SOME RESULTS IN THE THEORY OF RADICALS OF ASSOCIATIVE RINGS

by

ALLAN GEORGE HEINICKE

B.Sc., University of Manitoba, 1961

M.Sc., University of Manitoba, 1963

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department
of
MATHEMATICS

We accept this thesis as conforming
to the required standard

The University of British Columbia
December 1968

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and Study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date Feb 7/69

ABSTRACT

Supervisor: N.J. Divinsky

Several aspects of the theory of radical classes in associative ring theory are investigated.

In Chapter three, the Andrunakievic-Rjabuhin construction of radicals by means of annihilators of modules is employed to define several radical properties. One of these is shown to be the "weak radical" of Koh and Mewborn. The relations between these radicals, their properties and some of their applications to the study of classical quotient rings are investigated.

In Chapter four, the ideals of a ring K of the form $R(K)$, for a hereditary radical, R, are studied. A closure operation on the lattice of ideals is introduced, and the "closed" ideals are precisely the ideals of this type. It is proved that the ascending and descending chain conditions on the closed ideals of a ring imply that the ring has only a finite number of closed ideals.

In Chapter five, finite subdirect sums of rings are studied. The properties of hereditary radicals and of the various structure spaces, in a situation where one has a finite subdirect sum of rings, are investigated.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The author wishes to express his thanks to his research supervisor, Dr. N.J. Divinsky, for encouragement and advice tendered during the execution of the research herein described.

The author also wishes to thank Dr. T. Anderson for many helpful conversations.

The financial support of the H.R. MacMillan family, and of the National Research Council of Canada is gratefully acknowledged.

INTRODUCTION

In this thesis, by a ring, we mean an associative ring, not necessarily possessing a unity element. When K is a ring, a K-module will be a right K-module, and will usually be denoted by M or, when there is danger of ambiguity, by M_K. Of course, a ring K may be regarded as a right K-module, and, in this case, the submodules are just the right ideals of K.

If $f: L_1 \to L_2$ is a homomorphism of rings (resp. of right K-modules), we say that f is a monomorphism if $f(x) = f(y)$ implies $x = y$, and we say that f is a sur surjection (or a surjective mapping) if, for every element w of L_2 there is an element x in L_1 such that $f(x) = w$.

We are mainly concerned, in this thesis, with various aspects of radical theory. In Chapter one, the definitions and basic properties of radicals are discussed. And throughout the thesis, proofs of results are seldom given if the result is proved in the book by Divinsky (6). However, in such cases, explicit references are given to direct the reader to a proof.

It is well known that the Jacobson radical of a ring K can be described in terms of right K-modules. This result was generalized by Andrunakievic and Rjabuhin, who showed that any radical property can be described in terms of modules. The first part of Chapter two is devoted to an exposition of these results. The work in Chapter two then turns to a generalization (Theorem 2.2.1) of the well-known result that the Jacobson radical of a ring K is the intersection of the maximal modular right ideals of K.

In recent years, ring theory, from the point of view of radical theory, has diverged greatly from the other directions that have been taken. An attempt to bridge this gap is made in Chapter three. In this chapter, the techniques discussed in Chapter two are applied to define a number of new radical properties. The relations between these radicals and some of the more classical radicals are discussed, and two results (Theorems 3.7.1 and 3.7.2) given generalizations of the Jacobson density theorem. Also it is shown that one of the radicals discussed here coincides with the "weak radical" of Koh and Mewborn (2).

Chapter four marks a return to general radical theory. A closure operation on the lattice of two-sided ideals of a ring K is introduced for which the "closed" ideals of K are those ideals of the form H(K), where H is a hereditary radical. In Theorem 4.4.3 it is shown that a ring K has the ascending and descending chain conditions for closed ideals if and only if K possesses a finite number of closed ideals. Some attention is also given to the case where a ring is determined by its minimal closed ideals.

In the first part of Chapter five, a slight generalization of some results of Andrunakievic (1) is given. This leads to representations of certain rings as finite subdirect sums of prime rings. In the latter part of Chapter five, finite subdirect sums in general are examined. Let a ring K be a subdirect sum of the rings $K_1, K_2, \ldots K_n$, and let $S = \bigoplus_{i=1}^{n} K_i$. The relations between $H(K)$ and $H(S)$, where H is a hereditary radical, and the relations between the various structure spaces of K and of S are investigated (Theorems 5.4.5 and 5.5.1).

Throughout the thesis, definitions are indicated by underlining of the term being defined. Other things, such as theorems, corollaries, and examples are numbered by three integers. For example, Example 5.3.3 is found in Chapter five, §3, immediately following Lemma 5.3.2.

Most of the notation used is standard. A few exceptions are the following. If K is a ring, and S a subset of K, $|S>_K$ is the right ideal of K generated by S, and $<S>_K$ (or simply $<S>$ if there is no danger of ambiguity) denotes the two-sided ideal of K generated by S.

Also, suppose that F is a set whose members are all subsets of a given set S. If F is void, define $\prod F$ to be S, and otherwise define $\prod F$ to be $\bigcap_{X \in F} X$. In either case we also define $\coprod F$ to be $\bigcup_{X \in F} X$. Thus, for example, the well-known characterization of the Jacobson radical of a ring K as the intersection of all the modular maximal right ideals of K takes the form $J(K) = \prod F$, where $f = \{I$: I is a modular maximal right ideal of $K\}$. If F is void, then (both in fact and in this notation) $J(K) = K$.

CHAPTER ONE

GENERAL RADICAL THEORY

1.1 Radical Properties and Radical Classes

A class R of associative rings is called a radical class if the following conditions are satisfied:

(A) Any homomorphic image of a member of R is also in R,

(B) Any ring K has an ideal $R(K)$ which, as a ring, is in R, and which is maximum among the ideals of K which are in R.

(C) For any ring K, $K/R(K)$ has only one ideal in the class R, namely the ideal 0. (In other words, $R(K/R(K)) = 0$.)

As is well known, the classes J, consisting of all right quasi-regular rings, and N, consisting of all nil rings, are radical classes. On the other hand, the class of all nilpotent rings is not a radical class.

For other examples of radical classes, we refer to Divinsky (6). Also, for some of the elementary properties of radical classes, we refer to Chapter one of the same reference.

Many authors refer to a "radical property", rather than to a "radical class". If P is a property of rings, then P is a radical property iff the class S of all rings with property P is a radical class. Conversely, if R is a radical class, then the property "belonging to R" is a radical property. For example, "nil" is a radical property, while "nilpotent" is not.

If R is a radical class, and if K is a ring which belongs to R, then $R(K) = K$, and we call K an <u>R-radical ring</u>. If, on the other hand, $R(K) = 0$, then we say that K is <u>R-semisimple</u>.

For some of the more familiar radical classes, R-semisimple is sometimes given another name. For example, if J is the Jacobson radical class, consisting of all right quasi-regular rings, a ring K for which $J(K) = 0$ is sometimes referred to as a semiprimitive ring. Again, if B is the Baer Lower Radical (see Divinsky (6), sec. 3.3), then the B semisimple rings are called semiprime rings.

1.2 The Lower Radical Construction

Given a class of rings, M, it is reasonable to ask, if M is not itself a radical class, whether M is contained in a radical class, and, if so, whether there is a minimal or minimum radical class containing M. The answer to the first question is "yes", for the class of all rings is clearly a radical class.

The answer to the second question is also "yes". We shall show that, given any class M of rings, there is a minimum radical class containing M. This will be denoted

$S_o(M)$, and will be called the <u>lower radical class with respect to M</u>. The proof of the existence of $S_o(M)$ is due to Kurosh (16). The present construction is due to Sulinski, Anderson, and Divinsky (6).

The construction is achieved as follows. Suppose we are given a class of rings M. Define M_1 to be the class of all rings which are homomorphic images of members of M. Given any ordinal $\alpha > 1$, if M_β has been defined for all ordinals $\beta < \alpha$, then define M_α to be the class of all rings K for which every non-zero homomorphic image of K has a non-zero ideal in M_β, for some $\beta < \alpha$. It is easily seen that, for each ordinal α, M_α is homorphically closed (i.e. each homomorphic image of a member of M_α is in M_α), and also that, if α and γ are ordinals, where $\alpha < \gamma$, then $M_\alpha \subseteq M_\gamma$. Define $S_o(M) = \cup M_\alpha$, the union being taken over all ordinals.

Before proving that $S_o(M)$ is indeed the minimum radical class containing M, we quote the following lemma, whose proof is given in Divinsky (6). p.4.

<u>Lemma 1.2.1</u> A class R of rings is a radical class if and only if it satisfies the following conditions:

(A) Any homomorphic image of a member of R is in R

(D) If K is a ring such that every non-zero homomorphic image has a non-zero ideal in R, then K is itself a member of R.

With this lemma, we can now prove the following theorem.

<u>Theorem 1.2.2</u> $S_o(M)$ is a radical class which contains M. Furthermore, if R is any radical class containing M, R contains $S_o(M)$.

<u>Proof</u>: Clearly we have $M \subseteq S_o(M)$. Furthermore, since each M_α is homomorphically closed, $S_o(M)$ has condition (A).

Suppose K is a ring for which every non-zero homomorphic image has a non-zero ideal in $S_o(M)$. For each ideal I of K for which $I \neq K$, the factor ring K/I has a non-zero ideal in $S_o(M)$. For each such ideal I, we choose an ordinal α_I such that K/I has a non-zero ideal in M_{α_I}. Since the collection of all ideals of K is a proper set, there is an ordinal β such that $\beta > \alpha_I$ for all I. Now let K' be any non-zero homomorphic image of K. Then $K' \cong K/I$ for some ideal I of K. Since K/I has a non-zero ideal in $M_{\alpha_I} \subseteq M_\beta$, the image of this ideal under the isomorphism is a non-zero ideal of K' which is also in M_β. Thus we see that any non-zero homomorphic image of K has a non-zero ideal in M_β, which gives $K \in M_{\beta+1} \subseteq S_o(M)$. We have shown that

$S_o(M)$ satisfies condition (D), and thus is a radical class.

Now suppose that R is a radical class, and that $M \subseteq R$. Then, since R is homomorphically closed, $M_1 \subseteq R$. Suppose that α is an ordinal, and suppose that, for all $\beta < \alpha$, we have that $M_\beta \subseteq R$. Then, from the definition of M_α, and from the fact that R has condition (D), we obtain at once that $M_\alpha \subseteq R$. Thus $S_o(M) \subseteq R$. This proves the theorem.

It is perhaps worthwhile to note that $S_o(M)$ was shown to be a radical class without knowing whether or not the associated sequence of M_α's terminated in the sense that, for some ordinal α, $M_\alpha = M_{\alpha+1}$ (from which it follows that $M_\alpha = M_\beta$, for all $\beta \geq \alpha$). In Sulinski, Anderson, and Divinsky, (24), it was shown that this construction does indeed terminate, and that it terminates at ω_o, the first infinite ordinal. Thus, no matter what class M is taken, it is true that $S_o(M) = M_{\omega_o}$. Of course, it may be true that we have $S_o(M) = M_k$, where k is a finite ordinal, for some choices of the class M. In particular, if M is itself a radical class, then $S_o(M) = M_1 = M$. For M equal to the class of all nilpotent rings, denote $S_o(M)$ by B. This is the well-known Baer Lower radical. In Sulinski, Anderson, and Divinsky (24), it was shown that $B = M_2$.

The author has shown (Heinicke (10)) that there exists a class M for which the bound ω_o is attained. That is to say, there is a class M for which $S_o(M) \neq M_k$ for any finite ordinal k.

1.3 The Upper Radical Construction

A question which is "dual" to the one raised in §1.2 is: when can a given class of rings be the class of R-semisimple rings for some radical class R, and, if the given class C is not itself a semisimple class, is there a maximum radical class R, for which any member of C is R-semisimple?

In Chapter one of Divinsky (6) it is shown that a class C is the class of R-semisimple rings for a radical class R if and only if C satisfies both:

(E) Every non-zero ideal of a member of C can be homomorphically mapped onto a non-zero member of C, and

(F) If K is a ring for which every non-zero ideal can be homomorphically mapped onto a non-zero member of C, then K is itself in C.

Furthermore, it is also shown that, if C is a class of rings with (E), then the class $\overline{C}$, defined as the class of all rings K for which every non-zero ideal can be mapped onto a non-zero member of C, has both (E) and (F), and is therefore the class of R-semisimple rings for some radical property R. It turns out that R consists of all rings K

which cannot be mapped onto a non-zero member of C. Also, R is the largest radical class S for which every member of C is S-semisimple. The class R, defined in this manner, is the upper radical class with respect to C.

It might be pointed out that, given any class C of rings, it is possible to enlarge it to obtain a class C' which has (E) in such a way that the upper radical with respect to C' is, in fact, the largest radical class S for which all members of C are S-semisimple. For theorem 47 of Divinsky (6) shows that if I is an ideal of K, then, for any radical class R, $R(I)$ is also an ideal of K. An immediate corollary is that if K is R-semisimple, and if I is an ideal of K, then I is also R-semisimple. Thus, for any radical class R, the class of R-semisimple rings is closed under the taking of ideals. (Henceforth, a class closed under the taking of ideals will be called a hereditary class.) Clearly, any hereditary class has (E).

If K is a ring, and if S is a subring, we say that S is accessible to K if there is a finite chain

$$S \lhd S_1 \lhd S_2 \dots \lhd S_n = K,$$

(where $A \lhd B$ means that A is an ideal of B). If C is any class of rings, define C' to be the class of all rings isomorphic to an accessible subring of a member of C. It is simple to verify that C' is a hereditary class containing C. If R is the upper radical with respect to C', then any member of C is R-semisimple. Also, if S is a radical class for which every member of C is S-semisimple, it follows that any accessible subring of a member is also S-semisimple, and thus every member of C' is S-semisimple. Therefore, $S \subseteq R$. For this reason we are justified in referring to R as the <u>upper radical with respect to C</u>.

1.4 Hereditary Radicals

Many of the familiar radical classes, for example J, N, and B, are known to be hereditary classes. However, not all radical classes are hereditary (see Divinsky (6), p. 10). It is known, (see Divinsky, (6), p. 125), that a radical class R is hereditary if and only if, for any ideal of a ring K, $R(I) = I \cap R(K)$.

The following is due to Hoffman and Leavitt (12).

<u>Theorem 1.4.1</u> If M is a hereditary class, then $S_o(M)$ is a hereditary radical class.

<u>Proof</u>: It suffices to show that each M_α is hereditary. Let $K \in M_1$ and let I be an ideal of K. Then there is a homomorphism θ from a member L of M onto K. Now $\theta^{-1}(I)$ is an ideal of L, and, since M is hereditary, $\theta^{-1}(I)$ is in M. Then $I = \theta(\theta^{-1}(I))$ is in M_1, and M_1 is hereditary.

Let α be an ordinal > 1, and suppose that M_β is hereditary for all $\beta < \alpha$. Let $K \in M_\alpha$, and let I be an ideal of K, and let ψ be any non-zero homomorphism of I. If J is the kernel of ψ, then $\psi(I) \cong I/J$. We shall show that I/J has a non-zero ideal in M_λ for some $\lambda < \alpha$.

Let U be an ideal of K which is maximal with respect to having $I \cap U \subseteq J$. (Such a U exists by Zorn's lemma.) If $U = K$, then $I \subseteq J$, and ψ is a zero map, contrary to our assumption. Thus, $U \neq K$. Since K (and therefore K/U) is in M_α, K/U has a non-zero ideal W/U in M_λ for some $\lambda < \alpha$. W is strictly larger than U, and hence $W \cap I \not\subseteq J$. Therefore, $(W \cap I + J)/J$ is a non-zero ideal of I/J. We show that $(W \cap I + J)/J$ is in M_λ.

If $W \cap I \subseteq U$, then $W \cap I \subseteq U \cap I \subseteq J$, a contradiction. Therefore, $W \cap I \not\subseteq U$, and $(W \cap I + U)/U$ is a non-zero ideal of $W/U \in M_\lambda$. Since M_λ is hereditary, $(W \cap I + U)/U$ is in M_λ. Therefore, $(W \cap I)/(W \cap I \cap U)$ is in M_λ. Since $W \cap I \cap U \subseteq W \cap J$, we can map $(W \cap I)/(W \cap I \cap U)$ homomorphically onto $(W \cap I)/(W \cap J)$, and the latter is then in M_λ. Since $(W \cap I + J)/J \cong (W \cap I)/(W \cap I \cap J) = (W \cap I)/(W \cap J)$, we have that $(W \cap I + J)/J$ is in M_λ, as desired.

We have shown that M_1 is hereditary, and also that if M_λ is hereditary for all ordinals $1 < \lambda < \alpha$, then M_α is hereditary. It follows by transfinite induction that, for each ordinal $\alpha \geq 1$, M_α is hereditary. Therefore $S_o(M)$

is hereditary. Q.E.D.

The converse of this theorem is not true. In Michler (20) it is shown that the class of weakly regular rings is an example of a non-hereditary class whose lower radical is hereditary.

1.5 Supernilpotent, SP, and Dual Radicals

In many parts of ring theory, prime rings and prime ideals play an important role. An ideal I of a ring K is a prime ideal if $I \neq K$ and, if $aKb \in I$, then either $a \in I$ or $b \in I$. This latter condition is well known to be equivalent to: if A and B are both right (or both left, or both two-sided) ideals of K, and if $AB \subseteq I$, then either $A \subseteq I$ or $B \subseteq I$. A ring K is a prime ring if and only if 0 is a prime ideal. It is well known that the class of prime rings is a hereditary class, and one can then form the upper radical with respect to the class of prime rings. A well-known result of Levitzki (see chapter 3 of Divinsky (5)) says that this upper radical is the same as the Baer Lower radical class. This, it will be recalled, is the lower radical with respect to the class of nilpotent rings. Furthermore, Levitzki showed that, for any ring K, $B(K)$ is the intersection of all the prime ideals of K.

This sort of a result, where the R-radical of an arbitrary ring is the intersection of certain ideals is quite common in radical theory. This occurs, for example with the so-called SP-radicals, (called special radicals in the literature.)

A class of rings is said to be a <u>special class</u> if:

(X) Every member is a prime ring.

(Y) The class, together with the one-element ring 0, is a hereditary class.

(Z) If A is in the class, and if A is an ideal in K, then K/A^* is in the class, where $A^* = \{x \in K: \ Ax = xA = 0\}$.

Since a special class together with 0, is hereditary, the upper radical with respect to a special class C is the class of all rings which cannot be mapped onto a non-zero member of C. It was shown by Andrunakievic (1) (see also Divinsky (6), chapter 7) that such an upper radical is hereditary. Since any special class consists of prime rings, such an upper radical class must contain B, the Baer Lower radical. A radical that contains B and is hereditary is called a <u>supernilpotent radical</u>. Andrunakievic also proved that, for any ring K, if R is the upper radical with respect to a special class C, then $R(K) = \prod \{I: I \triangleleft K,$ and $R/I \in C\}$. This last statement implies that an R-semisimple ring is a subdirect sum of rings from the special class C. We shall call an upper radical with respect to a special class an <u>SP-radical class</u>. (In the literature, these radicals are called "special radicals". We prefer to avoid this term, because to use it would necessitate talking about "special radical classes" which are not special classes.)

A number of questions can be asked about SP-radicals. Since every SP-radical class is supernilpotent, the most obvious question is: is the converse true? Recently, Rjabuhin (22) claimed to have shown that the converse is not true. The author has not had access to this result, and cannot verify or disprove it.

A question more easily answered is: can distinct special classes define the same SP-radical? It is known (see Divinsky (6), chapter 7) that the class of right primitive rings is a special class which gives rise to the Jacobson radical. Since the Jacobson radical is right-left symmetric, the class of left primitive rings is also a special class whose upper radical is the Jacobson radical. Bergman (4) has shown that these are distinct special classes.

A ring is <u>subdirectly irreducible</u> if the intersection of the non-zero ideals is non-zero. In a subdirectly irreducible ring K, this intersection is called the <u>heart</u> of K. The heart H of a subdirectly irreducible ring is a unique minimal two-sided ideal, and either $H^2 = H$, or $H^2 = 0$. In the former case the heart is said to be <u>idempotent</u>. It is known that if $H^2 = H$, then H is a simple ring. Also, a subdirectly irreducible ring with an idempotent heart is a prime ring. (The proofs for these assertions, and for the ones that follow, can be found in chapter 7 of Divinsky (6).)

Suppose that P is a property of rings which is invariant under isomorphisms. That is, if $S \cong T$, then S has P if and only if T has P. Then it is true that the class of all subdirectly irreducible rings which have idempotent hearts, and whose hearts have P, is a special class. If we denote such a special class by C, then the upper radical with respect to C is called a <u>dual radical</u>.

Let R be any supernilpotent radical class. Define C_1 to be the class of all prime R-semisimple rings, and define C_2 to be the class of all subdirectly irreducible rings with idempotent hearts, whose hearts are R-semisimple. These are both special classes, and $C_2 \subseteq C_1$. If the corresponding upper radicals are denoted by R_φ, corresponding to C_2, and R_s, corresponding to C_1, then the following statements are true:

(1) $R \subseteq R_s \subseteq R_\varphi$

(2) R_s is the smallest special radical containing R.

(3) R_φ is the smallest dual radical containing R.

These results are due to Andrunakievic (1). Proofs may also be found in Divinsky (6).

CHAPTER TWO

RADICALS AND MODULES

2.1 Defining Radicals by Classes of Modules

It is well known that there are two ways to describe the Jacobson radical. The first method could be considered as "internal" or "element-wise". In this approach, the Jacobson radical class is the class of all right quasi-regular rings. (A ring is right quasi-regular if, for each x there is a y such that $x + y - xy = 0$.) The second method, an "external" method, concentrates on the right primitive rings. A ring is said to be right primitive if there is a simple right K-module M such that $(0:M) = \{x \in K: Mx = 0\} = 0$.

Suppose, for any ring K, we define Σ_K to be the class of simple right K-modules M for which $MK \neq 0$. (In the sequel, we shall call such a module an irreducible module.) Then, as is well known, the Jacobson radical of any ring K can be written $J(K) = \prod \{(0:M) : M \in \Sigma_K\}$.

In general, if A and B are subsets or a right K-module M, we define $(A:B) = \{k \in K : Bk \subseteq A\}$. If A is a submodule of M, then $(A:B)$ is a right ideal of K. If B is also a submodule of M, then $(A:B)$ is a two-sided ideal of K. Also, if I is a left ideal of K, we denote by I^M the set of all $m \in M$ such that $mI = 0$. This is a submodule of M.

Suppose that M is an irreducible right K-module. Then, for any non-zero m in M, $mK \neq 0$. (For if $mK = 0$, $m \in K^M$, and $K^M = M$, whence $MK = 0$, which is false.) Since M is simple, $MK = M$. We then have a homomorphism $(\theta(k) = mk)$ $\theta : K \to M$ of K-modules which is surjective. Any irreducible right K-module is therefore isomorphic to a module of the form K/I, where I is a right ideal of K. Such a right ideal must be a maximal right ideal, since M is simple, and, as is well known, must also be a modular right ideal. (A right ideal I of K is <u>modular</u> if and only if there is an element e in K, e not in I, such that $\{ex-x : x \in K\} \subseteq I$. If K has a left unity element, then every right ideal is modular.)

If M is any right K-module, then $(0:M) = \bigcap\{(0:m) : m \in M\}$. If M is irreducible, each $(0:m)$, where $m \neq 0$, is a maximal modular right ideal. We then obtain the result that, for any ring K, $J(K) = \bigcap\{(0:M) : M \in \Sigma_K\} = \bigcap\{(0:m) : 0 \neq m \in M, M \in \Sigma_K\}$. This shows that $J(K)$ can be expressed as an intersection of maximal modular right ideals. On the other hand, if I is a maximal modular right ideal, then K/I is an irreducible right K-module. It is well known that $(0:K/I) \subseteq I$. Since $J(K) \subseteq (0:K/I)$, we have that $J(K) \subseteq I$. Therefore, $J(K) \subseteq \bigcap\{I : I$ is a maximal modular right ideal of $K\}$. Combining this with the result above, we can conclude that $J(K) = \bigcap\{I : I$ is a maximal modular right ideal of $K\}$. In this chapter, we shall see how to generalize this to certain other radicals.

Suppose that $\varphi : K \to K'$ is a ring homomorphism. If M' is a right K'-module, then we can give M' a right K-module structure by defining $m' * k = m' \cdot \varphi(k)$, for all m' in M' and k in K. Also, if φ is a surjection, then any right K-module M for which $(0:M) \supseteq \ker(\varphi)$ can be given a right K'-module structure as follows: if $m \in M$ and $k' \in K'$, choose k in $\varphi^{-1}(k')$ and define $m * k' = m \cdot k$. This is well defined, since if $\varphi(k_1) = \varphi(k_2) = k'$, then $k_1 - k_2 \in \ker(\varphi) \subseteq (0:M)$, and $m \cdot k_1 = m \cdot k_2$. It is easily verified that, if φ is a surjection, then, in each of the two cases described above, the lattices of submodules of the original and of the induced module are isomorphic. Also, the induced module is trivial if and only if the original module is trivial.

In 1964, Andrunakievič and Rjabuhin (2), considered the following situation. Suppose that to every ring K there is assigned a (possibly empty) class Σ_K of right K-modules satisfying the following conditions:

(P.0) If $M \in \Sigma_K$, then $MK \neq 0$.

(P.1) If $B \triangleleft K$, and $M \in \Sigma_{K/B}$, then M, with the induced K-module structure described above, is in Σ_K.

(P.2) If $M \in \Sigma_K$, and $B \triangleleft K$ such that $B \subseteq (0:M)$, then M, as a K/B-module, is in $\Sigma_{K/B}$.

(P.3) If $0 = \prod\{(0:M) : M \in \Sigma_K\}$, and if B is a non-zero ideal of K, then $\Sigma_B \neq \emptyset$.

(P.4) If, for each non-zero ideal B of K, $\Sigma_B \neq \emptyset$, then $0 = \prod\{(0:M) : M \in \Sigma_K\}$.

Theorem 2.1.1

(1) If, for each ring K there is assigned a class Σ_K of right K-modules such that P.0 - P.3 are satisfied, then the class S, consisting of all rings K for which $\Sigma_K = \emptyset$, is a radical class.

(2) Under the assumptions of (1), condition P.4 is satisfied if and only if, for each ring K, $S(K) = \prod\{(0:M) : M \in \Sigma_K\}$.

Proof: (1) (This proof is due to I. Hentzel (10)). For each ring K, define $\ker(\Sigma,K) = \prod\{(0:M) : M \in \Sigma_K\}$. If $\ker(\Sigma,K) = 0$, we call the class Σ_K faithful. Also, define $L(\Sigma)$ to be the class of rings K for which there exists an $M \in \Sigma_K$ such that $(0:M) = 0$. (A K-module M for which $(0:M) = 0$ will be called a faithful module. Thus, if Σ_K has a faithful module, the class Σ_K is faithful.)

Recall, from §1.3, that if a class C of rings satisfies:

(E) Every non-zero ideal of a member of C can be mapped onto a non-zero member of C

then the upper radical with respect to C consists of all rings which cannot be mapped onto a non-zero member of C. We shall show that the class $L(\Sigma)$ satisfies (E). Suppose that K is in $L(\Sigma)$, and that $B \neq 0$ is an ideal of K. Since K is in $L(\Sigma)$, Σ_K is clearly faithful, and so, by P.3, there is an M in Σ_B. Let $I = (0:M) \triangleleft B$. Then $I \neq B$, and by P.2, M, as a B/I-module, is in $\Sigma_{B/I}$. An easy calculation shows that, in the ring B/I, $(0:M)_{B/I} = 0$. Thus B/I is a non-zero member of $L(\Sigma)$.

From the discussion in §1.3, the class S, consisting of all rings which cannot be mapped onto a non-zero member of $L(\Sigma)$ is a radical class. If K is a ring such that $\Sigma_K \neq \emptyset$, and if M is in Σ_K, the same argument as was used for B in the previous paragraph shows that $K/(0:M)$ is in $L(\Sigma)$. Conversely, if $I \triangleleft K$, and if $K/I \in L(\Sigma)$, then there is a right K/I module M in $\Sigma_{K/I}$. From P.1 we have that M, with the induced K module structure is in Σ_K. Thus we have that $\Sigma_K \neq \emptyset$ if and only if K can be mapped onto a non-zero member of $L(\Sigma)$, or, equivalently, the upper radical class with respect to $L(\Sigma)$ consists of all rings K for which $\Sigma_K = \emptyset$.

If $S(K) \nsubseteq \ker(\Sigma, K)$, then, for some $M \in \Sigma_K$, $S(K) \nsubseteq (0:M)$. Denoting $(0:M)$ by B, we have that $(S(K)+B)/B$ is a non-zero ideal of $K/B \in L(\Sigma)$. Thus, by condition (E), $(S(K)+B)/B$ can be mapped to a non-zero member of $L(\Sigma)$. Since $(S(K)+B)/B \cong S(K)/(S(K) \cap B)$, and since the latter is a homomorphic image of $S(K)$, we have that $S(K)$ can be mapped onto a non-zero member of $L(\Sigma)$. This homomorphic image T of $S(K)$ then has $\Sigma_T \neq \emptyset$. But T, being a homomorphic image of an S-radical ring, must be S-radical, and thus $\Sigma_T = \emptyset$. This contradiction shows that $S(K) \subseteq \ker(\Sigma, K)$.

(2) Suppose now that P.4 holds. $K/S(K)$ is S-semisimple, and (Divinsky (6), page 125) so is every ideal of $K/S(K)$. Thus, for every non-zero ideal B of $K/S(K)$, B is not S-radical, and so $\Sigma_B \neq \emptyset$. By P.4, $\ker(\Sigma, K/S(K)) = 0$.

For each M in $\Sigma_{K/S(K)}$, M can be given a K-module structure, and, as a K-module, M is in Σ_K. An easy calculation shows that $(0:M)_K = \{k \in K : M\,\theta(k) = 0\} = \theta^{-1}(\,(0:M)_{K/S(K)}\,)$, where θ is the natural homomorphism of K to $K/S(K)$. Since $0 = \ker(\Sigma, K/S(K))$, we obtain $S(K) = \theta^{-1}(0) = \theta^{-1}(\bigcap \{(0:M) : M \in \Sigma_{K/S(K)}\}) = \bigcap \{(0:M) : M$ is a K module induced from some $M' \in \Sigma_{K/S(K)}\}$. Since the latter intersection contains $\ker(\Sigma, K)$, we have $S(K) \supseteq \ker(\Sigma, K)$, and thus the two are equal.

Conversely, suppose that, for every ring K, $S(K) = \ker(\Sigma,K)$. If the hypothesis of P.4 is satisfied, no non-zero ideal of K can be S-radical, and so $0 = S(K) = \ker(\Sigma,K)$. This completes the proof of (2). Q.E.D.

It could be pointed out that any radical class can be described in this fashion. To be specific, Andrunakievic and Rjabuhin also proved that, if R is any radical class, then the assignment to a ring K the class of all nontrivial right K-modules M for which $K/(0:M)$ is R-semisimple gives classes of modules satisfying P.0 - P.4, and the radical class S of the theorem is the same as the class R.

We give two examples to show how some familiar radicals are defined in terms of modules.

Example 2.1.2 The Jacobson Radical

For each ring K, let Σ_K be the class of all irreducible right K-modules. It is easily seen that P.0, P.1, and P.2 are satisfied. Suppose that $\ker(\Sigma,K) = 0$, and that $B \neq 0$ is an ideal of K. Then for some M in Σ_K, $B \nsubseteq (0:M)$, and $MB \neq 0$. Since B is an ideal, MB is a K-submodule, and so $MB = M$. We can also consider M as a B-module. We show that, as a B-module, M is simple. If $m \in M$, and $m \neq 0$, then $mB \neq 0$, for otherwise, since B is a two-sided ideal, the set $\{n \in M : nB = 0\}$ would be a non-zero submodule, and thus would be all of M. This would give $MB = 0$, a contradiction. Now mB is a non-zero K-submodule of M, and so is all of M. Thus M, as a B module, is simple and non-trivial - i.e. M is in Σ_B. This shows P.3 is valid.

Suppose that K is a ring for which $\Sigma_B \neq \emptyset$ whenever B is a non-zero ideal of K. If $T = \ker(\Sigma, K) \neq 0$, then there is a simple non-trivial T-module M. Then there is a $t \in T$ such that $Mt \neq 0$. Let $m \in M$ be such that $mt \neq 0$. Then, (using the same argument as was used in the preceding paragraph) $mtT \neq 0$, and so $M(tK) \neq 0$. $m(tK)$ is clearly a T-submodule of M. We can give it the structure of a K-module by defining $m(tk)*k'$ to be $M(tkk')$. Furthermore, if x is in T, then $m(tk)*x = m(tk)x$, and so any K-submodule of $m(tK)$, with this multiplication, is a T-submodule of the original T-module M. This shows that $M(tK)$ is a simple K-module. Also, $m(tK)$ is not a trivial K-module, for, if $m(tK)*K = 0$, then $m(tT)T = 0$. Since $m(tT) \neq 0$, $\{n \in M: nT = 0\}$ is non-zero, is a submodule, and so must be all of M. This would give $MT = 0$, a contradiction. We have shown that the K-module $m(tK)$ is in Σ_K. Since $T = \ker(\Sigma, K)$, $T \subseteq (0: m(tK))$, and $0 = m(tK)*T = m(tK)T$. The argument above shows that this leads to a contradiction. Therefore $T = 0$, and P.4 is proved.

