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ON CHAIN MAPS INDUCING ISOMORPHISMS IN HOMOLOGY

by

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ABSTRACT

Let A be an abelian category, I the full subcategory of A consisting of injective objects of A, and $K(A)$ the category whose objects are cochain complexes of elements of A, and whose morphisms are homotopy classes of cochain maps.

In (5), lemma 4.6., p. 42., R. Hartshorne has proved that, under certain conditions, a cochain complex $X^{\bullet} \in |K(A)|$ can be embedded in a complex $I^{\bullet} \in |K(I)|$ in such a way that $I^{\bullet}$ has the same cohomology as $X^{\bullet}$.

In Chapter I we show that the construction given in the two first parts of Hartshorne's Lemma is natural i.e. there exists a functor $J : K(A) \longrightarrow K(I)$ and a natural transformation $i : Id_{K(A)} \longrightarrow EJ$ (where $E : K(I) \longrightarrow K(A)$ is the embedding functor) such that $i_{X^{\bullet}}$ is injective and induces isomorphism in cohomology. The question whether the construction given in the third part of the lemma is functorial is still open.

We also prove that J is left adjoint to E, so that $K(I)$ is a reflective subcategory of $K(A)$.

In the special case where A is a category ${}_{\Lambda}\mathfrak{M}$ of left Λ-modules, and $C({}_{\Lambda}\mathfrak{M})$ the category of cochain complexes in ${}_{\Lambda}\mathfrak{M}$ and cochain maps (not homotopy classes), we prove the existence of a functor $C({}_{\Lambda}\mathfrak{M}) \longrightarrow C(I)$.

In Chapter II we study the natural homomorphism

$$\nu : \mathrm{Hom}_A(M,L) \otimes_B N \longrightarrow \mathrm{Hom}_A(M, L \otimes_B N)$$

where A, B are rings, and M, L, N modules or chain complexes. In particular we give several sufficient conditions under which ν is an isomorphism, or

induces isomorphism in homology.

In the appendix we give a detailed proof of Hartshorne's Lemma. We think that this is useful, as no complete proof is, to our knowledge, to be found in the literature.

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CHAPTER I. COCHAIN COMPLEXES CONSISTING OF INJECTIVES.

We first introduce some notations which will be used in this chapter.

If $\mathcal{A}$ denotes an abelian category, we denote $K(\mathcal{A})$ the category whose objects are cochain complexes of objects of $\mathcal{A}$, and whose morphisms are homotopy classes of cochain maps.

A cochain complex $X^{\cdot}$ is said to be <u>bounded below</u> if there exists an integer N such $X^n = 0$ if $n > N$.

We denote by $K^+(\mathcal{A})$ the full subcategory of $K(\mathcal{A})$ consisting of the complexes bounded below.

According to (1) and (4), a subcategory $\mathcal{A}'$ of an abelian category $\mathcal{A}$ is <u>thick</u> (or is a <u>Serre subcategory</u> of $\mathcal{A}$) [1] if

(i) $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$.

(ii) $\mathcal{A}'$ is non empty.

(iii) For any short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ in $\mathcal{A}$, $B \in |\mathcal{A}'| \iff B', B'' \in |\mathcal{A}'|$ [2]

((iii) is equivalent to (iii)' : if $X \longrightarrow Y \longrightarrow Z$ is exact in $\mathcal{A}$, then $X, Z \in |\mathcal{A}'| \implies Y \in |\mathcal{A}'|$).

If $\mathcal{A}'$ is a thick subcategory of $\mathcal{A}$ we define $K_{\mathcal{A}'}(\mathcal{A})$ to be the full subcategory of $K(\mathcal{A})$ consisting of those complexes $X^{\cdot}$ whose cohomology objects $H^i(X^{\cdot})$ are in $\mathcal{A}'$. We define $K^+_{\mathcal{A}'}(\mathcal{A})$ by taking complexes bounded below.

[1] Note that the definition of a thick subcategory is not absolutely clear in (5)

[2] We denote by $|\mathcal{A}'|$ the class of objects of the category $\mathcal{A}'$.

If $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$, I is an <u>$\mathcal{A}$-injective object of $\mathcal{A}'$</u> if $I \in |\mathcal{A}'|$ and if I is injective, considered as an object of $\mathcal{A}$.

A cochain map $f : X^\cdot \longrightarrow Y^\cdot$ is called a <u>quasi-isomorphism</u> (abbreviated <u>quiso</u>) if $H^i(f) : H^i(X^\cdot) \longrightarrow H^i(Y^\cdot)$ are all isomorphisms. Similarly, we call a homotopy class a quasi-isomorphism if one (hence all) of its representatives induces isomorphisms in cohomology.

If A is isomorphic to B, we write $A \cong B$, and if A is homotopic to B, we write $A \simeq B$.

We now state the lemma we are going to discuss ((5), Lemma 4.6, p.42) :

<u>Lemma 1.</u>

Let $\mathcal{A}$ be an abelian category.

a) Let $\mathbb{P}$ be a subclass of $|\mathcal{A}|$, and assume

(i) every object of $\mathcal{A}$ admits a monomorphism into an element of $\mathbb{P}$.

Then every $X^\cdot \in |K^+(\mathcal{A})|$ admits a monic quasi-isomorphism i into a bounded below complex $I^\cdot$ of objects of $\mathbb{P}$.

b) Assume furthermore that $\mathbb{P}$ satisfies

(ii) If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a short exact sequence with $X \in \mathbb{P}$, then $Y \in \mathbb{P} \iff Z \in \mathbb{P}$.

