RADICAL CLASSES OF BOOLEAN ALGEBRAS

by

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ABSTRACT

This thesis obtains information about Boolean algebras by means of the radical concept. One group of results revolves about the concept, theorems, and constructions of general radical theory. We obtain some subdirect product representations by methods suggested by the theory. A large number of specific radicals are defined, and their properties and inter-relationships are examined. This provides a natural frame-work for results describing what epimorphs an algebra can have. Some new results of this nature are obtained in the process. Finally, a contribution is made to the structure theory of complete Boolean algebras. Product decomposition theorems are obtained, some of which make use of chains of radical classes.

TABLE OF CONTENTS

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INTRODUCTION

The first significant breakthrough in the study of structure by means of radicals was Cartan's classification of the finite-dimensional semi-simple Lie algebras over the field of complex numbers. Early in the twentieth century, Wedderburn obtained his structure theorem for finite-dimensional associative algebras. It took nearly forty years for the next major development: the definition of the Jacobson radical and the density theorem. The general procedure in these cases was the same: for a class of rings, define an ideal for each ring in the class, call it the radical, and see what can be said about radical-free (semi-simple) rings. In the early 1950's, Kurosh and Amitsur defined the general concept of a radical class, which became the subject of much subsequent research.

In 1939, Mostowski and Tarski introduced the notion of a superatomic Boolean algebra. A number of significant results concerning this class of algebras have been developed since, without exploiting the fact that it is a radical class. In 1972, Cramer generalized the superatomic radical to obtain a transfinite chain of radical classes of Boolean algebras. Again, the ideas and methods were not radical-theoretic in nature.

The aim of this thesis is to obtain information about Boolean algebras by means of the radical concept, and to place these results in the setting of general radical theory.

In general radical theory, it is necessary to begin with a class of rings closed under ideals and epimorphs. The class of Boolean algebras is not such a class: an ideal of a Boolean algebra is not a Boolean algebra. In Chapter One, starting with the more general concept of a Boolean ring, we provide a natural definition of a radical class of Boolean algebras. In the process, we obtain two interesting results concerning these radicals: first, every non-zero radical class of Boolean algebras contains the class of superatomic algebras; secondly, every radical class of Boolean algebras is hereditary.

In Chapter Two, we investigate the construction of the lower radical. In general, this is an infinite procedure; for Boolean algebras, the construction terminates at the second stage. If the generating class is closed under epimorphs, we find that a complete radical algebra must be a product of generating algebras. We define some lower radicals, the most important of which we call power-set radicals. For any power-set radical, we obtain necessary and sufficient conditions for a complete algebra to be in the radical, and for an algebra to generate the radical. The latter result proves useful in obtaining a product decomposition theorem for certain complete algebras.

In Chapter Three, we study the upper radical construction. The algebras which are semi-simple with respect to an upper radical are characterized as subdirect products of the algebras which determine the radical. An upper radical description of the superatomic radical yields a characterization of atomless Boolean algebras as subdirect products of separable, atomless algebras. Other upper radicals are defined, and we obtain a subdirect representation for the complete, atomless algebras.

Chapter Four concerns the radical defined by Cramer. Radical-theoretic methods are used to prove and extend some of his results. In particular, an upper radical description is obtained for some of his radicals.

Pierce has conjectured that any complete Boolean algebra is a product of homogeneous algebras. Chapter Five gives some partial results in this direction. It is shown that if any descending chain of principal ideals in a complete algebra has only finitely many isomorphism types, then the algebra is a product of homogeneous algebras. The main result asserts that certain complete algebras are products of unequivocal algebras (that is, algebras which must be either radical or semi-simple with respect to any radical class). Some of these results are re-stated in the language of cardinal properties, which provides some additional insight.

Chapter Six gives some closure properties of radical and semi-simple classes under the formation of products and coproducts. Not unexpectedly, power-set radicals are shown to be closed under suitably restricted products of complete algebras. A new product is defined which yields a radical algebra whenever all the algebras involved in the construction are radical. A number of radical classes are shown to be closed under finite coproducts. We obtain two results indicating that coproducts are strongly related to semi-simplicity. First, any coproduct of algebras, one of which is semi-simple, is itself semi-simple. Secondly, for any radical class, there is a cardinal κ such that for any algebra A of more than two elements, the coproduct of at least κ copies of A will be semi-simple.

In Chapter Seven, we regard radical classes as elements of a lattice. The structure of this lattice is investigated. The theme of Chapter One is re-iterated by showin the isomorphism of the lattice of Boolean ring radicals with the lattice of Boolean algebra radicals. Finally, we focus on the specific radicals defined in this work, with a view to locating them in the lattice.

PRELIMINARIES

This section outlines the terminology, notation, and basic facts to be used. Basic references are Sikorski [24] and Halmos [13]. Further references are given as needed, and details are provided for results not easily accessible or explicitly stated in the literature.

§1. Fundamental Notions

a) We assume familiarity with the concept of *Boolean ring*, both in the ring-theoretic and lattice-theoretic settings. We will use both the ring operations (+, ·), and the lattice operations of *join* ($\vee$), *meet* ($\wedge$), and *complementation* ($'$). The concepts of *ideal* in the two settings coincide. We will use the notation A_x for the *principal ideal* of the ring A generated by the element x. The term *Boolean algebra* (or simply *algebra*) will be used for a Boolean ring with a unity 1 distinct from the zero 0. A *subalgebra* of a Boolean algebra is a subring containing the unity. Any ideal of a Boolean algebra generates a subalgebra, consisting of the ideal together with the complements of elements of the ideal. Such a subalgebra will be called an *ideal-generated subalgebra*. An *algebra-homomorphism* is a *ring-homomorphism* which is 1-preserving.

b) The important connections between Boolean rings and algebras are the following:

i) every non-zero principal ideal of a Boolean ring is a Boolean algebra,

ii) any Boolean ring can be embedded as a maximal ideal in a Boolean algebra, which is unique up to isomorphism, and

iii) any non-zero epimorph of a Boolean algebra is itself an algebra, and the epimorphism must be 1-preserving.

c) We will use the symbol $\mathcal{A}$ for the class of all Boolean algebras, and the symbol $\mathcal{B}$ for the class of all Boolean rings.

d) We assume familiarity, also, with the Stone duality theory, which assigns to any Boolean algebra A, a topological space S(A), called its Stone space, which is Hausdorff, compact, and totally disconnected. The correspondence is reversible, and allows the following interchange of algebraic and topological concepts:

i) an element x of A (or the principal ideal A_x) corresponds to a clopen subset S(x) of S(A), and A is isomorphic to the algebra of clopen subsets of S(A);

ii) an ideal I of A corresponds to an open subset S(I) of S(A); namely, S(I) is the union of the clopen sets S(x) for $x \in I$;

iii) the epimorph A/I of A corresponds to the closed subset S(A) - S(I) of S(A);

iv) an embedding $A \rightarrowtail B$ corresponds to a continuous surjection $S(B) \twoheadrightarrow S(A)$;

v) an epimorphism $A \twoheadrightarrow B$ corresponds to a continuous injection $S(B) \rightarrowtail S(A)$.

e) Lattices will always be assumed to have extreme elements 0 and 1, which are distinct. An atom in a lattice is a non-zero element which contains only 0 and itself. A dual atom is an element distinct from 1 which is contained only in 1 and itself. In the Stone space of a Boolean algebra, atoms appear as isolated points. Elements x and y of a lattice are disjoint if $x \wedge y = 0$. A set D is disjointed if any two distinct elements of it are disjoint. The supremum of an arbitrary E in a lattice, when it exists, will be denoted by sup E. The terms complete (σ-complete) will be used to indicate that arbitrary (countable) suprema always exist.

f) The Axiom of Choice will be used without further mention, but the assumption of the Generalized Continuum Hypothesis (GCH) will always be explicit. For any cardinal κ, we will use the notation $\kappa+$ for the next largest cardinal, and exp κ for the cardinal more commonly denoted by 2^{κ}.

§2. Properties of Boolean Rings and Algebras

a) If I is an ideal of a Boolean ring A, and J is an ideal of I, then J is an ideal of A.

b) Every non-zero Boolean ring has a two-element epimorph. We will denote the two-element Boolean algebra by $\underline{2}$.

c) For any collection $\{A_i: i \in I\}$ of Boolean rings (algebras), the Cartesian product of the underlying sets together with the point-wise operations forms a Boolean ring (algebra), which we call the product of the A_i and denote by $A = \Pi(A_i: i \in I)$. Finite products are denoted by $A_1 \times A_2 \times \cdots \times A_n$. The Stone space of the product of algebras is the Stone-Cech compactification of the disjoint union of the $S(A_i)$.

d) The subset of the product A of algebras A_i, consisting of elements which are 0 in all but a finite number of coordinates is an ideal of A but is not an algebra. The subalgebra of A which it generates will be called the weak product of the A_i, and will be denoted by the symbol $w\Pi(A_i: i \in I)$. Its Stone space is the one-point compactification of the disjoint union of the $S(A_i)$.

e) An algebra A is a retract of an algebra B if there is an embedding f of A into B, and an epimorphism g of B onto A such that the composition gf is the identity on B. Any principal ideal of an algebra A is a retract of A. In particular, any factor algebra in a product or weak product A is a principal ideal of A and so is a retract of A.

f) Any Boolean ring A admits a product decomposition at any element x: namely, A is the product of A_x with the ideal of elements disjoint from x. If A is an algebra, we have $A \simeq A_x \times A_{x'}$.

g) A subset D of a Boolean algebra A is said to be <u>dense</u> if it consists of non-zero elements, and any non-zero element of A contains an element of D. In particular, we can speak of dense ideals and dense subalgebras of an algebra.

h) A Boolean algebra is said to be <u>separable</u> if it has a countable, dense subset. Any separable algebra has cardinality at most exp $\aleph_0$. Furthermore, there is only one complete, atomless, separable algebra: the quotient of the algebra of Borel sets of real numbers, modulo the ideal of all meagre Borel sets [16].

i) <u>Proposition 1</u>: Let A be a subalgebra of a Boolean algebra B. Then B has an epimorph which has a dense subalgebra isomorphic to A.

<u>Proof</u>: By Zorn's Lemma, choose an ideal I of B maximal with respect to the property $A \cap I = 0$. The restriction of the natural epimorphism $f: B \twoheadrightarrow B/I$ to A is an embedding, so B/I has a subalgebra isomorphic to A. Now suppose f(b) is a non-zero element of B/I. Then b is not in I, so the ideal J generated by I and b is strictly larger than I. By the maximality of I, there is a non-zero element a in $A \cap J$. Being in J, this element must have the form $a = x \vee c$, where $x \in I$

and $c \leq b$. Since f is an embedding, $0 \neq f(a) = f(c) \leq f(b)$. Hence the copy of A in B/I is dense.

§3. Complete Algebras

a) The algebra of all subsets of a set X will be denoted by P(X), and is called a <u>power-set algebra</u>. It is characterized as the product of $|X|$ copies of the two-element algebra, and its Stone space is the Stone-Cech compactification βX of the set X with the discrete topology. When $|X| = \aleph_\alpha$ is infinite, we will denote P(X) by P_α.

b) Any product of algebras over an index set I has P(I) as a retract.

c) Complete algebras are precisely the retracts of power-set algebras. They can also be characterized as the <u>injective</u> Boolean algebras. An algebra C is said to be injective if, whenever f is a homomorphism from A to C and g is an embedding of A in B, then there is a homomorphism h of B to C such that $f = hg$.

$$\begin{tikzcd} A \arrow[r, tail, "g"] \arrow[d, "f"'] & B \arrow[dl, dashed, "h"] \\ C & \end{tikzcd}$$

d) Pierce [22] has shown that an infinite cardinal $\aleph_\alpha$ is the cardinality of a complete Boolean algebra if and only if $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$.

e) Any Boolean algebra A has a <u>normal completion</u> $\overline{A}$ with the following properties:

i) $\overline{A}$ is complete, and contains A as a dense subalgebra. It is unique with respect to these two properties.

ii) If A is a subalgebra of a complete algebra B, then B has $\overline{A}$ as a subalgebra containing A.

Using i), it is easy to see that if $x \in A$, then the normal completion of A_x is the principal ideal of $\overline{A}$ generated by x. Hence $\overline{A}_x = \overline{A_x}$.

f) An algebra A satisfies the <u>countable chain condition</u> (c.c.c.) if any disjointed subset of A is at most countable.

<u>Proposition 2</u>: If A is an infinite algebra satisfying c.c.c., then

$$|\overline{A}| = |A|^{\aleph_0}.$$

<u>Proof</u>: Every element of $\overline{A}$ can be represented as a disjoint (hence, at most countable) supremum of elements from A. Hence $|\overline{A}| \leq |A|^{\aleph_0}$. Also, $|A|^{\aleph_0} \leq |\overline{A}|^{\aleph_0} = |\overline{A}|$, by d) above.

g) Any σ-complete algebra satisfies the following condition: If the algebra A is isomorphic to its principal ideal A_x, then it is also isomorphic to A_y for any $y \geq x$.

h) A Boolean algebra A is said to be <u>homogeneous</u> if it is isomorphic to each of its non-zero principal ideals.

i) The following proposition is suggested by Theorem 3.3 of Pierce [22]:

Proposition 3: The normal completion of a homogeneous algebra is homogeneous.

Proof: Let A be homogeneous and suppose x is a non-zero element of $\overline{A}$. Choose some non-zero y in A such that $y \leq x$. Since $A_y \approx A$, we get $\overline{A}_y \approx \overline{A}$ by the uniqueness of the normal completion. Using g), we conclude that $\overline{A}_x \approx \overline{A}$.

j) Pierce (see Cramer [5]) has shown that any infinite epimorph of a σ-algebra has P_o as a retract.

§4. Coproducts

a) If X_i is the Stone space of A_i for $i \in I$, then the topological product X of the X_i is also a Boolean space. The algebra A of clopen supsets of X is called the coproduct of the A_i and will be denoted by $\Sigma(A_i\colon i \in I)$. For finite coproducts, we will use the notation $A_1 + A_2 + \cdots + A_n$. The coproduct of algebras is unique up to isomorphism.

b) The projection map of X onto X_i provides a natural embedding of A_i in A. We identify the subalgebras of A so obtained with the A_i. Then the A_i form an independent family of subalgebras of A; that is, for any finite collection of non-zero elements x_i chosen from subalgebras with different indices, $x_1 \wedge x_2 \wedge \ldots \wedge x_n \neq 0$. Furthermore, every element of A is a finite join of elements of the above form.

c) Each A_i is, in fact, a retract of A. An easy topological argument yields a stronger result: if J is a subset of I and B_j is a retract of A_j for each $j \in J$, then $\Sigma(B_j: j \in J)$ is a retract of A. In particular, $\Sigma(A_j: j \in J)$ is a retract of A. We assume these partial coproducts are actually contained in A. Under this convention, A is the union of all its subalgebras which are finite coproducts of the A_i.

d) If F_i is a closed subset of X_i for each $i \in I$, then $\Pi(F_i: i \in I)$ is closed in $\Pi(X_i: i \in I)$. We spell this out algebraically:

<u>Proposition 4</u>: Let $A = \Sigma(A_i: i \in I)$ and let J_i be an ideal of A_i for each $i \in I$. Let J be the ideal of A generated by the union of the J_i. Then:

i) J consists of all finite joins of elements of the form $x_1 \wedge x_2 \wedge \ldots \wedge x_n$, where each x_i is chosen from one of the subalgebras A_k, and at least one x_i is chosen from the ideal J_k of A_k,

ii) $A/J \simeq \Sigma(A_i/J_i: i \in I)$.

The proof is a straight-forward verification. The proposition has some useful corollaries:

i) Let I be an ideal of A, J an ideal of B, and let K be the ideal of A + B consisting of finite joins of elements of the form $a \wedge b$, where $a \in A$, $b \in B$ and either $a \in I$ or $b \in J$. Then

$$A + B/K \simeq A/I + B/J.$$

ii) If L is the ideal of A + B consisting of all finite joins of elements of the form $a \wedge b$ where $a \in I$ and $b \in B$, then

$$A + B/L \simeq A/I + B.$$

iii) If $a \in A$ and $b \in B$, then $(A + B)_{a \wedge b} \simeq A_a + B_b$.

e) If B is finite with n atoms, then $A + B \simeq A^n$.

§5. Subdirect Products

a) If $A = \Pi(A_i: i \in I)$, then there is a natural epimorphism of A onto A_i for each $i \in I$. A subalgebra B of A is called a subdirect product of the A_i if each of these epimorphisms, restricted to B, still maps onto A_i.

b) If $\{J_i: i \in I\}$ is a collection of ideals of an algebra A whose intersection is the zero ideal, then A can be represented as a subdirect product of the A/J_i.

