of the intersection of two meta* ideals given above.) Ξ

DEFINITION: *A ring R is a U*-ring if each subring S of R is a meta* ideal of R.*

Theorem 30. Every subring of a U*-ring R is a U*-ring.

PROOF:

Suppose that R is a U*-ring and that S is a subring of R. If T is any subring of S, then T is a meta* ideal of R. Suppose the following is a D-chain in R: $R \supseteq D_2 \supseteq \dots \supseteq D_\beta = T$. Then there is also a D-chain in S which ends at T, namely: $S \supseteq D_2 \cap S \supseteq \dots \supseteq D_\beta \cap S = T$. Ξ

<u>Theorem 31</u>. A ring R is not a U*-ring iff R has a subring S which has a proper, non-zero subring Q with the property that Q is contained in no proper ideal of S.

PROOF:

Suppose R is not a U*-ring. Then R must have some subring Q which is not a meta* ideal of R. Consider all the ideals of R which contain Q. The intersection of all these ideals, I_1, is an ideal of R which contains Q. For the same reason there exists a smallest ideal of I_1, I_2, which contains Q. Define I_3, I_4, etc. in a similar fashion. If α is a limit ordinal, let $I_\alpha = \bigcap_{\gamma<\alpha} I_\gamma$. Eventually, $I_\beta = I_{\beta+1}$ for some ordinal number β. Since Q is not a meta* ideal, $I_\beta \neq Q$. Hence I_β is a subring of R in which Q is a proper, non-zero subring with the property that Q is contained in no proper ideal of I .

Suppose Q is a non-zero subring of R and Q is contained in a subring S of R but Q is contained in no ideal of S. Then Q is not a meta* ideal of S and therefore S is not a U*-ring. It follows from theorem 30 that R is not a U*-ring. Ξ

<u>Theorem 32</u>. A homomorphic image of a U*-ring is a U*-ring.

PROOF:

Let R/K be an arbitrary homomorphic image of a U*-ring R. Then if S* is any subring of R/K, S* is isomorphic to S/K for some subring S of the ring R. Since R is a U*-ring there exists a D-chain in R which ends at S. Suppose that $R \supset D_1 \supset \ldots \supset D_\beta = S$ is such a D-chain. Then the following is a D-chain in R/K which ends at S*: $R/K \supset D_1/K \supset \ldots \supset D_\beta/K \cong S^*$. Hence S* is a meta* ideal of R/K. It follows that R/K is a U*-ring. Ξ

<u>Remark K</u>. A power nilpotent ring need not be a U*-ring.

EXAMPLE:

The ring R defined in the example after remark J is a power nilpotent ring. However it has Q, the Zassenhaus Example modulo 2, as a homomorphic image. The ring Q is not a U*-ring since the subring of Q generated by the following set is not contained in any proper ideal of Q: $\{x_{q/2^n} : n, q \in N\}$. Theorem 32 shows that R cannot be a U*-ring. Ξ

Remark K is somewhat surprising since every weakly nilpotent ring is a U-ring. Since the ring R in example K is also an algebra it can be seen that not every power nilpotent algebra is a U*-algebra. However it is true that every power nilpotent semigroup is a U*-semigroup.

<u>Remark L</u>. A power nilpotent semigroup is a U*-semigroup.

PROOF:

Suppose that R is a power nilpotent semigroup and that S is a subsemigroup of R. Let R have the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = 0$. Then the following is a D-chain of subsemigroups in R: $R \supseteq S \cup K_1 \supseteq S \cup K_2 \supseteq \ldots \supseteq S \cup K_\beta = S$. Note that if α is a limit ordinal, then $S \cup K_\alpha = \bigcap_{\gamma<\alpha} (S \cup K_\gamma)$. Hence S is a meta* ideal of R and R is a U*-semigroup. ∎

6. PRIME PRODUCTS RINGS

The definition given below of prime products rings excludes all rings which have any idempotents other than 0. Consequently, the class of unique factorization domains is not a subset of the class of prime products rings. However, if the definition of prime element given below is modified to read "y is a prime in R if whenever u, $v \varepsilon R$ and $y = uv$, u or v is a unit in R", then a slightly modified definition of prime products ring can be given which includes all unique factorization domains as well as the rings defined to be prime products rings in this paper. Although the class of prime products rings does not include unique factorization domains, it is very large and includes rings of many types. In particular, the class of prime products rings includes all power nilpotent rings and all weakly nilpotent rings, but not all locally nilpotent rings.

DEFINITION: *An element $y \varepsilon R$ is a* <u>*prime element*</u> *if y cannot be written as a product of two elements in the ring R. (The two elements need not be distinct.)*

DEFINITION: *An element x in a ring R has a* <u>*prime factorization*</u> *if x may be written as the product of a finite number of prime elements in R.*

DEFINITION: *If* $x = y_1 y_2 \cdot \ldots \cdot y_n \varepsilon R$ *is a factorization of* x *in the ring* R *and if* $y_j = z_1 z_2$ *(where* z_1 *and* z_2 *are elements in* R *but neither is a unit in* R*), then* $y_1 \cdot \ldots \cdot y_{j-1} z_1 z_2 y_{j+1} \cdot \ldots \cdot y_n$ *is a* *__refinement of the factorization__* $y_1 \cdot \ldots \cdot y_n$ *of the element* x.

DEFINITION: *A* *__series of factorizations__* *of an element* $x \varepsilon R$*, a ring, begins with the trivial factorization* $x = x$*, and has the property that every other factorization in the series is a refinement of the previous factorization.*

DEFINITION: *A series of factorizations* *__ends__* *if a factorization of the form* $x = y_1 \cdot \ldots \cdot y_n$ *is obtained where* $y_1, \ldots, y_n$ *are all primes in the ring* R.

