AN AXI-SYMMETRIC CONTACT PROBLEM:
THE CONSTRICTION OF ELASTIC CYLINDERS UNDER AXIAL COMPRESSION

by

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ABSTRACT

The compression of fairly short solid cylinders under axial load is considered. A preliminary investigation examines the displacements produced by the superposition of a prescribed surface loading. This is followed by the more interesting problem in which the radial displacement is prescribed over part of the surface, the remaining part of the surface being stress free. Two types of elastic materials are considered; firstly, rubber-like materials governed by a strain energy function of the Mooney form, and secondly, metals which have a quadratic strain energy function. In the former case a finite axial compression is permitted prior to imposing any constraint on the outer curved surface of the cylinder. In all cases the irregularities introduced by the constraints are sufficiently small that they can be described by infinitesimal elasticity theory. The analysis utilizes displacement potential functions and the main problem is reduced to solving a set of dual cosine series. The particular case of the contact problem in which the cylinder height is equal to the radius is examined in detail and the contact stresses are given graphically.

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CHAPTER I

INTRODUCTION.

Some of the earliest contributions to the theory of symmetrically deformed cylinders were made by Chree (2), Filon (4) and Schiff (11). In particular Filon presented a very detailed study of the deformation of finite solid cylinders under various systems of surface loads and end conditions of practical interest with solutions derived via a standard Fourier analysis.

The same technique was used by Barton (1) in an attempt to solve the problem in which the radial displacement is prescribed. Using Filon's method to obtain the solution for a uniform pressure band, he applied superposition to derive the radial displacement of a cylinder loaded uniformly over a number of adjacent narrow bands. By choosing the magnitude of each pressure distribution to produce equal displacements at the centre of each band, his solution approximates the case in which the displacement is constant over a central surface area. This appears to be the first attempt at the solution to the problem of prescribing a uniform displacement across a band on the curved surface of a cylinder. Some of Filon's problems were later discussed by Tranter and Craggs (19) and the discontinuities occurring in the boundary values were sharpened by the use of integral transforms.

Sneddon (14) has also examined the case of an infinite cylinder loaded over half of its curved surface by a constant pressure. He utilizes the Heisenberg delta function and uses Fourier transform techniques to obtain a solution. The main problem we consider in the present investigation is illustrated in Figure 4. A circular cylinder is compressed by axial loads which produce shortening in the axial direction and expansion in the radial direction. Radial expansion is prevented over a central region of the curved surface by a rigid constraint. In reality this situation arises when a rivet is expanded into a circular hole in a relatively rigid plate. Thus we have a contact problem with mixed boundary values rather than a pressure band problem as initially examined by Barton.

As a preliminary investigation we consider the problem of an arbitrary symmetric pressure distribution applied across a band of finite width. In particular we obtain the displacements produced by certain distributions and so obtain qualitative properties which we might expect the solution of our main problem to exhibit. Solutions to the problems are obtained for two types of materials; incompressible rubber-like materials which have a strain-energy function of the Mooney form, and metals which have a quadratic strain-energy function. In the former case a finite uniform compression of the order of 25% is permitted prior to any surface constraint becoming effective. In the case of the main problem a small additional compression is then permitted to describe the non-uniformities caused by the constriction. For materials governed by the quadratic type of strain-energy function the deformation is assumed to be completely infinitesimal since elastic behaviour of metals ceases when the

strains are still extremely small. Thus for metals, the deformation can conveniently be described by the equations of classical elasticity.

Some related contact problems using the linear theory of elasticity have recently been examined. Spillers (17) has considered the infinite cylinder constricted over a central region by a tight collar. Severne (12) has examined numerically the shrink-fit stresses between hollow cylinders having a finite interval of contact. Numerical methods have also been applied by Conway and Farnham (3) to the problem of the shrink fit of a collar over an infinite cylinder. They consider the case in which there is no friction between the collar and cylinder and also the case in which there is no tangential slip. In the present investigation there is no shear between the cylinder and the constraint or between the ends of the cylinder and the load surfaces. The analysis extends the results of Spillers in the following ways. Firstly, the cylinder may be quite short rather than infinite in extent and secondly, the axial extension may be finite rather than infinitesimal, at least for rubber materials. In addition, the stress-strain relation for a constrained cylinder, and the load transmitted to the constricting band are obtained in closed form in terms of the axial load applied to the cylinder.

For the finite deformation of the rubber material the results of Green, Rivlin and Shield (6) are used. Their work, in a slightly modified form, is given in Green and Zerna (7) and is closely followed. The main problem here is reduced to solving a set of dual cosine series which in general are rather cumbersome. Attention is restricted to fairly short cylinders which enables asymptotic expansions to be used, thereby

simplifying the form of the series. The asymptotic forms of the series are brought to standard forms and solved using the methods given in Sneddon (15).

An essential feature of the solution to the main problem, even for the infinitesimal deformation associated with the quadratic strain-energy function, is the superposition of an unknown uniform deformation. This effectively makes the boundary conditions inhomogeneous when considering the non-uniform constriction. The solution of the dual series then implies the required uniform solution. A solution is obtained numerically for the particular case when the cylinder height and diameter are equal, and the contact width is half of the cylinder height. Transform methods are used to accelerate the slow convergence of resulting series which defines the radial stress. The stresses under the contact area are given graphically for various values of Poisson's ratio.

The analysis is separated into several main sections. Chapter 2 gives the main equations for the theory of finite deformations of an elastic solid. Chapter 3 contains the equations for the small deformation of a Mooney material superposed on a finite uniform extension and their use is exemplified in the problem of a prescribed surface pressure distribution. In Chapter 4 we obtain the particular solution for the contact problem being considered and reduce the problem to a set of dual series. The corresponding equations for a quadratic strain-energy function are given in chapter 5 together with the solution of the dual series equations for short cylinders. The coefficients occurring in the dual series are evaluated numerically for a particular geometry and are used to obtain the displacements and stresses at the surface of the cylinder.

CHAPTER II

THE THEORY OF FINITE ELASTIC DEFORMATIONS

For completeness and clarity we outline here the procedures developed by Green and Zerna (7) and follow their analysis closely. The theory is presented using a general system of co-ordinates and tensor notation.

2.1. STRAIN TENSOR, STRESS TENSOR AND EQUATIONS OF EQUILIBRIUM.

We introduce the definition of the unstrained state of a body, called briefly the body B_0 , as a standard state of uniform temperature and zero displacement, with reference to which all strains are specified. The points P_0 of B_0 are defined at the time $t = 0$ by a rectangular Cartesian co-ordinate system, x_i . B_0 is also described by a system of general curvilinear co-ordinates θ_i , so that

$$x_i = x_i(\theta_1, \theta_2, \theta_3) \quad (2.1.1)$$

We suppose that the points P_0 now move to new positions P , so that at time t they form a body B . The points P of B are defined by a new rectangular Cartesian system y_i and, by considering continuity in the physical situation, we may write

$$y_i = y_i^*(x_1, x_2, x_3; t) \quad (2.1.2)$$

where the functions y_i^* are single-valued, continuously differentiable with respect to x_i and t , and have unique inverses for each value of t . The convected co-ordinate system θ_i moves continuously with the

body and forms a curvilinear system θ_i in B , so that

$$y_i = y_i^*(\theta_1, \theta_2, \theta_3; t) \quad (2.1.3)$$

With respect to the curvilinear system θ_i , covariant and contravariant base vectors $\underline{g}_i$, $\underline{g}^i$ and associated metric tensors g_{ij} , g^{ij} are defined at each point P_0 of B_0 . Corresponding quantities G_i , G^i , G_{ij} and G^{ij} are defined at each point P of B . These tensors satisfy appropriate transformation laws under changes of the coordinates θ_i ; that is, including changes from co-ordinate systems in B to those in B_0 , and vice-versa.

The line element ds_0 of B_0 is given, using the usual summation conventions, by

$$ds_0^2 = dx^i dx^i = g_{ij} d\theta^i d\theta^j \quad (2.1.4)$$

If, as a result of the motion of points of B_0 , the relative positions of neighbouring points of the body are altered, so that the length ds_0 changes to ds , say, the body B_0 is said to be strained or deformed. The line element ds in the strained body B is given by

$$ds^2 = dy^i dy^i = G_{ij} d\theta^i d\theta^j \quad (2.1.5)$$

We assume that the strained state B is obtained from B_0 entirely by the action of external forces. The state of strain of the body B is characterized by the difference of the squared line elements. We write

$$ds^2 - ds_0^2 = 2\gamma_{ij} d\theta^i d\theta^j \quad (2.1.6)$$

where

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \quad (2.1.7)$$

The quantities γ_{ij} are the symmetrical components of a covariant tensor - the strain tensor. When $\gamma_{ij} = 0$ for all points of the body, $ds = ds_0$ and the displacement is rigid. It is shown in Green and Zerna that the covariant components of the strain tensor can also be expressed in terms of a displacement vector $\underline{v}$ with respect to base vectors in B_0 or B . We note here that equations corresponding to the equations of compatibility between the strain components of classical elasticity can be derived from the requirement that for B_0 and B to lie in Euclidean spaces, the components of the Riemann-Christoffel curvature tensor for each body must vanish. This leads to six independent equations which must be satisfied by the strain tensor. If the metric tensors are given as prescribed functions of the co-ordinates θ_i , the components of the strain tensor are then specified and such restrictions on the γ_{ij} ensure the existence of single-valued, continuous components of the displacement vector.

A stress vector $\underline{t}$, measured per unit area of the strained body B , and associated with the element of surface normal to the unit vector $\underline{n}$, can be defined at P in the strained body B , and stress quantities $-t_i$ can be associated with the elementary areas at P in the surfaces $\theta_i = \text{constant}$. Equilibrium requirements then necessitate the condition

$$\underline{t} = \sum_{i=1}^3 n_i \underline{t}_i \sqrt{G^{ii}} \quad (2.1.8)$$

where $\underline{n} = n_i \underline{G}^i$. For a given direction $\underline{n}$, $\underline{t}$, being a physical quantity, is an invariant whilst n_i is a covariant vector. Therefore, by

the Quotient theorem, $t_i \sqrt{G^{ii}}$ is a contravariant tensor of order 1 and we may write

$$t_i \sqrt{G^{ii}} = \tau^{ij} G_j = \tau_j^i G^j, \quad (2.1.9)$$

where τ^{ij} , τ_j^i are the contravariant and mixed components, respectively, of a symmetrical tensor called the stress tensor. The physical stress quantities referred to the θ_i -curves at P in B are

$$\sigma_{(ij)} = \tau^{ij} \sqrt{\frac{G_{jj}}{G^{ii}}} \quad (2.1.10)$$

and are not the components of a tensor.

If the body B is in a state of dynamic equilibrium under the action of a body force F per unit mass, and if f denotes the acceleration of each point of B , then the stress equations of motion can be expressed as

$$\tau^{ij} \parallel_i + \rho F^j = \rho f^j \quad (2.1.11)$$

where
$$F = F^j G_j, \quad f = f^j G_j, \quad (2.1.12)$$

and ρ is the density of the strained body B . The double line represents covariant differentiation with respect to the strained body B , that is, with respect to θ_i and the metric tensors G_{ij} , G^{ij} .

Finally, when the surface force vector P is prescribed at a boundary with normal n ,

$$t = P \quad (2.1.13)$$

where P is measured per unit area of the deformed surface. Expressing P in terms of components referred to base vectors G_i , G^i and using (2.1.8) and (2.1.9) leads to the boundary condition

$$\tau^{ij} n_i = P^j \quad \text{or} \quad \tau_j^i n_i = P_j. \quad (2.1.14).$$

2.2 STRAIN-ENERGY FUNCTION AND STRESS-STRAIN RELATIONS FOR AN ELASTIC BODY.

The above analysis is simply a mathematical characterization of the concepts of stress and strain, which provides functional expressions for these quantities without any assumption on the constitutive nature of the material. It is therefore applicable to all such media that can be represented with sufficient accuracy as continuous bodies. However, we are concerned only with an elastic body and these ideas are now related in the context of an elastic medium. A material is said to be elastic if it is able to recover its original state completely when the forces causing its deformation are slowly relaxed. This implies that when the stress tensor vanishes then so does the strain tensor.

On the basis of the first and second laws of thermodynamics and assuming the existence of an internal energy function depending only on the state of strain and physical properties of the material, Green and Zerna prove the existence of a strain-energy function W per unit volume of the unstrained body, for the cases in which the medium undergoes either isothermal or adiabatic reversible processes. In such cases, they further show that W , which is related to the internal energy function, and the stress and strain tensors are related by the equation

$$\tau^{ij} \dot{\gamma}_{ij} = \sqrt{g/G} \cdot \frac{\partial W}{\partial \gamma_{ij}} \cdot \dot{\gamma}_{ij} \quad (2.2.1)$$

where $g = |g_{ij}|$, $G = |G_{ij}|$ and the dot denotes differentiation with respect to time holding θ_i fixed.

For a compressible elastic body, the $\dot{\gamma}_{ij}$ are independent (except for symmetry requirements) and therefore

$$\tau^{ij} = \sqrt{g/G} \frac{\partial W}{\partial \gamma_{ij}} = \frac{1}{2} \sqrt{g/G} \cdot \left\{ \frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right\} . \quad (2.2.2)$$

When a suitable form for W is assumed, equation (2.2.2) provides a relation between stress and strain for a compressible elastic body.

When the body is incompressible, equation (2.2.1) is still valid but the $\dot{\gamma}_{ij}$ are no longer independent. However, since mass elements are conserved during deformation, it can be shown that $\rho_0 \sqrt{g} = \rho \sqrt{G}$, where ρ_0 is the density of B_0 . For an incompressible material $\rho_0 = \rho$, so that

$$G = g \quad (2.2.3)$$

at all points of the body. Thus

$$\dot{G} = \frac{\partial G}{\partial \gamma_{ij}} \cdot \dot{\gamma}_{ij} = 0 . \quad (2.2.4)$$

Now, from (2.1.7), $G = |G_{ij}| = |g_{ij} + 2\gamma_{ij}|$, so that

$$\frac{\partial G}{\partial \gamma_{ij}} = 2 \frac{\partial G}{\partial G_{ij}} = 2G G^{ij} .$$

The condition (2.2.4) becomes

$$G^{ij} \dot{\gamma}_{ij} = 0 . \quad (2.2.5)$$

Hence, for an incompressible material, equation (2.2.1) holds for all values of the $\dot{\gamma}_{ij}$ which satisfy (2.2.5). The coefficients of the $\dot{\gamma}_{ij}$ in (2.2.1) and (2.2.5) are therefore proportional so that

$$\tau^{ij} = \frac{1}{2} \left\{ \frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right\} + p G^{ij} , \quad (2.2.6)$$

where p is an arbitrary scalar invariant function of the co-ordinates θ_i for each value of the time t .

If we now assume the body to be homogeneous and isotropic, the strain-energy function W is an invariant function of the three invariants

of the second order strain tensor γ_{ij} . We therefore write

$$W = W(I_1, I_2, I_3) \quad (2.2.7)$$

where I_1, I_2, I_3 are strain invariants defined by

$$\left. \begin{aligned} I_1 &= 3 + 2\gamma^r_r = g^{rs} G_{rs} \\ I_2 &= 3 + 4\gamma^r_r + 2(\gamma^r_r \gamma^r_s - \gamma^r_s \gamma^s_r) = G^{rs} g_{rs} I_3 \\ I_3 &= |\delta^r_s + 2\gamma^r_s| = G/g \end{aligned} \right\} \quad (2.2.8)$$

The incompressibility condition can thus be expressed as $I_3 = 1$.