§6. Free Algebras

a) F is a free algebra on κ generators if F is generated by a set X of cardinality κ, and any function from X to an algebra A can be extended to a homomorphism of F to A. In case $\kappa = \aleph_\alpha$ is infinite, we will denote the free algebra on κ generators by F_α.

b) The free algebra on κ generators can be realized as the coproduct of κ copies of the four-element Boolean algebra, or equivalently, as the algebra of clopen subsets of the Cantor space 2^κ, the topological product of κ copies of the two-element discrete space.

c) For infinite free algebras, the cardinality of F_α is $\aleph_\alpha$. Furthermore, F_α is homogeneous and satisfies c.c.c. Hence its normal completion $\overline{F}_\alpha$ is homogeneous and has cardinality $\aleph_\alpha^{\aleph_0}$.

d) If A is a coproduct of κ algebras, each with more than two elements, then the free algebra on κ generators is a retract of A.

e) The countable free algebra F_0 is the only countable, atomless Boolean algebra. Its normal completion $\overline{F}_0$ is complete, atomless, and separable, and so it is isomorphic to the unique algebra with these properties (see §2, h).

f) Free algebras are examples of a more general concept. A Boolean algebra P is said to be <u>projective</u> if for any homomorphism f of P to an algebra B, and any epimorphism g of A onto B, there is a homomorphism h of P to A such that gh = f.

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "g"] & B \end{tikzcd}$$

§7. <u>Universal Mapping Properties</u>

a) Every algebra is an epimorph of a free algebra. There is an analogous result for complete algebras:

<u>Proposition 5</u>: If A is a complete algebra of cardinality at most $\aleph_\alpha$, then A is an epimorph of $\overline{F}_\alpha$.

Proof: A is an epimorph of F_α, which can be embedded in $\overline{F}_\alpha$. Then injectivity of A yields the result.

A form of this result was first proved in Efimov [10] by a fairly involved topological argument.

b) Hausdorff [15] has shown that if $\aleph_\beta = \exp \aleph_\alpha$, then F_β can be embedded in P_α. This has some important consequences.

Proposition 6: Let $\aleph_\beta = \exp \aleph_\alpha$. Then F_β is a subalgebra of a Boolean algebra A if and only if P_α is an epimorph of A.

Proof: One direction follows from the injectivity of P_α and the fact that it is an epimorph of F_β. The other direction follows from the projectivity of F_β and Hausdorff's result.

Proposition 7: Let $\aleph_\beta = \exp \aleph_\alpha$. Then $\overline{F}_\beta$ is a retract of P_α.

Proof: Since P_α is complete and F_β is a subalgebra of it, we get that $\overline{F}_\beta$ is also a subalgebra of it. But any complete subalgebra of an algebra is, in fact, a retract of that algebra.

c) The most useful form of the preceding results for our purposes is the following:

Proposition 8: Let $\aleph_\beta = \exp \aleph_\alpha$. Then P_α and $\overline{F}_\beta$ are epimorphs of one another.

Corollary: Any two complete algebras of cardinality exp $\aleph_0$ are epimorphs of one another.

Proof: It suffices to show that if A is complete of cardinality $\aleph_\gamma =$ exp $\aleph_0$, then A and P_0 are epimorphs of one another. By Prop. 5, A is an epimorph of $\overline{F}_\gamma$ and hence of P_0, by Prop. 8. By Pierce's result (§3, j), P_0 is an epimorph of A.

d) If $\alpha < \beta$, then P_α is a principal ideal (hence a retract) of P_β. The algebras are not isomorphic since they do not have the same number of atoms. They might, however, have the same cardinality. It is consistent with the usual axioms of set theory to assume, for example, that exp $\aleph_0$ = exp $\aleph_1$ (see Easton [9]). It is also consistent to assume that $\alpha < \beta$ implies exp $\aleph_\alpha <$ exp $\aleph_\beta$, since this is a consequence of GCH. In any case, we have the following:

Proposition 9 : P_α and P_β are epimorphs of one another if and only if exp $\aleph_\alpha$ = exp $\aleph_\beta$.

Proof: Assume exp $\aleph_\alpha$ = exp $\aleph_\beta = \aleph_\gamma$. By Prop. 8, P_α and $\overline{F}_\gamma$ are epimorphs of one another, as are P_β and $\overline{F}_\gamma$. Hence, the two power-set algebras are epimorphs of one another. The other direction is clear.

e) Pierce's result (§3, j) has useful consequences. If A is an infinite epimorph of a σ-complete algebra, then A has no infinite free epimorphs. In particular, no infinite complete algebra can have an infinite free epimorph.

CHAPTER ONE

RADICAL CLASSES OF BOOLEAN ALGEBRAS

The concept of a radical class of rings is well-known and has been studied extensively. The theory can be applied immediately to the class of Boolean rings, but some adjustment in the definitions and results is necessary for the class of Boolean algebras.

§1. General Radical Theory

This section is a review of the basic concepts of radical theory for associative rings. A general reference for this material is Divinsky [8].

1.1 Definition: A class of associative rings is called *universal* if it is closed under the formation of epimorphs and ideals.

In what follows, we assume that all classes of rings considered are subclasses of some fixed universal class.

1.2 Definition (Amitsur [2], Kurosh [17]): A non-empty class R of rings is a *radical class* (or simply a *radical*) if it satisfies the following properties;

i) every epimorph of an R-ring is an R-ring,

ii) every ring A contains an R-ideal, called the R-radical of A and denoted by $r(A)$, which contains every R-ideal of A, and

iii) for any ring A, $r(A/r(A)) = 0$.

1.3 <u>Definition</u>: If R is a radical class and A is a ring such that $r(A) = 0$, then A is called R-<u>semi-simple</u>.

When some fixed radical class is being discussed and there is no danger of ambiguity, we will simply use the terms "radical" and "semi-simple" without specific reference to the radical class.

It is obvious that the trivial ring $\{0\}$ is the only ring which can be simultaneously radical and semi-simple with respect to a radical class.

1.4 <u>Proposition</u>: For any radical class R and any ring A, $r(A)$ is the intersection of all ideals I of A for which A/I is R-semi-simple.

The following propositions characterize radical classes and give some of their closure properties.

1.5 <u>Proposition</u>: A class R is a radical class if and only if:

i) R is closed under epimorphs, and

ii) if A is a ring such that every non-zero epimorph of A has a non-zero R-ideal, then A is in R.

1.6 Proposition: If R is a radical class and I is an ideal of a ring A which is generated by R-ideals of A, then I is an R-ideal of A.

1.7 Corollary: If R is a radical class, then the weak direct product of R-rings is an R-ring.

1.8 Proposition: If R is a radical class and I is an R-ideal of a ring A such that A/I is in R, then A is in R.

1.9 Definition: For any class M of rings, we say a ring A is an approximate M-ring if every non-zero ideal of A has a non-zero epimorph in M.

1.10 Proposition: A class of rings M is the class of all R-semi-simple rings for some radical class R if and only if M is equal to the class of all approximate M-rings. In this case, R can be recovered from M as the class of all rings with no non-zero epimorph in M.

1.11 Definition: A class of rings is called hereditary if every ideal of a ring in the class is also in the class.

1.12 Proposition (Armanderiz [3]): A class M is the class of all semi-simple rings for some radical class if and only if:

i) M is hereditary,

ii) M is closed under subdirect products,

iii) if I is an M-ideal of a ring A for which A/I is in M, then A is in M, and

iv) if I is an ideal of a ring A such that I/B is a non-zero M-ring for some ideal B of I, then there is an ideal C of A contained in I such that I/C is a non-zero M-ring.

The fact that any semi-simple class of associative rings is hereditary was first proved by Anderson, Divinsky, and Sulinski [1].

1.13 <u>Proposition:</u> A radical R is hereditary if and only if $r(I) = I \cap r(A)$ for any ring A and any ideal I of A. Furthermore, if R is hereditary, then for any ring A, r(A) is the ideal of A generated by the principal R-ideals of A.

1.14 <u>Definition:</u> For any class C of rings and any ring A, define c(A) to be the ideal of A generated by the principal C-ideals of A.

This coincides with the definition of r(A) for a radical class R provided that R is hereditary. In the next section, we show that all radicals we consider are hereditary, so the notation will be unambiguous.

§2. <u>Radical Classes of Boolean Rings</u>

The class of Boolean rings is a universal class, so we can immediately apply the concepts and results of the previous section. We will prove only what is needed to facilitate the passage to Boolean algebras.

1.15 Proposition: If R is a non-zero radical class of Boolean rings, then the two-element Boolean algebra $\underline{2}$ is in R.

Proof: Any non-zero Boolean ring has $\underline{2}$ as an epimorph.

1.16 Corollary: If R is a non-zero radical class of Boolean rings, and A is a Boolean ring with a maximal ideal I in R, then A is in R.

Proof: Both I and $A/I \approx \underline{2}$ are in R. By Prop. 1.8, A is in R.

1.17 Proposition: Every radical class of Boolean rings is hereditary.

Proof: Let A be in the radical class R of Boolean rings. Any ideal I of A is generated by the principal ideals of A contained within it. Each of these is an epimorph of A and so is in R. By Prop. 1.6, I is in R.

We are now ready to prove the theorem which allows us to restrict attention to Boolean algebras.

1.18 Theorem: Let R be a non-zero radical class of Boolean rings, and let $S = A \cap R$, the class of Boolean algebras in R. Then for any Boolean ring B, the following are equivalent:

i) B is in R,

ii) every non-zero principal ideal of B is in S,

iii) C is in S, where C is the Boolean algebra containing B as a maximal ideal.

Proof: If B is in R, then every non-zero principal of B is in R by Prop. 1.17, and C is in R by Cor. 1.16. Being algebras, they are, in fact, in S. Thus i) implies ii) and iii). If ii) holds, then every principal ideal of B is in R. Since B is generated by its principal ideas, it is in R by Prop. 1.6. Thus ii) implies i). If iii) holds, then C is in R, so by Prop. 1.17, B is in R. So iii) implies i).

§3. Radical Classes of Boolean Algebras

1.19 Definition: A non-empty class S of Boolean algebras is called a radical class if and only if there is a radical class R of Boolean rings such that $S = A \cap R$.

The remainder of this section shows that all the concepts and results of radical theory can be expressed (with only minor modifications) entirely in the language of Boolean algebras. In keeping with Defn. 1.14, for any class S of algebras, the ideal $s(A)$ of the algebra A is the ideal generated by the S-ideals (necessarily principal) of A.

1.20 Proposition: A class S of Boolean algebras is a radical class if and only if:

i) S is closed under algebra epimorphs, and

ii) $s(A/s(A)) = 0$ for every Boolean algebra A.

Proof: If S is a radical class of algebras obtained from the Boolean ring radical R, then it is clear that $s(A) = r(A)$ for any algebra A. Hence, ii) follows immediately. Also, any algebra epimorphism is a ring epimorphism, so i) follows.

Suppose S satisfies i) and ii), and let R be the class of all Boolean rings A satisfying $A_x \in S$ for every non-zero $x \in A$. It is clear that S is precisely the class of all algebras in R, so all we need show is that R is a radical class of rings. Suppose that B is a ring epimorph of the R-ring A. Then any non-zero principal ideal of B is a ring epimorph of some non-zero principal ideal of A, which is an S-algebra. Moreover, a non-zero ring epimorphism on an algebra must preserve the unity and so is, in fact, an algebra epimorphism. By i), then, every non-zero principal ideal of B is an S-algebra, and so $B \in R$.

Now suppose A is a ring such that every non-zero epimorph of A has a non-zero R-ideal. If A is not in R, then there is a non-zero $x \in A$ such that A_x is not in S. But then $A_x/s(A_x)$ is a non-zero ring epimorph of A and so must have a non-zero R-ideal. But then the algebra $A_x/s(A_x)$ has an S-ideal, contradicting condition ii).

By Prop. 1.5, R is a radical class of Boolean rings.

1.21 Proposition: A class S of Boolean algebras is a radical class if and only if:

i) S is closed under algebra epimorphs, and

ii) if A is an algebra such that every algebra epimorph of A has an S-ideal, then A is in S.

Proof: Suppose S is a class satisfying i) and ii), define R as in the last proposition, and repeat the argument showing that R is closed under ring epimorphs. Now let A be a ring such that every non-zero ring epimorph of A has a non-zero R-ideal. In particular, every non-zero principal ideal of A satisfies this condition, and so has an S-ideal. By ii), the, every non-zero principal ideal of A is in S, and so A is in R. Thus, R is a radical class of Boolean rings and S consists of all the algebras in it. The opposite direction is clear.

1.22 Proposition: Let S be a radical class of algebras, obtained from the Boolean ring radical R. Then for any algebra (ring) A:

$$s(A) = r(A) = \{x: A_x \varepsilon R\} = \{x: x = 0 \text{ or } A_x \varepsilon S\}.$$

Proof: Let $I = \{x: A_x \varepsilon R\}$. All that needs proof is that I is an ideal of A. Since R is hereditary, I is closed under subelements. The fact that I is closed under finite joins follows from the fact that $A_{x \vee y}$ is generated by A_x and A_y, and Prop. 1.6.

This proposition makes precise what we will mean by the expression: the radical of an algebra consists of its radical elements.

1.23 Definition: If S is a radical class of algebras, we say an algebra A is S-semi-simple if $s(A) = 0$. A class of algebras is a semi-simple class if it consists of all S-semi-simple algebras for some radical class S of algebras.

1.24 <u>Definition</u>: For any class M of algebras, we say an algebra A is an approximate M-algebra if every non-zero principal ideal of A has an (algebra) epimorph in M.

1.25 <u>Proposition</u>: A class of algebras M is a semi-simple class if and only if M is equal to the class of all approximate M-algebras. In this case, the radical class associated with M is the class of all algebras with no (algebra) epimorph in M.

<u>Proof</u>: A straight-forward verification, similar in spirit to Prop. 1.20 and Prop. 1.21.

§4. <u>Conventions and Summary</u>

Unless otherwise stated, we will henceforth refer only to classes of algebras. The definitions of class properties will be modified in the obvious manner. For example:

1.26 <u>Definition</u>: A class of algebras is said to be <u>hereditary</u> if it is closed under the formation of non-zero principal ideals.

The term "epimorph" will henceforth mean "algebra epimorph."

We retrieve the symbol R for a radical class of Boolean algebras. We make the convention that an ideal I of an algebra A will be called an R-ideal of A if $A_x \in R$ for every $x \in I$. Hence an R-ideal need not be in R.

The following propositions summarize the properties of radical and semi-simple classes which we will find most useful. Some of these properties have already been proved, and the others are straight-forward extensions of known results.

1.27 Proposition: Let R be a radical class of Boolean algebras. Then:

i) $\underline{2} \in R$,

ii) R is hereditary,

iii) if A is an algebra with an R-ideal I such that A/I is in R, then A is in R,

iv) if I is an ideal of an algebra A generated by R-ideals of A, then I is an R-ideal of A,

v) R is closed under the formation of weak products.

1.28 Proposition: Let M be a semi-simple class of Boolean algebras. Then:

i) M is hereditary,

ii) M is closed under subdirect products,

iii) if I is an M-ideal of an algebra A such that A/I is in M, then A is in M.

CHAPTER TWO

THE LOWER RADICAL

Given any class of rings, it is possible to construct a smallest radical containing this class (see Divinsky [8]). We investigate the special features of the general construction in the case of Boolean algebra radicals.

§1. The Lower Radical Construction

2.1 Proposition: Let X be any class of rings in some universal class of rings. For any ordinal α, we define a class X_α as follows:

i) $X_o = X$,

ii) X_1 is the class of epimorphs of X-rings,

iii) for $\alpha > 1$, assuming X_β has been defined for all $\beta < \alpha$, let X_α be the class of all rings A such that every non-zero epimorph of A has a non-zero ideal belonging to X_β for some $\beta < \alpha$.

Let $L(X)$ be the union of the classes X_α. Then $L(X)$ is a radical class. Furthermore, if R is a radical class containing X, then R contains $L(X)$.

If H is a class of rings closed under epimorphs and $R = L(H)$, we have a construction due to Amitsur [2] which, for any ring A, yields an ideal $h^*(A)$ of A. Whenever H satisfies an additional condition, $h^*(A)$ coincides with $r(A)$, thus giving an internal, iterative construction of the radical of A in terms of H.

2.2 Definition (Pierce): A ladder in a ring A is a well-ordered chain of ideals of A, $0 = I_0 \leqslant I_1 \leqslant \ldots \leqslant I_\alpha \leqslant \ldots$, such that if α is a limit ordinal, then I_α is the union of the I_β for $\beta < \alpha$. We note that there is a least ordinal δ such that $I_\alpha = I_\delta$ for all $\alpha \geqslant \delta$, and we call I_δ the summit of the ladder $\{I_\alpha\}$.