(iii) There exists a positive integer n such that, if $X^o \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0$ is an exact sequence, and $X^o, X^1, \dots, X^{n-1} \in \mathbb{P}$, then $X^n \in \mathbb{P}$.

Then every $X^\cdot \in |K(\mathcal{A})|$ admits a quasi-isomorphism into a complex $I^\cdot$ of objects of $\mathbb{P}$.

c) Let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$, and assume that $\mathcal{A}'$ has

enough A-injectives. Then every $X^{\cdot} \in |K^{+}_{A'}(A)|$ admits a quasi-isomorphism into a bounded below complex $I^{\cdot}$ of A-injective objects of A'.

We give the construction of the quiso $i : X^{\cdot} \longrightarrow I^{\cdot}$ of Lemma 1 a). (For a complete proof of Lemma 1, see the appendix).

We may assume $X^p = 0$ for $p < 0$. Embed $X^o \rightarrowtail I^o$, with I^o in $\mathbb{P}$. Having defined $I^o \xrightarrow{d_I^o} I^1 \xrightarrow{d_I^1} \dots \xrightarrow{d_I^{p-1}} I^p$ and $i^k : X^k \longrightarrow I^k$ $(k=0,..,p)$, we choose I^{p+1} to be an element of $\mathbb{P}$ containing the pushout of

$$\begin{tikzcd} X^p \arrow[r, "d_X^p"] \arrow[d, "\bar{i}^p"'] & X^{p+1} \\ \mathrm{Cok}\, d_I^{p-1} & \end{tikzcd}$$

Then, d_I^p and i^{p+1} are defined by the following commutative diagram :

$$\begin{tikzcd} X^p \arrow[r, "d_X^p"] \arrow[d, "\bar{i}^p"] \arrow[dd, bend right=60, "i^p"'] & X^{p+1} \arrow[d] \arrow[ddr, dashed, bend left=60, "i^{p+1}"] & \\ \mathrm{Cok}\, d_I^{p-1} \arrow[r] & K^{p+1} & \\ I^p \arrow[rr, dashed, "d_I^p"'] & & I^{p+1} \end{tikzcd}$$

Now we prove that, if $\mathbb{P}$ is the full subcategory of A consisting of all injective objects of A, then the construction given in Lemma 1 a) is functorial.

More precisely : let A be an abelian category with enough injectives. Let I be the full subcategory of A consisting of all injective objects of A,

and let $E^+ : K^+(\mathcal{I}) \longrightarrow K^+(\mathcal{A})$ be the embedding functor. Then

<u>Proposition 2.</u>

There exists a functor $J^+ : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$ and a natural transformation

$$i : \mathrm{Id}_{K^+(\mathcal{A})} \longrightarrow E^+J^+$$

such that, for each $X^\bullet \in |K^+(\mathcal{A})|$:

(i) $i_{X^\bullet}$ is a quiso

(ii) $i_{X^\bullet}$ is monic.

($K^+(\mathcal{I})$ denotes the full subcategory of $K^+(\mathcal{A})$ whose objects $I^\bullet$ are such that $I^n \in |\mathcal{I}|$ for all n).

If $\mathcal{A}$ has enough injectives, Lemma 1 a) clearly holds for $\mathbb{P} = |\mathcal{I}|$. We prove that the construction is functorial. It suffices to prove that if $X^\bullet, Y^\bullet \in |K^+(\mathcal{A})|$ and if $I^\bullet$ (resp. $J^\bullet$) is a complex constructed from $X^\bullet$ (resp. $Y^\bullet$), then, for any $K^+(\mathcal{A})$-morphism $f : X^\bullet \longrightarrow Y^\bullet$, there exists a unique $\phi_{IJ} : I^\bullet \longrightarrow J^\bullet$ such that the diagram

$$\begin{tikzcd} X^\bullet \arrow[r, tail, "i"] \arrow[d, "f"'] & I^\bullet \arrow[d, "\phi_{IJ}"] \\ Y^\bullet \arrow[r, tail, "j"] & J^\bullet \end{tikzcd}$$

commutes.

($i : X^\bullet \rightarrowtail I^\bullet$ and $j : Y^\bullet \rightarrowtail J^\bullet$ are the embeddings of Lemma 1 a)). Indeed, let $L^\bullet$ be a complex constructed from $Z^\bullet \in |K^+(\mathcal{A})|$, and let $g : Y^\bullet \longrightarrow Z^\bullet$ be a morphism. One checks that

$$(1) \qquad \phi_{JL} \circ \phi_{IJ} = \phi_{IL}$$

and that $Id_{X^\cdot} : X^\cdot \longrightarrow X^\cdot$ yields

$$(2) \quad \phi_{II} = Id_{I^\cdot}.$$

Moreover, if $I_1^\cdot$ and $I_2^\cdot$ are two complexes constructed from $X^\cdot$, there exists a unique $\eta_I = \phi_{I_1 I_2} : I_1^\cdot \longrightarrow I_2^\cdot$ such that the diagram

$$\begin{tikzcd} X^\cdot \arrow[r, tail, "i_1"] \arrow[d, equal] & I_1^\cdot \arrow[d, "\eta_I"] \\ X^\cdot \arrow[r, tail, "i_2"] & I_2^\cdot \end{tikzcd}$$

commutes. But i_1 and i_2 are quiso's, so that η_I is a quiso from a bounded below complex consisting of injectives into I_2. Hence η_I is an isomorphism ((5), p. 41). Thus, we are allowed to identify $I_1^\cdot$ and $I_2^\cdot$ via the isomorphism η_I, and we write $J^+X^\cdot$ for $I^\cdot$.