Under the above assumptions, the stress-strain relations (2.2.2)

for a compressible body take the form

$$\tau^{ij} = \phi g^{ij} + \psi B^{ij} + p G^{ij}, \quad (2.2.9)$$

$$\left. \begin{aligned} \text{where} \quad \phi &= \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}, \quad \psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3} \frac{\partial W}{\partial I_3} \\ \text{and} \quad B^{ij} &= I_1 g^{ij} - g^{ir} g^{js} G_{rs} \end{aligned} \right\} \quad (2.2.10)$$

If the material is also incompressible, $I_3 = 1$ so that W is a function of I_1 and I_2 only. The stress-strain relations (2.2.6) also take the form

$$\tau^{ij} = \phi g^{ij} + \psi B^{ij} + p G^{ij},$$

$$\text{but now} \quad \phi = \frac{2\partial W}{\partial I_1}, \quad \psi = \frac{2\partial W}{\partial I_2} \quad (2.2.11)$$

and p is the arbitrary function introduced in (2.2.6). It arises from the fact that the material, being incompressible, is able to withstand a hydrostatic pressure without distortion and its value at the surface of the body is deduced from the prevailing stress conditions.

The usual method of procedure when solving problems in finite elasticity is to assume that the strained body B is obtained from B_0 by suitable displacements which are defined by relationships between the co-ordinate systems used to describe the initial and final states. The system of forces required to maintain this deformation is then found from the stress-strain relations and equilibrium condition. Unfortunately, the class of problems which lends itself readily to the above approach is limited owing to the non-linearity of the equations of equilibrium and, when it applies, the incompressibility condition. However, by restricting the types of deformation and also the geometry of the body in the initial and final states, the equations of equilibrium and the incompressibility condition reduce to ordinary differential equations in a single variable. Even after simplification to this extent, however, much useful information is obtained.

CHAPTER III

THE PERTURBATION OF AN ELASTIC SOLID CYLINDER
OF FINITE LENGTH UNDER A UNIFORM FINITE COMPRESSION.

We illustrate the procedures of the previous chapter in our initial analysis which establishes the uniform finite compression of an elastic isotropic incompressible solid cylinder under an axial load.

3.1 THE UNIFORM FINITE COMPRESSION.

For describing the case of a cylinder undergoing uniform compression it is convenient to introduce a set of cylindrical polar coordinates (r, θ, z) as the external reference frame. The convected coordinates θ_i are then chosen so that they coincide with the cylindrical polars in the deformed state.

With reference to the external frame, a point at (R, θ, Z) in the undeformed body moves to (r, θ, z) owing to the deformation, where $R = f(r)$ and $Z = g(z)$. In terms of the convected co-ordinates θ_i , it follows that $r = \theta_1$, $\theta = \theta_2$ and $z = \theta_3$ in the deformed body and $R = f(\theta_1)$, $\theta = \theta_2$ and $Z = g(\theta_3)$ in the undeformed state. The metric tensors are thus given by

$$G_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \theta_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad G^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/\theta_1^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.1.1)$$

$$g_{ij} = \begin{bmatrix} \left(\frac{df}{d\theta_1}\right)^2 & 0 & 0 \\ 0 & f^2 & 0 \\ 0 & 0 & \left(\frac{dg}{d\theta_3}\right)^2 \end{bmatrix} \quad g^{ij} = \begin{bmatrix} \left(\frac{df}{d\theta_1}\right)^{-2} & 0 & 0 \\ 0 & f^{-2} & 0 \\ 0 & 0 & \left(\frac{dg}{d\theta_3}\right)^{-2} \end{bmatrix} \quad (3.1.2)$$

With the above tensors, the incompressibility condition requires that

$$\theta_1^2 = \left(\frac{df}{d\theta_1}\right)^2 \cdot f^2 \left(\frac{dg}{d\theta_3}\right)^2.$$

That is

$$\frac{\theta_1}{f \cdot df/d\theta_1} = \frac{dg}{d\theta_3} = A, \text{ where } A \text{ is a constant.} \quad (3.1.3)$$

Integrating (3.1.3) gives $g = A\theta_3$ and $\theta_1^2 = Af^2$. Thus the incompressibility condition reduces to $r^2 z = \text{constant}$ which is just the constant volume condition using elementary considerations.

The cylinder has initial height $2h_0$ and diameter $2a_0$ and is compressed by a uniform axial load until its dimensions are $2h$ and $2a$. The compression ratio is thus given by $\lambda = h/h_0$ and it follows that $A = 1/\lambda$. The metric tensors in the undeformed state can now be expressed

as

$$g_{ij} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda \theta_1^2 & 0 \\ 0 & 0 & 1/\lambda^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda \theta_1^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

The strain invariants (2.2.8) are found as

$$I_1 = 2/\lambda + \lambda^2, \quad I_2 = 2\lambda + 1/\lambda^2 \quad (3.1.4)$$

and the tensor B^{ij} given in (2.2.10) becomes

$$B^{ij} = \begin{bmatrix} 1/\lambda^2 + \lambda & 0 & 0 \\ 0 & B^{11}/\theta_1^2 & 0 \\ 0 & 0 & 2\lambda \end{bmatrix} \quad (3.1.5)$$

The non-zero stresses (2.2.9) are

$$\begin{aligned}\tau^{11} &= \phi/\lambda + \psi(1/\lambda^2 + \lambda) + p, \quad \tau^{22} = \tau^{11}/\theta_1^2, \\ \tau^{33} &= \phi\lambda^2 + 2\psi\lambda + p.\end{aligned}$$

In this section the material is restricted to be of the Mooney form, that is, it is incompressible and has a strain energy function per unit volume given by

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \quad (3.1.6)$$

where C_1 and C_2 are material constants. From (2.2.11), we have $\phi = 2C_1$ and $\psi = 2C_2$. With no body forces or acceleration term, the stress equations of equilibrium (2.1.11) are satisfied identically for these values of ϕ and ψ . For the Mooney material we thus have

$$\begin{aligned}\tau^{11} &= \frac{2}{\lambda}(\lambda C_1 + C_2) + 2\lambda C_2 + p, \quad \tau^{22} = \tau^{11}/\theta_1^2, \\ \tau^{33} &= 2\lambda(\lambda C_1 + C_2) + p.\end{aligned}$$

In the deformed state there is no stress on the curved surface of the cylinder. From (2.1.14) this condition requires that $\tau^{11} = 0$, that is

$$p = -2\lambda C_2 - \frac{2}{\lambda}(\lambda C_1 + C_2).$$

The only non-zero component of the stress tensor is τ^{33} which is now given by

$$\tau^{33} = 2(\lambda^3 - 1)(C_1/\lambda + C_2/\lambda^2). \quad (3.1.7)$$

Using (2.1.10) this is also the only non-zero physical stress in the body.

The extension ratio λ is defined above in terms of the cylinder height before and after deformation. From the incompressibility condition,

$$a_{00}^2 h_0 = a_h^2, \quad \text{so that}$$

$$\lambda = h/h_0 = a_0^2/a^2 . \quad (3.1.8)$$

The total compressive load S is found by integrating (3.1.7) across the deformed surface $z = h$. Since τ^{33} is a constant, this integration gives

$$S = 2\pi a^2(1 - \lambda^3)(C_1/\lambda + C_2/\lambda^2) . \quad (3.1.9)$$

Equation (3.1.9) gives the total axial load required to compress a cylindrical specimen of initial radius a_0 to final radius a and maintain this compression. We note that this solution applies when there is no friction at the ends of the cylinder.

3.2 THE PERTURBATION IN TERMS OF DISPLACEMENT POTENTIAL FUNCTIONS.

The nature of the above uniform solution indicates that this problem is a special case of the general class of problems discussed originally by Green, Rivlin and Shield (6) and set out in Green and Zerna (7, §4). Following the method given in (7), a small non-dimensional parameter ε is introduced to describe the perturbation. With respect to the external reference frame, the displacement perturbation is denoted by $\varepsilon \underline{u} = \varepsilon(u, o, w)$, where u and w are respectively the radial and axial components of displacement and the circumferential displacement is identically zero. Displacement potential functions $\phi_1(r, z)$ and $\phi_2(r, z)$ are now introduced such that

$$u = \frac{\partial \phi_1}{\partial r} , \quad w = \frac{\partial \phi_2}{\partial z} . \quad (3.2.1)$$

The equations of equilibrium then show that it is sufficient to consider solutions of

$$\left. \begin{aligned} p' + d_{55} \nabla^2 \phi_1 + d_{44} \frac{\partial^2 \phi_1}{\partial z^2} &= 0 \\ p' + d_{55} \nabla^2 \phi_2 + d_{44} \frac{\partial^2 \phi_2}{\partial z^2} &= 0 \end{aligned} \right\} (3.2.2)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$, and where the constants d_{44} and d_{55} are given by

$$d_{44} = 2\lambda^2 (C_1 + C_2/\lambda), \quad d_{55} = d_{44}/\lambda^3. \quad (3.2.3)$$

The scalar function of position $\epsilon p'$ is the counterpart of p and denotes the perturbational hydrostatic pressure associated with the stresses in an incompressible material.

In addition, the incompressibility condition requires that $\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$. In terms of ϕ_1 and ϕ_2 this equation gives

$$\nabla^2 \phi_1 + \frac{\partial^2 \phi_2}{\partial z^2} = 0. \quad (3.2.4)$$

The perturbation in the stress tensor, resulting from the displacement perturbation is denoted by $\epsilon \tau^{ij}$. In terms of the displacement the stress increments are

$$\left. \begin{aligned} \tau^{11} &= p' + \alpha \frac{\partial u}{\partial r} + \beta \frac{u}{r}, \quad \tau^{22} = p' + \beta \frac{\partial u}{\partial r} + \alpha \frac{u}{r} \\ \tau^{13} &= d_{55} \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right), \quad \tau^{33} = p' + 2d_{55} \frac{\partial w}{\partial z} \end{aligned} \right\} (3.2.5)$$

where $\alpha = 2d_{55}$, $\beta = 4C_2(1 - \lambda^3)/\lambda^2$,

We further suppose that equilibrium is maintained by the addition of a hydrostatic pressure $\epsilon p'_0$, where p'_0 is a constant. This pressure arises from a particular integral

$$\phi_1^0 = -p'_0 \frac{z^2}{2d_{44}}, \quad \phi_2^0 = -p'_0 \frac{r^2}{2d_{44}} \quad (3.2.6)$$

of equation (3.2.2) and produces no distortion but gives rise to stresses

$$\tau'^{11} = r^2 \tau'^{22} = \tau'^{33} = p'_0 . \quad (3.2.7)$$

The perturbational surface forces are expressed in terms of the displacements (7; (4.1.23)) by

$$n_i (\tau'^{ij} + \tau'^{im} \omega^j{}_m) = p'^j , \quad (3.2.8)$$

where $\underline{n} = n_i \underline{G}^i$ is the outward normal with respect to θ_i in the uniformly deformed state, and $\epsilon \underline{P}' = p'^j \underline{G}_j$ is the perturbational surface force.

Eliminating p' from (3.2.2) and using (3.2.3) it follows that

ϕ_1 and ϕ_2 are related through the equations

$$\nabla^2 \chi + \lambda^3 \frac{\partial^2 \chi}{\partial z^2} = 0 , \quad (3.2.9)$$

where

$$\chi = \phi_1 - \phi_2 . \quad (3.2.10)$$

In terms of χ and ϕ_2 , the incompressibility condition (3.2.4) may be written as

$$\nabla^2 \chi + \nabla^2 \phi_2 + \frac{\partial^2 \phi_2}{\partial z^2} = 0 . \quad (3.2.11)$$

Equations (3.2.9), (3.2.10) and (3.2.11) must be solved generally for ϕ_1 and ϕ_2 . The function p' may then be found from either of (3.2.2) and the complete solution is then obtained by addition of (3.2.6). The general solution is then specialized by consideration of the appropriate boundary conditions.

3.3 INCOMPRESSIBLE CYLINDER UNDER FINITE COMPRESSION AND PRESCRIBED SURFACE LOADING.

The cylinder, which is initially stress-free in the absence of external forces, is subject to a finite uniform compression between flat frictionless plates. A small perturbation is then introduced corresponding to a pressure distribution, represented by the even function $\epsilon p(z)$, over the curved surface $r = a$. We further assume a state of zero shear stress over the curved surface for the whole length of the cylinder.

The boundary conditions over the curved surface $r = a$ lead in general to three conditions on the stress tensor and these are expressed by equation (3.2.8). With (3.2.5) and (3.2.7) and eliminating p' using (3.2.2), it follows that

$$p'_0 - d_{55} \left(\frac{\partial^2 \phi_2}{\partial z^2} + \lambda^3 \frac{\partial^2 \phi_1}{\partial z^2} + 2 \frac{\lambda^3}{v^3} \cdot \frac{1}{r} \frac{\partial \phi_1}{\partial r} \right) = p(z) \quad (3.3.1)$$

at $r = a$, for $-h \leq z \leq h$, where $v^3 = \lambda^2 (\lambda C_1 + C_2) / (C_1 + \lambda^2 C_2)$. Equation (3.3.1) expresses the condition that the physical stress normal to the surface $r = a$ be prescribed.

The condition that the shear stress vanishes on this surface is that

$$\frac{\partial^2 \phi_2}{\partial r \partial z} + \frac{\partial^2 \phi_1}{\partial r \partial z} = 0 \quad \text{at } r = a, -h \leq z \leq h. \quad (3.3.2)$$

Furthermore, there are no frictional forces at the top and bottom load surfaces. From (3.2.8) the condition is

$$\frac{\partial^2 \phi_2}{\partial r \partial z} + \lambda^3 \frac{\partial^2 \phi_1}{\partial r \partial z} = 0 \quad \text{at } z = \pm h, 0 \leq r \leq a. \quad (3.3.3)$$

Finally, we impose the boundary condition that there is no longitudinal displacement at the load surfaces, that is

$$w = 0 \quad \text{at } z = \pm h, 0 \leq r \leq a. \quad (3.3.4)$$

Physically, this condition means that the perturbational surface force merely changes the axial thrust necessary to maintain this height.

Solutions of equations (3.2.8), (3.2.9) and (3.2.10) for which surface tractions vanish everywhere and which also satisfy the conditions of zero displacements at the load surfaces have previously been used by Vaughan (20). The appropriate solutions in this case are

$$\begin{aligned}\phi_1(r, z) &= \phi_1^0 + \sum_{n=1}^{\infty} B_n \left\{ I_0(\beta_n r) - \frac{2I_1(\beta_n a)I_0(\alpha_n r)}{\lambda^{3/2}(1+\lambda^3)I_1(\alpha_n a)} \right\} \cos \beta_n z, \\ \phi_2(r, z) &= \phi_2^0 + \sum_{n=1}^{\infty} B_n \left\{ I_0(\beta_n r) - \frac{2\lambda^{3/2}I_1(\beta_n a)I_0(\alpha_n r)}{(1+\lambda^3)I_1(\alpha_n a)} \right\} \cos \beta_n z,\end{aligned}\quad (3.3.5)$$

where B_n are arbitrary constants, $\beta_n = n\pi/h$, $\alpha_n = \lambda^{3/2} \beta_n$, I_0 and I_1 are the modified Bessel functions of the first kind, and z is measured from the mid-height of the cylinder. The functions ϕ_1^0 and ϕ_2^0 are given in (3.2.6).