2.3 Lemma: Let R be a radical class and suppose $\{I_\alpha\}$ is a ladder in a ring A with summit I_δ, satisfying $I_{\alpha+1}/I_\alpha \in R$ for all ordinals α. Then I_δ is in R.

Proof: Using Prop. 1.6 and Prop. 1.8, an easy induction shows that I_α is in R for all ordinals α.

2.4 Definition: Let H be a class of rings closed under epimorphs. For any ring A, we define a ladder in A as follows:

i) $h_0(A) = 0$

ii) assuming that $h_\alpha(A)$ has been defined, $h_{\alpha+1}(A)/h_\alpha(A)$ is the ideal of $A/h_\alpha(A)$ generated by its H-ideals, and

iii) if α is a limit ordinal and $h_\beta(A)$ has been defined for all $\beta < \alpha$, then $h_\alpha(A)$ is the union of the $h_\beta(A)$ for $\beta < \alpha$.

We call $\{h_\alpha(A)\}$ the H-ladder in A and denote its summit by $h^*(A)$.

Using Prop. 1.6, we easily get:

2.5 Lemma: If H is closed under epimorphs and $R = L(H)$, then for all rings A, $h^*(A) \leq r(A)$.

2.6 Definition: A class of rings H, closed under epimorphs, is called an Amitsur class if $h^*(A) = 0$ implies $h^*(I) = 0$ for every ideal I of A.

2.7 Proposition: Let H be closed under epimorphs and let $R = L(H)$. Then $r(A) = h^*(A)$ for all rings A if and only if H is an Amitsur class.

The proof of the sufficiency is due to Amitsur [2]. The necessity follows from the fact that semi-simple classes are hereditary.

§2. The Lower Radical for Boolean Algebra Radicals

If X is a class of Boolean algebras, we can use it to generate a Boolean ring radical, and thence a radical class of algebras $L(X)$ which is minimal with respect to containing X. This assures us that there is a lower radical construction for Boolean algebra radicals.

Using the fact that ideals of ideals of an algebra are themselves ideals of that algebra, we can show that the general construction of Prop. 2.1 stops at the second stage for Boolean algebra radicals.

2.8 Proposition: Let X be any class of Boolean algebras, and let R be the class of all algebras A such that every epimorph of A has an ideal (necessarily non-zero principal) which is an epimorph of an X-algebra. Then $R = L(X)$.

Proof: Since $X \subseteq R \subseteq L(X)$, it suffices to show that R is a radical class. Clearly, R is closed under epimorphs. Now suppose that A is an algebra such that every epimorph of A has an ideal in R. We must show that A is in R. Suppose B is an epimorph of A. By assumption, B has an ideal C in R. Then C, being an epimorph of itself and in R, contains an ideal D which is an epimorph of an X-algebra. But then D is an ideal of B, and so every epimorph B of A contains an ideal which is an epimorph of an X-algebra. Thus A is in R as required, and R is a radical class.

The following definitions and lemma will be useful in determining the structure of algebras in a lower radical, particularly when the generating class is closed under epimorphs, and they will also be extremely useful in Chapter Five. First, recall (Preliminaries, §2, f) that any algebra has a product decomposition across any element and its complement. More generally, if D is any disjointed subset of an algebra A with sup D = 1, then A has a dense ideal-generated subalgebra isomorphic to $w\Pi(A_d\colon d \in D)$. When A is complete, we have $A \cong \Pi(A_d\colon d \in D)$. We extend these results as follows:

2.9 Definition: If P is a property of algebras, we say x is a P-element of A if A_x is a P-algebra. The property P is hereditary if, whenever $x \leq y$ are elements of A and y is a P-element of A, then x is also a P-element of A.

Whenever an algebra A has a dense subset of P-elements, for some hereditary property P, then any maximal disjoint subset D of P-elements must satisfy sup D = 1. Hence, we easily get the following:

2.10 Lemma: Let P be a hereditary property of Boolean algebras. If A is an algebra with a dense subset of P-elements, then A has a dense ideal-generated subalgebra isomorphic to a weak product of P-algebras. If A is a complete algebra, then A is a product of P-algebras if and only if the P-elements of A are dense in A.

Noting that the property of being a two-element algebra is hereditary, we immediately deduce the following well-known result:

2.11 Corollary: A complete algebra is a product of two-element algebras (that is, a power-set algebra) if and only if it is atomic (that is, the atoms are dense in the algebra).

2.12 Theorem: Suppose $\mathcal{H}$ is a class of algebras closed under epimorphs, and let $\mathcal{R} = \mathcal{L}(\mathcal{H})$. Then:

i) A is in $\mathcal{R}$ if and only if every epimorph of A has an ideal in $\mathcal{H}$,

ii) A is in R if and only if every epimorph of A has a dense subset of H-elements,

iii) any R-algebra A has a dense ideal-generated subalgebra isomorphic to a weak product of H-algebras,

iv) any complete algebra A in R is a product of H-algebras,

v) R contains atomless complete algebras if and only if H does.

Proof: The first assertion follows immediately from Prop. 2.8. Since principal ideal of an algebra are epimorphs of that algebra, ii) follows from i). Using Lemma 2.10, we immediately get iii) and iv) from ii), and v) follows from iv).

2.13 Proposition: Let $R = L(H)$ as in Prop. 2.12, and let A be a homogeneous algebra. Then A is in R if and only if A is in H.

Proof: If A is in R, then every epimorph of A and in particular, A itself, has an ideal in H. Since A is isomorphic to any principal ideal of itself, A is in H. The other direction is obvious.

Homogeneous algebras are a special case of the following more general concept:

2.14 Definition (Divinsky): A Boolean algebra A is unequivocal if for every radical class R, A is either in R or is R-semi-simple.

2.15 Proposition: An algebra A is unequivocal if and only if $\mathcal{L}(A_x) = \mathcal{L}(A)$ for every non-zero x in A.

Proof: We always have $\mathcal{L}(A_x)$ contained in $\mathcal{L}(A)$. If A is unequivocal, then it must be in $\mathcal{L}(A_x)$ since it cannot be semi-simple with respect to this radical. Hence we get $\mathcal{L}(A) = \mathcal{L}(A_x)$ for any non-zero x in A. Conversely, suppose this always holds for an algebra A, and suppose that A is not $\mathcal{R}$-semi-simple, for some radical $\mathcal{R}$. Then for some non-zero x in A, A_x is in $\mathcal{R}$. But then, $\mathcal{L}(A) = \mathcal{L}(A_x)$ is contained in $\mathcal{R}$, and so A must be in $\mathcal{R}$. Hence A is unequivocal.

2.16 Corollary: Every non-zero principal ideal of an unequivocal algebra is an unequivocal algebra.

2.17 Example: Unequivocal algebras need not be homogeneous. For example, let A be the product of non-isomorphic homogeneous algebras A_1 and A_2, which are epimorphs of one another. Such pairs exist: take $A_1 = \bar{F}_o$ and $A_2 = P_o/I$, where I is the ideal of finite sets in P_o. Clearly, in any such situation, $\mathcal{L}(A) = \mathcal{L}(A_1) = \mathcal{L}(A_2)$. Now let $x = (x_1, x_2)$ be a non-zero element of A, so that either x_1 or x_2 is non-zero. Then A_x has a principal ideal, hence an epimorph, isomorphic to either A_1 or A_2. So $\mathcal{L}(A_x) = \mathcal{L}(A)$ for all non-zero x in A, and A is unequivocal.

We now show that the Amitsur procedure for obtaining the radical is always valid for Boolean algebras.

2.18 Proposition: If H is a class of Boolean algebras closed under epimorphs, then H is an Amitsur class.

Proof: Suppose that A is an algebra for which $h^*(A) = 0$ and that I is an ideal of A. If I has any ideals in H, then A has these same ideals in H, contradicting $h^*(A) = 0$. Thus, $h^*(I) = 0$, and H is an Amitsur class by Defn. 2.6.

§3. The Superatomic Radical

2.19 Definition (Mostowski and Tarski [21]): A Boolean algebra is said to be superatomic if every epimorph has an atom.

The concept of a superatomic Boolean algebra was first studied by Mostowski and Tarski [21], and later by Yaqub [27] and Day [6], [7]. The following proposition from Day [7] summarizes the various characterizations of superatomic Boolean algebras.

2.20 Proposition: If A is a Boolean algebra, then the following are equivalent:

i) A is superatomic,

ii) every epimorph of A is atomic,

iii) every subalgebra of A has an atom,

iv) every subalgebra of A is atomic,

v) no subalgebra of A is an infinite free algebra,

vi) A has no subalgebra isomorphic to F_o,

vii) A has no chain of elements order-isomorphic with the chain of rational numbers,

viii) the Stone space of A is clairsemé; that is, every non-empty subspace of S(A) has an isolated point.

If we let H be the class consisting of the two-element algebra, then H is closed under epimorphs, and Defn. 2.19 can clearly be re-stated as follows: A is superatomic if and only if every epimorph of A has an ideal in H. By Theorem 2.12, we have:

2.21 Proposition: The class $\mathcal{O}$ of superatomic Boolean algebras is a radical class; namely, it is $L(\underline{2})$, the lower radical generated by the two-element Boolean algebra.

2.22 Corollary: Every radical class of Boolean algebras contains $\mathcal{O}$.

Proof: Every radical class of Boolean algebras contains $\underline{2}$.

2.23 Corollary (Day [7]): The weak product of superatomic Boolean algebras is a superatomic Boolean algebra.

By Prop. 2.18, we can use Amitsur's constuction to obtain the superatomic radical of any algebra. The following is clear:

2.24 Proposition: Let $H = \{\underline{2}\}$ and let A be any Boolean algebra. Then $h_{\alpha+1}(A)/h_\alpha(A)$ is the ideal of $A/h_\alpha(A)$ generated by its atoms; more precisely, it consists of all finite joins of atoms of $A/h_\alpha(A)$.

If A is superatomic, then it is the summit of its H-ladder. Let δ be the least ordinal for which $h_\delta(A) = A$.

2.25 Definition (Day [7]): The cardinal sequence of the superatomic Boolean algebra A is the sequence of order type δ whose α-term, for $\alpha < \delta$, is the cardinality of the set of atoms of $A/h_\alpha(A)$.

This specialization of the Amitsur construction has been used with success by Day in his study of superatomic algebras. We mention two of his more striking results.

2.26 Proposition (Day [7]): Two countable superatomic Boolean algebras are isomorphic if and only if they have the same cardinal sequence.

2.27 Proposition (Day [7]): If κ is an infinite cardinal, then there are more than κ non-isomorphic superatomic Boolean algebras of cardinality κ.

Superatomic Boolean algebras have arisen naturally in the study of free complete extensions of an algebra.

2.28 Definition: C is a free complete extension of B if:

i) C is complete,

ii) B is a subalgebra of C,

iii) any homomorphism of B to a complete algebra can be extended to a complete homomorphism (that is, one which preserves all suprema) of C to that algebra.

If B has a free complete extension, then it is unique up to isomorphism.

2.29 *Proposition* (Yaqub [27], Day [6]): B has a free complete extension if and only if B is superatomic.

Further results of a similar nature occur in these papers.

§4. *The Cardinality Radicals*

2.30 *Definition*: For any ordinal α, we call the lower radical generated by F_α, the free algebra on $\aleph_\alpha$ generators, a *cardinality radical* and denote it by $\mathcal{F}_\alpha$.

2.31 *Proposition*: $\mathcal{F}_\alpha$ is the lower radical generated by the class of Boolean algebras of cardinality at most $\aleph_\alpha$. Moreover, if $\alpha < \beta$, then $\mathcal{F}_\alpha$ is properly contained in $\mathcal{F}_\beta$.

Proof: The first assertion follows from the fact that every algebra of cardinality at most $\aleph_\alpha$ is an epimorph of F_α. If $\alpha < \beta$, F_α is an epimorph of F_β, so we get $\mathcal{F}_\alpha$ contained in $\mathcal{F}_\beta$. But F_β is in $\mathcal{F}_\beta$ and not in $\mathcal{F}_\alpha$, so the containment is proper.

In a natural sense, the superatomic radical is the first member of this chain, for it is the lower radical generated by all finite Boolean algebras.

We shall see in Chapter Seven that F_0 is an atom in the lattice of Boolean algebra radicals, so we have some interest in examples and some properties of F_0-algebras.

2.32 <u>Examples</u>: Of course, any weak product of countable algebras is in F_0. Furthermore, any atomless algebra in F_0 has a dense ideal-generated subalgebra isomorphic to a weak product of copies of F_0. The normal completion of any atomless algebra in F_0 is a power of $\overline{F}_0$, and any power can be so realized; that is, as the normal completion of an F_0- algebra.

Let S be a superatomic algebra with κ atoms. Since any element in an atomic algebra is the supremum of the atoms below it, we can embed S in $P(\kappa)$, which can be realized as the subalgebra of elements with $\{0, 1\}$-coordinates in any κ-product. In particular, we can assume that $P(\kappa)$ is embedded in F_0^{κ}. Let I be the ideal of F_0^{κ} generated by the atoms of S. Then any non-zero principal ideal of F_0^{κ} contained in I is isomorphic to F_0. Furthermore, if A is the subalgebra of F_0^{κ} generated by S and I, then I is also an ideal of A, and A/I is superatomic. In fact, A/I is isomorphic to the quotient of S by the ideal generated by its atoms. Hence, by Prop. 1.6 and Prop. 2.22, A is in F_0. Intuitively, we can regard this as the replacement of each atom in S by a copy of F_0.

In Chapter Six, we will describe another method of obtaining F_0-algebras from superatomic algebras.

A natural extension of countable Boolean algebras are the separable algebras, that is, those having a countable, dense subset.

2.33 <u>Definition</u>: K is the lower radical generated by the separable Boolean algebras.

We will find use for this radical in Chapter Seven.

§5. <u>The Power-Set Radicals</u>

2.34 <u>Definition</u>: The lower radical generated by P_α, the algebra of all subsets of a set of cardinality $\aleph_\alpha$, will be called a <u>power-set radical</u>, and denoted by P_α.

2.35 <u>Proposition</u>: Let $\aleph_\beta = \exp \aleph_\alpha$. Then $P_\alpha = L(\overline{F}_\beta)$.

<u>Proof</u>: By Prop. 0.8, P_α and $\overline{F}_\beta$ are epimorphs of one another.

2.36 <u>Proposition</u>: If $\alpha \leq \beta$, then P_α is contained in P_β. Furthermore, $P_\alpha = P_\beta$ if and only if $\exp \aleph_\alpha = \exp \aleph_\beta$.

<u>Proof</u>: The first assertion follows from the fact that P_α is an epimorph of P_β, and the second from the equivalence of the cardinality condition with the fact that P_α and P_β are epimorphs of one another, using Prop. 0.9.

2.37 <u>Theorem</u>: Let $\aleph_\beta = \exp \aleph_\alpha$. Let A be a complete algebra. Then the following are equivalent:

i) $A \in \mathcal{P}_\alpha$,

ii) A is an epimorph of P_α,

iii) A is an epimorph of $\overline{F}_\beta$,

iv) $|A| \leq \aleph_\beta$.

Proof: Using Prop. 0.5 and Prop. 0.8, we see that ii), iii) and iv) are equivalent, and they clearly imply i). Now assume i). Then by Theorem 2.12, A is a product of epimorphs of P_α, say $A = \Pi(A_i\colon i \in I)$. Let $\aleph_\beta = \aleph_\alpha \cdot |I|$. Then P_β can be represented as a product of $|I|$-copies of P_α, each of which has one of the A_i as an epimorph. Then P_β has their product A as an epimorph. We must now show that P_β is an epimorph of P_α. This is obvious if $\beta \leq \alpha$ so assume $\alpha < \beta$. Since $\aleph_\beta = \max\{\aleph_\alpha, |I|\}$, this means that $\aleph_\beta = |I|$. Since A is in $\mathcal{P}_\alpha$, and has $P(I) = P_\beta$ as a retract, this means that P_β is in $\mathcal{P}_\alpha$, hence that $\mathcal{P}_\beta = \mathcal{P}_\alpha$. But this is equivalent to the fact that P_α and P_β are epimorphs of one another.

2.38 Corollary: Let A be an infinite complete algebra. Then A is in $\mathcal{P}_0$ if and only if $|A| = \exp \aleph_0$. In this case, $\mathcal{L}(A) = \mathcal{P}_0$ and if A is atomless, then it is unequivocal.

Proof: Any infinite complete algebra has cardinality at least $\exp \aleph_0$. The rest follows from the Corollary to Prop. 0.8 and from Prop. 2.15.

It is interesting to note, here, that Monk and Solovay [20] have shown that there are $\exp \exp \aleph_\alpha$ isomorphism classes of complete Boolean algebras of cardinality $\exp \aleph_\alpha$.

We will be interested in Chapter Five with those complete algebras that generate a power-set radical.