Finally, we have to define, for a given $f : X^\cdot \longrightarrow Y^\cdot$, an induced morphism $J^+f : J^+X^\cdot \longrightarrow J^+Y^\cdot$. Or course, we define $J^+f = \phi_{IJ} : I^\cdot \longrightarrow J^\cdot$. If we do so, then the formulas (1) and (2) will ensure that J^+ is a functor. The only thing left to check is that the definition of J^+f is compatible with the identification made under η, i.e. that, if $I_1^\cdot$, $I_2^\cdot$ (resp. $J_1^\cdot$, $J_2^\cdot$) are constructed from $X^\cdot$ (resp. $Y^\cdot$), then $\phi_{I_2 J_2}\ \eta_I = \eta_J\ \phi_{I_1 J_1}$, which is clear since both make the following diagram commutative :

$$\begin{tikzcd} X^\cdot \arrow[r, tail, "i_1"] \arrow[d, "f"'] & I_1^\cdot \arrow[d, dashed] \\ Y^\cdot \arrow[r, tail, "j_2"] & J_2^\cdot \end{tikzcd}$$

a) Existence of ϕ_{IJ}.

We construct ϕ^p by induction : as J^o is injective, there exists a map $\phi^o : I^o \longrightarrow J^o$ such that

$$\begin{tikzcd} X^o \arrow[r, tail, "i^o"] \arrow[d, "f^o"'] & I^o \arrow[d, dashed, "\phi^o"] \\ Y^o \arrow[r, tail, "j^o"] & J^o \end{tikzcd}$$

commutes. Suppose now that $\phi^o, \ldots, \phi^p$ are defined, and consider

$$\begin{tikzcd}[column sep=small] & X^p \arrow[rr, "d_X^p"] \arrow[dr, "f^p"] \arrow[dd, "\bar{i}^p"] \arrow[ddll, "i^p"'] & & X^{p+1} \arrow[dr, "f^{p+1}"] \arrow[dd] & \\ & & Y^p \arrow[rr, "d_Y^p"] \arrow[dd, "\bar{j}^p"] & & Y^{p+1} \arrow[dd] \\ & \bar{I}^p \arrow[rr] \arrow[dr, dashed, "\bar{\phi}^p"] & & K^{p+1} \arrow[dr, dashed, "\psi"] & \\ I^p \arrow[rr] \arrow[dr, "\phi^p"'] \arrow[uurr] & & \bar{J}^p \arrow[rr] \arrow[dl] & & L^{p+1} \arrow[dl] \\ & J^p \arrow[rr] \arrow[uurr] \arrow[uu, "j^p"] & & I^{p+1} \arrow[dr, dashed, "\phi^{p+1}"] & \\ & & & & J^{p+1} \end{tikzcd}$$

where $\bar{I}^p = \mathrm{Cok}\, d_I^{p-1}$, $\bar{J}^p = \mathrm{Cok}\, d_J^{p-1}$, $K^{p+1} = \mathrm{pushout}(d_X^p; \bar{i}^p)$, $L^{p+1} = \mathrm{pushout}(d_Y^p; \bar{j}^p)$. Then, ϕ^{p-1}, ϕ^p induce $\bar{\phi}^p$ on cokernels. By universal property of the pushout, we get a map $\psi : K^{p+1} \longrightarrow L^{p+1}$ such that the upper cube commutes. As J^{p+1} is injective, ψ induces a map $\phi^{p+1} : I^{p+1} \longrightarrow J^{p+1}$ such that the square

$$\begin{tikzcd} K^{p+1} \arrow[r] \arrow[d, tail] & L^{p+1} \arrow[d] \\ I^{p+1} \arrow[r] & J^{p+1} \end{tikzcd}$$

commutes. It is then easy to check that the bottom square of the diagram p.6 commutes, so that ϕ is the required map.

b) Uniqueness.

Suppose that there exist ϕ_1 and ϕ_2 such that $\phi_1 i \simeq \phi_2 i$. Consider the short exact sequence (*) : $0 \longrightarrow X^\bullet \xrightarrow{i} I^\bullet \xrightarrow{p} Y^\bullet \longrightarrow 0$. From the long exact sequence of (*) in cohomology, one sees that $Y^\bullet$ is acyclic. In the diagram

$$\begin{tikzcd} X^\bullet \arrow[r, tail, "i"] & I^\bullet \arrow[r, two heads, "p"] \arrow[dr, "\phi_1 - \phi_2"'] & Y^\bullet \arrow[d, dashed, "f"] \\ & & J^\bullet \end{tikzcd}$$

we have $(\phi_1 - \phi_2)i \simeq 0$, so that $\phi_1 - \phi_2$ factors through $Y^\bullet$: there exists a map $f : Y^\bullet \longrightarrow J^\bullet$ such that $fp \simeq \phi_1 - \phi_2$. As $Y^\bullet$ is acyclic, and $J^\bullet$ consists of injectives and is bounded below, we have $f \simeq 0$ ((5), p.40). Thus, $\phi_1 - \phi_2 \simeq fp \simeq 0$, and this concludes the proof of Prop.2.

The functor J^+ has the other nice property to send quasi-isomorphisms into isomorphisms. More precisely :

Proposition 3.

If $f : X^\bullet \longrightarrow Y^\bullet$ is a quasi-isomorphism in $K^+(\mathcal{A})$, then $J^+f : J^+X^\bullet \longrightarrow J^+Y^\bullet$ is an isomorphism.

Proof.

From Prop.2, we have a commutative diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "i_X"] \arrow[d, "f"'] & E^+J^+X^\bullet \arrow[d, "E^+J^+f"] \\ Y^\bullet \arrow[r, "i_Y"] & E^+J^+Y^\bullet \end{tikzcd}$$

where i_X, i_Y and f are quiso's. Thus J^+f is a quiso. But $J^+X^\bullet$ is a bounded below complex consisting of injectives, so that J^+f is an isomorphism ((5), p.41).