From (3.2.1) and (3.3.5) it follows, as mentioned above, that $w = 0$ at the load surfaces $z = \pm h$. It remains to satisfy the boundary conditions on the curved surface of the cylinder on which the stress distribution is specified. This is obtained from equation (3.3.1). The above expressions for ϕ_1 and ϕ_2 necessitate expanding the pressure function $p(z)$, which we recall is even, as a Fourier cosine series. Assuming that $p(z)$ satisfies the necessary conditions (see, for example, (8, p.507)) we may write

$$p(z) = \frac{1}{2} \overline{a_0} + \sum_{n=1}^{\infty} a_n \cos \beta_n z, \quad (3.3.6)$$

where

$$a_n = \frac{2}{h} \int_0^h p(z) \cos \beta_n z dz, \quad n = 0, 1, 2, 3, \dots \quad (3.3.7)$$

The series (3.3.6) converges to the left hand side at all points of continuity of $p(z)$. With this expansion, the coefficients B_n are obtained by substituting (3.3.5) and (3.3.6) into (3.3.1). Carrying out these substitutions gives

$$p'_0 = \frac{1}{2} \overline{a_0} \quad (3.3.8)$$

$$B_n I_1(\beta_n a) \beta_n (1+\lambda^3) \beta_n \left\{ \frac{I_0(\beta_n a)}{I_1(\beta_n a)} - \frac{4\lambda^{3/2} I_0(\alpha_n a)}{(1+\lambda^3)^2 I_1(\alpha_n a)} + \frac{2\lambda^3 (1-\lambda^3)}{r^3 (1+\lambda^3) a} \right\} = \frac{\lambda^2 a_n}{2(\lambda C_1 + C_2)} \quad (3.3.9)$$

For any given incompressible material, the constants C_1 and C_2 are usually known. We note that a simple compression test based on equation (3.1.9) can be used to obtain these material constants if they are otherwise unknown. When the pressure function $p(z)$ is specified, its Fourier coefficients can be determined whence, for a given extension ratio λ , values for B_n are obtained from (3.3.9). The physical stresses produced by the superposed surface force are given in Green and Zerna, equation (4.5.9). In polar co-ordinates they take the form

$$\begin{aligned} \sigma_r &= \epsilon \left[-\frac{1}{2} \overline{a_0} + p' + 2d_{55} \frac{\partial^2 \phi_1}{\partial r^2} + \frac{\beta}{r} \frac{\partial \phi_1}{\partial r} \right] , \\ \sigma_\theta &= \epsilon \left[-\frac{1}{2} \overline{a_0} + p' + \beta \frac{\partial^2 \phi_1}{r^2} + \frac{2d_{55}}{r} \frac{\partial \phi_1}{\partial r} \right] , \\ \sigma_z &= \epsilon \left[-\frac{1}{2} \overline{a_0} + p' + 2d_{44} \frac{\partial^2 \phi_2}{\partial z^2} \right] , \\ \sigma_{rz} &= \epsilon \left[d_{44} \frac{\partial^2 \phi_1}{\partial r \partial z} + d_{55} \frac{\partial^2 \phi_2}{\partial r \partial z} \right] , \end{aligned} \quad (3.3.10)$$

where p' is given by (3.3.2).

The perturbational stresses and displacements can now be obtained by summing the appropriate series obtained by substituting the values for ϕ_1 and ϕ_2 into the appropriate equations of (3.2.1) and (3.3.10). The

total compressive load transmitted across a surface $z = \text{constant}$ is obtained in Appendix A.1 and is given by

$$S^* = S - \epsilon \pi a^2 p'_0, \quad (3.3.11)$$

where S is defined by (3.1.9) and is the load required to compress the material by an amount λ in the absence of the surface pressure distribution. From (3.3.8) it follows that

$$S^* = S - \frac{1}{2} \epsilon \pi a^2 \overline{a_0}, \quad (3.3.12)$$

so that the nature of the change in the axial thrust is determined by the sign of $\overline{a_0}$. We see that in the special cases when $\overline{a_0} = 0$, the axial thrust does not alter, so the applied stress distribution merely changes the geometry of the curved surface of the cylinder. This is not surprising as such cases occur for those functions $p(z)$ which satisfy

$$\int_0^h p(z) dz = 0. \quad (3.3.13)$$

This simply states that there is no resultant force applied to the curved surface.

At this stage, we recall that the purpose of this section is merely to give an indication of the nature of the displacements produced by a given surface pressure distribution. Having outlined above how physical quantities can be obtained for the cylinder under finite compression, we continue with the particular case for which the pre-stress is infinitesimal. Accordingly, we may express λ in the form

$$\lambda = 1 + \delta, \text{ where } \delta \ll 1. \quad (3.3.14)$$

Both the initial compression of the cylinder and the prescribed surface stress distribution are now perturbational so they can be evaluated independently and superposed, as in the classical theory,

to give the final state of the cylinder. It may be shown that a Mooney material behaves as a classical elastic material (a material for which W takes a quadratic form) for infinitesimal deformations if $6(C_1 + C_2)$ is identified with the Young modulus, E . Neglecting terms which would lead to products of order $O(\epsilon\delta)$, we find from (3.3.5) and (3.3.9) that under such conditions, ϕ_1 is given by

$$\begin{aligned} \phi_1(r, z) = \phi_1^0 - \frac{3a}{2E} \sum_{n=1}^{\infty} \frac{a_n}{\beta_n D(n) I_1^2(\beta_n a)} \{ \beta_n a I_0(\beta_n r) I_0(\beta_n a) \\ - \beta_n r I_1(\beta_n r) I_1(\beta_n a) + I_1(\beta_n a) I_0(\beta_n r) \} \cos(\beta_n z), \end{aligned} \quad (3.3.15)$$

$$\text{where} \quad D(n) = \{1 - \beta_n^2 a^2 [I_0^2(\beta_n a) / I_1^2(\beta_n a) - 1]\}. \quad (3.3.16)$$

This leads to an expression for the radial displacement

$$\begin{aligned} u(r, z) = - \frac{3a}{2E} \sum_{n=1}^{\infty} \frac{a_n}{D(n) I_1^2(\beta_n a)} \{ \beta_n a I_0(\beta_n a) I_1(\beta_n r) \\ - \beta_n r I_0(\beta_n r) I_1(\beta_n a) + I_1(\beta_n r) I_1(\beta_n a) \} \cos \beta_n z. \end{aligned} \quad (3.3.17)$$

Using equations (3.3.5), (3.3.9) and (3.3.10), we obtain similar expressions for ϕ_2 , the axial displacement and the stresses. It is easily shown that the above results, derived by a limiting process from the 'finite' solution, are merely special cases with Poisson's ratio $\sigma = \frac{1}{2}$ of corresponding quantities which could have been obtained by applying the results (5.1.8a), (5.1.8b) and (5.1.9) of the next chapter to the present problem. More general results, applicable to compressible materials, could thus have been formulated.

At the surface of the cylinder, the radial displacement is found by setting $r = a$ in (3.3.17), giving

$$u(a, z) = -\frac{3a}{2E} \sum_{n=1}^{\infty} \frac{a_n}{D(n)} \cos \beta_n z. \quad (3.3.18)$$

We consider the convergence of the series (3.3.18) to determine a suitable approach for computing numerical results. When $\beta_n a$ is fairly large (say > 10), I_0 and I_1 may be replaced by their semi-convergent expansions

$$I_0(x) = \sqrt{\frac{1}{2\pi x}} \cdot e^x \left\{ 1 + \frac{1}{8x} + \frac{9}{2(8x)^2} + O\left(\frac{1}{x^3}\right) \right\}, \quad (3.3.19)$$

$$I_1(x) = \sqrt{\frac{1}{2\pi x}} \cdot e^x \left\{ 1 - \frac{3}{8x} - \frac{15}{2(8x)^2} + O\left(\frac{1}{x^3}\right) \right\}. \quad (3.3.20)$$

We then find that

$$D(n) = -\beta_n a + O\left(\frac{1}{\beta_n a}\right) \text{ for large } n$$

so we rewrite (3.3.18) as

$$u(a, z) = \frac{3}{2E} \sum_{n=1}^{\infty} \frac{a_n}{\beta_n} \cos \beta_n z - \frac{3}{2E} \sum_{n=1}^{\infty} \frac{a_n}{\beta_n D(n)} \{\beta_n a + D(n)\} \cos \beta_n z. \quad (3.3.21)$$

The coefficients of $\cos \beta_n z$ in the second sum are order $O\left(\frac{a_n}{\beta_n^3}\right)$ so convergence is fairly rapid. Those of the first series are $O\left(\frac{a_n}{\beta_n}\right)$.

Thus, if $p(z)$ is integrable over $(0, h)$ we have, from the Riemann-Lebesgue lemma, that $a_n = O(n^{-\mu})$ where $\mu > 0$, and it follows that the first series will converge uniformly and absolutely to a continuous function. The physical requirement of a continuous displacement is therefore satisfied. The rate of convergence of the first series depends on the magnitude of μ and analytic methods are often needed to evaluate its sum.

3.4 NUMERICAL EXAMPLES.

In each of the following examples we consider a cylinder for which $h/a = 4$. Numerical results are obtained for pressure distributions $p(z)$ acting over the band: $-t < z < t$, where $t = h/4$. For the first case we consider a constant pressure band acting on the cylinder, and for the second, a distribution with parabolic profile; both are scaled to give a total force of magnitude P /unit length exerted on the surface of the cylinder across the band.

A. The Constant Pressure Band.

The pressure distribution is given by

$$p(z) = \begin{cases} -P/2t : |z| < t \\ 0 & |z| > t \end{cases}, \quad P \text{ constant.} \quad (3.4.1)$$

The Fourier coefficients for $p(z)$ are thus

$$\overline{a}_0 = \frac{2}{h} \int_0^h p(z) dz = -P/h. \quad (3.4.2)$$

$$a_n = \frac{2}{h} \int_0^h p(z) \cos\left(\frac{n\pi z}{h}\right) dz = -\frac{P}{n\pi t} \sin\left(\frac{n\pi t}{h}\right).$$

We see that the coefficient $\frac{1}{2} \overline{a}_0$ represents a uniform pressure distribution of total magnitude P acting over the whole length of the curved surface. Using (3.3.21), we find the radial displacement at the surface of the cylinder from

$$u(a, z) = \frac{3P}{2E(t/h)} \left[\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi t}{h}\right) \cos(n\pi \tau)}{n^2 \pi^2} - \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi t}{h}\right) \left\{ \frac{n\pi a}{h} + D(n) \right\}}{n^2 \pi^2 D(n)} \cos(n\pi \tau) \right] \quad (3.4.3)$$

where $\tau = z/h$

$$\text{Now, } \sin\left(\frac{n\pi t}{h}\right) \cos(n\pi \tau) = \frac{1}{2} [\sin(n\pi(t/h + \tau)) + \sin(n\pi\{\frac{t/h - \tau}{\tau - t/h}\})] ,$$

$$\begin{matrix} \tau < t/h \\ \tau > t/h \end{matrix} \quad (3.4.4)$$

so to evaluate the first sum, we make use of the known result (5)

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\log|2 \sin x/2| , \quad 0 < x < 2\pi \quad (3.4.5)$$

Furthermore,

$$\sum_{k=1}^n \sin(kx) = \frac{\cos x/2 - \cos(n+\frac{1}{2})x}{2 \sin x/2} , \quad 0 < x < 2\pi$$

so that integrating between the limits x and π , we find

$$|S_n(x)| = \left| \sum_{k=1}^n \frac{\cos(kx)}{k} \right| = \left| \int_x^{\pi} \frac{\cos x/2 - \cos(n+\frac{1}{2})x}{2 \sin x/2} dx + \sum_{k=1}^n \frac{(-1)^k}{k} \right|$$

$$\leq \left| \int_x^{\pi} \operatorname{cosec} x/2 dx \right| + C \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \right| = 2|\log|\tan x/4|| + C \log 2 \quad (3.4.6)$$

which is finite for $0 < x < 2\pi$. C is a sufficiently large, but finite, constant.

Thus the partial sums of (3.4.5) are bounded for $0 < x \leq \pi$.

We also note that $\log|\sin x/2|$ is integrable over any finite interval, even those including 0 and multiples of 2π . We now consider integrating both sides of relations of the form (3.4.5) between suitable limits with evaluation of the terms in (3.4.4) in mind. Using the above results, and treating individually the cases $\tau = 0$, $0 < \tau < t/h$, $\tau = t/h$ and $t/h < \tau \leq 1$, the Lebesgue dominated convergence theorem permits us to invert the order of summation and integration, so that for $0 < t < h$ we have

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi t}{h}\right) \cos(n\pi\tau) = \begin{cases} -\frac{1}{2} \left(\int_0^{\beta} \log\left|\sin\left(\frac{\alpha+x}{2}\right)\right| dx - \int_0^{\beta} \log\left|\sin\left(\frac{\alpha-x}{2}\right)\right| dx \right) \\ -2 \int_0^{\alpha/2} \log|\sin x| dx - \alpha \log 2 & \alpha \neq \beta \\ -\int_0^{\alpha} \log|\sin x| dx - \alpha \log 2 & \alpha = \beta \end{cases}$$

where $\alpha = \frac{\pi t}{h}$, $\beta = \pi\tau$.

After integration by parts the above expressions can be evaluated in terms of known functions and the integral

$$f(x) = \int_0^x y \cot y \, dy, \quad (3.4.7)$$

which was evaluated numerically.

The coefficients of the second series in (3.4.1) are $O(n^{-4})$ so convergence is rapid and we need sum only 20 terms to give accuracy to four significant figures. The numerical results obtained for the radial displacement are illustrated by the curve in figure 1.

B. Parabolic Pressure Band.

The pressure distribution here is given by

$$p(z) = \begin{cases} \frac{P}{\pi\sqrt{t^2 - z^2}} & |z| < t \\ 0 & |z| > t \end{cases} \quad (3.4.8)$$

Since the integral $\int_{-h}^h |p(z)| dz$ exists we may formally define

the Fourier coefficients

$$\begin{aligned} \overline{a_0} &= \frac{2}{h} \int_0^h p(z) dz = -P/h \\ a_n &= \frac{2}{h} \int_0^h p(z) \cos\left(\frac{n\pi z}{h}\right) dz = \frac{-2P}{\pi h} \int_0^t \frac{\cos\left(\frac{n\pi z}{h}\right)}{\sqrt{t^2 - z^2}} dz \end{aligned}$$

$$= -\frac{2P}{\pi h} \int_0^{\pi/2} \cos\left(\frac{n\pi t}{h} \sin \theta\right) d\theta = -\frac{P}{h} J_0\left(\frac{n\pi t}{h}\right) .$$

where J_0 is Bessel's function of the first kind of order zero and is given by

$$J_0(z) = \frac{2}{\pi} \int_0^{\pi/2} \cos(z \sin \theta) d\theta . \quad (3.4.9)$$

However, $p(z)$ is uniformly continuous and of bounded variation in any closed subinterval of $(-h, h)$ excluding the points $z = \pm t$. The series $\frac{1}{2} \overline{a_0} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ will therefore converge (8, p. 509) to $p(z)$ in such subintervals, and so reflects the behaviour of $p(z)$ at values of z arbitrarily close to $\pm t$.