2.39 Theorem: Let $\aleph_\beta = \exp \aleph_\alpha$. Let A be any algebra in $\mathcal{P}_\alpha$. Then the following are equivalent:

i) $\mathcal{L}(A) = \mathcal{P}_\alpha$,

ii) P_α is an epimorph of A,

iii) $\overline{F}_\beta$ is an epimorph of A,

iv) F_β is a subalgebra of A.

If A is any complete algebra, then A generates $\mathcal{P}_\alpha$ for its lower radical precisly when it has cardinality $\aleph_\beta$ and satisfies any one of ii), iii) or iv), which is equivalent to saying that A and P_α are epimorphs of one another.

Proof: Using Prop. 0.6 and Prop. 0.8, we see that ii), iii) and iv) are equivalent to one another, and it is clear that they imply i). Assuming i), we see that $\overline{F}_\beta$ is in $\mathcal{L}(A)$. Being homogeneous, it is an epimorph of A by Prop. 2.13. Thus i) implies iii). If A is a complete algebra, then A ε $\mathcal{P}_\alpha$ is equivalent to the inequality $|A| \leqslant \aleph_\beta$, which, in the presence of any of the conditions i) - iv), must actually be an equality. The rest follows immediately.

Finally, we introduce a related radical:

2.40 Definition: $\mathcal{P}$ is the lower radical generated by the class of all complete algebras.

2.41 Proposition: P is the lower radical generated by all power-set algebras, and so is the smallest radical class containing all the P_α.

Proof: Every complete algebra is a retract of a power-set algebra.

CHAPTER THREE

THE UPPER RADICAL

Given any class of rings satisfying a condition called regularity, it is possible to generate a semi-simple class which is minimal with respect to containing the given class.

§1. The Upper Radical Construction

We describe the construction immediately for classes of Boolean algebras. The method and proof of its validity are essentially the same as in general radical theory (see Divinsky [8]) with minor obvious modifications.

We recall that the two-element algebra must be in any radical class. This entails that semi-simple algebras must always be atomless.

3.1 Definition: A class M of Boolean algebras will be called regular if every M-algebras is atomless, and every M-algebra is an approximate M-algebra (Defn. 1.24).

Of course, any hereditary class of atomless algebras is a regular class.

3.2 Proposition: Let M be a regular class of Boolean algebras, and let $U(M)$ denote the class of all Boolean algebras with no epimorph in M. Then:

i) $U(M)$ is a radical class of algebras,

ii) every algebra in M is semi-simple with respect to $U(M)$,

iii) if R is a radical class such that every algebra in M is R-semi-simple, then R is contained in $U(M)$.

3.3 Definition: The radical class $U(M)$ is called the upper radical determined by the class M.

Given any class Y of atomless Boolean algebras, it is clear that the class M of all non-zero principal ideals of Y-algebras is a hereditary class of atomless algebras.

3.4 Definition: If Y is any class of atomless Boolean algebras and M is the class of all non-zero principal ideals of Y-algebras, then we will denote the radical class $U(M)$ by $U(Y)$ and call it the upper radical determined by the class Y.

§2. The Characterization Theorem

3.5 Theorem: Let M be a regular class of Boolean algebras, and let $U(M)$ be the upper radical determined by M. Then an algebra A is $U(M)$-semi-simple if and only if it is a subdirect product of M-algebras.

Proof: By Prop. 1.28, semi-simple classes are closed under subdirect products, so one direction is immediate. The other direction will follow if we can show that for $R = \mathcal{U}(\mathcal{M})$, and for any Boolean algebra A, r(A) is equal to the intersection J of all ideals I of A such that A/I is in $\mathcal{M}$. Then, for a semi-simple algebra, this intersection will be 0, and A will be a subdirect product of the A/I.

By Prop. 1.4, r(A) is the intersection of all ideals I of A for which A/I is R-semi-simple, and so it is contained in J. If they are not equal, let x be an element of J which is not in r(A). Then A_x is not in R, and so has an epimorph A_x/K in $\mathcal{M}$. Note that K is also an ideal of A. Let L denote the ideal of A generated by K and x'. Then, using §2, f) of the Preliminaries, we see that $A/L \simeq A_x/K$, which is in $\mathcal{M}$. Hence L is one of the ideals occurring in the definition of J, and so J is contained in L. But then x must be in L and, being disjoint from x', it must be in K. But this yields the contradiction $A_x/K = 0$. Hence J = r(A) as required.

§3. Atomless Boolean Algebras

The superatomic radical $\mathcal{O}$ is clearly the upper radical determined by the atomless Boolean algebras. We find a much smaller regular class which determines $\mathcal{O}$ as its upper radical.

3.6 Theorem: Let M be the class of all atomless, separable Boolean algebras. Then:

i) M is a hereditary class, and so determines an upper radical,

ii) this radical is, in fact, the superatomic radical, and so

iii) an algebra A is atomless if and only if it is a subdirect product of atomless, separable algebras.

Proof: It is clear that M is hereditary. By Prop. 2.20, an algebra is non-superatomic if and only if it has a subalgebra isomorphic to F_o. Hence, by Prop. 0.1, any non-superatomic algebra has an epimorph with a dense subalgebra isomorphic to F_o; that is, any non-superatomic algebra has an atomless, separable epimorph. Hence $U(M) = O$, and the rest follows from Theorem 3.5.

§4. Some Upper Radicals

It is clear that a class consisting of a single atomless, homogeneous algebra is a hereditary class, and so generates an upper radical. Any semi-simple algebra, then, can be represented as a subdirect power of this algebra. The radicals defined in this section all have this feature. In only one case do we make further mention of this fact, for it yields a subdirect power representation for atomless, complete algebras.

3.7 Definition: $\mathcal{E}_\alpha$ is the upper radical determined by the infinite free algebra F_α.

3.8 Proposition: If $\alpha < \beta$, then $\mathcal{E}_\alpha$ is properly contained in $\mathcal{E}_\beta$.

Proof: The containment follows easily from the fact that F_α is an epimorph of F_β. Furthermore, F_α is in $\mathcal{E}_\beta$ but not in $\mathcal{E}_\alpha$.

3.9 Definition: $\mathcal{G}_\alpha$ is the upper radical determined by $\overline{F}_\alpha$.

3.10 Proposition: If $\alpha \leq \beta$, then $\mathcal{G}_\alpha$ is contained in $\mathcal{G}_\beta$. Furthermore, $\mathcal{G}_\alpha = \mathcal{G}_\beta$ if and only if $\overline{F}_\alpha$ and $\overline{F}_\beta$ are epimorphs of one another.

Proof: $\overline{F}_\alpha$ is an epimorph of $\overline{F}_\beta$. The rest is obvious.

3.11 Corollary: Let $\aleph_\gamma = \exp \aleph_0$ and let $0 \leq \alpha, \beta \leq \gamma$. Then $\mathcal{G}_\alpha = \mathcal{G}_\beta$.

Proof: By the Corollary to Prop. 0.8, any two complete algebras of cardinality $\exp \aleph_0$ are epimorphs of one another.

For $\overline{F}_\alpha$ and $\overline{F}_\beta$ to be mutual epimorphs, it is necessary that their cardinalities be the same: $\aleph_\alpha^{\aleph_0} = \aleph_\beta^{\aleph_0}$. We do not know whether the cardinality condition is sufficient.

3.12 Theorem: Any atomless, complete algebra is a sudirect power of $\overline{F}_0$.

Proof: Using Theorem 3.5, it suffices to show that any atomless, complete algebra is G_o-semi-simple. Any principal ideal of an atomless, complete algebra is itself atomless and complete, and so, by §3, j) of the Preliminaries, it must have $\bar{F}_o$ as an epimorph and cannot be in G_o. Hence any atomless, complete algebra is G_o-semi-simple.

We shall see that G_o has many upper radical characterizations and so yields many subdirect product representations for atomless, complete algebras. Because of its special properties, however, $\bar{F}_o$ is an especially appropriate building block.

3.13 <u>Example</u>: Let Q be the algebra of all finite unions of left-closed, right-open subintervals of the unit interval $[0, 1)$ of the reals. It is clear that Q is an atomless, homogeneous algebra. The Stone space of Q is the set X obtained from the closed unit interval $[0, 1]$ of reals by splitting every interior point x into two parts, x^- and x^+. We consider X as an ordered set with the natural order: $0 < x^- < x^+ < y^- < y^+ < 1$ whenever $0 < x < y < 1$, and give it the order topology (see Sikorski [24], example §9, E). We will show that Q is in both G_o and E_o.

If Q is not in G_o, then it has P_o as an epimorph; in other words, we can embed βN in X. But $|X| = \exp \aleph_o$ and $|\beta N| = \exp \exp \aleph_o$, so this is impossible.

If Q is not in E_o, then we can embed the Cantor set $2^{\aleph_o}$ in X. To show that this is impossible, we show that any uncountable closed subspace of X has an uncountable base.

Let F be any uncountable closed subspace of X and let $\{G_\alpha\}$ be a base for F. Let F^+ (respectively F^-) be the points of F of the form x^+ (respectively x^-). Then one of F^+, F^- must be uncountable. Suppose F^+ is uncountable. For any x^+ in F^+, the interval $[x^+, 1]$ is clopen in X and so $F \cap [x^+, 1]$ is clopen in F. Thus there is a basic set G_x such that $x^+ \in G_x \subseteq F \cap [x^+, 1]$. If $x^+ < y^+$, then $x^+ \notin [y^+, 1]$ and so $x^+ \notin G_y$. Thus for distinct x^+, y^+ in F^+, we get distinct basic open sets G_x and G_y, and the base $\{G_\alpha\}$ must be uncountable. If F^- is uncountable, an obvious modification of the argument yields the same result.

§5. The Upper Radical Determined by Homogeneous Algebras

3.14 Definition: $\mathcal{J}$ is the upper radical determined by all atomless, homogeneous algebras.

$\mathcal{J}$ is contained in any upper radical determined by a class of atomless, homogeneous algebras. Thus it is contained in both $\mathcal{G}_o$ and $\mathcal{E}_o$. It is not equal to their intersection, however, for the algebra Q of the last example is $\mathcal{J}$-semi-simple. A natural question here is whether $\mathcal{J} = \mathcal{O}$. In Chapter Seven, we will discuss one consequence of a positive answer. We present some considerations which make a positive answer reasonable.

3.15 Definition: A monotonic cardinal property ν assigns a cardinal number $\nu(A)$ to each algebra A in such a way that $\nu(A_x) \leq \nu(A)$ for all non-zero x in A. If this inequality is an equality for all non-zero x in A, then A is called ν-homogeneous.

3.16 Lemma: If ν is a monotonic cardinal property, then the ν-homogeneous elements of any algebra A are dense in A.

Proof: For any non-zero x in A, pick $y \leq x$ such that $\nu(A_y)$ is minimal among the $\nu(A_z)$ for $0 \neq z \leq x$. By monotonicity of ν, A_y is ν- homogeneous.

3.17 Corollary: If ν is a monotonic cardinal property, then any complete algebra is a product of ν-homogeneous algebras.

Proof: It is clear that the property of being ν-homogeneous is hereditary, so we can apply Lemma 2.10.

This last result is due to Pierce [22] and provides support for his conjecture [23] that every complete algebra is a product of homogeneous algebras. We provide similar support for the conjecture that $J = 0$.

3.18 Proposition: Let ν be a monotonic cardinal property and let J_ν be the upper radical determined by the class of atomless, ν-homogeneous algebras. Then $J_\nu = 0$.

Proof: Any non-superatomic algebra has an atomless epimorph, which has an atomless, ν-homogeneous principal ideal. Hence J_ν can contain only superatomic algebras.

CHAPTER FOUR

CRAMER'S RADICALS

We investigate the radicals introduced by Cramer in [5].

§1. The Classes C_α

Superatomic Boolean algebras can be characterized as those having no countable free subalgebra. Cramer generalized this as follows:

4.1 Definition: The class C_α is the class of all Boolean algebras with no subalgebra isomorphic to F_α.

Cramer's proof that the C_α are radical classes uses topological methods. We present an algebraic proof.

4.2 Proposition: C_α is a radical class.

Proof: By projectivity, whenever F_α can be embedded in an epimorph of A, then F_α can be embedded in A. Thus C_α is closed under epimorphs. Now suppose that A is an algebra such that every epimorph of A has a principal ideal in C_α. If F_α can be embedded in A, then by Prop. 0.1, F_α can be densely embedded in some epimorph B of A. If x is any non-zero element of B, there is a non-zero $y \leq x$ such that $y \in F_\alpha$. Then the principal ideal of F_α generated by y can be embedded in B_y which can be embedded as a subalgebra in B_x. Since F_α is homogeneous, this says that no non-zero principal ideal of B can be in C_α, contradicting the assumption on A. Thus A is in C_α, and by Prop. 1.21, C_α is a radical class.

4.3 Proposition: If $\alpha < \beta$, then C_α is properly contained in C_β.

Proof: The containment follows from the fact that F_α can be embedded in F_β. It is proper since $F_\alpha \in C_\beta$ but $F_\alpha \notin C_\alpha$.

§2. The C_α as Upper Radicals

4.4 Definition: The class $\mathcal{D}_\alpha$ is the class of all Boolean algebras which do not have P_α as an epimorph.

4.5 Proposition (Cramer [5]): Let $\mathcal{H}_\beta = \exp \mathcal{H}_\alpha$. Then $\mathcal{D}_\alpha = C_\beta$.

Proof: This follows immediately from Prop. 0.6.

4.6 Corollary: $\mathcal{D}_\alpha$ is a radical class, and if $\alpha < \beta$, then $\mathcal{D}_\alpha$ is properly contained in $\mathcal{D}_\beta$.

Except for the fact that P_α is not atomless, the description of $\mathcal{D}_\alpha$ suggests an upper radical. If $\mathcal{H}_\beta = \exp \mathcal{H}_\alpha$, however, since P_α and $\overline{F}_\beta$ are epimorphs of one another, we immediately get $C_\beta = \mathcal{D}_\alpha = \mathcal{G}_\beta$. Actually, we can extend this result.

4.6 Proposition: C_α is contained in G_α, and $C_\alpha = G_\alpha$ if and only if $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$.

Proof: Since $\overline{F}_\alpha$ has F_α as a subalgebra, it is not in C_α and being homogeneous, it must be C_α-semi-simple. But G_α is the largest radical for which $\overline{F}_\alpha$ is semi-simple, so C_α is contained in G_α. If $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $|\overline{F}_\alpha| = \aleph_\alpha$, and so it is an epimorph of F_α. If F_α is embedded in an algebra A, then by injectivity of $\overline{F}_\alpha$, we get that A has $\overline{F}_\alpha$ as an epimorph. In other words, G_α is contained in C_α, and we get equality of the radicals. Conversely, suppose the radicals are equal, and suppose that $\aleph_\alpha^{\aleph_0} > \aleph_\alpha$. By cardinality, then, F_α cannot have $\overline{F}_\alpha$ as an epimorph and so F_α is in G_α. But $F_\alpha \notin C_\alpha$, so this contradicts the equality of the radical classes.

§3. The Radical $\mathcal{D}_o$

Since P_o and $\overline{F}_o$ are epimorphs of one another, $\mathcal{D}_o = G_o$. We have already seen that all atomless, complete algebras are $\mathcal{D}_o$-semi-simple. This section plays variations on the theme that $\mathcal{D}_o$-algebras are in a very strong sense the opposite of complete algebras. The basic fact we need is Pierce's result (§3, j) of Preliminaries) that any infinite epimorph of a σ-complete algebra must have P_o as an epimorph. An immediate consequence of this is the following:

4.7 Proposition: Let A be an infinite $\mathcal{D}_o$-algebra. Then:

i) A has no infinite σ-complete epimorphs,

ii) A is not the epimorph of any σ-complete algebra, and

iii) A has no infinite, complete subalgebra.

4.8 Proposition: Let $\mathcal{V}$ be any class of atomless algebras satisfying:

i) each algebra in $\mathcal{V}$ has P_o as an epimorph, and

ii) there is an algebra in $\mathcal{V}$ which is an epimorph of P_o.

Then the upper radical determined by $\mathcal{V}$ is $\mathcal{D}_o$.

Proof: By condition i), a $\mathcal{D}_o$-algebra cannot have an epimorph in $\mathcal{V}$, so $\mathcal{D}_o$ is contained in the upper radical determined by $\mathcal{V}$. By condition ii), any algebra in the upper radical determined by $\mathcal{V}$ cannot have P_o as an epimorph, and so is in $\mathcal{D}_o$.

4.9 Corollary: $\mathcal{D}_o$ is the upper radical determined by any of the following classes:

i) all atomless, complete algebras,

ii) all atomless, σ-complete algebras,

iii) all atomless, complete homogeneous algebras,

iv) the class consisting of all principal ideals of P_α/I, where I is the ideal of finite sets in P_α,

v) the class consisting of $\overline{F}_o$,

vi) the class consisting of any atomless, complete algebra of cardinality exp $\aleph_o$.