We are now able to state a first universal property for J^+ :

Corollary 4.

The functor $J^+ : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$ satisfies the following universal property :

For each quiso $f : X^\bullet \longrightarrow Y^\bullet$ there exists a unique homotopy class $\phi : Y^\bullet \to E^+J^+X^\bullet$ such that the diagram

$$\begin{tikzcd} X^\bullet \arrow[rr, "i_X"] \arrow[dr, "f"'] & & E^+J^+X^\bullet \\ & Y^\bullet \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

commutes.

Proof.

a) Existence.

As J^+f is an isomorphism by previous proposition, it is sufficient to take $\phi = (E^+J^+f)^{-1}i_Y$.

b) Uniqueness.

Suppose $\phi' : Y^{\cdot} \longrightarrow E^{+}J^{+}X^{\cdot}$ is such that $\phi' f = i_X$, and consider the diagram

$$\begin{tikzcd}
X^{\cdot} \arrow[r, tail, "i_X"] \arrow[dr, "f"'] & E^{+}J^{+}X^{\cdot} \arrow[r, tail, "i_{E^{+}J^{+}X^{\cdot}}"] & E^{+}J^{+}E^{+}J^{+}X^{\cdot} \\
& Y^{\cdot} \arrow[u, shift right, "\phi"'] \arrow[u, shift left, "\phi'"] \arrow[r, "i_Y"'] \arrow[ur, phantom, "(1)"] & E^{+}J^{+}Y^{\cdot} \arrow[u, shift right, "E^{+}J^{+}\phi"'] \arrow[u, shift left, "E^{+}J^{+}\phi'"]
\end{tikzcd}$$

As $\phi f = \phi' f$, we have $(J^{+}\phi)(J^{+}f) = (J^{+}\phi')(J^{+}f)$. But f is a quiso, so that $J^{+}f$ is an isomorphism; thus $J^{+}\phi = J^{+}\phi'$. By commutativity of (1), we get $i_{E^{+}J^{+}X^{\cdot}}.\phi = i_{E^{+}J^{+}X^{\cdot}}.\phi'$. $i_{E^{+}J^{+}X^{\cdot}}$ being monic, we have $\phi = \phi'$.

Proposition 5.

Notations being as in Prop.2, the functor J^{+} is left adjoint to the embedding functor $E^{+} : K^{+}(\mathcal{I}) \longrightarrow K^{+}(\mathcal{A})$. In other words, $K^{+}(\mathcal{I})$ is a reflective subcategory of $K^{+}(\mathcal{A})$.

Proof.

By Prop.2, we have a natural transformation

$$i : Id_{K^{+}(\mathcal{A})} \longrightarrow E^{+}J^{+}$$

Furthermore, for each $Y^{\cdot} \in |K^{+}(\mathcal{I})|$, we have $J^{+}E^{+}Y^{\cdot} = Y^{\cdot}$ by definition of J^{+}. Hence, we have a natural transformation

$$\delta : J^{+}E^{+} \longrightarrow Id_{K^{+}(\mathcal{I})}$$

which is the identity. Thus, it is sufficient to prove that

(i) $\delta_{J^{+}} \circ J^{+}i = Id_{J^{+}}$

(ii) $E^{+}\delta \circ i_{E^{+}} = Id_{E^{+}}$

(i) Apply Corollary 4 for $Y^\cdot = E^+J^+X^\cdot$:

$$\begin{tikzcd} X^\cdot \arrow[rr, "i_{X^\cdot}"] \arrow[dr, "i_{X^\cdot}"'] & & E^+J^+X^\cdot \\ & E^+J^+X^\cdot \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

The unique ϕ such that the diagram commutes is defined by

$$\phi = (E^+J^+i_{X^\cdot})^{-1} i_{E^+J^+X^\cdot}$$

But, also $\phi = Id_{E^+J^+X^\cdot}$ makes the diagram commutative. Thus $E^+J^+i_{X^\cdot} = i_{E^+J^+X^\cdot}$, and, by construction of J^+, $i_{E^+J^+X^\cdot} = Id_{E^+J^+X^\cdot}$, so that $E^+J^+i_{X^\cdot} = Id_{E^+J^+X^\cdot} = E^+Id_{J^+X^\cdot}$. Thus : $J^+i_{X^\cdot} = Id_{J^+X^\cdot}$. As $\delta_{J^+X^\cdot} = Id_{J^+X^\cdot}$, we get $\delta_{J^+X^\cdot} \circ J^+i_{X^\cdot} = Id_{J^+X^\cdot}$, and (i) is proved. Similarly, one proves (ii).

<u>Remark</u>.

Corollary 4 and Proposition 5 give two different universal properties for i_X :

$$\begin{tikzcd} X^\cdot \arrow[rr, "i_X"] \arrow[dr, "\text{quiso}"'] & & E^+J^+X^\cdot \\ & Y^\cdot \arrow[ur, dashed, "\phi"'] & \end{tikzcd} \qquad \begin{tikzcd} X^\cdot \arrow[rr, "i_X"] \arrow[dr] & & E^+J^+X^\cdot \arrow[dl, dashed, "E^+\psi"] \\ & E^+Y^\cdot & \end{tikzcd}$$

Corollary 4 Proposition 5

(Note that, in general, Prop.5 $\Longrightarrow$ Cor.4).