From (3.3.21), the radial displacement of the surface of the cylinder under the above pressure distribution is found as

$$u(a, z) = -\frac{3P}{2E} \left[\sum_{n=1}^{\infty} \frac{J_0\left(\frac{n\pi t}{h}\right) \cos(n\pi \tau)}{n\pi} - \sum_{n=1}^{\infty} \frac{J_0\left(\frac{n\pi t}{h}\right) \left\{ \frac{n\pi a}{h} + D(n) \right\}}{n\pi D(n)} \cos(n\pi \tau) \right] . \quad (3.4.10)$$

The terms of the second series are $O(n^{-7/2})$ for large n , and so converge fast enough for direct summation methods. Those of the first series, however are only $O(n^{-3/2})$ so we evaluate this sum by integral methods. From (3.4.9) we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{n\pi t}{h}\right) \cos n\pi \tau}{n\pi} &= \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi/2} \cos\left(\frac{n\pi t}{h} \sin \theta\right) \frac{\cos(n\pi \tau)}{n} d\theta \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\pi/2} \left\{ \cos \frac{n\pi}{h} (t \sin \theta + z) + \cos \frac{n\pi}{h} (t \sin \theta - z) \right\} d\theta . \end{aligned} \quad (3.4.11)$$

Since $\sum_{k=1}^n \frac{1}{k} \rightarrow 1$ as $n \rightarrow \infty$, we see that the inequality in (3.4.6)

in fact holds for $-2\pi < x < 2\pi$. Furthermore, the function $\log|\tan x/4|$ is absolutely integrable over an interval $-2\pi + \delta \leq x \leq 2\pi - \delta$, for any $\delta > 0$. By comparison, and provided $0 < t < h$, it is easily shown

that the partial sums $\sum_{k=1}^n \frac{1}{k} \cos \frac{k\pi}{h} (t \sin \theta + z)$ are bounded by an integrable function for $0 \leq \theta \leq \pi/2$ and for $0 \leq z \leq \pi$. By the Lebesgue dominated convergence theorem, we may invert the order of integration and summation in (3.4.11) and, using (3.4.5) we find

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{J_0\left(\frac{n\pi t}{h}\right) \cos(n\pi\tau)}{n\pi} &= -\frac{1}{2} \int_0^{\pi/2} \log\left|2 \sin\left[\frac{\pi}{h}(t \sin \theta + z)\right]\right| d\theta \\ &- \frac{1}{2} \int_0^{\pi/2} \log\left|2 \sin\left[\frac{\pi}{h}(t \sin \theta - z)\right]\right| d\theta. \end{aligned} \quad (3.4.12)$$

We observe that when $0 \leq z \leq t$, the second integrand is singular for θ satisfying: $\theta = \sin^{-1}\left(\frac{z}{t}\right)$.

It was necessary to evaluate numerically the integrals in (3.4.12). However, for certain values of τ , the method of summation described in Appendix A.4 was applied to the series in (3.4.12) and the results were found to agree. Figure 2 illustrates the surface deformation for this pressure distribution.

We may use these above results to anticipate the radial stress distribution produced when a constant displacement is prescribed over part of the curved surface. From figures 1 and 2 we see that although a constant pressure band might be adequate to describe the radial stress near the centre of a uniform constriction, it does not reproduce the required effects near the edge where we would expect the displacement to be angular.

Fig. 1. The variation of the surface radial displacement with z/h for the cylinder under a parabolic pressure band.

Fig. 2. The variation of the surface radial displacement with z/h for the cylinder under a uniform pressure band.

This property is however reflected in the case of the parabolic band, for which the resulting displacements are non-differentiable at the edge of the band, $z = t$. We might therefore expect a pressure distribution which is uniform across part of the band, tending towards an infinite discontinuity at the edges to be appropriate for describing the case of a finite cylinder subject to a constant radial displacement over a central band.

Before proceeding to discuss this latter problem, it is of interest to evaluate the shear stresses within the cylinder. From (3.3.10), (3.3.15) this is given for the parabolic band by

$$\sigma_{rz} = -\bar{P}\pi^2 \sum_{n=1}^{\infty} \frac{\left(\frac{a}{h}\right)^2 n^2}{D(n)} \cdot \frac{I_1\left(\frac{n\pi a}{h} \bar{r}\right)}{I_1\left(\frac{n\pi a}{h}\right)} \left[\frac{I_0\left(\frac{n\pi a}{h}\right)}{I_1\left(\frac{n\pi a}{h}\right)} - \bar{r} \frac{I_0\left(\frac{n\pi a}{h} \bar{r}\right)}{I_1\left(\frac{n\pi a}{h}\right)} \right] J_0\left(\frac{n\pi t}{h}\right) \sin(n\pi \tau) \quad (3.4.13)$$

where $\bar{P} = P/h$ and $\bar{r} = r/a$.

The convergence of the series is assured for $\bar{r} < 1$ by the term

$$e^{-\frac{n\pi a}{h}(1-\bar{r})} \quad \text{which appears in the asymptotic expansion of } \frac{I_1\left(\frac{n\pi a}{h} \cdot \bar{r}\right)}{I_1\left(\frac{n\pi a}{h}\right)}.$$

The results are illustrated in figure 3.

We notice that the shape of successive curves in figure 3 suggests that, as we approach the surface of the cylinder, the maximum and minimum values of the stress, to the right and left of $z = 0.25h$ respectively, might lead to some sort of discontinuity. In other words, though we have formally chosen our constants to make the shear stress vanish at $r = a$, the limit of the series (3.4.13) for σ_{rz} is not zero when r approaches a . We examine the series to determine the behaviour at $r = a$.

By considering the expansions (3.3.19), (3.3.20) for I_0 and I_1 , and making the substitutions

$$1 - \bar{r} = \delta, \quad \frac{n\pi a}{h} \cdot \bar{r} = k - d, \quad \text{where } k = \frac{n\pi a}{h}, \quad d = k\delta$$

successive coefficients of $J_0\left(\frac{n\pi t}{h}\right)\sin(n\pi\tau)$ in (3.4.13) are of the orders

$$de^{-d}, \quad \frac{de^{-d}}{n^2}, \quad \frac{de^{-d}}{n^3}, \quad \dots \quad (3.4.14)$$

Fig. 3. Variation with $\bar{r}$ and τ of the shear stress σ_{rz} .

Using the asymptotic form for the Bessel function J_0

$$J_0(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \pi/4) + O(z^{-3/2})$$

we may write

$$\begin{aligned} J_0\left(\frac{n\pi t}{h}\right) \sin\left(\frac{n\pi z}{h}\right) &= \frac{C}{n^{1/2}} [\sin\left[\frac{k}{a}(z+t) - \pi/4\right] \\ &+ \sin\left[\frac{k}{a}(z-t) + \pi/4\right]] + O(n^{-3/2}), \end{aligned} \quad (3.4.15)$$

where $C = \frac{1}{\pi} \sqrt{\frac{2h}{t}}$. By expanding the sine terms in (3.4.15) to remove $(\pi/4)$ from the arguments, it follows that any discontinuity will be determined by considering series involving products of the terms in (3.4.14) with the functions $n^{-1/2} \cdot \sin(n\alpha^+)$, $n^{-1/2} \cdot \cos(n\alpha^+)$ where $\alpha^+ = \frac{\pi}{h}(z+t)$.

Now the series $\sum \frac{de^{-d}}{n^2} \cdot \frac{\cos(n\alpha^+)}{n^{1/2}}$, and the corresponding ones for successive terms in (3.4.14) are of the order δ multiplied by a series which is finite and continuous up to and including the value $\delta = 0$. They tend therefore to the limit 0 with δ and introduce no discontinuities. Similar arguments hold for the corresponding sine series. We need only examine the series

$$\sum \frac{de^{-d}}{n^{1/2}} \left\{ \frac{\cos}{\sin} (n\alpha^+) \right\}. \quad (3.4.16)$$

To avoid fractional powers, we first consider the series

$$\sum \frac{de^{-d}}{n} \left\{ \frac{\cos}{\sin} (n\alpha^+) \right\} = \frac{\pi a \delta}{h} \sum e^{-k\delta} \left\{ \frac{\cos}{\sin} (n\alpha^+) \right\}.$$

For $\delta > 0$, we have the result

$$\sum_{n=1}^{\infty} e^{-\frac{n\pi a}{h} \cdot \delta} \cos(n\alpha^+) = \frac{e^{\frac{\pi a}{h} \cdot \delta} \cos(n\alpha^+) - 1}{\frac{2\pi a}{h} \cdot \delta - 2 e^{\frac{\pi a \delta}{h}} \cos(n\alpha^+) + 1}. \quad (3.4.17)$$

For $\alpha^+ \neq 0$, this series has limit zero as δ approaches zero. For

$\alpha^+ = 0$, the sum is $\frac{1}{\frac{\pi a \delta}{h} - 1}$, and the limit as $\delta \rightarrow 0$ of

$\frac{\pi a \delta}{h} \cdot \frac{1}{e^{\frac{\pi a \delta}{h}} - 1}$ is 1. The corresponding sine series has zero limit

for all values of α^+ .

Similarly, we consider,

$$\begin{aligned} & \frac{\pi a \delta}{h} \cdot \sum_{n=1}^{\infty} n e^{-\frac{n \pi a}{h} \delta} \cos(n \alpha^+) = \\ & \frac{\pi a \delta}{h} \cdot \frac{e^{\frac{\pi a \delta}{h}} \{e^{\frac{2 \pi a \delta}{h}} \cos(\alpha^+) - 2e^{\frac{\pi a \delta}{h}} + \cos(\alpha^+)\}}{\{1 - 2e^{\frac{\pi a \delta}{h}} \cos(\alpha^+) + e^{\frac{2 \pi a \delta}{h}}\}^2}, \end{aligned} \quad (3.4.18)$$

which also approaches zero as $\delta \rightarrow 0$, provided $\alpha^+ \neq 0$. However, if $\alpha^+ = 0$, the series approaches $\pm \infty$ as $\delta \rightarrow 0^+$. Again, the corresponding sine series has zero limit for all values of α^+ .

Comparing the behaviour of the last two series for small δ , we conclude that an infinite discontinuity is introduced into the shear stress at $r = a$ by the series in (3.4.16), at the points $\alpha^+ = 0$, that is, at $z = \pm t$, where the surface pressure distribution changes discontinuously. This is a result one would anticipate.

We further observe from the nature of the coefficients in the Fourier expansion for the constant pressure band that, as r approaches a , the behaviour of the series expansion for the shear stress depends on the terms $\frac{de^{-d}}{n} \cos(n \alpha^+)$. We can therefore expect a finite discontinuity in the shear stress at each edge of the band.

We finally give an indication of the magnitude of the applied pressure distribution for which the above analysis is valid. The perturbational physical surface force was taken as $\epsilon p(z)$ and in both cases we may write

$$\frac{\bar{a}}{a_o} = -\frac{P}{h} = -\frac{(\bar{P}/\epsilon)}{h} ,$$

where $\bar{P}$ is the constant corresponding to P which would appear in any experimental measurements of the surface distribution. We consider the special case of equation (3.3.12) when $S = \frac{\epsilon \pi a_o^2}{2}$, that is $S^* = 0$. The cylinder is thus allowed to extend under the applied surface pressure, there being no compressive load. The extension ratio λ_ϵ is obtained from (3.1.9) and given implicitly by

$$\frac{\pi a_o^2 \bar{P}}{2h} = \frac{2\pi a_o^2}{\lambda_\epsilon^2} (\lambda_\epsilon^3 - 1)(C_1 \lambda_\epsilon + C_2) . \quad (3.4.19)$$

Since the theory is based on the assumption that ϵ is small, (3.4.19) holds only when $\lambda_\epsilon = h/h_o$ is close to unity. To a first approximation therefore, we may rewrite this result as

$$\bar{P} = 2h_o \frac{(h-h_o)E}{h} . \quad (3.4.20)$$

where $6(C_1 + C_2)$ is again replaced by E for small displacements. Provided t is not close to h (in which case the perturbational stresses would not be $O(\epsilon)$ and the theory therefore invalid) a suitable (non-dimensional) choice for ϵ would be $\epsilon = \bar{P}/h_o E$. From (3.4.20), this corresponds to $\epsilon = \frac{2(h-h_o)}{h}$ which will be small compared to unity for small displacements.

CONSTRICTION OF INCOMPRESSIBLE CYLINDERS UNDER FINITE AXIAL STRAIN.

In this section, the mixed boundary value problem to be examined is illustrated in Figure 4. The restraining edges which cause constriction are rigid and may represent a hole in a plate into which the cylinder is symmetrically placed. The cylinder may fit loosely into the hole so that a certain amount of finite uniform compression occurs before contact is made. Once contact is established a small additional compression is permitted so that the resulting deformation (Figure 4c) is non-uniform.

Fig. 4. The constriction of an elastic cylinder under axial compression.

The solution is developed by introducing the intermediate hypothetical state shown in Figure 4b. The cylinder is compressed uniformly to its final height in the absence of any lateral restraint. A small

displacement perturbation is then introduced, keeping the height fixed, so that the resulting geometry corresponds to the constricted shape in Figure 4c. By taking this approach, it is possible to take $w = 0$ at the ends of the cylinder when introducing the perturbation. As we have previously found, this boundary condition permits simple variables-separable solutions to be used in the governing field equations.

There is no surface traction on any part of the body and on the curved surface outside of the contact area, the normal stress vanishes. The deformation in the axial direction is therefore produced by two smooth, flat load surfaces perpendicular to the axis.

4.1 SOLUTION IN TERMS OF DUAL SERIES.

As in §3.2, the equilibrium and incompressibility conditions require the potential functions for the perturbational displacements of this problem to satisfy equations (3.2.9), (3.2.10) and (3.2.11). We recall that appropriate solutions are

$$\begin{aligned}\phi_1 &= \phi_1^0 + \sum_{n=1}^{\infty} B_n \left\{ I_0(\beta_n r) - \frac{2I_1(\beta_n a)I_0(\alpha_n r)}{\lambda^{3/2}(1+\lambda^3)I_1(\alpha_n a)} \right\} \cos \beta_n z, \\ \phi_2 &= \phi_2^0 + \sum_{n=1}^{\infty} B_n \left\{ I_0(\beta_n r) - \frac{2\lambda^{3/2}I_1(\beta_n a)I_0(\alpha_n r)}{(1+\lambda^3)I_1(\alpha_n a)} \right\} \cos \beta_n z,\end{aligned}\tag{4.1.1}$$

where B_n are arbitrary constants, $\beta_n = \frac{n\pi}{h}$, $\alpha_n = \lambda^{3/2}\beta_n$, z is measured from the mid-height of the cylinder, and ϕ_1^0 and ϕ_2^0 are given in (3.2.6)

From (3.2.1) and (4.1.1) it follows, as mentioned above, that $w = 0$ at the load surfaces $z = \pm h$. Physically, this means that the cylinder is uniformly compressed between smooth load surfaces to a height h and the perturbation merely changes the axial thrust required to maintain this height. It is shown in Appendix A.1 that this change in the axial thrust is precisely $\epsilon p'_0 \times \pi a^2$, where $\epsilon p'_0$ is the unknown constant pressure introduced in (3.2.7).

The remaining boundary conditions to be satisfied refer to the curved surface of the cylinder. We require that ϵu be prescribed over the contact area so that the resulting displacements are compatible with the constraint. In addition, over the remaining part of the curved surface the normal stress vanishes.

With ϕ_1 given by (4.1.1), it follows from (3.2.1) that

$$u(a, z) = -\frac{(1-\lambda)^3}{1+\lambda} \sum_{n=1}^{\infty} B_n \beta_n I_1(\beta_n a) \cos \beta_n z. \quad (4.1.2)$$

Let $A_n = \frac{(1-\lambda)^3}{(1+\lambda)} B_n \beta_n^2 I_1(\beta_n a)$, and let $\bar{\epsilon u}(z)$ be the radial displacement at the surface to make a compatible contact with the rigid constraint over the region $0 \leq z < t$. The boundary requirement is then

$$\sum_{n=1}^{\infty} \frac{A_n}{n} \cos \beta_n z = -\frac{\pi}{h} \bar{\epsilon u}(z) \quad 0 \leq z < t \quad (4.1.3)$$

The normal stress over the curved surface is found from (3.2.5), (3.2.7) and (3.2.8). Eliminating p' from (3.2.5) by using (3.2.2), it follows that

$$\tau'_{11} = p'_0 - d_{55} \left(\frac{\partial^2 \phi_2}{\partial z^2} + \lambda^3 \frac{\partial^2 \phi_1}{\partial z^2} + \frac{2\lambda^3}{v} \frac{1}{r} \frac{\partial \phi_1}{\partial r} \right),$$

where

$$v^3 = \frac{(\lambda C_1 + C_2) \lambda^2}{(C_1 + \lambda^2 C_2)}.$$

Using the particular forms for ϕ_1 and ϕ_2 given by (4.1.1), we find that the normal stress vanishes outside of the contact area if

$$\sum_{n=1}^{\infty} A_n \left\{ \frac{(1+\lambda^3)^2}{(1-\lambda^3)} \cdot \frac{I_0(\beta_n a)}{I_1(\beta_n a)} - \frac{4\lambda^{3/2} I_0(\alpha_n a)}{(1-\lambda^3) I_1(\alpha_n a)} + \frac{2\lambda^3}{v^3 \beta_n a} \right\} \cos \beta_n z + p'_o/d_{55} = 0 \quad t < z \leq h. \quad (4.1.4)$$

Equations (4.1.3) and (4.1.4) are dual series for the determination of the coefficients A_n and hence B_n . In general, these equations are too cumbersome to solve but some interesting cases can be obtained by using asymptotic forms for the Bessel functions.