CHAPTER FIVE

DECOMPOSITIONS OF COMPLETE ALGEBRAS

The product decompositions of this chapter depend on finding a dense subset of P-elements in an algebra, for some hereditary property P. The search for P-elements below an arbitrary non-zero element leads naturally to the consideration of various chain-like conditions.

§1. The General Setting

We have already seen that Pierce's decomposition of complete algebras via cardinal properties is a special case of Lemma 2.10. The theorems of this chapter also make use of this lemma. We note that the properties of being homogeneous and unequivocal are hereditary properties. Hence, we immediately get the following:

5.1 Proposition: Let A be a complete algebra. Then A is a product of homogeneous (unequivocal) algebras if and only if the homogeneous (unequivocal) elements of A are dense in A.

Pierce's result also included a uniqueness feature (see[22]) which also holds for decompositions into homogeneous (unequivocal) algebras, whenever such decompositions exist. The following proposition includes all these uniqueness results as special cases.

5.2 Proposition: Let P be a hereditary property, and suppose A is a complete algebra with a dense subset of P-elements. Suppose there is an equivalence relation $\equiv$ on the P-elements of A such that:

(*) if x and y are P-elements of A and $x \not\equiv y$,

then $x \wedge y = 0$.

Then there is a unique decomposition $A \simeq \Pi(A_x : x \in X)$ with the following properties:

i) for any $x \in X$, A_x is a product of P-algebras A_y, $y \in Y_x$, where $y_1 \equiv y_2$ for any $y_1, y_2 \in Y_x$, and

ii) for $x \neq z$ in X, $y_1 \in Y_x$, $y_2 \in Y_z$, $y_1 \not\equiv y_2$.

Proof: The set X consists of the suprema of the equivalence classes of P-elements of A. Then X is disjointed by (*), and sup X = 1 by the density of P-elements in A. The rest of the proof is a straight-forward verification.

To see how this applies, we need to specify an equivalence relation for each of the properties we have considered:

i) ν-homogeneity: say $x \equiv y$ if $\nu(A_x) = \nu(A_y)$,

ii) homogeneity: say $x \equiv y$ if $A_x \simeq A_y$, and

iii) unequivocality: say $x \equiv y$ if $L(A_x) = L(A_y)$.

In each case, it is easy to see that condition (*) is satisfied, so that a suitable replacement of P in Prop. 5.2 will yield a uniqueness result in each of these three situations.

§2. Decompositions into Homogeneous Algebras

5.3 Definition: An algebra A will be called *near-homogeneous* if every descending chain of principal ideals of A contains only a finite number of isomorphism types of algebras.

Finite products of homogeneous algebras are a natural example of near-homogeneous algebras. Another class of examples are the power-set algebras. Any principal ideal of such an algebra is another power-set algebra, which is determined up to isomorphism by the cardinality of its atoms. Hence any descending chain of such ideals yields a descending chain of cardinals, which must be finite.

5.4 Theorem: If A is complete and near-homogeneous, then A is a product of homogeneous algebras.

Proof: By Prop. 5.1, it suffices to show that any non-zero principal ideal of A contains a non-zero homogeneous principal ideal. So let x be a non-zero element of A. If A_x is not homogeneous, there is a non-zero element $y \leq x$ such that A_y is not isomorphic to A_x. Proceeding inductively, we get a descending chain $A_x > A_y > \ldots > A_z > \ldots$ where no two adjacent algebras are isomorphic. By §3, g) of the

Preliminaries, if any two algebras in the chain are isomorphic, then they are also isomorphic to all the intervening ones. Hence no two algebras in the chain can be isomorphic. Because A is near-homogeneous, the chain must terminate in a finite number of steps, and the algebra thus obtained is a non-zero homogeneous principal ideal of A contained in A_x.

5.5 Example: We present an example of a complete algebra A which is a product of homogeneous algebras, but which is not near-homogeneous. Let $\{\kappa_n: n < \omega\}$ be a strictly increasing sequence of cardinals satisfying $\kappa_n^{\aleph_0} = \kappa_n$. Such sequences exist: for example, take $\kappa_0 = \exp \aleph_0$ and $\kappa_{n+1} = \exp \kappa_n$. For any such sequence, let A_n denote the normal completion of the free algebra on κ_n generators. Then A_n is complete, homogeneous, and its cardinality is κ_n. Let $A = \Pi(A_n: n < \omega)$. There is a natural isomorphism between certain principal ideals B_k of A and partial products of the A_n as follows: $B_k \simeq \Pi(A_n: k \leqslant n < \omega)$. Thus we get a descending chain of ideals of A: $B_0 > B_1 > \ldots > B_k > \ldots$. Any non-zero principal ideal of B_i must contain a copy of some A_n for $n \geqslant i$ and so must have cardinality at least κ_i. For $j < i$, howevery, B_j has a principal ideal isomorphic to A_j of cardinality κ_j. Since $\kappa_j < \kappa_i$, B_i cannot be isomorphic to B_j.

§3. Decompositions into Unequivocal Algebras

5.6 Definition: Let $\{R_\alpha\}$ be a well-ordered chain of radical classes. We say that an algebra A is R-layered if, for any non-zero x in A, either A_x is finite, or there is an ordinal β such that $L(A_x) = R_\beta$. A will be called layered if there exists some well-ordered chain of radical classes $\{R_\alpha\}$ such that A is R-layered.

Examples of layered algebras will be given in §4. It is clear that any principal ideal of an R-layered algebra is itself R-layered.

5.7 Theorem: If A is a complete layered algebra, then A is a product of unequivocal algebras.

Proof: By Prop. 5.1, it suffices to show that any non-zero principal ideal of A contains a non-zero unequivocal principal ideal. So let x be a non-zero element of A, and suppose $\{R_\alpha\}$ is the chain of radicals with respect to which A is layered. If A_x contains an atom, then it contains the unequivocal algebra $\underline{2}$. Otherwise, if A is atomless, let β be the least α such that $L(A_y) = R_\alpha$ for some non-zero $y \leq x$. Choose some y for which $L(A_y) = R_\beta$. Now for $0 \neq z \leq y$, $L(A_z)$ is contained in R_β. Since A is R-layered, $L(A_z)$ is some R_α, and by the minimality of β, $L(A_z) = R_\beta$. But then, by Prop. 2.15, A_y is an unequivocal principal ideal of A_x.

5.8 Theorem: Let A be a complete algebra, and suppose that for any non-zero x in A, there is a power-set algebra P such that A_x and P are epimorphs of one another. Then A is a product of unequivocal algebras.

Proof: Using Theorem 2.39, it is clear that the condition on A is precisely what is needed to make A a P-layered algebra for the chain $\{P_\alpha\}$.

Using Theorem 2.39, we see that is would be extremely useful, in determining the scope of this theorem, to know which algebras, other than power-set algebras and completions of free algebras, have large free subalgebras. Unfortunately, little is known. As a sample, we quote the following result:

5.9 Proposition (Efimov [11]): For any algebra A, let cA denote the supremum of the cardinalities of families of disjoint elements of A. Suppose A is an algebra such that $cA \leq \kappa$ and $|A| > \exp \exp \exp \kappa$. Then A has a free subalgebra on $(\exp \kappa)+$ generators.

We note that F_o can never be in the lower radical generated by an infinite complete algebra. Hence there is no point in attempting further results along these lines using the chains $\{F_\alpha\}$, $\{C_\alpha\}$, or $\{E_\alpha\}$.

§4. Connections with Cardinal Properties

It is possible to obtain Theorem 5.7 in a slightly more lengthy manner using Pierce's result (Cor. 3.17). There are some interesting additional results along the way, and the approach is better suited to presenting examples, so we proceed to develop it now.

5.10 Definition: For any well-ordered chain of radical classes $\{R_\alpha\}$, we say an algebra A is admissible with respect to the chain if it is in one of the radicals of the chain. For any admissible A, we define $\rho(A) = \min\{\aleph_\alpha : A \in R_\alpha\}$.

Then ρ is a cardinal property on admissible algebras, and the fact that it is monotonic follows easily from the fact that every radical class is hereditary. We note that $\rho(A)$ is always infinite.

5.11 Lemma: The admissible algebra A is ρ-homogeneous if and only if there is an ordinal β such that:

i) $A \in R_\alpha$ for all $\alpha \geq \beta$,

ii) A is R_α-semi-simple for all $\alpha < \beta$.

In this case, of course, $\rho(A) = \aleph_\beta$.

Proof: Let A be ρ-homogeneous with $\rho(A) = \aleph_\beta$. Then for all $\alpha \geq \beta$, A is in R_α. By ρ-homogeneity, $\rho(A_x) = \rho(A) = \aleph_\beta$ for all non-zero x in A. In other words, no non-zero principal ideal of A occurs in any R_α for $\alpha < \beta$. Hence A is R_α-semi-simple for all $\alpha < \beta$. Conversely, suppose such a β exists. Then clearly $\rho(A) = \aleph_\beta$. Since A is R_α-semi-simple for $\alpha < \beta$, A_x cannot be in any such R_α for any non-zero x in A. Since $\rho(A_x) \leqslant \rho(A) = \aleph_\beta$, we must have $\rho(A_x) = \aleph_\beta$. Hence A is ρ-homogeneous.

5.12 Lemma: Let $\{R_\alpha\}$ be a well-ordered chain of radicals. Suppose the algebra A is R-layered (hence admissible) and ρ-homogeneous. Then A is unequivocal.

Proof: Let $\rho(A) = \aleph_\beta$, and let x be a non-zero element of A. By ρ-homogeneity, A_x cannot be in R_α for any $\alpha < \beta$. Hence, for any such α, $L(A_x) \neq R_\alpha$. Since A is R-layered and $L(A_x)$ is contained in R_β, we must have $L(A_x) = R_\beta$ for any non-zero x in A. Hence A is unequivocal.

We are now ready to re-prove Theorem 5.7:

5.13 Theorem: If A is a complete layered algebra, then A is a product of unequivocal algebras.

Proof: Suppose A is R-layered. By Cor. 3.17, A is a product of ρ-homogeneous algebras. Being principal ideals of A, these algebras are also R-layered and so each is, in fact, unequivocal.

We are now ready to proceed with examples. We concentrate on the chains $\{P_\alpha\}$ and $\{D_\alpha\}$ which define cardinal properties π and δ

respectively. Note that every complete algebra is admissible with respect to these chains. For the remainder of this chapter, we assume GCH. Aside from the fact that GCH simplifies the examples we consider, the following proposition requires the assumption that if $\alpha < \beta$, then $\exp \aleph_\alpha < \exp \aleph_\beta$.

5.14 Proposition (GCH): For any admissible algebra A, $\delta(A) \leqslant \pi(A)+$. Furthermore, $\delta(A) = \pi(A)+$ if and only if $L(A) = P_\alpha$ where $\aleph_\alpha = \pi(A)$.

Proof: Since P_α cannot have $P_{\alpha+1}$ as an epimorph, we get $P_\alpha \subseteq D_{\alpha+1}$. This implies the first statement. Clearly, $\delta(A) = \pi(A)+$ if and only if $A \in P_\alpha$ but $A \notin D_\alpha$; that is, $A \in P_\alpha$ and A has P_α as an epimorph. By Theorem 2.39, this is equivalent to $L(A) = P_\alpha$.

5.15 Corollary (GCH): An algebra A is P-layered if and only if, for every non-zero x in A such that A_x is infinite, $\delta(A_x) = \pi(A_x)+$. A complete algebra with this property is a product of unequivocal algebras.

5.16 Examples (GCH): We note that $\pi(P_\alpha) = \aleph_\alpha$ and $\delta(P_\alpha) = \aleph_{\alpha+1}$. Hence every power-set algebra is P-layered.

Since $P_\alpha = L(\overline{F}_{\alpha+1})$, we have $\pi(\overline{F}_{\alpha+1}) = \aleph_\alpha$. Now suppose α is a limit ordinal. We always have that $\overline{F}_\alpha$ is an epimorph of P_α, so $\overline{F}_\alpha \in P_\alpha$. Suppose $\beta < \alpha$ and $\overline{F}_\alpha \in P_\beta$ Then $\overline{F}_\alpha$ is an epimorph of P_β so $\aleph_{\beta+1} \geqslant \aleph_\alpha^{\aleph_0}$. But $\beta < \alpha$ implies $\beta + 1 < \alpha$ and $\aleph_{\beta+1} < \aleph_\alpha^{\aleph_0}$.

So $\overline{F}_\alpha$ is not in P_β, $\beta < \alpha$, and $\pi(\overline{F}_\alpha) = \alpha$. In either case, $\pi(\overline{F}_\alpha) = \Sigma(\aleph_\gamma: \gamma < \alpha)$.

Since $F_{\alpha+1}$ has P_α as an epimorph, then, by injectivity of P_α, so does $\overline{F}_{\alpha+1}$. Hence $\delta(\overline{F}_{\alpha+1}) = \aleph_{\alpha+1}$. So, for successor ordinals. $\overline{F}_{\alpha+1}$ is P-layered.

For limit ordinals, the situation is unclear. For example, we do not know if $\overline{F}_\omega$ is P-layered. However, if α is a limit ordinal satisfying $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $|\overline{F}_\alpha| = \aleph_\alpha$, so $\overline{F}_\alpha$ cannot have P_α as an epimorph; that is, $\overline{F}_\alpha \in \mathcal{D}_\alpha$. If $\beta < \alpha$, then $\beta + 1 < \alpha$ and so F_α has P_β as an epimorph. So then does $\overline{F}_\alpha$, and $\overline{F}_\alpha \notin \mathcal{D}_\beta$ for $\beta < \alpha$. In this case, then, $\delta(\overline{F}_\alpha) = \aleph_\alpha = \pi(\overline{F}_\alpha)$. Hence, in this case, $\overline{F}_\alpha$ is not P-layered. We note that such ordinals exist; for example, take α to be the first uncountable ordinal.

CHAPTER SIX

CLOSURE PROPERTIES OF RADICAL AND SEMI-SIMPLE CLASSES

Every radical class is closed under finite products and weak products. Every semi-simple class is closed under subdirect products. We extend these results.

§1. Closure of Radical Classes under Products

Any $\mathcal{H}_\alpha$-product of algebras has P as an epimorph, so this gives us a crude negative result: whenever a radical does not contain P_α, then no $\mathcal{H}_\alpha$-product can be in the radical. One might hope to show the converse: whenever P_α is in a radical class, then it is closed under $\mathcal{H}_\alpha$-products. Cramer [5] has obtained a result which shows that this is false.

6.1 Proposition: For any ordinal α, there is a sequence $\{A_n : n < \omega\}$ of superatomic algebras whose product is not in C_α.

We shall see in Chapter Seven that any P_α is contained in some C_β, so we can find a sequence of superatomic algebras whose product is not in P_o. In Chapter Seven, we will present a weaker form of this (false) conjecture which has more likelihood of being true.

6.2 <u>Proposition:</u> If $\exp \aleph_\alpha = \exp \aleph_\beta$, then P_α is closed under $\aleph_\beta$-products of complete P_α-algebras.

<u>Proof:</u> Let $A = \Pi(A_i\colon i \in I)$ be a product of complete P_α-algebras with $|I| \leq \aleph_\beta$. By Theorem 2.37, $|A_i| \leq \exp \aleph_\alpha$ and so $|A| \leq (\exp \aleph_\alpha)^{\aleph_\beta} = \exp \aleph_\alpha$. Since this product is complete, it is in P_α by Theorem 2.37.

Halmos ([14], exercise 3, p. 118) has defined a "weak product" slightly different from our weak product. We give a generalization of his construction, which also includes our weak product and the construction of Example 2.32 as special cases. For any product A of algebras, P(I) is embedded in $A = \Pi(A_i\colon i \in I)$ as the elements of A with $\{0, 1\}$-coordinates. Let B be a subalgebra of P(I) which contains the atoms of P(I).

6.3 <u>Definition:</u> <u>The product of the A_i over B</u> is the subalgebra C of A consisting of all elements which differ from an element of B in at most a finite number of coordinates.

It is clear that C is the subalgebra of A generated by B and $w\Pi(A_i\colon i \in I)$. Halmos' "weak product" corresponds to choosing $B = P(I)$. For the weak product, choose B to be the finite-cofinite algebra on I. Our Example 2.32 used a superatomic subalgebra of P(I).

6.4 <u>Lemma:</u> Let J be the ideal of C generated by the A_i and let K be the ideal of B generated by its atoms. Then $C/J \cong B/K$.

Proof: It is clear that C contains the A_i. Let $x \in C$ and suppose that x differs from elements $b_1, b_2 \in B$ in at most a finite number of coordinates. Then b_1 differs from b_2 in at most a finite number of coordinates; that is, $b_1 + b_2 \in K$. Thus the map which sends x to the coset of b_1 in B/K is a well-defined epimorphism. Clearly, J is its kernel.