The corresponding radical property, as described in Theorem 2.1.1, is the upper radical property with respect to the class of all rings K with a faithful simple nontrivial module. These are the right primitive rings, and the radical property is the Jacobson radical property.

Example 2.1.3 The Baer Lower Radical

As was pointed out in Chapter one, the Baer Lower radical is also the upper radical with respect to the class of all prime rings. We show how this radical can be described in terms of modules. (See Andrunakievic and Rjabuhin (1).)

A right K-module is prime if and only if: $MK \neq 0$, and if $m \in M$ and $B \triangleleft K$ are such that $mB = 0$, then $m = 0$ or $MB = 0$.

Lemma 2.1.4 A right K-module is prime if and only if $MK \neq 0$, and for every non-zero submodule N of M, we have $(0:N) = (0:M)$.

Proof: If M is a prime right K-module, then $MK \neq 0$. Let N be a non-zero submodule of M. Clearly $(0:N) \supseteq (0:M)$. Also, $N(0:N) = 0$, and $(0:N)$ is an ideal of K. Since $N \neq 0$, we have $M(0:N) = 0$, i.e. $(0:N) \subseteq (0:M)$.

Conversely, suppose that M is a module satisfying the condition, and suppose that m in M and $B \triangleleft K$ are such that $mB = 0$. Then $mKB = 0$. If $mK \neq 0$, it is a non-zero submodule, and we have $B \subseteq (0:mK) = (0:M)$, and $MB = 0$. If $mK = 0$, then $m = 0$, for otherwise $\{n \in M : nK = 0\}$ is a non-zero submodule N such that $NK = 0$. This would give $MK = 0$, which is false. This proves the lemma. Q.E.D.

For each ring K, define Σ_K to be the class of all prime right K-modules. It is easy to verify that P.0, P.1, and P.2 are satisfied. If $\ker(\Sigma,K) = 0$, and if B is a non-zero ideal of K, then there is a member M of Σ_K such that $MB \neq 0$. We show that M is also a prime B module. Let $I \triangleleft B$ and $m \in M$ satisfy $mI = 0$. Then $mBI = 0$, and, since mB is a K submodule of M, either $MI = 0$ (by the lemma), or $mB = 0$. Since $MB \neq 0$, the latter situation implies that $m = 0$. Thus M is in Σ_B, and P.3 is proved.

We now prove P.4. Suppose that, for each non-zero ideal B of K, $\Sigma_B \neq \emptyset$. If $T = \ker(\Sigma,K) \neq 0$, then there is a prime T-module M. Proceeding as we did in the previous example, we can find m in M and t in T such that $mt \neq 0$. Then, since M is prime and $MT \neq 0$, $m(tT) \neq 0$, and so $m(tK) \neq 0$. As in the previous example, we can give $m(tK)$ a right K-module structure by defining $m(tk)*k'$ to be $m(tkk')$. We show that this K-module is prime. Since, as a T module, M is prime, we have $m(tT)T \neq 0$, and so $m(tK)*K \neq 0$. If $B \triangleleft K$, and $0 = m(tk)*B = m(tkB)$, then $m(tkTB) = 0$. Now, $TB \triangleleft B$, and, since M is a prime B-module, either $mtk = 0$ or $M(TB) = 0$. In the latter case, $m(tKB) = 0 = m(tK)*B$. Thus $m(tK) \in \Sigma_K$, and $(0\colon m(tK)) \supseteq T = \ker(\Sigma,K)$ - i.e. $m(tK)*T = 0$. Repeating the same argument that we have used before, the primeness of M as a T module leads to a contradiction. Therefore, $T = 0$, as desired.

We have shown that, if, to each ring K we assign the class Σ_K of prime right K modules, we have the conditions P.0 - P.4 satisfied. The corresponding radical is the upper radical with respect to the class $L(\Sigma)$ of all rings with a prime faithful right module. The next lemma shows that $L(\Sigma)$ is the class of all prime rings.

Lemma 2.1.5 K has a prime faithful right module M if and only if K is a prime ring.

Proof: Let M be a prime faithful right K-module, and suppose that $aKb = 0$, where a and b are in K. If $a \neq 0$, then $Ma \neq 0$, and $MaK \neq 0$, but $MaKb = 0$. MaK is a submodule, and therefore $b \in (0:MaK) = (0:M)$. (This last equality come from Lemma 2.1.4.) Thus $Mb = 0$, and $b = 0$, since M is faithful.

Conversely, suppose that K is a prime ring. It is straightforward to verify that K, considered as a right module over itself, is prime and faithful. Q.E.D.

This lemma shows that the radical associated with the classes of prime K modules (for all rings K) is the upper radical with respect to the class of all prime rings - i.e. the Baer Lower radical.

2.2 Intersections of One-sided Ideals

In an earlier paper than the one cited in the previous section, Andrunakievic and Rjabuhin (2) showed that any

SP-radical can be described as in the previous section in such a way that all members of Σ_K, for all K, are prime modules. Prime modules, therefore, occur in many situations.

In contrast to the situation for the Jacobson radical, it is not clear for special radicals in general, let alone for arbitrary radicals, whether or not the radical of a ring K can be expressed in terms of one-sided ideals. Of course, for the Jacobson radical case, J(K) is always the intersection of the maximal modular right ideals. We give next a generalization of this result.

In the sequel, the notation $I \underset{rt}{\triangleleft} K$ means I is a right ideal of K.

We consider classes Σ_K of K-modules which satisfy P.0 - P.4 and also the addition condition

(P.5) Every non-zero submodule of a member of Σ_K is also a member of Σ_K.

This condition is satisfied, for example, if Σ_K is the class of irreducible right K-modules, or if Σ_K is the class of prime right K-modules.

Theorem 2.2.1 Suppose that, for each ring K there is assigned a class Σ_K of right K-modules such P.0 - P.5 are satisfied. For any ring K, let $s(K) = \{I: I \underset{rt}{\triangleleft} K$ and $K/I \in \Sigma_K\}$. Then s(K) has a subset J such that, if S is the radical defined by the classes Σ_K (as in theorem 2.1.1), then $S(K) = \bigcap\{I: I \underset{rt}{\triangleleft} K \text{ and } I \in J\}$.

Furthermore, if each member of $s(K)$ is a modular right ideal, then we may take J to be all of $s(K)$, and thus $S(K) = \bigcap\{I: I \underset{rt}{\triangleleft} K \text{ and } K/I \in \Sigma_K\}$.

Proof: Let $m \neq 0$ be a member of M, where $M \in \Sigma_K$. Then $mK \neq 0$, for otherwise $N = \{x \in M : xK = 0\}$ would be a non-zero submodule of M, and, by P.5, N would be in Σ_K. However $NK = 0$, and this contradicts P.0.

Therefore $mK \neq 0$, and this is (by P.5) a member Σ_K. Since $mK \cong K/(0:m)$, we have $(0:m)$ is in $s(K)$. Now $S(K) = \bigcap_{M \in \Sigma_K} (0:M)$, by theorem 2.2.1, so we have

$S(K) = \bigcap_{M \in \Sigma_K} (0:M) = \bigcap_{M \in \Sigma_K} \bigcap\{(0:m) : 0 \neq m \in M\}$. Thus, if we set $J = \{I: I \underset{rt}{\triangleleft} K$ for which $\exists\ M \in \Sigma_K$, and $\exists\ m \neq 0$, $m \in M$ such that $I = (0:m)\}$, we have $J \subseteq s(K)$ and $S(K) = \bigcap J$.

Suppose now that each member of $s(K)$ is modular. That is, suppose that for each I in $s(K)$ there is an element e_I in K such that $e_I x - x \in I$ for each x in K. Let $m = e_I + I \in K/I$. Then $(0:m) = \{k \in K : e_I k \in I\} = \{k \in K : k \in I\} = I$. Thus I is in J and $J = s(K)$. Q.E.D.

CHAPTER THREE

THE UNIFORM, RATIONAL, AND WEAK RADICALS

In this chapter, we shall define, by means of modules, three radical classes, and some of the properties of these radicals will be investigated. The basic concepts used here are usually applied in the study of quotient rings, as found in the papers of Utumi, Johnson, and Goldie. A good reference which contains most of the basic results is Faith (7).

3.1 Essential and Rational Extensions

A right K-module M is an essential extension of a submodule N if $T \cap N \neq 0$ whenever T is a non-zero submodule of M. Under these circumstances, we also say that N is an essential submodule of M.

We say that M is a rational extension of a submodule N if, whenever we have $N \subseteq T \subseteq M$, T a submodule of M, and $f: T \to M$ a module homomorphism such that $f(N) = 0$, then $f = 0$. Following Faith (1), we write $M \triangledown N$ if M is an essential extension of N, and $M \blacktriangledown N$ if M is a rational extension of N.

Lemma 3.1.1 If $M \blacktriangledown N$, then $M \triangledown N$.

Proof: If T is a submodule of M such that $T \cap N = 0$, then define $f: N \oplus T \to M$ by $f(t) = t$, for $t \in T$, and

$f(n) = 0$ for $n \in N$. Then $f(N) = 0$, and f must therefore be the zero map - that is, $T = 0$, Q.E.D.

<u>Lemma 3.1.2</u> $M \triangledown N$ if and only if, whenever x and y are in M, with $y \neq 0$, there is a $k \in K$ and an integer n such that $xk + nx \in N$ and $yk + ny \neq 0$.

<u>Proof:</u> See Faith (7), page 58, for a proof of this result.

If M is a right K-module, an element m of M is said to be a <u>singular element</u> if $(0:m)$ is an essential submodule (i.e. right ideal) of K. If m is a singular element, and if $x \in K$, then mx is also singular. For let $0 \neq I \triangleleft_{rt} K$. If $xI = 0$, then $mxI = 0$, and $I \subseteq (0:mx)$. If $xI \neq 0$, then $(0:m) \cap xI \neq 0$, and there is an $i \neq 0$ in I such that $mxi = 0$. In either case, $I \cap (0:mx) \neq 0$, for all $0 \neq I \triangleleft_{rt} K$, and so mx is singular. Clearly a finite intersection of essential right ideals is essential, so, if x and y are singular elements, $(0:x-y) \supseteq (0:x) \cap (0:y)$, and $x-y$ is also singular. We see therefore that the set of singular elements of a module forms a submodule. This is known as the <u>singular submodule</u> of M, and is denoted by $Z(M)$. Clearly, if N is a submodule of M, then $Z(N) = N \cap Z(M)$. If $M = K$, as a right K-module, the singular submodule is a two-sided ideal of K, as is called the <u>right singular ideal</u> of K. This is usually denoted $Z_r(K)$. The left singular ideal is defined in the obvious way. The right and left singular ideals of a ring need not be the same. (See, for example, Small (25).)

The next lemma provides a partial converse to Lemma 3.1.1.

Lemma 3.1.3 If $Z(M) = 0$, and $M \triangledown N$, then $M \blacktriangledown N$.

Proof: Let x and $y \neq 0$ be in M. Since N is essential in M, it follows that $(N:x)$ is essential in K. (If I is a right ideal of K, either $xI = 0 \subseteq N$, and $I \subseteq (N:x)$, or $xI \neq 0$, in which case $xI \cap N \neq 0$ and therefore $I \cap (N:x) \neq 0$.) Since $Z(M) = 0$, $y \notin Z(M)$, and $(0:y)$ is not essential, whence $(N:x) \not\subseteq (0:y)$. There is therefore an element k of K in $(N:x)$ but not in $(0:y)$. Thus $xk \in N$, and $yk \neq 0$. The rest follows from Lemma 3.1.2. Q.E.D.

3.2 The Uniform, Rational, and Weak Radicals

A module is called uniform if it is an essential extension of every non-zero submodule. If, in addition, M is a rational extension of every non-zero submodule, we shall say that M is rationally uniform. We say that two modules M and M' are subisomorphic if there are monomorphisms $f:M \to M'$ and $g:M' \to M$. A module M which is subisomorphic to every non-zero submodule will be called a homogeneous module.

Lemma 3.2.1 A homogeneous module which is non-trivial is prime.

Proof: Let N be a non-zero submodule of a homogeneous -

module M. Then there is a monomorphism $f:M \to N$. Since f is one-to-one, it follows that $(0:M) = (0:f(M))$, and this contains $(0:N)$. Since $N \subseteq M$, $(0:M) \subseteq (0:N)$, and so $(0:N) = (0:M)$. By Lemma 2.1.4, M is therefore a prime module. Q.E.D.

We now define, by using these classes of modules, three radical properties. Two of them are new, and we shall show that the third coincides with the weak radical of Koh and Mewborn (17).

For each ring K, we assign the following classes of right K modules:

Σ_K^1 is the class of all prime, uniform right K-modules.

Σ_K^2 is the class of all prime, rationally uniform right K-modules.

Σ_K^3 is the class of all rationally uniform, homogeneous right K-modules.

<u>Theorem 3.2.2</u> The properties P.0 - P.5 are satisfied for each of the classes Σ^1, Σ^2, and Σ^3. (See Chapter two).

<u>Proof</u>: It is easily verified that the properties of uniformity, rational uniformity, and homogeneity, as well as primeness, are inherited by non-zero submodules, and so P.0, P.5 are seen to be true. The discussion of Example 2.1.3 showed that the property of primeness is preserved under the induced

module structures arising in P.1 and P.2. Uniformity is also preserved since it is a property of the lattice of submodules of a module, and the K- and K/B-modules occuring in P.1 and P.2 have isomorphic lattices of submodules. The other properties, rationality and homogeniety, are defined in terms of module homomorphisms from submodules of M to submodules of M. It is easily verified that these are preserved under the operations of P.1 and P.2. It follows then that each of the classes Σ^1, Σ^2, and Σ^3 satisfy P.1 and P.2.

We now establish P.3. Suppose that $0 = \prod\{(0:M) : M \in \Sigma_K^i\}$, where i is 1, 2, or 3, and let $B \neq 0$ be an ideal of K. Then there is an M in Σ_K^i such that $MB \neq 0$. The discussion in Example 2.1.3 showed that M, as a B-module, is prime.

If M is uniform as a K-module, and if U and V are two non-zero B-submodules of M, then (using the fact that M is prime) UB and VB are non-zero. These are K-submodules of M, and therefore have a non-zero intersection. Thus $U \cap V \neq \emptyset$, and M is a uniform B-module.

Suppose that M, as a K-module, is a rational extension of every non-zero submodule, and suppose that T and N are B-submodules of M, $0 \neq T \subseteq N$, and suppose that $f: N \to M$ is a B-homomorphism such that $f(T) = 0$. Then TB and NB are non-zero (since B is prime), and the restriction f'

of f to NB is a B-homomorphism for which $f'(TB) = 0$. We claim f' is a K-homomorphism. Any element of NB is of the form $x = \Sigma_{i=1}^{n} y_i b_i$, where $y_i \in N$, and $b_i \in B$. For $k \in K$, $f'(y_i b_i)k = (f'(y_i)b_i)k = f'(y_i)(b_i k) = f'(y_i b_i k)$, and it follows that $f'(x)k = f'(xk)$.

Since M is a rational extension of the K-submodule TB, and since $f'(TB) = 0$, we must have $f' = 0$, and therefore $f(NB) = 0$. For $n \in N$, $f(n)B = 0$. The primeness of M, and the fact that $MB \neq 0$, give us that $f(n) = 0$, and so f is the zero map. This shows that M, as a B module, is a rational extension of T, where T is any non-zero submodule.

Finally, suppose that M is subisomorphic to every non-zero K-submodule of M, and let N be any non-zero B-submodule. Then NB is a non-zero K-submodule, and NB is subisomorphic to M. The k-monomorphism $g: M \to NB \subseteq N$, and the embedding map of N into M are B-monomorphisms, and so M and N are subisomorphic. This completes the proof that each of the classes Σ^1, Σ^2, and Σ^3 satisfy P.3.

Now we establish P.4. Let i be 1, 2, or 3, and suppose that $\Sigma_B^i \neq \emptyset$ for any non-zero ideal B of K, but $0 \neq \prod\{(0:M) : M \in \Sigma_K^i\} = T$. Then there exists a prime T-module M in Σ_T^i. As in Example 2.1.3, there is m in M, t in T such that $m(tK) \neq 0$, and this has a K-module structure (defined by $(m(tk))*k' = m(tkk')$) for which $N = m(tK)$ is a prime K-module.

If M is uniform as a T module, since K-submodules of N are also T-submodules, N is also uniform as a K-module. Any non-trivial K homomorphism from a K-submodule of N to N is a T-homomorphism from a T-submodule of M into M, and must therefore have zero kernel if M, as a T-module, is a rational extension of every non-zero submodule. Thus this latter condition on the T-module M is passed on to the K-submodule N.

Finally, suppose that $M \in \Sigma_T^3$, and let L be a K-submodule of N. Then L is a T-submodule of M, and there is a T-monomorphism $f: M \to L$. We show that the restriction f' of f to $m(tK)$ is indeed a K-homomorphism. For any k and k' in K, we have

$$\begin{aligned} f(m(tk))*k' &= (f(m)\cdot tk)*k' && \text{(since } f \text{ is a T-homomorphism)} \\ &= (f(m)*tk)*k' && \text{(since } f(m) \text{ is in } L \subseteq N, \\ &&& \text{and, in } T \text{ the multiplications} \\ &&& \cdot \text{ and } * \text{ agree)} \\ &= f(m)*(tk\cdot k') && \text{(since } L \text{ is a K-module)} \\ &= f(m\cdot tkk') && \text{(since } tkk' \in T, \text{ and } f \\ &&& \text{is a T-homomorphism)} \\ &= f((m\cdot tk)*k') && \text{(from the definition of the} \\ &&& \text{multiplication } *) \end{aligned}$$

The map f, restricted to N, is therefore a K-homomorphism, and it is one-to-one. This map, together with the imbedding of L into N, are the maps which guarantee that N and L are subisomorphic, and $N \in \Sigma_K^3$. Q.E.D.

The previous theorem assures us that each of the classes Σ^1, Σ^2, and Σ^3 define radical properties as described in Theorem 2.1.1, and that, for any ring K, the radical is $\prod\{(0:M) : M \in \Sigma_K\}$, the Σ being Σ^1, Σ^2, or Σ^3, as the case may be. We shall call these radicals the <u>uniform radical</u>, the <u>rational radical</u>, and the <u>weak radical</u>, respectively, and we denote them by U, U*, and W. Thus U is the upper radical with respect to the class C_U of all rings K with a faithful, uniform, prime right K-module; U* is the upper radical with respect to the class C_{U^*} of all rings K with a faithful, rationally uniform and prime right K-module; and W is the upper radical with respect to the class C_H of all rings K with a faithful, rationally uniform, homogeneous right K-module. Since $C_W \subseteq C_{U^*} \subseteq C_U$, we have $U \subseteq U^* \subseteq W$. From Lemma 2.1.5, we see that any member of C_U is a prime ring. On the other hand, any right primitive ring (a ring with a simple non-trivial right module which is faithful) is clearly in C_W. Therefore we have $B \subseteq U \subseteq U^* \subseteq W \subseteq J$, where B and J are the Baer Lower and Jacobson radical classes.

Theorem 3.2.3 If K is a commutative ring, then $B(K) = U(K) = U^*(K) = W(K)$.

Proof: From Example 2.1.3, we have that $B(K) = \bigcap\{I \triangleleft K : K/I$ is a prime ring$\}$. Therefore, if we can show that any commutative prime ring is in C_W, then for a commutative prime ring K, we will have that $W(K)$ is contained in each ideal I for which K/I is prime, and then $W(K) \subseteq B(K)$.

Let K be a commutative prime ring. Then K is a domain ($xy = 0$ implies $x = 0$ or $y = 0$). K, as a right K module is clearly uniform, for if U and V are non-zero submodules (i.e. ideals), then $0 \neq UV = VU \subseteq U \cap V$. Since K has no zero divisors, $Z_r(K) = 0$, and, by Lemma 3.1.3 we have that K is rationally uniform. If U is a non-zero submodule, and if $0 \neq u \in U$, then $L_u: K \to U$, where $L_u(k) = uk$, is a K-monomorphism, and U is homogeneous. K is easily seen to be a faithful K-module, and so the ring K is in C_W. Q.E.D.

Corollary 3.2.3 The radicals J and W are distinct.

Proof: The subring S of the rationals consisting of numbers of the form (even integer)/(odd integer) is a commutative prime ring which is also right quasi-regular. Then $W(S) = 0$, and $J(S) = S$. Q.E.D.

Recall from §1.5, that a class C of rings is called a special class if it satisfies:

(X) Every member is prime.

(Y) $C \cup \{0\}$ is hereditary.

(Z) If $A \in C$, and $A \triangleleft K$, then K/A^* is in C, where $A^* = \{x \in K: xA = Ax = 0\}$.

The upper radicals with respect to the special classes are called the SP-radicals.

Theorem 3.2.4 Each of the classes C_U, C_{U^*} and C_W are special classes, and therefore U, U^* and W are all SP-radicals.

Proof: The proof is the same in all three cases. Let Σ_K denote one of Σ_K^1, Σ_K^2, or Σ_K^3, and let C be the corresponding class C_U, C_{U^*}, or C_W. We already know that any member of C is a prime ring. Suppose that $K \in C$ and $0 \neq B \triangleleft K$. Let M be a faithful member of Σ_K. M can be regarded as a B module, and $\{b \in B : Mb = 0\} = 0$. The proof that M, as a B module, is in Σ_B, proceeds exactly as in the verification of property P.3 in Theorem 3.2.2. This proves that B is in C and condition (Y) is established.

Suppose that A is in C, and that $A \triangleleft K$. In order to prove that condition (Z) is true, we must show that K/A^* is in C, where $A^* = \{k \in K : kA = Ak = 0\}$. We show that A^* is indeed an ideal of K. If x and y are in A^*, then $x-y$ is also. Suppose that $x \in A^*$ and $k \in K$.

Then $A(xk) = 0$. Also $(xk)A \subseteq xA = 0$; and thus $xk \in A^*$, and A^* is a right ideal of K. Similarly A^* is a left ideal.

Now A is in C, so there is a faithful member M of Σ_A. Proceeding exactly as in the verification of P.4, there are m in M, and a in A such that $m(aK) \neq 0$. Then $N = m(aK)$ has a K module structure defined by $m(ak)*k' = m(akk')$, and (as in the proof of P.4) this is a member of Σ_K. What is $(0:N_K)$? Since $aK \subseteq A$, clearly we have $A^* \subseteq (0:N_K)$. Suppose now that $x \in (0:N)$. Then $0 = m(aK)(AxA)$. Since $m(aK)$ is an A submodule of M, since $AxA \triangleleft A$, and since M is prime, $M(AxA) = 0$. But M is also faithful, and so $AxA = 0$. Thus $(Ax)^2 = (xA)^2 = 0$, and since A is a prime ring, $xA = Ax = 0$ i.e. $x \in A^*$. Thus $A^* = (0:N_K)$. Then, by P.2, $N \in \Sigma_{K/A^*}$ and $(0:N_{K/A^*}) = 0$.

3.3 Matrix Rings and the Radicals U, U^*, and W

For many radicals R, it is true that $R(K_n) = R(K)_n$, where S_n denotes the ring of $n \times n$ matrices with entries in a ring S. However, this is not true for all radicals - not even for all SP-radicals. For example, the generalized nil radical N_g, defined as the upper radical with respect to the class of all rings with no zero divisors, does not have this property. A field F is N_g semisimple, but the ring of 2×2 matrices over F is a simple ring with zero divisors, and so is N_g radical. We shall prove now that the radicals U, U^* and W do have this property.

Recall that U, U^* and W are all SP-radicals, and (see Divinsky, (6)), any SP-radical is hereditary.

Lemma 3.3.1 For a hereditary radical R, in order to prove that $R(K_n) = R(K)_n$ for all rings K, it is sufficient to prove it for all rings which have a unity element.

Proof: It is well known that if S is a ring with unity, there is a one-to one correspondence between the two-sided ideals of S and the two-sided ideals of S_n. An ideal I of S corresponds to I_n, and an ideal B of S_n corresponds to the set of all members of S which are an entry of some member of B. This correspondence is an isomorphism of the lattices of two sided ideals of S and of S_n.

It is equally well-known that any ring K can be embedded as an ideal in a ring $\overline{K}$ with unity. If R is any hereditary radical, then $R(K) = K \cap R(\overline{K})$. Under the lattice isomorphism, we get $R(K)_n = K_n \cap R(\overline{K})_n$. Therefore if all rings S with unity satisfy $R(S_n) = R(S)_n$, for any ring K we have $R(K)_n = K_n \cap R(\overline{K})_n = K_n \cap R(\overline{K}_n) = R(K_n)$, the last equality being due to the fact that R is hereditary, and $K_n \triangleleft \overline{K}_n$. Q.E.D.

In the sequel, I_{ij} will denote the matrix which, when multiplied on the right to a matrix M has the effect of interchanging the i'th and j'th columns of M. Thus $I_{ij} = (r_{st})$, where $r_{ii} = r_{jj} = 0$, $r_{ij} = r_{ji} = 1$, $r_{ss} = 1$ for $s \neq i$, $s \neq j$, and $r_{st} = 0$ for all other s and t. Also, for any element k of a ring K, $E_{ij}(k)$ will denote the matrix which has k in the (i,j) position, and zeros in all other positions.

<u>Lemma 3.3.2</u> Let C represent any of the classes C_u, C_{U*}, or C_W. If K is a ring with unity, and if K is in C, then K_n is in C.

<u>Proof</u>: Let M be a faithful member of the class Σ_K ($= \Sigma_K^i$, where $i=1,2$, or 3). Define $\overline{M}$ to be the Cartesian product of n copies of M. With componentwise addition and the obvious multiplication of an n-tuple $(m_1,m_2,\ldots m_n)$ on the right by an $n \times n$ matrix from the ring K_n, $\overline{M}$ becomes a right K_n-module.

Since M is in Σ_K, M is a prime module. We show that $\overline{M}$ is also prime. Let $\overline{N}$ be a non-zero submodule of $\overline{M}$, and let $0 \neq x = (x_1,x_2,\ldots,x_n) \in \overline{N}$. Suppose, say, that $x_j \neq 0$. Then $\overline{N}$ also contains xI_{1j}, and this n-tuple has its first entry non-zero. Let $\overline{B} \triangleleft K_n$, and let B be the set of members of K which are entries of members of $\overline{B}$. Then, as was mentioned earlier, $B \triangleleft K$, and $B_n = \overline{B}$. If $w = (w_1, w_2, \ldots, w_n)$ satisfies $x\overline{B} = 0$, where $w_j \neq 0$, say, then $\overline{B} = 0$. For let $y = (y_{st}) \epsilon \overline{B}$. Then $0 = wI_{1j}E_{1s}(r)yE_{t1}(1) = (w_j r y_{st}, 0, \ldots, 0)$ for all r in K. Thus $w_j K y_{st} = 0$ in M, which is a prime faithful K-module, and $w_j \neq 0$. We must have $y_{st} = 0$, and $y = 0$. This shows that $\overline{M}$ is prime and faithful.

Now suppose that M is uniform. Let $\overline{U}$ and $\overline{V}$ be non-zero submodules of $\overline{M}$, and let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be non-zero in $\overline{U}$ and $\overline{V}$ respectively. The remarks in the previous paragraph allow us to assume that u_1 and v_1 are not zero, and we so assume. Let k and k' be elements of K such that $u_1 k = v_1 k' \neq 0$. Then $uE_{11}(k) = vE_{11}(k') = (u_1 k, 0, \ldots, 0) \neq 0$, and this is in $\overline{U} \cap \overline{V}$. Therefore $\overline{M}$ is uniform, provided M is.

Now we consider the case where M is rationally uniform. Let $\overline{N}$ be a non-zero submodule in M, and suppose that $w = (w_1, \ldots, w_n) \neq 0$ is in $\overline{N}$. As we have seen, we can take $w_1 \neq 0$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots y_n) \neq 0$ be any members of $\overline{M}$. Suppose that $y_t \neq 0$. Since M is a rational extension of $w_1 K$, there is a k in K such that

$x_t k \in w_1 K$, and $y_t k \neq 0$. (This comes from Lemma 3.1.3). Suppose that $x_t k = w_1 k'$. Then $xE_{t1}(k) = (x_t k, 0, 0, \ldots, 0) = (w_1 k', 0, 0, 0) = wE_{11}(k') \in \overline{N}$, and $yE_{t1}(k) = (y_t k, 0, \ldots, 0) \neq 0$. By Lemma 3.1.2, $\overline{M}$ is a rational extension of $\overline{N}$. Thus, if M is rationally uniform, so is $\overline{M}$.

Suppose that M is homogeneous, and let $\overline{N}$ be a non-zero submodule of $\overline{M}$. If $x = (x_1, x_2, \ldots, x_n)$ is in $\overline{N}$, so is $xE_{t1}(1) = (x_t, 0, \ldots, 0)$. If we define N to be $\{m \in M : \quad x = (x_1, \ldots, x_n) \in \overline{N}$ such that $x_1 = m\}$, then N is a submodule of M which contains all elements of M appearing as entries of members of $\overline{N}$. Consider $\{(x_1, \ldots, x_n) \in \overline{M}: x_i \in N, i = 1, 2, \ldots, n\}$. This set $\overline{N}'$, we have seen, contains $\overline{N}$. Conversely, if $(x_1, \ldots, x_n) \in \overline{N}'$, each x_i is the first entry of some w_i in $\overline{N}$, and $w_1 E_{11}(1) + \ldots + w_n E_{1n}(1) = (x_1, 0, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \ldots + (0, 0, \ldots, x_n) = x$. Therefore $x \in \overline{N}$, and $\overline{N} = \overline{N}'$.

M is subisomorphic to N, and so there is a K-monomorphism $f: M \to N$. Define $\overline{f}: \overline{M} \to \overline{N}$ by $\overline{f}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$. This is one-to-one, and is a K_n-homomorphism. Also, $\overline{f}(\overline{M}) \subseteq \overline{N}' = \overline{N}$. This shows that $\overline{M}$ and $\overline{N}$ are subisomorphic. **Q.E.D.**

The converse of this lemma is also true.

<u>Lemma 3.3.3</u> If C represents any one of the classes C_U, C_{U*}, or C_W then, if K is a ring with unity, and if $K_n \in C$, then $K \in C$.

<u>Proof</u>: Let M be a faithful member of Σ_{K_n} ($= \Sigma^1_{K_n}$, $\Sigma^2_{K_n}$, or $\Sigma^3_{K_n}$ as the case may be), and define a K-module structure on M be defining $m*k$ to be $mE_{11}(k)$. Since M is faithful as a K_n module, $M*K = 0$. Choose m such that $m*K \neq 0$, and let $M' = m*K = \{mE_{11}(k): k \in K\}$. This will be a member of Σ_K, and will be faithful.

To show primeness, suppose that k' annihilates a non-zero submodule N of M'. Then there is a k in K such that $m*k \neq 0$, $m*k \in N$, and $m*k*K*k' = 0$. For any $x = (x_{ij})$ in K_n, $x = \Sigma^n_{i,j=1} E_{ij}(x_{ij})$, and $mE_{11}(k)xE_{11}(k') = mE_{11}(k)E_{11}(x_{11})E_{11}(k') = m*k*x_{11}*k' = 0$. Since $m*k = mE_{11}(k) \neq 0$, and since M, as a K_n module is faithful and prime, $E_{11}(k') \in (0: mE_{11}(k)K_n) = (0:M) = 0$, whence $k' = 0$. Thus every non-zero submodule of M' is faithful, and M' is prime and faithful.

Suppose now that M is a uniform K_n module, and let U' and V' be non-zero submodules of M'. Let $u = m*k$ and $v = m*k'$ be non-zero and in U' and V' respectively. Then $uK_n \cap vK_n \neq 0$. Let $x = (x_{ij})$ and

$y = (y_{ij})$ be such that $ux = vy \neq 0$. Writing x as $\Sigma_{i,j=1}^{n} E_{ij}(x_{ij})$, we see $ux = mE_{11}(k)x = m(\Sigma_{j=1}^{n} E_{1j}(kx_{1j}))$. If it were true that, for $t=1,2,\ldots,n$, $uxE_{t1}(1) = 0$, we would have $0 = \Sigma_{t=1}^{n} uxE_{t1}(1)E_{1t}(1) = ux$, which is false. Therefore, for some t, $uxE_{t1}(1) = mE_{11}(kx_{1t}) \neq 0$. Computation shows that $u^*x_{1t} = uxE_{t1}(1)$, and that $v^*y_{1t} = vyE_{t1}(1)$. Therefore $u^*x_{1t} = v^*y_{1t} \neq 0$, and $U' \cap V' \neq 0$. M' is shown to be uniform.

Assume that M is a rational extension of every non-zero submodule, and let N' be a non-zero submodule of M'. Suppose that $x = m^*k_1$ and $y = m^*k_2 \neq 0$ are in M', and let $z = m^*r$ be a non-zero member of N'. M is a rational extension of $zK_n \neq 0$, and, by Lemma 3.1.2, there is an $a = (a_{ij})$ such that $xa \in zK_n$ and $ya \neq 0$. Now, $ya = m(\Sigma_{j=1}^{n} E_{1j}(k_2a_{1j}))$, and, as in the previous paragraph, $yaE_{t1}(1) = mE_{11}(k_2a_{1t}) \neq 0$ for some t. Therefore $y^*a_{1t} = mE_{11}(k_2a_{1t}) \neq 0$. We also know that $xa \in zK_n$. Suppose that $xa = zw$, where $w = (w_{ij})$. Then $x^*a_{1t} = xE_{11}(a_{1t}) = mE_{11}(k_1)aE_{t1}(1) = xaE_{t1}(1) = zwE_{t1}(1) = mE_{11}(r)wE_{t1}(1) = mE_{11}(rw_{1t}) = mE_{11}(r)E_{11}(w_{1t}) = z^*w_{1t} \in z^*K \subseteq N'$. Therefore $y^*a_{1t} \neq 0$ and $x^*a_{1t} \in N'$ showing that M' is a rational extension of N'.

