A GENERAL CARTAN THEORY

by

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ABSTRACT

Recent results of Jacobson and Barnes indicate that Lie, Jordan and alternative algebras may have a common Cartan theory. In this thesis, we show this is indeed the case. We also show that for certain classes of non-associative algebras, called E-classes, that possess an Engel function, a general Cartan theory is possible.

In Chapter One, a generalization of nilpotence and solvability is introduced that permits our Cartan theory for E-classes. In Chapter Two, we construct Cartan subalgebras for alternative algebras based on a given Engel function. Jacobson's Cartan theory for Jordan algebras is given in Chapter Three along with our extensions of his results. We point out that the Engel function for alternative algebras and Jordan algebras coincides, and may be used to give the classical Cartan theory for Lie algebras

Commutative power associative algebras are discussed in Chapter Four, and some results are obtained.

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INTRODUCTION

The concept of a Cartan subalgebra plays a key role in the structure theory of Lie algebras. In 1966, Jacobson [16] introduced the notion of a Cartan subalgebra for Jordan algebras, and showed that an analogous Cartan theory is valid for Jordan algebras. His main results showed that in any finite dimensional Jordan algebra J over an infinite field F there do exist Cartan subalgebras. He also proved a conjugacy theorem for Cartan subalgebras of J when F is algebraically closed and of characteristic zero.

Because of the close relationship of Jordan, Lie and alternative algebras, the following question arises: does there exist a common Cartan theory for Lie, Jordan, and alternative algebras ? In this thesis, we give an affirmative answer to this question.

In Chapter One, we introduce a generalization of nilpotence and solvability. We find that for finite dimensional algebras, the generalized solvability is a radical property. We then define an Engel function. Using these functions and some linear algebra arguments of Barnes [8], we obtain a general Cartan theory. Finally, we collect those results of Chevalley [11] that are necessary for the conjugacy theorems.

In Chapter Two, we develop the Cartan theory for alternative algebras, and show that it parallels the theory for Lie algebras. Our Cartan subalgebras are characterized as minimal Engel subalgebras. As a result, we show that if the ground field F has "enough" elements and characteristic different than 2, then alternative algebras will always have Cartan subalgebras. Furthermore, if F is algebraically closed and of characteristic zero, any pair of Cartan subalgebras is conjugate.

Chapter Three contains a sketch of Jacobson's Cartan theory for Jordan algebras. We add to his theory our characterization of Cartan subalgebras as minimal Engel subalgebras, thus extending his existence theorem for Cartan subalgebras to Jordan algebras over finite fields having "enough" elements. We close Chapter Three with a discussion of associator solvable Jordan algebras, and introduce a class of nilpotent derivations.

Because of two recent results [6] and [22], we try in Chapter Four to extend the Cartan theory to commutative power associative algebras. We prove that if X is a commutative power associative algebra with unity and stable in the sense of Albert [4], and R is an A-nilpotent subalgebra containing 1, then R can be used to generate a nilpotent Lie algebra of linear transformation of X. If X is not stable, then we can only prove that this Lie algebra is solvable. An example shows that this result cannot be improved. There is also trouble in the existence of Cartan subalgebras. For to prove this, we need to know that nil algebras are nilpotent. This is a difficult unsolved problem.

We remark that no study was made into forms of uniqueness of Engel functions, or if different Engel functions could result in different Cartan theory.

PRELIMINARIES

Suppose U is a vector space over a field F. We will say that U is an <u>algebra</u> if and only if there is a multiplication defined on U, denoted by ab for $a,b \in U$, such that:

$$a(b+c) = ab + ac \qquad (a+b)c = ac + bc \qquad a,b,c \in U$$

$$\alpha(ab) = (\alpha a)b = a(\alpha b) \qquad \alpha \in F,\ a,b \in U$$

The <u>commutator</u> $[x,y]$ is defined $[x,y] = xy - yx$. If $[a,b] = 0$ for all $a,b \in U$, then U is commutative. The <u>associator</u> (x,y,z) is defined $(x,y,z) = xy \cdot z - x \cdot yz$. If $(a,b,c) = 0$ for all $a,b,c \in U$, then U is associative.

We note that in any algebra U we have the following identities:

$$\text{(P-1)} \quad (wx,y,z) - (w,xy,z) + (w,x,yz) - w(x,y,z) - (w,x,y)z = 0$$

$$\text{(P-2)} \quad [xy,z] - x[y,z] - [x,z]y - (x,y,z) - (z,x,y) + (x,z,y) = 0 .$$

We also note that the standard concepts of subalgebra, ideal, homomorphism, isomorphism, and anti-isomorphism carry over from associative algebras to non-associative algebras. If A is an ideal of U, we will write $A \triangleleft U$. Furthermore, the fundamental isomorphism theorems are valid. When we speak of a class $\mathcal{U}$ of algebras, we will assume that $\mathcal{U}$ is homomorphically closed. If K is an extension of F , by U_K we mean $U \otimes_F K$. For a general introduction to non-associative algebras, the reader is referred to Schafer [25].

An algebra L is a Lie algebra if and only if $[x,x] = 0$ and $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ where $[x,y]$ denotes multiplication in L . For a theory of Lie algebras, the reader is referred to Jacobson [18].

A linear transformation D on U is called a <u>derivation</u> if and only if for all $x,y \in U$, $xyD = xD \cdot y + x \cdot yD$. The set of derivations of U forms a Lie algebra, denoted by D(U) , and D(U) is called the <u>derivation algebra</u> of U .

Most of the notation in the thesis is standard. Definitions are indicated by underlining the term being defined. Theorems, lemmas,and corollaries are numbered with three integers to denote the chapter, section and order in which they appear.

CHAPTER ONE

GENERAL CARTAN THEORY

1.1 Generalized Solvable Radical Properties

Suppose P is a property that an algebra may possess. We say an algebra A is a P-algebra if it possesses the property P. An ideal of A is called a P-ideal if, as an algebra, it is a P-algebra. We say that P is a radical property if the following three conditions are satisfied:

(A) Any homomorphic image of a P-algebra is a P-algebra.

(B) Every algebra A contains a P-ideal which contains every other P-ideal of A. We denote this ideal by $P(A)$, and call $P(A)$ the P-radical of A.

(C) For every algebra A, $P(A/P(A)) = 0$.

If P is a radical property and A is an algebra such that $P(A) = 0$, then A is call P semi simple, whereas if $P(A) = A$, A is called a P radical algebra.

We recall that for finite dimensional algebras, solvability is a radical property. We will refer to this as the classical radical. We begin our study by introducing several new properties for finite dimensional algebras that generalize the concepts of solvable and nilpotence.

Throughout the rest of Chapter One, U will denote a finite dimensional algebra over an arbitrary field F.

Suppose $f(x_1,\ldots,x_n)$ is a linear homogeneous element of the free non-associative algebra on the n generators $x_1,\ldots,x_n$ over F. Then for n elements $u_1,\ldots,u_n$ of U, $f(u_1,\ldots,u_n)$ is an element of U.

We set $f^1(u_1,\ldots,u_n) = f(u_1,\ldots,u_n)$ for any set $\{u_1,\ldots,u_n\}$ of n elements from U, and for $k > 1$ and $k(n-1)+1$ elements $u_1,\ldots,u_{k(n-1)+1}$ in U,

$$f^k(u_1,\ldots,u_{k(n-1)+1}) =$$

$$f^1(f^{k-1}(u_1,\ldots,u_{(k-1)(n-1)+1}),u_{(k-1)(n-1)+2},\ldots,u_{k(n-1)+1}).$$

We will say U is <u>f-nilpotent</u> if there is a $k > 0$ such that for all sets $\{u_1,\ldots,u_{k(n-1)+1}\}$ of elements from U, we have $f^k(u_1,\ldots,u_{k(n-1)+1}) = 0$.

Next we set $f^{(1)}(u_1,\ldots,u_n) = f(u_1,\ldots,u_n)$ for any set $\{u_1,\ldots,u_n\}$ of n elements from U, and for $k > 1$ and n^k elements $u_1,\ldots,u_{n^k}$ from U,

$$(1)\qquad f^{(k)}(u_1,\ldots,u_{n^k}) =$$

$$f^{(1)}(f^{(k-1)}(u_1,\ldots,u_{n^{k-1}}),f^{(k-1)}(u_{n^{k-1}+1},\ldots,u_{2n^{k-1}}),$$

$$\ldots,f^{(k-1)}(u_{(n-1)n^{k-1}},\ldots,u_{n^k})).$$

We will call U <u>f-solvable</u> if there is a $k > 0$ such that for

all sets $u_1,\ldots,u_{n^k}$ of elements from U, $f^{(k)}(u_1,\ldots,u_{n^k}) = 0$.
Henceforth, we will write the right hand side of (1) as $f^{(1)}(f^{(k-1)}(\ \ ),f^{(k-1)}(\ \ ),\ldots,f^{(k-1)}(\ \ ))$.

We note that $f(x_1,x_2) = x_1x_2$ is an element in the free non-associative algebra on the two generators x_1 and x_2 for any field F. Hence, f-solvable and f-nilpotence with respect to this element are just ordinary solvability and nilpotence.

We begin our study of f-solvability with

<u>Lemma 1.1.1</u> Let $k > 0$ and $\{u_i\}_{i-1}^{n^k}$ be n^k elements from U. Then for any i, $1 \leq i < k$,

$$f^{(k)}(u_1,\ldots,u_{n^k}) = f^{(i)}(f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ )).$$

<u>Proof</u>: The proof is by induction on k. If $k = 1$ or $k = 2$, there is nothing to prove. By definition, we have

$$(1)\quad f^{(k)}(u_1,\ldots,u_{n^k}) = f^{(1)}(f^{(k-1)}(\ \ ),\ldots,f^{(k-1)}(\ \ )).$$

Hence, if $i = 1$, (1) is the desired result. Assuming $1 < i$, then $0 < i-1 < k-1$, and by the induction hypothesis,

$$f^{(k-1)}(\ \ ) = f^{(i-1)}(f^{((k-1)-(i-1))}(\ \ ),\ldots,f^{((k-1)-(i-1))}(\ \ ))$$
$$= f^{(i-1)}(f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ )).$$

By (1), we have

$$(2)\quad f^{(k)}(\ \ ) = f^{(1)}(f^{(i-1)}(f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ )),\ldots, f^{(i-1)}(f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ ))).$$

Since $1+(i-1) = i < k$, we may apply the induction hypothesis to the right hand side of (2) to obtain

$$f^{(k)}(\ \ ) = f^{(1+(i-1))}(f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ ))$$

$$= f^{(i)}f^{(k-i)}(\ \ ),\ldots,f^{(k-i)}(\ \ ))$$

which was to be proved. Q.E.D.

As an immediate consequence, we have

<u>Lemma 1.1.2</u> If A is an ideal of U and both U/A and A are f-solvable, then U is f-solvable.

<u>Proof</u>: Since U/A is f-solvable, there is a $k_1 > 0$ such that for all sets $\{u_i\}_{i=1}^{n^{k_1}}$ of elements from U,

$$f^{(k_1)}(u_1,\ldots,n_{n^{k_1}}) \in A .$$

But A is f-solvable, hence there is a $k_2 > 0$ such that for all sets $\{u_i\}_{i=1}^{n^{k_2}}$ of elements from A, $f^{(k_2)}(\ \ ) = 0$. Thus for all sets $\{u_i\}_{i=1}^{n^{k_1+k_2}}$ of elements from U,

$$f^{(k_1+k_2)}(\ \ ) = f^{(k_2)}(f^{(k_1)}(\ \ ),\ldots,f^{(k_1)}(\ \ )) = 0$$

which implies U is f-solvable. Q.E.D.

<u>Lemma 1.1.3</u> Suppose U is f-solvable and V is another finite dimensional algebra over F. If σ is a homomorphism of U onto V, then V is f-solvable.

<u>Proof</u>: Since U is f-solvable, there is a $k > 0$ such that for all sets of n^k elements $\{u_i\}_{i=1}^{n^k}$ from U, $f^{(k)}(u_1,\ldots,u_{n^k}) = 0$. Suppose $\{v_i\}_{i=1}^{n^k}$ is an arbitrary set of n^k elements from V. Since σ is onto V, there are elements w_i in U such that $v_i = \sigma(w_i)$ for $i = 1,\ldots,n^k$. Then,

$$f^{(k)}(v_1,\ldots,v_{n^k}) = f^{(k)}(\sigma(w_1),\ldots,\sigma(w_{n^k})) = \sigma f^{(k)}(w_1,\ldots,w_{n^k}) = 0$$

which implies V is f-solvable. Q.E.D.

As a result of Lemma 1.1.3, we have

<u>Lemma 1.1.4</u> The sum of two f-solvable ideals of U is an f-solvable ideal.

<u>Proof</u>: Suppose A and B are f-solvable ideals of U. By an isomorphism theorem, we have $(A+B)/B \cong A/(A\cap B)$. Applying Lemma 1.1.3, we see that $A/(A\cap B)$ is f-solvable, thus $(A+B)/B$ is f-solvable. By Lemma 1.1.2, $(A+B)$ is f-solvable. Q.E.D.

We now prove

<u>Theorem 1.1.5</u> For finite dimensional algebras over F, f-solvable is a radical property.

<u>Proof</u>: (A) is a consequence of Lemma 1.1.3. Now write $f(U)$

for the f-solvable ideal of U of maximum dimension. It follows from Lemma 1.1.4 that $f(U)$ contains all f-solvable ideals of U, and (B) follows. Now suppose $f(U/f(U))$ is not zero. Then there is a nonzero ideal $I/f(U)$ of $U/f(U)$ such that $I/f(U)$ is f-solvable. By Lemma 1.1.2, I is an f-solvable ideal of U, hence $I \subset f(U)$, and $I/f(U) = 0$, a contradiction. We conclude that $f(U/f(U)) = 0$, which proves (C). Q.E.D.

1.2 Properties of f-solvability

It is known that if U is nilpotent, then U is solvable. An easy induction argument on k shows that $f^{(k)}(U) \subseteq f^k(U)$, where $f^k(U)$ denotes the subspace of U spanned by $f^k(u_1,\ldots,u_{k(n-1)+1})$, and similarly for $f^{(k)}(U)$. Consequently, if U is f-nilpotent, then $f^k(U) = 0$ for some k, so $f^{(k)}(U) = 0$ and U is f-solvable. We have

Property 1.2.1 f-nilpotence implies f-solvability.

The converse of Property 1.2.1 is known to be false. Indeed, let $f(x_1,x_2) = x_1x_2$ in the case of Lie algebras.

Property 1.2.2 f-solvability is a hereditary radical in the sense that ideals of f-solvable algebras are f-solvable.

Proof: The proof is immediate, and notes that subalgebras of f-solvable algebras are f-solvable. Q.E.D.

Property 1.2.3 If K is an extension of F, then U is f-solvable (f-nilpotent) if and only if $U_K = K \otimes_F U$ is f-solvable (f-nilpotent).

Proof: Now $f(x_1,\ldots,x_n)$ is an element in the free non-associative algebra on the n generators $x_1,\ldots,x_n$ over F. Since K is an extension of F, certainly f is an element in the free non-associative algebra on the generators $x_1,\ldots,x_n$ over K. Therefore it makes sense to apply f to elements of U_K. Thus if U_K is f-solvable (f-nilpotent), clearly U is f-solvable (f-nilpotent).

Conversely, if $\{u_i\}_{i=1}^m$ is a basis for U_F, then $\{1 \otimes u_i\}_{i=1}^m$ is a basis for U_K. Therefore, since f is multilinear, we see that U_F f-solvable (f-nilpotent) implies U_K is f-solvable (f-nilpotent). Q.E.D.

Property 1.2.4 If U is a direct sum of ideals U_i $i = 1,\ldots,p$, then U is f-solvable (f-nilpotent) if and only if U_i is f-solvable (f-nilpotent) for all i.

Proof: If U is f-solvable (f-nilpotent), clearly each U_i is f-solvable (f-nilpotent). The converse follows because f is multilinear. Q.E.D.

Throughout the rest of this thesis, we will assume the degree n of $f(x_1,\ldots,x_n)$ is greater than one.

Property 1.2.5 If U is solvable, then U is f-solvable.

<u>Proof</u>: Suppose $U^2 = 0$. Then clearly U is f-solvable.

We now define $U^{(1)} = U$, and for $k > 1$, $U^{(k)} = U^{(k-1)}U^{(k-1)}$. Then, since U is solvable, the derived series terminates after k steps, that is,

$$U = U^{(1)} \triangleright U^{(2)} \triangleright \ldots \; U^{(k-1)} \triangleright U^{(k)} = 0$$

where $U^{(k-1)} \neq 0$. Observe that $(U^{(k-1)})^2 = 0$. Thus $U^{(k-1)}$ is f solvable. Now $U^{(k-2)} \triangleright U^{(k-1)}$ and since $(U^{(k-2)})^2 = U^{(k-1)}$, we have $(U^{(k-2)}/U^{(k-1)})^2 = 0$. Consequently $U^{(k-2)}/U^{(k-1)}$ is f-solvable, and by Lemma 1.2.2, $U^{(k-2)}$ is f-solvable. Repeating this process with $U^{(k-3)}$ and $U^{(k-2)}$, we see $U^{(k-3)}$ is f-solvable. Since this process must terminate after a finite number of steps, we conclude that U is f-solvable. Q.E.D.

Suppose L is a semi-simple Lie algebra, and let $f(x_1,x_2) = x_1x_2 + x_2x_1$. Then L is an f-radical algebra. This example shows that the converse of Property 1.2.5 is false.

What Property 1.2.5 shows is that if $S(U)$ is the solvable radical of U, then $S(U) \subseteq f(U)$. This example above shows that this inclusion will in general be proper. However, we can show

<u>Property 1.2.6</u> If $f(x_1,\ldots x_n)$ is a monomial, then $S(U) = f(U)$.

<u>Proof</u>: Let $f(x_1,x_2) = x_1x_2$. If $n = 2$, we may, without loss of generality, suppose $f = g$, and the result is immediate.

Suppose $n = 3$. Then without loss of generality, we may let $f(x_1, x_2, x_3) = x_1 x_2 \cdot x_3$. For elements u_1, u_2, u_3, u_4 of U, we observe that $g^{(2)}(u_1, u_2, u_3, u_4) = f(u_1, u_2, u_3, u_4)$, and indeed any 2^k elements $u_1, \ldots, u_{2^k}$ from U can be grouped into three elements y_1, y_2, y_3 from U such that $g^{(k)}(u_1, \ldots, u_{2^k}) = f(y_1, y_2, y_3)$. An easy induction argument shows this is true for any n. Hence, suppose U is f-solvable. Then there is a k_1 such that for all sets of k_1 elements from U, $f^{(k_1)}(\quad) = 0$. The above argument shows that there is a k_2 such that any set $\{u_i\}_{i=1}^{2^{k_2}}$ of 2^{k_2} elements from U can be grouped into a set $\{y_i\}_{i=1}^{n^{k_1}}$ of n^{k_1} elements from U such

$$g^{(k_2)}(u_1, \ldots, u_{2^{k_2}}) = f^{(k_1)}(y_1, \ldots, y_{n^{k_1}}) .$$

Since $f^{(k_1)}(\quad) = 0$, this shows $g^{(k_2)}(\quad) = 0$, and we have $U^{(2k_2)} = 0$. Q.E.D.

1.3 f-nil algebras

Suppose $u_1, \ldots, u_{n-1}$ are $n-1$ arbitrary elements from U. We will write $S(u_1, \ldots, u_{n-1})$ for the map from U to U defined by

$$xS(u_1, \ldots, u_{n-1}) = f(x, u_1, \ldots, u_{n-1})$$

for all x in U.