Up to now, we have only discussed applications of the first part of Lemma 1. Looking at the second part, we get the following result :

Let $\mathcal{A}$ be an abelian category of finite injective homological dimension, let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects of $\mathcal{A}$, and let $E : K(\mathcal{I}) \longrightarrow K(\mathcal{A})$ be the embedding functor. Then :

Proposition 6.

There exists a functor $J : K(\mathcal{A}) \longrightarrow K(\mathcal{I})$ and a natural transformation $i : Id_{K(\mathcal{A})} \longrightarrow EJ$ such that for each $X^{\bullet} \in |K(\mathcal{A})|$

(i) i_X is a quiso

(ii) i_X is monic.

The proof of this proposition depends heavily of the construction of the chain complex $I^{\bullet}$ of Lemma 1 b), so that it will be done in the appendix. The question whether J is left adjoint to the embedding E is still open.

Now we restrict our attention to the case where $\mathcal{A}$ is the category ${}_{\Lambda}\mathfrak{M}$ of left Λ-modules (we could get similar results for the category $\mathfrak{M}_{\Lambda}$ of right Λ-modules). In this category, we prove the existence of a natural embedding of any module into an injective module. As a consequence of this fact, we shall see that, in ${}_{\Lambda}\mathfrak{M}$, the functor J^{+} of Prop.2 may be considered as a functor $C^{+}({}_{\Lambda}\mathfrak{M}) \longrightarrow C^{+}(\mathcal{I})$, where $C^{+}(\mathcal{A})$ denotes the category whose objects are cochain complexes of objects of $\mathcal{A}$ and whose morphisms are cochain maps. Thus here we are working with cochain maps instead of homotopy classes of cochain maps.

Lemma 7.

For any $A \in |{}_{\Lambda}\mathfrak{M}|$, let A^{*} be the right Λ-module

$$A^{*} = \mathrm{Hom}_{\mathbb{Z}}(A\ ;\ \mathbb{Q}/\mathbb{Z})$$

Then the map $\psi : A \longrightarrow A^{**}$ defined, for $a \in A$, $f \in A^{*}$, by

$$\psi(a)(f) = f(a)$$

is a natural embedding.

Proof.

1. It is clear that $\psi \in \mathrm{Hom}_\Lambda(A;A^{**})$. (Note that $A \in |{}_\Lambda\mathfrak{M}|$ implies $A^* \in |\mathfrak{M}_\Lambda|$ which implies $A^{**} \in |{}_\Lambda\mathfrak{M}|$).

2. ψ is a monomorphism.

Let $a \neq 0$ be an element of A, and let (a) be the submodule of A generated by a. We define $\alpha : (a) \longrightarrow \mathbb{Q}/\mathbb{Z}$ as follows :

— If a is of infinite order, $\alpha(a)$ is any non-zero element of $\mathbb{Q}/\mathbb{Z}$.

— If the order of a is n, $\alpha(a)$ is an element of order dividing n.

Clearly, α is a well defined morphism.

Considering now the embedding $(a) \rightarrowtail A$ we get, since $\mathbb{Q}/\mathbb{Z}$ is injective, a non-zero map $f : A \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that

$$\begin{tikzcd} (a) \arrow[r, tail] \arrow[d, "\alpha"'] & A \arrow[dl, dashed, "f"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

commutes. Hence, for this particular morphism f, we have $\psi(a)(f) = f(a) = \alpha(a) \neq 0$, so that $a \neq 0$ implies $\psi(a) \neq 0$.

3. ψ is natural.

For any $f : A \longrightarrow B$, we define $f^{**} : A^{**} \longrightarrow B^{**}$ for $\alpha \in A^{**}$ and $\phi \in B^*$ by $f^{**}(\alpha)(\phi) = \alpha(\phi f)$. It is then easy to see that

$$\begin{tikzcd} A \arrow[r, "\psi_A"] \arrow[d, "f"'] & A^{**} \arrow[d, "f^{**}"] \\ B \arrow[r, "\psi_B"] & B^{**} \end{tikzcd}$$

commutes.

Theorem 8.

Every Λ-module can be embedded in a natural way into a cofree (hence injective) module. In other words, there exists a functor $L : {}_{\Lambda}\mathfrak{M} \longrightarrow {}_{\Lambda}\mathfrak{M}$ and a natural transformation $\mu : \mathrm{Id}_{{}_{\Lambda}\mathfrak{M}} \longrightarrow L$ such that, for each Λ-module M,

(i) LM is a cofree module,

(ii) μ_M is a monomorphism.

Proof.

We know that every Λ-module A is quotient of the free module generated by the underlying set of A. Hence, there exists a natural epimorphism $\phi : \bigoplus_{x \in M} \Lambda_{(x)} \twoheadrightarrow A$. Thus the composition

$$M \xrightarrow{\psi} M^{**} = \mathrm{Hom}_{\mathbb{Z}}(M^*;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{Hom}(\phi;\mathbb{Q}/\mathbb{Z})} \mathrm{Hom}_{\mathbb{Z}}\Big(\bigoplus_{x \in M^*} \Lambda_{(x)};\mathbb{Q}/\mathbb{Z}\Big) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \prod_{x \in A^*} \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{(x)};\mathbb{Q}/\mathbb{Z}) = LM$$

is a natural embedding and, by definition, LM is a cofree module.

We can now apply theorem 8 to Hartshorne's construction to get the following result :

Let $C^+({}_{\Lambda}\mathfrak{M})$ be the category whose objects are bounded below cochain complexes of elements in ${}_{\Lambda}\mathfrak{M}$, and whose morphisms are cochain maps. If I denotes the full subcategory of ${}_{\Lambda}\mathfrak{M}$ consisting of injective modules, and if $E^+ : C^+(I) \longrightarrow C^+({}_{\Lambda}\mathfrak{M})$ is the embedding functor, then

Proposition 9.