6.5 Proposition: Let R be a radical class and suppose the A_i and B are in R. Then the product C of the A_i over B is in R.

Proof: J is generated by radical ideals of C, and C/J, being isomorphic to an epimorph of a radical algebra, is radical. Thus $C \in R$.

6.6 Corollary: Radical classes are closed under weak products.

§2. Closure of Radical Classes under Coproducts

6.7 Basic Lemma: Let R be a radical class. Suppose A and B are Boolean algebras, and that A is the summit of a ladder $\{I_\alpha\}$ with the following property:

(*) for each α, each element of $I_{\alpha+1}/I_\alpha$ is a finite join of cosets [a] such that $(A/I\ )_{[a]} + B \in R$.

Then $A + B \in R$.

Proof: Let K_α be the ideal of A + B generated by I_α. Then $\{K_\alpha\}$ is a ladder in A + B with summit A + B, and so, by Lemma 2.3, it suffices to show that $K_{\alpha+1}/K_\alpha \in R$ for all α. Now $K_{\alpha+1}/K_\alpha$ is an ideal of $A + B/K_\alpha$, which, by Prop. 0.4, is isomorphic to $A/I_\alpha + B$. As an ideal of $A/I_\alpha + B$, the elements of $K_{\alpha+1}/K_\alpha$ can be represented as finite joins of elements of the form $[a] \wedge b$ where $[a] \in I_{\alpha+1}/I_\alpha$ and $b \in B$. By (*), $[a] = [a_1] \vee \ldots \vee [a_n]$ where $(A/I_\alpha)_{[a_i]} + B \in R$. This latter algebra is isomorphic to $(A + B/K_\alpha)_{[ai]}$ which contains $(A + B/K_\alpha)_{[a_i]\wedge b}$. Then, since each $[a_i] \wedge b$ is a radical element of $K_{\alpha+1}/K_\alpha$, we get that $K_{\alpha+1}/K_\alpha \in R$ as required.

6.8 <u>Theorem:</u> Let R be a radical class and let $B \in R$. Then, for any superatomic algebra A, $A + B \in R$.

<u>Proof:</u> By Prop. 2.24, A is the summit of a ladder $\{I_\alpha\}$ where every element of $I_{\alpha+1}/I_\alpha$ is a finite join of atoms $[p] \in A/I_\alpha$. Then $(A/I_\alpha)_{[p]} + B \approx \underline{2} + B \approx B \in R$.

6.9 <u>Corollary</u> (Day [7]): A finite coproduct of superatomic algebras is superatomic.

6.9 <u>Example:</u> Let Z be the Boolean space (under the order topology) of ordinals less than or equal to Ω, the first uncountable ordinal. Then the algebra S of clopen subsets of Z is superatomic, and so, by the last theorem, $A = F_o + S \in F_o$. We show that if I is the ideal of A generated by the elements x such that $A_x \approx F_o$, then A/I is not superatomic. Let $Y = S(F_o)$ be the Cantor set. Then $X = Y \times Z$ is the

Stone space of A. Let $U = S(I)$. For any clopen subset M of X, $M \cap U$ must have a countable base. Now suppose M is a clopen subset of X such that $(y, \Omega) \in M$ for some $y \in Y$. The projection $p_Z[M]$ of M onto Z is an open subset of Z containing Ω. This open set contains a clopen set N which is homeomorphic to Z, whose pre-image $p_Z^{-1}[N]$ is a clopen subset of X contained in M. Since Z has no countable base, neither, then, can M. Hence $U \cap \{(y, \Omega): y \in Y\} = \phi$. Clearly, $\{(y, \Omega): y \in Y\}$ is homeomorphic to Y, so $X - U$ has a closed subspace homeomorphic to Y. Algebraically, A/I has F_o as an epimorph, and so cannot be superatomic.

6.10 Lemma: Let H be a class of Boolean algebras closed under epimorphs. Suppose B is an algebra whose coproduct with any H-algebra is in $L(H)$. Then the coproduct of B with any $L(H)$-algebra is in $L(H)$.

Proof: Let $A \in L(H)$. Then it is the summit of its H-ladder $\{I_\alpha\}$, and every element of $I_{\alpha+1}/I_\alpha$ is a finite join of H-elements. By the assumption on B, its coproduct with any principal ideal generated by an H-element must be in $L(H)$. Hence, by Lemma 6.7, $A + B \in L(H)$.

6.11 Theorem: Let X be any class of algebras closed under finite coproducts. Then $L(X)$ is closed under finite coproducts.

Proof: Let H be the class of all epimorphs of X-algebras. Since the coproduct of epimorphs of two X-algebras is an epimorph of their coproduct, and since X is closed under finite coproducts, so then is H.

If C is an $\mathcal{H}$-algebra, then its coproduct with any other $\mathcal{H}$-algebra is in $\mathcal{H}$ and so in $L(\mathcal{H})$. By Lemma 6.10, the coproduct of C with any $L(\mathcal{H})$-algebra B is in $L(\mathcal{H})$. Since this is true for any C in $\mathcal{H}$, we have that the coproduct of B with any $\mathcal{H}$-algebra is in $L(\mathcal{H})$. Applying Lemma 6.10 again, we get that the coproduct of B with any $L(\mathcal{H})$-algebra A is in $L(\mathcal{H})$. Hence $L(\mathcal{H})$ is closed under finite coproducts. Since $L(X) = L(\mathcal{H})$, the result follows.

6.12 <u>Corollary</u>; F_α is closed under finite coproducts.

<u>Proof</u>: The class of algebras of cardinality at most $\aleph_\alpha$ is closed under finite (in fact, $\aleph_\alpha$-) coproducts.

One might hope to prove that F_α is closed under $\aleph_\alpha$-coproducts. We will show, in the next section, that it is not even closed under countable coproducts of superatomic algebra.

The following result is from Cramer [5]:

6.13 <u>Proposition</u>: Suppose that $\{A_i: i \in I\}$ is a collection of C_α-algebras, finite coproducts of which are in C_α. Suppose that $|I| = \aleph_\beta$ and that $\aleph_\alpha$ is $\aleph_\beta$-inaccessible (that is, $\aleph_\alpha$ cannot be expressed as the sum of $\aleph_\beta$ cardinals each of which is less than $\aleph_\alpha$). Then the coproduct $A = \Sigma(A_i: i \in I)$ is in C_α.

<u>Proof</u>: The coproduct A is the union of all its subalgebras B_j, $j \in J$, which are finite coproducts of the A_i (see Preliminaries, §4, c). Note that $|J| = |I| = \aleph_\beta$. If A has a free subalgebra generated by a set D

of cardinality $\aleph_\alpha$, then since $\aleph_\alpha$ is $\aleph_\beta$-inaccessible, $D \cap B_j$ must have cardinality $\aleph_\alpha$ for some $j \in J$. But $D \cap B_j$ generates a free subalgebra of B_j, contradicting $B_j \in C_\alpha$.

6.14 Corollary: Let $\{A_i : i \in I\}$ be a collection of F_α-algebras. Suppose that $|I| = \aleph_\beta$ and that $\aleph_{\alpha+1}$ is $\aleph_\beta$-inaccessible. Then the coproduct of the A_i is in $C_{\alpha+1}$.

Proof: Since F_α does not have $F_{\alpha+1}$ as a subalgebra, F_α is contained in $C_{\alpha+1}$, and F_α is closed under finite coproducts.

Intuitively, this says that small enough coproducts of small enough algebras cannot have large free subalgebras.

In view of the fact that finite products of projective algebras are projective, it is not unreasonable to ask if finite coproducts of complete algebras are complete. It is relevant in this context since a positive answer would have consequences concerning the closure of P and possibly the P_α under finite coproducts. Unfortunately, the answer is almost always negative.

6.15 Proposition: If A and B are infinite algebras, then A + B is not complete.

Proof: Choose infinite disjoint collections $\{a_i : i < \omega\}$ and $\{b_i : i < \omega\}$ in A and B respectively. Set $x_i = a_i \wedge b_i$ in A + B. Let $x \in A + B$ be an upper bound of $\{x_i : i < \omega\}$. We show that x cannot be a least upper bound. We note that x can be represented in the form $x = (c_1 \wedge d_1) \vee \dots \vee (c_n \wedge d_n)$ where $c_i \in A$ and $d_i \in B$. Then there exist k and $i \neq j$ such that $x_i \wedge (c_k \wedge d_k) \neq 0$ and $x_j \wedge (c_k \wedge d_k) \neq 0$, for otherwise, x would intersect only finitely many x_i. In other words, $(a_i \wedge c_k) \wedge (b_i \wedge d_k) \neq 0$ and $(a_j \wedge c_k) \wedge (b_j \wedge d_k) \neq 0$. Then, since A and B are independent subalgebras of $A + B$, we get that $y = (a_i \wedge c_k) \wedge (b_j \wedge d_k) \neq 0$. For any $m < \omega$, since $i \neq j$, either $a_i \wedge a_m = 0$ or $b_j \wedge b_m = 0$. Hence $x_m \wedge y = 0$ for all $m < \omega$. But $0 \neq y \leq x$ so $x \wedge y'$ is an upper bound of $\{x_n: n < \omega\}$ which is strictly smaller than x.

This proposition generalizes Exercise 6N of Gillman and Jerison [12], where it is asserted that $P_\alpha + P_\alpha$ is not complete.

6.16 <u>Corollary</u>: The coproduct of two algebras is complete if and only if one is finite and the other complete.

<u>Proof</u>: This follows immediately from the proposition and the fact that if A is finite with n atoms, then $A + B \simeq B^n$.

Using the fact that $\mathcal{O}$ is closed under finite coproducts, we are able to prove the following:

6.17 <u>Proposition</u>: $\mathcal{E}_o$ is closed under finite coproducts.

<u>Proof</u>: Suppose that A and B do not have F_o as an epimorph. If f is an epimorphism of $A + B$ onto F_o with kernel K, let $I = K \cap A$ and

$J = K \cap B$ be the corresponding ideals in A and B. Let L be the ideal of A + B generated by I and J. Since L is contained in K, A + B/L has $A + B/K \simeq F_o$ as an epimorph. The epimorphism f maps A onto the subalgebra A/I of F_o. Hence A/I is countable, so the only atomless epimorph it could have is F_o. Because $A \in \mathcal{E}_o$, this cannot occur, so A/I is superatomic. Similarly, B/J is superatomic, and so, then, is their coproduct A/I + B/J. However, $A/I + B/J \simeq A + B/L$, and we have already shown that this algebra has F_o as an epimorph. This contradiction shows that $A + B \in \mathcal{E}_o$.

6.18 Corollary: Let X be the product of the Boolean spaces $X_1, \ldots X_n$. Then X has a subspace homeomorphic to the Cantor set $2^{\aleph_o}$ if and only if one of the X_i does.

§3. Coproducts and Semi-Simplicity

The results of this section indicate that coproducts are far more likely to be semi-simple than radical.

6.19 Theorem: Let $\mathcal{R}$ be a radical class and suppose the collection $\{A_i: i \in I\}$ contains at least one $\mathcal{R}$-semi-simple algebra. Then the coproduct of the A_i is $\mathcal{R}$-semi-simple.

Proof: Clearly, it suffices to show that if A is $\mathcal{R}$-semi-simple, then so is A + B for any B. By Prop. 1.25, we must show that any non-zero principal ideal of A + B has an $\mathcal{R}$-semi-simple epimorph, and it is clear that we can restrict our attention to non-zero elements of the form $a \wedge b$ where $a \in A$ and $b \in B$. But $(A + B)_{a \wedge b} \cong A_a + B_b$ and A_a, which is R-semi-simple, is a retract of this coproduct. Hence $A + B$ is R-semi-simple.

Any infinite coproduct of algebras is atomless; that is, O-semi-simple. We are able to obtain analogous results for any radical, provided we restrict ourselves to the coproduct of infinitely many copies of the same algebra.

6.20 Definition: Let κ be any cardinal and A any algebra. We write κA for the coproduct of κ copies of A and call it a κ-multiple of A.

6.21 Proposition: If A is any algebra and κ any infinite cardinal, then κA is unequivocal.

Proof: Let $X = S(A)$ and suppose M is a clopen subset of $X^{\kappa} = S(\kappa A)$. Then $M = \Pi(M_\alpha: \alpha < \kappa)$ where M_α is a clopen subset of X and $M_\alpha = X$ for all but a finite number of α. But then the partial product $\Pi(M_\alpha: M_\alpha = X)$ is a retract of M and is homeomorphic to X^{κ}. Algebraically, any non-zero principal ideal of κA has a retract isomorphic to κA. Thus the lower radical generated by any such principal ideal is the same as the lower radical generated by κA. In other words, κA is unequivocal.

6.22 Corollary: If A is not in a radical class R, then κA is R-semi-simple for any infinite κ.

<u>Proof</u>: Since A has a non-radical epimorph A, it cannot be in R. Being unequivocal, it is R-semi-simple.

6.23 <u>Corollary</u>: For any α, there is a superatomic algebra S such that $\aleph_0 S$ is F_α-semi-simple.

<u>Proof</u>: Let S be the finite-cofinite algebra on a set of cardinality $\aleph_{\alpha+1}$. Since every non-zero principal ideal of $\aleph_0 S$ has cardinality $\aleph_{\alpha+1}$, it must be F_α-semi-simple.

6.24 <u>Corollary</u>: Every algebra is a retract of an unequivocal algebra.

The last corollary is a generalization of the fact that every algebra is an epimorph of a free algebra. Grätzer [13] has announced a stronger result: for any algebra A, there is an algebra B such that A + B is homogeneous. Thus any algebra is a retract of a homogeneous one.

6.25 <u>Definition</u>: A radical class R is <u>proper</u> if it does not contain all Boolean algebras. Then it cannot contain all free algebras. For any proper radical class R, let $\sigma(R) = \aleph_\alpha$ where α is the least ordinal β such that $F_\beta \notin R$. Note that $\sigma(R)$ is always infinite.

6.26 <u>Theorem</u>: Let R be a proper radical class and let A be an algebra with more than two elements. Then κA is R-semi-simple for all $\kappa \geqslant \sigma(R)$.

<u>Proof</u>: Let $\kappa = \aleph_\alpha \geqslant \sigma(R)$. By §6, d) of the Preliminaries, κA has F_α

as a retract. By definition of $\sigma(\mathcal{R})$, $F_\alpha \notin \mathcal{R}$, so $\kappa A \notin \mathcal{R}$. By Cor. 6.22, κA is $\mathcal{R}$-semi-simple.

6.27 Examples: We list $\sigma(\mathcal{R})$ for known $\mathcal{R}$:

i) $\sigma(\mathcal{O}) = \aleph_0$,

ii) $\sigma(\mathcal{P}_\alpha) = \aleph_0$,

iii) $\sigma(\mathcal{P}) = \aleph_0$,

iv) $\sigma(\mathcal{E}_\alpha) = \aleph_\alpha$,

v) $\sigma(\mathcal{F}_\alpha) = \aleph_{\alpha+1}$,

vi) $\sigma(\mathcal{C}_\alpha) = \aleph_\alpha$,

vii) $\sigma(\mathcal{G}_\alpha) = \aleph_\beta$ where $\aleph_\beta = \aleph_\alpha^{\aleph_0}$,

viii) $\sigma(\mathcal{J}) = \aleph_0$,

ix) $\sigma(\mathcal{K}) = \aleph_1$.

CHAPTER SEVEN

THE LATTICE OF RADICALS

We can partially order Boolean algebra radicals by the relation of containment. If we extend the term "lattice" to include structures defined on classes as well as sets, we find that the Boolean algebra radicals form a lattice with some interesting algebraic properties.

§1. Lattice-Theoretic Preliminaries

7.1 Definition: An abstract algebra $\langle L; \vee, \wedge, *, 0, 1\rangle$ is called a pseudo-complemented distributive lattice (with 0 and 1) if $\langle L; \vee, \wedge, 0, 1\rangle$ is a distributive lattice (with 0 and 1) and $*$ is a unary operation on L satisfying $a \wedge b = 0$ if and only if $b \leq a^*$. Thus a^* is the maximum of the elements disjoint from a.

A more general concept is the following:

7.2 Definition: An abstract algebra $\langle L; \vee, \wedge, 0, 1\rangle$ is a Brouwerian lattice if it is a lattice (with 0 and 1) in which, for any $a, b \in L$, there is $c \in L$ such that $a \wedge x < b$ if and only if $x \leq c$. We denote the element c by $(b:a)$.

Setting $a^* = (0:a)$, we see that any Brouwerian lattice is pseudo-complemented, and it can be shown (see Birkhoff [4]) that any Brouwerian lattice is distributive.