We observe that U is f-nilpotent if and only if $S(u_1,\ldots,u_{n-1})\cdot S(v_1,\ldots,v_{n-1})\cdot\ldots\cdot S(w_1,\ldots,w_{n-1}) = 0$ for all $u_i,v_i,\ldots,w_i$ in U. In particular, if U is f-nilpotent, then $S(u_1,\ldots,u_{n-1})$ is a nilpotent map on U. Therefore, we will call an element u of U f-nilpotent if and only if $S(u,\ldots,u)$ is a nilpotent map. U is said to be _f-nil_ if and only if each element of U is f-nilpotent.

Lemma 1.3.1 If the dimension of U is m, then u is f-nilpotent if and only if $S(u,\ldots,u)^m = 0$.

Proof: If $S(u,\ldots,u)^m = 0$, then u is f-nilpotent. Conversely, suppose $S(u,\ldots,u)^{m'} = 0$. Then the characteristic polynomial $\varphi(\lambda,u)$ of $S(u,\ldots,u)$ must be λ^m. If not, there is an irreducible factor $\pi(\lambda) \neq \lambda$ in the factorization over F of $\varphi(\lambda,u)$, which implies there is a $v \in U$ such that $vS(u,\ldots,u)^k \neq 0$ for all k, which is impossible. Since $S(u,\ldots,u)$ is a root of $\varphi(\lambda,u) = \lambda^m$, we have $S(u,\ldots,u)^m = 0$.

Q.E.D.

It is clear that if U is f-nilpotent, then U is f-nil. We are interested in the converse, that is, when does f-nil imply f-nilpotent. To study this problem, we need to know that under suitable conditions on F, f-nil is preserved under field extensions. As in Property 1.2.3, we see that if K extends F, it makes sense to talk about U_K being f-nil.

Lemma 1.3.2 If the dimension of U is m and F has at least nm+1 elements, then U is f-nil if and only if U_K is f-nil for all extensions K of F.

Proof: If U_K is f-nil, it is clear that U is itself f-nil.

Conversely, suppose $u_1, \ldots, u_m$ is a basis of U over F. We consider the map

$$S(\alpha_1 u_1 + \ldots + \alpha_m u_m, \alpha_1 u_1 + \ldots + \alpha_m u_m, \ldots, \alpha_1 u_1 + \ldots + \alpha_m u_m)$$

where $\alpha_i \in F$ and $u = \alpha_1 u_1 + \ldots + \alpha_m u_m$. Since S is linear in each of its arguments, we have

$$0 = S(u, \ldots, u)^m =$$

$$= \Big(\sum_{i_1, \ldots, i_m = 1}^{m} \alpha_{i_1} \cdots \alpha_{i_m} S(u_{i_1}, \ldots, u_{i_m}) \Big)^m = T$$

Let T'_{i_1} equal the sum of the terms of T where α_1 appears to the i_1 power. Set $T'_{i_1} = (\alpha_1^{i_1}) T_{i_1}$. Then

$$T = \sum_{i_1 = 0}^{mm} \alpha_1^{i_1} T_{i_1} .$$

However, $T_{nm} = S(u_1, \ldots, u_1)^m = 0$ and

$$T_o = S(\alpha_2 u_2 + \ldots + \alpha_m u_m, \ldots, \alpha_2 u_2 + \ldots + \alpha_m u_m)^m = 0 .$$

Therefore $T = \sum_{i_1 = 1}^{nm-1} \alpha_1^{i_1} T_{i_1} = 0$. Since F has at least nm+1 elements, we choose nm-1 different non-zero values for α_1.

This yields a system of nm-1 homogeneous equations in T_{i_1}, $i_1 = 1,\ldots,nm-1$, whose matrix of coefficients is a Vandermonde matrix. Thus we conclude $T_{i_1} = 0$ for $i_1 = 0,\ldots,nm$.

Now let $T'_{i_1 i_2}$ be the sum of the terms of T_{i_1} where α_2 appears to the i_2 power and set $T'_{i_1 i_2} = \alpha_2^{i_2} T_{i_1 i_2}$. Hence $T_{i_1} = \sum_{i_2=0}^{nm-i_1} \alpha_2^{i_2} T_{i_1 i_2} = 0$. If $i_1 \neq 0$, we choose $nm-i_1+1$ different values for α_2 and conclude as in the previous case that $T_{i_1 i_2} = 0$. If $i_1 = 0$, we have $T_0 = 0$, or $S(\alpha_2 u_2+\ldots+\alpha_m u_m,\ldots,\alpha_2 u_2+\ldots+\alpha_m u_m)^m = 0$. Repeating the above process on this map, we conclude $T_{0\, i_2} = 0$.

Continuing this method, we have $T_{i_1\ldots i_m} = 0$ for all m-tuples $(i_1,\ldots,i_m)$ such that $i_1+\ldots+i_m = nm$.

Now $\{1\otimes u_i\}_{i=1}^m$ is a basis of U_K. Thus, if $v \in U_K$, we may write $v = \sum_{i=1}^m \xi_i(1\otimes u_i) = \sum_{i=1}^m \xi_i u_i$, where $\xi_i \in K$.

Then $S(v,\ldots,v)^m = S(\xi_1 u_1+\ldots+\xi_m u_m,\ldots,\xi_1 u_1+\ldots+\xi_m u_m)^m$

$$= \sum_{\substack{i_1,\ldots,i_m \\ i_1+\ldots+i_m=nm}} \xi_i^{i_1}\ldots\xi_m^{i_m} T_{i_1\ldots i_m} .$$

But $T_{i_1\ldots i_m} = 0$. Consequently, $S(v,\ldots,v)^m = 0$ for all v in U_K, and U_K is f-nil. Q.E.D.

As a consequence of the method of proof of this lemma, we have

Lemma 1.3.3 If V is a subalgebra of U, where U and F satisfy the hypothesis of Lemma 1.3.2, and if $S(v,\ldots,v)$ is a nilpotent transformation of U for all v in V, then for all extensions K of F, $S(v',\ldots,v')$ is a nilpotent transformation of U_K for all v' in V_K.

We now set $B_{u,f} = \{x \in U : xS(u,\ldots,u)^m = 0\}$ where the dimension of U is m. When no ambiguity over the f arises, we will simply write B_u. We see that u is f-nilpotent if and only if $B_u = U$.

Lemma 1.3.4 Suppose every maximal subalgebra of U is an ideal of U and that B_u is a subalgebra of U containing u. Then $B_u = U$.

Proof: We note that the only ideal of U that contains B_u is U itself. For $S(u,\ldots,u) = S$ is a transformation of U, hence by Fitting's lemma, $U = U_{0S} + U_{1S}$ where $U_{0S} = \{x \in U : xS^m = 0\} = B_u$, U_{iS}, $i = 0,1$, are invariant under S, and S restricted to U_{1S} is an isomorphism. Now, suppose $B_u \subset I \triangleleft U$. Since $u \in B_u$, $IS \leq I$. But $U_{1S}S \leq I$ as $u \in I$. Consequently, $U = U_{0S} \oplus U_{1S} \leq I$, which implies

$U = I$. Thus B_u is a maximal subalgebra, and $B_u = U$. Q.E.D.

A class $\mathcal{U}$ of finite dimensional algebras is called an E-class over F if and only if the following three conditions are satisfied:

(A) If $U \in \mathcal{U}$, then the ground field of U is either F or an extension K of F .

(B) If $U \in \mathcal{U}$, then $U_K \in \mathcal{U}$ for all extensions K of F.

(C) If V is a subalgebra of an algebra in $\mathcal{U}$, then $V \in \mathcal{U}$.

We will say that f is an Engel function for the E-class $\mathcal{U}$ if and only if for all algebras $U \in \mathcal{U}$,

(D) U is f-nilpotent if and only if U is f-nil.

(E) B_u is a subalgebra of U containing u .

If f is an Engel function, the B_u are called Engel subalgebras.

We now give a corollary to Lemma 1.3.4.

Corollary 1.3.5 If f is an Engel function for the E-class $\mathcal{U}$, $U \in \mathcal{U}$, and every maximal subalgebra of U is an ideal, then U is f-nilpotent.

Proof: By Lemma 1.3.4, $B_u = U$ for all $u \in U$. Consequently U is f-nil. But f is an Engel function, so U is f-nilpotent.

Q.E.D.

1.4 Cartan Subalgebras

Suppose L is a nilpotent Lie algebra of linear transformations of an m-dimensional vector space W over F. Then we can write $W = W_0 \oplus W_1$ where W_0 and W_1 are respectively the Fitting null and one component of W relative to L. We have

$$W_0 = \{w \in W : w\ell^m = 0 \text{ for all } \ell \in L\}$$

and

$$W_1 = \bigcap_i W(L^*)^i$$

where L^* is the subalgebra generated by L and the identity in the enveloping associative algebra $C(L)$ of L [18]. If K is an extension of F, then L_K is a nilpotent Lie algebra of linear transformations on W_K. Since $(W_0)(L^*)^m = 0$ and elements of L^*_K are K linear combinations of elements of L^*, we have that $(W_0)_K = (W_K)_0$. Similarly, $(W_1)_K = (W_K)_1$. Furthermore, we have

Lemma 1.4.1 (Jacobson) W_1 can be characterized as any complimentary subspace of W_0 which is invariant under L.

Proof: Suppose N is such a compliment. Then $W = W_0 \oplus N$. However, L acts on N as a nilpotent Lie algebra of linear transformations. Therefore N has a Fitting decomposition $N = N_0 \oplus N_1$ relative to L restricted to N. But N_0 must be zero. For if $n \in N_0$ then $n\ell_N^{m'} = 0$ where m' is the dimension of N and ℓ_N is the restriction to N of $\ell \in L$.

This implies that $n \in W_0 \cap N_0 = 0$ since $N_0 \subset N = W$. Thus $N = N_1 \subseteq W_1$ and $N = W_1$. Q.E.D.

Let $\mathcal{U}$ be an E-class of algebras over F, and $f = f(x_1,\ldots,x_n)$ an Engel function for $\mathcal{U}$. Let $U \in \mathcal{U}$ and V be a subalgebra of U. By $\underline{L_U(V)}$ we will mean the Lie algebra of linear transformations on the vector space U generated by $S(v_1,\ldots,v_{n-1})$, $v_i \in V$, $i = 1,\ldots,n-1$.

We will be interested in subalgebras V of U for which $L_U(V)$ is nilpotent. Indeed, if $L_U(V)$ is nilpotent, we can decompose U into $U_0 \oplus U_1$ where U_0 and U_1 are the Fitting null and one components of U relative to $L_U(V)$. We would like to be able to say that under these circumstances, $V \subset U_0$. What will happen in our theory is that if V is f-nilpotent, then $L_U(V)$ is nilpotent and $V \subset U_0$. From the remarks preceding Lemma 1.4.1, we see we will be able to study this problem by extending F to its algebraic closure.

Motivated by these observations, we make the following definition. We say a subalgebra H of U is a <u>Cartan subalgebra</u> if and only if:

(A) H is f-nilpotent.

(B) $L_U(H)$ is nilpotent.

(C) H coincides with the Fitting null component of U relative to $L_U(H)$.

In addition, if U contains a unity, then

(D) H contains the unity of U.

The class of finite dimensional Lie algebras is an E-class for any F, and $f(x_1,x_2) = x_1x_2$ is an Engel function for all members of this class. As is well-known, (A), (B), and (C) give the definition of Cartan subalgebras of Lie algebras.

Let $\mathcal{U}$ be an E-class over F and f an Engel function for $\mathcal{U}$. The subalgebra B_u will be called <u>minimal Engel in U</u> if the dimension of B_u is minimal. Clearly if B_u is minimal Engel in U and $B_v \subset B_u$, then $B_v = B_u$.

In the Lie theory, Barnes [8] has shown that if L is a Lie algebra of dimension m and F has at least m elements, then H is a Cartan subalgebra of L if and only if H is minimal Engel in L, where the Engel function for L is the one given above. We would like to develop a similar theory for our Engel class $\mathcal{U}$.

To study this problem, we need some more information on extending F. If V is a subalgerba of $U \in \mathcal{U}$ and if K is an extension of F, then $(L_U(V))_K = L_{U_K}(V_K)$. Hence, if $L_U(V)$ is nilpotent, then $L_{U_K}(V_K)$ is nilpotent. Since we have $(U_0)_K = (U_K)_0$ and V f-nilpotent implies V_K is f-nilpotent, it follows that V is a Cartan subalgebra of U if and only if V_K is a Cartan subalgebra of U_K.

1.5 Construction of Cartan Subalgebras

As in the case with Lie algebras, a fundamental problem is constructing Cartan subalgebras. We begin our investigation as follows.

Let $\{u_i\}_{i=1}^m$ be a basis for an algebra U in the E-class $\mathcal{U}$ over F. For $u \in U$, write $u = \alpha_1 u_1 + \dots + \alpha_m u_m$. Let $\varphi(\lambda, u)$ be the characteristic polynomial of $S(u,\dots,u)$, where $S(u,\dots,u)$ is the map defined above for the Engel function f. An easy, but tedious, computational argument shows

<u>Lemma 1.5.1</u> $u_i S(u,\dots,u) = \sum_{k=1}^{m} \omega_k^{(i)}(\alpha_1,\dots,\alpha_m) u_k$ where $\omega_k^{(i)}(\alpha_1,\dots,\alpha_m)$ is a homogeneous polynomial of total degree n-1 in $\alpha_1,\dots,\alpha_m$.

We now consider the matrix of $S(u,\dots,u)$ acting on U. Since the matrix is determined by the action of $S(u,\dots,u)$ on the basis elements u_i, $i = 1,\dots,m$, we have, letting $\omega_i^{(j)} = \omega_i^{(j)}(\alpha_1,\dots,\alpha_m)$

$$\begin{vmatrix} \omega_1^{(1)} & \omega_2^{(1)} & \dots & \omega_m^{(1)} \\ \omega_1^{(2)} & \omega_2^{(2)} & \dots & \omega_m^{(2)} \\ \vdots & & & \vdots \\ \omega_1^{(m)} & \cdot & \cdot & \omega_m^{(m)} \end{vmatrix} = \text{matrix of } S(u,\dots,u)$$

Lemma 1.5.2 $\varphi(\lambda,u) = \lambda^m + \beta_1(u)\lambda^{m-1}+\dots+\beta_s(u)\lambda^{m-s}$ where either $\beta_i(u) = \beta_i(\alpha_1,\dots,\alpha_m)$ is a homogeneous polynomial of total degree $(n-1)i$ or the zero polynomial.

Proof: $\varphi(\lambda,u)$ is the characteristic polynomial of the matrix of $S(u,\dots,u)$. Hence $\beta_j(u)$ are just products of the $\omega_i^{(k)}$. Therefore, $\beta_j(u)$ is either zero or homogeneous of total degree $(n-1)j$ in the $\alpha_1,\dots,\alpha_m$. Q.E.D.

For an algebra $U \in \mathcal{U}$, let s' be the maximal integer such that $\beta_{s'}(u) \neq 0$ for some $u \in U$. An element $v \in U$ is called f-regular if and only if $\beta_{s'}(v) \neq 0$.

Lemma 1.5.3 Suppose $U \in \mathcal{U}$. An element $v \in U$ is f-regular if and only if B_v is minimal Engel in U.

Proof:
$$\begin{aligned}\varphi(\lambda,u) &= \lambda^m + \beta_1(u)\lambda^{m-1} +\dots+ \beta_{s'}(u)\lambda^{m-s'}\\ &= \lambda^{m-s'}(\lambda^{s'}+\dots+\beta_{s'}(u)) .\end{aligned}$$

If u is f-regular $\beta_{s'}(u) \neq 0$. Thus the multiplicity of the root 0 in $\varphi(\lambda,u)$ is $m-s'$, so the dimension of B_u is $m-s'$, where $s' \leq m-1$ since $u \in B_u$. If B_u is not minimal Engel, then there is a $v \in U$ such that the dimension of B_v is less than $m-s'$, say $\dim B_v = m-s''$ where $s'' > s'$. Then $\varphi(\lambda,v) = \lambda^m + \beta_1(v)\lambda^{m-1} +\dots+ \beta_{s''}(v)\lambda^{m-s''}$ where $\beta_{s''}(v) \neq 0$ contradicting our choice of s'. Hence B_u is minimal Engel in U.

Conversely, suppose B_u is minimal Engel in U.

Then there is an $s'' \leq m-1$ such that $\dim B_u = m-s''$ and $\varphi(\lambda,u) = \lambda^m + \beta_1(u)\lambda^{m-1} + \ldots + \beta_{s''}(u)\lambda^{m-s''}$ where $\beta_{s''}(u) \neq 0$. If $s'' > s'$, then we have a contradiction of our choice of s', whereas if $s'' < s'$, then clearly B_u is not minimal Engel. Consequently, $s'' = s'$, and u is f-regular. Q.E.D.

We will now show that if our ground field is sufficiently large, then U contains f-regular elements. Recall that $\mathcal{U}$ is an E-class over F where f is an Engel function for $\mathcal{U}$.

Lemma 1.5.4 Suppose $U \in \mathcal{U}$ and $\dim U = m$. If F has at least $(n-1)(m-1)$ elements, then U contains f-regular elements.

Proof: We will prove first that if $g(x_1,\ldots,x_k)$ is a non-zero homogeneous polynomial in $x_1,\ldots,x_k$ of total degree $(m-1)s'$ over a field K of at least $(m-1)s'$ elements, then there are elements $\xi_1,\ldots,\xi_k$ in K such that

$$g(\xi_1,\ldots,\xi_k) \neq 0 . \qquad [8]$$

The proof is by induction on k.

If $k = 1$, we may take $g(x_1) = x_1$ and clearly there is a non-root of g. Suppose we have verified the result for all $k' < k$, and suppose next that $g(x_1,\ldots,x_{k-1},x_k)$ is not identically zero. Thus there are elements $\xi_1,\ldots,\xi_{k-1}$ in K such that $g(\xi_1,\ldots,\xi_{k-1},x_k)$ has a non-zero coefficient. This is simply a polynomial in x_k, and if the maximal power of x_k appearing is less than $s'(m-1)$, then there is a non-root ξ_k

in K. Hence $g(\xi_1,\ldots,\xi_k) \neq 0$. If the maximal power of x_k is $s'(m-1)$, then we have

$$g(x_1,\ldots,x_k) = \beta x_k^{s'(m-1)} + \text{terms}$$

where "terms" involves lower powers of x_k, each term of "terms" has at least one x_i, $i \neq k$, in it, and $0 \neq \beta \in K$. Then $g(0,\ldots,0,1) = \beta \neq 0$. In either case, we see g has a non-root, which was to be proved.

Now $\beta_{s'}(u) = \beta_{s'}(\alpha_1,\ldots,\alpha_m)$ is a homogeneous polynomial of degree $s'(m-1)$, and is not identically zero by assumption. Since $s'(m-1) \leq (n-1)(m-1)$, the above result shows there are elements $\xi_1,\ldots,\xi_m$ in F such that $\beta_{s'}(\xi_1,\ldots,\xi_m) \neq 0$. Thus the element $u = \xi_1 u_1+\ldots+\xi_m u_m$ is an f-regular element of U, where $u_1,\ldots,u_m$ is a basis for U. Q.E.D.

Lemma 1.5.5 Suppose F has at least $m(n-1)$ elements and V is a subalgebra of U. If B_u is minimal with respect to dimension in the set $\{B_v : v \in V\}$ and $V \subseteq B_u$, then $B_u \subseteq B_v$ for all $v \in V$.