DEFINITION: *An element* $x \varepsilon R$ *has (the property)* *__FF__* *if every series of factorizations of* x *ends after a finite number of steps.*

__Remark M__. A non-zero idempotent may be a finite product of primes. However, an idempotent cannot have FF.

EXAMPLE:

Let S be the semigroup consisting of two elements s and t where all products are s. Let R be the algebra over the field of integers modulo 2 which has S for a basis. Then t is a prime in R since it cannot be written as a product of two elements in R. Also, $s = t \cdot t$ and hence s is a product of primes. However, s also equals $s \cdot s$ and hence the series of factorizations of s of the form $s = s^n$ does not end after a finite number of steps. ∃

Note that in the ring R defined in the example above $s = t \cdot t$ and $s = t \cdot t \cdot t$ also. However, the second factorization is not a refinement of the first since $t \neq t \cdot t$.

DEFINITION: *A* <u>*prime products ring*</u> *is a ring in which every non-zero element has FF.*

The following theorems give some indication of the structure of the class of prime products rings. It is interesting to note that these are nearly the same results as were obtained for power nilpotent rings.

<u>Theorem 33</u>. Every subring S of a prime products ring R is a prime products ring.

PROOF:

Let x be a non-zero element of S, a subring of R. Then every series of factorizations in R of x ends after a finite number of steps since R is a prime products ring. But every series of factorizations in S of x is also part of a series of factorizations in R of x. Consequently, every series of factorizations in S of x is finite and S is a prime products ring. ∃

<u>Theorem 34</u>. A complete direct sum of prime products rings is a prime products ring.

PROOF:

Suppose $R = \bigoplus_{\gamma\varepsilon C} A_\gamma$ where each A_γ is a prime products ring. If x is any non-zero element of R and $x = \sum_{\gamma\varepsilon F} x_\gamma$ where x_γ is a non-zero element of A_γ for all $\gamma\varepsilon F$ and F is a subset of C, then every series of factorizations in R of x consists of the products of the sums of corresponding terms in the series of factorizations in A_γ of x_γ. That is, if $x_\gamma = y_\gamma z_\gamma$ for all $\gamma\varepsilon F$, then $x = (\sum_{\gamma\varepsilon F} y_\gamma)(\sum_{\gamma\varepsilon F} z_\gamma)$ is a refinement of the trivial factorization of x and every refinement of the trivial factorization of x has this form. Since the series of factorizations of x ends when any one of the series of factorizations of $x_\gamma(\gamma\varepsilon F)$ ends, x must have FF. Hence R is a prime products ring. Ξ

<u>Corollary</u>. A discrete direct sum of prime products rings is a prime products ring and a subdirect sum of prime products rings is a prime products ring.

PROOF:

This follows from theorems 33 and 34. Ξ

Remark N. A prime products ring may be a subring of a ring which has no prime elements.

EXAMPLE:

The ring of rational numbers, Q, has no prime elements since Q has an identity. The subring R consisting of the set of even integers is a prime products ring in which every integer not divisible by four is a prime. Ξ

Remark N shows that a ring R which is not a prime products ring may have a subring which is a prime products ring. The following theorem indicates that such subrings cannot be too closely related to the ring R itself.

Theorem 35. If I is a subring of R, if I is a prime products ring, and if $R^p \subseteq I$ for some integer p, then R is a prime products ring.

PROOF:

Due to the fact that a subring of a prime products ring is a prime products ring it is sufficient to prove the case when $R^2 = I$. Suppose that $x \varepsilon R$ does not have FF in R. Then x is not a prime in R and hence $x \varepsilon R^2 = I$. Let $\{s_n : n \varepsilon N\}$ be an infinite series of refinements of x in R where s_n: $x_{n,1} x_{n,2} \cdots x_{n,n} = x$. An infinite series of refinements of x in I, $\{t_n : n \varepsilon N\}$ where t_n: $y_{n,1} y_{n,2} \cdots y_{n,n} = x$, can be constructed by defining the $y_{i,j}$ in terms of suitable $x_{\ell,m}$'s. Let $y_{1,1} = x_{1,1}$ and let $y_{2,1} = x_{4,1}x_{4,2}\varepsilon R^2 = I$ while $y_{2,2} = x_{4,3}x_{4,4}\varepsilon I$. Then t_2: $y_{2,1}y_{2,2} = x$ is a refinement of t_1. In general, the $y_{n+1,j}$'s can be defined in the following special way (where the g_i's are chosen so that $y_{n+1,1}\cdot\cdots\cdot y_{n+1,n+1}$ is a refinement of $y_{n,1}\cdot\cdots\cdot y_{n,n}$): let $y_{n+1,1} = x_{4n,1}\cdot\cdots\cdot x_{4n,g_1}$, $y_{n+1,2} = x_{4n,g_1+1}\cdot\cdots\cdot x_{4n,g_1+g_2}$, $\ldots$, $y_{4n+1,n+1} = x_{4n,\sum_{i=1}^{n-1} g_i+1}\cdot\cdots\cdot x_{4n,\sum_{i=1}^{n} g_i}$ where $g_i\varepsilon N$ for all i in [1,n] and $\sum_{i=1}^{n} g_i = 4n$. It has already been shown that the $y_{n+1,j}$'s can be chosen in this way when n = 1. Suppose that they can be so chosen when n = n. Then s_{4n+4}: $x_{4n+4,1}\cdot\cdots\cdot x_{4n+4,4n+4}$ has the property that there exist numbers h_i, i = 1,...,n+1, such that $y_{n+1,1} = x_{4n+4,1}\cdot\cdots\cdot x_{4n+4h_1}$, $y_{n+1,2} = x_{4n+4,h_1+1}\cdot\cdots\cdot x_{4n+4,h_1+h_2}$, $\ldots$, $y_{n+1,n+1} = x_{4n+4,\sum_{i=1}^{n} h_i+1}\cdot\cdots\cdot x_{4n+4,\sum_{i=1}^{n+1} h_i}$ where $h_i\varepsilon N$ for all i in [1,n+1] and $\sum_{i=1}^{n+1} h_i = 4n+4$.