4.2 ASYMPTOTIC SOLUTION FOR SHORT CYLINDERS.

For large arguments the Bessel functions, I_0 and I_1 , may be replaced to any degree of accuracy by their asymptotic expansions given by (3.3.19) and (3.3.20) respectively. Hence for sufficiently large values of $(\beta_n a)$

$$\frac{I_0(\beta_n a)}{I_1(\beta_n a)} = 1 + \frac{1}{2\beta_n a} + O\left(\frac{1}{\beta_n^2 a^2}\right). \quad (4.2.1)$$

In particular, if $\beta_1 a \geq \pi$, that is $a/h \geq 1$, the error in the approximation (4.2.1) is less than 5% and decreases very rapidly for increasing n . Therefore the above approximation can be made in (4.1.4) and the error can be expected to be small if the diameter exceeds the height of the cylinder. Carrying out this substitution gives

$$\sum_{n=1}^{\infty} A_n \left\{ \frac{(1+\lambda^3)^2 - 4\lambda^{3/2}}{1 - \lambda^3} + \frac{1}{2\beta_n a} \left(\frac{4\lambda^3}{3} - \lambda^3 - 3 \right) + 0 \left(\frac{1}{\beta_n^2 a^2} \right) \right\} \cos \beta_n z$$

$$+ p'_0/d_{55} = 0 . \quad (4.2.2)$$

The cases of practical interest occur when λ is fairly close to unity. Accordingly, we express λ in the form

$$\lambda = 1 - \delta , \text{ where } \delta^2 \ll 1 \quad (4.2.3)$$

If $\delta < \frac{1}{4}$, then the axial strain is limited to about 25%. This should be sufficiently large to accomodate most physical situations arising from the deformation of rubber blocks. Substituting (4.2.3) into (4.2.2) and ignoring δ^2 in comparison with unity leads to the following equation

$$\sum_{n=1}^{\infty} A_n \left\{ -\left(2 - \frac{9\delta}{2}\right) - \frac{3\delta}{2\beta_n a} \cdot \frac{(3C_2 - C_1)}{(C_1 + C_2)} + 0 \left(\frac{1}{\beta_n^2 a^2} \right) \right\} \cos \beta_n z$$

$$+ p'_0/d_{55} = 0 \quad t < z \leq h .$$

For most materials $C_1 > C_2 \geq 0$, so that $\left| \frac{3C_2 - C_1}{C_1 + C_2} \right| \leq 1$.

Also if we consider short cylinders such that $\beta_n a > 2\pi$ (say) for all n , that is diameter $> 2 \times$ height, the above equation may be represented to within 3% error in the coefficients of A_n by

$$\sum_{n=1}^{\infty} A_n \cos \beta_n z - \frac{p'_0}{2d_{55}} \left(1 + \frac{9\delta}{4} \right) = 0 \quad t < z \leq h . \quad (4.2.4)$$

For longer cylinders the restriction on $(\beta_n a)$ may be removed as described later in chapter 5.

The dual series equations to be solved for the determination of A_n are given by (4.1.3) and (4.2.4). These equations are brought to a more standard form by making the following changes in notation,

$$x = \pi(1-z/h) , c = \pi(1-t/h) , A_n = (-1)^n n a_n . \quad (4.2.5)$$

Equation (4.1.3) may now be written in the form

$$\sum_{n=1}^{\infty} a_n \cos nx = -\frac{\pi}{h} \bar{u}(x) \quad c < x \leq \pi \quad (4.2.6a)$$

and equation (4.2.4) becomes

$$\sum_{n=1}^{\infty} n a_n \cos nx = \frac{p'_0}{2d_{55}} (1+9\delta/4) \quad 0 \leq x < c \quad (4.2.6b)$$

In general $\bar{u}(x)$ is determined from the shape of the contact area. However in this investigation, $\bar{u}(x)$ is now restricted to be a constant so that the contact area corresponds to a flat indentation. With $\bar{u}$ a constant, we make the following changes in notation:

$$\frac{1}{2} \bar{a}_0 = \frac{\bar{u}\pi}{h} , \frac{1}{2} \mu \bar{a}_0 = -\frac{p'_0}{2d_{55}} (1+9\delta/4) . \quad (4.2.7)$$

Equations (4.2.6) may now be written in the interesting form

$$\frac{1}{2} \mu \bar{a}_0 + \sum_{n=1}^{\infty} n a_n \cos nx = 0 \quad 0 \leq x < c , \quad (4.2.8a)$$

$$\frac{1}{2} \bar{a}_0 + \sum_{n=1}^{\infty} a_n \cos nx = 0 \quad c < x \leq \pi . \quad (4.2.8b)$$

The solution of these equations is non-trivial since, from (4.2.7), $\bar{a}_0$ is non-trivial. The equations would of course be inhomogeneous had $\bar{u}$ been allowed to depend on z , but the method of solving them would be the same. The solution to these equations is obtained from the more general results of Sneddon (15, p. 161). The procedure begins by assuming that the function

$$g(x) = \frac{1}{2} \bar{a}_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad 0 \leq x \leq \pi$$

has component $g_1(x) = \cos \frac{1}{2}x \int_0^c \frac{h(t)}{x \sqrt{(\cos x - \cos t)}} dt$,

in the interval $0 \leq x < c$. From the theory of Fourier series it follows that

$$\begin{aligned} \bar{a}_0 &= \frac{2}{\pi} \left\{ \frac{\pi}{\sqrt{2}} \int_0^c h(t) dt \right\}, \\ a_n &= \frac{2}{\pi} \left\{ \frac{\pi}{2\sqrt{2}} \int_0^c h(t) [P_n(\cos t) + P_{n-1}(\cos t)] dt \right\}. \end{aligned} \quad (4.2.9)$$

where we have identified P_n , the Legendre polynomial of degree n , from Mehler's integral formula (10). If we substitute these values for a_n into the integrated form

$$\frac{1}{2} \mu \bar{a}_0 x + \sum_{n=1}^{\infty} a_n \sin nx = 0 \quad 0 \leq x < c$$

of equation (4.2.8a) and interchange the order of summation and integration, it follows that $h(t)$ is the solution of the integral equation

$$\int_0^c \frac{h(t)}{\sqrt{2}} \sum_{n=1}^{\infty} [P_n(\cos t) + P_{n-1}(\cos t)] \sin nx \, dt = -\frac{1}{2} \mu \bar{a}_0 x. \quad (4.2.10)$$

Sneddon (15) shows that $\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} [P_n(\cos t) + P_{n-1}(\cos t)] \sin nx = \frac{\cos \frac{1}{2}x H(x-t)}{\sqrt{(\cos t - \cos x)}}$, where $H(x)$ is the Heaviside step function. Using

this result, (4.2.10) becomes

$$\int_0^x \frac{h(t) dt}{\sqrt{(\cos t - \cos x)}} = -\frac{1}{2} \bar{a}_0 \mu x \sec \frac{1}{2}x \quad 0 \leq x < c,$$

which has the solution (15, p. 41)

$$h(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{(-\frac{1}{2} \mu \bar{a}_0) x \sin \frac{1}{2}x \, dx}{\sqrt{(\cos x - \cos t)}}. \quad (4.2.11)$$

Using (4.2.7) and (4.2.9) we now have

$$\bar{a}_0 = p'_0 \frac{I(c)}{d_{55}} (1+9\delta/4) \quad (4.2.12)$$

where

$$I(c) = \frac{\sqrt{2}}{\pi} \int_0^c \frac{x \sin \frac{1}{2}x \, dx}{\sqrt{(\cos x - \cos t)}} \quad .$$

From (4.2.7) and (4.2.12) it follows that

$$p'_0 = \frac{2\bar{u}\pi}{hI(c)} d_{55}(1-9\delta/4) \quad (4.2.13)$$

Equation (4.2.13) establishes the value of the unknown constant p'_0 for a given value of $\bar{u}$.

The remaining a_n are given by

$$a_n = \frac{\bar{u}\pi}{hI(c)} \int_0^c [P_n(\cos t) + P_{n-1}(\cos t)] \frac{dI(t)}{dt} \cdot dt \quad (4.2.14)$$

The solution is now complete if $\bar{u}$ is specified or alternatively, from (4.2.13), if p'_0 is specified. In this particular problem it is convenient to consider p'_0 as the independent variable since this quantity specifies the axial load. In particular, denoting the total compressive load acting on the cylinder by S^* , it is shown in Appendix A.1 that

$$S^* = S - \pi a_0^2 \epsilon p'_0 \quad ,$$

where S would be the load required to compress the cylinder to the same height in the absence of the constriction (see Figure 4). In view of the restriction on the magnitude of λ , it follows that the initial radius, a_0 , of the cylinder and the radius, s , of the constricting circle must satisfy

$$s/a_0 < 1.15 \quad .$$

CHAPTER V

CONSTRICTION OF FINITE COMPRESSIBLE MATERIALS UNDER AXIAL COMPRESSION

In this chapter the constriction of compressible cylinders is considered. As for the incompressible case examined in Chapter 4, the cylinder is compressed axially, and radial expansion is prevented over a central region. For incompressible rubber-like materials it is meaningful to consider very large axial extensions and to assume a strain-energy function of the form (3.1.6). However for compressible materials, the classical equations of elasticity, derivable from a quadratic form for W , are used. In terms of the strain tensor, strain invariants are given by (2.2.8). The most general quadratic form in the strain invariants is

$$W = c_1 I_1 + c_2 I_1^2 + c_3 I_2 + c_4 \quad (5.0.1)$$

In order that (5.0.1) be homogeneous of degree two in the strains it follows that

$$3c_1 + 9c_2 + 3c_3 + c_4 = 0, \quad 2c_1 + 12c_2 + 4c_3 = 0 \quad (5.0.2)$$

Now, for a classical, isotropic, homogeneous compressible material, the strain-energy function is given by Sokolnikoff (16) as

$$W = \frac{\sigma E \gamma_r^r \gamma_s^s}{2(1+\sigma)(1-2\sigma)} + \frac{E}{2(1+\sigma)} \gamma_s^r \gamma_r^s,$$

where E and σ are respectively Young's modulus and Poisson's ratio.

Comparing coefficients of quadratic terms shows that

$$4c_2 + 2c_3 = \frac{\sigma E}{2(1+\sigma)(1-2\sigma)}, \quad 2c_3 = -\frac{E}{2(1+\sigma)} \quad (5.0.3)$$

Solving (5.0.2) and (5.0.3) for the coefficients c_1 , c_2 , c_3 and c_4 leads to the expression

$$W = -\frac{EI_1}{4(1-2\sigma)} + \frac{E(1-\sigma)I_1^2}{8(1+\sigma)(1-2\sigma)} - \frac{EI_2}{4(1+\sigma)} + \frac{3E}{8(1-2\sigma)} . \quad (5.0.4)$$

In addition, for metals it is not physically meaningful to impose large axial strains since elastic behaviour ceases when the strains exceed about one half of one percent. Hence in this chapter we consider infinitesimal deflections.

5.1. Solution in Terms of Dual Series.

As in chapter 4, potential functions ϕ_1 and ϕ_2 are introduced where

$$u = \frac{\partial \phi_1}{\partial r} , \quad w = \frac{\partial \phi_2}{\partial z} . \quad (5.1.1)$$

Since the displacements are entirely perturbational, the stress equations of equilibrium may be found by evaluating the appropriate expressions in §4.4 of Green and Zerna, with W given by (5.0.4), and then letting $\lambda \rightarrow 1$. Alternatively, the equations can be derived from the finite theory developed in chapter 2 by retaining only first order terms in the strains and displacements. In terms of the displacement potential functions, these equilibrium conditions are satisfied if

$$\begin{aligned} 2(1-\sigma)\nabla^2\phi_1 + (1-2\sigma)\frac{\partial^2\phi_1}{\partial z^2} + \frac{\partial^2\phi_2}{\partial z^2} &= 0 , \\ 2(1-\sigma)\frac{\partial^2\phi_2}{\partial z^2} + (1-2\sigma)\nabla^2\phi_2 + \nabla^2\phi_1 &= 0 . \end{aligned} \quad (5.1.2)$$

It similarly follows from general expressions given in Green and Zerna §4.4 (or directly, from Sokolnikoff) that the stresses are

$$\begin{aligned}
\sigma_r &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{\partial u}{\partial r} + \sigma \frac{u}{r} + \sigma \frac{\partial w}{\partial z} \right] \\
\sigma_\theta &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{u}{r} + \sigma \frac{\partial u}{\partial r} + \sigma \frac{\partial w}{\partial z} \right] \\
\sigma_z &= \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{\partial w}{\partial z} + \sigma \frac{\partial u}{\partial r} + \sigma \frac{u}{r} \right] \\
\tau_{rz} &= \frac{E}{2(1+\sigma)} \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right]
\end{aligned} \tag{5.1.3}$$

A solution of (5.1.2) corresponding to a uniform compression of the cylinder is found by setting

$$\phi_1 = Ar^2 + Bz^2, \quad \phi_2 = Cr^2 + Dz^2,$$

where A, B, C and D are constants. Substituting these expressions into (5.1.2) leads to the requirement that

$$\begin{aligned}
4A(1-\sigma) + B(1-2\sigma) + D &= 0, \\
D(1-\sigma) + C(1-2\sigma) + A &= 0.
\end{aligned} \tag{5.1.4}$$

Denoting the displacement and radial stress at $r = a$ by u_o and p_o respectively, it follows from (5.1.1) and (5.1.3) that

$$2Aa = u_o, \quad \frac{2E}{(1+\sigma)(1-2\sigma)} (A+\sigma D) = p_o. \tag{5.1.5}$$

The solution of (5.1.4) and (5.1.5) is

$$\begin{aligned}
A &= \frac{u_o}{2a}, \quad B = -\frac{p_o(1+\sigma)}{2\sigma E} + \frac{u_o(1-2\sigma)}{2a\sigma}, \\
C &= \frac{u_o}{2a\sigma} - \frac{(1-\sigma)(1+\sigma)}{2\sigma E} p_o, \quad D = \frac{p_o(1+\sigma)(1-2\sigma)}{2\sigma E} - \frac{u_o}{2a\sigma}.
\end{aligned}$$

Using (5.1.1) and (5.1.3), this solution gives the following stresses and displacements at the surfaces of the cylinder,

$$\sigma_r^o = p_o, \quad u = u_o \quad \text{at} \quad r = a,$$

$$\sigma_z^o = \frac{(1-\sigma)}{\sigma} p_o - \frac{Eu_o}{a\sigma}, \quad w_o = \frac{hp_o(1+\sigma)(1-2\sigma)}{\sigma E} - \frac{hu_o}{a\sigma} \quad \text{at } z = h. \quad (5.1.6)$$

As is to be expected the solution fails when $\sigma = 0$ since radial expansion cannot be produced by an axial load in this case. Since constriction cannot then occur, the situation becomes trivial.