In a complete lattice, an obvious candidate for $(b:a)$ is the supremum of all x such that $a \wedge x \leqslant b$. If the lattice also satisfies the infinite distributive law: $a \wedge \sup\{a_i: i \in I\} = \sup\{a \wedge a_i: i \in I\}$, then this will suffice to show that the supremum c in questions does indeed satisfy $a \wedge c \leqslant b$, and the lattice will be Brouwerian.

7.3 Proposition (see Lakser [18]): Let $\langle L; \vee, \wedge, *, 0, 1\rangle$ be a pseudo-complemented distributive lattice. Then for any $a, b \in L$:

a) i) $a \leqslant a^{**}$,

ii) $a \leqslant b$ implies $b^* \leqslant a^*$,

iii) $a^* = a^{***}$,

iv) $a = a^{**}$ if and only if $a = b^*$ for some $b \in L$,

v) $a = a^{**}$, $b = b^{**}$ implies $a \wedge b = (a \wedge b)^{**}$,

vi) $0^* = 1$, $1^* = 0$, $0 = 0^{**}$, $1 = 1^{**}$.

b) Let $L^* = \{a^*: a \in L\} = \{a \in L: a = a^{**}\}$, called the skeleton of L. Then $0, 1 \in L^*$ and L^* is closed under $\wedge$ and $*$. If we define $a \cup b = (a^* \wedge b^*)^* = (a \vee b)^*$, then L^* is closed under $\cup$, and $\langle L^*; \cup, \wedge, *, 0, 1\rangle$ is a Boolean algebra, which is complete if L is.

c) Let $D = \{a \in L: a^* = 0\}$, called the set of dense elements of L. Then D is a filter in L; that is, it is closed under $\wedge$ and larger elements.

7.4 Lemma: Let p be an atom in a pseudo-complemented distributive lattice L. Then:

i) for any $a \in L$, $p \leqslant a \vee a^*$, and

ii) $p \leqslant a$ if and only if $p \leqslant a^{**}$.

Proof: If p is not contained in a, then since it is an atom, $p \wedge a = 0$. Then $p < a^*$ and i) follows. If $p \leqslant a^{**}$ but p is not contained in a, then again we get $p \leqslant a^*$ so that $p \leqslant a \wedge a^{**} = 0$, contradicting the fact that atoms are non-zero. The other direction of ii) is obvious.

7.5 Proposition: Let L be a complete pseudo-complemented distributive lattice, and let t be the supremum of the atoms of L (we assume there are some). Let $L_t = \{a \in L: a \leqslant t\} = \{a \wedge t: a \in L\}$. For any $a \in L$, define $a^\circ = a^* \wedge t$. Then:

i) $\langle L_t; \vee, \wedge, {}^\circ, 0, t\rangle$ is a Boolean algebra; in fact, it is a power-set algebra,

ii) L_t is an epimorph of L^*, and the epimorphism is an isomorphism if and only if $t \in D$.

Proof: Note first that by Lemma 7.4, $t \leqslant a \vee a^*$ for any $a \in L$. Thus $t = (a \vee a^*) \wedge t = (a \wedge t) \vee (a^* \wedge t) = (a \wedge t) \vee a^\circ$ for any $a \in L$. If $a \in L_t$, then $a = a \wedge t$, so we get $t = a \vee a^\circ$ for all $a \in L_t$. Clearly, $a \wedge a^\circ = 0$, so L_t is a Boolean algebra. Since it is complete and atomic, it is a power-set algebra. Define $f: L^* \rightarrow L_t$ by $f(a) = a \wedge t$. Then f clearly preserves $\wedge$. Also, $f(a^*) = a^* \wedge t = a^\circ$. By Lemma 7.4,

$a \wedge p = a^{**} \wedge p$ for all $a \in L$. We extend this to t by showing that $a \wedge t = \sup \{a \wedge p\colon p \text{ is an atom}\}$. It is clear that $a \wedge t$ is an upper bound for this set. Suppose c is any other upper bound. Then for any atom p, $p \leq (a \vee a^*) \wedge (p \vee a^*) = (a \wedge p) \vee a^* \leq c \vee a^*$. But then $t \leq c \vee a^*$ and $a \wedge t \leq a \wedge (c \vee a^*) = a \wedge c \leq c$. Hence $a \wedge t$ is, in fact, the least upper bound of the set. It easily follows that $a \wedge t = a^{**} \wedge t$ for all $a \in L$. Thus $f(a \cup b) = (a \vee b)^{**} \wedge t = (a \vee b) \wedge t = (a \wedge t) \vee (b \wedge t) = f(a) \vee f(b)$. Furthermore, for any $a \wedge t \in L_t$, $f(a^{**}) = a \wedge t$. Hence f is an epimorphism. Clearly, a is in the kernel of f if and only if $a \wedge t = 0$ if and only if $a \leq t^*$. Thus f is an isomorphism if and only if $t^* = 0$; that is, $t \in D$.

7.6 Proposition: Let L be as in Prop. 7.5. Then L is atomic if and only if $t \in D$.

Proof: Suppose $t \in D$ and that a is a non-zero element of L containing no atoms. Then $a \wedge p = 0$ for all atoms p, so that $a \wedge t = 0$. But then $a \leq t^* = 0$. Conversely, if L is atomic and $t \notin D$, then there is an atom p contained in t^*. But then $p \leq t \wedge t^* = 0$.

Recalling that for any $a \in L$, $t = (a \wedge t) \vee (a^* \wedge t)$, we see that if $a \in D$, then $t \leq a$. Hence t is a lower bound for D, and in case $t \in D$, then D is the principal filter generated by t. In this case, L splits at t into a principal filter above t and a power-set algebra below it.

§2. The Lattice of Radicals for Associative Rings

Snider [25, 26], using results of Leavitt [19], gave the first account of the lattice of radical for associative rings. The class of such radicals forms a complete lattice under the natural ordering. The meet of any collection of radicals is their intersection and the join is the lower radical generated by their union. The join is also determined by its semi-simple class, which is the intersection of the semi-simple classes of the radicals in the collection. The class of hereditary radicals forms a complete sublattice of the lattice of radicals and is shown to satisfy an infinite distributive law which makes the remarks following Defn. 7.2 pertinent. We conclude that the lattice of hereditary radicals is Brouwerian, distributive, and pseudo-complemented. Snider shows that this lattice is atomic, the atoms being the lower radicals generated by a single simple ring. Hence, using Prop. 7.5 and Prop. 7.6, we can extend his results as follows:

7.7 Proposition: Let T be the lower radical of associative rings generated by the class of simple rings. Then:

i) T is hereditary and $T^* = 0$,

ii) the class of hereditary radicals contained in T form a power-set Boolean algebra under the natural order, and

iii) this algebra is isomorphic to the skeleton of the lattice of hereditary radicals.

Snider characterizes $(\mathcal{S}:\mathcal{R})$, but there is an intuitively more obvious candidate for it than he gives. Unfortunately, it is not, in general, hereditary. It will, however, yield a nice characterization of $(\mathcal{S}:\mathcal{R})$ in universal classes for which every radical is hereditary.

7.8 Proposition: Let $\mathcal{R}$ and $\mathcal{S}$ be hereditary radicals, let $\mathcal{M}$ be the class of $\mathcal{R}$-rings which are $\mathcal{S}$-semi-simple, and let $\mathcal{W}$ be $\mathcal{U}(\mathcal{M})$. If $\mathcal{W}$ is hereditary, then $\mathcal{W} = (\mathcal{S}:\mathcal{R})$.

Proof: Since $\mathcal{R}$ is hereditary and semi-simple classes are hereditary, the class $\mathcal{M}$ is hereditary and so determines an upper radical. First, suppose A is a ring in $\mathcal{R} \wedge \mathcal{W}$. $\mathcal{R}$ is closed under epimorphs, so by definition of $\mathcal{W}$, A can have no non-zero epimorph which is $\mathcal{S}$-semi-simple. But then $A \in \mathcal{S}$, so $\mathcal{R} \wedge \mathcal{W} \leq \mathcal{S}$. Now suppose $\mathcal{V}$ is a radical such that $\mathcal{R} \wedge \mathcal{V} \leq \mathcal{S}$. Let A be in $\mathcal{M}$. Then $0 = s(A) \geq r(A) \cap v(A) = A \cap v(A) = v(A)$. So A is $\mathcal{V}$-semi-simple. Since $\mathcal{W}$ is the largest radical for which $\mathcal{M}$-rings are semi-simple, $\mathcal{V} \leq \mathcal{W}$. Thus $\mathcal{W} = (\mathcal{S}:\mathcal{R})$.

7.9 Corollary: Let $\mathcal{R}$ be a hereditary radical and let $\mathcal{W}$ be the upper radical generated by $\mathcal{R}$. If $\mathcal{W}$ is hereditary, then $\mathcal{W} = \mathcal{R}^*$.

Proof: $\mathcal{R}^* = (0:\mathcal{R})$ is the upper radical determined by $\mathcal{R}$-rings which are semi-simple with respect to the zero radical; that is, the class $\mathcal{R}$.

§3. The Lattice of Radicals for Boolean Algebras

Snider's results can be appled immediately to radicals of Boolean rings, and since Boolean ring radicals are hereditary, we can use the descriptions of Prop. 7.8 and Cor. 7.9.

7.10 Proposition: The class of Boolean ring radicals forms a complete, Brouwerian, pseudo-complemented distributive lattice with extreme elements. If R and S are Boolean ring radicals, then $(S:R)$ is the upper radical generated by R-rings which are S-semi-simple, and R^* is the upper radical generated by R.

We recall that any non-zero radical class must contain the two-element Boolean algebra. Thus $R \wedge R^* = 0$ entails that either R or R^* must be 0, so that pseudo-complementation is trivial. However, this same fact means that we can discard 0 and the Boolean ring radical $L(\underline{2})$ will serve as a zero for the new lattice. In order to see that the lattice-theoretic properties are essentially unchanged, all we need do is verify the following:

7.11 Lemma: If R and S are non-zero Boolean ring radicals, then $(S:R)$ is non-zero.

Proof: Since R and S are non-zero, $\underline{2}$ is in both of them, and then $R \wedge L(\underline{2}) = L(\underline{2}) \leq S$. Then $L(\underline{2}) \leq (S:R)$.

Then Prop. 7.10 holds without change for <u>non-zero</u> Boolean ring radicals, except for the description of pseudo-complements, which now becomes $R^* = (L(\underline{2}):R)$; that is, R^* is now the upper radical generated by atomless R-rings.

7.12 <u>The Isomorphism Theorem</u>: Let Lat(B) be the class of non-zero radical classes of Boolean rings and Lat(A) the class of Boolean algebra radicals. Let f be the map which sends any non-zero Boolean ring radical R into the class of Boolean algebras in R. Then f is a one-to-one correspondence between Lat(B) and Lat(A) which preserves order in both directions, and so

$$\langle \mathrm{Lat}(B); \vee, \wedge, *, L(\underline{2}), B\rangle \simeq \langle \mathrm{Lat}(A); \vee, \wedge, *, 0, A\rangle.$$

<u>Proof</u>: By definition of radical classes of algebras, f is onto. Since we can recover R from f(R) as the class of Boolean rings, all of whose principal ideals are in f(R), f is one-to-one. The rest is obvious.

For the sake of completeness, we give a description of R^* and R^{**} for Boolean algebra radicals.

7.13 <u>Proposition</u>: Let R be a radical class of Boolean algebras. Then R^* is the class of all algebras with no atomless epimorphs in R, and R^{**} is the class of algebras A such that any atomless epimorph of A has an atomless epimorph in R.

7.14 Corollary: If A is atomless, then $L(A)^* \leq U(A)$.

Proof: If B has no atomless epimorph in $L(A)$, then it certainly has no principal ideal of A as an epimorph.

We now have all the notions required to state the conjecture mentioned in the discussion following Prop. 6.1. The conjecture is that if R is a radical class and $A = \Pi(A_i: i \in I)$ is a product of R-algebras such that $P(I) \in R$, then $A \in R^{**}$. Recalling that $P(I)$ is a retract of A, let M be the ideal of A such that $A/M \cong P(I)$. If J is any ideal of A such that A/J is atomless, let K(J) be the ideal of A generated by M and J. The conjecture would be proved if we could show that A/K(J) is infinite for any such J. (We use Pierce's result of the Preliminaries 3, j) and the definition of R^{**}.)

We also present some considerations related to the conjecture that $J = 0$. Suppose H_1 is a class of homogeneous algebras closed under epimorphs (hence containing $\underline{2}$) and that H_2 consists of all other homogeneous algebras. Then any radical R for which H_1-algebras are radical and H_2-algebras are semi-simple must satisfy $L(H_1) \leq R \leq U(H_2)$. Taking $H_1 = \{\underline{2}\}$, we see that the conjecture $J = 0$ is a special case of the conjecture that $L(H_1) = U(H_2)$. One can easily extend Prop. 7.8 to show that if $S = U(M)$, then $(S:R) = U(R \cap M)$, from which it follows that if R and S have the same homogeneous algebras in them, then

$(J:R) = (J:S)$. Thus we get:

7.15 Proposition: If $J = 0$ and R and S are radical classes with the same homogeneous algebras in them, then $R^* = S^*$.

§4. Dual Atoms and Complements in Lat(A)

Snider's proof [25] that the lattice of hereditary radicals for associative rings has no dual atoms can be considerably simplified for Boolean algebra radicals.

7.16 Proposition: If R is a proper radical class (that is, not every algebra is radical), then R is properly contained in a proper radical class.

Proof: Since R cannot contain all free algebras, let α be some ordinal such that $F_\alpha \notin R$. Then $F_\alpha \in F_\alpha \vee R$, so this is a radical class properly containing R. For any $\beta > \alpha$, F_β is not in R or in F_α. Being unequivocal, it is semi-simple with respect to both radicals, and so it is $F_\alpha \vee R$-semi-simple. Hence $F_\alpha \vee R$ is proper.

Snider [26] gives a characterization of complemented hereditary radicals which we can use to deduce that 0 and A are the only complemented elements of Lat(A). We choose to deduce this from the following stronger result:

7.17 Proposition: The supremum of any set of proper radical classes is proper.

Proof: Let $\{R_i : i \in I\}$ be a set of radical classes. For each $i \in I$, choose $\alpha(i)$ such that $F_{\alpha(i)} \notin R_i$. Since I is a set, the set of $\alpha(i)$ has a supremum α. Then F_α is R_i- semi-simple for all $i \in I$, and so is semi-simple with respect to the supremum of these radicals.

7.18 Corollary: $\mathcal{O}$ and $\mathcal{A}$ are the only complemented elements of Lat($\mathcal{A}$).

§5. Locating Known Radicals in Lat($\mathcal{A}$)

We have already obtained some lattice-theoretic relationships between our radicals, and they will not be repeated here as they are summarized in the diagram which comprises §7.

7.19 Proposition: For all α, $\mathcal{P}_\alpha{}^* = \mathcal{D}_o$. Also, $\mathcal{P}^* = \mathcal{D}_o$.

Proof: Let $\mathcal{Y}$ be any one of the following classes: atomless $\mathcal{P}_\alpha$-algebras, for any α, or atomless $\mathcal{P}$-algebras. Then $\mathcal{Y}$ satisfies the conditions of Prop. 4.8 and so the upper radical determined by $\mathcal{Y}$ is $\mathcal{D}_o$. But $\mathcal{R}^*$, for any radical $\mathcal{R}$, is the upper radical generated by atomless $\mathcal{R}$-algebras.

Corollary: $\mathcal{D}_o$ is in the Boolean algebra of skeletal elements of Lat($\mathcal{A}$). Furthermore, $\mathcal{P} \leq \mathcal{D}_o{}^*$ and for each α, $\mathcal{P}_\alpha \leq \mathcal{D}_o{}^*$

7.20 Proposition: Let β be the least ordinal such that $\exp \aleph_\beta > \exp \aleph_\alpha$. Then β is the least ordinal such that $\mathcal{P}_\alpha \leq \mathcal{D}_\beta$. In this case, the containment is proper.

Proof: By cardinality, P_α does not have P_β as an epimorph, but all smaller power-set algebras are epimorphs of P_α. If the radicals are equal, then $\mathcal{D}_o = \mathcal{D}_o \wedge \mathcal{D}_\beta = \mathcal{D}_o \wedge \mathcal{P}_\alpha = \mathcal{P}_\alpha{}^* \wedge \mathcal{P}_\alpha = \mathcal{O}$, which is a contradiction.

7.21 Proposition: Let β be the least ordinal such that $\exp \aleph_\beta > \exp \aleph_o$. Then for all $\alpha \geqslant \beta$, $\mathcal{D}_\alpha{}^* = \mathcal{O}$.