Hence $h_j \geq 4$ for at least one j in [1,n]. Suppose $h_k \geq 4$. Then by letting $y_{n+2,i} = y_{n+1,i}$ if $i < k$, by letting $y_{n+2,k} = x_{4n+4,\sum_{i=1}^{k-1} h_i+1}\cdot x_{4n+4,\sum_{i=1}^{k-1} h_i+2}$ and $y_{n+2,k+1} = x_{4n+4,\sum_{i=1}^{k-1} h_i+3}\cdot\cdots\cdot x_{4n+4,\sum_{i=1}^{k} h_i}$, and by letting $y_{n+2,i} = y_{n+1,i-1}$ if $i > k+1$, the $y_{n+2,j}$'s are chosen the given special way so that t_{n+2} is a refinement of t_{n+1}. Hence, by induction on n, t_n can be chosen for all $n \in N$ and therefore x does not have FF in I. Ξ

The following theorems and remarks compare the class of prime products rings with some of the other classes of rings defined above.

<u>Theorem 36</u>. If R is a weakly nilpotent ring, then R is a prime products ring.

PROOF:

Suppose that y_o is a non-zero element of R which has a series of refinements of the trivial factorization which does not end after a finite number of steps. Suppose that $y_o = x_1 z_1$ where either x_1 or z_1 does not have FF. Let $y_1 = z_1$ if x_1 has FF; otherwise let $y_1 = x_1$. A sequence $\{y_n\}$ of elements, each of which does not have FF, can be defined recursively. In general, let $y_{n-1} = x_n z_n$ where either x_n or z_n does not have FF and let $y_n = z_n$ if x_n has FF; otherwise let $y_n = x_n$. Let $G = \{n \in N\colon y_n = x_n\}$ and let $H = \{n \in N\colon y_n = z_n\}$. Then at least one of the two sets G and H has an infinite number of elements. Suppose that G does. Let $w_o = y_o$ and define w_n in the following way: if $n \in G$, $w_n = (x_1, x_{i_2}, \ldots, x_{i_{n_i}})y_n$ where $i_1 < i_2 < \ldots < i_{n_i} < n$ and $\{i_1, i_2, \ldots, i_{n_i}\} = (N \sim G) \bigcap [1,n]$;

otherwise $w_n = w_{n-1}$. Then $w_{n-1} = w_n z_n$ for every $n \varepsilon G$. To eliminate repetitions let $F: N \to G$ be an order preserving mapping onto G and define $u_n = w_{F(n)}$ and $v_n = z_{F(n)}$ for every $n \varepsilon N$. Then $y_o = u_1 v_1$ and $u_n = u_{n+1} v_{n+1}$ for every $n \varepsilon N$. Let R have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_\beta = J_{\beta+1}$. Then any ordered product of the form $v_n v_{n-1} \cdot \ldots \cdot v_1 \notin J_1$ since $u_n(v_n v_{n-1} \cdot \ldots \cdot v_1) = y_o \neq 0$. Suppose that $v_n v_{n-1} \cdot \ldots \cdot v_1 \notin J_\alpha$ for every $n \varepsilon N$. Then since $v_{n+1}(v_n v_{n-1} \cdot \ldots \cdot v_1) \notin J_\alpha$, $(v_n v_{n-1} \cdot \ldots \cdot v_1) \notin J_{\alpha+1}$. Hence by transfinite induction (the step at limit ordinals is obvious) $v_n v_{n-1} \cdot \ldots \cdot v_1 \notin J_\beta$ for every $n \varepsilon N$. Hence R is not weakly nilpotent. The case when H has an infinite number of elements can be handled in an analogous way. Ξ

<u>Theorem 37</u>. If R is a power nilpotent ring, then R is a prime products ring.

PROOF:

Let R have the K-chain K: $R \supset K_1 \supset K_2 \supset \ldots \supset K_\beta = 0$. If $x \notin K_1 = R^2$, then x is a prime and therefore has only the trivial factorization. Suppose that $x \notin K_\gamma$ implies that x has FF. If $y \notin K_{\gamma+1}$, then either y is a prime or y has factorizations of the form $y = y_1 y_2$. In every factorization of this form neither y_1 or y_2 are elements in K_γ, since otherwise y lies in $K_\gamma R + RK_\gamma = K_{\gamma+1}$ which is a contradiction.

Hence y_1 and y_2 have FF by assumption and therefore every series of refinements of the factorization $y = y_1 y_2$ must end after a finite number of steps. It follows that every series of refinements of the trivial factorization $y = y$ must end after a finite number of steps. Hence every element $x \notin K_{\gamma+1}$ has FF. By transfinite induction (the step at limit ordinals is obvious) every element $x \notin K_\beta$, i.e. every non-zero element in R, has FF. Ξ

<u>Remark O</u>. A homomorphic image of a prime products ring need not contain any primes.

EXAMPLE:

The ring R defined in the example for remark J is power nilpotent and therefore a prime products ring. The Zassenhaus Example modulo 2, Q, is a homomorphic image of R and Q contains no primes since every element in Q may be written as the product of its square root times its square root. Ξ

The ring R defined in the example for remark J is power nilpotent and therefore a prime products ring. The fact that the Zassenhaus Example is a homomorphic image of R shows that R is not a U-ring. Hence not every prime products ring is a U-ring.

<u>Remark P</u>. A prime products ring need not be power nilpotent.

EXAMPLE:

Let R be the commutative ring generated by the set $\{x_n : n \in N\}$ with the following set of generating relations: $x_n = x_{4n}x_{4n+1} + x_{4n+2}x_{4n+3}$ for all $n \in N$. Then R is not power nilpotent since every element in the generating set for R lies in R^2 and hence $R = R^2$. However, R is a prime products ring since if z is a non-zero element in R and $z = \sum_{i=1}^{h} (\prod_{j=1}^{k_i} x_{n_{i,j}})$, then every series of factorizations of z ends in $\leq$ k steps where $k = \max\{k_i : i \text{ in } [1,h]\}$. This is true since each of the three basic type of substitutions made possible by the generating relations also has this property. Examples of these basic types of substitutions are: $x_n = x_{4n}x_{4n+1} + x_{4n+2}x_{4n+3}$, $x_{4n}x_{4n+1} = x_n - x_{4n+2}x_{4n+3}$, $x_{4n+2}x_{4n+3} = x_n - x_{4n}x_{4n+1}$.

Ξ

Remark Q. A prime products ring need not be a U*-ring.