Proof: We will consider U, B_u, and $U-B_u$ as vector spaces over F. For a fixed element $c \in V$, we write $u_\mu = u + \mu c$, $\mu \in F$. Since $V \subseteq B_u \subseteq U$ and $u_\mu \in B_u$, we may consider $S(u_\mu,\ldots,u_\mu)$ as a linear transformation on U, B_u, and $U-B_u$. Let $\theta(\lambda,u_\mu)$ be the characteristic polynomial of the linear transformation induced by $S(u_\mu,\ldots,u_\mu)$ on B_u, $\psi(\lambda,u_\mu)$ be the characteristic polynomial of the linear transformation induced by $S(u_\mu,\ldots,u_\mu)$ on $U\text{-}B_u$.

Since $u_\mu \in B_u$, B_u is invariant under $S(u_\mu,\ldots,u_\mu)$. Let $\{u_i'\}_{i=1}^{m'}$ be a basis for B_u and extend it to a basis $\{u_i'\}_{i=1}^{m}$ for U . Relative to this basis, the matrix of $S(u_\mu,\ldots,u_\mu)$ is block triangular.

$$A^* = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix}$$

where A may be regarded as the matrix of the linear transformation $S(u_\mu,\ldots,u_\mu)$ induces on B_u . Hence the characteristic polynomial of the block A is $\theta(\lambda,u_\mu)$. We have $\theta(\lambda,u_\mu) = \lambda^{m'} + \alpha_1^*(\mu)\lambda^{m'-1}+\ldots+\alpha_{m'}^*(\mu)$ where $\alpha_i^*(\mu)$ is a polynomial in μ of degree at most $(n-1)i$. Now $\{u_i' + B_u\}_{i=m'+1}^{m}$ is a basis for $U\text{-}B_u$, so C may be regarded as the matrix of the transformation induced on $U\text{-}B_u$ by $S(u_\mu,\ldots,u_\mu)$. Hence the characteristic polynomial for C is

$$\psi(\lambda,u_\mu) = \lambda^{m-m'} + \beta_1^*(\mu)\lambda^{m-m'-1}+\ldots+\beta_{m-m'}^*(\mu)$$

where $\beta_i^*(\mu)$ is a polynomial of degree at most $(n-1)i$ in μ . Since the characteristic polynomial of A^* is $\varphi(\lambda,u_\mu)$ it follows that $\varphi(\lambda,u_\mu) = \theta(\lambda,u_\mu)\psi(\lambda,u_\mu)$.

We claim that $\alpha_i^*(\mu)$ is identically zero for all i . By construction, 0 is not a characteristic root of the linear transformation induced on $U\text{-}B_u$ by $S(u,\ldots,u)$. Therefore, $\beta_{m-m'}^*(0) \neq 0$. Since $\beta_{m-m'}^*(\mu)$ has degree at most $(n-1)(m-m')$, $\beta_{m-m'}^*(\mu)$ has at most $(n-1)(m-m')$ roots in F . Consequently there are $p = m'(n-1)$ distinct elements $\xi_1,\ldots,\xi_p$ in F such that $\beta^*_{m-m'}(\xi_j) \neq 0$. Hence $B_{u+\xi_j c} \subseteq B_u$ since 0 is not a characterstic root of $S(u+\xi_j c,\ldots,u+\xi_j c)$ on $U-B_u$. By the minimality of B_u , we have $B_u = B_{u+\xi_j c}$. Therefore, $\theta(\lambda,u\xi_j) = \lambda^{m'}$, or $\alpha^*_i(\xi_j) = 0$ for $i = 1,\ldots,m'$ and $j = 1,\ldots,p$. But $\alpha^*_i(\mu)$ has at most $i(n-1)$ distinct roots in F . Therefore, if $i < m'$, then $(n-1)i < p$, and it follows that $\alpha^*_i(\mu)$ is identically zero. Since $u + \mu c$ is in B_u and $B_{u+\mu c}$, $\alpha^*_{m'}(\mu)$ must be identically zero.

Now $\alpha^*_i(\mu)$ identically zero for all i implies that $B_u \subseteq B_{u+\mu c}$ for all $c \in V$, $\mu \in F$. Hence for $b \in V$, let $c=u-b$. Then $B_u \subseteq B_{u-c} = B_b$. Q.E.D.

Barnes [8] shows the following. Let $f(x_1,x_2) = x_1x_2$ be the Engel function for the class of Lie algebras, and suppose L is a Lie algebra over F . If the dimension of L is m , then Lemma 1.5.4 says that if F has at least $m-1$ elements, then L has regular elements. Lemma 1.5.5 in this case says that if F has at least m elements and if L_1 is a subalgebra of L , B_a is minimal in $\{B_b : b \in L_1\}$ and $L_1 \subseteq B_a$, then $B_a \subseteq B_b$ for all $b \in L_1$. He then gives an example to show that for these lemmas, the restrictions on F cannot be removed.

Now suppose B_a is minimal Engel in L . Then B_a is certainly minimal in $\{B_b : b \in B_a\}$, and hence $B_a \subseteq B_b$ for all b in B_a . Thus $S(b)$ is nilpotent on B_a for all b in B_a and it follows since f is an Engel function that B_a

is nilpotent. In this case, it is clear that $L_L(B_a)$ is nilpotent and B_a coincides with the Fitting null component of L relative to $L_L(B_a)$. Consequently, if F has at least m elements and B_a is minimal Engel in L , then B_a is a Cartan subalgebra.

In chapter two and three, we will use Lemma 1.5.4 and Lemma 1.5.5 to construct Cartan subalgebras for alternative and Jordan algebras using essentially the same method described above for the Lie case.

1.6 The Inner Automorphism Group of an Algebra

To show that Cartan subalgebras of U are conjugate under a certain class of automorphisms on U seems to be a difficult problem. Chevalley [11] has solved the problem in the case of Lie algebras over algebraically closed fields of characteristic zero. In this section, we construct a method that parallels Chevalley's that will allow us to solve the problem in Chapters Two and Three when the ground field is algebraically closed and of characteristic zero. Where proofs are not given, the reader is referred to Chevalley [11].

We begin by defining the Zariski topology on an m-dimensional vector space V over F where the characteristic of F is zero. Suppose $v_1,\ldots,v_m$ is a basis for V . Then for each $v \in V$, there is a unique set $\xi_1,\ldots,\xi_m$ of elements in F such that $v = \xi_1 v_1+\ldots+\xi_m v_m$. Now suppose $f(\lambda_1,\ldots,\lambda_m)$ is an element in the polynomial ring $F[\lambda_1,\ldots,\lambda_m]$. Then

$f(\lambda_1,\ldots,\lambda_m)$ and the basis $v_1,\ldots,v_m$ determine a map from V to F defined

$$f(v) = f(\xi_1,\ldots,\xi_m) .$$

We call f a <u>polynomial function on V</u>. If f and g are two polynomial function on V and $\alpha \in F$, we set

$$(f+g)(v) = f(v) + g(v)$$
$$(fg)(v) = f(v)g(v)$$
$$(\alpha f)(v) = \alpha(f(v)) .$$

Thus the set $F[V]$ of polynomial functions on V is an F-algebra. Moreover, since the characteristic of F is zero, the map

$$f(\lambda_1,\ldots,\lambda_m) \to f$$

is an isomorphism of $F[\lambda_1,\ldots,\lambda_m]$ onto $F[V]$. This map sends λ_i into the polynomial function π_i where

$$\pi_i(\xi_1 v_1+\ldots+\xi_i v_i +\ldots+\xi_m v_m) = \xi_i .$$

Since π_i is a homomorphism from V to F, it follows that $V^* = \mathrm{Hom}_F(V,F)$ and the constant functions generate $F[V]$.

If W is a non-empty subset of V, the polynomial functions on W, $F[W]$, are defined to be the restriction to W of elements in $F[V]$.

Suppose E is a subset of V. We say E is an <u>algebraic set</u> if and only if there is a subset B of $F[V]$ such that

$$E = \{x \in V : \ b(x) = 0 \ \text{ for all } \ b \in B\} .$$

The union of two algebraic sets is algebraic, and the intersection of a family of algebraic sets is algebraic [11; III, page 169]. Furthermore, V and $\emptyset$ may be considered algebraic. We will call a subset of V closed if and only if it is algebraic. The above properties on algebraic sets show that these closed sets determine a topology on V. This topology is called the Zariski topology.

If E is a subset of V, the closure of E, $\bar{E}$, is defined

$$\bar{E} = \{x \in V : \ p(x)=0 \ \text{ for all } p \in F[V] \text{ such that } p(y)=0 \text{ for all } \ y \in E\} .$$

Now if $v_1',\ldots,v_m'$ is another basis for V, then for $v \in V$, there are sets of elements $\xi_1,\ldots,\xi_m,\xi_1',\ldots,\xi_m'$ in F such that

$$v = \xi_1 v_1+\ldots+\xi_m v_m = \xi_1' v_1'+\ldots+\xi_m' v_m' .$$

But $v_i' = \eta_{i1}v_1+\ldots+\eta_{im}v_m$ where $\eta_{i1},\ldots,\eta_{im} \in F$. Consequently, if $f(\lambda_1,\ldots,\lambda_m)$ is an element in $F[\lambda_1,\ldots,\lambda_m]$, then there is another element $f'(\lambda_1,\ldots,\lambda_m)$ in $F[\lambda_1,\ldots,\lambda_m]$ such that

$$f(\xi_1,\ldots,\xi_m) = f'(\xi_1',\ldots,\xi_m') .$$

If $F[V]$ and $F[V]'$ are the polynomial functions on V relative to $v_1,\ldots,v_m$ and $v_1',\ldots,v_m'$ respectively, it follows for f an element in $F[V]$, there is an element f' in $F[V]'$ such that $f(v) = f'(v)$ for all v in V. In this respect, polynomial functions are independent of the basis chosen for V, hence the Zariski topology for V is independent of the basis.

If W is a subspace of V, it follows from the remarks above that $F[W]$ is the restriction to W of $F[V]$. It is clear that W is algebraic, thus closed, and the Zariski topology on W is the topology on W induced by the Zariski topology on V.

Suppose E is a non-empty subset of V. Then E is <u>irreducible</u> if and only if $F[E]$ is an integral domain. Under the topology induced on E by the Zariski topology on V, we observe that E is irreducible if and only if all non-empty relatively open subsets of E are dense in E. [11; III, page 175].

Since F is of characteristic zero, we claim that all open subsets of V are Zariski dense in V. To see this, we begin with

<u>Lemma 1.6.1</u> Suppose f and g are elements in $F[\lambda_1,\ldots,\lambda_m]$ with g non-zero. Let $C = \{(\alpha_1,\ldots,\alpha_m)\colon \alpha_i \in F$ and $g(\alpha_1,\ldots,\alpha_m) \neq 0\}$. If $f(\alpha_1,\ldots,\alpha_m) = 0$ for all $(\alpha_1,\ldots,\alpha_m) \in C$, then f is identically equal to zero.

<u>Proof</u>: For $m = 1$, the result is clear, since g has infinitely many non-roots. For $m > 1$, we may write f and g as polynomials in λ_m with coefficients from $F[\lambda_1,\ldots,\lambda_{m-1}]$. We may then proceed by induction on m to complete the proof. Q.E.D.

<u>Corollary 1.6.2</u> If E is a non-empty Zariski open subset of V, then E is Zariski dense in V.

<u>Proof</u>: The compliment of E, cE, is closed. Consequently, there is a subset B of $F[V]$ such that $b(a) = 0$ for $a \in cE$ and all $b \in B$. Since E is non-empty, B is non-empty. Let b be a non-zero element in B. Then $E_1 = \{v \in V: b(v) \neq 0\}$ is an open set of V contained in E. Suppose $p \in F[V]$ and $p(v) = 0$ for all $v \in E_1$. By Lemma 1.6.1, it follows that $p(v) = 0$ for all $v \in V$. Hence, $\bar{E} = V$, and E is Zariski dense in V. Q.E.D.

<u>Corollary 1.6.3</u> If W is a non-zero subspace of V, then W is irreducible.

<u>Proof</u>: Since the Zariski topology on W coincides with the topology induced by the Zariski topology on V, the result follows immediately from Corollary 1.6.2 and the remarks preceding Lemma 1.6.1. Q.E.D.

We will call a subset E of V <u>epais</u> if and only if E is irreducible and contains a non-empty relatively open subset of its Zariski closure. We have

<u>Corollary 1.6.4</u> If W is a non-empty subspace of V then W is epais.

<u>Proof</u>: W is closed and irreducible since W is a subspace. Since W is non-empty, $F[W] \neq 0$, Thus if p is a non-zero

element in $F[W]$, the set $\{v \in W : p(v) \neq 0\}$ is open and the result follows. Q.E.D.

We return for a moment to our algebra U over F . Recall that an element $u\,U$ is f-regular if and only if $\beta_{s'}(u) \neq 0$ [Lemma 1.5.4]. But $\beta_{s'} \in F[U]$, and we have

<u>Corollary 1.6.5</u> The set of f-regular elements of U form a dense open subset of U .

We will denote the set of f-regular elements of U by U_1 .

Since subalgebras of U are subspaces, we have

<u>Corollary 1.6.6</u> If H is a subalgebra of U , then H is epais in U .

To see how the Zariski topology will help us to solve the conjugacy problem, we must define what we mean by the group of inner automorphisms of U . We begin as follows.

Let $C = \mathrm{Hom}_F(V,V)$. Then $F[C]$ is generated by the constant functions and $C^* = \mathrm{Hom}_F(C,F)$. A group G of automorphisms of V is called an <u>algebraic group</u> if and only if there is a subset B of $F[C]$ such that

$$G = \{\sigma : \sigma \text{ is an automorphism of } V \text{ and } b(\sigma) = 0 \text{ for all } b \in B\} .$$

We wish to define what is meant by the Lie algebra of an algebraic group G [11; II, page 125-136]. We observe that $F[C]$ can be made into a two-sided C module by making the following definitions: for $e, x \in C$ and $p \in F[C]$,

$$(p \cdot e)(x) = p(ex) \qquad \text{right translation}$$
$$(e \cdot p)(x) = p(xe) \qquad \text{left translation.}$$

If F_1 is the set of constant functions, then $F_1 \cap C^* = \{0\}$. Thus, in $F[C]$, the sum $F_1 + C^*$ is direct. Consequently, the linear map D''_e, $e \in C$, which is left translation by e on C^* can be extended to a linear map D'_e on $F_1 + C^*$ by setting $D'_e(k) = 0$ for all $k \in F_1$. Chevalley [11; II, page 21-26] has shown this map can be uniquely extended to a derivation D_e of $F[C]$. If e_1 and e_2 are elements of C, since $D_{e_1+e_2}$ and $D_{e_1} + D_{e_2}$ both map F_1 onto 0 and agree on C^*, it follows that $D_{e_1+e_2} = D_{e_1} + D_{e_2}$, or the map $e \to D_e$ is linear. Similarly, since $D_{[e_1,e_2]}$ and $[D_{e_1}, D_{e_2}]$ are derivations of $F[C]$ which coincide on C^* and map F_1 onto 0, we have $D_{[e_1,e_2]} = [D_{e_1}, D_{e_2}]$.

Now let G be an algebraic group and let Q be the subset of $F[C]$ such that if q is an element of Q, then q restricted to G is zero. Clearly Q is an ideal of $F[C]$, and Q is called the <u>ideal of polynomial functions associated with G</u>. If Q is a prime ideal, G is called <u>irreducible</u>. It is easy to see that G is irreducible if and only if $F[G]$

is an integral domain.

If G is an algebraic group and Q is its associated ideal of polynomial functions, we see that the set

$$\{e \in C : D_e(Q) \subseteq Q\}$$

is a Lie subalgebra of the Lie algebra C_L of C. This Lie algebra, denoted G_L, is called the <u>Lie algebra of G</u>.

Now let $L(U)$ denote the Lie algebra over F generated by R_u, L_u for $u \in U$. $L(U)$ is called the <u>Lie multiplication algebra</u> of U. A derivation D of U is called an <u>inner derivation</u> if and only if $D \in L(U)$. It is known that the set $\mathcal{D}'(U)$ of inner derivations of U is an ideal in the derivation algebra $\mathcal{D}(U)$ of U [25].

Now let $g(x_1,\ldots,x_p)$ be a non-zero linear homogeneous element in the free non-associative algebra over F in p generators $x_1,\ldots,x_p$. Suppose, for all elements $u_1,\ldots,u_{p-1}$ in U, the map

$$xD(u_1,\ldots,u_{p-1}) = g(x,u_1,\ldots,u_{p-1}) \qquad x \in U$$

is a derivation of U. Then $D(u_1,\ldots,u_{p-1}) \in \mathcal{D}(U)$. We have

$$\mathcal{D}'(U) \supset \{\sum_i D(u_1^{(i)},\ldots,u_{p-1}^{(i)}) : u_i^j \in U\} \neq \emptyset$$

We set

$$\mathcal{G} = \{G : G \text{ is an algebraic group of automorphisms on the vector space } U \text{ such that } G_L \geq \mathcal{D}'(U)\} .$$

Chevalley [11; II, page 179] shows that if A is the group of automorphisms of the algebra U, then $A_L = \mathcal{D}(U)$. Of course, $A \in \mathcal{G}$. Then $I(U) = \cap\{G : G \in \mathcal{G}\}$ is an irreducible algebraic group such that $I(U)_L \supseteq \mathcal{D}'(U)$ [11; II, page 165-172] and $A \supset I(U)$. Consequently elements in $I(U)$ are automorphisms of the algebra U, and we call $I(U)$ the <u>group of inner automorphisms of U</u>.

We now return to our arbitrary vector space V. Recall that $F[V]$ and $F[\lambda_1,\ldots,\lambda_m]$ are isomorphic. If $a = \xi_1 v_1+\ldots+\xi_m v_m$ is an element in V where $v_1,\ldots,v_m$ is a basis for V, and if f is an element in $F[V]$, we define a linear function $d_a f$ on V by

$$d_a f(\eta_1 v_1+\ldots+\eta_m v_m) = \left(\frac{\partial f}{\partial \lambda_1}\right)_a \eta_1+\ldots+\left(\frac{\partial f}{\partial \lambda_m}\right)_a \eta_m .$$

We call $d_a f$ the <u>differential of f at a</u>. This map has the following properties:

(A) $d_a(f+g) = d_a f + d_a g$

(B) $d_a(\alpha f) = \alpha)d_a f)$ $\alpha \in F$

(C) $d_a(fg) = f(a)d_a g + g(a)d_a f$.

Furthermore, for $a \in V$, the map

$$y \to d_a f(y)$$

is linear.

Let E be an irreducible subset of V and $x \in E$. The tangent space to x at E is defined to be

$$T(E;x) = \{y \in V : d_x f(y) = 0 \text{ for all } f \in F[V] \text{ such that } f(a) = 0 \text{ for all } a \in E\}$$

Since $y \to d_a f(y)$ is linear, $T(E;x)$ is a subsapce of V. Furthermore, we have

Lemma 1.6.7 If W is a subspace of V, then $W \subset T(W;x)$ for all $x \in W$.