To represent the non-uniform constriction, a variables-separable solution of (5.1.2) is now sought of the form

$$\phi_1(r, z) = \bar{\phi}_1(r) \cos \beta_n z, \quad \phi_2(r, z) = \bar{\phi}_2(r) \cos \beta_n z,$$

where $\beta_n = \frac{n\pi}{h}$. The first equation of (5.1.2) gives

$$\bar{\phi}_2 = \frac{2(1-\sigma)}{\beta_n^2} \nabla^2 \bar{\phi}_1 - (1-2\sigma) \bar{\phi}_1 \quad (5.1.7)$$

and when this is substituted into the second of (5.1.2), it is found that $\bar{\phi}_1$ satisfies the equation

$$(\nabla^2 - \beta_n^2) \bar{\phi}_1 = 0.$$

It follows that

$$\phi_1 = \{\bar{A}_n I_0(\beta_n r) + B_n \beta_n r I_1(\beta_n r)\} \cos \beta_n z \quad (5.1.8a)$$

and hence from (5.1.7) that

$$\phi_2 = \{[\bar{A}_n + 4(1-\sigma)B_n] I_0(\beta_n r) + B_n \beta_n r I_1(\beta_n r)\} \cos \beta_n z. \quad (5.1.8b)$$

It follows from (5.1.1) and (5.1.3) that the axial displacement and shear stress, corresponding to any solution of the form (5.1.8), vanish at the ends of the cylinder $z = \pm h$.

The constants $\bar{A}_n$ and B_n in (5.1.8) are found from the conditions prevailing on the curved surface. In particular there is to be no shear stress acting on this surface. From (5.1.3) it follows that $\tau_{rz} = 0$ on $r = a$ if

$$\frac{\partial^2 \phi_1}{\partial r \partial z} + \frac{\partial^2 \phi_2}{\partial r \partial z} = 0 \quad \text{at } r = a .$$

This equation is satisfied when

$$\bar{A}_n = -B_n \left\{ \frac{\beta_n a I_0(\beta_n a)}{I_1(\beta_n a)} + 2(1-\sigma) \right\} . \quad (5.1.9)$$

The values of B_n are to be found from the mixed boundary conditions over the curved surface. Over the central region $0 \leq z < t$, $0 \leq \theta \leq 2\pi$, the radial displacement is governed by the constraint, whereas over the remaining curved surface the normal stress vanishes.

The sum of all such solutions (5.1.8) for all integers n are now superposed together with the uniform solution (5.1.6). From (5.1.1), (5.1.6), (5.1.8) and (5.1.9) the total radial displacement at $r = a$ is found to be

$$u(a, z) = u_0 - 2(1-\sigma) \sum_{n=1}^{\infty} B_n \beta_n I_1(\beta_n a) \cos \beta_n z .$$

A new constant, A_n , is now defined by

$$A_n = 2(1-\sigma) \beta_n^2 B_n I_1(\beta_n a) . \quad (5.1.10)$$

so that

$$u(a, z) = u_0 - \frac{h}{\pi} \sum_{n=1}^{\infty} \frac{A_n}{n} \cos \beta_n z .$$

As mentioned above, the total radial displacement must vanish for $0 \leq z < t$ at $r = a$. Thus

$$\sum_{n=1}^{\infty} \frac{A_n}{n} \cos \beta_n z = \frac{\pi u_0}{h} \quad 0 \leq z < t , \quad (5.1.11)$$

where u_0 is a constant. The radial stress is found from (5.1.3),

(5.1.6) and (5.1.1) in terms of ϕ_1 and ϕ_2 and is

$$\sigma_r = p_0 + \frac{E}{(1+\sigma)(1-2\sigma)} \left[(1-\sigma) \frac{\partial^2 \phi_1}{\partial r^2} + \frac{\sigma}{r} \frac{\partial \phi_1}{\partial r} + \frac{\partial^2 \phi_2}{\partial z^2} \right] . \quad (5.1.12)$$

Substituting (5.1.8) and (5.1.9) into (5.1.12) and letting $r = a$, the surface value of σ_r is found to be

$$\sigma_r = p_o + \frac{aE}{(1+\sigma)} \sum_{n=1}^{\infty} \beta_n^3 B_n \left\{ \left[1 + \frac{2(1-\sigma)}{\beta_n^2 a^2} \right] I_1(\beta_n a) - \frac{I_0^2(\beta_n a)}{I_1(\beta_n a)} \right\} \cos \beta_n z.$$

Using (5.1.10) this equation may be written in the form

$$\sigma_r = p_o - \frac{E}{2(1+\sigma)(1-\sigma)} \sum_{n=1}^{\infty} A_n (1+\alpha_n) \cos \beta_n z \quad \text{at } r = a, \quad (5.1.13)$$

where

$$\alpha_n = \beta_n a \left\{ \frac{I_0^2(\beta_n a)}{I_1^2(\beta_n a)} - \frac{2(1-\sigma)}{\beta_n^2 a^2} - \frac{1}{\beta_n a} - 1 \right\}. \quad (5.1.14)$$

The boundary condition not yet satisfied is that requiring the normal stress to vanish outside of the contact area. From (5.1.13) it follows that

$$\sum_{n=1}^{\infty} A_n \cos \beta_n z - \frac{2p_o}{E} (1+\sigma)(1-\sigma) = - \sum_{m=1}^{\infty} A_m \alpha_m \cos \beta_m z, \quad t < z \leq h. \quad (5.1.15)$$

Finally, equations (5.1.11) and (5.1.15) are brought to standard form by making the following substitutions:

$$x = \pi(1-z/h), \quad c = \pi(1-t/h), \quad A_n = (-1)^n \frac{2p_o}{E} (1+\sigma)(1-\sigma) n a_n.$$

The equations to be solved are thus

$$-1 + \sum_{n=1}^{\infty} n a_n \cos nx = - \sum_{m=1}^{\infty} m a_m \alpha_m \cos mx \quad 0 \leq x < c, \quad (5.1.16a)$$

$$\frac{1}{2} \bar{a}_o + \sum_{n=1}^{\infty} a_n \cos nx = 0 \quad c < x \leq \pi, \quad (5.1.16b)$$

where

$$\frac{1}{2} \bar{a}_o = \frac{-\pi E u_o}{2h p_o (1+\sigma)(1-\sigma)}. \quad (5.1.17)$$

In terms of a_n , the surface displacement is

$$u(a, x) = -\frac{2hp_0}{\pi E} (1+\sigma)(1-\sigma) \left\{ \frac{1}{2} \bar{a}_0 + \sum_{n=1}^{\infty} a_n \cos nx \right\} \quad (5.1.18)$$

and the normal surface stress is

$$\sigma_r(a, x) = p_0 \left\{ 1 - \sum_{n=1}^{\infty} n a_n (1+\alpha_n) \cos nx \right\} . \quad (5.1.19)$$

The equations (5.1.16) may be solved by the standard methods given in Sneddon. In the next section, the solution is derived for fairly short cylinders.

5.2 Analytic Solution of the Dual Series.

The method of solution follows that described in §4.2. In particular the coefficients are

$$\begin{aligned} \bar{a}_0 &= \sqrt{2} \int_0^c h(t) dt, \\ a_n &= \frac{1}{\sqrt{2}} \int_0^c h(t) [P_n(\cos t) + P_{n-1}(\cos t)] dt. \end{aligned} \quad (5.2.1)$$

In this case, however, $h(t)$ is given by

$$h(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\sin x/2}{\sqrt{(\cos x - \cos t)}} dx \left\{ x - \int_0^x \sum_{m=1}^{\infty} m a_m \alpha_m \cos mudu \right\} .$$

Formally changing the order of summation and integration in the second integral of the above expression, and performing this latter integration, we obtain

$$\begin{aligned} h(t) &= \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{x \sin x/2}{\sqrt{(\cos x - \cos t)}} dx \\ &+ \frac{2}{\pi} \sum_{m=1}^{\infty} a_m \alpha_m \frac{d}{dt} \int_0^t \frac{1}{2} \left\{ \frac{\cos(m+\frac{1}{2})x - \cos(m-\frac{1}{2})x}{\sqrt{(\cos x - \cos t)}} \right\} dx . \end{aligned}$$

Using Mehler's integral formula for the Legendre function, this reduces to

$$h(t) = \sqrt{2} \frac{dI}{dt}(t) + \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} a_m \alpha_m \frac{d}{dt} [P_m(\cos t) - P_{m-1}(\cos t)] , \quad (5.2.2)$$

where

$$I(t) = \frac{\sqrt{2}}{\pi} \int_0^t \frac{x \sin x/2}{\sqrt{(\cos x - \cos t)}} dx .$$

It is shown in appendix A.3 that

$$I(t) = 2 \log |\sec t/2| . \quad (5.2.3)$$

Substituting from equations (5.2.2) and (5.2.3) into (5.2.1) and interchanging integral and summation signs formally, gives the following expression for the determination of the coefficients a_n :

$$\begin{aligned} a_n &= \int_0^c [P_n(\cos t) + P_{n-1}(\cos t)] \tan t/2 dt \\ &+ \frac{1}{2} \sum_{m=1}^{\infty} \alpha_m a_m \int_0^c [P_n(\cos t) + P_{n-1}(\cos t)] \frac{d}{dt} [P_m(\cos t) - P_{m-1}(\cos t)] dt . \end{aligned}$$

Mehler's formula for $P_n(\cos t)$ enables the first integral to be written as

$$\frac{4}{\pi} \int_0^c \cos nx \cdot \ln |\cos x/2 \sec c/2 + \sqrt{(\cos x/2 \sec c/2)^2 - 1}| dx .$$

This form is most fitting for numerical evaluation. The second integral can be expressed in closed form as a finite sum of terms comprising products of pairs of Legendre functions.

We rewrite (5.2.4) as

$$a_n = b(n) - \sum_{m=1}^{\infty} c'(n,m) a_m , \quad n = 1, 2, \dots \quad (5.2.5)$$

where $b(n) = \frac{4}{\pi} \int_0^c \cos nx \cdot \ln | \cos x/2 \sec c/2 + \sqrt{(\cos x/2 \sec c/2)^2 - 1} | dx$

and where $c'(n,m) = \frac{1}{2} \alpha_m \int_{\cos(c)}^1 [P_n(u) + P_{n-1}(u)] \cdot [P'_m(u) - P'_{m-1}(u)] du$. (5.2.6)

The coefficients a_n are now calculated by truncating the series in (5.2.5) at $m = M$. After a slight rearrangement, we obtain the following system of M linear equations in M unknowns,

$$\sum_{m=1}^M c(n,m) a_m = b(n), \quad n = 1, 2, \dots, M. \quad (5.2.7)$$

where $c(n,m) = c'(n,m)$, $n \neq m$, and $c(n,n) = 1 + c'(n,n)$. These equations (5.2.7) can be written in matrix form as

$$\tilde{C} \tilde{a}^T = \tilde{b}, \quad (5.2.8)$$

where the matrix C has pronounced diagonal dominance. An iterative method will be most suitable for their solution and may be expected to converge very rapidly.

Returning now to the evaluation of $\bar{a}_0$, (5.2.1) and the truncated form of (5.2.2) give the approximate result

$$\frac{1}{2} \bar{a}_0 \doteq I(c) + \sum_{m=1}^M c'(0,m) a_m,$$

where $c'(0,m) = \frac{1}{2} \alpha_m [P_m(\cos c) - P_{m-1}(\cos c)]$

The above equation gives our approximate value for $\bar{a}_0/2$ whence follows, from (5.1.7), a relation between u_0 and p_0 . In particular we obtain

$$\frac{u_0}{h} \doteq - \frac{2p_0}{E} (1+\sigma)(1-\sigma) \left\{ I(c) + \sum_{m=1}^M c'(0,m) a_m \right\}. \quad (5.2.9)$$

In this particular problem it is convenient to give results in terms of the total axial load T . From (5.1.6) and (5.2.9) the uniform stress σ_z^0 is found in terms of p_0 as follows

$$\sigma_z^o = \frac{(1-\sigma)}{\sigma} p_o c_1, \quad (5.2.10)$$

where

$$c_1 = 1 + \frac{2h(1+\sigma)}{\pi a} \left\{ I(c) + \sum_{m=1}^M c'(o,m) a_m \right\}.$$

and the axial displacement at the load surfaces $z = \pm h$ is

$$w_o = \frac{(1+\sigma) h p_o c_2}{\sigma E}, \quad (5.2.11)$$

where

$$c_2 = 1 - 2\sigma + \frac{2h(1-\sigma)}{\pi a} \left\{ I(c) + \sum_{m=1}^M c'(o,m) a_m \right\}.$$

It is shown in Appendix A.2 that the axial stress distribution arising from the solution (5.1.8) is self-equilibrating so that the total axial compressive load P^* is given by

$$P^* = -T = -\pi a^2 \sigma_z^o.$$

Hence, from (5.2.10)

$$p_o = -\frac{\sigma P^*}{\pi a^2 (1-\sigma) c_1} \quad (5.2.12)$$

and so, from (5.2.11)

$$\frac{w_o}{h} = -\frac{P^*}{\pi a^2 E} \cdot \frac{(1+\sigma)}{(1-\sigma)} \cdot \frac{c_2}{c_1}. \quad (5.2.13)$$

Equation (5.2.13) is the stress-strain relation for the constricted cylinder. When $\sigma = 0$, $c_1 = c_2$ and (5.2.13) reduces to Hooke's law for uniaxial compression. This is to be expected since, when $\sigma = 0$, there is no radial expansion under axial compression and thus no constriction. For this same reason, the radial stress must also vanish when $\sigma = 0$.

The radial displacement given by (5.1.18) can now be expressed in terms of P^* by using (5.2.12) and is

$$\frac{u(a,x)}{h} = \frac{2\sigma P^* (1+\sigma)}{\pi a^2 E c_1} \left\{ \frac{1}{2} \frac{1}{a_o} + \sum_{n=1}^{\infty} a_n \cos nx \right\}. \quad (5.2.14)$$

Similarly, the radial stress at the surface is

$$\sigma_r(a, x) = - \frac{\sigma P^*}{\pi a^2 (1-\sigma) c_1} \left\{ 1 - \sum_{n=1}^{\infty} \alpha_n (1+\alpha_n) \cos nx \right\} . \quad (5.2.15)$$

An important quantity in this problem is the membrane stress transmitted to the restraining plate. As is shown later, the distribution of the radial stress across the contact width is highly non-uniform, being singular at the edges of the constraint. However, the membrane stress is

$$N_r = 2 \int_0^t \sigma_r(z) dz = \frac{2h}{\pi} \int_c^{\pi} \sigma_r(x) dx .$$

Using (5.2.15) it follows that

$$N_r = - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \int_c^{\pi} \left\{ 1 - \sum_{n=1}^{\infty} \alpha_n (1+\alpha_n) \cos nx \right\} dx .$$

Formally integrating within the summation leads to the expression

$$N_r = - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \left\{ (\pi - c) + \sum_{n=1}^{\infty} \alpha_n (1+\alpha_n) \sin(nc) \right\}$$

which, with equation (5.2.1) gives

$$\begin{aligned} N_r = & - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \left\{ (\pi - c) + \sum_{n=1}^{\infty} \alpha_n a_n \sin(nc) \right. \\ & \left. + \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \int_0^c h(t) [P_n(\cos t) + P_{n-1}(\cos t)] \sin(x) \cdot dt \right\} . \end{aligned}$$

A formal change in the order of integration and summation then gives

$$\begin{aligned} N_r = & - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \left\{ (\pi - c) + \sum_{n=1}^{\infty} \alpha_n a_n \sin(nc) \right. \\ & \left. + \frac{1}{\sqrt{2}} \int_0^c h(t) \sum_{n=1}^{\infty} [P_n(\cos t) + P_{n-1}(\cos t)] \sin(nc) dt \right\} \\ = & - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \left\{ (\pi - c) + \sum_{n=1}^{\infty} \alpha_n a_n \sin(nc) + \int_0^c \frac{h(t) \cos \frac{1}{2}c H(c-t) dt}{\sqrt{\cos t - \cos c}} \right\} , \end{aligned}$$

and, using (5.2.2), we formally obtain

$$N_r = - \frac{2\sigma h P^*}{\pi a^2 (1-\sigma) c_1} \left\{ (\pi - c) + \sqrt{2} \cos c/2 \int_0^c \frac{dI(t)}{dt} \cdot \frac{1}{\sqrt{(\cos t - \cos c)}} dt \right\}.$$

Finally, from (5.2.3), an elementary integration gives

$$N_r = - \frac{2\sigma P^* h}{\pi a^2 (1-\sigma) c_1}, \quad (5.2.16)$$

where P^* is the total axial compressive load and c_1 is a constant defined in (5.2.10). Since the non-uniformities in σ_r decay very rapidly, (5.2.16) could be used to obtain a good approximation to the stresses in a fairly rigid plate when loaded by a soft rivet. The existence of N_r in the restraining plate also implies a circumferential hoop stress $N_\theta = -N_r$.