EXAMPLE:

The ring R defined in the example for remark P is a prime products ring. The subring S of R generated by the set $\{x_{2n} : n \in N\}$ is a proper subring of R. However $x_n = x_{4n}x_{4n+1} + x_{4n+2}x_{4n+3} \in SR + SR$. Hence for all $n \in N$, $x_n \in SR \subset I(S)$, the ideal of R generated by S. It follows that R is not a U*-ring since S is not contained in any proper ideal of R. Ξ

7. RING TYPES

Meldrum (12) has recently completed the study of group types. An analogous definition of ring type is given below and the theory of ring types is nearly completed in this paper except for the determination of the class of possible ring types for weakly nilpotent rings.

DEFINITION: *A ring R has* <u>*type* (j,k)</u> *if its J-chain terminates after j steps and its K-chain terminates after k steps.*

<u>Theorem 38</u>. If R is the direct sum of the rings A and B and A has ring type (j,k) while B has ring type (ℓ,m), then R has ring type $(\max\{j,\ell\},\max\{k,m\})$.

PROOF:

Let A have the J-chain J_A and the K-chain K_A and let B have the J-chain J_B and the K-chain K_B. Then if R's J-chain is J and R's K-chain is K, $J_\alpha = (J_A)_\alpha \oplus (J_B)_\alpha$ and $K_\alpha = (K_A)_\alpha \oplus (K_B)_\alpha$ for every ordinal number α. Hence R's J-chain must have length $h = \max\{j,\ell\}$ and R's K-chain must have length $n = \max\{k,m\}$. Ξ

Theorem 38 shows that the direct sum of rings with given ring types may be a ring with a quite different ring type. Furthermore, it makes it easy to see how rings of all possible ring types may be constructed from the examples of rings of various types given below.

The next theorem characterizes the class of possible ring types for nilpotent rings.

<u>Theorem 39</u>. Suppose R is a nilpotent ring and $R^n \neq 0$ while $R^{n+1} = 0$. Then R has ring type (n,n).

PROOF:

The K-chain for R has length n since it is the following: $R \supset R^2 \supset \ldots \supset R^n \supset 0$. On the other hand, there exists $x_1, \ldots, x_n \varepsilon R$ such that $x_1 \cdot \ldots \cdot x_n \neq 0$. It follows that $x_2 \cdot \ldots \cdot x_n$ does not lie in J_1, that $x_3 \cdot \ldots \cdot x_n$ does not lie in J_2, ... , and that x_n does not lie in J_{n-1}, where R has the J-chain J. However, $R^n \subseteq J_1$, $R^{n-1} \subseteq J_2$, ... , and $R \subseteq J_n$. So R's J-chain must end at J_n and have length n. Ξ

Since there are nilpotent rings of every index, it follows from theorem 39 that there are rings of type (n,n) for every natural number n.

<u>Remark R</u>. For every ordinal number $\beta \geq \omega$, there are rings of type (β,ω) which are both weakly nilpotent and power nilpotent.

EXAMPLE:

Let $\beta = \gamma+n$ where γ is a limit ordinal number and n is a non-negative integer. Let R be the commutative ring generated by the set $S = \{x_\alpha$: α is an ordinal number, but not a limit ordinal and $\alpha \leq \beta\}$, with the relations:

(1) $x_\alpha{}^2 = 0$ for all $x_\alpha \varepsilon S$ where $\alpha < \beta$; if β is not a limit ordinal, then $x_\beta{}^n \neq 0$ while $x_\beta{}^{n+1} = 0$.

(2) Suppose that $x_{\delta_i} \varepsilon S$ for all i in [1,m]. Then the product $x_{\delta_1} \ldots x_{\delta_m} = 0$ if α is the smallest of the ordinal numbers $\{\delta_1,\ldots,\delta_m\}$, if $\alpha = \eta+k$ where η is a limit ordinal number and $k \varepsilon N$, and if $k < m$. (Hence a product of elements from S is 0 if the number of factors exceeds the "finite" part of any of the subscripts of elements in the product).

Let R have the K-chain K and the J-chain J. Then $K_\omega = 0$ since if $x_{\alpha_{i,j}}$ stands for an arbitrary element in S, if z is a non-zero element in R, and if $z = \sum_{i=1}^{q} (\prod_{j=1}^{\ell_i} x_{\alpha_{i,j}})$, then z has only prime factorizations with fewer than h+1 prime factors where $h = \max\{\ell_1,\ldots,\ell_q\}$. It follows that $z \notin R^{h+1} = K_h$ and hence $z \notin K_\omega$. Now let $\eta+k$ be any non-limit ordinal number smaller than β where η is a limit ordinal or zero and k is a natural number. Then it can be shown by induction on $\eta+k$ that $J_{\eta+k}$ is generated by $J_{\eta+k-1}$ and the following set of elements: $\{x_{\eta+k};\ x_{\eta+k+1}\cdot x_\delta$ where $\delta > \eta+k+1;\ \ldots\ ;\ x_{\eta+k+m}x_{\delta_1}\cdot\ldots\cdot x_{\delta_m}$ where $\delta_1,\ldots,\delta_m > \eta+k+m;\ \ldots\ \}$. Note that $x_{\eta+\omega+\delta_1}\cdot\ldots\cdot x_{\eta+\hat{\omega}+\delta_h} \notin J_{\eta+k}$ since if $p > k+h$, then $x_{\eta+p}\cdot x_{\eta+\omega+\delta_1}\cdot\ldots\cdot x_{\eta+\omega+\delta_h} \notin J_{\eta+k-1}$. It follows from this that x_α first occurs in R's J-chain at J_α for all ordinal numbers $\alpha \le \beta$. Hence R's J-chain ends at $J_\beta = R$ and the ring R does have type (β,ω). Ξ

<u>Remark S</u>. There exist power nilpotent rings of type $(n,\gamma+n)$ where n is any non-negative integer and γ is any limit ordinal number.