Proof: Let $v_1,\ldots,v_{m'}$ be a basis for W, and extend it to a basis $v_1,\ldots,v_m$ for V. Relative to this basis, if $f \in F[V]$ and $f(v) = 0$ for all $v \in W$, it follows that $\frac{\partial f}{\partial \lambda_i} \equiv 0$ for $i = 1,\ldots,m'$. If $x \in W$, $y = \eta_1 v_1 + \ldots + \eta_m v_m$,

$$d_x f(y) = \left(\frac{\partial f}{\partial \lambda_1}\right)_x \eta_1 + \ldots + \left(\frac{\partial f}{\partial \lambda_m}\right)_x \eta_m .$$

If $y \in W$, $y = \eta_1 v_1 + \ldots + \eta_{m'} v_{m'}$, and if $f(v) = 0$ for all $v \in W$, $d_x f(y) = 0$. Thus $W \subseteq T(W;x)$ for all $x \in W$. Q.E.D.

If G is an irreducible group of automorphisms of V and E is a subset of V, then

$$\Omega = \{y \in V : y = x\sigma \text{ for some } x \in E, \sigma \in G\}$$

is called the orbit of E with respect to G. If E is irreducible, then Ω is irreducible [11; III, page 192-193].

In order to apply the theory given by Chevalley in what follows, we must assume F is algebraically closed. Suppose H is a subalgebra of our algebra U. Since H is epais, by Lemma 1.6.7 and Chevalley [11; III, page 192-193], the orbit Ω of H under $I(U)$ is irreducible and $T(\Omega;x)$ contains $x\mathcal{D}'(U) + H$ for all $x \in H$ where $x\mathcal{D}'(U) = \{xD\colon X \in \mathcal{D}'(U)\}$. From Chevalley [11; III, Proposition 13, page 180 and the Corollary, page 192], we have

<u>Theorem 1.6.8</u> If H contains a non-empty relatively open subset O such that $T(\Omega;a) = U$ for all $a \in O$, then Ω contains a non-empty open subset of U.

We now give

<u>Theorem 1.6.9</u> Suppose minimal Engel subalgebras of U are Cartan subalgebras. If H_1 and H_2 are two Cartan subalgebras of U such that H_1 and H_2 satisfy the hypothesis of Theorem 1.6.8, then H_1 and H_2 are conjugate under $I(U)$ in the sense that there is an element $\sigma \in I(U)$ such that $H_1 = H_2\sigma$.

<u>Proof</u>: Let Ω_1 and Ω_2 be the orbits of H_1 and H_2 under $I(U)$. By Theorem 1.6.8, Ω_1 and Ω_2 contain non-empty open subsets of U. It follows that $U_1 \cap (\Omega_1 \cap \Omega_2) \neq \emptyset$. Let b be a non-zero element in this intersection. As $b \in \Omega_i$, $i = 1,2$, $b \in H_i\sigma_i$ for some σ_i in $I(U)$. But b is f-regular, so B_b is a Cartan subalgebra. Therefore $B_b \subseteq H_i\sigma_i$, hence $B_b = H_i\sigma_i$. Thus $H_1 = H_2\sigma$ where $\sigma = \sigma_2\sigma_1^{-1} \in I(U)$.

Q.E.D.

Consequently, when F is algebraically closed and of characteristic zero, to show Cartain subalgebras are conjugate, we must show that minimal Engel subalgebras are Cartan subalgebras, and that Cartan subalgebras satisfy the hypothesis of Theorem 1.6.8.

With respect to the conjugacy problem, we feel there must be a self-contained approach. We feel this could be developped following the methods given by Jacobson in [18; Chapter IX]. However, to date we have been unsuccessful.

CHAPTER TWO

ALTERNATIVE ALGEBRAS

2.1 Introduction

An algebra A over a field F is called an <u>alternative algebra</u> if and only if $(x,x,y) = (y,x,x) = 0$ for all elements x and y of A. From this definition, it is clear that homomorphic images of alternative algebras are alternative and that a direct sum of alternative algebras is alternative. Since the associator is multilinear, we have that A_K is alternative for all extensions K of F. Throughout Chapter Two, we will assume our alternative algebras are finite dimensional.

In this section we will give the necessary theory to develop a Cartan theory for alternative algebras. In what follows, where proofs are not provided, the reader is referred to Schafer [25].

We begin by linearizing the defining identities of A, and obtain

(1) $(x,y,z) = -(x,z,y) = (z,x,y)$

(2) $(x,y,x) = 0$

For right and left multiplication R_x and L_x, $x \in A$, of A, these become

$$(3)\quad R_xR_y = R_{xy} = L_{xy} - L_yL_x = L_yR_x - R_xL_y$$
$$= L_xL_y - L_{yx} = R_yL_x - L_xR_y = R_{yx} - R_yR_x$$

and

$$(4)\quad R_xL_x = L_xR_x$$

We next give

Theorem 2.1.1 (Artin) The subalgebra generated by any two elements of A is associative.

As a consequence of this result, we have

Corollary 2.1.2 (i) Alternative algebras are power-associative in the sense that the subalgebra generated by a single element is associative. (ii) For all $x \in A$, $R_{x^i} = (R_x)^i$ and $L_{x^i} = (L_x)^i$ for $i = 0,1,\ldots$

Now suppose e is an idempotent of A. By (4) and the above corollary, R_e and L_e are commuting idempotent operators. It follows that A is a vector space direct sum

$$A = A_{00} + A_{10} + A_{01} + A_{11}$$

where $A_{ij} = \{x \in A : ex_{ij} = ix_{ij} \quad x_{ij}e = jx_{ij}\}$ $i,j = 0,1$. Hence if $x \in A$, we write the decomposition of x:

$$x = exe + (ex - exe) + (xe - exe) + (x - ex - xe + exe).$$

where $exe \in A_{11}$, $ex - exe \in A_{10}$, $xe - exe \in A_{01}$, and $x - ex - xe - exe \in A_{00}$.

A set $e_1, \ldots e_s$ of idempotents is called <u>pairwise orthogonal</u> in case $e_i e_j = e_j e_i = 0$ for $i \neq j$, $i,j = 1,\ldots,s$. If A has a unity and $e_1,\ldots,e_s$ is a set of pairwise orthogonal idempotents whose sum is the unity of A, we get a refined decomposition of A as a vector space direct sum

$$(2.1.3) \qquad A = \sum_{i,j=1}^{s} \oplus A_{ij}$$

$$A_{ij} = \{x_{ij} \in A \;;\; e_k x_{ij} = \delta_{ki} x_{ij} \quad x_{ij} e_k = \delta_{jk} x_{ij} \quad k = 1,\ldots,s\}$$

, where δ_{ij} is the Kronecker delta.

This decomposition is called the <u>Pierce decomposition</u> of A relative to $e_1,\ldots,e_s$, and we have

<u>Theorem 2.1.4</u> Let (2.1.3) be the Pierce decomposition of A relative to the pairwise orthogonal idempotents $e_1,\ldots,e_s$. Then:

(i) $A_{ij} A_{jk} \subseteq A_{ij}$ $\quad i,j,k = 1,\ldots,s$

(ii) $A_{ij} A_{ij} \subseteq A_{ji}$ $\quad i,j = 1,\ldots,s$

(iii) $A_{ij} A_{kl} = 0$ $\quad j \neq k \quad (i,j) \neq (k,l)$
$i,j,k,l = 1,\ldots,s$

(iv) $(x_{ij})^2 = 0$ $\quad x_{ij} \in A_{ij} \quad i \neq j$.

We now need some information on nilpotent alternative algebras. We begin with

Theorem 2.1.5 The following are equivalent:

(i) A is nilpotent

(ii) A is solvable

(iii) A is nil

An element $z \in A$ is called properly nilpotent if and only if za is nilpotent for all $a \in A$. If z is properly nilpotent, by Theorem 2.1.1, we see az is nilpotent for all $a \in A$. It is known that the radical of A can be characterized as the set of properly nilpotent elements of A.

Suppose A is semi-simple in the classical sense. Then A has a unity, and we have

Theorem 2.1.6 A non-zero alternative algebra is semi-simple if and only if it is a direct sum of simple ideals.

Thus the study of semi-simple alternative algebras is reduced to studying the simple ones.

We say an idempotent $e \in A$ is primitive if and only if e cannot be written $e = e' + e''$ where e' and e'' are non-zero orthogonal idempotents. If A has a unity, then this unity element can be expressed as a sum of pairwise orthogonal primitive idempotents. Let t be the maximal integer such that the unity of A is expressible as a sum of t pairwise orthogonal primitive idempotents. Then the degree of A is defined to be t.

Now suppose F is algebraically closed and (2.1.3) is the Pierce decomposition of A relative to a set $e_1,\ldots,e_t$ of pairwise orthogonal primitive idempotents. Since e_i is the unity of A_{ii} and in fact is the only idempotent in A_{ii}, we have if $x \in A_{ii}$ and x is not nilpotent then the subalgebra of A_{ii} generated by x, $F[x]$, is commutative, associative, and contains e_i. By Wedderburn's theorem, $F[x] = F_1 + N$ where N is a nil algebra and F_1 is semi-simple and, in this case, simple. Then F_1 is a matrix algebra over a division algebra. Since e_i is the only idempotent in F_1 and F is algebraically closed, it follows $F_1 = Fe_i$. Consequently, if $x \in A_{ii}$, we may write $x = \alpha e_i + n$ where $\alpha \in F$ and n is nilpotent. Now let $n \in A_{ii}$ and n be nilpotent. We claim if $a \in A_{ii}$, then na is nilpotent. If na is not nilpotent, by the above remarks it follows that na has an inverse $(na)^{-1}$. Let p be the integer such that $n^p = 0 \neq n^{p-1}$. Then $0 \neq n^{p-1} = n^{p-1}[(na)(na)^{-1}] = n^p a(na)^{-1} = 0$, a contradiction. Similarly an is nilpotent, and thus the set of nilpotent elements of A_{ii} form an ideal. It follows that $A_{ii} = Fe_i + N$ where N is nilpotent.

Jacobson calls an arbitrary algebra U over a field K almost nil if and only if U has a unity 1 and $U = F1 + N$ where N is a nil ideal.

Hence, we have shown

<u>Lemma 2.1.7</u> If F is algebraically closed and (2.1.3) is the Pierce decomposition of A relative to pairwise orthogonal primitive idempotents $e_1, \ldots, e_s$, then the A_{ii} are almost nil.

Now suppose A is simple. If the degree of A is 1, then A is a division algebra. If the ground field is closed then Lemma 2.1.7 says that $A = F1$. If the degree of A is greater than two, then A is associative. If the degree of A is two, then A is either associative or a Cayley algebra [25].

To define the inner automorphism group of an alternative algebra, we require some information on inner derivations of alternative algebras. We begin by defining the nucleus $N(A)$ of an alternative algebra as the set

$$\{g \in A : (g,x,y) = 0 \text{ for all } x,y \in A\} .$$

If the characteristic of F is different from 3, then $R_g - L_g$ is a derivation of A if and only if $g \in N(A)$ [25, page 76]. If we set

$$(5) \quad D(b,c) = [L_b, L_c] + [L_b, R_c] + [R_b, R_c]$$

then $D(b,c)$ is a derivation of A [25, page 77]. Schafer [25, page 78] shows that if the characteristic of F is different than 2 and 3 and if A has a unity, then any inner derivation D of A can be written

$$D = R_g - L_g + \sum_{i=1}^{p} D(b_i, c_i)$$

$g \in N(A)$, $b_i, c_i \in A$.

2.2 The Universal Multiplication Envelope of an Alternative Algebra.

Suppose A is an alternative algebra over F and A' is its anti-isomorphic image. If $a \in A$, we will write a' for the anti-isomorphic image of a . We set $B_1 = B = A \oplus A'$ and inductively $B_n = B_{n-1} \otimes B_1$. Let $T(B)$ be the associative algebra defined by

$$T(B) = B_1 \oplus B_2 \oplus B_3 \oplus \cdots$$

where the vector space operations in $T(B)$ are as usual and multiplication in $T(B)$ is denoted by $\otimes$. Let S be the ideal of $T(B)$ generated by elements of the form

(1) $a_1' \otimes a_2 - a_2 \otimes a_1' - a_1 a_2 + a_1 \otimes a_2$

(2) $a_1 a_2 - a_1 \otimes a_2 - (a_2 a_1)' + a_1' \otimes a_2'$

(3) $(a_2 a_1)' - a_1' \otimes a_2' - a_1 \otimes a_2' + a_2' \otimes a_1$

where $a_i \in A$.

The associative algebra $U(A) = T(B)/S$ is called the <u>universal multiplication envelope</u> of A . If i' is the canonical homomorphism from $T(B)$ into $U(A)$, then the restriction of i' to $B = B_1$ defines a linear map from B into $U(A)$. We call this map i, and if $a \in A$, we will write $ai = \bar{a}$, $a'i = \bar{a}'$ where $\bar{a}$, $\bar{a}'$ are the cosets of a and a' in $U(A)$. From (1), (2) and (3), we have the following relations in $U(A)$:

$$(4) \quad \bar{a}_1' \otimes \bar{a}_2 - \bar{a}_2 \otimes \bar{a}_1' = \overline{a_1 a_2} - \bar{a}_1 \otimes \bar{a}_2$$
$$= \overline{(a_2 a_1)'} - \bar{a}_1' \otimes \bar{a}_2'$$
$$= \bar{a}_1 \otimes \bar{a}_2' - \bar{a}_2' \otimes \bar{a}_1 .$$

Lemma 2.2.1 Let ρ be a linear map from B into an associative algebra V such that

$$[a_1'\rho, a_2\rho] = (a_1 a_2)\rho - (a_1\rho)(a_2\rho)$$
$$= (a_2 a_1)'\rho - (a_1'\rho)(a_2'\rho) = [a_1\rho, a_2'\rho] .$$

Then there is a unique homomorphism ρ^* from $U(A)$ into V such that $\bar{a}\rho^* = a\rho$ and $\bar{a}'\rho^* = a'\rho$.

Proof: Suppose $\{b_i : i \in I\}$ is a basis for B. Then the distinct elements $b_{i_1} \otimes b_{i_2} \otimes \ldots \otimes b_{i_n}$ form a basis for B_n where $b_{i_1} \otimes, \ldots \otimes b_{i_n} = b_{i_j} \otimes \ldots \otimes b_{j_n}$ if and only if $i_k = j_k$ $k = 1, \ldots, n$. Consequently, the set of all these elements for a basis for $T(B)$.

We now define the map ρ'' from $T(B)$ into V where $(b_{i_1} \otimes \ldots \otimes b_{i_n})\rho'' = (b_{i_1}\rho) \ldots (b_{i_n}\rho)$. Clearly this is a homomorphism from $T(B)$ into V such that $a\rho'' = a\rho$ if $a \in B$.

By hypothesis, the generators of S are mapped into the kernel of ρ'', consequently ρ'' induces a homomorphism ρ^* of $U(A)$ into V. If $a \in A$, then $ai\rho^* = \bar{a}\rho^* = a\rho'' = a\rho$; similarly $a'i\rho^* = a'\rho$, and $\rho = i\rho^*$ as desired.

Since $T(B)$ is generated by B, $U(A)$ is generated by Bi, and it follows that ρ^* is unique. Q.E.D.

Lemma 2.2.2 If K is an ideal of A and D is the ideal in $U(A)$ generated by $(K \oplus K')i$, then there exists an isomorphism of $U(A/K)$ onto $U(A)/D$ such that $\overline{a + (K \oplus K')}$ is mapped onto $\bar{a} + D$ and $\overline{a' + (K \oplus K')}$ is mapped onto $\bar{a}' + D$, $a \in A$.

Proof: We define a map $\alpha^*: B \to U(A)/D$ where, if $b \in B$, then $b\alpha^* = bi + D$. Since $(K \oplus K')i \subseteq D$, α^* maps $K \oplus K'$ onto 0. Thus α^* induced a map $\alpha : B/K \oplus K' \to U(A)/D$ such that $(b + (K \oplus K'))\alpha = bi + D$.

If θ is a linear map from $B/K \oplus K'$ into an associative algebra V where $B/K \oplus K'$ and θ satisfy the conditions of Lemma 2.2.1, it follows by examining homomorphism and using Lemma 2.2.1 that there is a unique homomorphism θ' from $U(A)/D$ into V.

If i_K is the natural map from $B/K \oplus K'$ into $U(A/K)$ it now follows that the diagram

$$\begin{tikzcd} U(A/K) \arrow[dr, dashed, shift left, "i_k^*"'] & \\ B/K + K' \arrow[u, "i_k"] \arrow[r, "\alpha"] & U(A)/D \arrow[ul, dashed, shift left, "\alpha^*"'] \end{tikzcd}$$

is commutative, and i_k^* is the desired isomorphism. Q.E.D.

We will use the universal multiplication envelope of A to prove the following result. Suppose A_1 is a subalgebra of A, and V is the subalgebra of $\mathrm{Hom}_F(A,A)$ generated by the maps R_a, L_a, $a \in A_1$, acting on A. Then if J is a solvable ideal of A_1, the maps R_a, L_a, $a \in J$, acting on A generated a nilpotent ideal in C. We claim that it is sufficient to show that $(J \oplus J')i_1$ generates a nilpotent ideal in $U(A_1)$, where i_1 is the natural map from $A_1 \oplus A_1'$ into $U(A)$. Indeed, by (3) in §2.1, the linear map ρ from $A_1 \oplus A_1'$ into C where, if $a \in A_1$, $a\rho = R_a$ and $a'\rho = L_a$, and C satisfy the conditions of Lemma 2.2.1. Thus there is a homomorphism from $U(A_1)$ into C. Since C is generated by R_a and L_a, it follows that this homomorphism is onto, and maps the ideal generated by $(J \oplus J')i$ onto the ideal of C generated by R_a, L_a, $a \in J$. Consequently if $(J \oplus J')i_1$ generates a nilpotent ideal in $U(A_1)$, R_a, L_a, $a \in J$ generated a nilpotent ideal in C.

We begin by showing

Lemma 2.2.3 Suppose I is an ideal of A and I^* is the subalgebra of $U(A)$ generated by $(I \oplus I')i$. Then $D = I^* \otimes U(A)$ is an ideal of $U(A)$.

Proof: It is clear that D is a right ideal. Hence, suppose $a, b \in A$. From (4), we obtain the following identities:

$$(5) \quad \bar{a} \otimes \bar{b}' = \bar{b}' \otimes \bar{a} - \overline{ba} + \bar{b} \otimes \bar{a}$$

$$(6) \quad \bar{a}' \otimes \bar{b}' = (\overline{ba})' + (\overline{ba}) - \bar{b} \otimes \bar{a}$$

$$(7) \quad \bar{a} \otimes \bar{b} = (\overline{ab}) + (\overline{ba}) - \bar{b} \otimes \bar{a}$$

$$(8) \quad \bar{a}' \otimes \bar{b} = \bar{b} \otimes \bar{a}' + \bar{b} \otimes \bar{a} - (\overline{ba}) .$$

Since $(A \oplus A')i$ generates $U(A)$ and $(I \oplus I')i$ generates I^*, it follows from (5) - (8) that D is a two-sided ideal of $U(A)$. Q.E.D.

Corollary 2.2.4 If I^* is nilpotent, D is nilpotent.

Proof: Since $U(A) \otimes I^* \subseteq I^* + I^* \otimes U(A)$, an easy induction argument shows that $D^n \subseteq (I^*)^n + (I^*)^n \otimes U(A)$. The result is now immediate. Q.E.D.

Lemma 2.2.5 If A is solvable and the dimension of A is 1, then $(U(A))^3 = 0$.

Proof: If $\dim A = 1$, then $A = Fe$ where $e^2 = 0$. From (5) - (8), we obtain $\bar{e}\bar{e} = \bar{e}'\bar{e}' = 0$ and $\bar{e}'\bar{e} = \bar{e}\bar{e}'$. Since $(A \oplus A')i$ generates $U(A)$, $\bar{e}$ and $\bar{e}'$ generate $U(A)$ in this case. Consequently, $(U(A))^3 = 0$. Q.E.D.