Proof: For any α, we have $\mathcal{D}_\alpha{}^* \leqslant \mathcal{D}_o{}^*$. If $\alpha \geqslant \beta$, then by Prop. 7.20, $\mathcal{P}_o \leqslant \mathcal{D}_\beta \leqslant \mathcal{D}_\alpha$ so that $\mathcal{D}_\alpha{}^* \leqslant \mathcal{P}_o{}^* = \mathcal{D}_o$. But then $\mathcal{D}_\alpha{}^* \leqslant \mathcal{D}_o \wedge \mathcal{D}_o{}^* = \mathcal{O}$.

7.22 Corollary: Let β be as above, and let $\gamma = \exp \aleph_\beta$. Then for $\alpha \geqslant \gamma$, $\mathcal{C}_\alpha{}^* = \mathcal{O}$.

Proof: This follows from the fact that $\mathcal{C}_\gamma = \mathcal{D}_\beta$.

7.23 Proposition: Let $\aleph_\beta = \aleph_\alpha{}^{\aleph_o}$. Then $\mathcal{C}_\alpha \leqslant \mathcal{G}_\alpha \leqslant \mathcal{C}_\beta$.

Proof: The first containment was proved in Prop. 4.6. Note that F_β has $\overline{F}_\alpha$ as an epimorph. Then if A has F_β as a subalgebra, it must, by injectivity of $\overline{F}_\alpha$, have $\overline{F}_\alpha$ as an epimorph. This proves the second containment.

7.24 Corollary: Let γ be as in Cor. 7.22. Then for all $\alpha \geqslant \gamma$, $\mathcal{G}_\alpha{}^* = \mathcal{O}$.

7.25 Proposition: Let $\aleph_\beta = \exp \aleph_\alpha$. Then $\mathcal{P}_\alpha \leqslant \mathcal{F}_\beta$.

Proof: By cardinality, $P_\alpha \in \mathcal{F}_\beta$.

7.26 <u>Proposition</u>: $\mathcal{F}_\alpha \leq \mathcal{C}_{\alpha+1}$.

<u>Proof</u>: Clearly, $F_\alpha \in \mathcal{C}_{\alpha+1}$.

7.27 <u>Corollary</u>: $\mathcal{F}_\alpha \leq \mathcal{D}_\alpha$.

<u>Proof</u>: Let $\mathcal{H}_\beta = \exp \mathcal{H}_\alpha \geq \mathcal{H}_{\alpha+1}$. Then $\mathcal{F}_\alpha \leq \mathcal{C}_{\alpha+1} \leq \mathcal{C}_\beta = \mathcal{D}_\alpha$.

7.28 <u>Proposition</u>: $\mathcal{C}_\alpha \leq \mathcal{E}_\alpha$.

<u>Proof</u>: F_α is $\mathcal{C}_\alpha$-semi-simple and $\mathcal{E}_\alpha$ is the largest radical for which this is true.

7.29 <u>Corollary</u>: $\mathcal{F}_\alpha \leq \mathcal{E}_{\alpha+1}$.

7.30 <u>Proposition</u>: $\mathcal{F}_\alpha{}^* \leq \mathcal{E}_\alpha$.

<u>Proof</u>: Apply Cor. 7.14.

7.31 <u>Proposition</u>: For $\alpha > 0$, $\mathcal{E}_\alpha{}^* = \mathcal{O}$.

<u>Proof</u>: By Cor. 7.29, $\mathcal{F}_o \leq \mathcal{E}_1 \leq \mathcal{E}_\alpha$. Then $\mathcal{E}_\alpha{}^* \leq \mathcal{F}_o{}^*$. By Prop. 7.30, $\mathcal{F}_o{}^* \leq \mathcal{E}_o \leq \mathcal{E}_\alpha$. Then $\mathcal{E}_\alpha{}^* \leq \mathcal{E}_\alpha$, and so $\mathcal{E}_\alpha{}^* = \mathcal{O}$.

7.32 <u>Proposition</u>: $\mathcal{F}_o{}^* = \mathcal{E}_o$.

<u>Proof</u>: By Prop. 7.30, $\mathcal{F}_o{}^* \leq \mathcal{E}_o$. Suppose A does not have F_o as an epimorph, and let B be an atomless epimorph of A. If $B \in \mathcal{F}_o$, then it has a countable principal ideal which would be isomorphic to F_o, contradicting the assumption on A. Hence A has no atomless epimorph in $\mathcal{F}_o$; that is, $\mathcal{E}_o \leq \mathcal{F}_o$.

7.33 Corollary: $\mathcal{D}_o^* \leq \mathcal{E}_o$.

Proof: By Cor. 7.27, $F_o \leq \mathcal{D}_o$.

7.34 Corollary: $\mathcal{E}_o$ is in the Boolean algebra of skeletal elements of Lat(A). So also is the radical $\mathcal{E}_o \wedge \mathcal{D}_o$, which is distinct from $\mathcal{E}_o$, $\mathcal{D}_o$, and $\mathcal{O}$.

Proof: The meet of skeletal elements is skeletal. Note that F_o is in $\mathcal{D}_o$ but not in $\mathcal{E}_o$, and P_o is in $\mathcal{E}_o$ but not in $\mathcal{D}_o$. The algebra Q of Example 3.13 is an atomless algebra in $\mathcal{E}_o \wedge \mathcal{D}_o$.

7.35 Proposition: K contains P_o and F_o. $K^* = \mathcal{O}$. If $\aleph_\gamma = \exp \aleph_o$, then $K \leq F_\gamma$, and for all $\alpha \geq \gamma$, $F_\alpha^* = \mathcal{O}$.

Proof: Since P_o and F_o are separable, the first assertion follows immediately. The second statement is simply a re-statement of Theorem 3.6. Since every separable algebra has cardinality at most $\aleph_\gamma$, we get that $K \leq F_\gamma$. Then if $\alpha \geq \gamma$, $K \leq F_\gamma \leq F_\alpha$ and $F_\alpha^* \leq K^* = \mathcal{O}$.

7.36 Corollary: If γ is as in Prop. 7.35, and $\alpha \geq \gamma + 1$, then $C_\alpha^* = \mathcal{O}$.

Proof: Use Prop. 7.26 and Prop. 7.35. This sharpens Cor. 7.22.

7.37 Corollary: If $\aleph_1 = \exp \aleph_o$, then $C_\alpha^* = \mathcal{O}$ for all $\alpha \geq 2$. Note that in this case, $C_1 = \mathcal{D}_o$, so Cor. 7.36 is a best possible result.

§6. Atoms in the Lattice

7.38 Theorem: Let R be an atom in Lat(A). Then:

i) for any non-superatomic algebra A ε R, $R = L(\mathrm{A})$,

ii) there is an atomless, separable algebra A in R, such that $R = L(\mathrm{A})$,

iii) any atomless algebra A in R is unequivocal, and

iv) for any atomless algebra A in R, $L(\mathrm{A})^* = U(\mathrm{A})$ so that $U(\mathrm{A})^* \neq 0$.

Conversely, if A is an atomless unequivocal algebra such that $U(\mathrm{A})^* \neq 0$, then $L(\mathrm{A})$ is an atom in the lattice of radicals.

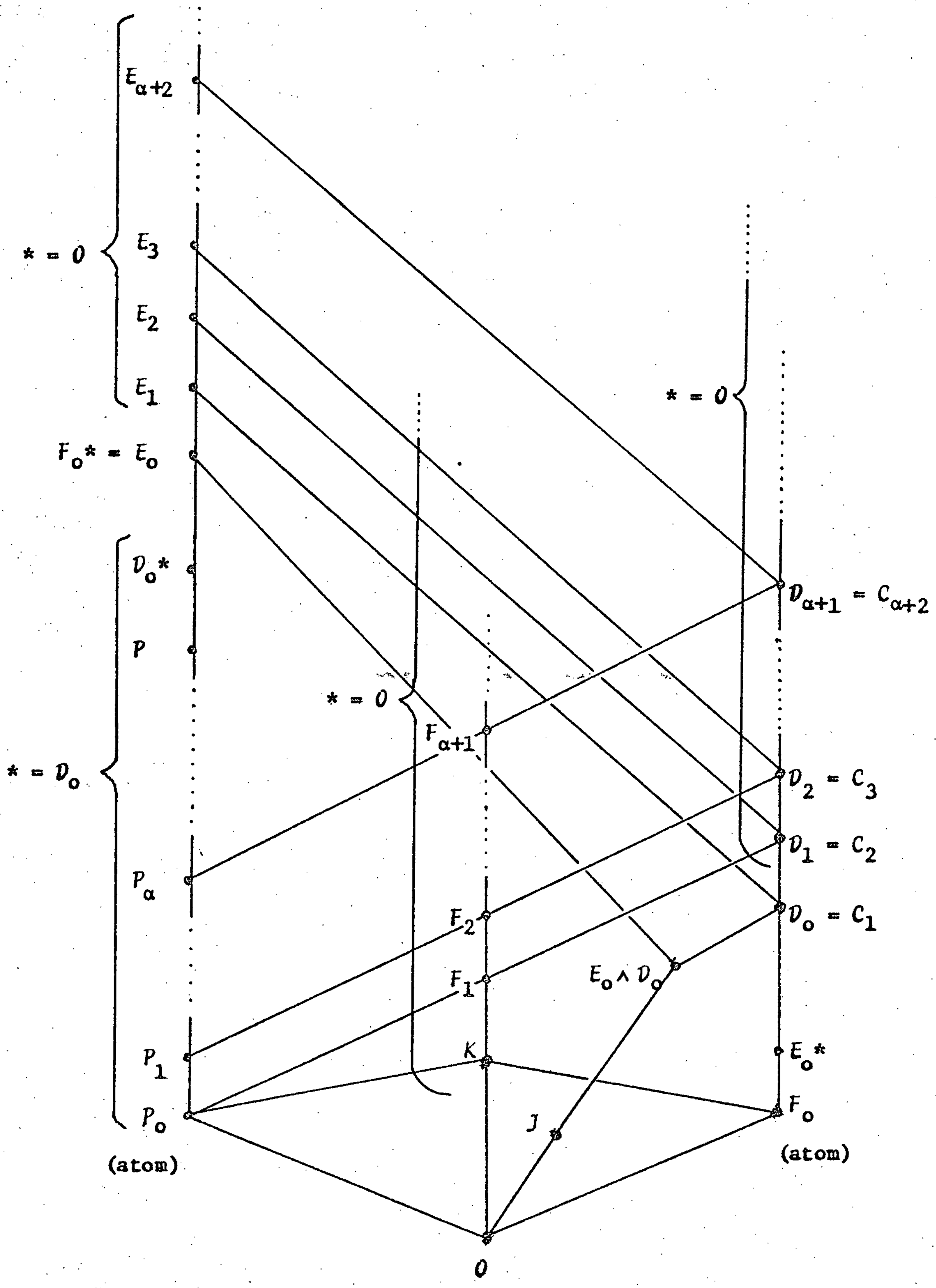
Proof: The first assertion is obvious, and ii) follows from the fact that any non-superatomic algebra has an atomless, separable epimorph. Using Prop. 2.15, iii) follows from i). If A is an atomless R-algebra, then $L(\mathrm{A})^* \leq U(\mathrm{A})$ by Cor. 7.14. Since $L(\mathrm{A})$ is an atom, either $L(\mathrm{A}) \leq U(\mathrm{A})$ or $L(\mathrm{A}) \wedge U(\mathrm{A}) = 0$. The first is impossible, and the second implies $U(\mathrm{A}) \leq L(\mathrm{A})^*$. Then iv) follows. For the converse, suppose A is unequivocal and $U(\mathrm{A})^* \neq 0$. If A were $U(\mathrm{A})^*$-semi-simple, we would have $U(\mathrm{A})^* \leq U(\mathrm{A})$, contradicting $U(\mathrm{A})^* \neq 0$. Thus, since it is unequivocal, A ε $U(\mathrm{A})^*$ and $L(\mathrm{A}) \leq U(\mathrm{A})^*$, so $U(\mathrm{A}) \leq U(\mathrm{A})^{**} \leq L(\mathrm{A})^* \leq U(\mathrm{A})$. Hence $U(\mathrm{A}) = L(\mathrm{A})^*$. Now let S be any radical and suppose A $\notin$ S. Then A is S-semi-simple, so $S \leq U(\mathrm{A}) = L(\mathrm{A})^*$ and $L(\mathrm{A}) \wedge S = 0$.

7.39 Theorem: $\mathcal{P}_o$ and $\mathcal{F}_o$ are atoms in Lat(A).

Proof: Write $\mathcal{P}_o = L(\overline{F}_o)$ and $\mathcal{F}_o = L(F_o)$. Then $\overline{F}_o$ and F_o are unequivocal. Also $\mathcal{U}(\overline{F}_o)^* = \mathcal{D}_o^* \neq 0$, and $\mathcal{U}(F_o)^* = \mathcal{E}_o^* \neq 0$. Hence, by the last theorem, $\mathcal{P}_o$ and $\mathcal{F}_o$ are atoms.

§7. A Diagram of the Lattice

On the next page, we present a diagram of Lat(A), which summarizes the results of the last two sections. For simplicity, we assume GCH and we omit mention of the G_α, which, by Prop. 7.23, are interspersed among the chain $\{C_\alpha\}$.


$E_{\alpha+2}$
$* = 0$
E_3
E_2
E_1
$F_o* = E_o$
D_o*
P
$* = D_o$
P_α
P_1
P_o
(atom)
$* = 0$
$F_{\alpha+1}$
F_2
F_1
K
J
0
$E_o \wedge D_o$
$* = 0$
$D_{\alpha+1} = C_{\alpha+2}$
$D_2 = C_3$
$D_1 = C_2$
$D_o = C_1$
E_o*
F_o
(atom)

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INDEX OF NOTATION

Page references are given where they might be helpful.

Ordinals: $\alpha, \beta, \gamma, \ldots$ Well-ordered chains: $\{I_\alpha\}$ (29), $\{h_\alpha(A)\}$ (29), $\{R_\alpha\}$ (63).

Cardinals: κ, $\aleph_\alpha$, $|A|$; $\kappa+$ = next largest cardinal after κ; $\exp \kappa = 2^\kappa$.

Rings and Algebras: A, B, C, ... ; $\dot{A}_x$ = principal ideal of A generated by x; $h^*(A)$ (29); $\overline{A}$ = normal completion of the algebra A (10-11); $\underline{2}$ = two-element Boolean algebra; F_α = free algebra on $\aleph_\alpha$ generators; P_α = power-set algebra on a set of cardinality $\aleph_\alpha$; Q (49).

Topological Spaces: X, Y, Z, ... ; βX = Stone-Cech compactification of the space X; βN = Stone-Cech compactification of a countable set with the discrete topology; 2^κ = product of κ copies of the two-element discrete space, a Cantor space; $2^{\aleph_0}$ = the Cantor set; S(A), S(x), S(I) = concepts associated with the Stone duality (6).

Constructions: Product of algebras: $\Pi(A_i: i \in I)$, $A \times B$, A^κ (8); weak product of algebras: $w\Pi(A_i: i \in I)$ (8); coproduct of algebras: $\Sigma(A_i: i \in I)$, $A + B$ (12), κA (75); product of toplogical spaces: $\Pi(X_i: i \in I)$, X^κ.

Lattice operations: (b:a), a* (78-79).

Cardinal properties: ν (51); ρ (63); π, δ(64); $\sigma(\mathcal{R})$ (76).

Classes of Rings and Algebras: $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{M}$, $\mathcal{H}$, ... ; radical classes with corresponding radical ideal: $\mathcal{R}$, r(A), $\mathcal{S}$, s(A), ... (18-19); lower radical: $\mathcal{L}(\mathcal{X})$ (28); upper radical $\mathcal{U}(\mathcal{Y})$ (45); $\mathcal{B}$ = the class of Boolean rings; $\mathcal{A}$ = the class of Boolean algebras; Lat($\mathcal{B}$) = the class of radical classes of Boolean rings; Lat($\mathcal{A}$) = the class of radical classes of Boolean algebras.

Radical Classes of Boolean Algebras:

$\mathcal{O}$ = the superatomic Boolean algebras (36),

$\mathcal{F}_\alpha$ = lower radical generated by F_α (38),

$\mathcal{K}$ = lower radical generated by separable algebras (40),

$\mathcal{P}_\alpha$ = lower radical generated by P_α (40),

$\mathcal{P}$ = lower radical generated by complete algebras (42),

$\mathcal{E}_\alpha$ = upper radical determined by F_α (48),

$\mathcal{G}_\alpha$ = upper radical determined by $\overline{F}_\alpha$ (48),

$\mathcal{J}$ = upper radical determined by atomless homogeneous algebras (50),

$\mathcal{J}_\nu$ = upper radical determined by atomless ν-homogenenous algebras (51),

C_α = algebras without F_α as a subalgebra (53),

$\mathcal{D}_\alpha$ = algebras without P_α as an epimorph (54).