Suppose now that M is homogeneous, and let $N' \neq 0$ be a submodule of M'. Let $z = m^*k \neq 0$ be in N'. Then zK_n is a non-zero submodule of the K_n module M, and there is a K_n homomorphism $f: M \to zK_n$. The restriction g of f to M' is one-to-one and is a K-homomorphism. It remains only to show that $g(M') \subseteq N'$. Recall that $M' = m^*K$. Let $f(m) = zx \in zK_n$, where $x = (x_{ij})$. For any m' in M', $m' = m^*r = mE_{11}(r)$, and $g(m') = f(mE_{11}(r)) = zxE_{11}(r) = mE_{11}(k)xE_{11}(r) = mE_{11}(kx_{11}r) = mE_{11}(k)E_{11}(x_{11}r) = z^*(x_{11}r) \in z^*K \subseteq N'$. The proof is complete. Q.E.D.

<u>Theorem 3.3.4</u> If K is a ring with unity, and if R is any one of the radicals U, U^*, or W, then $R(K_n) = R(K)_n$.

<u>Proof</u>: By theorem 2.1.1, for any ring K, $R(K) = \prod\{(O:M): M \in \Sigma_K\}$. From properties P.1 and P.2, it follows that an ideal I of K is $(O:M)$ for some $M \in \Sigma_K$ if and only if K/I is in C. Also, if K has a unity, the lattices of two-sided ideals of K and of K_n are isomorphic, and, for $I \triangleleft K$, $K_n/I_n \cong (K/I)_n$ - that is, they are isomorphic rings.

It follows then that $K/I \in C$ if and only if $K_n/I_n \in C$. Therefore $R(K) = \prod\{I \triangleleft K: K/I \in C\} = \prod\{I \triangleleft K: K_n/I_n \in C\}$. Using the lattice isomorphism, this gives

$R(K)_n = [\bigcap\{I \triangleleft K\colon K_n/I_n \in C\}]_n = \bigcap\{I_n \triangleleft K_n\colon K_n/I_n \in C\}$,

and this last intersection is just $R(K_n)$. Q.E.D.

Corollary 3.3.5 If K is any ring, and if R is any one of U, U^*, or W, then $R(K_n) = R(K)_n$.

Proof: Apply Lemma 3.3.1 and Theorem 3.3.4. Q.E.D.

3.4 Relations Between the classes Σ^1, Σ^2, Σ^3

In order to clarify the relations between these classes, and between the corresponding radicals, we introduce two new radicals. For any ring K, define Σ_K^H to be the class of all non-trivial homogeneous right K-modules. Recall, from Lemma 3.2.1, that every member of Σ_K^H is a prime module.

Also, for any ring K, we define Σ_K^{UH} to be $\Sigma_K^1 \cap \Sigma_K^H$. An examination of the proofs of Theorems 3.2.2 and 3.2.4 will show that these classes satisfy P.0 - P.5, and that the radicals they define are SP-radicals. We shall call these radicals the homogeneous radical, corresponding to Σ^H, which which will be denoted by H, and the uniformly homogeneous radical, denoted by H^U, corresponding to Σ^{UH}. From the definition of Σ^3, we see that $\Sigma_K^3 = \Sigma_K^2 \cap \Sigma_K^H$ for any ring K. The relation between these classes can be expressed in the following diagram (where the arrows represent inclusions).

$$\begin{tikzcd}
 & \Sigma_K^1 \arrow[dl, tail] & \Sigma_K^2 \arrow[l, tail] & & \\
\text{Prime K-modules} & & \Sigma_K^{UH} \arrow[ul, tail] \arrow[dl, tail] & \Sigma_K^3 \arrow[ul, tail] \arrow[l, tail] & \text{Irreducible K-modules} \arrow[l, tail] \\
 & \Sigma_K^H \arrow[ul, tail] & & &
\end{tikzcd}$$

Fig. 3.4.1

<u>Theorem 3.4.1</u> Any member of C_H, the class of rings K with a faithful member of Σ_K^H, has no non-zero nil right ideals and no non-zero nil left ideals.

<u>Proof</u>: Suppose that $K \in C_H$, and let L be a non-zero nil left ideal. If $a \in L$, then, for any x in K, (xa) is in L, and $(xa)^n = 0$ for some n. Then $(ax)^{n+1} = 0$, and aK is a nil right ideal. Since K is a prime ring, by Lemma 2.1.5, $aK \neq 0$. Therefore, if there is a non-zero nil left ideal in K, there is also a non-zero nil right ideal. We denote such a right ideal by T.

Let M be a faithful member of Σ_K^H. Then $MT \neq 0$, and so $mT \neq 0$ for some m in M. Now mT is a K-submodule of M, and so there is a K-monomorphism $f: M \to mT$ (since M is homogeneous). Suppose that $f(m) = mt$. By induction we obtain $f^n(m) = mt^n$ for every positive integer n. The element t is nilpotent, and so $f^{n_o}(m) = 0$ for some n_o.

Since f is one-to-one, we obtain $m = 0$, a contradiction. Therefore there cannot be any non-zero nil right or left ideals in K. Q.E.D.

<u>Corollary 3.4.2</u> The nil radical N is contained in H.

<u>Proof</u>: For any ring K, let $M \in \Sigma_K^H$, and let $A = (0:M)$. Then $(N(K)+A)/A$ is a nil ideal of K/A, a member of C_H, and therefore $N(K)+A = A$, and $N(K) \subseteq A$. Therefore $N(K) \subseteq \prod\{(0:M) : M \in \Sigma_K^H\} = H(K)$. Q.E.D.

Recall that the generalized nil radical N_g is the upper radical with respect to the class of all rings with no zero divisors.

<u>Lemma 3.4.3</u> A ring with no zero divisors is in C_H.

<u>Proof</u>: Consider K, a ring with no zero divisors, as a right module over itself. For any non-zero submodule (right ideal) T, let $t \neq 0$ be in T. The map $L_t : K \to T$, where $L_t(k) = tk$, is a K-monomorphism, which, together with the natural imbedding of T into K, gives a pair of maps which shows K and T are subisomorphic. Therefore K is in Σ_K^H, and is faithful. Q.E.D.

<u>Theorem 3.4.4</u> The radical H is contained in N_g.

<u>Proof</u>: From §1.3 and the definition of H, an H-radical ring K is one which cannot be homomorphically mapped onto a non-zero member of C_H, and, by the previous lemma, such a ring cannot be mapped onto a non-zero ring with no zero divisors. K is therefore N_g radical. Q.E.D.

The relation between the various radicals is illustrated in the following diagram. (C.f. Divinsky, (6), page 156.)

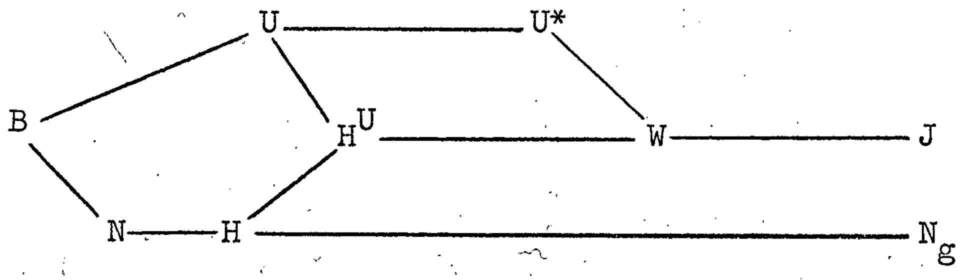

Fig. 3.4.2

By considering the singular submodule $Z(M)$ of a module M, we can obtain more information about the relations between these module classes.

If M is a right K-module, and if $A \triangleleft K$, $A \subseteq (0:M)$, then the additive abelian group of M can be given a K/A-module structure as described in the property P.1. Suppose we denote these modules by M_K and $M_{K/A}$. It makes sense, if we consider these two modules as having the same underlying set, to compare $Z(M_K)$ and $Z(M_{K/A})$.

Lemma 3.4.5 If M is a right K-module, and if $A \triangleleft K$, $A \subseteq (0:M)$, then $Z(M_{K/A}) \subseteq Z(M_K)$.

Proof: For m in M, $\{k \in K : mk = 0\} = (0:m)_K$ is a right ideal of K which contains A, and which, under the natural homomorphism $K \to K/A$ maps to $\{\bar{k} \in K/A : m\bar{k} = 0\} = (0:m)_{K/A}$. If $m \in Z(M_{K/A})$, then $(0:m)_{K/A}$ is an essential right ideal of K/A. Let I be a non-zero right ideal of K. If $I \cap A \neq 0$, then $I \cap (0:m)_K \neq 0$. If $I \cap A = 0$, then $(I+A)/A$ is a non-zero right ideal in K/A, and $(I+A)/A \cap (0:m)_{K/A} \neq 0$. This implies $(I+A) \cap (0:m)_K \supsetneqq A$. Let $x = i+a$ be in $(I+A) \cap (0:m)_K$ but not in A. Then $i \neq 0$, and $i = x - a$ is in $(0:m)_K + A = (0:m)_K$ and $i \in I$. This proves that $(0:m)_K$ is essential in K, and $m \in Z(M_K)$. Q.E.D.

Theorem 3.4.6 Let K be a ring, and let M be a right K-module such that $M \in \Sigma_K^2$, and such that $M \neq Z(M)$. Then

(1) M has a submodule $N \neq 0$ such that $N \in \Sigma_K^3$. Either $N \subseteq Z(M)$ or $Z(M) = 0$.

(2) If K is semiprime, then $Z(M) = 0$.

(3) If K is a prime ring, then $Z_r(K) = Z_1(K) = 0$.

Proof: (1) Let M and K satisfy the assumptions of the theorem, and let $(0:M) = A$. From Lemma 2.1.5, K/A is a prime ring, and so $B(K)$, the Baer Lower radical of K, is contained in A. Let $\overline{K} = K/B(K)$. Using property P.2, M can be considered a $\overline{K}$ module, and, as such, M is in $\Sigma_{\overline{K}}^2$. From the previous lemma, we have that $Z(M_{\overline{K}}) \subseteq Z(M_K) \neq M$, and so there is an m in M such that $(0:m)_{\overline{K}}$ is not an essential right ideal of $\overline{K}$. There is, therefore, a non-zero right ideal I of $\overline{K}$ such that $I \cap (0:m)_{\overline{K}} = 0$. It follows that the $\overline{K}$-module homomorphism $f:\overline{K} \to M$, where $f(\overline{k}) = m\overline{k}$, is one-to-one on I. I is therefore isomorphic to a $\overline{K}$-submodule of $M_{\overline{K}}$. We show that I (and hence $f(I)$) is in $\Sigma_{\overline{K}}^3$. From P.5, $f(I)$ is in $\Sigma_{\overline{K}}^2$, and thus so is I. Suppose that V is a non-zero $\overline{K}$-submodule of I - that is, V is a right ideal of $\overline{K}$ contained in I. Since $\overline{K}$ is semiprime, $V^2 \neq 0$, and so, for some v in V, $vI \neq 0$. Since I is a rational extension of every non-zero submodule, the $\overline{K}$-module homomorphism of I to V which maps i to vi must be one-to-one. Therefore V and I are subisomorphic. Since V is any submodule of I, this shows that $I \in \Sigma_{\overline{K}}^3$. The $\overline{K}$-submodule $f(I)$ is then in $\Sigma_{\overline{K}}^3$, and, by property P.1, $f(I)$, as a K-module, is in Σ_K^3. This is the desired module N. If $Z(N) \neq 0$ then N is subisomorphic to $Z(N)$, and it follows that $N = Z(N) \subseteq Z(M)$. If, on the other hand, $Z(N) = 0$, then $0 = Z(N) = N \cap Z(M)$, and, since M is uniform, $Z(M) = 0$.

(2) Suppose now that K is semiprime. In the notation of the previous paragraph, $K = \overline{K}$. In order to show that $Z(M) = 0$ it is sufficient to show that $Z(N) = 0$, that is, to show $Z(I) = 0$. But if $u \in Z(I)$, then $(0:u)$ is essential in K, and $(0:u) \cap I \neq 0$, and so there is $i \in I$ $i \neq 0$, such that $ui = 0$. Then the map $L_u : I \to I$, where $L_u(i) = ui$, has a non-zero kernel, and, since I is rational over this kernel, $L_u = 0$, and $uI = 0$. Thus $Z(I) \subseteq \{x \in K : xI = 0\} = la(I)$. Then $Z(I)^2 \subset la(I) \cdot I = 0$. Since K is semiprime, $Z(I) = 0$.

(3) Suppose that K is a prime ring. If T is any two-sided ideal, and if I is any right ideal, then $0 \neq IT \subseteq I \cap T$. This shows that any two-sided ideal must meet in a non-zero fashion any right ideal. Now $Z_r(K)$ is a two-sided ideal, and $Z(I) = I \cap Z_r(K) = 0$. Therefore $Z_r(K) = 0$.

In order to show that $Z_1(K) = 0$ also, we first show that, for any non-zero u in I, $(0:u)$ is a maximal right annihilator. Suppose $b \in K$, and $(0:b) \supsetneqq (0:u)$. If J is a non-zero right ideal of K, either $J \cap (0:u)$ is zero or it is non-zero. In the latter case, $J \cap (0:b) \neq 0$, while in the former case, $(0:u) + J \supsetneqq (0:u)$. Since $K/(0:u) \cong uK \subseteq I$, $K/(0:u)$ is uniform, and it follows that $((0:u) + J) \cap (0:b) \supsetneqq (0:u)$. Suppose that $a \in (0:u)$ and $j \in J$ are such that $a + j = x$ is in $(0:b)$ but is not in $(0:u)$. Then $j \neq 0$, and $j = x-a$ is in $(0:b) \cap J$. Thus, again in this case $J \cap (0:b) \neq 0$. Therefore $(0:b)$ is essential as a right ideal of K and $b \in Z_r(K) = 0$. This shows that $(0:u)$ is a maximal right annihilator in K.

We choose u in I, $u \neq 0$. If $Z_1(K) \neq 0$, then, since K is prime, there is an s in $Z_1(K)$ such that $usu \neq 0$. Then $0 \neq su \in Z_1(K)$, and so $la(su)$ is an essential left ideal, and $la(su) \cap Ku \neq 0$. Thus there is a k in K such that $ku \neq 0$, but $kusu = 0$. Then $su \in (0:ku) = (0:u)$. (The last equality comes from that fact that $(0:u)$ is a maximal right annihilator, $(0:ku) \supseteq (0:u)$, and $ku \neq 0$.) Thus $usu = 0$, a contradiction. Therefore $Z_1(K) = 0$. Q.E.D.

<u>Corollary 3.4.7</u> If $M \in \Sigma_K^1$, and if $Z(M) = 0$, the conclusions of the theorem hold for M and K.

<u>Proof:</u> M is uniform, and so, by Lemma 3.1.3, M is a rational extension of every non-zero submodule. Thus $M \in \Sigma_K^2$. We then apply the theorem. Q.E.D.

<u>3.5 Rings With the Ascending Chain Condition</u>

Let P be a property of submodules of a right K-module M. We say that M satisfies the <u>ascending chain condition for P-submodules</u> if, for every increasing sequence $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_i \ldots$ of submodules which all have property

P, there is an integer t such that $N_t = N_{t+i}$ for all $i \geq 0$. This is equivalent to the statement: every non-empty set of submodules, each having property P, has a maximal (relative to inclusion) member.

If P is merely the property of being a submodule of M, and if M has the ascending chain condition for P-submodules, we say that M is Noetherian. If K, as a right K-module over itself, is Noetherian, we say that K is a right Noetherian ring. A K-module is finitely generated if there is a finite set $x_1, x_2, \ldots x_n$ of elements from M such that the smallest submodule of M containing $x_1, x_2, \ldots x_n$ (which we denote by $|x_1,x_2,\ldots,x_n\rangle$) is M itself. It is well known that a module is Noetherian if and only if every submodule is finitely generated.

A submodule N of a right K-module M is called a complement submodule if there is a submodule A of M such that N is maximal among the submodules T satisfying $T \cap A = 0$. Also we say that N is a complement of A. By Zorn's Lemma, it follows that every submodule has a complement N. Also, A is essential in M if and only if 0 is a complement for A. The ascending chain condition for complement submodules of a right K-module will be denoted by max.-rc.

The next few results are due to Goldie (8).

Theorem 3.5.1 A right K-module has max-rc if and only if M contains no infinite direct sums of non-zero submodules.

Proof: Let M have max-rc, and suppose that $V = \oplus_{j=1}^{\infty} B_j$ is a direct sum of submodules of M. Let T_0 be a complement of V, and suppose we have defined T_i, $0 \leq i \leq n$, such that, for $i \geq 1$, T_i is a complement of $\oplus_{j=i+1}^{\infty} B_j$, and $T_i \supseteq T_{i-1} + B_i$. Then $T_n \cap \Sigma_{j=n+1}^{\infty} B_j = 0$, and hence the sum $T_n + B_{n+1}$ is direct, and $(T_n + B_{n+1}) \cap \oplus_{j=n+2}^{\infty} B_j = 0$. By Zorn's Lemma, there is a submodule T_{n+1} which is maximal among the submodules X satisfying $X \cap \oplus_{j=n+2}^{\infty} B_j = 0$, $X \supseteq T_n + B_{n+1}$. It is clear that T_{n+1} is a complement of $\oplus_{j=m+2}^{\infty} B_i$. The sequence $T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_n$ of complement submodules can thus be extended to a sequence $T_0 \subsetneqq T_1 \subsetneqq T_2 \cdots \subsetneqq T_n \subsetneqq T_{n+1}$, where T_{n+1} is also a complement submodule. It follows by induction that there is an infinite strictly increasing sequence of complement submodules, contrary to our assumption. Therefore max-rc implies that there are no infinite direct sums of submodules.

Conversely, suppose there was a strictly increasing sequence $T_1 \subsetneqq T_2 \subsetneqq \cdots \subsetneqq T_n \subsetneqq \cdots$ of complement submodules, and suppose that T_i is a complement of A_i. Now $B_i = T_{i+1} \cap A_i \neq 0$, and $B_n \cap \Sigma_{i=1}^{n-1} B_i \subseteq A_n \cap T_n = 0$ for each n. It follows that the sum $\Sigma_{i=1}^{\infty} B_i$ is a direct sum. Therefore, if there are no infinite direct sums, M must have max-rc. Q.E.D.

Lemma 3.5.2 Let M be a right K-module with max-rc. Then every submodule contains a uniform submodule.

Proof: Suppose that N is a submodule which contains no uniform submodules. Then N is itself not uniform, and N contains non-zero submodules N_1 and N'_1 such that $N_1 \cap N'_1 = 0$, and the sum $N_1 + N'_1$ is direct. Suppose that, for an integer n, we have a direct sum $N_1 + N_2 + \ldots N_n + N'_n$ of non-zero submodules of N. Since N'_n is not uniform, there submodules N_{n+1} and N'_{n+1} of N'_n whose intersection is zero. It follows that the sum $N_1 + N_2 + \ldots N_n + N_{n+1} + N'_{n+1}$ is direct. It follows that we can construct a sequence $N_1, N_2, \ldots$ of submodules for which the sum is an infinite direct sum. By the previous theorem, this contradicts max-rc. Q.E.D.

If M is a right K-module with max-rc, the previous lemma assures us that M contains uniform submodules. If we let $\{U_\alpha : \alpha \in \Lambda\}$ be the set of non-zero uniform submodules of M, an application of Zorn's Lemma guarantees that there is a subset F of Λ maximal among the subsets X of Λ for which $\Sigma_{\alpha \in X} U_\alpha$ is a direct sum. This F is a finite set, by Theorem 3.5.1. Suppose that $\{U_\alpha : \alpha \in F\} = \{U_1, U_2, \ldots U_n\}$. Then $V = U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is essential, for if $V \cap A = 0$, for some submodule $A \neq 0$, then A contains a uniform submodule $W \neq 0$. If $W = U_{\lambda_0}$, where $\lambda_0 \in \Lambda$, then

$G = F \cup \{\lambda_0\} \supsetneqq F$, and $\sum_{\alpha \in G} U_B$ is direct. This contradicts

the maximality if F. These remarks show that, if M has max-rc, there is a family $U_1, U_2, \ldots, U_n$ of uniform submodules such that $V = \oplus_{i=1}^{n} U_i$ is a direct sum, and is essential in M.

<u>Remark</u> The integer n appearing in the previous paragraph is an invariant of the module, called the <u>dimension</u> of M. It is the maximum possible length of a direct sum of submodules of M. (See Goldie (8), Chapter three.) Neither the invariance of n, nor the fact that there are no direct sums of more than n submodules will be needed for our purposes.

<u>Theorem 3.5.3</u> Let K be a semiprime ring with a family $\{W_\alpha : \alpha \in \Lambda\}$ of uniform right ideals such that $\sum_{\alpha \in \Lambda} W_\alpha$ is an essential right ideal of K. Then $U(K) = 0$.

<u>Proof</u>: Any uniform right ideal W of a semiprime ring K is a prime right K-module (and hence $W \in \Sigma_K^1$). For if $V \neq 0$ is a submodule of W, and if $k \in (0:V)$, then $Vk = 0$, $VKk = 0$, and $V \subset la(Kk) = \{x : xKk = 0\}$. Now Kk is a left ideal, and so $la(Kk)$ is easily seen to be a two-sided ideal of K. If $WkK \neq 0$, then $T = V \cap WkK \neq 0$, since W is uniform, but

$T^2 \subseteq V \cdot WkK \subseteq la(Kk) \cdot KkK = 0$. Since K is semiprime, T must be zero, and so $WkK = 0$, and $Wk \subseteq la(K)$. But $la(K) = 0$, so $Wk = 0$. Thus, if V is a non-zero submodule of W, then $(0:V) \subseteq (0:W)$. The other inclusion is always true, so $(0:V) = (0:W)$. Also $WK \neq 0$, since $W^2 \neq 0$. By Lemma 2.1.4, W is thus a prime K-module.

If $T = \sum_{\alpha \in \Lambda} W_\alpha$ is essential in K, where each W is uniform, and hence in Σ_K^1, we have $U(K) \subseteq \bigcap_{\alpha\in\Lambda} (0:W_\alpha) \subseteq (0:T)$. The proof will be complete if we can show that $(0:T) = 0$. But T is essential, so if $(0:T) \neq 0$, then $(0:T) \cap T \neq 0$, and this intersection is a right ideal whose square is zero. Since K is semiprime, this is a contradiction. Therefore $(0:T) = 0$. Q.E.D.

<u>Corollary 3.5.4</u> If K is a ring for which $K/B(K)$ has max-rc, then $U(K) = B(K)$.

<u>Proof</u>: If $\overline{K} = K/B(K)$ has max-rc, then as we have seen, there is a finite family of uniform right ideals whose sum is an essential right ideal of $\overline{K}$. By the theorem, $U(\overline{K}) = 0$, which implies $U(K) \subseteq B(K)$. The other inclusion is always true, so we have $U(K) = B(K)$. Q.E.D.

<u>Theorem 3.5.5</u> If K is a semiprime ring with max-rc and $Z_r(K) = 0$, then $W(K) = 0$.

Proof: The condition max-rc guarantees that there is a family of uniform right ideals $W_1, W_2, \ldots, W_n$ such that $T = \Sigma_{i=1}^{n} W_i$ is essential. The argument in Theorem 3.5.3 shows that each W_i is in Σ_K^1. Since $Z(W_i) = W_i \cap Z_r(K) = 0$, Lemma 3.1.3 guarantees that each W_i is a rational extension of every non-zero submodule, and so $W_i \in \Sigma_K^2$. If V is a submodule of W_i (i.e. a right ideal of K contained in W_i), $V^2 \neq 0$ and so there is a v in V such that $vW_i \neq 0$. Since W_i is rational over all submodules, the map $L_v : W_i \to V$, where $L_v(w) = vw$, must have kernel zero and hence is one-to-one. Thus W_i and V are subisomorphic, and, since V was any non-zero submodule, W_i is homogeneous. This shows that each $W_i \in \Sigma_K^3$. Thus $W(K) \subseteq \bigcap_{i=1}^{n} (0:W_i) \subseteq (0:T)$, and, as in Theorem 3.5.3, $(0:T) = 0$. Q.E.D.

Corollary 3.5.6 If K is a right Noetherian ring, then $B(K) = W(K)$ and this is a nilpotent ideal.

Proof: It is known that, in a right Noetherian ring K, $B(K)$ is nilpotent. The ring $\overline{K} = K/B(K)$ is also right Noetherian, and hence has max-rc.

We now show that $Z_r(\overline{K}) = 0$. For x in $\overline{K}$, $(0:x^i) \subseteq (0:x^{i+1})$. Since $\overline{K}$ is right Noetherian, for each x there is an n such that $(0:x^n) = (0:x^{n+1})$. Suppose that x is in $Z_r(\overline{K})$. Then, if n is as before, and if $x^n \neq 0$, $x^n\overline{K} \neq 0$ (since $\overline{K}$ is semiprime), and $x^n\overline{K} \cap (0:x) \neq 0$ (since x is in $Z_r(\overline{K})$). Therefore there is a $\overline{k}$ in $\overline{K}$ such that $x^n\overline{k} \neq 0$ but $xx^n\overline{k} = 0$, and $(0:x^n) \neq (0:x^{n+1})$. This contradiction shows that, for x in $Z_r(\overline{K})$, $x^n = 0$, and so $Z_r(\overline{K})$ is a nil ideal. By Levitzki's theorem, (see Faith (1), Chapter ten) $Z_r(\overline{K})$ is nilpotent, and thus $Z_r(\overline{K}) = 0$.

We can now apply Theorem 3.5.5 to conclude that $W(\overline{K}) = 0$, and so $W(K) \subseteq B(K)$. Since we always have $W(K) \supseteq B(K)$, we are done. Q.E.D.

<u>Remark</u> It is well known (see, for example, Faith (7), Chapter nine, Theorem 7) that a semiprime ring K with max-rc and $Z_r(K) = 0$ has a classical right quotient ring Q which is semiprime and right Artinian. Conversely, it is also true that if K has a semiprime right Artinian right classical quotient ring Q, then K is semiprime with max-rc and $Z_r(K) = 0$. Thus the conditions in Theorem 3.5.5 are familiar ones in ring theory.

The concept of homogeneous modules is related to the concept of basic modules, as introduced by Goldie in (9). For Noetherian modules over right Noetherian modules, Goldie defined a <u>basic module</u> M to be one which is homogeneous and

and has $Z(M) = 0$. By Lemma 3.5.2, a Noetherian module M contains a uniform submodule N. If M is also homogeneous, M is isomorphic to a submodule of N, and thus M is itself uniform. Also $Z(M) = 0$ and M uniform imply (Lemma 3.1.3) that M is a rational extension of each non-zero submodule. Thus a basic module, by Goldie's definition, is in Σ_K^3. For our purposes, we shall define a <u>basic module</u> to be one which is uniform, homogeneous, and which has $Z(M) = 0$. This is equivalent to saying a basic module is a member of Σ_K^3 with singular submodule zero.

The next result is due to Goldie (9).

<u>Theorem 3.5.7</u> Let K be a right Noetherian ring, and let M be any right K-module. Either $Z(M)$ is essential in M or M contains a basic module.

<u>Proof</u>: If $Z(M)$ is not essential in M, there is a submodule N of M for which $Z(N) = N \cap Z(M) = 0$. Since K is right Noetherian, there is a non-zero submodule N' of N such that $(0:N')$ is a maximal member of $\{(0:X)\colon X$ is a non-zero submodule of $N\}$. Then $N'K \neq 0$, for $N'K = 0$ implies $N' \subseteq N \cap Z(M) = 0$, which is false. Also, if T is a non-zero submodule of N', then $(0:T) \supseteq (0:N')$, and the maximality of $(0:N')$ gives $(0:T) = (0:T) = (0:N')$. N' is therefore a prime K-module, by Lemma 2.1.4. For any $n \neq 0$ in N', $nK \neq 0$ (elsewise $n \in N \cap Z(M) = 0$), and nK is a Noetherian

right K-module which, by Lemma 3.5.2, must contain a non-zero uniform submodule M'. Thus M' is prime, uniform, and has $Z(M') \subseteq Z(N) = 0$. By Corollary 3.4.7 M' contains a non-zero submodule M'' in Σ_K^3. This submodule is basic. Q.E.D.

In the reference (9), Goldie showed that, for the cases he was considering, the uniform right ideals of a ring serve as examples of basic modules. One can show that, in a sense, these are the only examples. This is the importance of the next theorem.

Theorem 3.5.8

(1) A basic K-module M is isomorphic, as a K-module, to a uniform right ideal of K.

(2) If K is semiprime and if $Z_r(K) = 0$, any uniform right ideal of K is a basic K-module.

Proof: (1) If M is a basic K-module, $Z(M) = 0$, and for $m \neq 0$ in M, $(0:m)$ is not essential in K. Therefore there is a right ideal I of K such that $I \cap (0:m) = 0$. The K-module homomorphism $f: K_K \to M$, where $f(k) = mk$ is one-to-one on I, so I is isomorphic to a submodule of M. The homogeneity of M guarantees that there is a K-monomorphism of M into $f(I)$. Composing this mapping with the inverse of the restriction of f to I gives a K-monomorphism g from M into K. M is basic, and therefore uniform, so $g(M)$ is a uniform right ideal of K.

(2) If K is semiprime, and $Z_r(K) = 0$, any uniform right ideal I satisfies $Z(I) = 0$. Proceeding just as in Theorem 3.5.5, I is in Σ_K^3, and so I is indeed basic. Q.E.D.

3.6 Quasi-injective Modules

In this section, we introduce the concept of a quasi-injective module, and sketch some of the results known about them. Most of the results of this section are due to Johnson and Wong and their proofs may be found in Faith (7). Later we shall employ these results to obtain generalizations of the Jacobson density theorem.

A right K-module E is said to be <u>injective</u> if, given a K-module M and a submodule N, and given a K-homomorphism $f:N \to E$, there is a K-homomorphism $g:M \to E$ such that the restriction of g to N is f. It is well known that, given any right K-module M, there is an injective K-module $E(M)$ and a K-monomorphism $i:M \to E(M)$ such that $E(M)$ is an essential extension of $i(M)$. Furthermore, if there is another injective K-module $E'(M)$, and a K-monomorphism $i':M \to E'(M)$ such that $E'(M)$ is an essential extension of $i'(M)$, it is known that there is a K-isomorphism $f:E(M) \to E'(M)$ such that $f_i(m) = i'(m)$ for all m in M. Therefore the pair $(E(M), i)$ is, in a sense, unique up to an isomorphism which respects the imbedding of M. The pair $(E(M), i)$ is called the <u>injective hull of M</u>.

A right K-module Q is <u>quasi-injective</u> if, whenever we have N a submodule of Q, and $f: N \to Q$ a K-homomorphism, there is a K-homomorphism $g: Q \to Q$ whose restriction to N is f. Clearly any injective module is quasi-injective, and any simple module is quasi-injective. However, injectives and simple modules do not exhaust the class of quasi-injective modules. For example, let K be the ring of integers, and let M be the K-module (i.e., an abelian group) of integers modulo 4. It is well known that, for this choice of K, the K-injectives are the divisible abelian groups, so M is neither simple nor injective. The only submodules of M are M, 0, and $\{\overline{0}, \overline{2}\}$. Clearly, to show M is quasi-injective, it is sufficient to show that any homomorphism from N $(= \{\overline{2}, \overline{0}\})$ to M can be extended to an endomorphism of M. Let f be such a homomorphism. Then $2f(\overline{2}) = f(\overline{4}) = f(\overline{0}) = \overline{0}$, and so $f(\overline{2}) = \overline{2}$ or $\overline{0}$. In the former case, let $g(\overline{i}) = \overline{i}$, and in the latter case let $g(\overline{i}) = \overline{0}$, where $i = 0,1,2,$ or 3. This gives the desired extension, so M is quasi-injective.

<u>Theorem 3.6.1</u> Let M be any right K-module, and let $(E(M), i)$ be the injective hull of M. (If we identify M and $i(M)$, we can regard M as being a submodule, and an essential submodule, of $E(M)$.) If we define $H = \mathrm{Hom}_K(E(M), E(M))$, then, writing the operators of H on the left, HM is a quasi-injective K-module which is an essential extension of M. M is itself quasi-injective if

and only if $M = HM$.

Proof: Clearly HM is a K-submodule of $E(M)$, and it contains M. Since $E(M)$ is an essential extension of M, it follows that HM is also an essential extension of M. Suppose that N is a submodule of HM, and that $f: N \to HM$ is a K-homomorphism. We can regard f as mapping N to $E(M)$. From the injectivity of $E(M)$, there is an h in H such that $h(n) = f(n)$ for all n in N. If $w \in HM$, $w = \Sigma_{i=1}^{t} h_i(m_i)$, where $h_i \in H$ and $m_i \in M$. Then $h(w) = \Sigma_{i=1}^{t} hh_i(m_i) \in HM$, and so $h(HM) \subseteq HM$. Therefore the restriction $\overline{h}$ of h to HM is an endomorphism of HM which extends f. This proves that HM is quasi-injective.