Lemma 2.2.6 If A is solvable, then $U(A)$ is nilpotent.

Proof: The proof is by induction on the dimension of A. By Lemma 2.2.5, we assume $\dim A = n > 1$, Since A is solvable, there is an $n-1$ dimensional subspace I such that $A^2 \subset I \subsetneqq A$, and in fact, I is an ideal of A. Consequently, as $\dim A/I = 1$ and $(A/I)^2 = 0$, $U(A/I)$ is nilpotent. Let I_1 be the ideal in $U(A)$ generated by $(I \oplus I')i$. By Lemma 2.2.2, $U(A/I)$ is isomorphic to $U(A)/I_1$. If I^* is the subalgebra of $U(A)$ generated by $(I \oplus I')i$, then $I_1 = I^* + I^* \otimes U(A)$. Since $\dim I < n$, $U(I)$ is nilpotent. It follows from Lemma 2.2.1 that I^* is nilpotent since I^* is a homomorphic image of $U(I)$. Consequently, I_1 is nilpotent, and since $U(A)/I_1$ is too, $U(A)$ is nilpotent. Q.E.D.

Theorem 2.2.7 Suppose I is a solvable ideal of A. Then $(I \oplus I')i$ generates a nilpotent ideal in $U(A)$.

Proof: Suppose I^* is the subalgebra of $U(A)$ generated by $(I \oplus I')i$. Since I is solvable, $U(I)$ is nilpotent, thus I^* is nilpotent, and the result follows by Corollary 2.2.4. Q.E.D.

Corollary 2.2.8 If A_1 is a subalgebra of A, I is a solvable ideal of A_1, and C is the subalgebra of $\mathrm{Hom}_F(A,A)$ generated by R_a, L_a, $a \in A_1$, then R_b, L_b, $b \in I$, generate a nilpotent ideal in C.

Proof: The proof is immediate by the remarks preceding Lemma 2.2.3. Q.E.D.

2.3 Existence of an Engel Function for Alternative Algebras.

We will now apply the theory developed in Chapter One to alternative algebras. In this section we will show that, if the ground field F of an alternative algebra A has enough elements, then there is an Engel function for the algebra. We will assume throughout the rest of the chapter that A is finite dimensional and contains a unity element 1, and that the characteristic of F is different than 2.

We define

$$(1)\quad a(x_1,x_2,x_3) = \tfrac{1}{2}\{x_3\cdot x_1x_2 + x_2x_1\cdot x_3 - x_2x_3\cdot x_1 - x_1\cdot x_3x_2\}$$

and observe that $a(x_1,x_2,x_3)$ is a linear homogeneous element in the free non-associative algebra on the generators x_1, x_2, x_3 over F.

We begin our study with a test for the a-nilpotence of an alternative algebra.

Lemma 2.3.1 If F is algebraically closed, then A is a-nilpotent if and only if A is a direct sum of almost nil ideals.

Proof: Assume first that A is a-nilpotent and 2.1.3 is the Pierce decomposition of A where the idempotents e_i are primitive. By Lemma 2.1.8, the A_{ii} are almost nil, and from Theorem 2.1.4, $A_{ii}A_{jj} = A_{jj}A_{ii} = 0$, $i \neq j$. We claim $A_{ij} = 0$ when $i \neq j$. Indeed if $x_{ij} \in A_{ij}$, $i \neq j$, then

$$a(x_{ij},e_i,e_j) = \tfrac{1}{2}(e_j \cdot x_{ij}e_i + e_i x_{ij} \cdot e_j - e_i e_j \cdot x_{ij} - x_{ij} \cdot e_j e_i)$$
$$= \tfrac{1}{2}x_{ij} \ .$$

Consequently, $a^k(x_{ij},e_ie_j,\ldots,e_i,e_j) = (\frac{1}{2})^k x_{ij}$. Since A is a-nilpotent, for some k' , $a^{k'}(x_{ij},e_ie_j,\ldots,e_ie_j) = (\frac{1}{2})^{k'} x_{ij} = 0$, hence $x_{ij} = 0$. Thus $A_{ij} = 0$, and it follows $A = \Sigma \oplus A_{ii}$.

Conversely, suppose $A = \Sigma \oplus B_i$ where the B_i are almost nil. By the linearity of $a(x_1,x_2,x_3)$, it is sufficient to consider $a(x_1,x_2,x_3)$ on one B_i . Since $B_i = Ff_i + N_i$ where f_i is a primitive idempotent, for $b_i,c_i \in B_i$, we compute $a(f_i,b_i,c_i) = a(b_i,f_i,c_i) = a(b_i,c_i,f_i) = 0$. Again by the linearity of $a(x_1,x_2,x_3)$, we see it is sufficient to consider $a(x_1,x_2,x_3)$ acting on N_i . Since N_i is nilpotent by Theorem 2.1.5, it follows that N_i is a-nilpotent, hence A is a-nilpotent. Q.E.D.

For the alternative algebra, A , the maps S introduced in §1.3 become

$$xS(b,c) = a(x,b,c) \quad \text{for all } b,c \in A \ .$$

We can now show that $a(x_1,x_2,x_3)$ satisfies the first condition to be an Engel function. [See 1.3. We also note that alternative algebras form an E-class over F].

<u>Lemma 2.3.2</u> Suppose $\dim A = n$ and F has at least $2n+1$ elements. Then A is a-nilpotent if and only if A is a-nil.

Proof: Since F has at least $2n+1$ elements, by Lemma 1.3.2, it is sufficient to prove the result when F is algebraically closed. Clearly if A is a-nilpotent, then $S(b,b)^n = 0$ for all $b \in B$ and thus A is a-nil.

Conversely, suppose A is a-nil and (2.1.3) is the Pierce decomposition of A where the idempotents e_i are primitive. Then the A_{ii} are almost nil, and we claim that $A_{ij} = 0$, $i \neq j$. Indeed, if $x_{ij} \in A_{ij}$, $i \neq j$, we compute

$$x_{ij}S(e_i,e_i) = -(\tfrac{1}{2})x_{ij} .$$

Then $x_{ij}S(e_i,e_i)^k = (-\tfrac{1}{2})^k x_{ij}$. Since $S(e_i,e_i)^n = 0$, it follows that $x_{ij} = 0$. Thus A is a direct sum of almost nil algebras, and by Lemma 2.3.1, A is a-nilpotent. Q.E.D.

Recall that if R is a subalgebra of A, by $L_A(R)$ we mean the Lie algebra of linear transformations on A generated by $S(b,c)$, $b,c,\in R$. Furthermore, $B_b = \{x \in A : xS(b,b)^n = 0\}$ where $\dim A = n$. We have

Theorem 2.3.3 Suppose R is an a-nilpotent subalgebra of A containing 1. Then $L_A(R)$ is nilpotent, and if $A = A_0 \oplus A_1$ is the Fitting decomposition of A relative to $L_A(R)$, then

(i) A_0 is a subalgebra of A containing R

(ii) $A_0A_1 \subseteq A_1$ $\qquad A_1A_0 \subseteq A_1$

Moreover, if F is algebraically closed,

(iii) $A_0 = \cap\{B_b : b \in R\}$

Proof: By the remarks in §1.4, we see that we may assume F is algebraically closed. Since $1 \in R$, there is a set of pairwise orthogonal primitive idempotents $e_1,\ldots,e_s$ such that $R = \sum_{i=1}^{s} \oplus R_i$ where $R_i = Fe_i + N_i$. Then $N = \sum_{i=1}^{s} \oplus N_i$ is the radical of R.

Let C be the subalgebra of $\mathrm{Hom}_F(A,A)$ generated by R_b, L_b, $b \in R$. By Corollary 2.2.8, R_c, L_c, $c \in N$ generate a nilpotent ideal in C. Therefore, if $z \in N$, $b \in R$ it follows that $S(z,b) = \frac{1}{2}(R_zL_b + L_zR_b - L_{zb} - R_{bz})$ is in this nilpotent ideal. Similarly $S(b,z)$ is in this ideal. Thus $S(z,b)$ and $S(b,z)$ are in the radical N^* of C.

If (2.1.3) is the Pierce decomposition of A relative to $e_1,\ldots,e_s$, we see that $R_i \subseteq A_{ii}$, $i = 1,\ldots,s$. Furthermore, by Theorem 2.1.4, we see that A_{ij} is invariant relative to R_b, L_b, $b \in R$. Therefore, to show $L_A(R)$ is nilpotent, we claim it is sufficient to show that for all i and j, the restriction $S(b,c)^{ij}$ of $S(b,c)$, $b,c \in R$ to A_{ij} generates a nilpotent Lie algebra of linear transformations on A_{ij}. For (2.1.3) is a vector space direct sum of A, hence there is a basis for A relative to which the matrix for $S(b,c)$ is block diagonal with each block representing $S(b,c)^{ij}$ for some i and j. When the Lie product of $S(b,c)$ and $S(d,e)$, $b,c,d,e \in R$, is considered, we see the product is determined by the Lie product on the individual blocks. Therefore,

$$L_A(R) \cong \Sigma \oplus L_{A_{ij}}(R)$$

where by $L_{A_{ij}}(R)$ we mean the Lie algebra of linear transformations on A_{ij} generated by $S(b,c)^{ij}$, $b,c \in R$. Consequently, if $L_{A_{ij}}(R)$ is nilpotent for each i and j, it follows that $L_A(R)$ is nilpotent.

If $b,c \in R$, we write

$$b = \Sigma(\beta_k e_k + z_k) \qquad \beta_k, \gamma_k \in F$$

$$c = \Sigma(\gamma_k e_k + w_k) \qquad z_k, w_k \in N .$$

Using the linearity of $S(b,c)$, we compute

$$S(b,c) = S(\Sigma(\beta_k e_k + z_k), \Sigma(\gamma_k e_k + w_k))$$

$$= \sum_{k,\ell} \beta_k \gamma_\ell S(e_k e_\ell) + T$$

where, by the previous remarks, $T \in N^*$. Thus

$$(2) \quad S(b,c)^{ij} = \sum_{k,\ell} \beta_k \gamma_\ell S(e_k, e_\ell)^{ij} + T^{ij} .$$

Suppose $x_{ij} \in A_{ij}$. Then we compute

$$x_{ij} S(e_k, e_\ell)^{ij} = \rho x_{ij} \qquad \text{where } \rho \text{ is a scalar.}$$

From (2), we have $S(b,c)^{ij} = \mu_{ij} I^{ij} + T^{ij}$ where $\mu_{ij} \in F$ and I is the identity transformation of A. Since $T^{ij} \in N^*$, it is clear that $S(b,c)^{ij}$ generate a nilpotent Lie algebra of linear transformations on A_{ij}, hence $L_A(R)$ is nilpotent.

From Theorem 2.1.4, we readily compute that $S(e_k, e_l)^{ii} = 0$ for all i, k, and l. By (2) we have

$$S(b,c)^{ii} = T^{ii} \in N^* .$$

Since N^* is the radical of C, it follows that each element of $L_A(R)$ acts nilpotently on A_{ii}. Therefore

$$R \subseteq \Sigma A_{ii} \subseteq A_0 .$$

If $b \in R$, then $S(b,b) \in L_A(R)$. By definition of A_0, if $x \in A_0$ then $xS(b,b)^n = 0$ where $\dim A = n$. Since $e_1, \ldots, e_s$ are elements of R, we have

$$R \subseteq \Sigma A_{ii} \subseteq A_0 \cap \{B_b : b \in R\} \subseteq \bigcap_{i=1}^{s} B_{e_i}$$

Suppose $x \in \bigcap_{i=1}^{s} B_{e_i}$. By (2.1.3), we write

$$x = \sum_{i=1}^{s} x_{ii} + \sum_{i \neq j} x_{ij} .$$

We compute

$$xS(e_k, e_k) = \sum_i x_{ii} S(e_k, e_k) + \sum_{i \neq j} x_{ij} S(e_k, e_k) .$$

Now $x_{ii} S(e_k, e_k) = 0$ for all i. For $i \neq j$, we compute

$$x_{ij} S(e_k, e_k) = \begin{cases} 0 & i, j \neq k \\ -\frac{1}{2} x_{kj} & i = k \neq j \\ -\frac{1}{2} x_{ik} & i \neq j = k \end{cases}$$

Thus $xS(e_k, e_k) = (-\frac{1}{2}) \sum_{\substack{i=k\neq j \\ i\neq j=k}} x_{ij}$.

But $x \in B_{e_k}$ implies $xS(e_k, e_k)^n = (-\frac{1}{2})^n \sum_{\substack{i=k\neq j \\ i\neq j=k}} x_{ij} = 0$. Since (2.1.3) is direct, it follows that $x_{kj} = 0$, $k \neq j$ and $x_{ik} = 0$, $i \neq k$. Since $x \in B_{e_k}$ for all k , it follows that $x = \Sigma\, x_{ii}$, which implies $\bigcap_{i=1}^{s} B_{e_i} \subseteq \Sigma\, A_{ii}$. Consequently,

$$R \subseteq A_0 = \Sigma\, A_{ii} = \cap\{B_b : b \in R\} = \bigcap_{i=1}^{s} B_{e_i} .$$

Since $\sum_{i\neq j} A_{ij}$ is invariant under $L_A(R)$ and since $A = A_0 \oplus \sum_{i\neq j} A_{ij}$, we have, by Lemma 1.4.1, that $A_1 = \sum_{i\neq j} A_{ij}$.

That A_0 is a subalgebra of A and $A_0A_1 \subseteq A_1$ and $A_1A_0 \subseteq A_1$ follow immediately from the properties of the Pierce decomposition. Q.E.D.

Recall that we wish to show that $a(x_1, x_2, x_3)$ is an Engel function for A . We have yet to show that for $b \in A$, B_b is a subalgebra of A containing b . Since $a(b,b,b) = 0$ implies $bS(b,b) = 0$, it is immediate that $b \in B_b$. Using Theorem 2.3.3, we can now show

<u>Lemma 2.3.4</u> For any $b \in A$, B_b is a subalgebra of A .

<u>Proof</u>: From (P-2) and (2) of §2.1 we have the following identity for all $x,y,z \in A$:

$$(3)\quad [xy,z] = x[y,z] + [x,z]y + 3(x,y,z)\ .$$

Interchanging the x and y, and subtracting gives

$$(4)\quad 6(x,y,z) = [xy,z] - x[y,z] - [x,z]y - [yx,z]$$
$$+ y[x,z] + [y,z]x$$
$$= [[x,y],z] + [[y,z],x] + [[z,x],y]\ .$$

Now let $x,b \in A$. By Theorem 2.1.1, x and b generate an associative subalgebra of A, and hence, from (4) we compute

$$(5)\quad xS(b^i,b^j) = \tfrac{1}{2}(b^j x b^i + b^i x b^j - b^i b^j x - x b^j b^i)$$
$$= -\tfrac{1}{2}(-b^i x b^j - b^j x b^i + x b^j b^i + b^i b^j x)$$
$$= -\tfrac{1}{2}[[x,b^i],b^j]\ .$$

From (4) we compute

$$[[x,b^j],b^i] + [[b^j,b^i],x] + [[b^i,x],b^j] = 6(x,b^i b^j)$$

and hence $[[x,b^i],b^j] = [[x,b^j],b^i]$. Consequently

$$(6)\quad S(b^i,b^j) = S(b^j,b^i)\ .$$

Now suppose $i \geq 2$. Then

$$xS(b^i,b^j) = -\tfrac{1}{2}[[x,b^i],b^j]$$
$$= \tfrac{1}{2}[[b^{i-1}b,x],b^j]$$
$$= \tfrac{1}{2}\{[b^{i-1}[b,x],b^j] + [[b^{i-1},x]b,b^j]\} \text{ by (3)}$$
$$= \tfrac{1}{2}\{b^{i-1}[[b,x],b^j] + [[b^{i-1},x],b^j]b\} \text{ by (3)}$$

$$= -\tfrac{1}{2}\, b^{i-1}[[x,b],b^j] - \tfrac{1}{2}[[x,b^{i-1}],b^j]b$$
$$= x\{S(b,b^j)L_{b^{i-1}} + S(b^{i-1},b^j)R_b\} .$$

We now have

$$(7)\quad S(b^i,b^j) = S(b,b^j)L_{b^{i-1}} + S(b^{i-1},b^j)R_b \qquad i \geq 2 .$$

We observe that when the exponent i is lowered in (7) that j remains unchanged. Also, since $R_{b^t} = (R_b)^t$ and $L_{b^t} = (L_b)^t$ we may apply (6) and (7) to obtain

$$(8)\quad S(b^i,b^j) = S(b,b)\varphi(R_b,L_b)$$

where $\varphi(u,v)$ is a polynomial in u and v .

Next we compute

$$xR_bS(b^i,b^j) = (xb)S(b^i,b^j)$$
$$= -\tfrac{1}{2}[[xb,b^i],b^j] = -\tfrac{1}{2}[[x,b^i]b,b^j]$$
$$= -\tfrac{1}{2}[[x,b^i],b^j]b = xS(b^i,b^j)R_b .$$

Consequently,

$$(9)\quad R_bS(b^i,b^j) = S(b^i,b^j)R_b$$

and similarly

$$(10)\quad L_bS(b^i,b^j) = S(b^i,b^j)L_b .$$

From (9) and (10) we see that $S(b^i,b^j)$ commutes with any polynomial in R_b and L_b . From (8) it follows

$$(11) \quad S(b^i,b^j)^k = S(b,b)\varphi'(R_b,L_b)$$

where $\varphi'(u,v)$ is a polynomial in u and v.

We now define $B_b^* = \{x \in A : xS(b^i,b^j)^n = 0$, $i,j = 0,\dots,n\}$ where $\dim A = n$. From (11) we have immediately that $B_b^* = B_b$.

Now let $F[b]$ be the subalgebra of A generated by b and 1. Since $F[b]$ is commutative and associative, it is a-nilpotent. Therefore, by Theorem 2.3.3, $L_A(F[b])$ is nilpotent and the Fitting null component A_0 of A relative to $L_A(F[b])$ is a subalgebra containing $F[b]$. But elements of $L_A(F[b])$ are sums of products of $S(b^i,b^j)$. Hence it follows that $A_0 = B_b^* = B_b$ and B_b is a subalgebra of A. Q.E.D.

Summarizing our results, we have

Theorem 2.3.5 Let $\mathfrak{A}$ be the class of alternative algebras such that

(i) if $A \in \mathfrak{A}$, then A has a unity

(ii) if $A \in \mathfrak{A}$, $\dim A = n$ and F is the ground field, then F has at least $2n+1$ elements.

Then $a(x_1,x_2,x_3)$ is an Engel function for $\mathfrak{A}$.

As a corollary, we give

Corollary 2.3.6 If A is restricted as in Theorem 2.3.5 and every maximal subalgebra of A is an ideal, then A is a-nilpotent.

Proof: The proof is immediate from Corollary 1.3.5. Q.E.D.

2.4 Cartan Subalgebras of Alternative Algebras

We will now assume A is an n-dimensional alternative algebra with unity 1 where the ground field F has at least N elements, where N is the maximum of $2n+1$ and 2^{n-1}. We note, therefore, that the results of 2.3 are valid. In this section we will show that the class of alternative algebras described above have Cartan subalgebras [§1.4].