5.3. Numerical Results.

Numerical results are now given for the particular case $h/a = 1$, $t = h/2$; that is, $c = \pi/2$. From (5.2.6) the elements $c(n, m)$ of the matrix C are determined from

$$c'(n, m) = \frac{1}{2} \alpha_m \int_0^1 \{P_n(t) + P_{n-1}(t)\} \cdot \{P'_m(t) - P'_{m-1}(t)\} dt. \quad (5.3.1)$$

Using the recurrence relations for the Legendre polynomials we find that

$$P'_m(t) - P'_{m-1}(t) = \sum_{k=1}^m (-1)^{k+1} (2m-2k+1) P_{m-k}(t). \quad (5.3.2)$$

From (5.3.2) and the results (21) that

$$\int_0^1 P_m(t) P_n(t) dt = \begin{cases} 1/(2n+1) & , m = n \\ 0 & , m-n \text{ even} \\ \frac{(-1)^{\mu+\nu}}{2^{m+n-1} (n-m) (n+m+1)} \cdot \frac{m! n!}{(\nu!)^2 (\mu!)^2} & , n = 2\nu+1, m = 2\mu \end{cases}$$

it follows that (5.3.1) can be expressed in the form

$$c'(n,m) = \frac{1}{2^m} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{1/2(3m-k+n-\delta+1)} (2m-2k+1) (m-k)! (n-\delta)!}{2^{m-k+n-\delta-1} (n-m-\delta+k) (n+m-k-\delta+1) \left\{ \left(\left[\frac{n-\delta}{2} \right] \right) ! \right\}^2 \left\{ \left(\left[\frac{m-k}{2} \right] \right) ! \right\}^2} \right. \\ \left. + E_{n,m} \right\}, \quad (5.3.3)$$

where

$$\delta = \begin{cases} 1 & \text{if } (m+n+k) \text{ even} \\ 0 & \text{if } (m+n+k) \text{ odd} \end{cases}, \quad E_{n,m} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and where $[x]$ represents the integral part of the real number x .

The matrix elements $c(n,m)$ can now be computed. The constant terms $b(n)$ appearing in the truncated series (5.2.7) are evaluated numerically from the integral expression (5.2.6). For n sufficiently large (say, $n \geq 20$) the asymptotic expansion developed in Appendix A.3 is used, giving results accurate to six significant figures. Solutions for the dual series coefficients a_n are calculated for M , the number of terms in the truncated expression (5.2.7), ranging from 50 to 150 and for Poisson's ratio taking the values 0.3, 0.4 and 0.5.

Since $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, the error in the truncation approximation decreases as M increases. In fact, the truncation may be interpreted as setting equal to zero the coefficients α_n for $n > M$. It therefore follows that $|\alpha_{M+1}|$ provides an upper bound for the relative error in the calculated coefficients a_n .

For $n > M$, the coefficients a_n are determined from equation (5.2.5), truncated at $m = M$. This may be further truncated at $\bar{M} \leq M$ as for the asymptotic evaluation described below; that is, when n is sufficiently large, say $n > N (\geq M)$, an asymptotic expansion of (5.2.5) is

appropriate. Using (5.3.2) we may write

$$\sum_{m=1}^M c'(n,m) a_m = \begin{cases} (-1)^{v+1} \sum_{j=1}^M \frac{(-1)^{\frac{j+\delta}{2}} (2j-1)(2v-\delta)!(j-1)!}{2^{2v+j-\delta-2} (2v-j-\delta+1)(2v+j-\delta) \{(v-\delta)!\}^2 \{(\frac{j+\delta-2}{2})!\}^2} \\ \times \sum_{p=j}^M (-1)^{p+\delta-1} \frac{\alpha_p a_p}{2}, & \text{if } n = 2v, \end{cases} \quad (5.3.4a)$$

$$\begin{cases} (-1)^v \sum_{j=1}^M \frac{(-1)^{\frac{j+\delta}{2}} (2j-1)(2v+\delta)!(j-1)!}{2^{2v+j+\delta-2} (2v-j+1+\delta)(2v+j+\delta)(v!)^2 \{(\frac{j+\delta-2}{2})!\}^2} \\ \times \sum_{p=j}^M (-1)^p \frac{\alpha_p a_p}{2}, & \text{if } n = 2v+1, \end{cases} \quad (5.3.4b)$$

$$\text{where } \delta = \begin{cases} 1 & \text{if } j \text{ odd} \\ 0 & \text{if } j \text{ even} \end{cases}.$$

Using Stirling's formula for the asymptotic approximation of the factorial function we find that, for each v and j , the terms of the inner series oscillate in sign and decrease in magnitude, so that the error in truncation at $p = \bar{M} < M$ is of the order of magnitude of the first neglected term. Furthermore, for $j = 1, \dots, M$ the leading term of the corresponding inner series decreases as j increases. Thus for a given value M , which determines the accuracy of the dual series approximation, we truncate both sums in each of the expressions (5.3.4) at $\bar{M} \leq M$ when a degree of accuracy compatible with the initial approximation is attained. We therefore write the asymptotic solution for the coefficients a_n for $n > N$ as

$$a'_n \sim b(n) - E(n; M; \bar{M}), \quad (5.3.5)$$

where $b(n)$ is given asymptotically in Appendix A.3. $E(n; M; \bar{M})$ represents a correction term and may be expressed using Stirling's formula in (5.3.4) as

$$E(n; M; \bar{M}) = \begin{cases} (-1)^{v+1} \frac{2}{\pi} \sum_{j=12}^{\bar{M}} \frac{(-1)^{\frac{j+\delta}{2}} (2j-1)(j-1)! \left(1 - \frac{1}{2v}\right)^{-\delta/2} (2v)^{\delta-1/2}}{2^{j+\delta-2} (2v-j-\delta+1)(2v+j-\delta) \left(1 - \frac{1}{v}\right)^{-\delta} \left\{ \left(\frac{j+\delta-2}{2}\right)! \right\}^2} \\ \times \sum_{p=j}^M (-1)^{p+\delta-1} \frac{\alpha_p a_p}{2} , & \text{if } n = 2v . \end{cases} \quad (5.3.6a)$$

$$E(n; M; \bar{M}) = \begin{cases} (-1)^v \frac{2}{\pi} \sum_{j=1}^{\bar{M}} \frac{(-1)^{\frac{j+\delta}{2}} (2j-1)(j-1)! \left(1 - \frac{1}{2v+1}\right)^{(\delta-1)/2} (2v+1)^{\delta-1/2}}{2^{j+\delta-2} (2v-j+\delta+1)(2v+j+\delta) \left(\frac{j+\delta-2}{2}\right)!^2} \\ \times \sum_{p=j}^M (-1)^p \frac{\alpha_p a_p}{2} , & \text{if } n = 2v + 1 . \end{cases} \quad (5.3.6b)$$

$$\text{where } \delta = \begin{cases} 1 & \text{if } j \text{ odd} \\ 0 & \text{if } j \text{ even} \end{cases} .$$

We observe that the terms in each of these expressions are of the same order as those in the asymptotic expansion of $b(n)$ and so the correction term cannot be neglected entirely. However, the leading coefficient in the expansion of $b(n)$ is dominant, from which it follows that the dependence of the a_n on Poisson's ratio is slight. Furthermore, for a given value of $\bar{M}$, we expect a smaller relative error in the a'_n as Poisson's ratio approaches its limiting value $\frac{1}{2}$. For example, when $N = M = 50$ and $\bar{M} = 3$, a'_N approximates a_N with relative error less than 2.4×10^{-3} for all values of σ .

From (5.3.5), (5.3.6) and (A.3.6) we see that the a_n are of order $O(n^{-3/2})$. The series for the radial displacement and, *a fortiori*, the series for the radial stress will therefore be slowly convergent. Furthermore, we expect the stress series to diverge at the edge of the constriction. To increase the accuracy of our numerical result

we therefore employ the method for accelerating convergence illustrated in Appendix A.4.

Setting $y = \pi/2 - x$, we rewrite equation (5.2.14) for the displacement of the curved surface outside the contact area as

$$\frac{u(a, x) \cdot \pi a^2}{2h\sigma P^*(1+\sigma)} = \frac{1}{2} \bar{a}_0 + \sum_{m=1}^{\infty} (-1)^m a'_{2m} \cos(2my) + \sum_{m=0}^{\infty} (-1)^m a'_{2m+1} \sin(2m+1)y$$

$$+ \sum_{n=1}^{\infty} (a_n - a'_n) \cos nx \quad . \quad \begin{array}{l} 0 \leq x \leq \pi/2 \\ \pi/2 \geq y \geq 0 \end{array}$$

Inserting the asymptotic expressions for a'_n , the first two series can be summed to any degree of accuracy using the appropriate values for the parameter k in the results of (A.4.4) and (A.4.5). The third series is summed after truncation at $n = N$. The relative difference between a'_n and a_n leads to an estimate of the reduction of the error compared to a simple substitution and truncated sum in (5.2.14). A similar but more complicated analysis is applied to the series (5.2.15) for the surface radial stress under the constraint.

The values of a_n for $0 \leq n \leq 10$ obtained for $N = M = 100$ and $\bar{M} = 3$ are given in Table 1.

Table 1. Variation of a_n with n and Poisson's ratio.

	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 0.5$
$\bar{a}_0$	1.3131	1.2528	1.1980
a_1	0.9500	0.8993	0.8584
a_2	0.2301	0.2153	0.2020
a_3	-0.1639	-0.1596	-0.1555
a_4	-0.0900	-0.0857	-0.0818
a_5	0.0729	0.0712	0.0696
a_6	0.0502	0.0481	0.0462
a_7	-0.0432	-0.0422	-0.0413
a_8	-0.0330	-0.0317	-0.0306
a_9	0.0293	0.0286	0.0280
a_{10}	0.0238	0.0229	0.0221

The values of c_1 and c_2 defined by equations (5.2.10) and (5.2.11) are 1.543, 1.558, 1.572, and 0.693, 0.439, 0.191, for $\sigma = 0.3, 0.4$ and 0.5 respectively. Using these values of c_1 the radial stress over the contact area is evaluated in non-dimensional form and is illustrated in Figure 5. At the extremities of the contact area, $z/h = 0.5$, the stress is indeed singular as anticipated. Thus a region of plastic deformation will always be present and is localized to the outer regions of the contact area for low axial loads. The radial displacement exterior to the contact area is shown in Figure 6. It should be observed that the high dependence upon Poisson's ratio does not affect the curve but appears in the definition of the ordinate.

Fig. 5. The variation of the surface radial stress under the constriction with Poisson's ratio.

Fig. 6. The variation of the surface radial displacement with Poisson's ratio.

Comparing Figure 5 to the radial stress distribution given by Barton, we find that the latter result does not exhibit uniformity over the central region of the band. His method is based on the displacement produced by a uniform pressure distribution and this inconsistency probably arises through ignoring edge effects in this solution.

We finally remark that a preliminary solution to the dual series equations (5.1.16) was obtained by taking $M = 1$ and $\alpha_n = 0$ for $n \geq 2$. It follows from the magnitude of α_2 , that such an approximation leads to solutions for the a_n which are within 5% of actual values. In fact, a comparison of numerical results shows that the error in the stress distribution and displacements is still smaller, so that satisfactory results are achieved even when using this simpler approximation.

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Appendix A.1.

LOAD ACROSS AN AXIAL SECTION FOR MOONEY MATERIAL
UNDER FINITE AXIAL COMPRESSION

The cylinder has been uniformly compressed by an amount λ by the action of two smooth load surfaces acting at the ends of the cylinder. From (3.1.9) the load required to maintain this compression is

$$S = 2\pi a^2(1 - \lambda^3)(C_1/\lambda + C_2/\lambda^2)$$

A displacement perturbation has been introduced to account for the constriction of the cylinder over the central region of the curved surface where the load surfaces have been held fixed. Thus a change in the pressure exerted by the load plates is anticipated. The perturbation stress in the axial direction is denoted by $\epsilon P'_3$ and is obtained from (3.2.8). In particular

$$P'_3 = \tau'^{33} + \tau^{33} \frac{\partial w}{\partial z}.$$

Eliminating τ'^{33} by using (3.2.5) and (3.2.7) gives

$$P'_3 = p'_0 + p' + \frac{\partial^2 \phi_2}{\partial z^2} (2d_{44} - \tau^{33}).$$

From the first of (3.2.2) and from (3.2.4) it is seen that

$$p' = d_{55} \frac{\partial^2 \phi_2}{\partial z^2} - d_{44} \frac{\partial^2 \phi_1}{\partial z^2},$$

$$\text{giving } P'_3 = p'_0 + \frac{\partial^2 \phi_2}{\partial z^2} (2d_{44} + d_{55} - \tau^{33}) - d_{44} \frac{\partial^2 \phi_1}{\partial z^2}.$$

Eliminating τ^{33} by using (3.1.7) finally gives

$$P'_3 = p'_0 + 2(C_1 + C_2/\lambda) \{ (\lambda^2 + 2/\lambda) \frac{\partial^2 \phi_2}{\partial z^2} - \lambda^2 \frac{\partial^2 \phi_1}{\partial z^2} \}.$$

Finally, using the expressions for ϕ_1 and ϕ_2 given by (3.3.5) it follows that the perturbational stress across the surface $z = z_0$ is

$$P'_3 = p'_0 + \frac{4}{\lambda} (C_1 + C_2/\lambda) \sum_{n=1}^{\infty} \frac{\beta_n^2 B_n F_n(r)}{I_1(\lambda^{3/2} \beta_n a)} \cos \beta_n z_0, \quad (\text{A.1.1})$$

where

$$F_n(r) = \lambda^{3/2} I_1(\beta_n a) I_0(\lambda^{3/2} \beta_n r) - I_0(\beta_n r) I_1(\lambda^{3/2} \beta_n a).$$

The total load across the surface $z = z_0$ is

$$S^* = S - 2\pi \int_0^a \epsilon P'_3 r dr.$$

Using the standard result $\int_0^1 I_0(\xi t) t dt = I_1(\xi)/\xi$ it follows, after a simple change of variable, that

$$\int_0^a F_n(r) r dr = 0.$$

Thus integrating within the summation in (A.1.1) gives

$$S^* = S - \epsilon \pi a^2 p'_0. \quad (\text{A.1.2})$$

Thus the unknown hydrostatic pressure $\epsilon p'_0$ determines the increase in the axial load due to the effect of constricting the cylinder.