Clearly, if $M = HM$, then M is quasi-injective. Conversely, suppose that M is quasi-injective, and let $h \in H$. If $h = 0$, then $E(M) = h^{-1}(M)$. If $h \neq 0$, then $h(E(M)) \neq 0$, $h(E(M)) \cap M \neq 0$ (since $E(M)$ is an essential extension of M), and so $h^{-1}(M) \neq 0$, and $W = M \cap h^{-1}(M) \neq 0$. The restriction of h to W is a map $g: W \to M$ which, since M is quasi-injective, can be extended to a map $\overline{g}: M \to M$. By the injectivity of $E(M)$ there is $\overline{h}$ in H such that $\overline{h}(m) = \overline{g}(m)$ for all m in M. Consider $(h-\overline{h})M$. If this were non-zero, then $(h-\overline{h})M \cap M \neq 0$, and there are m and m' in M such that $m = \overline{h}(m') - h(m') \neq 0$. Then $h(m') = \overline{h}(m') - m = \overline{g}(m') - m \in M$, and so $m' \in W$. But this gives

$g(m') = h(m')$, and we obtain $h(m') = g(m') = \bar{g}(m') = \bar{h}(m')$, and $m = 0$, a contradiction. Therefore $(h-\bar{h})M = 0$. Then, for m in M, $h(m) = \bar{h}(m) = \bar{g}(m) \in M$, and so $h(M) \in M$, and $HM \subseteq M$. Since $M \subseteq HM$, we have $HM = M$. Q.E.D.

In the future the quasi-injective module HM obtained from the injective hull $E(M)$ of M will be denoted $\underline{Q(M)}$.

Remark Using the same notation as in the theorem, for any h in H, we have $h(HM) \subseteq HM$. Therefore the restriction of h to HM is in $\Lambda = \mathrm{Hom}(\ Q(M), Q(M)\ )$. It follows then that we also have $Q(M) = \Lambda M$.

Lemma 3.6.2 Any complement submodule of a quasi-injective module M is a direct summand of M.

Proof: Let M be quasi-injective, and let N be a complement submodule of M. Suppose that N is a complement of L. Then N is maximal among the submodules X of M such that $X \cap L = 0$. Consider the map $g_0 : N + L \to N$ defined by

$g_0(n + a) = n$, where $n \in N$ and $a \in L$. We claim that

this can be extended to a homomorphism $f : M \to N$. This is sufficient to prove the theorem, for then $M = N \oplus (1-f)M$, for any m in M may be written $m = f(m) + (m - f(m))$, and if $n = x - f(x)$ is in the intersection of these submodules, $f(x) \in N$, $ff(x) = g_0 f(x) = f(x)$, and we have $n = f(n) = f(x) - ff(x) = f(x) - f(x) = 0$.

Consider the set of all ordered pairs (Y,t), where

Y is a submodule of M containing $N \oplus L$, and t is a K-homomorphism from Y into N extending g_o. We partially order this set by saying $(Y_1,t_1) \succ (Y_2,t_2)$ if and only if $Y_1 \supseteq Y_2$, and $t_1(y) = t_2(y)$ for all y in Y_2. It is easily verified that this partial ordering is inductive, and so we can apply Zorn's Lemma to obtain a maximal extension (W,g), relative to this ordering, of g_o. Then $W \geq N + L$, $g: W \to N$ is a K-homomorphism which is an extension of g_o.

We claim that $W = M$, and g is the desired map. For suppose that $W \neq M$. Since M is quasi-injective, the map $g: W \to N \subseteq M$ can be extended to a homomorphism $f: M \to M$, and $f(M) \not\subset N$. (For $f(M) \subset N$ implies that $(M,f) \succ (W,g)$, contradicting the maximality of (W,g)). Then $f(M) + N \supsetneq N$, and so $(F(M) + N) \cap L \neq 0$. Let $f(m) + n = a \neq 0$, where a is in L. If $m \in W$, then $f(m) = g(m)$ is in N, and a is in $L \cap N = 0$, which is false. Thus $m \notin W$. If we set $T = f^{-1}(N + L)$, we see that m is in T, and that $T \supseteq W$. Therefore $T \supsetneq W$. If we let f' be the restriction of f to T, it is easily verified that $(T, g_of') \succ (W,g)$, contradicting the maximality of (W,g). Thus $W = M$, and $g: M \to N$ extends g_o. Q.E.D.

A right K-module is said to be <u>indecomposable</u> if and only if M cannot be written as $A \oplus B$, where A and B are non-zero submodules.

Corollary 3.6.3 A quasi-injective module M is uniform if and only if it is indecomposable.

Proof: Clearly any uniform module is indecomposable. Conversely, if M is not uniform, there are submodules S and T, both non-zero, such that $S \cap T = 0$. Zorn's Lemma can be applied to obtain a complement S' of S containing T. By the theorem, S' is a direct summand of M, and $S' \neq M$, $S' \neq 0$. This shows that M is not indecomposable. Q.E.D.

Theorem 3.6.4 If M is any right K-module, the following are equivalent:

(1) M is uniform

(2) $Q(M)$ is uniform

(3) $Q(M)$ is indecomposable.

Proof: (1), together with the fact that $Q(M)$ is an essential extension of M implies (2), and (2) clearly implies (1). Since $Q(M)$ is quasi-injective, (2) and (3) are equivalent by Corollary 3.6.3. Q.E.D.

A ring is said to be regular if, for every x there is a y such that $x = xyx$. If $x = xyx$, then $e = xy$ is idempotent, and $|x\rangle = |e\rangle$. (Recall that $|w\rangle$ means the submodule - in this case, the right ideal - generated by w.) Consequently, in a regular ring, every principal right ideal is generated by an idempotent. Conversely, if every principal right ideal in a ring K is generated by an idempotent, then, for any x, $|x\rangle = |e\rangle$, and $e = xy$ for some y in K. But $x = ew$, for some w, and $ex = eew = ew = x$. Therefore $x = ex = xyx$, and K is a regular ring.

If y is a left quasi-regular element in a ring K with unity, then $x + y - xy = 0$, and $(1-x)(1-y) = 1$, so $1-y$ has a left inverse. Conversely, if K is a ring with unity, and if $1-y$ is an element with a left inverse z, then writing $x = 1-z$, we have $(1-x)(1-y) = z(1-y) = 1$. This implies that $x + y - xy = 0$, and y is left quasi-regular.

A unit of a ring K with unity is an element with a (two-sided) inverse. If K is a ring for which the non-units form an ideal I, we call K a local ring. If K is a local ring, no element of $J(K)$ can be a unit (for if it were otherwise $J(K)$ would contain 1, and 1 is clearly not right or left quasi-regular), and so $J(K) \subseteq I$. Conversely, if $x \in I$, then $1-x \notin I$ (for otherwise $1 = 1-x + x$ would be in I, and hence a non-unit, which is absurd) and so $(1-x)^{-1}$ exists. As above, this implies that x is left quasi-regular, and I is a left quasi-regular left ideal, that is $I \subseteq J(K)$. Consequently, in a local ring K, the set of non-invertible elements of K is precisely $J(K)$. It follows at once that, in this case, $K/J(K)$ is a division ring.

It is a well known theorem, known as Schur's Lemma, that, if M is a simple right K-module, then $\mathrm{Hom}_K(M,M)$ is a division ring. Simple modules are clearly indecomposable quasi-injectives, so the second part of the following theorem provides a generalization of Schur's Lemma.

Theorem 3.6.5 Suppose that Q is a quasi-injective right K-module. Denote $\Lambda = \mathrm{Hom}_K(Q,Q)$. Then $J(\Lambda) = \{\alpha\epsilon\Lambda\colon\ \mathrm{Ker}(\alpha)$ is essential in $Q\}$ and $\Lambda/J(\Lambda)$ is a regular ring. Furthermore, Λ is a local ring if and only if Q is indecomposable, and, in this case, $J(\Lambda) = \{\alpha\epsilon\Lambda\colon\ \ker(\alpha) \neq 0\}$.

Proof: Let $I = \{\alpha\epsilon\Lambda\colon\ \ker(\alpha)$ is essential in $Q\}$. If $\lambda\epsilon\Lambda$, let L be a maximal member of the set of all submodules X of Q which satisfy $X \cap \ker(\lambda) = 0$. Then $L \oplus \ker(\lambda)$ is essential in Q. For if $T \neq 0$ is a submodule of Q either $T \subseteq L$ or $T + L \supsetneqq L$, in which case $(T+L) \cap \ker(\lambda) \neq 0$. In either case it follows that $T \cap (\ker(\lambda) \oplus L) \neq 0$. Define $f\colon \lambda L \to Q$ by setting $f(\lambda x) = x$. If $\lambda x = \lambda y$, where x and y are in L, then $x-y \in L \cap \ker(\lambda) = 0$, so f is well defined on L. Since Q is quasi-injective, f can be extended to $\theta\colon Q \to Q$. A simple calculation shows that $(\lambda - \lambda\theta\lambda)(L \oplus \ker(\lambda)) = 0$, so $\lambda - \lambda\theta\lambda \in I$.

Since $\ker(\alpha - \beta) \supseteq \ker(\alpha) \cap \ker(\beta)$, and since $\ker(\lambda\alpha) \supseteq \ker(\alpha)$ it follows that I is a left ideal of Λ. If $\alpha\epsilon I$, since $\ker(1-\alpha) \cap \ker(\alpha) = 0$, we have $\ker(1-\alpha) = 0$. If $T = (1-\alpha)Q$, then the map $g\colon T \to Q$, where $g((1-\alpha)q) = q$, is well defined and can be extended to a map $\delta\epsilon\Lambda$. Then $\delta(1-\alpha) = 1$, the identity map on Q, so $(1-\alpha)$ has a left inverse, and α is left quasi-regular. Therefore I is a left quasi-regular left ideal, and $I \subseteq J(\Lambda)$.

Suppose now that $\lambda \in J(\Lambda)$. Then, as above, there is a $\theta \in \Lambda$ such that $\lambda - \lambda\theta\lambda \in I$. Since $\lambda\theta \in J(\Lambda)$, $\lambda\theta$ is left and right quasi-regular, and $(1-\lambda\theta)^{-1}$ exists. Since I is a left ideal, $\lambda = (1-\lambda\theta)^{-1}(1-\lambda\theta)\lambda = (1-\lambda\theta)^{-1}(\lambda-\lambda\theta\lambda) \in I$. Therefore $J(\Lambda) \subseteq I$, and $J(\Lambda) = I$. We have proved that $J(\Lambda) = I$, and that $\Lambda/J(\Lambda)$ is a regular ring.

Suppose now that $\lambda + J(\Lambda)$ is an idempotent element of $\Lambda/J(\Lambda)$. Then $\lambda^2 - \lambda \in J(\Lambda)$, and $X = \ker(\lambda^2-\lambda)$ is essential in Q. Now we claim that the sum $\lambda X + (1-\lambda)X$ is direct. For if $\lambda x = x' - \lambda x'$ for x, x' in X, then $\lambda^2 x = (\lambda-\lambda^2)x' = 0$. Also, $(\lambda^2-\lambda)x = 0$ so we have $\lambda x = 0$. Consequently, if Q is indecomposable (and hence uniform, by Corollary 3.6.3) and if $\lambda + J(\Lambda)$ is idempotent in $\Lambda/J(\Lambda)$, then either $\lambda X = 0$, and $\lambda \in J(\Lambda)$ (since X is essential), or $(1-\lambda)X = 0$, and $(1-\lambda) \in J(\Lambda)$. Thus the indecomposability of Q implies that $\Lambda/J(\Lambda)$ has no idempotents other than 0 and 1. Since $\Lambda/J(\Lambda)$ is regular, given any $x \neq 0$ in $\Lambda/J(\Lambda)$, there is a y such that $xyx = x$. Therefore xy and yx are non-zero idempotents and so these are both 1, showing that $\Lambda/J(\Lambda)$ is a division ring. Now, if α is a non-unit in Λ, then, for some $\beta \in \Lambda$, we have $\alpha-\alpha\beta\alpha = (1-\alpha\beta)\alpha \in J(\Lambda)$. Since $\Lambda/J(\Lambda)$ is a division ring, this implies that either α or $(1-\alpha\beta)$ is in $J(\Lambda)$. The latter implies that $1 - (1-\alpha\beta) = \alpha\beta$ is invertible, so α would be invertible, which is false. Therefore, if α is a non-unit in Λ, we have $\alpha \in J(\Lambda)$. Since any member of $J(\Lambda)$ is a non-unit, the non-units form an ideal and Λ is a local ring, provided Q is indecomposable.

Conversely, assume that Λ is a local ring. If Q is not indecomposable, we can write $Q = S \oplus T$ where S and T are non-zero. The map $p: Q \to Q$, where $p(s+t) = s$, is an idempotent element of Λ, and so $p(1-p) = 0$. Since $\Lambda/J(\Lambda)$ is a division ring, either p or $1-p$ is in $J(\Lambda)$ and either $\ker(p) = T$ or $\ker(1-p) = S$ is essential in Q. Since both S and T are not essential, Q must be indecomposable.

If Q is indecomposable, and therefore uniform, (by Corollary 3.6.3), any non-zero submodule of Q is essential, and so it follows from the first part of the theorem that $J(\Lambda) = \{\alpha: \ker(\alpha) \neq 0\}$. Q.E.D.

<u>Theorem 3.6.6</u> If M is a prime module, then $Q(M)$ is prime, and $(0:M) = (0:Q(M))$.

<u>Proof</u>: Recall that $Q(M) = HM$, where $H = \mathrm{Hom}_K(E(M),E(M))$, and where $E(M)$ is the injective hull of M. Suppose that T is a non-zero submodule of $Q(M)$. $T \cap M \neq 0$, since $Q(M)$ is an essential extension of M. From the primeness of M, we have $(0:M) = (0: T \cap M) \supseteq (0:T) \supseteq (0:Q(M))$. However,

it is clear from the fact that $Q(M) = HM$ that $(0{:}Q(M)) \supseteq (0{:}M)$. Therefore $(0{:}T) = (0{:}Q(M)) = (0{:}M)$, and $Q(M)$ is prime by Lemma 2.1.4. Q.E.D.

For a right K-module M, and N a subset of M, let $N^K = \{k \in K\colon Nk = 0\}$, and for T a subset of K, let $T^M = \{m \in M\colon mT = 0\}$. $(T^M)^K$ is denoted T^{MK} and $(N^K)^M$ is denoted N^{KM}. We note for future reference that, if M is a prime K-module, then $K^M = 0$.

Theorem 3.6.7 Let Q be a quasi-injective right K-module, and let $\Lambda = \mathrm{Hom}_K(Q,Q)$. Then Q has a left Λ-, right K-bimodule structure. If N is any Λ-submodule satisfying $N^{KQ} = N$, then, for any x in Q, $(N+\Lambda x)^{KQ} = N + \Lambda x$.

Proof: Let $\Lambda = N^K$ and $B = x^K$. Then $A^Q = N$. Simple computation shows $B = x^K = (\Lambda x)^K$.

Clearly $N + \Lambda x \subseteq (N+\Lambda x)^{KQ}$, and so it suffices to prove $(N+\Lambda x)^{KQ} \subseteq N+\Lambda x$. Now $(N+\Lambda x)^K = N^K \cap (\Lambda x)^K = A \cap B$, so we only need show $(A \cap B)^Q \subseteq N + \Lambda x$. Suppose $y \in (A \cap B)^Q$, and consider $\theta\colon xA \to yA$ defined by $\theta(xa) = ya$. If $xa = xa'$, where a and a' are in A, then $a-a' \in A \cap x^K = A \cap B$. Since $y \in (A \cap B)^Q$, $y(a-a') = 0$, and the mapping is well-defined. This is a K-homomorphism from xA to $yA \subseteq Q$, and the quasi-injectivity of Q implies that θ may be extended to $\lambda : Q \to Q$. Then $\lambda xa = ya$ for all a in A, so $(\lambda x-y) \in$ $(\lambda x-y) \in A^Q = N$, and $y \in N + \Lambda x$. Q.E.D.

<u>Corollary 3.6.8</u> Suppose that Q is quasi-injective, and that $K^Q = 0$. If $x_1, x_2, \dots, x_n$ are any elements from Q, then $(\Sigma_{i=1}^{n} \Lambda x_i)^{KQ} = \Sigma_{i=1}^{n} \Lambda x_i$.

<u>Proof</u>: Taking $N = 0$ in the theorem, since $0^{KQ} = K^Q = 0$, we have $(\Lambda x_1)^{KQ} = \Lambda x_1$. The rest follows by the obvious induction. Q.E.D.

We shall say that the elements $x_1, x_2, \dots, x_n$ of Q are <u>Λ-independent</u> if, for each i, $x_i \notin \Sigma_{\substack{j=1 \\ j\neq i}}^{n} \Lambda x_j$.

<u>Theorem 3.6.9</u> (Density theorem for quasi-injective modules)

If Q is a quasi-injective right K-module such that $K^Q = 0$, and if $x_1, x_2, \dots, x_n$ are Λ-independent elements of Q (where $\Lambda = \mathrm{Hom}_K(Q,Q)$), then there are right ideals $A_1, A_2, \dots, A_n$ in K such that $x_i A_i \neq 0$, $x_i A_j = 0$ for $j \neq i$. Furthermore, if $y_1, y_2, \dots, y_n$ are elements of Q such that $y_i \in x_i A_i$ for each i, then there exists k in K such that $x_i k = y_i$, for $i = 1, 2, \dots, n$.

<u>Proof</u>: Let $A_i = (\Sigma_{\substack{j=1 \\ j\neq i}}^{n} \Lambda x_j)^K$. This is a right ideal in K,

and, by Corollary 3.6.8, $A_i^Q = \Sigma_{\substack{j=1 \\ j \neq i}}^n \wedge x_j$, which does not contain x_i. Therefore $x_i A_i \neq 0$, but $x_i A_j = 0$ for $j \neq i$. Suppose now that y_i is in $x_i A_i$, $i = 1,2,\ldots,n$, and suppose $y_i = x_i a_i$, where $a_i \in A_i$. Then $k = \Sigma_{i=1}^n a_i$ will satisfy $x_i k = y_i$ for all i. Q.E.D.

<u>Remark</u> If M is a faithful, irreducible right K-module, then M is simple and therefore quasi-injective. Also, $K^M = 0$, and $\wedge$ is a division ring. Furthermore, if x $x_1, x_2, \ldots, x_n$, and $A_1, \ldots, A_n$ are as in the theorem, then the simplicity of M gives $x_i A_i = M$. Thus, in this case, if $x_1, x_2, \ldots, x_n$ are $\wedge$-independent, and if $y_1, \ldots, y_n$ are arbitrary, there is a k in K such that $x_i k = y_i$ for $i = 1,2,\ldots,n$. This is the classical Jacobson Density Theorem.

<u>Remark</u> If M is a prime right K-module, $Q(M)$ is quasi-injective and prime (Theorem 3.6.6), and so $K^{Q(M)} = 0$, and Theorem 3.6.9 is applicable to $Q(M)$. We shall make use of this fact later.

In Theorems 3.6.4 and 3.6.6, we have seen that the properties of primeness and uniformity are transferred from M to $Q(M)$, and from $Q(M)$ to M. We now show that this is also true for the property of being a rational extension of every non-zero submodule.

<u>Theorem 3.6.10</u> Let M be a right K-module, $E(M)$ the injective hull of M, $H = \mathrm{Hom}_K(E(M), E(M))$, and $Q = HM$. The following are equivalent;

(1) Q is a rational extension of every non-zero submodule.

(2) M is a rational extension of every non-zero submodule.

(3) $\Lambda = \mathrm{Hom}_K(Q,Q)$ is a division ring.

<u>Proof</u>: The property of being a rational extension of every non-zero submodule is inherited by non-zero submodules, so (1) implies (2).

We show (2) implies (3). Since rational extensions are essential extensions, (2) implies that M is uniform. By Theorems 3.6.4 and 3.6.5, Λ is a local ring, and $J(\Lambda) = \{\alpha\colon \ker(\alpha) \neq 0\}$. We will prove that $J(\Lambda) = 0$.

Let $\alpha \in J(\Lambda)$, and let $L = \ker(\alpha)$. First of all $M \subseteq L$. For otherwise, $\alpha(M) \neq 0$, $M \cap \alpha(M) \neq 0$, and so $W = \{m \in M\colon \alpha(m) \in M\}$ has $\alpha(W) \neq 0$. Now the restriction f of α to W has kernel $L \cap M \neq 0$ (since M is essential in Q). Since M is a rational extension of $L \cap M$, we have $0 = f(W) = \alpha(W)$ a contradiction. This shows that $\alpha \in J(\Lambda)$ implies $\alpha(M) = 0$. Recall now that $Q = \Lambda M$. (See the remark following Theorem 3.6.1.) Thus any q in Q may be written $q = \Sigma_{i=1}^{n} \lambda_i m_i$, and $\alpha(q) = \Sigma_{i=1}^{n} \alpha\lambda_i(m_i)$. Since $\alpha \in J(\Lambda)$, $\alpha\lambda_i \in J(\Lambda)$, $\alpha\lambda_i(m_i) = 0$, and so $\alpha(q) = 0$. Thus $J(\Lambda) = 0$, and Λ is a division ring.

Finally we show (3) implies (1). Suppose $0 \neq X \subseteq T$, where X and T are submodules of Q, and suppose $f: T \to Q$ if a K-homomorphism with $f(X) = 0$. Q is quasi-injective, and so f can be extended to a member λ of Λ. Now if $\lambda \neq 0$, λ is invertible, since Λ is a division ring, and so λ is one-to-one. But $f(X) = \lambda(X) = 0$. Therefore $\lambda = 0$, and so $f(T) = 0$. This shows that Q is a rational extension of the submodule X. Since X was an arbitrary non-zero submodule of Q, we are done. Q.E.D.

3.7 Density Theorems for the Radicals H^U and W

In this section, we prove some results which generalize the well-known Jacobson Density Theorem. Also we will prove that the radical W coincides with the "weak radical" of Koh and Mewborn (17).

We shall say a ring K is an L.W.-transitive ring if and only if:

(1) There is a local ring Λ and a left Λ-module V such that K acts faithfully as a subring of $\mathrm{Hom}_\Lambda(V,V)$. That is, V has a left Λ-, right K-bimodule structure, and V_K is faithful.

(2) V contains a K-submodule M which is uniform, and for which $\Lambda M = V$.

(3) If $x_1, x_2, \ldots, x_n$ is any set of Λ-independent elements of V, and if $y_1, y_2, \ldots, y_n$ is any set of elements from M (note - not from anywhere in V) then there exist k in K and an invertible element λ in Λ such that $x_i k = \lambda y_i$, for $i = 1, 2, \ldots, n$.

Furthermore we shall say that a ring K is <u>W-transitive</u> if K is L.W.-transitive, and if the local ring Λ is a division ring.

<u>Theorem 3.7.1</u> A ring K is an L.W.-transitive ring if and only if K has a faithful module M in Σ_K^{UH}. Thus an H^U-semisimple ring is a subdirect sum of L.W.-transitive rings.

<u>Proof</u>: Suppose first that $M \in \Sigma_K^{UH}$, and that $(0:M) = 0$. Let $V = Q(M)$ and $\Lambda = \mathrm{Hom}_K(V,V)$. Since M is uniform, Theorems 3.6.4 and 3.6.5 tell us that Λ is a local ring. V, as a right K-module, if faithful by Theorem 3.6.6 and the fact that M is prime and faithful, and the remark following Theorem 3.6.1 says that $\Lambda M = V$. Thus (1) and (2) of the definition of L.W.-transitive rings are satisfied. Suppose now that $x_1, x_2, \ldots, x_n$ are Λ-independent elements of V, and $y_1, y_2, \ldots, y_n$ are any elements of M. Since V is a prime K-module (Theorem 3.6.6), we can apply Theorem 3.6.9 to find right ideals A_i, $i = 1, 2, \ldots, n$, such that $x_i A_i \neq 0$, but $x_i A_j = 0$ for $i \neq j$. Since M is

uniform, V is uniform by Theorem 3.6.4, and so

$P = M \cap (\bigcap_{i=1}^{n} x_iA_i) \neq 0$. M is homogeneous, so there is a K-monomorphism $f: M \to P$ which, by the quasi-injectivity of V, can be extended to a member λ of Λ. If $\ker(\lambda) \neq 0$, then $0 \neq M \cap \ker(\lambda) \subseteq \ker(f)$, which is false. Therefore $\ker(\lambda)$ is zero, and λ is a monomorphism. By Theorem 3.6.5, $\lambda \notin J(\Lambda)$ and so λ^{-1} exists. Now $\lambda(y_i) = f(y_i) \in P \subseteq x_iA_i$ for each i, so, by the second part of Theorem 3.6.9 there is an element k of K such that $x_ik = \lambda y_i$ for $i = 1,2,\ldots,n$. K is therefore L.W.-transitive.

Conversely suppose that K is L.W.-transitive, and suppose that V, Λ, and M satisfy the conditions. We choose $m \neq 0$ in M, and let $I = \{k \in K: mk = 0\}$. Consider the right K-module K/I. This is isomorphic to $mK \subseteq M$, and so K/I is uniform. We show that K/I is homogeneous. If X/I is a non-zero submodule of K/I, where X is a right ideal of K, $X \supsetneqq I$, then $mX \neq 0$. Suppose $x \in X$ is such that $mx \neq 0$. By condition (3), there are k in K and λ in Λ, λ invertible, such that $mxk = \lambda m$. Consider the K-module homomorphism $f: K \to X/I$, where $f(s) = xks + I$. The kernel of f is $\{s: xks \in I\} = \{s: mxks = 0\}$. Since λ is invertible, this kernel is precisely $\{s: ms = 0\} = I$. Then f induces a K-module monomorphism $\bar{f}: K/I \to X/I$, and so K/I and X/I are subisomorphic. This shows that K/I

is homogeneous. If $s \in (0:K/I)$, then $Ks \subseteq I$, and $mKs = 0$. For any y in M there are θ in $\wedge$, θ invertible, and r in K such that $mr = \theta y$. Then $\theta ys = mrs = 0$, and so $ys = 0$. Thus if $s \in (0:K/I)$, $Ms = 0$, and so $Vs = \wedge Ms = 0$. Since K acts faithfully on V, this implies $s = 0$. Thus K/I is a faithful right K-module which is homogenous and therefore prime (Lemma 3.2.1). K/I is therefore uniform, prime, homogeneous and faithful, and is therefore a faithful member of Σ_K^{UH}.

Now, if K is an H^U semisimple ring, K is a subdirect sum of rings K_α with a faithful member of $\Sigma_{K_\alpha}^{UH}$. Q.E.D.

<u>Theorem 3.7.2</u> A ring K is W-transitive if and only if K has a faithful member of Σ_K^3. Also, a W-semisimple ring is a subdirect sum of W-transitive rings.

<u>Proof</u>: Suppose M is a faithful member of Σ_K^3. Then, as in the previous theorem, we can show that K acts in an L.W.-transitive fashion on $Q(M)$. In order to show that K is W-transitive, it suffices to show that $\wedge = \mathrm{Hom}_K(Q(M), Q(M))$ is a division ring. But this follows immediately from the fact that M is a rational extension of each non-zero submodule and from Theorem 3.6.10.

Conversely, suppose that K is W-transitive, and that

V, Λ, and M satisfy (1) - (3) of the definition. As in the previous theorem, we can choose $m \neq 0$ in M and show that K/I, where $I = (0:m)$, is a faithful member of Σ_K^{UH}. In order to complete the proof it is sufficient to prove that K/I is a rational extension of every non-zero submodule. Suppose that $f: T/I \to K/I$ is a K-homomorphism with kernel J/I, where J and T are right ideals of K, and $I \subsetneqq J \subseteq T$.

We will show $f = 0$. If $f \neq 0$, let $f(T/I) = X/I \neq 0$, where $X \supsetneqq I$, and X is a right ideal of K. Then $mX \neq 0$, and there exist r in K, x in X, and λ in Λ, λ invertible, such that $mxr = \lambda m \neq 0$. Since λ is invertible, $(0:mxr) = (0:m) = I$, and $xr \not\in I$. But $mxrI = \lambda mI = 0$, whence $xrI \subseteq I$. Now $0 \neq xr + I \in X/I = f(T/I)$, and so we can let $xr + I = f(t + I)$ for some t in T, t not in I. Then $f(tI + I) = [f(t+I)]I = (xr+I)I = xrI + I = 0$ (in K/I), so $tI \subseteq J$. (Recall J/I is the kernel of f).

Either mt and m are dependent or they are independent. If mt and m are Λ-independent, there are s in K and θ in Λ, θ invertible, such that $mts = 0$, $ms = \theta m \neq 0$. Then $ts \in I \subseteq J$, but $s \not\in I$.

If mt and m are Λ-dependent there is a β in Λ (a division ring) such that $\beta(mt) = m$. The $(0:mt) = (0:m) = I$, and it follows that $tI \subseteq I$. However, $tJ \not\subset I$, for $tJ \subset I$ implies $mtJ = 0$, whence $mJ = 0$, and $J \subset I$, which is false. Thus it follows that $(tJ+I) \gneqq I$ and that

$(tJ+I)/I \cap J/I \neq 0$ in K/I (recall K/I is uniform). Therefore $Y = (tJ+I) \cap J \supsetneqq I$, and $mY \neq 0$. As we have seen before, there are y in Y, d in K, and γ in Λ such that $myd = \gamma m \neq 0$, and $y' = yd$ satisfies $y'I \subset I$ but $y' \not\subset I$. Now $y' \in Y$, so we can write $y' = tj + i \in J$, where $tj \not\subset I$. If $j \in I$, then $tj \in tI \subseteq I$, which is false. Again in this case we have an element $s\ (=j)$ such that $s \not\subset I$, but $ts \in J$.

Now $0 = f(ts+I) = f(t+I)s = (xr+I)s = xrs + I$, so $xrs \in I$. But then $s \in (0:mxr) = (0:m) = I$, a contradiction. This shows that $f = 0$, and therefore K/I is a rational extension of an arbitrary submodule $J/I \neq 0$.

Now, if K is a W-semisimple ring, K is a subdirect sum of rings K_α, where each K_α has a faithful member of $\Sigma^3_{K_\alpha}$. Q.E.D.

A ring A is a <u>right order in a ring B</u> if A is a subring of B, if every element of A which is not a zero-divisor in A is invertible in B, and if every element of B can be written as uv^{-1}, where u, v are in A, and v is not a zero-divisor in A.

In (17), Koh and Mewborn defined a ring to be a <u>weakly transitive</u> ring if:

(1) K acts faithfully as a ring of linear transformations on a left vector space V over a division ring D,

(2) There is a right order S in D, and a left S-, right K-submodule M of V which is a bimodule, which is uniform as a K-module, and which satisfies $V = DM$,

(3) If $x_1, x_2, \ldots, x_n$ is a finite D-independent set of elements from M, and if $y_1, y_2, \ldots, y_n$ are any elements of M, then there are k in K and s in S such that $x_i k = s y_i$, for $i = 1, 2, \ldots, n$.

Furthermore, they defined a radical class, which we will denote by W^*, by defining $W^*(K) = \prod\{I \triangleleft K\colon K/I$ is weakly transitive$\}$.

We now show that a ring acts in a W-transitive fashion on some vector space if and only if it acts in a weakly transitive fashion on some (perhaps different) vector space. It is an immediate consequence of this fact, the previous theorem, and Theorem 2.1.1 that $W(K) = W^*(K)$ for all rings K.

<u>Theorem 3.7.3</u> A ring K is weakly transitive if and only if it has a faithful member of Σ_K^3.

<u>Proof</u>: If K acts in a weakly transitive fashion on V, the same arguments as in Theorems 3.7.1 and 3.7.2 give a faithful member of Σ_K^3.

Conversely, suppose that M is a faithful member of Σ_K^3. As in the first part of the proof of Theorem 3.7.2,

$\Lambda = \mathrm{Hom}_K(Q(M),Q(M))$ is a division ring, and K acts in a W-transitive fashion on $Q(M)$. We will show that K acts in a weakly transitive fashion on $Q(M)$.

Let $S = \mathrm{Hom}_K(M,M)$. Since M is a rational extension of every non-zero submodule, every member of S is a monomorphism, and S has no zero divisors. The quasi-injectivity of $Q(M)$ guarantees that each s in S can be extended to a λ in Λ. If λ_1 and λ_2 are two extensions of s in S, then $M \subseteq \ker(\lambda_1 - \lambda_2)$. By Theorem 3.6.5, $\lambda_1 - \lambda_2 \in J(\Lambda) = 0$. Thus any element s in S has a unique extension $\overline{s}$ in Λ. In this way we have induced a mapping from S to Λ which is easily verified to be a ring monomorphism.

We now show that $\overline{S} = \{\overline{s}: s \in S\}$ is a right order in Λ. Let $\lambda \in \Lambda$, $\lambda \neq 0$. Then $W = \{m \in M: \lambda(m) \in M\}$ is a non-zero K-submodule of M, and $\lambda(W) \subseteq M$. Since W and M are subisomorphic, there is a module monomorphism $f:M \to W$. We can regard f as a mapping from M to M, that is, as a member of S. Then $\lambda \circ f: M \to W \to \lambda(W) \subseteq M$, and λf can also be considered as a member of S. Now $\overline{\lambda f}$ and $\lambda \overline{f}$ agree on M, and so $\lambda \overline{f} = \overline{\lambda f} = \overline{s}$. Thus $\lambda = \overline{s}\ (\overline{f})^{-1}$. Since Λ is a division ring, any non-zero member of $\overline{S}$ clearly has an inverse in Λ, so $\overline{S}$ is a right order in Λ.