There exists a functor $J^+ : C^+({}_\Lambda\mathfrak{M}) \longrightarrow C^+(I)$ together with a natural transformation

$$i : \mathrm{Id}_{C^+({}_\Lambda\mathfrak{M})} \longrightarrow E^+J^+$$

such that, for each $X^\cdot \in |C^+({}_\Lambda\mathfrak{M})|$,

(i) i_X is a quiso

(ii) i_X is monic.

Remark.

Note that, if we work in the category $C^+({}_\Lambda\mathfrak{M})$ (and not in $K^+({}_\Lambda\mathfrak{M})$), the functor J^+ is no more left adjoint to the embedding functor: if J^+ were left adjoint to E^+, then the counit

$$\delta : J^+E^+ \longrightarrow \mathrm{Id}_{C^+(I)}$$

of the adjunction would be an isomorphism since E^+ is full and faithful ((8), p.88). As $E^+\delta \circ i_{E^+} = \mathrm{Id}_{E^+}$, i_{E^+} would be an isomorphism; whence, to prove that J^+ is not left adjoint to E^+, it is sufficient to show that i_{E^+} is not an isomorphism.

First, note that if $\mathrm{Hom}(\beta;\mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(A;\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(B;\mathbb{Q}/\mathbb{Z})$ is epic, then $\beta : B \longrightarrow A$ is monic. Indeed, suppose that x is a non-zero element of B such that $\beta(x) = 0$. By an argument similar to the one used in the proof of Lemma 7, one shows that there exists $f : B \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that $f(x) \neq 0$. As $\mathrm{Hom}(\beta;\mathbb{Q}/\mathbb{Z})$ is epic, there exists $g : A \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that $g\beta = f$. Then, $f(x) = g(\beta(x)) = 0$ since $\beta(x) = 0$, contradicting the fact that $f(x) \neq 0$.

Also, if A is a non-zero Λ-module, the canonical epimorphism

$$\phi : \bigoplus_{x \in A} \Lambda_{(x)} \twoheadrightarrow A$$

is not monic, so that, for a non-zero Λ-module M,

$$\mathrm{Hom}_{\mathbb{Z}}(\phi;\mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(M^*;\mathbb{Q}/\mathbb{Z}) \rightarrowtail \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{x \in M^*} \Lambda_{(x)};\mathbb{Q}/\mathbb{Z})$$

is not epic, and the map $\mu_M : M \longrightarrow LM$ of theorem 8 is not an isomorphism. Thus, the map $i_{E^+X^\cdot} : E^+X^\cdot \longrightarrow E^+J^+E^+X^\cdot$ is not an isomorphism.

In Chapter II, we shall use the dual of Lemma 1; so let's make some remarks about the dual situation.

As the proof of Lemma 1 invokes only properties of abelian categories and as the dual of any abelian category is abelian, the dual of lemma 1 holds. We rewrite it in details :

<u>Lemma 1^d</u>.

Let $\mathcal{A}$ be an abelian category.

a) Let $\mathbb{P}$ be a subclass of $|\mathcal{A}|$, and assume

(i) For each object X of $\mathcal{A}$, there exists an element P of $\mathbb{P}$ and an epimorphism $P \twoheadrightarrow A$.

Then, for every $X. \in |K^+(\mathcal{A})^{op}|$ (where $\mathcal{C}^{op}$ represents the dual of $\mathcal{C}$), there exists a bounded below chain complex P. of elements of $\mathbb{P}$ together with a quasi-isomorphism $\pi : P. \longrightarrow A.$, with each π_n epic.

b) Assume furthermore that $\mathbb{P}$ satisfies

(ii) If $0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$ is a short exact sequence with $X \in \mathbb{P}$, then $Y \in \mathbb{P} \Longleftrightarrow Z \in \mathbb{P}$, and

(iii) There exists a positive integer n such that, if $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_o$ is an exact sequence, and

$X_o, \ldots, X_{n-1} \in \mathbb{P}$, then $X_n \in \mathbb{P}$.

Then, for every $X. \in |K(\mathcal{A})^{op}|$ there exists a complex P. of elements of $\mathbb{P}$ and a quiso $P. \longrightarrow X.$.

c) Let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$, and assume that $\mathcal{A}'$ has enough $\mathcal{A}$-projectives. Then, for every $X. \in |K^+_{\mathcal{A}'}(\mathcal{A})^{op}|$ there exists a bounded below complex P. of $\mathcal{A}$-projective objects of $\mathcal{A}'$ and a quiso $P. \longrightarrow X.$.

Remarks.

1. Consider the particular case $\mathcal{A} = {}_\Lambda\mathfrak{M}$. If $\mathbb{P}$ is the class of projective modules of ${}_\Lambda\mathfrak{M}$, then the condition (i) holds, since ${}_\Lambda\mathfrak{M}$ has enough projectives. Furthermore, if the sequence $0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$ is exact and if X is projective, then the sequence splits, so that $Y = Z \oplus X$. Thus, Y is projective if and only if Z is projective, and (ii) holds. Also, by definition, (iii) holds if the left homological dimension of Λ is finite.

Finally, the category $\mathcal{A}'$ of finitely generated Λ-modules is a thick subcategory of ${}_\Lambda\mathfrak{M}$ since if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact, then A' and A'' are finitely generated if and only if A is finitely generated. Also, $\mathcal{A}'$ has enough $\mathcal{A}$-projectives since every finitely generated module is a quotient of a finitely generated projective module. Hence, we may write Lemma 1^d c) as follows for the case of modules :

For every bounded below chain complex X. such that all $H_n(X.)$ are finitely generated, there exists a quasi-isomorphism $P. \longrightarrow X.$, where P. is a bounded below complex, with P_n projective and finitely generated.