EXAMPLE:

Let R be the ring of all $(\gamma+n)$ by $(\gamma+n)$ matrices with only a finite number of non-zero integer entries and with only zeros on the main diagonal and to the left of the main diagonal. Addition and multiplication in R are the usual matrix addition and multiplication: if $X = (x_{\alpha,\beta})$ and $Y = (y_{\alpha,\beta})$ are matrices in R, then $X + Y = (x_{\alpha,\beta}+y_{\alpha,\beta})$ and $XY = (\sum_{\eta=0}^{\gamma+n-1} x_{\alpha,\eta}y_{\eta,\beta})$.

Let R have the K-chain K: $R \supset K_1 \supset \ldots \supset K_{\gamma+n} = 0$. Computation shows that K_α is the ring of all matrices in R in which all the entries are zeros on the α diagonals parallel to the main diagonal and just to the right of the main diagonal. Since there are exactly $\gamma+n$ such diagonals, $K_{\gamma+n} = 0$, while $K_{\gamma+n-1} \neq 0$. Let R have the J-chain J. Computation shows that whenever $m \leq n$, J_m is the subring of R consisting of all matrices in R in which all the entries are zeros on the last m diagonals parallel to the main diagonal and to the right of it. The subring J_n of R is the end of R's J-chain since in every matrix in R there are no last n+1 diagonals parallel to the main diagonal and to the right of it. Hence R is power nilpotent and has type $(n,\gamma+n)$. Ξ

It follows from remarks R and S and theorem 38 that there are power nilpotent rings of type (α,β) for any non-finite ordinal numbers α and β since every non-finite ordinal number β has the form $\gamma+n$ where n in a non-negative integer and γ is a limit ordinal number.

<u>Theorem 40</u>. A power nilpotent ring must have one of the following ring types: (n,n) where n is a non-negative integer; or $(m, \gamma+n)$ where n is a non-negative integer, m is an ordinal number $\geq n$, and γ is a limit ordinal number.

PROOF:

Suppose that R is a power nilpotent ring. If R has a finite K-chain, then R is nilpotent and therefore has type (n,n) for some non-negative integer n. Suppose that R's

K-chain has length $\gamma+n$ where n is a non-negative integer and γ is a limit ordinal number. Then $K_{\gamma+n-1} \subseteq J_1$, the first term of R's J-chain. Hence $J_1 \neq 0$. It is easy to see that $K_\gamma \subseteq J_n$. However, if $K_\gamma \subseteq J_{n-1}$, then $K_{\gamma-1} \subseteq J_{n-2}, \ldots, K_{\gamma+n-1} = 0$, which is a contradiction. Hence $J_n \neq J_{n-1}$ and R's J-chain has length at least n. It follows that R has type $(m,\gamma+n)$ where m is an ordinal number $\geq n$. Ξ

It is easy to see from Remarks R and S and from theorems 38 and 39 and from the fact that a direct sum of power nilpotent rings is a power nilpotent ring that there are power nilpotent rings of all the types given in the statement of theorem 40.

<u>Remark T</u>. There exist rings of type $(0,\beta)$ for every ordinal number β.

EXAMPLE:

Let A_0 be the class of all ordinal numbers. Let $\lambda_0 = 1$, define $\lambda_{\alpha+1} = \lambda_\alpha\omega$, and if α is a limit ordinal, let $\lambda_\alpha = \inf\{\delta\varepsilon A_0: \delta > \lambda_\gamma \text{ for all } \gamma<\alpha\}$. Let A_α be the smallest subclass of A_0 with the properties:

(1) $\lambda_\alpha \varepsilon A_\alpha$,

(2) if $\delta,\eta\varepsilon A_\alpha$, then $\delta+\eta\varepsilon A_\alpha$, and

(3) if $B \subset A_\alpha$ and B is a set, then the $\inf\{\delta\varepsilon A_0: \delta > \eta$ for all $\eta\varepsilon B\}\varepsilon A_\alpha$.

Note that $A_0 \supset A_1 \supset \ldots \supset A_\alpha \supset A_{\alpha+1} \supset \ldots$, and that if α is a limit ordinal, then $A_\alpha = \bigcap_{\gamma<\alpha} A_\gamma$ since $\lambda_\alpha = \inf\{\delta \varepsilon A_0 : \lambda_\gamma < \delta$ for all $\gamma < \alpha\}$ is an element in A_γ for all $\gamma < \alpha$.

Lemma T1. Suppose that Q is a commutative ring and that Q has the K-chain K: $Q \supset K_1 \supset \ldots \supset K_\beta = K_{\beta+1}$. Then $K_\alpha K_\gamma \subseteq K_{\alpha+\gamma}$ for all ordinal pairs (α,γ) where α is a limit ordinal.

PROOF:

Let α be any fixed limit ordinal. Suppose that $\gamma = 1$. Then $K_\alpha K_1 \subseteq K_\alpha Q = K_{\alpha+1}$. Suppose that $K_\alpha K_\gamma \subseteq K_{\alpha+\gamma}$. Then $K_\alpha K_{\gamma+1} = K_\alpha K_\gamma Q \subseteq K_{\alpha+\gamma} Q = K_{\alpha+\gamma+1}$. If $K_\alpha K_\gamma \subseteq K_{\alpha+\gamma}$ for all $\gamma < \mu$, and μ is a limit ordinal, then $K_\alpha \bigcap_{\gamma<\mu} K_\gamma \subseteq \bigcap_{\gamma<\mu} K_{\alpha+\gamma} = K_{\alpha+\mu}$. Hence by transfinite induction the lemma is true.

Lemma T2. Suppose that Q is a commutative ring with K-chain K. Then $\bigcap_{n\varepsilon N} (K_{\lambda_\alpha})^n \subseteq K_{\lambda_{(\alpha+1)}}$.

PROOF:

By lemma T1, $(K_{\lambda_\alpha})^n \subseteq K_{(\lambda_\alpha)n}$ for every $n \varepsilon N$. Since $\lambda_{\alpha+1} = (\lambda_\alpha)\omega$, it follows that $\bigcap_{n\varepsilon N} (K_{\lambda_\alpha})^n \subseteq \bigcap_{n\varepsilon N} K_{(\lambda_\alpha)n} = K_{\lambda_{\alpha+1}}$.