Finally suppose that $x_1, x_2, \dots x_n$ are Λ-independent members of M, and that $y_1, y_2, \dots, y_n$ are in M. In the proof of Theorem 3.7.1, it was shown that $x_i k = \lambda y_i$ for some k

in K and $\lambda \in \Lambda$, where λ was an extension of a map $f: M \to P \subseteq M$. Clearly, in this case, $\lambda = \overline{f}$. This completes the proof. Q.E.D.

3.8 Prime Rings with Zero Singular Ideal and a Uniform Right Ideal

In Faith (7), page 129, the following problem is posed: if K is a prime ring with a uniform right ideal U and if $Z_r(K) = 0$ then $S = \mathrm{Hom}_K(U,U)$ is a right Ore domain (that is, S is a right order in a division ring). If $x_1, x_2, \ldots, x_n$ are S-independent elements of U, and if $y_1, y_2, \ldots, y_n$ are in U, does there exist $s \neq 0$ in S and k in K such that $x_i k = s y_i$, $i = 1,2,\ldots,n$. ?

If K and U are as above, then U, as a K-module, is a faithful member of Σ_K^3. This follows because the argument in Theorem 3.5.5 shows that U is in Σ_K^3, and since $U \cdot (0:U) = 0$, and since K is a prime ring, $(0:U) = 0$. In Theorem 3.7.3 it was shown that the conclusion of the problem is valid provided we regard U as a submodule of $Q(U)$, and choose our x_i's to be Λ-independent ($\Lambda = \mathrm{Hom}_K(Q(U),Q(U))$ not S-independent.

The following example shows that this is the best that can be done, and that the conjecture in the form stated above is false.

Example 3.8.1

Let K be a ring with 1 and with no zero-divisors, and for which K, as a right module over itself, is uniform, but for which K is not uniform as a left module over itself. Such a ring, or rather one with "right" and "left" interchanged is given in Example 9, page 71, of Divinsky (6).

K_K is uniform, and, since K has no zero divisors, $Z_r(K) = 0$. The ring K is a right order in a division ring D, and $Q(K_K) = D$, as a right K-module. K is isomorphic to $\mathrm{Hom}_K(K,K)$, and the embedding of $\mathrm{Hom}_K(K,K)$ into $\mathrm{Hom}_D(D,D) \cong D$ is just the embedding of K into D. The ring D is a division ring, and <u>any</u> two elements of D are left D-dependent.

However, since K, as a left K-module, is not uniform, there are x and y non-zero in K such that $Kx \cap Ky = 0$, and x and y are left S (i.e. K) independent. If the conjecture were true, there would exist k in K and s in S (i.e. K) such that $xk = sy$ and $ys = sy$, where $sy \neq 0$. But then $(x-y)k = 0$, and, since K has no zero divisors, either $x = y$ or $k = 0$, both of which are false. Therefore the conjecture in its original form is false.

The following results show how Goldie's Theorem on prime rings may be placed in a context similar to that of the classical density theorem for primitive rings.

Suppose that M is a faithful member of Σ_K^3, and that $Z(M) = 0$. (That is, suppose that M is a faithful basic module.) Then $D = \mathrm{Hom}_K(\ Q(M), Q(M)\ )$ is a division ring, and K acts in a W-transitive fashion on ${}_DQ$ (Theorem 3.7.2). The next theorem shows what happens when ${}_DQ$ is finite-dimensional.

<u>Theorem 3.8.2</u> Let M_K be a faithful basic module, and suppose that ${}_DQ$ is finite-dimensional, where $Q = Q(M)$, and $D = \mathrm{Hom}_K(Q,Q)$. Then K is a right order in the simple artinian ring $L = \mathrm{Hom}_D(Q,Q)$.

<u>Proof</u>: The ring L is the ring of all endomorphisms of a finite-dimensional vector space over D, and so L is indeed simple artinian.

We show first that K_K is essential in L_K. Let $x_1, x_2, \ldots, x_n$ be any basis for ${}_DQ$, and suppose that $s \neq 0$ is in L. Then $x_i s \neq 0$ for some i, which we may take to be 1. Since Q_K is uniform, $T = \bigcap_{i=1}^{n} x_i K$ is non-zero and is an essential submodule of Q_K. For each i, $E_i = \{k \epsilon K: (x_i s)k \epsilon T\}$ is an essential right ideal of K, and so $E = \bigcap_{i=1}^{n} E_i$ is essential in K. Now $x_1 sE \neq 0$ (otherwise $x_1 s \in Z(Q_K) = 0$) and so there is an e in E such that

$se \neq 0$. For each i, $x_i se \in T$, and so, by Theorem 3.6.9, there is an element k in K such that $x_i k = x_i se$ for $i = 1,2,\ldots,n$. Since $x_1, x_2, \ldots x_n$ is a basis for ${}_D Q$, $k = se \neq 0$. Therefore $sK \cap K \neq 0$ for any $s \neq 0$ in L, and K_K is an essential submodule of L_K.

Also, K is a subring of the left and right artinian ring L, and so K has the descending chain condition on both left and right annihilators. Since K is also prime, we can apply the recent result of Johnson and Levy (14) to conclude that any essential right ideal of K contains a regular element. Using the notation of the last paragraph, for $s \neq 0$ in L, the ideal E has a regular element c. As above, there is an element k in K such that $k = sc$.

The proof will be complete when we show that any regular element of K has an inverse in L. Suppose that k is a regular element in K. Then if s in L is such that $ks = 0$, then $k(sK \cap K) = 0$, and so $sK \cap K = 0$, which implies that $s = 0$, by the first paragraph. Therefore a regular element in K has no right annihilator (except 0) in L. Since L is artinian, this is sufficient for this element to have an inverse in L. Q.E.D.

Corollary 3.8.3. For a prime ring K, the following are equivalent.

(1) K is a right order in a simple artinian ring.

(2) $Z_r(K) = 0$, and K has max-rc.

(3) K has a faithful basic module M, and K satisfies the descending chain condition on right annihilators of subsets of M.

(4) $Z_r(K) = 0$, K has a uniform right ideal U, and K satisfies the descending chain condition on right annihilators of U.

Proof: The equivalence of (1) and (2) is well known. Furthermore, these conditions imply that K has a uniform right ideal U. Since K has an artinian quotient ring, and since the descending chain condition on right annihilators is inherited by subrings, (4) is easily seen to be true.

If (4) is satisfied, then the uniform right ideal U is a basic module (Theorem 3.5.8). Also U_K is faithful, since K is a prime ring. Therefore (3) is implied by (4).

Now if (3) is satisfied, then $Q(M)$ is finite-dimensional over the division ring $D = \mathrm{Hom}_K(\ Q(M), Q(M)\ )$. For if it were otherwise, since $Q(M) = DM$, we would be able to find a sequence $x_1, x_2, \ldots$ of D-independent elements in M. By the chain condition in (3), we obtain $\{x_1, x_2, \ldots, x_n\}^K =$ $\{x_1, x_2, x_n, x_{n+1}\}^K$ for some n. But then, by Corollary 3.6.8, we obtain $\sum_{i=1}^{n} Dx_i = \sum_{i=1}^{n+1} Dx_i$, contradicting the D-independence of the x_i's.

Thus $Q(M)$ is finite dimensional, and (1) follows from the previous theorem. Q.E.D.

It is well-known that a ring which acts transitively on a finite-dimensional vector space must be the endomorphism ring of that vector space, and consequently is a simple artinian ring. The next theorem is the analogue of this result for W-transitive rings.

Theorem 3.8.4 Let K be a W-transitive ring, and suppose that ${}_DV$ is finite dimensional. Then K has a simple artinian classical right quotient ring.

Proof: We are assuming that we have D, ${}_DV_K$, and M_K as in the definition of W-transitivity, and that ${}_DV$ is finite-dimensional.

We show first that $Z(V_K) = 0$. For if $v \in Z(V_K)$, $v \neq 0$, then for any m in M, there are k in K and d in D, $d \neq 0$, such that $vk = dm$. Then $(0:m) = (0:dm) = (0:vk)$, and the latter is essential since $v \in Z(V_K)$. Now $V = DM$, and this is finite-dimensional, so there is a basis $m_1, m_2, \ldots, m_n$ consisting of elements of M. But then $E = \bigcap_{i=1}^{n} (0:m_i)$ is essential, and so is non-zero. But $VE = (\sum_{i=1}^{n} Dm_i)E = 0$, and this contradicts the fact that K acts faithfully on V. Hence $Z(V_K) = 0$, and so $Z(M_K) = 0$.

Therefore M is a uniform K-module for which $Z(M) = 0$. Since $V = DM$ is faithful as a K-module, M is also faithful. Also M is a prime K-module. For suppose that $mI = 0$ for some m in M and $I \triangleleft K$. By Theorem 3.7.2, K has <u>some</u> faithful member of Σ_K^3, and so K is a prime ring. Therefore either $I = 0$, or I, being a two-sided ideal in a prime ring, is an essential right ideal, and $m \in Z(M) = 0$. This establishes the primeness of M.

Thus M is a faithful member of Σ_K^2, and $Z(M) = 0$. By Theorem 3.4.6, M has a submodule $N \neq 0$ which is a member of Σ_K^3. Also N_K is faithful and $Z(N) = 0$.

Let $V' = Q(N)$ and $D' = \mathrm{Hom}_K(V',V')$. The latter is a division ring, by Theorem 3.6.10 and Lemma 3.1.3.

We will now show that $\dim({}_{D'}V') \leq \dim({}_D V)$. Once this has been established, we may apply Theorem 3.8.2 to conclude that K is a right order in $\mathrm{Hom}_{D'}({}_{D'}V', {}_{D'}V')$, which is a simple artinian ring.

Suppose now that $\dim({}_D V) = n$, but $\dim({}_{D'}V') > n$. Then, since $D'N = V'$, we can find $x_1, x_2, \ldots, x_{n+1}$, a set of D'-independent elements of N. However, since $N \subseteq M \subseteq V$, these elements are not D-independent. Without loss of generality, we may assume that we may write $x_{n+1} = \sum_{j=1}^{n} d_j x_j$, where $d_1, d_2, \ldots, d_n$ are in D. Then $(0:x_{n+1}) \supseteq \bigcap_{j=1}^{n} (0:x_j)$, from which it follows that $\{x_1, x_2, \ldots, x_n\}^K = \{x_1, x_2, \ldots, x_n, x_{n+1}\}^K$. We may then apply Corollary 3.6.8 and conclude that $\sum_{j=1}^{n} D'x_j = \sum_{j=1}^{n+1} D'x_j$, contradicting the D'-independence of the x_j's. Thus $\dim({}_{D'}V') \leq \dim({}_{D}V)$. Q.E.D.

Remark: Under the conditions of the theorem, K is not, in general, a right order in $\mathrm{Hom}_D({}_DV, {}_DV)$. For example, let D be any division ring which is not commutative, and let C be the centre of D. Let $K = C$, ${}_DV_K = {}_DD_C$, and $M_K = C_C$. It is simply verified that K acts W-transitively on the vector space V, and that V is one-dimensional over D. But K is the centre of a division ring, and therefore is a field, and is its own classical right quotient ring. Also, $\mathrm{Hom}_D(V,V) = D$, which is not commutative, so $D \neq K =$ right quotient ring of K.

CHAPTER FOUR

HEREDITARY RADICAL IDEALS OF A RING

In this chapter we focus attention on the ideals I of a ring K for which $I = R(K)$ for some hereditary radical R. These, as we shall see, will be the closed ideals for a closure operation defined on the lattice of all ideals of the ring K. The properties of this collection of closed ideals will be investigated when certain conditions are imposed on K.

4.1 An Equivalence Relation for Rings

Recall from Chapter one that a radical class is said to be hereditary if and only if any ideal of a member of R is also a member of R. Also recall that a subring T of a ring K is called an accessible subring of K if there is a finite chain

$$T = T_0 \subseteq T_1 \subseteq T_2 \quad \cdots \subseteq T_n = K ,$$

where T_i is an ideal of T_{i+1}. It follows that, if R is a hereditary radical, and if $K \in R$, then any accessible subring of K is in R.

In Theorem 1.4.1, we saw that if M is a hereditary class of rings which is also closed under the taking of homomorphic images, then $S_0(M)$, the lower radical class with respect to M, is a hereditary radical class. Suppose that we are given any class M of rings. Does there necessarily exist a smallest hereditary radical class containing M? This question is answered affirmatively by the next theorem.

<u>Theorem 4.1.1</u> Suppose M is an arbitrary class of rings, and define M^* to be the class of all homomorphic images of accessible subrings of members of M. Then M^* is a homomorphically closed hereditary class, and $S_0(M^*)$ is the smallest hereditary radical class containing M.

<u>Proof</u>: M^* is clearly homomorphically closed. We show that M^* is hereditary. If $K \in M^*$, then there is a ring S in M, a chain $T = T_1 \subseteq T_2 \subseteq \dots \subseteq T_n = S$, where T_i is an ideal of T_{i+1}, and there is a homomorphism $f: T \to K$ which maps T onto K. Now, if I is an ideal of K, then $f^{-1}(I) = J$ is an ideal of T, and hence J is an accessible subring of S. Also, the restriction of f to J is a ring homomorphism from J onto I. This shows that $I \in M^*$, and thus M^* is hereditary.

By Theorem 1.4.1, $S_0(M^*)$ is hereditary. If R is any hereditary radical class, and if $M \in R$, then, since R is closed under taking of accessible subrings and homomorphic images, $M^* \subseteq R$. From the properties of the lower radical, we obtain $S_0(M^*) \subseteq R$. Q.E.D.

For a given class M of rings, we will denote $S_0(M^*)$

by $H(M)$. If M consists of a single ring K, we shall denote $H(M)$ by H_K.

We define an equivalence relation for rings by saying the rings K_1 and K_2 are <u>hereditarily equivalent</u> if and only if $H_{K_1} = H_{K_2}$, a situation which we denote by $K_1 \sim K_2$. This is easily seen to be an equivalence relation.

For example, any ring is hereditarily equivalent to a direct sum of copies of itself. For if $S = \bigoplus_{\alpha\in} K_\alpha$, where $K_\alpha \cong K$ for each α, then K is isomorphic to an ideal of S, and $K \in H_S$. On the other hand, S is a direct sum of copies of K, and so $S \in H_K$. That this guarantees $S \sim K$ follows from the next result.

<u>Proposition 4.1.2</u> For any rings K and K', $K \sim K'$ if and only if $K \in H_{K'}$ and $K' \in H_K$.

<u>Proof</u>: If $K \sim K'$, then $K \in H_K = H_{K'}$ and $K' \in H_{K'} = H_K$. Conversely, if $K \in H_{K'}$, then, by Theorem 4.1.1 we have $H_K \subseteq H_{K'}$. Similarly $H_{K'} \subseteq H_K$, and we obtain $H_K = H_{K'}$, that is $K \sim K'$. Q.E.D.

<u>Proposition 4.1.3</u> If K and K' are rings, then $K \sim K'$ if and only if, for every hereditary radical R, the conditions $R(K) = K$ and $R(K') = K'$ are equivalent.

Proof: Suppose first that $K \sim K'$. If R is a hereditary radical, and if $R(K) = K$, then $K \in R$, and $H_{K'} = H_K \subseteq R$, by Theorem 4.1.1. Thus $K' \in R$, or $R(K') = K'$. Similarly $R(K') = K'$ implies $R(K) = K$.

Conversely, if the conditions $R(K) = K$ and $R(K') = K'$ are equivalent for all hereditary radicals R, then $H_{K'}(K') = K'$ implies $H_{K'}(K) = K$, or $K \in H_{K'}$. Similarly $K' \in H_K$. By the previous proposition we have $K \sim K'$. Q.E.D.

For any ring K, we shall denote by $E(K)$ the class of all rings K' for which $K \sim K'$. Clearly, if $K' \sim K$, then $K' \in H_K$ so we have at once that $E(K) \subseteq H_K$. The opposite inclusion is not always true, for the one-element ring 0 is always in H_K but, unless $K = 0$, $0 \notin E(K)$.

A slightly less trivial example is the following. Let $K = C_\infty$, the zero ring on the (additive) infinite cyclic group, and let $K' = C_2$, the zero ring on the (additive) cyclic group of order 2. Then K' is a homomorphic image of K, and so $K' \in H_K$. However, $K' \notin E(K)$. This can be seen as follows: the class R of all rings whose (additive) underlying abelian group is a torsion group is easily seen to be a radical class, and is a hereditary radical class. Now $K' \in R$ but $K \notin R$. Thus $K \not\sim K'$.

It is of some interest to ask the question: when is is $E(K) \cup \{0\}$ equal to the class H_K? The next few results lead up to the answer to this question. We shall see that $E(K) \cup \{0\} = H_K$ if and only if K is hereditarily equivalent to a simple ring.

Lemma 4.1.4 Let R be any radical class (not necessarily hereditary), and let K be a ring. Then:

(1) If $B \triangleleft K$, and if both B and K/B are in R, then K is in R.

(2) If $\{B_\alpha : \alpha \in \Lambda\}$ is a family of ideals of K, and if $B_\alpha \in R$ for each α, then $\sum_{\alpha \in \Lambda} B_\alpha$ is in R.

Proof: (1) Clearly $R(K) \supseteq B$, and so $K/R(K)$ is a homomorphic image of K/B. Since K/B is in R, so must $K/R(K)$ be in R. Therefore, $K/R(K)$ is both R-radical and R-semisimple, so $K/R(K) = 0$, $K = R(K)$, and $K \in R$.

(2) Let $S = \sum_{\alpha \in \Lambda} B_\alpha$. Then, for each α, $B_\alpha \in R$, $B_\alpha \triangleleft S$, so $B_\alpha \subseteq R(S)$. Therefore $S = \sum_{\alpha \in \Lambda} B_\alpha \subseteq R(S)$, so $S = R(S)$, and $S \in R$. Q.E.D.

The next lemma is, in essence, due to Divinsky, Anderson, and Sulinski. (c.f. Divinsky (6), Theorem 47).

Lemma 4.1.5 Let R be any radical class, and let K be a ring. Suppose that $K \triangleright J \triangleright I$, and that $I \in R$.

(1) If $x \in K$, then $I + xI$ is an ideal of J, and

$I + xI$ is in R.

(2) The ideal of K generated by I, $I + KI + IK + KIK = \langle I \rangle_K$, is in R.

Proof: $I + xI$ is closed under addition. Also $(I+xI)J \subseteq I+xI$, and $J(I+xI) \subseteq JI + (Jx)I \subseteq I$. Therefore $I+xI$ is an ideal of J. Consider the map $f: I \to (I+xI)/I$, where $f(y) = xy+I$. This satisfies $f(y+y') = f(y) + f(y')$, and $f(yy') + I = (xy)y' + I = I$ (since $xy \in J$), while $f(y)f(y') = (xy)(xy') + I = (xyx)y' + I = I$. Therefore f is a ring homomorphism and is a surjection. Since $I \in R$, we have that I and $(I+xI)/I$ are in R, whence, by Lemma 4.1.4, $I+xI$ is in R.

(2) In a similar fashion one can show that $I+Ix$ is an ideal in J and is a member of R. If we replace I by $I+zI$, for $z \in K$, we also obtain $(I+zI) + (I+zI)x$ is an ideal in J which belongs to R. Now we can write $\langle I \rangle_K$ as

$$I + \sum_{x \in K} (I+xI) + \sum_{y \in K} (I+Iy) + \sum_{z, w \in K} \{(I+zI) + (I+zI)w\}.$$ By

part (1) this is a sum of ideals of J which are all members of R. By Lemma 4.1.4, $\langle I \rangle_K$ is in R. Q.E.D.

Corollary 4.1.6 Let R be any radical class, and let I be an accessible subring of a ring K. If $I \in R$, then $\langle I \rangle_K$ is also in R.

Proof: We have seen that the statement is true whenever $I \triangleleft J \triangleleft K$. Suppose that we have proved the result for all accessible subrings I' which satisfy $I' \triangleleft J'_2 \triangleleft J'_3 \triangleleft \dots \triangleleft J'_{n-1} = K$, and suppose that we have $I \triangleleft J_2 \triangleleft \dots \triangleleft J_n = K$. Then let $T = \langle I \rangle_{J_3} = I + J_3 I + IJ_3 + J_3 IJ_3$. By the lemma we have that $T \in R$. Now, we have $T \triangleleft J_3 \triangleleft J_4 \triangleleft \dots \triangleleft J_n = K$, and so, by our induction assumption, $\langle T \rangle_K$ is in R. Since $I \subseteq T$, we have $\langle I \rangle_K \subseteq \langle T \rangle_K$. On the other hand, it is easily seen that $T \subseteq \langle I \rangle_K$, and so $\langle T \rangle_K \subseteq \langle I \rangle_K$. Therefore $\langle T \rangle_K = \langle I \rangle_K$, and this in R. Q.E.D.

<u>Theorem 4.1.7</u> Let M be a homomorphically closed class of rings. Then a ring K is in $S_0(M)$ if and only if every non-zero homomorphic image of K has an accessible subring which is a non-zero member of M.

<u>Proof</u>: We use the notation of §1.2. Suppose first that K is in $S_0(M)$, and let $K' \neq 0$ be a homomorphic image of K. Then $K' \in S_0(M)$, and, by the definition of $S_0(M)$, $K' \in M_\alpha$ for some ordinal α. If $\alpha \neq 1$, then K' has a non-zero ideal K'_1 which belongs to M_{α}, for some ordinal $\alpha_1 < \alpha$. Suppose that we have found $K'_1, K'_2, \dots, K'_n$, where $K'_i \triangleleft K'_{i-1}$, and $K'_j \in M_{\alpha_j}$ where $\alpha_n < \alpha_{n-1} < \dots \alpha_1$. If $\alpha_n \neq 1$, then K'_n has a non-zero ideal K'_{n+1} which is in $M_{\alpha_{n+1}}$ for some $\alpha_{n+1} < \alpha_n$. Thus, if K' had no accessible

subrings in M_1 ($=M$), we would be able to find an infinite descending sequence of ordinal numbers. Since the ordinals are well ordered, this is impossible, and so K' must have an accessible subring which is a member of M.

Conversely, suppose the condition holds. If $S_0(M)(K) \neq K$, then $K/S_0(M)(K) = K' \neq 0$, and, by our assumption, this has a non-zero accessible subring I which is a member of $S_0(M)$. By Corollary 4.1.6, $\langle I \rangle_{K'}$ is also in $S_0(M)$. If J is the inverse image of $\langle I \rangle_{K'}$ under the natural homomorphism from K to K', Lemma 4.1.4 guarantees that $J \in S_0(M)$. This is a contradition since $J \supsetneqq S_0(M)(K)$. Therefore $K = S_0(M)(K)$, or $K \in S_0(M)$. Q.E.D.

<u>Theorem 4.1.8</u> For any non-zero ring K, the following are equivalent:

(1) $K \sim T$ whenever T is a non-zero homomorphic image of an accessible subring of K.

(2) $E(K) \cup \{0\} = H_K$.

(3) $E(K) \cup \{0\}$ is a hereditary radical class.

(4) $E(K) \cup \{0\}$ is hereditary and is homomorphically closed.

(5) For any ring L, and for any hereditary radical R, if $0 \neq R(L) \subseteq H_K(L)$, then $R(L) = H_K(L)$. (i.e. $H_K(L)$ is either zero or a minimal non-zero

hereditary radical ideal of L).

(6) $E(K)$ contains a simple ring.

<u>Proof</u>: (1) implies (2). We already know that $E(K) \cup \{0\} \subseteq H_K$. Suppose that L is a non-zero member of H_K. By Theorem 4.1.1 and the previous theorem, L has a non-zero accessible subring L' in $\{K\}^*$. By definition of $\{K\}^*$, L' is a homomorphic image of an accessible subring of K, and so (1) implies $L' \sim K$. Now L' is an accessible subring of L and so $L' \in H_L \subseteq H_K$, and $H_K = H_{L'} \subseteq H_L \subseteq H_K$. Therefore $K \sim L$, and thus $H_K \subseteq E(K) \cup \{0\}$.

That (2) implies (3), and that (3) implies (4), are both obvious.

(4) implies (1). Suppose that we have $K \triangleright K_1 \triangleright K_2 \ldots \triangleright K_n$, and a homomorphism f of K_n onto a non-zero ring T. Now $K \in E(K)$, and by using (4) we see that $K_1, K_2, \ldots K_n$ and finally T are in $E(K) \cup \{0\}$. Since $T \neq 0$, we have $T \in E(K)$, or $K \sim T$.

(2) implies (5). Suppose that R is a hereditary radical, and that L is a ring for which $0 \neq R(L) \subseteq H_K(L)$. Then the ring $T = R(L)$ is in H_K and therefore is in $E(K)$, so $H_K = H_T$. In particular, $H_K(L) = H_T(L)$. Since $T = R(K)$, $T \in R$, and $H_T \subseteq R$. Therefore $H_K(L) = H_T(L) \subseteq R(L)$, and we have $R(L) = H_K(L)$.

(5) implies (2). Suppose that L is a non-zero member of H_K. Let K' be a ring isomorphic to K, and consider the (external) direct sum $K' \oplus L$. Since K' and L are both in H_K, we have $0 \neq L \subseteq H_L(K' \oplus L) \subseteq H_K(K' \oplus L) = K' \oplus L$. By condition (5), $K' \oplus L = H_L(K' \oplus L)$, which implies $K' \in H_L$. Since $K' \cong K$, we have $K \in H_L$ and $L \in H_K$, which gives, by Proposition 4.1.2, $K \sim L$.

(6) implies (2). Suppose that $E(K)$ contains a simple ring S. Then S certainly satisfies (1), and, by a previous part of this proof, $H_S = E(S) \cup \{0\}$. Since $H_S = H_K$ and since $E(S) = E(K)$, the result follows.

(4) implies (6). We choose $x \neq 0$ in K. By Zorn's Lemma, there is an ideal U of K maximal among those ideals which do not contain x. It follows then that, if H is the image of the (two-sided) ideal generated by x under the natural homomorphism of K onto K/U, H is the intersection of all non-zero ideals of K/U, and $H \neq 0$. By condition (4), H is in $E(K)$. If $H^2 = H$ then H is well known to be a simple ring. If $H^2 \neq H$, then $H^2 = 0$. In this case let $h \neq 0$ be in H. The additive subgroup T generated by h is an ideal of H, and this is a cyclic abelian group. T can therefore be homomorphically mapped onto C_p (the zero ring on the additive cyclic group of order p) for some prime p. Again using (4), we have that T and C_p are in $E(K)$. But C_p is a simple ring. Q.E.D.

4.2 A Closure Operation

We introduce in this section a closure operation in the lattice of all ideals of a ring K, and show that the "closed" ideals are precisely those ideals of the form $R(K)$ for some hereditary radical R. Some of the properties of this lattice are considered.

A mapping c of a lattice to itself is called a <u>closure operation</u> if it satisfies:

C 1. $A \leq B$ implies $c(A) \leq c(B)$.

C 2. $c(c(X)) = c(X)$ for all X in the lattice.

C 3. $X \leq c(X)$ for all X in the lattice.

The elements of the lattice for which $X = c(X)$ are called <u>closed</u> elements (relative to the closure operation in question).

<u>Theorem 4.2.1</u> The mapping c of the lattice of two-sided ideals of a ring K into itself, defined by $c(I) = H_I(K)$, is a closure operation. The closed ideals of K are precisely those ideals of the form $R(K)$ for some hereditary radical R.

<u>Proof</u>: If A, B are ideals of K, and if $A \subseteq B$ then $A \trianglelefteq B$, and so $A \in H_B$ whence $H_A \subseteq H_B$. Therefore $c(A) = H_A(K) \subseteq H_B(K) = c(B)$.

Clearly C 3 is satisfied. We now establish C 2. If $B = R(K)$ for some hereditary radical R, then $B \in R$, $H_B \subseteq R$, and so $c(B) = H_B(K) \subseteq R(K) = B$, and $B = c(B)$.

Thus any ideal of the form $R(K)$ is closed. In particular, this shows that C 2 is satisfied.

Finally, if B is a closed ideal, $B = c(B) = H_B(K)$, where H_B is, of course, a hereditary radical. Q.E.D.

In the sequel, for B an ideal of K, we shall denote $c(B)$ by $\overline{B}$.

<u>Theorem 4.2.2</u> If A and B are ideals of a ring K, then $\overline{A} = \overline{B}$ if and only if $A \sim B$.

<u>Proof</u>: If $\overline{A} = \overline{B}$, then $A \triangleleft \overline{A} = \overline{B} = H_B(K)$, so $A \in H_B$. Similarly, $B \in H_A$, and thus $A \sim B$, by Proposition 4.1.2. Conversely, if $A \sim B$ then $H_A = H_B$, and clearly $\overline{A} = \overline{B}$. Q.E.D.

<u>Theorem 4.2.3</u> Let K be any ring. Then:

(1) 0 is a closed ideal of K.

(2) The intersection of any set of closed ideals is closed.

(3) If $\{A_\alpha: \alpha\in\Lambda\}$ is a family of ideals of K, then

$$\overline{\sum_{\alpha\in\Lambda} A_\alpha} = \overline{\sum_{\alpha\in\Lambda} \overline{A_\alpha}}.$$

<u>Proof</u>: Since the class consisting of the single ring 0 is a hereditary radical class, (1) is satisfied.

Suppose that $\{C_\alpha: \alpha\in\Lambda\}$ is a set of closed ideals of K. Since $\bigcap_{\alpha\in\Lambda} C_\alpha \subseteq C_\beta$ for each β, we have $\overline{\bigcap_{\alpha\in\Lambda} C_\alpha} \subseteq \overline{C_\beta} = C_\beta$, and hence $\overline{\bigcap_{\alpha\in\Lambda} C_\alpha} \subseteq \bigcap_{\beta\in\Lambda} C_\beta$. Together with property C 1, this implies (2).

Suppose now that $\{A_\alpha\colon \alpha\in\Lambda\}$ is a set of ideals of K. Then $A_\alpha \subseteq \overline{A}_\alpha$ for each α, and so $\overline{\sum_{\alpha\in\Lambda} A_\alpha} \subseteq \overline{\sum_{\alpha\in\Lambda} \overline{A}_\alpha}$. On the other hand, $A_\alpha \subseteq \sum_{\beta\in\Lambda} A_\beta$ and so $\overline{A}_\alpha \subseteq \overline{\sum_{\beta\in\Lambda} A_\beta}$; which gives $\sum_{\alpha\in\Lambda} \overline{A}_\alpha \subseteq \overline{\sum_{\beta\in\Lambda} A_\beta}$. Applying the closure operation to this last inequality establishes (3). Q.E.D.

The set of closed ideals of a ring K has a partial ordering (inclusion) and this induces a lattice structure on the set of closed ideals of K. If we denote these lattice operations by $\wedge$ and $\vee$, then this lattice is a complete lattice, where, if $\{C_\alpha\colon \alpha\in\Lambda\}$ is a set of closed ideals,

$$\bigwedge_{\alpha\in\Lambda} C_\alpha = \bigcap_{\alpha\in\Lambda} C_\alpha, \quad\text{and}\quad \bigvee_{\alpha\in\Lambda} C_\alpha = \overline{\sum_{\alpha\in\Lambda} C_\alpha}.$$

A lattice is said to be <u>modular</u>, if, whenever a, b and c are in the lattice, where $b \leq a$, we have $a \wedge (b\vee c) = b \vee (a\wedge c)$. Many of the common lattices in algebraic systems, such as the lattice of ideals in a ring, are modular lattices, and this fact is crucial in proving results as the Jordan-Hoelder Theorem. The following example shows that the lattice of closed ideals of a ring need not be a modular lattice. In particular, this example also shows that the lattice operations ($\vee$ and $\wedge$) in the lattice of closed ideals are not necessarily the same as the lattice operations $(+$ and $\cap)$ in the lattice of all ideals of a ring. That is, the lattice of closed ideals is not necessarily a sublattice of the lattice of all ideals of a ring.

Example 4.2.4

Let S be a simple non-trivial Jacobson radical ring of characteristic 2. (Such rings are known to exist. See Divinsky (6), p. 112, or Sasiada and Cohn (23).) We embed S into a ring S' with unity by putting a ring structure on the Cartesian product $S \times Z_2$ (Z_2 being the ring of integers modulo 2) by defining addition componentwise and multiplication by $(s,n)(s_1,n_1) = (ss_1 + ns_1 + n_1 s, nn_1)$. It is well known that S' has an associative ring structure, that $(0,1)$ is a unity for S', and that $\{(s,0): s \in S\}$ is an ideal of S' which is isomorphic to S. Let Z be the ring of integers, and let $K = Z \oplus S'$. We claim that Z, S, S', K and 0 are all closed ideals of K.

We have mentioned that the class T of all rings whose underlying (additive) abelian groups are torsion groups is a hereditary radical class. (See §4.1.) Clearly for the ring K, $T(K) = S'$. Since $J(K) = S$, we have that K, 0, S and S' are all closed ideals.

If Z was not closed in K, we would have $Z \neq H_Z(K)$, and therefore $H_Z(K/Z) \cong H_Z(S') \neq 0$. By Theorem 4.1.7 and the definition of H_Z this means that S' would have a non-zero accessible subring L which is a homomorphic image of an accessible subring of Z, and L would be commutative. Since S is the unique maximal ideal of S', and since S is a simple ring, we would have to have $L = S'$ or $L = S$, and thus S would be a commutative simple ring. Such rings are well known to be either fields or rings for which the multiplication is trivial. On the other hand S is a non-trivial Jacobson radical ring, and thus can be neither a field nor a trivial ring. Therefore Z must be a closed ideal of K.