As an immediate corollary of Theorem 2.3.3, we give

Lemma 2.4.1. Suppose F is algebraically closed. Then H is a Cartan subalgebra of A if and only if there is a set $e_1, \ldots, e_s$ of pairwise orthogonal primitive idempotents whose sum is 1 and $H = \sum_{i=1}^{s} A_{ii}$.

From Lemma 1.5.4, we see that A contains a-regular elements, hence by Lemma 1.5.3, A contains minimal Engel subalgebras. Furthermore, by Lemma 1.5.5, if A' is a subalgebra of A and B_b is minimal in $\{B_c : c \in A'\}$ where $A' \subseteq B_b$, then $B_b \subseteq B_c$ for all $c \in A'$.

We can now prove

<u>Theorem 2.4.2</u> H is a Cartan subalgebra of A if and only if H is minimal Engel in A.

<u>Proof</u>: Suppose $H = B_b$ is minimal Engel in A. Clearly $1 \in B_b$. Now B_b is minimal in $\{B_c : c \in H\}$ and since $H \subseteq B_b$, $B_b \subseteq B_h$ for all $h \in H$. Thus $S(h,h)$ is nilpotent on H for all $h \in H$. It follows from Lemma 2.3.2 that H is a-nilpotent. By Theorem 2.3.3, $L_A(H)$ is nilpotent and if A_0 is the Fitting null component of A relative to $L_A(H)$, then $B_b = H \subseteq A_0$. However, $A_0 \subseteq \cap\{B_h : h \in H\} \subseteq B_b$ since $S(h,h) \in L_A(H)$ for $h \in H$. Thus $A_0 = B_b$ and B_b is a Cartan subalgebra.

Conversely, suppose H is a Cartan subalgebra of A and B_b is minimal with respect to dimension in $\{B_h : h \in H\}$. We claim $H = B_b$. Since H is a-nilpotent, we have $H \subseteq B_b$, consequently $B_b \subseteq B_h$ for all $h \in H$.

We now extend F to its algebraic closure K and make the following observations:

(i) $H_K \subseteq (B_b)_K$

(ii) H_K is a Cartan subalgebra of A_K [§1.4].

Consequently from Theorem 2.3.3, $H_K = \cap\{B_{h'} : h' \in H_K\}$. Since $B_b \subseteq B_h$ for all $h \in H$, $S(h,h)$ is nilpotent on B_b for all $h \in H$. By Lemma 1.3.3, we have that $S(h',h')$ acts nilpotently on $(B_b)_K$ for all $h' \in H_K$. Therefore $(B_b)_K \subseteq H_K$, and by (i) $H_K = (B_b)_K$. But $\dim H_K = \dim(B_b)_K$ implies $\dim H = \dim B_b$. Since $H \subseteq B_b$, it follows $H = B_b$ as required. Q.E.D.

We have as a corollary

Corollary 2.4.3 A contains Cartan subalgebras.

Proof: The proof is immediate from the theorem and the remarks preceding Lemma 2.4.1. Q.E.D.

2.5 Properties of Cartan Subalgebras

The problem we wish to study is the one concerning the conjugacy of Cartan subalgebras. Barnes [8] gives a successful proof for solvable Lie algebras. We feel that a similar result could be obtained for a-solvable alternative algebras, but we have not been successful. In any event, such a proof appears to require the following two lemmas.

Lemma 2.5.1 Suppose H is a Cartan subalgebra of A and I is an ideal of A. Then $(H+I)/I$ is a Cartan subalgebra of A/I.

Proof: For extensions of K of F we have $(A/I)_K = A_K/I_K$ and $((H+I)/I)_K = (H_K+I_K)/I_K$. Consequently $(H+I)/I$ is a Cartan subalgebra of A/I if and only if $((H+I)/I)_K$ is a Cartan subalgebra of $(A/I)_K$. It follows we may assume F is algebraically closed.

Since $(H+I)/I = H/H\cap I$, we have that $(H+I)/I$ is a-nilpotent. Thus by Theorem 2.3.3, $L_{A/I}((H+L)/I) = \hat{L}$ is nilpotent and if $\hat{A}_0$ is the Fitting null component of A/I relative to $\hat{L}$, then $(H+I)/I \subseteq \hat{A}_0$.

By Lemma 2.4.1 there is a set of primitive idempotents $e_1,\ldots,e_t$ such that $H = \sum_{i=1}^{t} A_{ii}$ where $1 = e_1+\ldots+e_t$. Arrange the e_i so that $e_1,\ldots,e_{t'} \notin I$ and $e_{t'+1},\ldots,e_t \in I$. It follows $(e_1+I)+\ldots+(e_{t'}+I) = \hat{1}$, where $\hat{1}$ is the unity of A/I. Since $I \triangleleft A$, if $e_i \in I$, $\sum_{j=1}^{t} A_{ij} \subseteq I$.

Let $\hat{e}_i = e_i+I$, $i = 1,\ldots,t'$. Using the argument given in Lambeck [20], we see that we can lift the $\hat{e}_i$ to idempotents in A, and it follows that the $\hat{e}_i$ are pairwise orthogonal primitive idempotents in A/I. Let $A/I = \sum_{i,j} \hat{A}_{ij}$ be the Pierce decomposition of A/I relative to the $\hat{e}_i$, and write $B_{\hat{e}_i} = \{\hat{x} \in A/I : \hat{x}S(\hat{e}_i,\hat{e}_i)^m = 0\}$. By Theorem 2.3.3, we have $(H+I)/I \subseteq \hat{A}_0 \subseteq \bigcap_{1}^{t'} B_{\hat{e}_i}$. As in the proof of Theorem 2.3.3 if we choose $\hat{x} \in \bigcap B_{\hat{e}_i}$ and write $\hat{x} = \sum \hat{x}_{ii} + \sum_{i \neq j} \hat{x}_{ij}$ we can show $\hat{x}_{ij} = 0$ for $i \neq j$, and it follows that $\hat{A}_0 = (H+L)/I$. Therefore $(H+I)/I$ is a Cartan subalgebra of A/I. Q.E.D.

<u>Lemma 2.5.2</u> Suppose I is an ideal of A, J is a subalgebra of A containing 1 and such that $I \subseteq J \subseteq A$, and J/I is a Cartan subalgebra of A/I. If H is a Cartan subalgebra of J, then H is a Cartan subalgebra of A.

<u>Proof:</u> Since H is a Cartan subalgebra of J, H is a-nilpotent

and contains 1 . Let A_0 and $\hat{A}_0$ be the Fitting null components of A and A/I respectively relative to $L_A(H)$ and $L_{A/I}((H+I)/I)$. Since (H+I)/I is a Cartan subalgebra of J/I and since J/I is itself a-nilpotent, it follows that (H+I)/I = J/I , hence H+I = J and $\hat{A}_0 = J/I$. If $x+I \in A_0$ then $x \in J$. But H is a Cartan subalgebra of J and it follows that $x \in H$. Consequently, $A_0 = H$ and H is a Cartan subalgebra of A . Q.E.D.

With respect to the conjugacy problem, we must be content with

<u>Theorem 2.5.3</u> Suppose F is algebraically closed and of characteristic zero. If H_1 and H_2 are two Cartan subalgebras of A , then there is an inner automorphism $s \in I(A)$ such that $H_1 = H_2 s$.

<u>Proof</u>: In §1.6, we defined the group I(A) of inner automorphisms of A . From (5) in §2.1, we see that $D(b,c) = [L_b,L_c] + [L_b,R_c] + [R_b,R_c]$ is in the ideal D′(A) of inner derivations of the derivation algebra D(A) of A . Hence, to apply Theorem 1.6.9, we must show that H_1 and H_2 each contain relatively open subsets O_1 and O_2 such that $T(\Omega_i : a) = A$ for all $a \in O_i$, $i = 1,2$, where Ω_i is the orbit of H_i under I(A) .

By Lemma 2.4.1, we write $H_1 = \sum_{i=1}^{t} A_{ii}$ where $e_1,\ldots,e_t$ are primitive idempotents in $A_{ii} = Fe_i + N_i$. Let

$O_1 = \{a \in H_1 : a = \Sigma\, \alpha_i e_i + z_i \quad \alpha_i \in F\ ,\ z_i \in N_i\ ,\ \text{and} \prod_{i \neq j} (\alpha_i - \alpha_j) \neq 0\}$.

If $f(x_1,\ldots,x_t) = \prod_{i \neq j} (x_i - x_j)$, it follows $f(x_1,\ldots,x_t) \to f \in F[H_1]$ and O_1 is an open subset of H_1.

Let (2.1.3) be the Pierce decomposition of A relative to $e_1,\ldots,e_t$. For $b_{k\ell} \in A_{k\ell}$, $k \neq \ell$, and $a = \Sigma(\alpha_i e_i + z_i) \in O_1$, we compute, using Theorem 2.1.4.

$$aD(e_k, b_{k\ell}) = (\alpha_k - \alpha_\ell) b_{k\ell} + z_k b_{k\ell} - b_{k\ell} z_\ell\ .$$

Since $D(e_k, b_{k\ell}) \in D'(A)$, $aD(e_k, b_{k\ell}) \in T(\Omega_1; a)$ by the remarks preceding Lemma 1.6.8. Define

$$S_{k\ell} = (\alpha_k - \alpha_\ell) I + L_{z_k} - R_{z_\ell}$$

where I is the identity transformation on A. Then the above computations show $b_{k\ell} S_{k\ell} \in T(\Omega_1; a)$. Now $S_{k\ell}$ maps $A_{k\ell}$ into itself since $A_{kk} A_{k\ell} \in A_{k\ell}$ and $A_{k\ell} A_{\ell\ell} \subseteq A_{k\ell}$. From §2.2, L_{z_k} and R_{z_ℓ} are nilpotent, hence $L_{z_k} - R_{z_\ell}$ is nilpotent, and since $\alpha_k - \alpha_\ell \neq 0$, it follows that $S_{k\ell}$ is invertible. Therefore $A_{k\ell} \subseteq T(\Omega_1; a)$ for $k \neq \ell$. By Lemma 1.6.7, we have $H_1 \subseteq T(\Omega_1; a)$, hence $T(\Omega_1; a) = A$. Thus H_1 and O_1 satisfy the hypothesis of Lemma 1.6.8. Similarly there is a relatively open subset O_2 of H_2 satisfying the hypothesis of Lemma 1.6.8. The proof is now immediate by Theorem 1.6.9.

Q.E.D.

CHAPTER THREE

JORDAN ALGEBRAS

3.1 Introduction

An algebra J over a field F is called a (commutative) Jordan algebra if and only if $[x,y] = (x^2,y,x) = 0$ for all elements x and y of J. From the definition, it is clear that homomorphic images of Jordan algebras are Jordan algebras and a direct sum of Jordan algebras is a Jordan algebra. Since the commutator and associator are multilinear, it follows that J_K is a Jordan algebra for all extensions K of F. Throughout Chapter Three, we will assume our Jordan algebras are finite dimensional.

In this section, we will give the necessary theory of Jordan algebras to develop a Cartan theory for these algebras. In what follows, where proofs are not provided, the reader is referred to Albert [2].

We begin by linearizing $(x^2,y,x) = 0$ and obtain

$$(1)\quad J(w,x,y,z) = (wx,y,z) + (xz,y,w) + (zw,y,x) = 0$$

Then computing $0 = J(x,y,w,z) - J(x,y,z,w)$, we have

$$(2)\quad D(w,x,y,z) = (w,xy,z) - x(w,y,z) - (w,x,z)y = 0 .$$

As a consequence of (2), we have that the map

(3) $xD(a,b) = (a,x,b) \quad x \in J$ for all $a,b \in J$

is a derivation of J. In terms of right multiplication R_x, (3) implies that $[R_a,R_b]$ is a derivation of J, and furthermore, by (2), we have $[R_y,[R_w,R_z]] = R_{(w,y,z)}$. Hence, the Lie multiplication algebra $L(J)$ of J is $R(J) + [R(J),R(J)]$ where $R(J) = \{R_a : a \in J\}$. If D is an arbitrary derivation of J, then for $x \in J$, $[R_x,D] = R_{xD}$. Suppose D is an inner derivation of J. Then $D = R_x + \sum_i [R_{y_i},R_{x_i}]$. If J has a unity, it follows that $x = 0$, since $1D = 0$. Therefore, for Jordan algebras with unity the inner derivations are expressible in the form $\sum_i D(b_i,c_i)$ by (3).

We next give

<u>Theorem 3.1.1</u> Jordan algebras are power associative in the sense that single elements generate an associative subalgebra.

In the course of the proof, we note

(4) $R_{a^i}R_{a^j} = R_{a^j}R_{a^i} \quad i,j = 0,1,\ldots$

Also, we have

<u>Theorem 3.1.2</u> The following are equivalent in J:

(i) J is nil

(ii) J is solvable

(iii) J is nilpotent

Now suppose e is an idempotent of J. We compute $0 = J(a,e,e,e) = 2(ae\cdot e)e - 3ae\cdot e + ae$. Consequently, $a(2R_e^3 - 3R_e^2 + R_e) = 0$ and the transformation R_e satisfied the polynomial $2\lambda^3 - 3\lambda^2 + \lambda = \lambda(2\lambda-1)(\lambda-1)$. Thus the characteristic roots of R_e are among $0, \frac{1}{2}$, and 1. It follows that J can be written as a vector space direct sum $J = J_0(e) + J_{\frac{1}{2}}(e) + J_1(e)$ where $J_\lambda(e) = \{x \in J : xe = \lambda x\}$ $\lambda = 0, \frac{1}{2}$, and 1. If $x \in J$, then we write $x = (2e\cdot ex - ex) + (4ex - 4e\cdot ex) + (x + 2e\cdot ex - 3ex)$.

If J has a unity and $e_1, \ldots, e_s$ are a set of pairwise orthogonal idempotents whose sum is the unity of J, we get a refined decomposition of J as a vector space direct sum:

$$(3.1.3) \qquad J = \sum_{i \leq j = 1}^{s} J_{ij}$$

$$J_{ii} = J_1(e_i), J_{ij} = J_{\frac{1}{2}}(e_i) \cap J_{\frac{1}{2}}(e_j) \quad i \neq j.$$

This decomposition is called the <u>Pierce decomposition</u> of J relative to $e_1, \ldots, e_s$. We have

<u>Theorem 3.1.4</u> If (3.1.3) is the Pierce decomposition of J relative to $e_1, \ldots, e_s$, then

(i) $J_{ii}^2 \subseteq J_{ii} \quad J_{ii}J_{jj} = 0 \quad i \neq j$

(ii) $J_{ij}J_{ii} \subseteq J_{ij} \quad J_{ij}^2 \subseteq J_{ii} + J_{jj} \quad i \neq j$

(iii) $J_{ij}J_{jk} \subseteq J_{ik} \quad J_{ii}J_{jk} = 0 \quad J_{ij}J_{k\ell} = 0$

if i,j,k,ℓ are distinct.

Suppose J is semi-simple. Then J contains a unity. Furthermore, we have

<u>Theorem 3.1.5</u> A non-zero Jordan algebra is semi-simple if and only if J is a direct sum of simple ideals.

Thus the study of semi-simple Jordan algebras is reduced to the study of simple algebras. If J is simple and contains a unity, then in the case when the characteristic of F is 0, the corresponding J_{ij}, $i \neq j$, have a common dimension, necessarily greater than zero.

If F is algebraically closed and (3.1.3) is the Pierce decomposition of J relative to a set of pairwise orthogonal primitive idempotents $e_1, \ldots, e_s$, we may argue as in the alternative case that if $x \in J_{ii}$, then $x = \alpha e_i + n$ where $\alpha \in F$ and n is nilpotent. By McKrimmon [21], we have

<u>Lemma 3.1.6</u> If F is algebraically closed and (3.1.3) is the Pierce decomposition of J relative to a set of primitive idempotents, then the J_{ii} are almost nil.

3.2 The Universal Multiplication Envelope of a Jordan Algebra

In [16], Jacobson proved the analogue for Jordan algebras of Corollary 2.2.8. We will now sketch the proof.

Suppose J is a finite dimensional Jordan algebra over F . We define inductively $J_1 = J$ and $J_n = J_{n-1} \otimes J_1$, $n>1$. Let $T(J)$ be the associative algebra defined by

$$T(J) = J_1 \oplus J_2 \oplus \cdots$$

and let S be the ideal of $T(J)$ generated by elements of the form

$$(1) \quad x \otimes x^2 - x^2 \otimes x$$

$$(2) \quad x^2y + 2x \otimes y \otimes x - y \otimes x^2 - 2x \otimes xy$$

where $x,y \in J$. The associative algebra $U(J) = T(J)/S$ is called the <u>universal multiplication envelope</u> of J . If i' is the canonical homomorphism from $T(J)$ into $U(J)$, then the restriction of i' to $J = J_1$ defines a linear map from J into $U(J)$. We call this map i , and if $a \in J$, write $ai = \bar{a}$ where $\bar{a}$ is the coset of a in $U(J)$.

From (1) and (2) we have the following identies in $U(J)$:

$$(3) \quad \bar{x} \otimes \overline{x^2} = \overline{x^2} \otimes \bar{x}$$

$$(4) \quad 2\bar{x} \otimes \bar{y} \otimes \bar{x} = \bar{y} \otimes \overline{x^2} + 2x \otimes \overline{xy} \quad \overline{x^2y}$$

The analogues of Lemmas 2.2.1 and 2.2.2 now follow, and we state

<u>Lemma 3.2.1</u> If ρ is a linear map from J into an associative algebra V such that

$$(5) \quad (x\rho)(x^2\rho) = (x^2\rho)(x\rho)$$

$$(6)\quad 2(x\rho)(y\rho)(x\rho) = (y\rho)(x^2\rho) + 2(x\rho)(xy\rho) - (x^2y)\rho$$

then there is a unique homomorphism ρ^* from $U(J)$ into V such that $xi\rho^* = x\rho$ for all $x \in J$.

<u>Lemma 3.2.2</u> If K is an ideal of J and D is the ideal in $U(J)$ generated by Ki , then there exists an isomorphism of $U(J/K)$ onto $U(J)/D$ such that $\overline{x+K}$ is mapped onto $\bar{x}+D$.

Furthermore, it is known that $U(J)$ is finite dimensional [14, page 519].

We now give

<u>Theorem 3.2.3</u>. If I is a solvable ideal of J , then Ii generates a nilpotent ideal in $U(J)$.

<u>Proof</u>: The proof is by induction on the dimension of I , and we assume $I \neq 0$. By Penico [23], we know that there is an ideal I' of J such that $I^2 \subseteq I' \subsetneqq I$. By the induction hypothesis, $I'i$ generates a nilpotent ideal $\bar{I}'$ in $U(J)$, and by Lemma 3.2.2, $U(J/I') = U(J)/\bar{I}'$. Now suppose the image of I/I' generates a nilpotent ideal in $U(J/I')$. Then by the isomorphism noted above, Ii will generate a nilpotent ideal in $U(J)$. Therefore, it is sufficient to prove the theorem for the case when $I^2 = 0$, since $(I/I')^2 = 0$.

From the identities (3) and (4) we can prove the following results [16]:

(A) $U(I)$ is nilpotent

(B) If $e_1,\ldots,e_m$ is a basis for I and $e_1,\ldots,e_m,\ldots,e_n$ is a basis for J, and k is a positive integer, then any monomial in $U(J)$ of the form $\bar{e}_{j_1} \ldots \bar{e}_{j_\ell}$ in which $k+m$ of the j's are in the range $Q = \{1,2,\ldots,m\}$ is a linear combination of monomials of the form $\bar{e}_{i_1} \ldots \bar{e}_{i_k} \ldots$ where $i_1,\ldots,i_k \in Q$.