LOAD ACROSS AN AXIAL SECTION FOR CLASSICAL MATERIAL
UNDER INFINITESIMAL AXIAL EXTENSION

The cylinder is deformed uniformly by a radial stress p_o acting over the curved surface and a radial displacement u_o is also imposed at the outer surface. These conditions imply that the cylinder is deformed axially by an amount

$$w_o = \frac{z}{\sigma} \left\{ \frac{p_o}{E}(1+\sigma)(1-2\sigma) - \frac{u_o}{a} \right\} .$$

and loaded axially by a uniform stress

$$\sigma_z^o = \frac{-Eu_o}{\sigma a} + \frac{p_o(1-\sigma)}{\sigma} .$$

In addition, due to the constriction, there is a non-uniform stress distribution given by

$$\sigma_z = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) + \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \cdot \frac{\partial w}{\partial z} .$$

Expressing σ_z in terms of ϕ_1 and ϕ_2 by using (5.1.1) and then using the expression for ϕ_1 and ϕ_2 given by (5.1.8) it may be shown that

$$\sigma_z = \frac{E}{(1+\sigma)} \cdot \sum_{n=1}^{\infty} B_n \beta_n^2 \left\{ \beta_n a \frac{I_o(\beta_n r) I_o(\beta_n a)}{I_1(\beta_n a)} - \beta_n r I_1(\beta_n r) - 2I_o(\beta_n r) \right\} \cos(\beta_n z) .$$

The contribution of σ_z towards the total axial load is $2\pi \int_0^a \sigma_z \cdot r dr$

Using the result that

$$\int_0^a r^2 I_1(\beta_n r) dr = \frac{a^2}{\beta_n} I_o(\beta_n a) - \frac{2}{\beta_n} \int_0^a I_o(\beta_n r) \cdot r dr ,$$

and putting $\xi = \beta_n a$, $r = ta$ in the standard result

$$\int_0^1 I_1(\xi t) \cdot t dt = I_1(\xi) / \xi ,$$

it follows that $\int_0^a \sigma_z \cdot r dr = 0$.

Thus the only contribution to the axial load comes from the uniform solution and is

$$T = \pi a^2 \sigma_z^0$$

where σ_z^0 is given above.

Appendix A.3.

INTEGRAL EVALUATION AND ASYMPTOTIC ANALYSIS

(i) An integral which arises in our solution of the dual series equations of sections §4.2 and §5.2 is $I(t)$ defined by the equation

$$I(t) = \frac{\sqrt{2}}{\pi} \int_0^t \frac{x \sin x/2 \, dx}{\sqrt{(\cos x - \cos t)}} .$$

It is easily shown that this integral only exists for $-\pi < t < \pi$.

The evaluation is greatly simplified by considering

$$\frac{dI(t)}{dt} = \frac{\sqrt{2}}{\pi} \cdot \frac{d}{dt} \int_0^t \frac{x \sin x/2 \, dx}{\sqrt{(\cos x - \cos t)}} ,$$

which we first rewrite in the form

$$\frac{dI(t)}{dt} = -\frac{\sqrt{2}}{\pi} \frac{d}{dt} \int_0^t x \sec x/2 \left\{ \frac{d}{dx} (\cos x - \cos t)^{1/2} \right\} dx .$$

Integrating by parts and then performing the differentiation with respect to t leads to the expression

$$\frac{dI(t)}{dt} = \frac{\sqrt{2} \sin t}{2\pi} \int_0^t \frac{(\sec x/2 + x/2 \cdot \sec x/2 \tan x/2) dx}{\sqrt{(\cos x - \cos t)}} .$$

Letting $y = \tan x/2$, we find

$$\begin{aligned} \frac{dI(t)}{dt} &= \frac{1}{\pi} \sin t (1+\alpha^2)^{1/2} \int_0^\alpha \frac{1+y \tan^{-1} y}{\sqrt{(\alpha^2-y^2)}} dy , \text{ where } \alpha = \tan t/2 \\ &= \frac{1}{\pi} \sin t (1+\alpha^2)^{1/2} \left\{ \int_0^\alpha \frac{dy}{\sqrt{(\alpha^2-y^2)}} + \int_0^\alpha \frac{y \tan^{-1} y}{\sqrt{(\alpha^2-y^2)}} dy \right\} \\ &= \frac{1}{\pi} \sin t (1+\alpha^2)^{1/2} \int_0^\alpha \frac{(1+y^2)}{(1+y^2)\sqrt{(\alpha^2-y^2)}} dy , \text{ after an} \end{aligned}$$

integration by parts. Substituting $y = \alpha \sin u$ reduces this latter integral to

$$\begin{aligned}\frac{dI(t)}{dt} &= \frac{1}{\pi} \cdot \sin t (1+\alpha^2)^{3/2} \int_0^{\pi/2} \frac{du}{1+\alpha^2 \sin^2 u} \\ &= \frac{2}{\pi} \sin t (1+\alpha^2)^{3/2} \int_0^{\pi/2} \frac{du}{(2+\alpha^2)-\alpha^2 \cos 2u}\end{aligned}$$

from which, it follows that

$$\frac{dI(t)}{dt} = \frac{1}{\pi} \sin t (1+\alpha^2)^{3/2} \int_0^{\pi} \frac{dx}{(2+\alpha^2)-\alpha^2 \cos x}.$$

Using the standard result, $\int_0^{\pi} \frac{d\theta}{a+b \cos \theta} = \frac{\pi}{\sqrt{(a-b)(a+b)}}$, $|b| < |a|$, we finally obtain

$$\frac{dI(t)}{dt} = \tan t/2 \quad (\text{A.3.1})$$

$$\text{Therefore} \quad I(t) = 2 \log |\sec t/2| \quad (\text{A.3.2})$$

(ii) We now consider the value of the integral defined in (5.2.6) by

$$b(n) = \frac{4}{\pi} \int_0^c \cos(nx) \cdot \frac{\log \cos x/2 \sec c/2 + \sqrt{(\cos x/2 \sec c/2)^2 - 1}}{\log |\cos x/2 \sec c/2 + \sqrt{(\cos x/2 \sec c/2)^2 - 1}} dx \quad (\text{A.3.3})$$

This representation (A.3.3) can be further simplified by an integration by parts, whereupon we obtain

$$b(n) = \frac{2}{n\pi} \int_0^c \frac{\sin(nx) \sin x/2 dx}{\sqrt{\cos^2 x/2 - \cos^2 c/2}}$$

In particular, we require the asymptotic behaviour of $b(n)$ as $n \rightarrow \infty$ for the case $c = \pi/2$. With this value for c , we have

$$b(n) = \frac{2\sqrt{2}}{n\pi} \int_0^{\pi/2} \frac{\sin(nx) \sin x/2}{\sqrt{\cos x}} dx$$

We may therefore consider the asymptotic development of the complex Fourier integral

$$F(n) = \int_0^{\pi/2} \frac{e^{inx} \sin x/2}{\sqrt{\cos x}} dx \quad (\text{A.3.4})$$

and our result will follow by taking the imaginary part.

We observe that the main contribution to $F(n)$ comes from the neighbourhood of $x = \pi/2$ due to the singularity of the integrand at that point. The asymptotic behaviour does, however, in general depend on both end points. The analysis is simplified by the introduction of a neutralizer $v(x)$ (see, for example, Sirovich (13)), which isolates each endpoint and allows its contribution to be evaluated independently. If $v(x)$ is a neutralizer for (A.3.4) at $x = 0$, then v satisfies the conditions

$$v(0) = 1, \quad v(\pi/2) = 0,$$

$$\left. \frac{d^n}{dx^n} v(x) \right|_{x=0} = 0, \quad \text{for all } n \geq 1 \quad \text{as } 0, \pi/2 \text{ are}$$

$$\left. \frac{d^n}{dx^n} v(x) \right|_{x=\pi/2}$$

approached from the interior. It follows that $(1-v(x))$ is a neutralizer for $x = \pi/2$.

$F(n)$ is now written in the form

$$F(n) = \int_0^{\pi/2} \frac{e^{inx} \sin x/2}{\sqrt{\cos x}} v(x) dx + \int_0^{\pi/2} \frac{e^{inx} \sin x/2}{\sqrt{\cos x}} \{1-v(x)\} dx$$

$$= F_1(n) + F_2(n).$$

The asymptotic contribution from $F_2(n)$, for example, now depends entirely on the behaviour of its integrand near $x = \pi/2$. Accordingly,

we set $t = \pi/2 - x$ and obtain

$$F_2(n) = \frac{e^{in\pi/2}}{\sqrt{2}} \int_0^{\pi/2} \frac{e^{-int(\cos t/2 - \sin t/2)}}{\sqrt{\sin t}} \{1 - v(\pi/2 - t)\} dt .$$

Writing $\frac{\cos t/2 - \sin t/2}{\sqrt{\sin t}} = t^{-1/2} \phi(t)$, $0 \leq t < \infty$

$$\text{where } \phi(t) = \begin{cases} \frac{t^{1/2}(\cos t/2 - \sin t/2)}{\sqrt{\sin t}} \cdot \{1 - v(\pi/2 - t)\} & 0 \leq t \leq \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

it follows that $\phi \in C_0^\infty[0, \infty)$ and

$$F_2(n) = \frac{e^{in\pi/2}}{\sqrt{2}} \int_0^\infty e^{-int} t^{-1/2} \phi(t) dt .$$

Applying the standard result given in Sirovich (p. 78) we find

$$F_2(n) = \frac{e^{in\pi/2}}{\sqrt{2}} \sum_{k=0}^{N-1} \frac{\phi^{(k)}(0) (k-1/2)!}{k! n^{k+1/2}} e^{-i(k+1/2)\pi/2} + O(n^{-N}) \quad (\text{A.3.5})$$

as $n \rightarrow \infty$, for any $N \geq 1$.

A similar analysis applied to $F_1(n)$ leads to an asymptotic development which is entirely real and so contributes nothing to the expansion of $b(n)$ for large n .

It therefore follows that the first few terms of the asymptotic expansion of $b(n)$ are

$$\begin{aligned} b(2n) &= \frac{(-1)^{n+1}}{(2n)} \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{(2n)^{1/2}} - \frac{1}{4(2n)^{3/2}} + \frac{1}{32(2n)^{5/2}} - \dots \right\} , \\ b(2n+1) &= \frac{(-1)^n}{(2n+1)} \sqrt{\frac{2}{\pi}} \cdot \left\{ \frac{1}{(2n+1)^{1/2}} + \frac{1}{4(2n+1)^{3/2}} + \frac{1}{32(2n+1)^{5/2}} + \dots \right\} . \end{aligned} \quad (\text{A.3.6})$$

Appendix A.4.

ACCELERATING THE CONVERGENCE OF SERIES USING THE MELLIN TRANSFORM.

The essence of this method which transforms a slowly convergent series into a new series which is rapidly convergent is given in Titchmarsh (18) and is further exemplified by Macfarlane (9). The results are derived quite simply from the Mellin inversion formulae.

If a function f satisfies certain conditions (18, §1.29), the Mellin transform, F , of f is defined by

$$F(s) = \int_0^{\infty} f(x)x^{s-1}dx \quad . \quad (A.4.1)$$

The reciprocal formula is

$$f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)x^{-s}ds, \quad \sigma_1 < \sigma < \sigma_2 \quad . \quad (A.4.2)$$

The variable s is complex with real part σ and the limits σ_1, σ_2 define a vertical strip in the complex s -plane within which the integral (A.4.1) is convergent.

Replacing x by $(n+a)$ in (A.4.2) we have

$$f(n+a) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \cdot (n+a)^{-s} ds.$$

Summing over n and formally interchanging summation and integration, it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} f(n+a) &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \cdot \sum_{n=0}^{\infty} (n+a)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s) \zeta(s, a) ds, \quad \sigma_1 < \sigma < \sigma_2 \end{aligned} \quad (A.4.3)$$

provided $\sigma > 1$, where $\zeta(s, a)$ is the generalized Riemann Zeta function (21). The contour of integration is now distorted to include the poles of the integrand and the integral evaluated using Cauchy's theorem.

In particular we consider the series

$$(i) \quad \sum_{n=1}^{\infty} \frac{\cos(2nx)}{(2n)^k}, \quad 0 < x < \pi \quad ; \quad (ii) \quad \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^k}, \quad 0 < x < \pi$$

where $k > 0$ is a real parameter. We first observe the following property deduced from (A.4.1) and (A.4.2), that if $f(x)$ has Mellin transform $F(s)$ for $\sigma_1 < \sigma < \sigma_2$, then

$$x^{\rho} f(x), \quad F(s+\rho) \quad \sigma_1 - \rho < \sigma < \sigma_2 - \rho$$

$$\text{and} \quad f(ax), \quad a^{-s} F(s) \quad \sigma_1 < \sigma < \sigma_2$$

are Mellin pairs, where a and ρ are complex constants.

(i) The Mellin transform of

$$f(y) = \cos y \quad \text{is} \quad F(s) = \frac{2^{s-1} \sqrt{\pi} \Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2} - \frac{1}{2}s)} \quad 0 < \sigma < 1$$

so that

$$\frac{\cos y}{y^k} \quad \text{and} \quad \frac{2^{s-k-1} \sqrt{\pi} \Gamma(\frac{s-k}{2})}{\Gamma(\frac{1+k}{2} - \frac{1}{2}s)} \quad k < \sigma < 1+k$$

are reciprocal transforms. Thus

$$\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2nx)^k} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \sqrt{\pi} 2^{s-k-1} \Gamma(\frac{s-k}{2}) (2x)^{-s} \zeta(s) ds$$

where $\zeta(s, 1) = \zeta(s)$ is the Zeta function of Riemann.

Now, $\Gamma(\frac{s-k}{2})$ has poles at $s = -2n + k$, $n = 0, 1, 2, \dots$ with

residues $\frac{2(-1)^n}{\Gamma(n+1)}$; $\zeta(s)$ has a pole at $s = 1$ with residue 1. Furthermore,

the integrand tends to zero as $s \rightarrow -\infty$. Therefore, by Cauchy's theorem,

$$\sum_{n=1}^{\infty} \frac{\cos(2nx)}{(2n)^k} = 2^{-k-1} \sqrt{\pi} \left\{ \frac{\Gamma(\frac{1-k}{2}) x^{k-1}}{\Gamma(\frac{1}{2}k)} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \zeta(k-2n)}{\Gamma(n+\frac{1}{2}) \Gamma(n+1)} \right\} \quad (\text{A.4.4})$$

which is rapidly convergent. The condition $\sigma > 1$ in (A.3.3) is satisfied provided $k > 0$, a result we anticipate since this also implies the convergence of the cosine series (for $x \neq 0, \pm 2\pi, \dots$). When $0 < k \leq 1$, the series is divergent at $x = 0$ and this behaviour is reflected in the first term of (A.4.4).

(ii) We similarly obtain the Mellin pairs

$$\frac{\sin(2n+1)x}{[(2n+1)x]^k}, \quad (2n+1)^{-s} \cdot \frac{2^{s-k-1} \sqrt{\pi} \Gamma(\frac{1-k}{2} + \frac{s}{2})}{\Gamma(\frac{2+k}{2} - \frac{s}{2})}, \quad -1 + k < \sigma < 1 + k$$

from which follows the result

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^k} &= \sqrt{\pi} \left\{ \frac{2^{-k-1} \Gamma(1-k/2) x^{k-1}}{\Gamma(\frac{1}{2} + \frac{1}{2}k)} \right. \\ &+ \sum_{n=0}^{\infty} (-1)^n \frac{2^{-2n-1} x^{2n+1} [1-2^{2n+1-k}]}{\Gamma(n+1) \Gamma(n+3/2)} \zeta(k-2n-1) \left. \right\}, \quad (\text{A.4.5}) \end{aligned}$$

provided $0 < x < 2\pi$, when $0 < k \leq 1$.