Since $K/(Z+S) \cong S'/S \cong Z_2$, a homomorphic image of Z, we have $K/(Z+S) \in H_Z \subseteq H_{Z+S}$. By Lemma 4.1.4, we then obtain $K \in H_{Z+S}$ or $\overline{Z+S} = K$. We can now see that the lattice of closed ideals of K is not modular, since $S' \cap \overline{(S+Z)} = S' \cap K = S'$, while $\overline{S + (S' \cap Z)} = \overline{S+0} = \overline{S} = S$. Also, since $\overline{S+Z} \neq S+Z$, the lattice of closed ideals is not a sublattice of the lattice of all ideals.

<u>Theorem 4.2.5</u> Let A and B be ideals of a ring K, and let $B \subseteq A$. Then $\overline{(A/B)} \subseteq (\overline{A})/B$.

<u>Proof</u>: A/B is a homomorphic image of A, so $H_{A/B} \subseteq H_A$. Therefore $\overline{(A/B)} = H_{A/B}(K/B) \subseteq H_A(K/B) = H_A(K)/B = \overline{A}/B$. Q.E.D.

<u>Corollary 4.2.6</u> If A and B are ideals of K, where $B \subseteq A$, then, if A is closed, A/B is a closed ideal of K/B.

Proof: $\overline{(A/B)} \subseteq \overline{A}/B = A/B$. Q.E.D.

The converse of the corollary is false. The following example shows that closed ideals are not preserved under extensions. We give an example of a ring K with ideals A and B, where $A \supseteq B$, such that B is a closed ideal of K, A/B is a closed ideal of K/B, but A is not closed in K.

Example 4.2.7

Let F be the field of two elements, and let $A = Fx \oplus Fy$, where $x^2 = 0$ and $y^2 = y$. Let $K = \{(a,f): a\in A, f\in F\}$ with addition defined componentwise and multiplication defined by $(a,f)(a',f') = (aa' + fa' + f'a, ff')$. We can identify the elements of A with $\{(a,0): a\in A\} \subseteq K$, and thus regard A as a subring of K. Under this identification, it is easily seen that A and $B = Fy$ are ideals in K. It is equally easy to see that $K/B \cong \{(fx,g): f,g \in F\}$ with addition defined componentwise and multiplication defined by $(fx,g)(f'x,g') = ((gf'+g'f)x, gg')$. This latter ring has four elements, and the only non-trivial accessible subring is the one isomorphic to A/B, and this is a nilpotent subring. It follows, since B has a unity element and has two elements, that K/B has no non-zero accessible subrings which are homomorphic images of accessible subrings of B. By Theorem 4.1.7, $H_B(K/B) = 0$, and so $B = \overline{B}$. In K/B, the ideal A/B is nilpotent, and is, in fact the Baer Lower Radical of K/B.

Therefore A/B is closed in K/B.

It is easy to see that $K/A \cong F \cong B \in H_A$. Thus, by Lemma 4.1.4, $\overline{A} = H_A(K) = K$, and A is not closed.

4.3 Minimal Closed Ideals of a Ring

We shall call a minimal (non-zero) closed ideal of a ring an atom. Then Theorem 4.1.8 asserts that $E(K)$ contains a simple ring if and only if, for every ring L, $H_K(L)$ is either zero or an atom.

Recall that B denotes the Baer Lower Radical. It is shown in Divinsky (6) (page 43) that $B = S_0(\{C_\infty\})$, which implies at once, since B is hereditary, that $B = H_{C_\infty}$.

Lemma 4.3.1 Let K be a ring for which $B(K)$ is non-zero and has no elements of finite additive order. Then $B(K)$ is an atom of K.

Proof: If T is any non-zero ideal of K contained in $B(K)$, then $T \in B$. By Theorem 4.1.7, T then has an accessible subring T' which is a non-zero homomorphic image of an accessible subring of C_∞. But $B(K)$, and therefore T, has no elements of finite additive order, and C_∞ is isomorphic to every subring of itself. It follows that T' must be isomorphic to C_∞. Since T' is an accessible subring of T, $T' \in H_T$, and we have $B = H_{C_\infty} = H_{T'} \subseteq H_T \subseteq B$. Therefore $B(K) = H_T(K) = \overline{T}$, and $B(K)$ can not properly contain any non-zero closed ideals. Q.E.D.

For any ring K, and any prime p, let $F_p(K)$ be the set of all elements of K whose additive order is a power of p. It is easily verified that $F_p(K)$ is an ideal of K, and that $F_p(K/F_p(K)) = 0$. From this it follows that the class F_p of all rings whose elements all have additive order a power of p is a radical class. Clearly this class is hereditary.

<u>Theorem 4.3.2</u> If K is a ring for which $B(K) \neq 0$, then K contains an atom.

<u>Proof</u>: If the additive group of $B(K)$ has no elements of finite order, Lemma 4.3.1 says that $B(K)$ is itself an atom. If $B(K)$ does have elements of finite order, it follows from standard arguments of abelian group theory that $T = F_p(B(K)) \neq 0$ for some prime p. Now $T = F_p(B(K)) = F_p(K) \cap B(K)$, and this is an ideal of K. Also, T is in B, and so, by Lemma 4.1.4, T has a non-zero accessible subring T' which is a homomorphic image of C_∞. Since T also is in F_p, so is T', and so $T' \cong C_{p^n}$ for some n. But then T' has an ideal T'' which is isomorphic to C_p, and T'' is also an accessible subring of K. By Corollary 4.1.6, $\langle T''\rangle_K \in H_{T''} = H_{C_p}$ and so $H_{C_p}(K) \neq 0$. By Theorem 4.1.8,

and the fact that C_p is a simple ring, $H_{C_p}(K)$ is an atom of K. Q.E.D.

Theorem 4.3.3 Let I be a minimal ideal of a ring K. Then $\bar{I}$ contains an atom.

Proof: If I is a minimal ideal of K, either $I^2 = I$ or $I^2 = 0$. If $I^2 = I$, it is well known that I is a simple ring. Then, by Theorem 4.1.8, $\bar{I} = H_I(K)$ is an atom.

If $I^2 = 0$, consider the additive group of I. If this has elements of finite order, then, for some prime p, $I \cap F_p(K) \neq 0$. By the minimality of I, $I \subseteq F_p(K)$. The same arguments as in the proof of the previous theorem guarantees that I has an accessible subring T'' isomorphic to C_p. We then obtain $H_{T''} \subseteq H_I$, and so $\bar{I} \supseteq H_{T''}(K) = H_{C_p}(K) \neq 0$. By Theorem 4.1.8, $H_{C_p}(K)$, since it is not zero, is an atom. On the other hand, if $I^2 = 0$, and if I has no elements of finite additive order, choose $x \neq 0$ from I. Then the additive subgroup of I generated by x is an ideal of I, and is isomorphic to C_∞. Therefore $B = H_{C_\infty} \subseteq H_I$, and $B(K) \subseteq H_I(K) \subseteq \bar{I}$. By Theorem 4.3.2, this contains an atom. Q.E.D.

Remark It is not necessarily true that $\bar{I}$ is itself an atom.

For example, let F be the field of rational numbers, and $A = Fx+Fy$, where $x^2 = 0$, $xy = x = yx$, $y^2 = y$. If we set $K = A \oplus C_p$, Fx is an ideal of K. In fact, Fx is a minimal ideal of K. For if L is an ideal of K contained in A, L is an ideal of A, and, since A has a unity, L must be a subalgebra of A. Since Fx is of dimension 1 over F, Fx is a minimal subalgebra of A, and hence is a minimal ideal of K. Now Fx contains a copy of C_∞ as an ideal, and so it follows that $H_{Fx} = B$, and $\overline{Fx} = B(K) = Fx \oplus C_p$. This is not an atom, for it properly contains $C_p = F_p(K)$.

4.4 Rings with Chain Conditions on Closed Ideals

In this section we shall consider rings which have either or both of the ascending and descending chain conditions on closed ideals. Since the lattice of closed ideals is not modular, we have no reason to expect a result analogous to the Jordan-Hoelder Theorem. On the other hand, a closed ideal C of a ring must contain all homomorphic images of C which are ideals of K, so distinct closed ideals must differ in some definite way from one another. In the presence of, say, both the ascending and descending chain conditions for closed ideals, we might hope that there would not be too many different closed ideals. We shall see that this is the case. Indeed, we shall prove that the two chain conditions are necessary and sufficient conditions to guarantee that there are only a finite number of closed ideals in the ring.

First of all, we show that the ascending chain condition on closed ideals is not a property which is preserved under homomorphisms.

Example 4.4.1

We give an example of a ring K which has the ascending chain condition for closed ideals, but for which there is a homomorphic image of K which does not have this property.

Let $K = \bigoplus_{n=1}^{\infty} K_n$, where $K_i \cong Z$, the ring of integers, and let $A = \bigoplus_{n=1}^{\infty} p_n K_n$, where $\{p_i : i=1,2,\ldots\}$ is the sequence of prime numbers. Then $A \triangleleft K$, and $K/A \cong \bigoplus_{i=1}^{\infty} Z_{p_i} = B$, where Z_{p_i} is the ring of integers module p_i. Let $W_m = \bigoplus_{i=1}^{m} Z_{p_i}$; this is an ideal of B. For $x \in W_m$, $(\prod_{i=1}^{m} p_i)\, x = 0$, and the same is true if x is any element of a homomorphic image of an accessible subring of W_n. On the other hand, if $y \in B/W_m$, and if $(\prod_{i=1}^{m} p_i) y = 0$, it follows that $y = 0$ (since y has additive order relatively prime to $\prod_{i=1}^{m} p_i$). Therefore B/W_m has no non-zero accessible subrings which are homomorphic images of accessible subrings of W_m. From Theorem 4.1.8 we obtain $H_{W_m}(B/W_m) = 0$, and so W_m is closed in B. Since

$W_i \subsetneqq W_{i+1}$ we have that $B \cong K/A$ does not have the ascending chain on closed ideals.

We now show that K does have the ascending chain condition on closed ideals. Let C_i, $i = 1,2,\ldots$ be an ascending chain of closed ideals of K. For each j there is an i such that $0 = C_jK_i \subseteq C_j \cap K_i$. For any i and j consider the i'th projection π_i of K onto Z, and consider the set $S = \{\pi_i(C_j \cap K_i) : i$ and j are positive integers$\}$. Since Z has the ascending chain on all ideals, there are i_o and j_o such that $\pi_{i_o}(C_{j_o} \cap K_{i_o})$ is a maximal member of S. We note that, if θ_i is the embedding of Z into K which takes Z onto K_i, the fact that C_j is closed guarantees that $C_j \supseteq \theta_i\pi_{i'}(C_j \cap K_{i'})$ for all i, i', and j.

Consider $n > j_o$. If $C_n \supsetneqq C_{j_o}$, there is an element x in C_n which does not belong to C_{j_o}. Writing $x = x_1+x_2+\ldots+x_t$, where $x_i \in K_i$, we must have $x_i \notin C_{j_o}$ for some i. Now K_i has a unity e_i, and $x_i = xe_i \in C_n$. Therefore $x_i \in C_n \cap K_i$, and $\pi_i(x_i) \in \pi_i(C_n \cap K_i)$. We have $\theta_i\pi_{i_o}(C_{j_o} \cap K_{i_o}) \subseteq C_{j_o} \cap K_i \subseteq C_n \cap K_i$, and so $\pi_{i_o}(C_{j_o} \cap K_{i_o}) = \pi_i\theta_i\, \pi_{i_o}(C_{j_o} \cap K_{i_o}) \subseteq \pi_i(C_n \cap K_i)$. By the maximality of $\pi_{i_o}(C_{j_o} \cap K_{i_o})$, we must have equality, and thus $\pi_i(x_i) \in \pi_{i_o}(C_{j_o} \cap K_{i_o})$. But then $x_i = \theta_i\pi_i(x_i) \in \theta_i\pi_{i_o}(C_{j_o} \cap K_{i_o}) \subseteq C_{j_o}$. This is a contradiction. Therefore $C_n = C_{j_o}$ for all $n > j_o$. This proves that K has the ascending chain condition on closed ideals.

<u>Lemma 4.4.2</u> Let $\{K_\alpha : \alpha\in\Lambda\}$ be a family of rings indexed by some set Λ, and suppose that each K_α is an ideal in a ring T. (This is no real limitation, for we could take $T = \bigoplus_{\alpha\in\Lambda} K_\alpha$.) Then $H_{\sum_{\alpha\in\Lambda} K_\alpha} = S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$.

<u>Proof</u>: Since each H_{K_α} is hereditary, $\bigcup_{\alpha\in\Lambda} H_{K_\alpha}$ is a hereditary homomorphically closed class, and so $S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$ is a hereditary radical class, by Theorem 1.4.1. Since each $K_\alpha \in S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$, Lemma 4.1.4 gives $\sum_{\alpha\in\Lambda} K_\alpha \in S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$, and so $H_{\sum_{\alpha\in\Lambda} K_\alpha} \subseteq S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$.

Now let B be a member of $S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})$. By Theorem 4.1.7, every non-zero homomorphic image B' of B has a non-zero accessible subring W' in $\bigcup_{\alpha\in\Lambda} H_{K_\alpha}$, and thus $W' \in H_{K_\alpha}$ for some $\alpha \in \Lambda$. We apply Theorem 4.1.7 again, and recall the definition of H_{K_α}, and we can conclude that

W' has an accessible subring W'' which is a homomorphic image of an accessible subring Y of K_α. Since $K_\alpha \triangleleft \sum_{\alpha\in\Lambda} K_\alpha$, Y is an accessible subring of $\sum_{\alpha\in\Lambda} K_\alpha$. Also W'' is an accessible subring of B'. Therefore B' has a non-zero accessible subring (W'') which is a non-zero homomorphic image of an accessible subring of $\sum_{\alpha\in\Lambda} K_\alpha$. Again applying Theorem 4.1.7 and the definition of H_T, $B \in H_{\sum_{\alpha\in\Lambda} K_\alpha}$. Q.E.D.

<u>Corollary 4.4.3</u> Under the same conditions as in the Lemma, let V be a ring for which $H_{\sum_{\alpha\in\Lambda} K_\alpha}(V) \neq 0$. Then, for some $\alpha\in\Lambda$, $H_{K_\alpha}(V) \neq 0$.

<u>Proof</u>: By the theorem, $S_0(\bigcup_{\alpha\in\Lambda} H_{K_\alpha})(V) \neq 0$. By Theorem 4.1.7, V has an accessible subring W which is a non-zero member of $\bigcup_{\alpha\in\Lambda} H_{K_\alpha}$ and $W \in H_{K_\alpha}$ for some α. By Corollary 4.1.6, $\langle W\rangle_V \in H_{K_\alpha}$, and so $0 \neq \langle W\rangle_K \subseteq H_{K_\alpha}(V)$. Q.E.D.

If A and B are closed ideals of a ring K, we shall say that <u>A covers B</u> if $A \supsetneqq B$, but there no closed ideals T for which $A \supsetneqq T \supsetneqq B$.

<u>Lemma 4.4.4</u> Let K be a ring with the ascending chain

condition for closed ideals. Then there are only a finite number of closed ideals which cover a given closed ideal.

Proof: Let B be a closed ideal of K, and let $\{A_\alpha : \alpha \in \Lambda\}$ be the set of closed ideals which cover B. By the assumed chain condition there is a finite subset F_0 of Λ for which $\overline{\sum_{\beta \in F_0} A_\beta}$ is a maximal member of $\{\overline{\sum_{\gamma \in G} K_\gamma} : G$ is a finite subset of $\Lambda\}$. Suppose that $F_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. For each $\lambda \in \Lambda$, $\overline{K_\lambda + \Sigma_{i=1}^n K_{\alpha_i}} = \overline{\Sigma_{i=1}^n K_{\alpha_i}}$, and therefore $K_\lambda \subseteq \overline{\Sigma_{i=1}^n K_{\alpha_i}}$. This implies that $K_\lambda / B \in H_{\Sigma_{i=1}^n K_{\alpha_i}}$. For some i, by Corollary 4.4.3, we have $H_{K_{\alpha_i}}(K_\lambda / B)$ is non-zero. But then $0 \neq H_{K_{\alpha_i}}(K_\lambda / B) = K_\lambda / B \cap H_{K_{\alpha_i}}(K/B) = K_\lambda / B \cap K_{\alpha_i}/B$. This last equality is due to the fact that $B \subseteq K_{\alpha_i}$, and the fact that K_{α_i} is closed. Therefore $0 \neq (K_\lambda \cap K_{\alpha_i})/B$. Since K_α and K_{α_i} are both closed, so is their intersection. However, since both K_λ and K_{α_i} cover B, we must have $K_\lambda = K_{\alpha_i}$. Therefore $\Lambda = F_0$. Q.E.D.

Theorem 4.4.3 A ring K has both the ascending and descending chain conditions on closed ideals if and only if there is at most a finite number of closed ideals.

<u>Proof</u>: Obviously, if there is only a finite number of closed ideals, the chain conditions must hold.

Suppose that both chain conditions hold. We assume there is an infinite number of closed ideals and obtain a contradiction. If there is an infinite number of closed ideals, then $P = \{I\colon\ I$ is a closed ideal, and there are infinitely many closed ideals J such that $J \supsetneqq I\}$ is a non-void set. (For the ideal $0 \in P$.) By the ascending chain condition, there is a maximal member T_0 of P. Let $L_1, L_2, \ldots, L_n$ be the (finite) set of closed ideals which cover T_0. (The previous Lemma guarantees that this is a finite set.) Each L_i is not in P, so there is just a finite set $L_{i,j}$, $j = 1,2,\ldots,N_i$ of closed ideals W satisfying $W \supseteq L_i$. We claim that any closed ideal which properly contains T_0 must belong to the finite set $\bigcup_{i=1}^{n} \bigcup_{j=1}^{N_i} \{L_{i,j}\}$, and this will contradict $T_0 \in P$.

Let J be a closed ideal of K such that $J \supsetneqq T_0$. By the descending chain condition for closed ideals, there is a closed ideal L of K minimal with respect to the property $T_0 \subsetneqq L \subseteq J$. Clearly L covers T_0, and so $L = L_i$ for some i. But then $J = L_{i,j}$ for some j. Q.E.D.

4.5 Rings Determined by Their Atoms

In module theory, the concept of a completely reducible module is well established. (See, for example, Jacobson (13), Chapter three.) The analogous concept for rings is also well-known. The main result in this direction is the following theorem, which is due to Blair (5).

Theorem 4.5.1 For a given ring K, the following are equivalent:

(1) $K = \sum_{\alpha\in\Lambda} S_\alpha$, where each S_α is a simple ring and an ideal of K.

(2) $K = \bigoplus_{\gamma\in\Gamma} S_\gamma$, where each S_γ is a simple ring and an ideal of K.

(3) For any ideal I of K, there is an ideal J such that $I \oplus J = K$.

Proof: Clearly (2) implies (1). We show that (1) implies (2). If we choose $\alpha\in\Lambda$, the set $\{S_\alpha\}$ is trivially an independent family of ideals, that is the sum of all the ideals in the set is a direct sum. By Zorn's Lemma there is a subset Γ of Λ which is maximal with respect to the property that $\sum_{\gamma\in\Gamma} S_\gamma$ is a direct sum. For any $\alpha\in\Lambda$, if $S_\alpha \cap (\bigoplus_{\gamma\in\Gamma} S) = 0$, then $\sum_{\beta\in\Gamma\cup\{\alpha\}} S_\beta$ is direct, contradicting the maximality of Γ. Therefore, for any $\alpha\in\Lambda$, $S_\alpha \cap (\bigoplus_{\gamma\in\Gamma} S_\gamma) \neq 0$. Since S_α is a simple ring, the intersection is all of S_α. Therefore, $S_\alpha \subseteq \bigoplus_{\gamma\in\Gamma} S_\gamma$, and $K = \bigoplus_{\gamma\in\Gamma} S_\gamma$.

(2) implies (3). Let I be an ideal of K. If $I = K$, there is no more to be done. If $I \neq K$, then, for some $\gamma\in\Gamma$, $I \not\supseteq S_\gamma$, and $I \cap S_\gamma = 0$, since S_γ is simple. Therefore the collection of all subsets Δ of Γ for which $I \cap (\bigoplus_{\delta\in\Delta} S_\delta) = 0$ is not empty. Zorn's Lemma can be applied to give a subset Δ_0 of Λ maximal with this property. Then the sum $I + D$, where $D = \bigoplus_{\delta\in\Delta_0} S_\delta$, is direct. If $I \oplus D \neq K$, then $S_\gamma \not\subseteq I + D$ for some $\gamma \in \Gamma$. Since S_γ is simple, $S_\gamma \cap (I\oplus D) = 0$, and we obtain a contradiction to the maximality of Δ_0. Therefore $I \oplus D = K$, as desired.

(3) implies (1). First of all, we show that K has minimal ideals which are simple rings. Let $x \in K$, $x \neq 0$. By Zorn's Lemma, there is an ideal J of K maximal with respect to not containing x. We claim that J is a maximal ideal of K. If $B \triangleleft K$, $B \supsetneqq J$, then $x \in B$. There is an ideal C of K such that $B \oplus C = K$. Since the lattice of ideals of K is modular, $J = J + (B\cap C) = B \cap (J+C)$. Now, if $J \subsetneqq J + C$, then x is in both B and $J+C$, and therefore $x \in J$, which is false. Therefore $J = J+C$, or $C \subseteq J$. But $J \subseteq B$, and therefore $C \subseteq C \cap B = 0$. This shows that

$B = K$, and J is therefore a maximal ideal.

By condition (3), we can write $K = J \oplus J'$. If J' is not a simple ring, it has a proper ideal L', which is an ideal of K, since J' is a direct summand of K. We then obtain $J \subseteq J \oplus L' \subsetneqq K$, contradicting the maximality of K. Therefore J' is a simple ring and a minimal ideal of K.

Let S be the sum of all the minimal ideals of K which are simple rings. Then, if $S \neq K$, $K = S \oplus S'$, for some ideal $S' \neq 0$. Let $s' \neq 0$ be in S', and let J be an ideal of K maximal with respect to containing S and not containing s'. The same proof as above shows that J is a maximal ideal of K. If $K = J \oplus J'$, J' is then a minimal ideal of K, J' is a simple ring, and $J' \subseteq S$. But then $J' \subseteq J' \cap J = 0$. This shows $S = K$, and (1) is satisfied. Q.E.D.

<u>Corollary 4.5.2</u> If K satisfies the conditions of the theorem, any ideal I of K is of the form $\underset{\theta \in}{\oplus} S_\theta$ for some subset of Λ.

<u>Proof</u>: We have seen that $K = I \oplus D$. Applying the arguments used in the proof of the previous theorem, we can write $K = D \oplus T$ where $T = \underset{\theta \in}{\oplus} S_\theta$. Then $I \cong K/D \cong T = \underset{\theta \in}{\oplus} S_\theta$. If $f\colon T \to I$ is the isomorphism, we have $I = \underset{\theta \in}{\oplus} f(S_\theta)$, and each $f(S_\theta) \in \{S_\alpha \colon \alpha \in \Lambda\}$. Q.E.D.

We remark at this point that, in the proof of Theorem 4.5.1, the modularity of the lattice of two-sided ideals of K was used. We have seen (in Example 4.2.4) that the lattice of closed ideals of a ring is not necessarily a modular lattice. We shall attempt to see how much of Theorem 4.5.1 remains true when we discuss the lattice of closed ideals, rather than the lattice of all ideals. For the rest of this chapter, $\{S_\alpha : \in \Lambda\}$ will denote the family of atoms of K, and

$S_1(K) = \overline{\sum_{\alpha \in \Lambda} S_\alpha}$.

<u>Theorem 4.5.3</u> Let K be a ring, and let L be a closed ideal of K such that $L \subseteq S_1(K)$. Then there is a family $\{S_\delta : \delta \in \Delta\}$ of atoms such that $S_1(K) = \overline{L \oplus (\oplus_{\delta \in \Lambda} S)}$.

<u>Proof</u>: If $L = S_1(K)$, let $\Delta = \emptyset$. Otherwise, for some atom S_α of K, $S_\alpha \not\subseteq L$. Then $S_\alpha \cap L$ is a closed ideal of K properly contained in S_α, and therefore $S_\alpha \cap L = 0$. The collection of all subsets Γ of Λ for which $\{S_\gamma : \gamma \in \Gamma\}$ is an independent family, and for which $L \cap (\oplus_{\gamma \in \Gamma} S_\gamma) = 0$, is not empty. Zorn's Lemma can be applied to give a maximal such family Δ_o. We claim $S_1(K) = \overline{L \oplus (\oplus_{\delta \in \Delta_o} S_\delta)}$.

If this equality were not true, then $\overline{L \oplus (\oplus_{\delta \in \Delta_o} S_\delta)}$ must fail to contain, and hence must have zero intersection with, some atom S_α. But then $S_\alpha \cap (L \oplus (\bigoplus_{\delta\in\Delta_0} S_\delta)) = 0$, and we obtain a contradiction to the maximality of Δ_0. Q.E.D.

<u>Corollary 4.5.4</u> If K is a ring, then

(1) the family of all atoms is an independent family

(2) for each closed ideal $L \subseteq S_1(K)$, there is a closed ideal L' of K of the form $\overline{\bigoplus_{\delta\in\Delta} S_\delta}$ such that $S_1(K) = \overline{L \oplus L'}$.

<u>Proof</u>: We show (2) first. Under the present hypotheses, the statement of the previous theorem becomes $S_1(K) = \overline{L \oplus (\bigoplus_{\delta\in\Delta_0} S_\delta)}$. If we let $W = \bigoplus_{\delta\in\Delta_0} S_\delta$ we have $S_1(K) = \overline{L \oplus W}$. We do not know that W is necessarily closed. However, it follows from Theorem 4.2.3 that we also have $S_1(K) = \overline{L \oplus \overline{W}}$.

We now establish (1). By applying the previous theorem, taking $L = 0$, we obtain $S_1(K) = \overline{\bigoplus_{\gamma\in\Gamma} S_\gamma}$ for some subset Γ of Λ. It remains only to show that $\Gamma = \Lambda$. For any $\alpha \in \Lambda$, $S_\alpha \subseteq S_1(K) = \overline{\bigoplus_{\gamma\in\Gamma} S_\gamma} = H_{\bigoplus_{\gamma\in\Gamma} S_\gamma}(K)$. It follows by an application of Corollary 4.4.3 that, for some $\gamma \in \Gamma$, $0 \neq H_{S_\gamma}(S_\alpha) = S_\alpha \cap H_{S_\gamma}(K) = S_\alpha \cap S_\gamma$. Since S_α and S_γ are both atoms, this implies that $S_\alpha = S_\gamma$, and $\alpha = \gamma$. Q.E.D.

<u>Remark</u> In the case where $K = S_1(K)$, part (2) of the corollary says that for any closed ideal L of K there is a closed ideal L' of the form $\overline{\underset{\delta\in\Delta}{\oplus} S_\delta}$ such that $S_1(K) = K = \overline{L \oplus L'}$.

In an arbitrary ring there may be more that one closed ideal L' (for a given closed ideal L) such that $K = \overline{L \oplus L'}$. For example, consider the ring K of Example 4.2.4. In this ring, Z, S and S' are closed ideals, and $K = \overline{Z \oplus S} = \overline{Z \oplus S'}$, where $S \neq S'$. This ring K does not satisfy $K = S_1(K)$. It would be interesting to know whether, if $K = S_1(K)$, one can have L, L', and L'' closed, $L' \subsetneq L''$, and $K = \overline{L \oplus L'} = \overline{L \oplus L''}$.

In this example, since S is an atom, S is clearly minimal among the closed ideals L' such that $K = \overline{L' \oplus Z}$. The same can be said sometimes in a more general situation.

<u>Theorem 4.5.5</u> If K is a ring, L a closed ideal of K, and if $\{S_\omega : \omega\in\Omega\}$ is a family of atoms of K such that $K = \overline{L \oplus (\underset{\omega\in\Omega}{\oplus} S_\omega)}$, then for any closed ideal L' such that $K = \overline{L \oplus L'}$, then $L' \supseteq \overline{\underset{\omega\in\Omega}{\oplus} S_\omega}$. In other words, $\overline{\underset{\omega\in\Omega}{\oplus} S_\omega}$ is a unique minimal closed L' such that $K = \overline{L \oplus L'}$.

<u>Proof</u>: Suppose that $K = \overline{L \oplus L'}$. For each $\omega \in \Omega$, $S_\omega \subseteq \overline{L \oplus L'}$, and $S_\omega \in H_{L \oplus L'}$. By Corollary 4.4.3, either

$H_L(S_\omega) \neq 0$, or $H_{L'}(S_\omega) \neq 0$. But $H_L(S_\omega) = S_\omega \cap H_L(K) = S_\omega \cap L = 0$, and so $0 \neq H_{L'}(S_\omega) = S_\omega \cap L'$. Since S_ω is an atom, $S_\omega \subseteq L'$. Thus $\bigoplus_{\omega\in\Omega} S_\omega \subseteq L'$, and, since L' is closed, $\overline{\bigoplus_{\omega\in\Omega} S_\omega} \subseteq L'$. Q.E.D.

Theorem 4.5.6 The following are equivalent for any ring K:

(1) $K = S_1(K)$

(2) (a) For each closed ideal L there is a closed ideal L' such that $K = \overline{L \oplus L'}$.

(b) Every non-zero closed ideal contains an atom.

(3) (a) For each closed ideal $L \neq K$ there is a closed ideal $L' \neq 0$ such that $L \cap L' = 0$.

(b) Every closed ideal contains an atom.

Proof: (1) implies (2). That (1) implies (2) (a) follows from Corollary 4.5.4. Also, we have seen in the same corollary that the set of atoms is an independent family of ideals. Suppose $L \neq 0$ is a closed ideal, $\{S_\alpha: \alpha\in\Lambda\}$ is the family of atoms of K, and that $K = \overline{L \oplus (\bigoplus_{\delta\in\Delta} S_\delta)}$. Since $L \neq 0$, there is an α in Λ but not in Δ. By Corollary 4.4.3, either $H_L(S_\alpha) \neq 0$ or, for some δ, $H_{S_\delta}(S_\alpha) \neq 0$. In the latter case, we would have $0 \neq H_{S_\delta}(K) \cap S_\alpha = S_\delta \cap S_\alpha$. Since S_δ and S_α are both atoms, this would imply that $S_\alpha = S_\delta$, which is false. Therefore we have $0 \neq H_L(S_\alpha) = S_\alpha \cap H_L(K) = S_\alpha \cap L$. Since S_α is an atom, this gives $S_\alpha \subseteq L$, as desired.

Clearly (2) implies (3). To show that (3) implies (1), suppose that (3) holds, but that $K \neq S_1(K)$. By (3)(a) there is a closed ideal L' such that $S_1(K) \cap L' = 0$, and, by (3)(b), L' contains an atom T. Then $T \subseteq S_1(K) \cap T \subseteq S_1(K) \cap L' = 0$, a contradiction. Therefore $K = S_1(K)$. Q.E.D.

<u>Theorem 4.5.7</u> Let K be a ring, and $\{S_\alpha : \alpha \in \Lambda\}$ the family of atoms of K. Suppose also that $K = S_1(K)$. Then the lattice of closed ideals of K is modular if and only if, for every closed ideal L of K there is a subset Λ_L of Λ such that $L = \overline{\bigoplus_{\lambda \in \Lambda_L} S_\lambda}$. In this case the lattice of closed ideals is distributive. (That is, if A, B, and L are closed ideals, then $L \cap \overline{(A+B)} = \overline{L \cap A + L \cap B}$.)

<u>Proof</u>: Suppose first that the lattice of closed ideals is modular, and let L be a closed ideal of K. By Corollary 4.5.4, there is a subset Δ of Λ such that $\overline{L \oplus (\overline{\bigoplus_{\delta \in \Delta} S_\delta})} = K$. Also by the same corollary, the family of all atoms is independent. If we set $\Lambda_L = \Lambda \sim \Delta$ then we also have $\overline{(\bigoplus_{\lambda \in \Lambda_L} S_\lambda) \oplus \overline{(\bigoplus_{\delta \in \Delta} S_\delta)}} = \overline{\bigoplus_{\alpha \in \Lambda} S_\alpha} = K$. From Theorem 4.5.5, we

obtain $\overline{\bigoplus_{\lambda\in\Lambda_L} S_\lambda} \subseteq L$. If we denote $\overline{\bigoplus_{\lambda\in\Lambda_L} S_\lambda}$ by B, then the assumed modularity of the lattice of closed ideals gives
$L = L \cap K = L \cap (\overline{B \oplus (\overline{\sum_{\delta\in\Delta} S_\delta})}) = \overline{B + (L \cap (\overline{\sum_{\delta\in\Delta} S_\delta}))} = \overline{B+0} = B,$
as desired.