From (A) it follows that if I^* is the subalgebra of $U(J)$ generated by Ii, then I^* is nilpotent. Suppose $(I^*)^p = 0$. From (B) we have that any product of elements a $U(J)$ which include $p+m$ elements of I is zero. Consequently, if $\bar{I}$ is the ideal in $U(J)$ generated by I^*, $(\bar{I})^{p+m} = 0$.

Q.E.D.

<u>Corollary 3.2.4</u> If J_1 is a subalgebra of J, I is a solvable ideal of J_1, C is the subalgebra of $\mathrm{Hom}_F(J,J)$ generated by R_a, $a \in J_1$ and $\bar{C}$ is the ideal of C generated by R_b, $b \in I$, then $\bar{C}$ is nilpotent.

<u>Proof</u>: The map $\rho : a \to R_a$ is a linear map from J_1 into C. In deriving (1) of 3.1, we note that ρ satisfied the hypothesis of Lemma 3.2.1. Consequently the diagram

$$\begin{tikzcd}
U(J_1) \arrow[dr, dashed, "\rho^*"] & \\
J_1 \arrow[u, "i_1"] \arrow[r] & C
\end{tikzcd}$$

is commutative where i_1 is the natural map from J_1 into $U(J_1)$. It follows that ρ^* is onto C , and by Theorem 3.2.3, as Ii_1 generates a nilpotent ideal in $U(J_1)$, C is nilpotent.

Q.E.D.

3.3 Cartan Subalgebras of Jordan Algebras

We will now apply the theory developed in Chapter One to Jordan algebras. Since the development is virtually identical with the alternative case, we will furnish proofs only when the method does not follow by an obvious modification of the alternative proof.

In this section, J will denote an n-dimensional Jordan algebra with unity 1 where the ground field F has at least N elements, where N is the maximum of 2^{n-1} and $2n+1$.

We define

(1) $A(x_1,x_2,x_3) = x_1x_2 \cdot x_3 - x_1 \cdot x_2x_3$

and observe that $A(x_1,x_2,x_3)$ is a linear homogeneous element in the free non-associative algebras on the generators x_1 , x_2 and x_3 over F . We also note that the maps S introduced in §1.3 become $xS(b,c) = A(x,b,c)$ for all $b,c \in J$.

We now give

Lemma 3.3.1 If F is algebraically closed, then J is A-nilpotent if and only if J is a direct sum of almost nil algebras.

Lemma 3.3.2 (Engel) J is A-nilpotent if and only if J is A-nil.

Theorem 3.3.3 Suppose R is an A nilpotent subalgebra of J containing 1. Then $L_J(R)$ is nilpotent, and if $J = J_0 \oplus J_1$ is the Fitting decomposition of J relative to $L_J(R)$,

(i) J_0 is a subalgebra of J containing R

(ii) $J_0 J_1 \subseteq J_1$

Moreover, if F is algebraically closed,

(iii) $J_0 = \cap\{B_b : b \in R\}$

Corollary 3.3.4 If F is algebraically closed, then H is a Cartan subalgebra of J if and only if there is a set $e_1, \ldots, e_t$ of pairwise orthogonal primitive idempotents whose sum is 1 and $H = \sum_{i=1}^{t} U_{ii}$.

We can now prove

Lemma 3.3.5 For all $a \in J$, B_a is a subalgebra of J.

Proof: The proof is patterned after the proof of Lemma 2.3.4.

By (4) in §3.1, we have that R_{a^i} and R_{a^j} commute for all i and j, hence we have

(2) $S(a^i,a^j) = S(a^j,a^i) \quad i,j = 0,1,\ldots$

Computing $D(x,a^{i-2},a,a^j)$ for $i \geq 2$ gives

(3) $S(a^i,a^j) = S(a,a^j)R_{a^{i-1}} + S(a^{i-1},a^j)R_a$.

From (P-1) in the Introduction, we compute $(a^k x,a^i,a^j) - (a^k,xa^i,a^j) + (a^k,x,a^{i+j}) - a^k(x,a^i,a^j) - (a^k,x,a^i)a^j = 0$. From (4) in §3.1, we have $(a^k,xa^i,a^j) = (a^k,x,a^{i+j}) = (a^k,x,a^i)a^j = 0$. Hence $(xa^k,a^i,a^j) - (x,a^i,a^j)a^k = 0$ and we have

(4) $R_{a^k}S(a^i,a^j) = S(a^i,a^j)R_{a^k} \quad i,j,k \geq 0$.

By an easy induction proof on i , $i \geq 2$, we can prove

(5) $S(a^i,a^k) = S(a,a^k)\,\varphi_i$ where

$\varphi_i = \sum_{j=0}^{i-2} R_{a^{i-j}}R_a^{i-j} + 2R_a^{i-1} \quad i > 2, R_a^0 = I$, and if

$i = 2$, $\varphi_i = 2R_a$.

From (2) we have that $S(a,a^k) = S(a,a)\varphi_k$. Therefore,

(6) $S(a^i,a^k) = S(a,a)\varphi_i\varphi_k$.

By (4), φ_i and φ_k commute with $S(a^p,a^q)$, hence

(7) $S(a^i,a^k)^m = S(a,a)^m(\varphi_i\varphi_k)^m$.

If $B_a^* = \{x \in J : xS(a^i,a^j)^n = 0 , \quad i,j = 0,\ldots,n\}$ then from (7) we have $B_a = B_a^*$.

The rest of the proof is the same as the conclusion of the proof of Lemma 2.3.4. Q.E.D.

Summarizing the above results, we have

Theorem 3.3.6 Let $\mathfrak{J}$ be the class of Jordan algebras such that

(i) if $J \in \mathfrak{J}$, then J contains a unity

(ii) if $J \in \mathfrak{J}$, $\dim J = n$, and F is the ground field, then F has at least $2n+1$ elements.

Then $A(x_1, x_2, x_3)$ is an Engel function for $\mathfrak{J}$.

Applying Lemmas 1.5.3, 1.5.4, 1.5.5 and following the proof of Theorem 2.4.2, we have

Theorem 3.3.7 H is a Cartan subalgebra of J if and only if H is minimal Engel in J.

Corollary 3.3.8 J contains Cartan subalgebras.

We remark that we have the counterparts of Lemmas 2.5.1 and 2.5.2 for Jordan algebras. As in the alternative case, our attempt to follow Barnes [8] in obtaining conjugacy results was not successful. Again, we must be content with

Theorem 3.3.9 Suppose F is algebraically closed and of characteristic zero. If H_1 and H_2 are two Cartan subalgebras of J , then there is an inner automorphism $s \in I(J)$ of J such that $H_1 s = H_2$.

Proof: Using $D(b,c) = [R_b, R_c]$ as our inner derivations, the proof proceeds identically with the alternative case. Q.E.D.

Remark: It should be noted that if A is an alternative algebra, then we can make A into a Jordan algebra A^+ by defining a new product $a \circ b$ of A by the equation $a \circ b = \frac{1}{2}(ab + ba)$. Using $A(x_1, x_2, x_3)$ in A^+ , we might expect to develop a Cartan theory for alternative algebras. We compute $(x_1 \circ x_2) \circ x_3 - x_1 \circ (x_2 \circ x_3) = 2a(x_1, x_2, x_3) + 2A(x_1, x_2, x_3)$. Therefore, there is no reason to expect that the Cartan theory for A developed in Chapter Two arises from iterating the associator in A^+ .

Remark: Using the defining identities, we find that the Engel functions are used for alternative and Jordan algebras are identical. In the case of Lie algebras, $a(x_1, x_2, x_3) = -x_1 x_3 \cdot x_2$. By virtue of Property 1.2.6, it follows that $a(x_1, x_2, x_3)$ is an Engel function for Lie algebras and that the Cartan theory developed from this function coincides with the classical theory. Thus we have the one function that gives the Cartan theory for Lie, Jordan and alternative algebras.

3.4 A-solvable Jordan Algebras

Suppose J' is a finite dimensional Jordan algebra over a field F where F is algebraically closed and of characteristic zero. If J' is semi-simple in the classical sense, then J' has a unity $1'$ and is expressible as a direct sum of simple Jordan algebras (see §3.1). Let J be one of these simple summands.

Let 1 be the unity of J, and let $\sum_{i \leq j} J_{ij}$ be the Pierce decomposition of J relative to a set $e_1, \ldots, e_t$ of pairwise orthogonal primitive idempotents whose sum is 1. Albert [2] has shown that if $i \neq j$, then $J_{ij} \neq 0$ and in fact all the J_{ij}, $i \neq j$ have a common dimension δ. Furthermore, each J_{ij}, $i \neq j$, has an orthonormal basis, that is, a basis $x_1, \ldots, x_\delta$ such that $x_k x_\ell = 0$ is $k \neq \ell$, and $x_k^2 = e_i + e_j$ [2]. Of course, if $t = 1$, then $J = F1$. Therefore, we will let $t > 1$.

Consider J_{ij}, $i \neq j$, and let x_{ij} be an element in an orthonormal basis for J_{ij}. Consequently, $x_{ij}^2 = e_i + e_j$. Let $y = e_i - e_j$. We compute

$$(x_{ij}, y, y) = (x_{ij}, e_i - e_j, e_i - e_j) = -x_{ij}$$

$$(x_{ij}, x_{ij}, y) = (x_{ij}, x_{ij}, e_i - e_j) = y .$$

Suppose p is an arbitrary positive integer. From the above computations we see we may construct a 3^p-tuple $(z_1, \ldots, z_{3^p})$

where the z_i are either x_{ij} or y such that $A^{(p)}(z_1,\ldots,z_{3^p}) = x_{ij}$. Similarly, we may construct a 3^p-typle $(w_1,\ldots,w_{3^p})$ such that $A^{(p)}(w_1,\ldots,w_{3^p}) = y$. It follows that J is not A solvable, hence J' is not A-solvable. We have shown

<u>Lemma 3.4.1</u> Let F be an algebraically closed field of characteristic zero and J a Jordan algebra over F. Then J is semi-simple with respect to A-solvability if and only if J is a direct sum of simple Jordan algebras, each one of which has degree greater than one.

<u>Proof</u>: If J is semi-simple with respect to A-solvability, by Property 1.2.5, J is semi-simple in the classical sense. Consequently J is a direct sum of simple algebras, and the above calculation shows that at least one of these must have degree greater than one (since simple Jordan algebras of degree 1 over F are associative). The converse is immediate. Q.E.D.

As a corollary, we have

<u>Corollary 3.4.2</u> Let F and J be as in Lemma 3.4.1. If J is A-solvable, and semi-simple in the classical sense, then J is a direct sum of copies of F.

In what follows J will denote an n-dimensional Jordan algebra with unity 1 over a field F where F is of characteristic zero. From (3), §3.1, we know that $D(b,c) = [R_b,R_c]$ is a derivation of J for all $b,c \in J$. If $D(b,c)^p = 0$ for some p , then $\exp D(b,c) = \sum_{k=0}^{\infty} \frac{1}{k!} D(b,c)^k$ is an automorphism of J . [18].

Albert [2] has shown that if K is an extension of F, since char $F = 0$, J_F is semi-simple in the classical sense if and only if J_K is semi-simple in the classical sense. Since $D(b,c)^k$ is multilinear for all k , we see it is sufficient to investigate the nilpotence of $D(b,c)$ when F is algebraically closed.

We now prove

Theorem 3.4.3 If J' is an A-solvable subalgebra of J , then the derivations $D(b,c)$, $b,c \in J'$, of J are nilpotent.

Proof: By the remarks preceding the theorem, we see it is sufficient to prove the theorem when F is algebraically closed. Let us denote the classical radical of J' by $S(J')$.

Case 1: $S(J') = 0$. By Lemma 3.4.2, there is a set of pairwise orthogonal primitive idempotents $e_1,\ldots,e_s \in J'$ such that $J' = Fe_1 \oplus \cdots \oplus Fe_s$. From the identity $J(w,x,y,z) = 0$ [(1), §(3.1)], we compute $J(e_i,e_i,x,e_j) = (e_i^2,x,e_j) + 2(e_ie_j,x,e_j) = 0$. If $i \neq j$, it follows that $D(e_i,e_j) = 0$, Of course, $D(e_i,e_i) = 0$. Thus for all $b,c \in J'$, $D(b,c) = 0$.

Case 2: $S(J') \neq 0$. By the Wedderburn principal theorem for Jordan algebras [23], we have $J' = J_1' \oplus S(J')$ where

$J_1' \cong J'/S(J')$. Consequently by Corollary 3.4.2, $J_1' = Fe_1 \oplus \ldots \oplus Fe_s$ where the e_i are pairwise orthogonal primitive idempotents whose sum is the unity of J' [20; page 72]. Thus $J' = Fe_1 \oplus \ldots \oplus Fe_s \oplus S(J')$.

Let C be the subalgebra of $\mathrm{Hom}_F(J,J)$ generated by R_a , $a \in J'$. Then the maps R_b , $b \in S(J')$ generate a nilpotent ideal N^* in C , by Corollary 3.2.4. Consequently, if $a \in J'$ and $b \in S(J')$, $D(a,b) = [R_a, R_b] \in N^*$. From Case 1, we see that $D(e_i, e_j) = 0$ for all i and j . Therefore if $a,b \in J$ and we write

$$a = \Sigma\, \alpha_i e_i + a_1 \qquad \alpha_i, \beta_i \in F$$

$$b = \Sigma\, \beta_i e_i + b_1 \qquad a_1, b_1 \in S(J')$$

it follows that $D(a,b) = \sum_{k,\ell} \alpha_k \beta_\ell D(e_k, e_\ell) + T = T \in N^*$. Consequently, for some p , $D(a,b)^p = 0$, and the derivations are nilpotent as desired. Q.E.D.

As a consequence of this theorem, we have that if J is A-solvable, the for all $b,c \in J$, $D(b,c)$ is nilpotent. Suppose we let $I^*(J)$ be the group of automorphisms of J generated by $\exp D(b,c)$, $b,c \in J$. We have not been able to answer the following:

(i) What is the relationship between $I(J)$ and $I^*(J)$?

(ii) Are Cartan subalgebras of J conjugate under $I^*(J)$?

If the answer to (ii) is yes , it may be possible to use the techniques given in Barnes [8] to improve our conjugacy results.

CHAPTER FOUR

COMMUTATIVE POWER ASSOCIATIVE ALGEBRAS

4.1 Introduction

A commutative algebra X over a field F is called power associative if and only if the subalgebra generated by each element of X is associative. Examples of such algebras are the Jordan algebras studies in the previous chapter. X is called strictly power associative if and only if X_K is power associative for all extensions K of F. In this chapter, we will let X denote a finite dimensional power associative algebra over a field F where F has at least four elements and characteristic different than 2.

We note that $(x^2,x,x) = 0$ is an identity in X. Since F has at least four elements, we may linearize this identity and obtain

$$(1)\quad 4wx \cdot x^2 = 2(wx \cdot x)x + (wx^2 \cdot x) + wx^3$$

and

$$(2)\quad 4[wx \cdot yz + wy \cdot xz + wz \cdot xy] = (wx \cdot y + xy \cdot w + yw \cdot x)z + (xy \cdot z + yz \cdot x + zx \cdot y)w + (yz \cdot w + zw \cdot y + wy \cdot z)x + (zw \cdot x + wx \cdot z + xz \cdot w)y .$$

We begin our discussion by showing that solvable commutative power associative algebras are nilpotent [6]. We define $X_1 = X$, and inductively $X_n = X_{n-1} \otimes X_1$. Let $T(X)$ be the associative algebra defined $T(X) = X_1 \oplus X_2 \oplus \cdots$, where the vector space operations are as usual, and multiplication is denoted by $\otimes$. Let S be the ideal of $T(X)$ generated by elements of the form

$$(3) \quad 4x \otimes x^2 - 2x \otimes x \otimes x - x^2 \otimes x - x^3$$

$$\begin{aligned}(4) \quad & 4[x \otimes yz + y \otimes xz + z \otimes xy] - [x \otimes y + xy + y \otimes x] \otimes z \\ & - [xy\cdot z + yz\cdot x + zx\cdot y] - [yz + y \otimes z + z \otimes y] \otimes x \\ & - [z \otimes x + x \otimes z + xz] \otimes y \qquad x,y,z \in X\end{aligned}$$

The associative algebra $U(X) = T(X)/S$ is called the <u>universal multiplication envelope</u> of X . If i' is the canonical homomorphism from $T(X)$ into $U(X)$ then the restriction of i' to $X = X_1$ defines a linear map from X into $U(X)$. We call this map i , and if $x \in X$, we write $xi = \bar{x}$ where $\bar{x}$ is the coset of x in $U(X)$. From (3) and (4), we have the following identities in $U(X)$:

$$(5) \quad 4\bar{x} \otimes \overline{x^2} = 2\bar{x} \otimes \bar{x} \otimes \bar{x} - \overline{x^2} \otimes \bar{x} - \overline{x^3}$$

$$\begin{aligned}(6) \quad & 4[\bar{x} \otimes \overline{yz} + \bar{y} \otimes \overline{xz} + \bar{z} \otimes \overline{xy}] = [\bar{x} \otimes \bar{y} + \overline{xy} + \bar{y} \otimes \bar{x}] \otimes \bar{z} \\ & + [\overline{xy}\cdot\bar{z} + \overline{yz}\cdot\bar{x} + \overline{zx}\cdot\bar{y}] + [\overline{yz} + \bar{y} \otimes \bar{z} + \bar{z} \otimes \bar{y}] \otimes \bar{x} \\ & + [\bar{z} \otimes \bar{x} + \overline{xz} + \bar{x} \otimes \bar{z}] \otimes \bar{y} .\end{aligned}$$

The analogue of Lemma 2.2.1 now follows, and we state

<u>Lemma 4.1.1</u> Let ρ be a linear map from X into an associative algebra V such that

(i) $4(x\rho)(x\rho) = 2(x\rho)^3 + (x^2\rho)(x\rho) + (x^3\rho)$

(ii) $4[(x\rho)(yz)\rho + (y\rho)(xz\rho) + (z\rho)(xy\rho)] =$

$$[(x\rho)(y\rho) + (xy)\rho + (y\rho)(x\rho)](z\rho) + [xy\cdot z + yz\cdot x + zx\cdot y]\rho$$

$$+ [(yz)\rho + (y\rho)(z\rho) + (z\rho)(y\rho)](x\rho) +$$

$$[(z\rho)(x\rho) + (zx)\rho + (x\rho)(z\rho)](y\rho) .$$

Then there is a unique homomorphism ρ^* from $U(X)$ into V such that $\bar{x}\rho^* = x\rho$ for all $x \in X$.

We also observe that $U(X)$ is generated by Xi .

We now prove

Lemma 4.1.2 Let P be an arbitrary associative algebra over F and M a subalgebra of P such that $P^3 \subseteq M + PM + MP$. Then for each integer $k \geq 1$, $P^{3^k} \subseteq M^k + PM^k + M^kP + PM^kP$.

Proof: [6] The proof is by induction on k , the result being obvious for $k = 1$.

In general, we have $P^{3^{k+1}} = P^{3^k}P^{3^k}P^{3^k} \subseteq P^{3^k}P^3P^{3^k}$:
Applying the induction hypothesis to P^{3^k} , we obtain

$P^{3^{k+1}} \subseteq M^kP^aM^k + M^kP^bM^kP + PM^kP^cM^k + PM^kP^dM^kP$ where a,b,c and d are integers greater than 2 . Since $P^n \subseteq M + PM + MP$ for $n \geq 3$, the above relation implies $P^{3^{k+1}} \subseteq M^{k+1}+PM^{k+1}+M^{k+1}P+PM^{k+1}P$, which completes the induction proof. Q.E.D.