2. The dual of this statement is false since in general a finitely generated

module cannot be embedded in a finitely generated injective module (take for instance $\Lambda = \mathbb{Z}$).

3. Even in ${}_{\Lambda}\mathfrak{M}$, the quiso of lemma 1^d a) is not a homotopy equivalence in general, as we see in the following example : take

$$X. \quad : \quad \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Then, Hartshorne's construction yields a chain complex P. and a quiso π as follows :

$$\begin{tikzcd}
P. \arrow[d, "\pi"'] & : & \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, two heads, "p"] & 0 \\
X. & : & \ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Then, $\mathrm{Hom}(P.;\mathbb{Z}) \cong P.$ and $\mathrm{Hom}(X.;\mathbb{Z}) = 0$, since $\mathbb{Z}/2\mathbb{Z}$ is a torsion group. Hence, $H^n(\mathrm{Hom}(P.;\mathbb{Z})$ is not isomorphic to $H^n(\mathrm{Hom}(X.;\mathbb{Z})$, which proves that P. and X. are not homotopy equivalent.

As a final remark, note that the dual of Propositions 2, 3 and Corollary 4 hold. There is also a dual construction for theorem 8, since every Λ-module M is a quotient of $\bigoplus_{x \in M} \Lambda_{(x)}$. Hence, the dual of Proposition 9 also holds.

CHAPTER II. THE GROUP HOMOMORPHISM $\nu : \mathrm{Hom}_A(M;L) \otimes_B N \longrightarrow \mathrm{Hom}_A(M;L \otimes_B N)$.

In this chapter, we denote ${}_A\mathfrak{M}$ the category of left A-modules, and ${}_A\mathfrak{M}_B$ the category of (A,B)-bimodules. ($M \in |{}_A\mathfrak{M}_B|$ means that M is a left A-module, a right B-module, and that $(\alpha x)\beta = \alpha(x\beta)$ for all $\alpha \in A$, $x \in M$, $\beta \in B$).

We first prove the existence of a group homomorphism

$$\nu : \mathrm{Hom}_A(M;L) \otimes_B N \longrightarrow \mathrm{Hom}_A(M;L \otimes_B N) :$$

Let A,B be two rings, $M \in |{}_A\mathfrak{M}|$, $L \in |{}_A\mathfrak{M}_B|$, $N \in |{}_B\mathfrak{M}|$. Then

<u>Proposition 1.</u>

The map $\nu : \mathrm{Hom}_A(M;L) \otimes_B N \longrightarrow \mathrm{Hom}_A(M;L \otimes_B N)$ defined by $\nu(u \otimes y)(x) = u(x) \otimes y$ $(u \in \mathrm{Hom}_A(M;L),\ y \in N,\ x \in M)$ is a group homomorphism, natural in M, L, N.

<u>Proof.</u>

ν is well defined : for $y \in N$ and $u \in \mathrm{Hom}_A(M;L)$, define

$$\nu'(u,y) : M \longrightarrow L \otimes_B N$$

by $\nu'(u,y)(x) = u(x) \otimes y$ $(x \in M)$. Then, by definition of the left A-module structure of $L \otimes_B N$, $\nu'(u,y) \in \mathrm{Hom}_A(M;L \otimes_B N)$. Furthermore, one checks directly that

$$\begin{aligned} \nu' : \mathrm{Hom}_A(M;L) \times N &\longrightarrow \mathrm{Hom}_A(M;L \otimes_B N) \\ (u,y) &\longmapsto \nu'(u,y) \end{aligned}$$

is bilinear and B-balanced. By the universal property of $\otimes_B$, ν' induces the group homomorphism ν.

ν is natural in M. Indeed, for each $f: M' \longrightarrow M$, the commutativity of the diagram

$$(1)\qquad \begin{tikzcd} \mathrm{Hom}_A(M;L) \otimes_B N \arrow[r, "\nu_M"] \arrow[d, "\mathrm{Hom}_A(f;L) \otimes_B N"] & \mathrm{Hom}_A(M;L \otimes_B N) \arrow[d, "\mathrm{Hom}_A(f;L \otimes_B N)"] \\ \mathrm{Hom}_A(M';L) \otimes_B N \arrow[r, "\nu_{M'}"] & \mathrm{Hom}_A(M';L \otimes_B N) \end{tikzcd}$$

results immediately of the definitions.

Similarly, one verifies the naturality in L and N.

Convention.

To the end of this chapter, M,L,N,A,B,ν will have the same meaning as in Proposition 1.

In general, ν is neither an epimorphism, nor a monomorphism. We give a series of sufficient conditions under which ν is an isomorphism. These conditions will be summarized at the end of the chapter.

Proposition 2.

If N is projective (resp. finitely generated projective), then ν is a monomorphism (resp. ν is an isomorphism).

Proof.

We first prove the proposition when N is free (resp. finitely generated free). In this case, N can be identified to a direct sum of copies of the ring B : $N = \bigoplus_I B$ (if N is finitely generated, the set I of indices can be chosen finite).