Conversely, suppose that any closed ideal L is of the form $\overline{\sum_{\lambda\in\Lambda_L} S_\lambda}$ for some $\Lambda_L \subseteq \Lambda$. We note first that if Θ and Δ are subsets of Λ, then $\overline{(\overline{\sum_{\theta\in\Theta} S_\theta}) + (\overline{\sum_{\delta\in\Delta} S_\delta})} = \overline{\sum_{\gamma\in\Delta\cup\Theta} S_\gamma}$ and $(\overline{\sum_{\theta\in\Theta} S_\theta}) \cap (\overline{\sum_{\delta\in\Delta} S_\delta}) = \overline{\sum_{\beta\in\Theta\cap\Delta} S_\beta}$. The first equality is obvious. Also $\overline{\sum_{\beta\in\Theta\cap\Delta} S_\beta}$ is clearly contained in both $\overline{\sum_{\theta\in\Theta} S_\theta}$ and $\overline{\sum_{\delta\in\Delta} S_\theta}$. By our assumption, we may write the closed ideal $\overline{\sum_{\theta\in\Theta} S_\theta} \cap \overline{\sum_{\delta\in\Delta} S_\delta}$ as $\overline{\sum_{\gamma\in\Gamma} S_\gamma}$ for $\Gamma \subseteq \Lambda$. For each $\gamma \in \Gamma$, $S_\gamma \subseteq \overline{\sum_{\theta\in} S_\theta}$, and, by Corollary 4.4.3, for some θ in Θ, we have $0 \neq H_{S_\theta}(S_\gamma) = S_\theta \cap S_\gamma$. Since these are both atoms, $S_\gamma = S_\theta$ and $\gamma = \theta$. Similarly, γ is in Δ. Therefore we have $\overline{\sum_{\theta\in\Theta} S_\theta} \cap \overline{\sum_{\delta\in\Delta} S_\delta} \subseteq \overline{\sum_{\alpha\in\Delta\cap\Theta} S_\alpha}$.

An immediate consequence of these remarks is that if $L = \overline{\sum_{\lambda\in\Lambda_L} S_\lambda}$ and $L_1 = \overline{\sum_{\alpha\in\Lambda_{L_1}} S_\beta}$ are closed ideals in K, and if $L \subseteq L_1$, then $\Lambda_L \subseteq \Lambda_{L_1}$.

We can now show that the lattice of closed ideals of K is distributive. The above remarks show that the mapping from subsets of Λ to closed ideals, where Δ maps to $\overline{\sum_{\delta\in\Delta} S_\delta}$, is a lattice homomorphism. Our assumption just says that this map is <u>onto</u> the lattice of closed ideals. Since the lattice of subsets of any set is a distributive lattice, and since distributivity is preserved under lattice homomorphims, the result follows at once. Q.E.D.

In conclusion, we examine what occurs if we impose a stronger condition than just $K = S_1(K)$.

<u>Theorem 4.5.8</u> Suppose that K is a ring, $\{S_\alpha : \alpha\in\Lambda\}$ the set of atoms of K, and suppose that $K = \sum_{\alpha\in\Lambda} S_\alpha$. Then:

(1) the family of atoms is independent,

(2) for any closed ideal L there is a <u>unique</u> closed ideal L' such that $L \oplus L' = K$

(3) for each closed ideal L there is a subset Λ_L of Λ such that $L = \sum_{\lambda\in\Lambda_L} S_\lambda$.

(4) the lattice of closed ideals of K is a distributive sublattice of the lattice of ideals of K.

<u>Proof</u>: (1) is true for any ring K, by Corollary 4.5.4. We now prove (2). If L is a closed ideal of K, by Corollary 4.5.4, there is a subset Δ of Λ such that $K = \overline{L \oplus (\sum_{\delta\in\Delta} S_\delta)} =$

$\overline{L \oplus (\sum_{\delta\in\Delta} S_\delta)}$. For each $\lambda\in\Lambda\sim\Delta$, from Corollary 4.4.3 we obtain that either $H_L(S_\lambda) \neq 0$, or, for some $\delta\in\Delta$, $H_{S_\delta}(S_\lambda) \neq 0$. As we have seen before, the latter would imply that $S_\delta = S_\lambda$, and $\lambda = \delta$, which is false. Therefore, for $\lambda\in\Lambda\sim\Delta$, $0 \neq H_L(S_\lambda)$, and, as before, we obtain $S_\lambda \subseteq L$. Therefore, $K = \bigoplus_{\lambda\in\Lambda} S =$ $(\bigoplus_{\lambda\in\Lambda\sim\Delta} S_\lambda) \oplus (\sum_{\delta\in\Delta} S_\delta) \subseteq L \oplus (\sum_{\delta\in\Delta} S_\delta) \subseteq K$, and thus $K = L \oplus (\bigoplus_{\delta\in\Delta} S_\delta)$. We claim that $K = L \oplus (\overline{\bigoplus_{\delta\in\Delta} S_\delta})$. For if $L \cap (\overline{\bigoplus_{\delta\in\Delta} S_\delta}) \neq 0$, then $0 \neq H_{\bigoplus_{\delta\in\Delta} S_\delta}(L)$, and, by Corollary 4.4.3, we would have $H_{S_\delta}(L) \neq 0$ for some $\delta\in\Delta$. As we have seen previously, this would imply $S_\delta \subseteq L$, which is false. Thus we have $K = L \oplus L'$, where $L' = \overline{\bigoplus_{\delta\in\Delta} S_\delta}$ is a closed ideal. If we also had $K = L \oplus L''$, where L'' is also closed, then $L' \cong K/L \cong L''$. Since L' and L'' are isomorphic closed ideals, they are the same. This proves (2).

Using the notation of the previous paragraph, if L is a closed ideal, we have $K = (\bigoplus_{\lambda\in\Lambda\sim\Delta} S_\lambda) \oplus (\bigoplus_{\delta\in\Delta} S_\delta) = L \oplus (\bigoplus_{\delta\in\Delta} S_\delta)$, where $L \supseteq \bigoplus_{\lambda\in\Lambda\sim\Delta} S_\lambda$. It follows immediately that $L = \bigoplus_{\lambda\in\Lambda\sim\Delta} S_\lambda$. (In general, if $K = U \oplus V = U' \oplus V$, where $U \subseteq U'$, then $U = U'$.) This proves (3).

Now we show (4). Let L and L' be closed ideals of K. In order to show that the lattice of closed ideals of K is a sublattice of the lattice of all ideals of K, we must show that $\overline{L+L'} = L + L'$. We may write $\overline{L+L'} = \sum_{\gamma\in\Gamma} S_\gamma$ by (3). Then, for each $\gamma\in\Gamma$, $H_{L+L'}(S_\gamma)$ is non-zero. Therefore, by Corollary 4.4.3, either $H_L(S_\gamma)$ or $H_{L'}(S_\gamma)$ is non-zero. Since S_γ is an atom, this implies that either $S_\gamma \subseteq L$ or $S_\gamma \subseteq L'$ and, in either case, $S_\gamma \subseteq L + L'$. Therefore $\overline{L+L'} \subseteq L + L'$.

The lattice of all ideals of K is a modular lattice, and so is any sublattice. Theorem 4.5.7 can now be applied to conclude that the lattice of closed ideals is indeed distributive. Q.E.D.

CHAPTER FIVE

RINGS WITH CHAIN CONDITIONS ON CHARACTERISTIC IDEALS

5.1 Characteristic Ideals

An ideal I of a ring K is a characteristic ideal of K if, whenever K is an ideal of a ring S, I is also an ideal of S.

Characteristic ideals are not uncommon in a ring. Clearly, in a ring K, K and 0 are characteristic ideals. If K has a unity element e, whenever K is an ideal in a ring S we can write $S = eSe + (1-e)Se + eS(1-e) + (1-e)S(1-e)$. However, eS and Se are in K, and so $eS = eSe = Se$, which gives $eS(1-e) = (1-e)Se = 0$. Also we obtain $eSe = K$. It follows then that $S = K \oplus T$, where T is a two-sided ideal of S, as, of course, is K. From this we see that any ideal of K is an ideal of S, which implies that any ideal of K is a characteristic ideal of K.

From the Andrunakievic Lemma (Divinsky (6), page 107) we can see that any idempotent ideal I of K is a characteristic ideal of K. For if $I \triangleleft K$, and $I^2 = I$, then, whenever $K \triangleleft S$, the ideal $\langle I\rangle_S$ of S generated by I satisfies $I = I^3 \subseteq \langle I\rangle_S{}^3 \subseteq I$, from which we see that $I = \langle I\rangle_S{}^3$ is an ideal of S.

The relevance of characteristic ideals to the study of radicals comes from the fact that, for any radical class R, and for any ring K, $R(K)$ is a characteristic ideal of K. (See Divinsky (6), page 124.)

Another example of a characteristic ideal is an ideal I of a ring K which is such that K/I is a semiprime ring. For if K is an ideal of a ring S, then $\langle I\rangle_S \subseteq K$, and, by the Andrunakievic Lemma, $\langle I\rangle_S{}^3 \subseteq I$. Since K/I is semiprime, we obtain $\langle I\rangle_S \subseteq I$, and thus $I = \langle I\rangle_S$, an ideal of S. Therefore I is a characteristic ideal of K.

As a final example of characteristic ideals, we note that, in a semiprime ring K, for any ideal I of K, $I^* = \{x \in K\colon Ix = 0\}$ is a two-sided characteristic ideal of K. First of all, I^* is easily seen to be a right ideal of K. It is a left ideal of K because, if $x \in I^*$ and $k \in K$, then $I(kx) \subseteq Ix = 0$. Here we have used the fact that I is a two-sided ideal of K. Since $II^* = 0$, we have $(I^*I)^2 = 0$, which gives, since K is semiprime, $I^*I = 0$. It follows that $I^* = \{x \in K\colon xI = Ix = 0\}$. Suppose now that K is an ideal in a ring S. Then $I^*S \subseteq K$, and $I(I^*S) = (II^*)S = 0$, so $I^*S \subseteq I^*$. Also, $SI^* \subseteq K$, and $(I\,SI^*)^2 = 0$. Using the semiprimeness of K, we have $I\,SI^* = 0$, or $SI^* \subseteq I^*$. This proves that I^* is a characteristic ideal of K.

Clearly, if K is a semiprime ring and if I is an ideal of K, then $I \subseteq I^{**}$. Since taking annihilators reverses inclusions, we have $I^* \supseteq I^{***}$. However, if we replace I by I^* in the equation $I \subseteq I^{**}$ we see that $I^* \subseteq I^{***}$, and thus $I^* = I^{***}$. It is an immediate consequence of these remarks that, in a semiprime ring K, either of the ascending or descending chain conditions on ideals of the form I^*, where I is an ideal of K, implies the other chain condition, and also that these conditions are implied by either the ascending or descending chain condition on characteristic ideals.

An ideal of the form I^*, where I is an ideal of K, will be called an <u>annihilator ideal</u> of K.

<u>Theorem 5.1.1</u> Let K be a ring, and let I be a characteristic ideal of K. If K has the ascending (resp. descending) chain condition on characterisitic ideals, then K/I also has the ascending (resp. descending) chain condition on characteristic ideals.

<u>Proof</u>: Let C_n, $n = 1,2,\ldots$ be an ascending chain of characteristic ideals of K/I. If we let T_n be the inverse image of C_n under the natural homomorphism $f:K \to K/I$, then T_n is an ideal of K, and $T_n \subseteq T_{n+1}$.

We claim that T_n is a characteristic ideal of K. For if $K \triangleleft S$, since I is characteristic, $I \triangleleft S$, and we can form the factor ring S/I. This has K/I as an ideal. Also, the natural homomorphism f can be extended to $g:S \to S/I$.

Since $C_n = T_n/I$ is a characteristic ideal of K/I,

C_n is an ideal of S/I, and thus $T_n = f^{-1}(C_n) = g^{-1}(C_n)$ is an ideal in S. This shows that T_n is indeed a characteristic ideal of K.

By the assumed chain condition, there is an integer m such that $T_m = T_{m+i}$ for all $i \geq 0$. Therefore $C_m = C_{m+i}$ for all $i \geq 0$, and this proves K/I has the ascending chain condition on characteristic ideal. The descending chain condition case is proved in the same way. Q.E.D.

5.2 Some Structure Theorems

In this section, some results of Andrunakievic (1) are applied to show that certain rings can be described as finite subdirect sums of prime rings. These results of Andrunakievic were also noted by Levy (19).

Recall from Chapter one the definitions of supernilpotent, SP, and dual radicals. If R is any supernilpotent radical, it was asserted in Chapter one that we have $R \subseteq R_S \subseteq R_\varphi$, where R_S is the upper radical with respect to the class of prime R-semisimple rings, and R_φ is the upper radical with respect to the class of all subdirectly irreducible rings with R-semisimple hearts. The radicals R_S and R_φ are, respectively, the smallest SP-radical, and the smallest dual radical, containing the radical class R.

Theorem 5.2.1 (c.f. Andrunakievic (1), Lemma 16)

Let A be a non-zero ideal of a semiprime ring K. If K has the ACC on annihilator ideals, then A contains a subring A' which is a prime ring, and which is an ideal of K.

Proof: Suppose the statement is false. Then A is not itself a prime ring, and so A has non-zero ideals A_1' and B_1' such that $A_1'B_1' = 0$. Let $A_1 = \langle A_1'\rangle_K{}^3$, and $B_1 = \langle B_1'\rangle_K{}^3$. Since K is semiprime, neither A_1 nor B_1 is zero, and, by the Andrunakievic Lemma, $A_1B_1 \subseteq A_1'B_1' = 0$.

Suppose that, for a positive integer n we have found A_i, B_i, $i = 1,2,\ldots,n$, such that A_{i+1} and B_{i+1} are non-zero ideals of K contained in A_i and $A_iB_i = 0$. By our assumption, A_n is not a prime ring, and so it contains ideals (of A_n) U and V which are non-zero but which satisfy $UV = 0$. If we let $A_{n+1} = \langle U\rangle_K{}^3$ and $B_{n+1} = \langle V\rangle_K{}^3$, then these are non-zero ideals of K (since K is semiprime) contained in A_n. Also $A_{n+1}B_{n+1} \subseteq UV = 0$. Thus the sequences of A_i's and B_i's, $i = 1,2,\ldots,n$, can be extended by adjoining A_{n+1} and B_{n+1}. By induction, there are infinite sequences A_i and B_i, i a positive integer, such that, for each i, A_{i+1} and B_{i+1} are non-zero ideals of K contained in A_i and $A_iB_i = 0$ for all i. Since $A_i \supseteq A_{i+1}$, we have $A_i^* \subseteq A_{i+1}^*$. By the ACC on annihilator ideals, there is an m such that $A_m^* = A_{m+1}^*$. Since $A_{m+1}B_{m+1} = 0$, $B_{m+1} \subseteq A_{m+1}^* = A_m^*$, and therefore $B_m^2 \subseteq A_mA_m^* = 0$. The semiprimeness of K then gives $B_m = 0$, a contradiction. Q.E.D.

Lemma 5.2.2 (c.f. Andrunakievic (1), Corollary 8)

Let R be a supernilpotent radical, and K an R-semisimple ring with the ACC on annihilator ideals. Denote by C_1 the class of prime R-semisimple rings. Then K contains an ideal B such that $B^* \neq 0$, and K/B is in C_1.

Proof: Since R is a supernilpotent radical, K is also a semiprime ring. By the previous theorem, there is a non-zero ideal A of K which is a prime ring. From Divinsky (6), Corollary 2 of Theorem 47 (page 125), A is also R-semisimple, and so A is in C_1. From Chapter one, C_1 is a special class, and so K/A^* is in C_1. Since $0 \neq A \subseteq A^{**}$, we see that $B = A^*$ is an ideal of the desired form. Q.E.D.

Theorem 5.2.3 (c.f. Andrunakievic (1), Theorem 14)

Let R be a supernilpotent radical, and let K be an R-semisimple ring with ACC on annihilator ideals. Then K is a subdirect sum of a finite number of prime R-semisimple rings.

<u>Proof</u>: Consider $\{P_\alpha : \alpha\epsilon\Lambda\}$ where, for each $\alpha\epsilon\Lambda$, P_α is an ideal of K such that $P_\alpha^* \neq 0$ and K/P_α is a prime R-semisimple ring. By the previous lemma, this is a non-void family of ideals of K. Let $A = \bigcap_{\alpha\epsilon\Lambda} P_\alpha$. We show first that $A = 0$.

From Theorem 5.2.1, if $A \neq 0$, A contains a non-zero ideal B of K which is a prime ring. Proceeding as in the proof of Lemma 5.2.2, we obtain the result that $B^* \epsilon \{P_\alpha : \alpha\epsilon\Lambda\}$. Suppose that $B^* = P_{\alpha_0}$. Then $B^* \supseteq A$.

Since $B \subseteq A$, we have $B^* \supseteq A^*$, and hence $B^* \supseteq A+A^*$. Taking annihilators gives $B^{**} = P_{\alpha_0}{}^* \subseteq (A+A^*)^* \subseteq A^* \cap A^{**}$. Then $(P_{\alpha_0}{}^*)^2 \subseteq A^*A^{**} = 0$, and, since K is semiprime, $P_{\alpha_0}{}^* = 0$. This contradiction of the definition of the P_α's shows $A = 0$.

It remains only to show that A is the intersection of a <u>finite</u> number of P_α's.

First of all, each P_α satisfies $P_\alpha = P_\alpha^{**}$. For P_α is a prime ideal of K, and $0 = P_\alpha^* P_\alpha^{**} \subseteq P_\alpha$. Thus either $P_\alpha^* \subseteq P_\alpha$ (which would imply $(P_\alpha^*)^2 = 0$, which is false), or $P_\alpha^{**} \subseteq P_\alpha$. Since P_α is always contained in P_α^{**}, the equality follows.

We now show that, for any finite subset F of Λ, $(\bigcap_{\beta\in F} P_\beta)^{**} = \bigcap_{\beta\in F} P_\beta$. We have $\bigcap_{\beta\in F} P_\beta = \bigcap_{\beta\in F}(P_\beta^{**}) = (\sum_{\beta\in F} P_\beta^{*})^{*} \supseteq (\bigcap_{\beta\in F} P_\beta)^{**}$. The last inequality comes from the easily verified fact that $(\bigcap_{\beta\in F} P_\beta)^{*} \supseteq \sum_{\beta\in F}(P_\beta^{*})$, and the second equality comes from the fact (also easily verified) that, for ideals A_1, $A_2, \ldots A_n$ of K, $\bigcap_{i=1}^{n} A_i^{*} = (\sum_{i=1}^{n} A_i)^{*}$. Since $(\bigcap_{\beta\in F} P_\beta)^{**}$ always contains $\bigcap_{\beta\in F} P_\beta$, the desired equality follows.

Since K is semiprime, the remarks in §5.1 show that K also has the DCC on annihilator ideals. Also, for any finite subset F of Λ, $\bigcap_{\beta\in F} P_\beta$ is its own "double annihilator", and is therefore an annihilator ideal. By the DCC on annihilator ideals, there is a finite subset F_0 of Λ such that $\bigcap_{\beta\in F_0} P_\beta$ is a minimal member of $\{\bigcap_{\gamma\in G} P_\gamma$: G is a finite subset of $\Lambda\}$. It follows then that, for each λ in Λ, $P_\lambda \cap (\bigcap_{\beta\in F_0} P_\beta) = \bigcap_{\beta\in F_0} P_\beta$, or $\bigcap_{\beta\in F_0} P_\beta \subseteq P_\lambda$. Therefore

$\bigcap_{\beta\in F_0} P_\beta \subseteq A = 0$.

We have found a finite subset $\{P_\beta\colon \beta\in F_0\}$, the intersection of whose members is zero. The ring K is therefore a subdirect sum of the finite number of rings $\{K/P_\beta\colon \beta\in F_0\}$, and each of these rings is a prime R-semisimple ring. Q.E.D.

<u>Theorem 5.2.4</u> Let K be a ring with the ACC (resp. DCC) on characteristic ideals, and let R be any supernilpotent radical class. Then $R(K) = R_s(K)$, and $K/R(K)$ is a finite subdirect sum of prime R-semisimple rings satisfying the same chain condition.

<u>Proof</u>: As we have seen in §5.1, $R(K)$ is a characteristic ideal of K, and so, by Theorem 5.1.1, $K/R(K)$ also has the ACC (resp. DCC) on characteristic ideals. It was also shown in §5.1 that either of these chain conditions in a semiprime ring implies both the ACC and DCC for annihilator ideals. Therefore $K/R(K)$ satisfies the hypotheses of Theorem 5.2.3, and we can conclude that $K/R(K)$ is a subdirect sum of prime R-semisimple rings $K_1, K_2, \ldots, K_n$. Each K_i is a homomorphic image of $K/R(K)$, and thus of K. Suppose $K_i = K/I_i$. Then each I_i is a prime ideal of K, and is thus a characteristic ideal of K (see §5.1). By Theorem 5.1.1, K_i has the ACC (resp. D.C.C.) on characteristic ideals.

Since each K_i is prime and R-semisimple, it is R_s semisimple. Since, as is well known, for any radical class P, a subdirect sum of P-semisimple rings is P-semisimple, we have that $K/R(K)$ is R_s-semisimple. Since $R(K) \subseteq R_s(K)$, this implies $R(K) = R_s(K)$. Q.E.D.

<u>Lemma 5.2.5</u> Let S be a prime ring with DCC on characteristic ideals. Then S is a subdirectly irreducible ring with

heart H satisfying $H^2 = H$.

Proof: Since S, and therefore S^n for all integers n, is a characteristic ideal of S, the DCC on characterisitc ideals guarantees that $S^n = S^{n+1}$ for some n. Since S is prime $S^n \neq 0$. Thus S^n is a non-zero idempotent ideal. We saw in §5.1 that any idempotent ideal is characteristic. Therefore we can apply the DCC on characterisitc ideals to find a minimal non-zero idempotent ideal I.

Suppose that I' is any minimal non-zero idempotent ideal. Then $I \cap I'$ is a characteristic ideal which, if it is not zero, has some power of itself, say $(I \cap I')^m$ idempotent. But the minimality of both I and I' must then give $I = (I \cap I')^m = I'$. On the other hand, we cannot have $I \cap I' = 0$, for this would imply $II' = 0$, contradicting the fact that S is a prime ring. Therefore we conclude that there is a unique non-zero minimal idempotent ideal I.

Now let B be any non-zero ideal of S. Then SBS is a characteristic ideal, and using the DCC and the fact that S is prime we see that some power of SBS is idempotent. Again using the DCC on characteristic ideals, we can conclude that there is a minimal idempotent ideal of S contained in SBS, and hence in B. Therefore $I \subseteq B$. Since I is an ideal of S contained in every non-zero ideal of S, S is indeed subdirectly irreducible, and I, the heart of S is idempotent. Q.E.D.

Theorem 5.2.6 Let K be a ring with DCC on characteristic ideals, and let R be a supernilpotent radical property. Then $R(K) = R_\varphi(K)$ and $K/R(K)$ is a subdirect sum of rings K_i, $i = 1,2,\ldots,n$, where each K_i is prime, R-semisimple, subdirectly irreducible, and satisfies the DCC on characteristic ideals.

Proof: By Theorem 5.2.4, $K/R(K)$ can be represented as a subdirect sum of rings $K_1,K_2,\ldots,K_n$, where each K_i is prime, R-semisimple, and satisfies the DCC on characteristic ideals. By the previous Lemma, each K_i is subdirectly irreducible. Let the heart of K_i be H_i. Since H_i is an ideal of the R-semisimple ring K_i, H_i is also R-semisimple. From the definition of R_φ we see that each K_i is also R_φ-semisimple. $K/R(K)$ is a subdirect sum of R_φ-semisimple rings, and is therefore R_φ-semisimple. Since $R(K) \subseteq R_\varphi(K)$, we obtain the desired equality. Q.E.D.

In what follows, L will denote the Levitzki (locally nilpotent) radical class, and B will denote the Baer Lower radical.

Lemma 5.2.7 In a ring K, let $\{N_\alpha: \alpha\in\Lambda\}$ be the family of all nilpotent ideals of K, and let $T = \sum_{\alpha\in\Lambda} N_\alpha$. Then T is a characteristic ideal.

Proof: Suppose K is an ideal of a ring S, and consider $\langle N_\alpha \rangle_S$, the ideal in S generated by N_α. This is an ideal of S contained in K, and so is an ideal of K. By the Andrunakievic Lemma, $\langle N_\alpha \rangle_S^{\,3} \subseteq N_\alpha$, so $\langle N_\alpha \rangle_S$ is nilpotent, and $\langle N_\alpha \rangle_S \subseteq T$. Then $\langle T \rangle_S \subseteq \sum_{\alpha \in \Lambda} \langle N_\alpha \rangle_S \subseteq T$, so $T = \langle T \rangle_S$, or T is an ideal of S. Q.E.D.

Theorem 5.2.8 Let K be a ring with either the ACC or the DCC on characteristic ideals. Then $B(K) = T$, and this is nilpotent.

Proof: It is well known that the nilpotence of T will imply that $T = B(K)$. It is sufficient, therefore, to prove that T is nilpotent.

Suppose first that K has the DCC on characteristic ideals. If T is not nilpotent, some power of T, say T^n, is non-zero and idempotent (since T is characteristic). Then $I = T^n$ satisfies $I^2 = I^3 = I$. By the DCC on characteristic ideals, there is a minimal member of the set of all characteristic ideals J for which $IJI \neq 0$. Let this minimal member of J_0. Then, for some x in J_0, $IxI \neq 0$. Now IxI is a characteristic ideal of K contained in J_0, and $I(IxI)I = I^2xI^2 = IxI \neq 0$. By the minimality of J_0, $J_0 = IxI$, and we have $x = \sum_{i=1}^{n} y_i x z_i$ for $y_1, y_2, \ldots, y_n$,

$z_1, \ldots, z_n$ in I. Since $T \subseteq B(K) \subseteq L(K)$, the subring W of K generated by the y_i's, the z_i's and x is nilpotent of index, say m. Repeated substitution for x m times in the right hand side of $x = \sum_{i=1}^{n} y_i x z_i$ gives $x = 0$, a contradiction of $IxI \neq 0$. In the DCC case, therefore, T is nilpotent.

Consider now the case where K has the ACC on characteristic ideals. The set of all nilpotent characteristic ideals is not empty, for it contains the ideal 0, and so it contains a maximal member H. For any nilpotent ideal N of K, KNK is a nilpotent characteristic ideal, and so is H + KNK. By the maximality of H, H = H + KNK, or $KNK \subseteq H$. Then $T^3 \subseteq KTK = K(\sum_{\alpha \in \Lambda} N_\alpha)K \subseteq \sum_{\alpha \in \Lambda} KN_\alpha K \subseteq H$, so T^3, and therefore T, is nilpotent.

<u>Corollary 5.2.9</u> If K is a ring with the DCC on characteristic ideals, L(K) = B(K), and this is nilpotent.

<u>Proof</u>: From Theorem 5.2.8, B(K) is nilpotent. From Theorem 5.2.6, $B(K) = B_\varphi(K)$, and (see Divinsky (1), Theorem 67) $L(K) \subseteq B_\varphi(K)$. Therefore $B(K) = L(K) = B_\varphi(K)$, and this is nilpotent. Q.E.D.

We would like to have a converse to Theorem 5.2.4 or to Theorem 5.2.6. The problem in obtaining one is that, if

K is a ring, and A and B are characteristic ideals. where $A \supseteq B$, we do not know whether or not A/B is a characteristic ideal of K/B. If this question could be answered affirmatively, the techniques used in the following theorem could be used to obtain converses to Theorems 5.2.4 and 5.2.6.

<u>Theorem 5.2.10</u> Let R be a supernilpotent radical property. A ring K is R-semisimple and has the ACC (resp. DCC) on all two-sided ideals if and only if it is a finite subdirect sum of prime R-semisimple rings with the ACC (resp. DCC) on all two-sided ideals.

<u>Proof</u>: We shall prove the result in the ACC case. The proof in the DCC case is similar.

Suppose first that K is R-semisimple and has the ACC on all ideals. By Theorem 5.2.4, K is a subdirect sum of rings $K_1,\ldots,K_n$, and each K_i is prime and R-semisimple. From the properties of subdirect sums, each K_i is a homomorphic image of K. It follows that K_i also has the ACC on all ideals.

Conversely suppose that K is a subdirect sum of the prime R-semisimple rings $K_1,K_2,\ldots,K_n$, where each K_i has the ACC on all ideals. Then K, being a subdirect sum of R-semisimple rings, is R-semisimple.

It remains only to show that K has the ACC on all ideals. Suppose that $I_1, I_2, \ldots, I_n$ are ideals of K such that $\bigcap_{i=1}^{n} I_i = 0$, and $K_i \cong K/I_i$. (Such ideals exist, since K is a subdirect sum of the K_i's.)

Let $\{L_t\colon\ t$ a positive integer$\}$, be an ascending chain of ideals of K. We can use the ACC in K_i to find, for $j = 1,2,\ldots,n$, an integer N_j such that $(L_{N_j} \cap I_0 \cap I_1 \cap \ldots \cap I_{j-1}) + I_j = (L_{N_j+r} \cap I_0 \cap \ldots \cap I_{j-1}) + I_j$ for all $r \geq 0$, where we denote K by I_0. If we set $N = \max\{N_1, N_2, \ldots, N_n\}$, then, for all $r \geq 0$, we have

(1) $L_N + I_1 = L_{N+r} + I_1$

(2) $L_N \cap I_1 + I_2 = L_{N+r} \cap I_1 + I_2$

$\vdots \qquad\qquad \vdots$

(n) $L_N \cap I_1 \cap I_2 \cap \ldots \cap I_{n-1} + I_n = L_{N+r} \cap I_1 \cap \ldots \cap I_{n-1} + I_n$.

We show that $L_N = L_{N+r}$ for all $r \geq 0$. For any x in L_{N+r} there is an x_1 in L_N such that $s_1 = x - x_1 \in I_1$. Then $s_1 \in L_{N+r} \cap I_1$ and there is an x_2 in $L_N \cap I_1$ such that $s_2 = s_1 - x_2 \in I_2$. Also, x_1 and x_2, and hence s_2 are in $L_{N+r} \cap L_1$, and so $s_2 \in L_{N+r} \cap I_1 \cap I_2$.

Continuing in this manner, we find, for $j = 1,2,\ldots,n$ elements $x_j \in L_N \cap I_1 \cap \ldots \cap I_{j-1}$ and $s_j = s_{j-1} - x_j \in L_{N+r} \cap I_1 \cap I_2 \cap \ldots \cap I_j$.

We have

$$s_1 = x - x_1$$

$$s_2 = s_1 - x_2$$

$$s_3 = s_2 - x_3$$

$$\vdots$$

$$s_{n-1} = s_{n-2} - x_{n-1}$$

$s_n = s_{n-1} - x_n$, where $s_n \in L_{N+r} \cap I_1 \cap I_2 \ldots \cap I_n = 0$, and where each $x_j \in L_N$. Then $s_{n-1} = x_n \in L_N$. Climbing back up, we see that s_{n-1}, $s_{n-2},\ldots,s_2$, and finally s_1 are in L_N. But then $x = s_1 + x_1$ is also in L_N. This shows that $L_{N+r} \subseteq L_N$ for all $r \geq 0$. Since the L_i's are an ascending sequence, we have $L_N = L_{N+r}$ for all $r \geq 0$. This proves that K has the ACC on all ideals. Q.E.D.

5.3 Finite Subdirect Sums of Rings

Suppose that a ring K is a subdirect sum of a family of rings $\{K_\alpha: \alpha\epsilon\Lambda\}$. More precisely, suppose that K has a family of ideals $\{I_\alpha: \alpha\epsilon\Lambda\}$ such that, for each α, there is an isomorphism $\theta_\alpha: K/I_\alpha \to K_\alpha$, and $\bigcap_{\alpha\in\Lambda} I_\alpha = 0$, and denote by i the monomorphism of K into $\prod_{\alpha\in\Lambda} K_\alpha$, where $i(x) = (\theta_\alpha(x+I_\alpha))_{\alpha\in\Lambda}$. Following Levy (19), we shall say that the subdirect embedding i is <u>irredundant</u> if each K_α is necessary in the sense that, for each $\beta\in\Lambda$, $\bigcap_{\alpha\in\Lambda\sim\{\beta\}} I \neq 0$. Also, we will say that K is an <u>irredundant subdirect sum</u> of the K_α's.

Not all subdirect sums are irredundant. For example, if K is the ring of even integers, then K can be represented as a subdirect sum of the nilpotent rings $2Z/2^nZ$, $n = 1,2,\ldots$, (where Z is the ring of integers), and any finite number of the rings $2Z/2^nZ$ may be omitted.

On the other hand, if K is a subdirect sum of a finite number of rings $K_1,K_2,\ldots,K_n$, then K may be represented as an irredundant subdirect sum of some of the K_i's. For suppose the corresponding ideals of K (such that $K/I_j \cong K_j$) are K_i, $i = 1,2,\ldots,n$. Then $\bigcap_{i=1}^{n} I_i = 0$, and we can find a subset F of $\{1,2,\ldots,n\}$ minimal with respect to having the property that $\bigcap_{\gamma\in F} I_\gamma = 0$. Then K is an irredundant subdirect sum of $\{K_\gamma : \gamma\in F\}$.

The next observation is due to Levy (19).

Lemma 5.3.1 Let $i: K \to \prod_{\beta \in \Lambda} K_\beta$ be a subdirect embedding. For each α we identify K_α with the ideal $\{ (x_\beta)_{\beta\in\Lambda}: x_\beta = 0$ for $\beta \neq \alpha\}$ of $\prod_{\beta\in\Lambda} K_\beta$. Then i is an irredundant subdirect embedding if and only if, for each $\alpha \in \Lambda$, $i(K) \cap K_\alpha$ is a non-zero ideal of $\prod_{\beta\in\Lambda} K_\beta$.