Lemma 4.1.3 If X is solvable and the dimension of X is 1, then $U(X)$ is nilpotent.

Proof: Since $X = Fx$ where $x^2 = 0$, from (5) we obtain $\bar{x} \otimes \bar{x} \otimes \bar{x} = 0$. Since $\bar{x}$ generates $U(X)$ in this case, we have $(U(X))^3 = 0$. Q.E.D.

Lemma 4.1.4 If X is solvable, then $U(X)$ is nilpotent.

Proof: The proof is by induction on the dimension n of X. By Lemma 4.1.3, we assume $n > 1$ and that $U(X')$ is nilpotent for all solvable algebras X' of dimension less than n.

Since $X^2 \subsetneqq X$, there is an $n-1$ dimensional subspace B such that $X^2 \subseteq B \subseteq X$, consequently $B \triangleleft X$. Since B is solvable, $U(B)$ is nilpotent.

Let B^* be the subalgebra of $U(X)$ generated by Bi, and $D = B^* + B^* \otimes U(X) + U(X) \otimes B^*$. We claim

$$(7) \qquad U(X) \otimes U(X) \otimes U(X) \subseteq D$$

Since $U(X)$ is generated by Xi and $X = B + Fw$ for some $w \notin B$, to prove (7), it is sufficient to show $\bar{x} \otimes \bar{y} \otimes \bar{z} \in D$ where $x,y,z = B \cup \{w\}$.

If x or z is in B, then $\bar{x} \otimes \bar{y} \otimes \bar{z} \subseteq D$. Therefore we may assume $x = z = w$. From (6) and the fact that $X^2 \subseteq B$, we have

$$2\bar{w} \otimes \bar{y} \otimes \bar{w} \equiv -2\bar{y} \otimes \bar{w} \otimes \bar{w} - 2\bar{w} \otimes \bar{w} \otimes \bar{y} \pmod{D}.$$

If $y \in B$, then $\bar{w} \otimes \bar{y} \otimes \bar{w} \in D$, and if $y = w$, from (5) we have $\bar{w} \otimes \bar{w} \otimes \bar{w} \in D$. Consequently (7) is valid.

Consider the diagram

$$\begin{array}{ccc} U(B) & & \\ {\scriptstyle j}\uparrow & & \\ B & \xrightarrow[i]{} & B^* \end{array}$$

where j is the map from B into $U(B)$ and $i : X \to U(X)$. Since i satisfies the hypothesis of Lemma 4.1.1, it follows there is a homomorphism j^* from $U(B)$ into B^* such that $(bj)j^* = bi$. Thus j^* is onto, and it follows B^* is nilpotent. From (7) and Lemma 4.1.2, it follows that $U(X)$ is nilpotent. Q.E.D.

<u>Theorem 4.1.5</u> If X is solvable, then X is nilpotent.

<u>Proof</u>: Let $M(X)$ be the subalgebra of $\mathrm{Hom}_F(X,X)$ generated by R_a, $a \in X$. The linear map $\rho : X \to M(X)$ where $\rho(a) = R_a$, $a \in X$, is a linear map from X into $M(X)$ which, by (1) and (2), satisfies the conditions of Lemma 4.1.1. Consequently,

$$\begin{tikzcd} U(X) \arrow[dr, dashed, "\rho^*"] & \\ X \arrow[u, "i"] \arrow[r, "\rho"'] & M(X) \end{tikzcd}$$

is commutative, and ρ^* is onto. Therefore $M(X)$ is nilpotent, and by Schafer [25; p. 18], X is nilpotent. Q.E.D.

Corollary 4.1.6 If x is a nilpotent element of X, then R_x is nilpotent.

Proof: Since the subalgebra $F[x]$ of X generated by x is nilpotent, $U(F[x])$ is nilpotent. Consequently $M(F[x])$ is nilpotent, and it follows that R_x is nilpotent. Q.E.D.

We have seen that nil Jordan algebras are solvable. However, for an arbitrary commutative power associative algebra X, it is not known if X nil implies X nilpotent.

Let e be an idempotent of X. From (1) we see that R_e satisfies the equation $2R_e^3 - 3R_e^2 + R_e = 0$. As with Jordan algebras, we write X as a vector space direct sum $X = X_0(e) + X_{\frac{1}{2}}(e) + X_1(e)$ where $X_i(e) = \{x \in X : xe = ix\}$ $i = 0,\frac{1}{2},1$. Albert [4] shows the following multiplicative relations: $X_i(e)X_i(e) \subseteq X_i(e)$, $i = 0,1$, $X_1(e)X_0(e) = 0$, $X_{\frac{1}{2}}(e)X_0(e) \subseteq X_{\frac{1}{2}}(e) + X_1(e)$, and $X_{\frac{1}{2}}(e)X_1(e) \subseteq X_{\frac{1}{2}}(e) + X_0(e)$. We see these relations are weaker than in the Jordan case. We say the idempotent e is stable if and only if $X_{\frac{1}{2}}(e)X_0(e) \subseteq X_{\frac{1}{2}}(e)$ and $X_{\frac{1}{2}}(e)X_1(e) \subseteq X_{\frac{1}{2}}(e)$. X is called stable if and only if every idempotent of X is stable.

If X has a unity 1 and $e_1,\ldots,e_s$ are pairwise orthogonal idempotents whose sum is 1, we have a refined decomposition

Theorem 4.1.7 [Albert] X is a vector space direct sum $X = \sum_{i\leq j} \oplus X_{ij}$ where $X_{ii} = X_1(e_i)$, $X_{ij} = X_{\frac{1}{2}}(e_i) \cap X_{\frac{1}{2}}(e_j)$ $i \neq j$, and

(i) $X_{ii}X_{ii} \subseteq X_{ii}$ $\qquad X_{ii}X_{jj} = 0 \quad i \neq j$

(ii) $X_{ij}X_{jk} \subseteq X_{ik}$ $\qquad i \neq j,k \quad j \neq k$

(iii) $X_{ij}X_{pq} = 0$ $\qquad p \neq i,j \quad q \neq i,j$

(iv) $X_{ii}X_{ij} \subseteq X_{ij} + X_{jj}$ $\quad i \neq j$

The decomposition in 4.1.7 is called the Pierce decomposition of X relative to $e_1,\dots,e_s$.

For our Cartan theory, we can expect difficulty in not knowing whether nil commutative power associative algebras are solvable and in the fact that idempotents in X need not be stable. Finally, it is known that if X is power associative X_K need not be power associative. To circumvent this difficulty, we will assume in the rest of the chapter that X is strictly power associative.

4.2 Cartan Theory of Commutative Power Associative Algebras.

In this section, X will denote an $n < \infty$ dimensional commutative, strictly power associative algebra with unity 1 over a field F of at least 4 elements, char $F \neq 2$. Thus the results of §4.1 are valid. Our study begins with

Lemma 4.2.1 If F is algebraically closed and 1 is the only idempotent in X, then X is almost nil.

Proof: Using a recent result of Oehmke [22], if X is simple, then $X = F \cdot 1$ and we are done. Therefore we assume X is not simple, hence $n > 1$. Let N be a maximal ideal in X where $N \neq X$. Clearly $1 \notin N$, and since 1 is the only idempotent in X, it follows that N is nil. Therefore N is the nil radical of X, and X/N is simple.

Let $\hat{1}$ be the image of 1 in X/N. We claim $\hat{1}$ is the only idempotent in X/N. For suppose $\hat{e} = e_1 + N$ is an idempotent in X/N. By Lambek [20, p. 72], we may lift $\hat{e}$ to an idempotent e in X such that the image of e in X/N is $\hat{e}$. But $e = 1$, consequently $\hat{1} = \hat{e}$ as desired. By Oehmke $X/N = F \cdot \hat{1}$, and it follows $X = F \cdot 1 + N$, or X is almost nil.

Q.E.D.

As in the Jordan case, we define

$$(1) \quad A(x_1, x_2, x_3) = x_1x_2 \cdot x_3 - x_1 \cdot x_2x_3$$

We can now prove

Lemma 4.2.2 If X is nil and A-nilpotent, then X is nilpotent.

Proof: The proof is by induction on the dimension of X. We may assume X is not associative, hence $n > 1$. Let $N_1 = \{g \in X : (g,x,y) = 0 \text{ for all } x,y \in X\}$ be the nucleus of X.

We observe, since X is commutative, if $g \in N_1$, then $(x,g,y) = (x,y,g) = 0$ for all $x,y \in X$. Since X is A-nilpotent, $N_1 \neq \{0\}$. As N_1 is associative, N_1 is solvable.

For $u \in N_1$, we note that $uX \triangleleft X$. If $uX = X$, since X is commutative, $X^n = (uX)^n = u^n X^n$. But $u^n = 0$, hence $X^n = 0$ as desired.

If $uX = 0$ for all $u \in N_1$, then $N_1 X = 0$ implies $N_1 \triangleleft X$. Since $\dim X/N_1 < n$, X/N_1 is solvable by the induction hypothesis. Thus X is solvable, and by Theorem 4.1.5, X is nilpotent.

If $0 \neq uX \neq X$, we treat X/uX similarly to conclude X is nilpotent. Q.E.D.

Lemma 4.2.3 If F is algebraically closed and X is A-nilpotent, then X is a direct sum of almost nil algebras. If N is the nil radical of X, then N is nilpotent.

Proof: The proof is identical to the proof given in Lemma 3.3.1, since the X_{ii} in the Pierce decomposition of X relative to pairwise orthogonal primitive idempotents are almost nil. That N is nilpotent is an immediate consequence of Lemma 4.2.2.

Q.E.D.

As in the case of Jordan algebras, we define maps $S(b,c)$ by

(2) $xS(b,c) = A(x,b,c)$ for all $b,c \in X$.

We now give

Lemma 4.2.4 If X is nil, then X is A-nil.

Proof: For $x \in X$, we wish to show that $S(x,x)$ is nilpotent. Since x is nilpotent, the subalgebra of X generated by x, denoted by $F[x]$, is nilpotent. If $M(F[x])$ is the subalgebra of $\mathrm{Hom}_F(X,X)$ generated by R_b, $b \in F[x]$, we have seen from Corollary 4.1.6 that

$$\begin{tikzcd} U(F[x]) \arrow[drr, dashed, "\rho^*"] & & \\ F[x] \arrow[u, "i"] \arrow[rr, "\rho"'] & & M(F[x]) \end{tikzcd}$$

commutes, where $\rho : b \to R_b$, $b \in F[x]$. Since $S(x,x) = R_x R_x - R_{x^2}$ is in $M(F[x])$, it follows $S(x,x)$ is nilpotent. Thus X is A-nil. Q.E.D

We would like to show that the analogue of Lemma 3.3.2, (Engel) holds in X. If we assume F has at least $2n+1$ elements, we know it is sufficient to study the problem when F is algebraically closed. If X is A-nil, we can write $X = \Sigma X_{ii}$ where the X_{ii} are almost nil. But to say that X is A-nilpotent, we need the converse of Lemma 4.2.3.

Next we wish to discuss whether A-nilpotent subalgebras of X generate nilpotent Lie algebras of linear transformations of X. We will suppose that F is algebraically closed, and that R is an A-nilpotent subalgebra of X containing 1. Let C be the subalgebra of $\mathrm{Hom}_F(X,X)$ generated by R_a, $a \in R$. By Lemma 4.2.3, there is a set $e_1,\ldots,e_t$ of pairwise orthogonal primitive idempotents such that $R = \Sigma\, Fe_i + N_i$, and $N = \Sigma\, N_i$ is nilpotent. Consequently, by Theorem 4.1.5, the subalgebra N' of C generated by R_b, $b \in N$, is nilpotent. Thus there is an integer p such that $(N')^p = 0$. We have

<u>Lemma 4.2.5</u> N' generates a nilpotent ideal of C.

<u>Proof</u>: Let $X = \sum_{i \leq j} X_{ij}$ be the Pierce decomposition of X relative to $e_1,\ldots,e_t$. Hence $Fe_i + N_i \subseteq X_{ii}$. If $x \in X$, we write $x = \Sigma\, x_{ii} + \sum_{i<j} x_{ij}$, and if $a \in R$, $a = \Sigma\, \alpha_i e_i + a_i$, $\alpha_i \in F$, $a_i \in N_i$. Since R_a is linear in a, it follows that C is generated by R_b, $b \in N$, and R_{e_i}, $i = 1,\ldots,t$.

Let $b \in N$ and write $b = \Sigma\, b_i$, $b_i \in N_i$. We observe that $x_{ii}R_b = x_{ii}R_{b_i}$ and $x_{ij}R_b = x_{ij}(R_{b_i} + R_{b_j})$, $i \neq j$.

Let $b^{(1)},\ldots,b^{(p)} \in N$ where $b^{(\ell)} = \Sigma\, b_i^{(\ell)}$, and let $T_1 = R_{b^{(1)}}\cdots R_{b^{(p)}} \in N'$. Then $x_{ii}T_1 = 0$ since $xT_1 = 0$.

Suppose $0 \neq x_{ij} \in X_{ij}$, $i \neq j$. Then $x_{ij}R_{b^{(1)}} =$

$x_{ij}(b_i^{(1)} + b_j^{(1)}) = (x_{ij}b_i^{(1)})_{jj} + (x_{ij}b_i^{(1)})_{ij} + (x_{ij}b_j^{(1)})_{ij} + (x_{ij}b_j^{(1)})_{ii}$. Writing $T_2 = R_{b^{(2)}} \cdots R_{b^{(p)}}$ we claim $(x_{ij}b_i^{(1)})_{jj}T_2 = (x_{ij}b_j^{(1)})_{ii}T_2 = 0$. Indeed, we have $(x_{ij}b_i^{(1)})_{jj}T_2 = (\cdots((x_{ij}b_i^{(1)})_{jj}b_j^{(2)})_{jj}b_j^{(2)})\cdots b_j^{(p)})$. But $0 = (\cdots((x_{ij}b_i^{(1)})b_j^{(2)})\cdots b_j^{(p)}) \in X_{ii} + X_{ij} + X_{jj}$. We observe that the component in X_{jj} is exactly $(x_{ij}b_i^{(1)})_{jj}T_2$. Since the Pierce decomposition is direct, $(x_{ij}b_i^{(1)})_{jj}T_2 = 0$. Similarly, $(x_{ij}b_j^{(1)})_{ii}T_2 = 0$ as desired.

If $T_3 = R_{b^{(3)}} \cdots R_{b^{(p)}}$, we may prove in a similar manner that $((x_{ij}b_i^{(1)})_{ij}b_i^{(2)})_{jj}T_3 = \cdots = ((x_{ij}b_j^{(1)})_{ij}b_j^{(2)})_{ii}T_3 = 0$, and this process continues.

Hence, by induction, for any $k > 0$, we have $x_{ij}T_1 = \Sigma(\cdots(x_{ij}b_{m_1})_{ij}\cdots b_{m_k})_{ij}T_k$ where the summation is over all possible combinations of $m_q = i$ or j , and $T_k = R_{b^{(k)}} \cdots R_{b^{(p)}}$.

Suppose next that T is a product of $R_{b^{(\ell)}}$, $\ell = 1,\ldots,p$, and some R_{e_k}'s. Clearly, $x_{ij}T = 0$ if $k \neq i$ or $k \neq j$ for some k . Therefore, assume $k = i$ or $k = j$, and that

$$T = T'R_{e_{k_1}} \cdots R_{e_{k_z}} T''$$

where $T' = R_{b^{(1)}} \cdots R_{b^{(\ell)}}$ and $T'' = R_{b^{(\ell+1)}}T^*$. From the computations above, we have

$$x_{ij}T = (\tfrac{1}{2})^z \ \Sigma(\ldots(x_{ij}b_{m_1}^{(1)})_{ij}\ldots b_{m_\ell}^{(\ell)})_{ij}T''$$

where the summation is over possible combinations of $m_q = i$ or j. Repeating this process by splitting T'', and continuing, we conclude $x_{ij}T = 0$.

What we have shown is that if $T \in C$ and each term of T contains at least p elements R_b , $b \in N$, then $xT = \Sigma\, x_{ii}T + \sum_{i<j} x_{ij}T = 0$. Thus if $a \in R$ and $b \in N$, it follows that $(R_aR_b)^p = (R_bR_a)^p = 0$. As the ideal generated by N' is $N^* = N' + CN' + N'C + CN'C$, it now follows that $(N^*)^p = 0$.

Q.E.D.

We now have

<u>Theorem 4.2.6</u> If R is an A-nilpotent subalgebra of X containing 1 , then $L_X(R)$ is solvable.

<u>Proof</u>: By the remarks in §1.4, we may assume that F is algebraically closed. Hence, let R, N, N', N^*, and C be as defined in Lemma 4.2.5.

Now if $b \in N$, $a \in R$, we see that $S(a,b)$ and $S(b,a) \in N^*$. Furthermore, $S(e_i,e_j)S(a,b)$ and $S(a,b)S(e_i,e_j) \in N^*$, for the idempotents e_i and e_j . Since $[S(e_i,e_j),S(e_k,e_\ell)] = 0$ it follows that if T_1 , $T_2 \in L_X(R)$, then $[T_1,T_2] \in N^*$. As $(N^*)^p = 0$, it follows immediately that $(L_X(R))^{(2p)} = 0$.

Q.E.D.

<u>Theorem 4.2.7</u> Suppose K is the algebraic closure of F, and X_K is stable. If R is an A-nilpotent subalgebra of X containing 1, then $L_X(R)$ is nilpotent, and if $X = X_0 \oplus X_1$ is the Fitting decomposition of X relative to $L_X(R)$, then

(i) X_0 is a subalgebra of X containing R

(ii) $X_0 X_1 \subseteq X_1$.

Moreover, if $F = K$, then

(iii) $X_0 = \cap \{B_b : b \in R\}$

<u>Proof</u>: The proof is identical with that of Theorem 3.3.3.

Q.E.D.

The following example shows that for A-nilpotence, the analogue of Theorem 3.3.3 for general commutative power associative algebras is false.

<u>Example 4.2.8</u> Let X' be the algebra with basis u,f,g,h over a field F whose characteristic is prime to 30. Let X' be commutative, and the multiplication table determined by the following: $u^2 = u$, $uf = f$, $ug = \frac{1}{2}g$, $fg = h$, and all other products zero. Albert [4] shows that X' is power associative. Let X be the algebra obtained from X' by adjoining a unity 1. Then X is commutative and power associative. Let $v = 1-u$. Then u and v are pairwise orthogonal primitive idempotents, and $X_{11} = X_1(u) = Fu + Ff$, $X_{12} = X_{\frac{1}{2}}(u) \cap X_{\frac{1}{2}}(v) = Fg$, and $X_{22} = Fv + Fh$. Clearly $R = X_{11} + X_{22}$ is an A-nilpotent

subalgebra of X containing 1. Now $L_X(R)$ is generated by $S(b,c)$, $b,c \in R$. Thus $S(f,v)$ and $S(u,v) \in L_X(R)$. We compute $gS(f,v)S(u,v) = 0$ and $g(S(u,v)S(f,v) = \frac{1}{4}h$. Consequently, $g[S(f,v),S(u,v)] = -\frac{1}{4}h$. By induction, we show

$$g[[\ldots[S(f,v),S(u,v)],\ldots,S(u,v)],S(u,v)] = (-\tfrac{1}{4})^p h .$$

Therefore, there does not exist a positive integer k such that $[L_X(R)]^k = 0$, so $L_X(R)$ is not nilpotent.

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