Let γ be any non-zero ordinal number. Let α_1 be the largest ordinal number such that $\lambda_{\alpha_1} \leq \delta$, and let δ_1 be the largest ordinal number in A_{α_1} such that $\delta_1 \leq \delta$. Let α_2 be the largest ordinal number such that $\delta_1+\lambda_{\alpha_2} \leq \delta$, and let δ_2 be the largest ordinal number in A_{α_2} such that $\delta_1+\delta_2 \leq \delta$. In general let α_j be the largest ordinal number such that $\delta_1+\ldots+\delta_{j-1}+\lambda_{\alpha_j} \leq \delta$, and let δ_j be the largest ordinal number in A_{α_j} such that $\delta_1+\ldots+\delta_j \leq \delta$. Eventually, for some natural number n, $\delta_1+\ldots+\delta_n = \delta$ since $\alpha_1,\alpha_2,\ldots,\alpha_n$ is a strictly decreasing set of ordinal numbers.

DEFINITION: *The representation $\delta = \delta_1+\ldots+\delta_n$ of an ordinal number given immediately above is its* <u>*limit form*</u>.

<u>Lemma T3</u>. The limit form of every non-zero ordinal number is unique.

PROOF:

Given an ordinal number δ, the ordinal number δ_1, the first term in δ's limit form, is uniquely determined. The ordinal number δ_j is uniquely determined once the ordinals $\delta_1,\ldots,\delta_{j-1}$ have been determined. Hence the sum $\delta = \delta_1+\ldots+\delta_n$ is composed of uniquely determined terms.

There exists an ordinal number ρ such that $\lambda_\rho > \beta$. Let G be the set of all ordinal numbers less than λ_ρ. Let R be the ring generated by the set $\{x_\delta: \delta\varepsilon G\}$ with the defining relations:

(1) R is commutative.

(2) Let δ have the limit form $\delta = \delta_1+\ldots+\delta_m$ where $\delta_m \varepsilon A_{\alpha_m} \sim A_{\alpha_{m+1}}$.

Then δ_m must be the n-th ordinal in the usual ordering of the ordinals in A_{α_m} where n is a natural number. The generator x_δ satisfies the relationship: $(x_\delta)^{n+1} = x_{\delta_1+\ldots+\delta_{m-1}}$.

<u>Lemma T4</u>. Let R have the K-chain K. Let α be an ordinal number. Then $x_\delta \varepsilon K_{\lambda_\alpha}$ if $\delta\varepsilon G \bigcap A_\alpha$.

PROOF:

If $\alpha = 1$, then $\delta\varepsilon G\bigcap A_1$ implies that δ is a limit ordinal number. Since $x_\delta = (x_{\delta+n})^{n+1}$ for every $n\varepsilon N$, it follows that $x_\delta \varepsilon K_\omega = K_{\lambda_1}$. Suppose that $x_\delta \varepsilon K_\alpha$ for all $\delta\varepsilon G\bigcap A_\alpha$. Suppose also that $\Theta\varepsilon G\bigcap A_{\alpha+1}$. Then Θ must be the γ-th ordinal in A_α. where γ is a limit ordinal. Let μ_n be the $(\gamma+n)$-th ordinal in A_α, namely, $\Theta+(\lambda_\alpha)n$. Then $(x_{\mu_n})^{n+1} = x_\Theta$ and hence $x_\Theta \varepsilon (K_{\lambda_\alpha})^{n+1}$ for all $n\varepsilon N$. By lemma T2, $x_\Theta \varepsilon K_{\lambda_{\alpha+1}}$. If α is a limit ordinal, then $A_\alpha = \bigcap_{\gamma<\alpha} A_\gamma$. Hence $\delta\varepsilon G\bigcap A_\alpha$ implies that $\delta\varepsilon G\bigcap A_\gamma$ for all $\gamma < \alpha$, and therefore $x_\delta \varepsilon \bigcap_{\gamma<\alpha} K_{\lambda_\gamma} = K_{\lambda_\alpha}$. By transfinite induction on α the lemma follows.

Note that if $y \varepsilon R$, then y can be expressed in terms of $\{x_\delta: \delta \varepsilon G\}$, the generators of R, by an equation of the form $y = \sum_{j=1}^{p} L_j y_j$ where L_j is a non-zero integer for all j in [1,p] and where the y_j are all distinct elements of R of the form $y_j = \prod_{m=1}^{j_m} x_{\delta_{j,m}}$ where $\delta_{j,m} \varepsilon G$ for all j in [1,p] and all m in $[1, j_m]$. If K_Θ lies in R's K-chain and if $y_\ell \notin K$ for at least one ℓ in [1,p], then $y \notin K_\Theta$. For $y \varepsilon K_\Theta$ implies that a sum of distinct terms, each of which lies in $R \sim K_\Theta$ and has the form $L_j \prod_{m=1}^{j_m} x_{\delta_{j,m}}$, must equal a sum of distinct terms of the same general form in K_Θ. This is impossible since there are no additive relations given in the definition of the ring R.

Let $L_1 = R$, and let $L_n = K_{n-1}$ for all $n \varepsilon N$. Let $L_\alpha = K_\alpha$ for all non-finite ordinal numbers α.

Lemma T5. Suppose that $L_{(\lambda_\alpha)n} = (L_{\lambda_\alpha})^n$. Then $L_{(\lambda_\alpha)n} L_\Theta = L_{(\lambda_\alpha)n+\Theta}$ if $\Theta \leq \lambda_\alpha$. Hence $L_{(\lambda_\alpha)(n+1)} = (L_{\lambda_\alpha})^{n+1}$.

PROOF:

If $\Theta = 1$, $L_{(\lambda_\alpha)n} L_1 = L_{(\lambda_\alpha)n} R = L_{(\lambda_\alpha)n+1}$. Suppose that $L_{(\lambda_\alpha)n} L_\eta = L_{(\lambda_\alpha)n+\eta}$ and $\eta < \lambda_\alpha$. Then $L_{(\lambda_\alpha)n} L_{\eta+1} = L_{(\lambda_\alpha)n} L_\eta R = L_{(\lambda_\alpha)n+\eta} R = L_{(\lambda_\alpha)n+\eta+1}$. Let Θ be a limit ordinal number $\leq \lambda_\alpha$.