Proof: First of all, $i(K) \cap K_\alpha$ is an ideal of $\prod_{\alpha\in\Lambda} K_\alpha$. For suppose $x = (x_\delta)_{\delta\in\Lambda} \in i(K) \cap K_\alpha$. Then $x_\delta = 0$ for all $\delta \neq \alpha$. Now suppose that $(y_\delta)_{\delta\in\Lambda}$ is an element of $\prod_{\alpha\in\Lambda} K_\alpha$. From the properties of subdirect embeddings, there is an element a in K such that the α'th component of $i(a)$ is y_α. Then we see that $xy = (x_\delta y_\delta)_{\delta\in\Lambda}$ and that $x_\delta y_\delta = 0$ for $\delta \neq \alpha$. Since $x \in i(K)$, let $x = i(b)$ for some $b \in K$. Simple computation shows that $xy = i(b)y = i(b)i(a) = i(ba)$, and that this is in $i(K) \cap K_\alpha$. Therefore $(i(K) \cap K_\alpha)\prod_{\beta\in\Lambda} K_\beta \subseteq i(K) \cap K_\alpha$, and $i(K) \cap K_\alpha$ is a right ideal of $\prod_{\beta\in\Lambda} K_\beta$. In a similar fashion, $i(K) \cap K_\alpha$ is a left ideal.

Suppose now that the map i is an irredundant subdirect embedding. Then, for each α, there is a non-zero element a in $\bigcap_{\substack{\beta\in\Lambda \\ \beta\neq\alpha}} I_\beta$. Simple computation shows that $0 \neq i(a) \in i(K) \cap K_\alpha$.

Conversely, suppose that, for each α, $i(K) \cap K_\alpha \neq 0$.

Then, for a given α, if a in K is such that $0 \neq i(a) \in i(K) \cap K_\alpha$, it is easily seen that $a \in \bigcap_{\substack{\beta \in \Lambda \\ \beta \neq \alpha}} I_\beta$. Q.E.D.

We have seen, in Theorem 5.2.6, that, under certain conditions, we can express a ring K as a finite subdirect sum of subdirectly irreducible rings. As noted above, K can then be expressed as an irredundant subdirect sum of some of these rings. Under these circumstances, the following result holds.

<u>Lemma 5.3.2</u> Let $i: K \to \bigoplus_{i=1}^{n} K_i = s$ be an irredundant subdirect embedding, and let each K_i be subdirectly irreducible with heart H_i.

Then we have $H = \bigoplus_{i=1}^{n} H_i \subseteq \bigoplus_{i=1}^{n} (i(K) \cap K_i) \subseteq i(K) \subseteq S$, and H and $\bigoplus_{i=1}^{n} (i(K) \cap K_i)$ are ideals in S, and hence in $i(K)$.

<u>Proof</u>: By the previous lemma, for each i, $i(K) \cap K_i$ is a non-zero ideal of S. It is therefore a non-zero ideal of K_i, and therefore contains H_i. The rest follows immediately. Q.E.D.

Under the conditions described in the Lemma, it is natural to ask whether more can be said about $i(K)$. For example, is $i(K)$ an ideal of S, or is

$i(K) = \overset{n}{\underset{i=1}{\oplus}} (i(K) \cap K_i)$? The following example shows that neither of these statements need be true.

Example 5.3.3

Let L be any ring of characteristic p, and let L' be the ring $\{ (x,n): x \in L, n \in Z_p$ (the integers modulo $p) \}$ with addition defined componentwise and multiplication defined by $(x,n)(x',n') = (xx' + n'x + nx', nn')$. (c.f. Example 4.2.4) This is a ring with unity element $(0,1)$ and which has a copy of L as an ideal.

Let S be a simple non-trivial Jacobson radical ring of characterisitic p (see Example 4.2.4), and form the ring $T = S_1' \oplus S_2'$, where $S_1 = S_2 = S$. Also, let $W = (S_1 \oplus S_2)'$. There is a ring monomorphism $i: W \to T$, defined by $i[((s_1,s_2),n)] = ((s_1,n),(s_2,n))$. This map i, composed with the projections of T onto S_1' and S_2' gives mappings from W onto S_1' and S_2', so it follows that i is a subdirect embedding.

It is easily verified that S' is subdirectly irreducible with heart S, and also that $S_1 \oplus S_2 = (i(W') \cap S_1') \oplus (i(W') \cap S_2') \subsetneq i(W')$. Since $S_1 = i(W') \cap S_1'$, and similarly for S_2, Lemma 5.3.1 assures us that i is an irredundant subdirect embedding. Also, $i(W')$ is not an ideal of T, for, if it were, we would have $i(W')$ must contain $((0,1),(0,1))\cdot((0,1),(0,0)) = ((0,1),(0,0))$ and this is not true.

5.4 Hereditary Radicals of Finite Subdirect Sums of Rings

If H is a hereditary radical class, and if I is an ideal of a ring S, then $H(I) = I \cap H(S)$ (Divinsky (6), page 125). This is not in general true if I is just a subring of S, not even if I is subdirectly embedded into S. For example, we have seen that the ring of even integers, which is J-semisimple (where J is the Jacobson radical) is a subdirect sum of nilpotent rings. It is easily verified that, if each K_α is a right quasi-regular ring, then $\prod_{\alpha\in\Lambda} K_\alpha$ is also right quasi-regular. The even integers, therefore, can be subdirectly embedded into $\prod_{n=1} (2Z/2^nZ)$, which is Jacobson radical.

In this section, we shall see that the situation is different for finite subdirect sums. We shall prove that, if $i: K \rightarrow \bigoplus_{j=1}^{n} K_j = S$, is a subdirect embedding, then, for every hereditary radical H, $H(i(K)) = i(K) \cap H(S)$. Note that, in Example 5.3.3, we saw that $i(K)$ need not be an ideal of S.

Lemma 5.4.1 If $S = \bigoplus_{j=1}^{n} K_j$, and if H is any hereditary radical, then $H(S) = \bigoplus_{j=1}^{n} H(K_j)$.

Proof: Since $K_j \triangleleft S$ for each j, $H(K_j) \triangleleft S$, since $H(K_j)$ is a characteristic ideal of K_j. Therefore we have $\bigoplus_{j=1}^{n} H(K_j) \subseteq H(S)$. Now let $x = \sum_{j=1}^{n} x_j$ be in $H(S)$, where $x_j \in K_j$ for each j, and consider the projection $p_j: S \to K_j$. Under this map we have $x_j = p_j(x) \in p_j(H(S)) \subseteq H(K_j)$. Therefore $x \in \sum_{j=1}^{n} H(K_j)$. Q.E.D.

Lemma 5.4.2 Let $i: K \to S = \bigoplus_{j=1}^{n} K_j$ be a subdirect embedding. Then $i(H(K)) \subseteq i(K) \cap H(S)$.

Proof: Let p_j be the projection of S onto K_j. From the properties of subdirect sums, the composition $p_j i$ maps K onto K_j for each j, and so $p_j i(H(K))$ is an ideal of K_j, and is therefore contained in $H(K_j)$. Then, for $y \in H(K)$, we have $i(y) = \sum_{j=1}^{n} p_j i(y) \in \sum_{j=1}^{n} H(K_j) = H(S)$. Since $i(H(K)) \subseteq i(K)$, we have $i(H(K)) \subseteq H(S) \cap i(K)$. Q.E.D.

Lemma 5.4.3 Using the same notation as above, $H(S) \cap i(K)$

is a subdirect sum of the H-radical rings $p_j(H(S) \cap i(K))$, $j = 1,2,\ldots,n$.

Proof: $H(S) \cap i(K)$ is an ideal of $i(K)$, so $p_j(H(S) \cap i(K))$ is an ideal of $p_j(i(K)) = K_j$. Also, $p_j(H(S) \cap i(K)) \subseteq p_j(H(S)) = H(K_j)$, where this last equality follows from Lemma 5.4.1. Therefore $p_j(H(S) \cap i(K))$ is indeed an H-radical ring.

For any x in $H(S) \cap i(K)$, we can write $x = \sum_{j=1}^{n} p_j(x)$, and so we have $H(S) \cap i(K) \subseteq \bigoplus_{j=1}^{n} p_j(H(S) \cap i(K))$. It is clear that the inclusion mapping is a subdirect embedding. Q.E.D.

Lemma 5.4.4 Let P be a subdirect sum of rings $P_1, P_2, \ldots, P_n$, and suppose that each P_j is H-radical, where H is a hereditary radical. Then P is also an H-radical ring.

Proof: We know that there are ideals $I_1, I_2, \ldots, I_n$ such that $P_j \cong P/I_j$, and $\bigcap_{j=1}^{n} I_j = 0$. We proceed by induction on n.

When $n = 1$, the result is trivially true. Suppose that we know the result is true for $n = k-1$, and consider the case where $n = k$. If we let $T_1 = I_1$, and $T_2 = \bigcap_{j=2}^{n} I_j$, it follows that $P/T_1 \cong P_1$, so P/T_1 is H-radical. Also it is easily seen that P/T_2 is a subdirect sum of the rings $(P/T_2)/(I_j/T_2) \cong P/I_j$, $j = 2,3,\ldots,n$, and, by our induction assumption, P/T_2 is H-radical. Also $T_1 \cap T_2 = 0$, so P is a subdirect sum of the rings P/T_1 and P/T_2. It is therefore sufficient to prove the result when $k = 2$.

Since $T_1 \cap T_2 = 0$, $T_1 \cong (T_1)/(T_1 \cap T_2) \cong (T_1+T_2)/T_2 \triangleleft P/T_2$. Since P/T_2 is H-radical, and since H is hereditary, T_1 is therefore an H-radical ring. Therefore $T_1 \subseteq H(P)$, which implies $H(P/T_1) = H(P)/T_1$. But $H(P/T_1) = P/T_1$, and it follows that $H(P) = P$, and P is H-radical. Q.E.D.

Theorem 5.4.5 Let $i\colon K \to S = \bigoplus_{j=1}^{n} K_j$ be a subdirect embedding. Then, for H a hereditary radical, $H(i(K)) = i(K) \cap H(S)$.

Proof: Since i is a monomorphism, we have $H(i(K)) = i(H(K))$, and this is contained in $i(K) \cap H(S)$ by Lemma 5.4.2. Also, $i(K) \cap H(S)$ is an ideal of $i(K)$, and, by Lemmas 5.4.3 and 5.4.4, $i(K) \cap H(S)$ is an H-radical ring. Therefore $i(K) \cap H(S) \subseteq H(i(K))$. Q.E.D.

Remark: It is not necessarily true that $H(i(K)) = \bigoplus_{j=1}^{n} (H(K_j) \cap i(K))$. In Example 5.3.3, let H be the radical F_p. (See §4.3). Then $F_p(i(W')) = i(W')$, while $(F_p(S_1') \cap i(W')) \oplus (F_p(S_2') \cap i(W')) = S_1 \oplus S_2$, and these are not equal.

5.5 Finite Subdirect Sums and Structure Spaces

Let P be a property of rings such that if S has P, and if S' is isomorphic to S, then S' also has P. For any ring K, let $F(K:P)$ be $\{I: I$ is a prime ideal of $I \neq K$, and K/I has $P\}$. For example, if P is the property of being a right primitive ring, $F(K:P)$ is the set of all ideals I of K for which K/P is a right primitive ring.

As is well known, at least in special cases, it is possible to define a topology on $F(K:P)$ by introducing a closure operation in the following manner: if $\mathfrak{U} \subseteq F(K:P)$, then $\mathrm{cl}(\mathfrak{U})$ is defined to be $\{I \in F(K:P): I \supseteq \prod \mathfrak{U}\}$, where $\prod \mathfrak{U} = \bigcap_{U \in \mathfrak{U}} U$. This closure operation satisfies:

Cl 1: $\mathrm{cl}(\emptyset) = \emptyset$

Cl 2: For $\mathfrak{U} \subseteq F(K:P)$, $\mathfrak{U} \subseteq \mathrm{cl}(\mathfrak{U})$.

Cl 3: For $\mathfrak{U} \subseteq F(K:P)$, $\mathrm{cl}\ \mathrm{cl}(\mathfrak{U}) = \mathrm{cl}(\mathfrak{U})$.

cl 4: For $\mathfrak{U}$ and $\mathfrak{V}$ contained in $F(K:P)$, $\mathrm{cl}(\mathfrak{U} \cup \mathfrak{V}) = \mathrm{cl}(\mathfrak{U}) \cup \mathrm{cl}(\mathfrak{V})$.

It is a standard result of topology that, when we have a closure operation satisfying Cl 1 - Cl 4, the subsets $\mathfrak{U}$ which satisfy $\mathfrak{U} = \mathrm{cl}(\mathfrak{U})$ form the closed sets for a topology. (Kelley (15), Chapter one)

We show that Cl 1 - Cl 4 are satisfied. By definition, for $\mathfrak{U} = \emptyset$, we set $\prod \mathfrak{U} = K$. With this convention, Cl 1 and Cl 2 are easily seen. From Cl 2, if $\mathfrak{U} \subseteq F(K:P)$, we have $\mathrm{cl}(\mathfrak{U}) \subseteq \mathrm{cl}\ \mathrm{cl}(\mathfrak{U})$. Now, if $I \in \mathrm{cl}\ \mathrm{cl}(\mathfrak{U})$, then

$I \supseteq \prod \mathrm{cl}(\mathcal{U}) = \prod\{J \in F(K:P) : J \supseteq \prod\mathcal{U}\}$. It follows that $\prod\mathcal{U} \subseteq \prod\mathrm{cl}(\mathcal{U}) \subseteq I$, from which we see $I \in \mathrm{cl}(\mathcal{U})$ and Cl 3 is proved. If $\mathcal{U}$ and $\mathcal{W}$ are subsets of $F(K:P)$, and if $\mathcal{U} \subseteq \mathcal{W}$, then $\prod\mathcal{U} \supseteq \prod\mathcal{W}$ and so $\mathrm{cl}(\mathcal{U}) \subseteq \mathrm{cl}(\mathcal{W})$. Therefore $\mathrm{cl}(\mathcal{U}) \cup \mathrm{cl}(\mathcal{V}) \subseteq \mathrm{cl}(\mathcal{U} \cup \mathcal{V})$ for all subsets $\mathcal{U}$ and $\mathcal{V}$ of $F(K:P)$. Now let $I \in \mathrm{cl}(\mathcal{U} \cup \mathcal{V})$. Then $I \supseteq \prod(\mathcal{U} \cup \mathcal{V}) = (\prod\mathcal{U}) \cap (\prod\mathcal{V}) \supseteq (\prod\mathcal{U}) \cdot (\prod\mathcal{V})$. Since I is a prime ideal of K, we have either $I \supseteq \prod\mathcal{V}$ and $I \in \mathrm{cl}(\mathcal{V})$, or $I \supseteq \prod\mathcal{U}$, and $I \in \mathrm{cl}(\mathcal{U})$. In either case $I \in \mathrm{cl}(\mathcal{U}) \cup \mathrm{cl}(\mathcal{V})$. This establishes Cl 4.

Various properties of structure spaces, and the relations between topological properties of the structure space and the ring-theoretic properties of the ring have been investigated by several authors. (See, for example, Michler (21), and the references cited there.)

Our present goal is to prove the following:

<u>Theorem 5.5.1</u> Let K be a ring, and let $i: K \to S = \bigoplus_{j=1}^{n} K_j$ be a subdirect embedding. if P is a property of rings which respects isomorphisms, then the map $f: F(S:P) \to F(i(K):P)$, where $f(I) = I \cap i(K)$, is a continuous closed mapping onto $F(i(K):P)$. Furthermore, for $I \in F(S:P)$, $S/I \cong i(K)/f(I)$. From these results it follows that $F(i(K):P)$ (and therefore $F(K:P)$) has the topology given by a quotient topology for $F(S:P)$.

We shall break the proof into a number of steps.

Lemma 5.5.2

(1) If $I \neq S$ is a prime ideal of S, then $I \cap i(K)$ is a prime ideal of $i(K)$, and $i(K) \cap I \neq i(K)$.

(2) $S/I \cong i(K)/(i(K) \cap I)$, under the conditions of (1).

(3) If $I \in F(S:P)$, then $I \cap i(K) \in F(i(K):P)$.

Proof: Clearly (3) will follow from (2). If $I \neq S = \bigoplus_{j=1}^{n} K_j$, then, for some j_0 we have $K_{j_0} \not\subset I$. For any $j \neq j_0$ we have $0 = K_j K_{j_0} \subseteq I$, so if I is a prime ideal of S, $K_j \subseteq I$ for all $j \neq j_0$. Then we see that we can write $I = (I \cap K_{j_0}) \oplus (\bigoplus_{j \neq j_0} K_j)$. Therefore $S = K_{j_0} + I$.

The map $p_{j_0} i\colon K \to K_{j_0}$ is onto K_{j_0} (where p_j is the projection of S onto K_j), so that, for $x = \sum_{j=1}^{n} x_j$ in S, where each x_j is in K_j, there is a k in K such that $p_{j_0} i(k) = x_{j_0}$. Since $I \supseteq \sum_{j \neq j_0} K_j$, it follows that $x \in i(K) + I$, and hence $S = i(K) + I$. It is a consequence of this that $S/I = (i(K) + I)/I \cong i(K)/(i(K) \cap I)$.

From this isomorphism, both (1) and (2) follow. Q.E.D.

Lemma 5.5.3 The mapping $f: F(S:P) \to F(i(K):P)$, where $f(I) = I \cap i(K)$, is a continuous mapping.

Proof: By Lemma 5.5.2, the map f as defined is indeed a map into $f(i(K):P)$. To show that it is continuous, it suffices to show that the inverse image of a closed set is closed.

Let C be a closed subset of $F(i(K):P)$. Then $f^{-1}(C) = \{I \in F(S:P): I \cap i(K) \supseteq \prod C\}$. If $I' \in cl(f^{-1}(C))$, then I' contains $\prod f^{-1}(C)$, and $I' \cap i(K) \supseteq (\prod f^{-1}(C)) \cap i(K) = (\bigcap_{I \in f^{-1}(C)} I) \cap i(K) = \bigcap_{I \in f^{-1}(C)} I \cap i(K) \supseteq \prod C$. Thus $I' \cap i(K) \supseteq \prod C$, or $f(I') = I' \cap i(K) \in cl(C) = C$. Thus $I' \in f^{-1}(C)$, and $f^{-1}(C)$ is closed. Q.E.D.

Lemma 5.5.4 Let $i: K \to S = \bigoplus_{j=1}^{n} K_j$ be subdirect embedding, and let $f: F(S:P) \to F(i(K):P)$ be the continuous mapping which sends I to $I \cap i(K)$. For $i(Q) \in F(i(K):P)$, $f^{-1}(Q)$ is a finite non-empty set. In particular, f is a surjection.

Proof: Let D_j, $j = 1,2,\ldots,n$, be the kernel of the map $p_j i: K \to K_j$, where p_j is the j'th projection of S onto K_j. It is easily verified that $i(D_j) = i(K) \cap (\sum_{m \neq j} K_m)$.

Suppose that Q is a prime ideal in K. Since

$D_1 D_2 \ldots D_n \subseteq \bigcap_{j=1}^{n} D_j = 0$, we have that $Q \supseteq D_j$ for least one j. Let $T = \{j : 1 \leq j \leq n$ and $D_j \subseteq Q\}$. If $Q'' \in f^{-1}(i(Q))$, then $Q'' \cap i(K) = i(Q) \neq i(K)$, so Q'' must fail to contain K_{j_o} for some j_o. We show that j_o is in T.

Since Q'' is a prime ideal of S, and since $K_j K_{j_o} = 0 \subseteq Q''$, for $j \neq j_o$, it follows that $K_j \subseteq Q''$, and so $\sum_{j \neq j_o} K_j \subseteq Q''$. Then $f(Q'') = i(Q) = Q'' \cap i(K) \supseteq (\sum_{j \neq j_o} K_j) \cap i(K) = i(D_{j_o})$, whence $D_{j_o} \subseteq Q$ and $j_o \in T$. Thus, if $Q'' \in f^{-1}(i(Q))$, there is a unique j_o in T such that $Q'' \supseteq \sum_{j \neq j_o} K_j$. Therefore $f^{-1}(i(Q)) = \bigcup_{t \in T} \{Q'' \in f^{-1}(i(Q)) : Q'' \supseteq \sum_{j \neq t} K_j\}$.

We know that T is a finite non-void set. To complete the proof, we will show that, for t in T, $\{Q'' \in f^{-1}(i(Q)) : Q'' \supseteq \sum_{j \neq t} K_j\}$ consists of the single member $Q' = p_t i(Q) + \sum_{j \neq t} K_j$. First of all, $S/Q' \cong K_t/p_t i(Q) = p_t i(K)/p_t i(Q)$. Since $D_t = \ker(p_t i) \subseteq Q$, we have $S/Q' \cong (K/D_t)/(Q/D_t) \cong K/Q$. Therefore Q' is indeed a prime ideal of S if Q is a prime ideal of K, and $Q' \in F(S:P)$ if $Q \in F(K:P)$.

Now we show that Q' is indeed in $f^{-1}(i(Q))$. For $q \in Q$, we have $i(q) = \sum_{j=1}^{n} p_j i(q) \in (p_t i(Q) + \sum_{j \neq t} K_j) \cap i(K)$, and thus $I(Q) \subseteq Q' \cap i(K)$. Let $x = i(k) \in Q' \cap i(K)$. We may write $x = \sum_{j=1}^{n} p_j i(k)$. Since $x \in Q'$, there is a q in Q such that $p_t i(q) = p_t i(k)$. Then $i(q-k) = i(q) - i(k) \in i(K) \cap (\sum_{j \neq t} K_j) = i(D_t) \subseteq i(Q)$. Then $i(k)$ is in $i(Q)$, and thus $i(K) \cap Q' \subseteq i(Q)$. We have now shown that $f(Q') = i(Q)$, as desired. At this point, we have shown that, for any Q in $F(K:P)$, $f^{-1}(i(Q))$ is not empty, and thus f maps onto $F(i(K):P)$.

Suppose now that $Q'' \in f^{-1}(i(Q))$, and $Q'' \supseteq \sum_{j \neq t} K_j$. We shall show $Q'' = Q'$. Now $f(Q'') = Q'' \cap i(K) = i(Q)$, so $p_t i(Q) \subseteq p_t(Q'') \subseteq Q''$. (This last inequality comes from the fact that $Q'' \supseteq \sum_{j \neq t} K_j$.) From the definition of Q', we have $Q' \subseteq Q''$.

If $x = \sum_{j=1}^{n} x_j$ is in Q'', where $x_j \in K_j$, then $Q'' \ni x - \sum_{j \neq t} x_j = x_t$. Since i is a subdirect embedding, there is a k in K such that $p_t i(k) = x_t$. Then $i(k) = \sum_{j=1}^{n} p_t i(k) \in Q'' + \sum_{j \neq t} K_j = Q''$. Therefore $i(k) \in i(K) \cap Q'' = i(Q) = i(K) \cap Q' \subseteq Q'$. Since Q' also contains $\sum_{j \neq t} K_j$, we obtain x_t as a member of Q'. But then $x = x_t + \sum_{j \neq t} x_j$ is also in Q', or $Q'' \subseteq Q'$. Thus $Q'' = Q'$, as desired. Q.E.D.

<u>Lemma 5.5.5</u> Under the same assumptions as before, f is a closed mapping.

<u>Proof</u>: As in the previous lemma, if I is a prime ideal of S $(I \neq S)$ there is a unique j_o in $\{1,2,\ldots n\}$ such that $I \supseteq \sum_{j \neq j_o} K_j$. It follows that we can write $I = (I \cap K_{j_o}) + \sum_{j \neq j_o} K_j$. If we set $F_j = \{I \in F(S:P) : I \supseteq \sum_{m \neq j} K_m\}$ it is easy to verify that $F(S:P) = \bigcup_{j=1}^{n} F_j$, and that this is a disjoint union. From the definition of the closure operation in the topology, each F_j is a closed set in $F(S:P)$. If C is any closed subset of $F(S:P)$, we can write $C = \bigcup_{j=1}^{n} (F_j \cap C)$, and $f(C) = \bigcup_{j=1}^{n} f(F_j \cap C)$. Since C is closed, so is $F_j \cap C$. In order to show that f is a closed mapping, it is therefore sufficient to show that each $f(F_j \cap C)$ is closed if C is closed.

Let $i(Q) \in cl(f(F_j \cap C))$. Then $i(Q) \supseteq \bigcap_{Q'' \in F_j \cap C} (Q'' \cap i(K)) \supseteq (\sum_{m \neq j} K_m) \cap i(K) = i(D_j)$. From the previous lemma, or rather the proof of the lemma, we have that $Q' = p_j i(Q) + \sum_{m \neq j} K_m$

satisfies $f(Q') = i(Q)$. We show that $Q' \in cl(F_j \cap C)$.

Let $x = \sum_{m=1}^{n} x_m \in \prod(F_j \cap C)$. Since i is a subdirect embedding, there is a k in K such that $p_j i(k) = x_j$. Then $i(k) = x_j + \sum_{m \neq j} p_m i(k)$. Since each Q'' in $F_j \cap C$ contains $\sum_{m \neq j} K_m$ it follows that x_j and then $i(k)$ are in $\prod(F_j \cap C)$, and so $i(k) \in i(K) \cap (\prod(F_j \cap C))$. In other words, for Q'' in $F_j \cap C$, $i(k) \in i(K) \cap Q'' = f(Q'')$. Therefore $i(k) \in \prod\{f(Q'') : Q'' \in C \cap F_j\}$. Since $i(Q) \in cl(f(F_j \cap C))$ we have $i(k)$ is in $i(Q)$, so $k \in Q$. Therefore $x_j = p_j i(k) \in p_j i(Q)$, and $x = \sum_{m=1}^{n} x_m \in p_j i(Q) + \sum_{m \neq j} K_m = Q'$. This shows $Q' \supseteq \prod(F_j \cap C)$, or $Q' \in cl(F_j \cap C) = F_j \cap C$.

Now $i(Q) = f(Q') \in f(F_j \cap C)$, and so $f(F_j \cap C)$ is closed. Q.E.D.

Corollary 5.5.6 The topology on $F(K:P)$ has the topology of a quotient space of $F(S:P)$ induced by the map f.

Proof: We have seen that the map $f: F(S:P) \to F(i(K):P)$ is continuous, closed, and surjective. From Kelley (15) (Theorem 3.8, page 95) $F(i(K):P)$ has the quotient topology induced by f. Q.E.D.

Remark: This completes the proof of Theorem 5.5.1.

We have seen that the mapping $f: F(S:P) \to F(K:P)$ is a closed mapping. The following example shows that f need not be an open mapping.

Example 5.5.7

We consider the rings W' and T as defined in Example 5.3.3. Recall that, using the notation of the example, we had an irredundant subdirect embedding $i: (S_1 \oplus S_2)' \to S_1' \oplus S_2'$. Let P be the property of being a prime ring. It is easily verified that i and the homomorphism between $F(W:P)$ and $F(i(W):P)$ induce a continuous closed mapping $f: F(T:B) \to F(W:B)$ such that $f(S_1' \oplus S_2) = f(S_1 \oplus S_2') = S_1 \oplus S_2$, $f(S_1') = S_1$, and $f(S_2') = S_2$. The set $\{S_1, S_1 \oplus S_2\}$ is a closed but not open set in $F(W:B)$, and this is the image under the mapping f of $\{S_1', S_1' \oplus S_2\}$ which is both closed and open in $F(T:B)$.

We consider briefly the case where a ring K is a finite subdirect sum of prime rings. Then, as we have seen, there exists an irredundant subdirect embedding $i: K \to \bigoplus_{i=1}^{n} K_i$, where each K_i is a prime ring. It follows from Levy (19), that if a ring K is an irredundant subdirect sum of rings $\{K_\alpha: \alpha \in \Lambda\}$, then the rings K_α are determined, up to isomorphism, by the ring K. We consider the opposite question:

if $i: K \to \bigoplus_{j=1}^{n} K_j$ is an irredundant subdirect embedding, to what extent is K determined by $K_1, K_2, \ldots, K_n$? A partial answer is found in the following results.

<u>Lemma 5.5.8</u> Let $L \xrightarrow{i'} K \xrightarrow{i} S = \bigoplus_{j=1}^{n} K_j$, where i and i' are ring monomorphisms such that i and ii' are subdirect embeddings, and each K_j is a prime ring. Then $K = i'(L)$ if and only if, for every prime ideal Q' of S, $Q' \cap i(K) = Q' \cap ii'(L)$.

<u>Proof</u>: Clearly, if $K = i'(L)$, the condition is satisfied.

To show that the condition is sufficient, it suffices to show that $i(K) = ii'(L)$, since i and ii' are monomorphisms. Let p_j denote the projection of S onto K_j. Then any element $i(k)$ of $i(K)$ may be written $i(k) = \sum_{j=1}^{n} p_j i(k)$. Since ii' is a subdirect embedding, there is an element w in L such that $p_1 ii'(w) = p_1 i(k)$, and so $i(k - i'(w))$ is in $\sum_{j=2}^{n} K_j$. Now $Q' = \sum_{j=2}^{n} K_j$ is a prime ideal of S, and we have $i(k - i'(w)) \in Q' \cap i(K) = Q' \cap ii'(L)$. Therefore there is an element z in L such that $i(k - i'(w)) = ii'(z)$. It follows that $k = i'(w) + i'(z) = i'(w+z) \in i'(L)$. Since k is arbitrary in K, we have $K = i'(L)$. Q.E.D.

<u>Theorem 5.5.9</u> Let $S = \bigoplus_{j=1}^{n} K_j$, where each K_j is prime, and let $i: K \to S$ and $i': K \to S'$ be subdirect embeddings. Then $i(K) = i'(K')$ if and only if the following are satisfied:

(1) For each prime ideal Q' of S, $i(K) \cap Q' = i'(K') \cap Q'$.

(2) The subring $i(K) \cap i'(K')$, under the natural embedding into S, is subdirectly embedded into S.

<u>Proof</u>: Clearly these two conditions follow if $i(K) = i'(K')$. Conversely suppose these conditions hold. Then, if we let $T = i(K) \cap i'(K')$, and if we let $f: T \to i(K)$ and $g: i(K) \to S$ be the obvious embeddings, we can apply the previous lemma and obtain $i(K) = f(T) = T = i(K) \cap i'(K')$. Similarly we find $i(K) \cap i'(K') = i'(K')$. Q.E.D.

BIBLIOGRAPHY

1. Andrunakievic, V.A., Radicals of Associative Rings, Mat. Sb., 44(1958), 179-212, translated in A.M.S. Translations, Ser. 2, 52(1966), 95-128.

2. Andrunakievic, V.A., and Rjakuhin, Ju.M., Modules and Radicals, Sov. Math., 5(1964), 728-732.

3. Andrunakievic, V.A., and Rjakuhin, Ju.M., Special Modules and Special Radicals, (Russian), Dokl. Akad. Nauk. SSSR, 147(1966), 1274-1277.

4. Bergman, G., A Ring Primitive on the Right but Not on the Left, Proc. A.M.S., 15(1964), 473-475, Erratum.

5. Blair, R.L., Ideal Lattices and the Structure of Rings, Trans. A.M.S., 75(1953), 136-153.

6. Divinsky, N., Rings and Radicals, Toronto: University of Toronto Press, 1965.

7. Faith, C., Lectures on Injective Modules and Quotient Rings, Berlin, Heidelberg, New York: Springer Verlag, 1967.

8. Goldie, A., Rings with Maximum Condition, Multigraphed Lecture Notes, New Haven: Yale University Press, 1961.

9. Goldie, A., Torsion-free Modules and Rings, J. Algebra, 1(1964), 268-287.

10. Heinicke, A, A Note on Lower Radical Constructions for Associative Rings, Cdn. Math. Bull., 11(1968), 23-30.

11. Hentzel, I., A Note on Modules and Radicals, to appear in Proc. A.M.S.

12. Hoffman, A.E. and Leavitt, W.G., Properties Inherited by the Lower Radical, to appear.

13. Jacobson, N., Structure of Rings, A.M.S. Coll. Publ. 37, (Rev. ed.) 1964.

14. Johnson, P.E. and Levy, L., Regular Elements in Semiprime Rings, Proc. A.M.S., 19(1968), 961-963.

15. Kelley, J.L., General Topology, Princeton: D. Van Nostrand Co., Ind., 1955.

16. Koh, K. and Mewborn, A.C., A Class of Prime Rings, Cdn. Math Bull., 9(1966), 63-72.

17. Koh, K. and Mewborn, A.C., The Weak Radical of a Ring, Proc. A.M.S., 18(1967), 554-559.

18. Kurosh, A.G., Radicals of Rings and Algebras, Mat. Sb., 33(1953), 13-26.

19. Levy, L., Unique Subdirect Sums of Prime Rings, Trans. A.M.S., 106(1963), 64-76.

20. Michler, G., Radicale und Sockel, Math. Ann., 167(1966), 1-48.

21. Michler, G., Radicals and Structure Spaces, Algebra 4(1966), 199-219.

22. Rjabuhin, Ju.M., Overnilpotent and Special Radicals, Studies in Alg. and Math. Anal. (Russ.), Izdat. "Karta Moldovenjaske", Kishinev, (1965), 65-72.

23. Sasiada, and Cohn, P.M., An Example of a Simple Radical Ring, J. Alg. 5(1967), 373-377.

24. Sulinski, A., Anderson, C., and Divinsky, N., Lower Radical Properties for Associative and Alternative Rings, J. Lond. Math. Soc., 41(1966), 417-424.

25. Small, L., On Some Questions in Noethesian Rings, Bull. A.M.S., 72(1966), 853-857.