Suppose that $L_{(\lambda_\alpha)n} \cdot L_{\hat{\eta}} = L_{(\lambda_\alpha)n+\eta}$ for all ordinal numbers $\eta < \Theta$. It will be shown that $\bigcap_{\eta<\Theta} L_{(\lambda_\alpha)n} L_\eta = \bigcap_{\eta<\Theta} L_{(\lambda_\alpha)n+\eta} = L_{(\lambda_\alpha)n+\Theta}$. If $y \varepsilon (L_{\lambda_\alpha})^{n+1}$, then $y \varepsilon L_{(\lambda_\alpha)(n+1)} \subseteq L_{(\lambda_\alpha)n+\Theta}$. If $y \notin (L_{\lambda_\alpha})^{n+1}$, then y expressed in terms of the generators, $\{x_\delta : \delta \varepsilon G\}$, equals $\sum_{j=1}^{p} L_j y_j$ where each y_j is distinct, where some $y_j = x_{\delta_1} \cdot \ldots \cdot x_{\delta_n} x_{\phi_1} \cdot \ldots \cdot x_{\phi_s}$ where $x_{\delta_i} \varepsilon L_{\lambda_\alpha}$ for all i in [1,n] and $x_{\phi_1} \cdot \ldots \cdot x_{\phi_s} \notin L_{\lambda_\alpha}$. Suppose that y_j also equals $x_{\delta'_1} \cdot \ldots \cdot x_{\delta'_n} x_{\phi'_1} \cdot \ldots \cdot x_{\phi'_t}$ where $x_{\delta'_i} \varepsilon L_{\lambda_\alpha}$ for all i in [1,n] and $x_{\phi'_1} \cdot \ldots \cdot x_{\phi'_t} \notin L_{\lambda_\alpha}$. Then $\delta_i' = \delta_{\sigma(i)}$ for all i in [1,n] (where σ is a symmetric permutation on n letters) since $y_j \notin (L_{\lambda_\alpha})^{n+1}$, and $x_{\phi_1} \cdot \ldots \cdot x_{\phi_s} = x_{\phi'_1} \cdot \ldots \cdot x_{\phi'_t}$ since R has no divisors of zero. Hence $y_j \varepsilon L_{(\lambda_\alpha)n} \bigcap_{\eta<\Theta} L_\eta = L_{(\lambda_\alpha)n} L_\Theta$. Every y_i is either an element in $(L_{\lambda_\alpha})^{n+1} \supseteq L_{(\lambda_\alpha)n} L_\Theta$, or y_i has the same properties as y_j, and therefore $y_i \varepsilon L_{(\lambda_\alpha)n} L_\Theta$ for all i in [1,p]. Hence $y = \sum_{i=1}^{p} y_i \varepsilon L_{(\lambda_\alpha)n} L_\Theta$ and $L_{(\lambda_\alpha)n+\Theta} = L_{(\lambda_\alpha)n} L_\Theta$. By lemma T1, $L_{(\lambda_\alpha)n} L_\Theta \subseteq L_{(\lambda_\alpha)n+\Theta}$. Hence by transfinite induction, $L_{(\lambda_\alpha)n} L_\Theta = L_{(\lambda_\alpha)n+\Theta}$ if Θ is an ordinal number $\leq \lambda_\alpha$.

<u>Lemma T6</u>. $L_{(\lambda_\alpha)n} = [I(\{x_\delta : \delta \varepsilon A_\alpha\})]^n$ for all ordinal numbers α and all $n \varepsilon N$.

PROOF:

$L_{(\lambda_1)1} = L_\omega = I(\{x_\delta : \delta \varepsilon A_1\})$. Suppose that $L_{(\lambda_\theta)1} = I(\{x_\delta : \delta \varepsilon A_\theta\})$ for all ordinal numbers $\theta \leq \alpha$. Suppose also that $L_{(\lambda_\alpha)n} = [I(\{x_\delta : \delta \varepsilon A_\alpha\})]^n = [L_{\lambda_\alpha}]^n$. Then by lemma T5, $L_{(\lambda_\alpha)(n+1)} = [L_{\lambda_\alpha}]^{n+1} = [I(\{x_\delta : \delta \varepsilon A_\alpha\})]^{n+1}$. Hence by induction on n, $L_{(\lambda_\alpha)n} = [I(\{x_\delta : \delta \varepsilon A_\alpha\})]^n$ for all $n \varepsilon N$. Hence $L_{\lambda_{\alpha+1}} = L_{(\lambda_\alpha)\omega} = \bigcap_{n \varepsilon N} L_{(\lambda_\alpha)n} = \bigcap_{n \varepsilon N} [I(\{x_\delta : \delta \varepsilon A_\alpha\})]^n = I(\{x_\delta : \delta \varepsilon A_{\alpha+1}\})$.

Hence by transfinite induction on α, (the step at limit ordinals is obvious), $L_{(\lambda_\alpha)1} = I(\{x_\delta : \delta \varepsilon A_\alpha\})$ for all ordinals α.

Hence R's K-chain does not end until after K_ρ and therefore $K_\beta \neq K_{\beta+1}$.

Let S be the ring generated by the set $\{x_\delta : \delta \varepsilon G\}$ with the defining relations (1) and (2) given for the ring R above, and (3) let K_β be the β-th term in R's K-chain, let n be a natural number and let $\delta_i \varepsilon A_{\alpha_i} \sim A_{\alpha_{i+1}}$ for all i in [1,n] where

$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. If $x_{\delta_1} x_{\delta_2} \cdot \ldots \cdot x_{\delta_n} \varepsilon K_\beta$ and if $x_{\phi_1} x_{\phi_2} \cdot \ldots \cdot x_{\phi_t} = x_{\delta_n}$ where $\phi_t \varepsilon A_0 \sim A_1$, then the relation $x_{\delta_1} \cdot \ldots \cdot x_{\delta_n} = x_{\delta_1} \cdot \ldots \cdot x_{\delta_{n-1}} \cdot x_{\varepsilon_1} \cdot \ldots \cdot x_{\varepsilon_{t-1}} \cdot (x_{\gamma+4n} x_{\gamma+4n+1} + x_{\gamma+4n+2} x_{\gamma+4n+3})$ holds where $\phi_t = \gamma+n$, and $\gamma = 0$ or $\gamma \varepsilon A_1$, the set of limit ordinals.

It is easy to see that elements in the ring R of the form $(x_{\delta_1} \cdot \ldots \cdot x_{\delta n-1} \cdot x_{\phi_1} \cdot \ldots \cdot x_{\phi_{t-1}} \cdot x_{\gamma+4n} x_{\gamma+4n+1})$ lie in $K_{\beta+1}$ in R's K-chain. Hence if S has the K-chain H, then $H_\alpha = K_\alpha$ if $\alpha \leq \beta$. However $H_{\beta+1} = H_\beta$ due to the additive relations in S defined by relations (3). Hence the ring S has type $(0,\beta)$ since S also has a trivial J-chain. S is a prime products ring since x_0 has FF due to the fact that every decreasing chain of ordinal numbers is finite. Also, every x_δ, $\delta \varepsilon G$, has FF since x_δ occurs as a factor in a series of factorizations of x_0. Ξ

Remark U. There are prime products rings of type $(n,n-1)$ for every natural number n.

EXAMPLE:

Let R(k) be the commutative ring generated by the set $\{y_n : n \varepsilon N\}$ with the relations:

(1) $y_1^{k+1} = 0$.

(2) $y_1 y_s = y_1^2$ for all $s \varepsilon N$.

(3) if $n_k > n_1, \ldots, n_{k-1}$, then $y_{n_1} \cdot y_{n_2} \cdot \ldots \cdot y_{n_k} = y_{n_1} y_{n_2} \cdot \ldots \cdot y_{4n_k} y_{4n_k+1} + y_{n_1} y_{n_2} \cdot \ldots \cdot y_{4n_k+2} y_{4n_k+3}$.

Due to relations of type (3), $R(k) = R(k)^{k+1}$. However $y_2 y_3 \ldots y_{k+1} \notin R(k)^{k-1}$ and hence $R(k)$'s K-chain has exactly k-1 steps. Let $R(k)$ have the J-chain J: $0 \subset J_1 \subset J_2 \subset \ldots \subset J_k = J_{k+1}$. This J-chain ends at J_k since $y_1^k \varepsilon J_1$, $y_1^{k-1} \varepsilon J_2$, ... , $y_1 \varepsilon J_k$. The next theorem shows that $R(k)$'s J-chain cannot have length greater than k since its K'chain has length k-1. Hence the ring $R(k)$ has type $(k,k-1)$ and is a prime products ring for the same reasons that the ring R given in the example for remark P is a prime products ring. Ξ

The following theorem together with remarks R, S, T, U gives complete information for determining the possible ring types of all commutative rings.

<u>Theorem 41</u>. There are no commutative rings of type (m,n) where n is a natural number and m is any ordinal number $> n+1$.

PROOF:

Let R be a commutative ring. If R's K-chain ends after n steps, then $R^{n+1} = R^{n+2}$. Let R have the J-chain J and suppose $x \varepsilon J_{n+2}$. Then $xR \subseteq J_{n+1}$, $xR^2 \subseteq J_n$, ... , $xR^{n+1} \subseteq J_1$ and $xR^{n+2} = 0$. Hence $xR^{n+1} = 0$ and this implies that $x \varepsilon J_{n+1}$. Hence $J_{n+2} = J_{n+1}$ and R's J-chain has less than n+2 steps. ∎

<u>Theorem 42</u>. There are no rings of type (m,n) where n is a natural number and m is any ordinal number > 2n+2.

PROOF:

Let R be any ring such that $R^{n+1} = R^{n+2}$. Let R have the J-chain J and note that $x \varepsilon J_p$ iff $\sum_{s=o}^{p} R^s x R^{p-s} = 0$ where R^0 means that R does not appear on that side. If $x \varepsilon J_{2n+3}$, then $\sum_{s=o}^{2n+3} R^s x R^{2n+3-s} = 0$. But in each case for s = 0,1,...,2n+3 either s $\geq$ n+2 or 2n+3-s $\geq$ n+2. Hence the equation above may be rewritten $\sum_{s=o}^{2n+2} R^s x R^{2n+2-s} = 0$ since $R^{n+2} = R^{n+1}$. From this it follows that $x \varepsilon J_{2n+2}$ and hence R's J-chain can be no longer than 2n+2 steps. ∎

<u>Corollary</u>. Every weakly nilpotent ring has a ring type similar to a power nilpotent ring. Hence the theory of ring types cannot decide whether or not every weakly nilpotent ring is power nilpotent.

PROOF:

There are no weakly nilpotent rings of type (n,β) where n is a natural number and β is any ordinal number other than n. For suppose the R has the J-chain: $0 \quad J_1 \quad J_2 \quad \ldots \quad J_n = R$. Then $R^2 \subseteq J_{n-1}$, $R^3 \subseteq J_{n-2}$, $\ldots$, and $R^{n+1} = 0$. Hence R is nilpotent and has type (n,n). Theorem 40 shows that there are power nilpotent rings of all types of the form (α,β) where α and β are both non-finite ordinal numbers. Theorem 42 shows that there are no rings of the type (α,k) where k is a non-negative integer and α is an non-finite ordinal number. Ξ

CONCLUSION

This paper establishes some important relationships between the different generalizations of nilpotence defined above. In particular, every weakly nilpotent ring is a U-ring and a prime products ring. Also, every power nilpotent ring is a prime products ring, but not necessarily a U*-ring, and not every meta* ideal is a meta ideal.

A few conjectures are suggested by the results in the paper. The section on ring types provides information that suggests that every weakly nilpotent ring may be power nilpotent. Theorem 19 suggests that every nil ring which is a U-ring may also be a prime products ring. Also, theorem 22 resulted from an attempt to prove that every nil U_2-ring is weakly nilpotent. Finally, theorem 29 makes it seem probable that not every U*-ring is a U-ring. These are challenging conjectures that hopefully will be resolved by further research.

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