Consider the following diagram :

$$(2)\qquad \begin{tikzcd} \mathrm{Hom}_A(M;L)\otimes_B \bigoplus_I B \arrow[r, "\nu"] \arrow[d, tail, "\alpha"'] & \mathrm{Hom}_A(M;L\otimes_B \bigoplus_I B) \arrow[d, tail, "\beta"] \\ \bigoplus_I \mathrm{Hom}_A(M;L) \arrow[r, tail, "j"] & \prod_I \mathrm{Hom}_A(M;L) \end{tikzcd}$$

where α is the canonical isomorphism, defined by $\alpha(u\otimes(y_i)_I) = (uy_i)_I$, j the embedding, and β is defined, for $f\in \mathrm{Hom}_A(M;L\otimes_B \bigoplus_I B)$ such that $f(x) = u(x)\otimes(y_i)_I$, by $\beta(f) = (f_i)_I$, with $f_i(x) = u(x)y_i$. β is a monomorphism, as composition of the following natural maps :

$$\mathrm{Hom}_A(M;L\otimes_B \bigoplus_I B) \xrightarrow{\cong} \mathrm{Hom}_A(M;\bigoplus_I (L\otimes_B B) \xrightarrow{\cong} \mathrm{Hom}_A(M;\bigoplus_I L) \rightarrowtail$$

$$\xrightarrow{\text{embedding}} \mathrm{Hom}_A(M;\prod_I L) \xrightarrow{\cong} \prod_I \mathrm{Hom}_A(M;L).$$

One checks that (2) commutes, so that $\beta\nu = j\alpha$ is a monomorphism. Hence, ν is a monomorphism.

If N is finitely generated, the set I is finite, thus since $\bigoplus_I X \cong \prod_I X$, j and β are isomorphisms, and ν is also an isomorphism, and the proposition is proved when N is free.

Suppose now N projective. Then N is a direct summand in a free module, i.e. there exist a module R and a free module F such that $F = N \oplus R$. But ν_F is an isomorphism if and only if ν_N and ν_R are isomorphisms. Indeed, consider the following diagram :

$$
(3)\qquad
\begin{tikzcd}
\mathrm{Hom}_A(M;L)\otimes_B (N\oplus R) \arrow[r, "\cong", "\phi"'] \arrow[d, "\nu_F"'] & \left[\mathrm{Hom}_A(M;L)\otimes_B N\right]\oplus\left[\mathrm{Hom}_A(M;L)\otimes_B R\right] \arrow[d, "\nu_N\oplus\nu_R"] \\
\mathrm{Hom}_A(M;L\otimes_B (N\oplus R)) \arrow[r, "\cong", "\phi'"'] & \left[\mathrm{Hom}_A(M;L\otimes_B N)\right]\oplus\left[\mathrm{Hom}_A(M;L\otimes_B R)\right]
\end{tikzcd}
$$

where ϕ and ϕ' are the canonical isomorphisms.

One checks that (3) commutes, hence ν_F is an isomorphism if and only if $\nu_N\oplus\nu_R$ is an isomorphism. But $\nu_N\oplus\nu_R$ is an isomorphism if and only if ν_N and ν_R are both isomorphisms. This concludes the proof of Proposition 2.

We have an analogous result when we put the assumptions on the module M :

<u>Proposition 3</u>.

If M is finitely generated and projective, then ν is an isomorphism.

<u>Proof</u>.

As for Proposition 2, we first suppose that M is finitely generated free, so that we can write $M = \bigoplus_{i=1}^{n} A$. Then, if α, β and γ are the canonical isomorphisms, the following diagram commutes :

$$
(4)\qquad
\begin{tikzcd}
\mathrm{Hom}_A(\bigoplus_{i=1}^{n} A\ ;\ L)\otimes_B N \arrow[r, "\nu"] \arrow[d, tail, "\alpha"'] & \mathrm{Hom}_A(\bigoplus_{i=1}^{n} A\ ;\ L\otimes_B N) \arrow[d, tail, "\beta"] \\
\left[\bigoplus_{i=1}^{n} L\right]\otimes_B N \arrow[r, tail, two heads, "\gamma"] & \bigoplus_{i=1}^{n}(L\otimes_B N)
\end{tikzcd}
$$

Hence, ν is an isomorphism if M is free.

We pass from the case M free to the case M projective by the same argument as in the proof of Proposition 2.

Now, we "spread" the two conditions onto M and N. In order to prove our next result, we need a

Lemma :

If M is finitely generated, then there exists a natural isomorphism

$$\phi : \mathrm{Hom}_A(M ; \bigoplus_I L) \longrightarrow \bigoplus_I \mathrm{Hom}_A(M;L)$$

defined, for each $f \in \mathrm{Hom}_A(M ; \bigoplus_I L)$ such that $f(x) = (y_i)_I$ by $\phi(f) = (f_i)_I$, where f_i is defined by $f_i(x) = y_i$.

Proof.

ϕ is well defined : if M is generated by $x_1,\ldots,x_n$, and if $f : M \longrightarrow \bigoplus_I L$ is such that $f(x) = (y_i)_I$, then f is entirely determined by $f(x_j) = (y_{ij})_{\substack{i \in I \\ j=1,\ldots,n}}$. But, by definition of $\bigoplus_I L$, y_{ij} is nonzero for only finitely many $i \in I$. This implies that f_i is nonzero for only finitely many indices $i \in I$, thus $(f_i)_I \in \bigoplus_I \mathrm{Hom}_A(M;L)$.

One checks directly that ϕ is a monomorphism and an epimorphism.

Proposition 4.

If M is finitely generated and if N is projective, then ν is an isomorphism.

Proof.

If N is free : $N = \bigoplus_I B$, the proposition follows from the commutativity of the diagram

$$(5)\qquad \begin{tikzcd} \mathrm{Hom}_A(M;L) \otimes_B \bigoplus_I B \arrow[r, "\nu"] \arrow[d, tail, two heads] & \mathrm{Hom}_A(M;L \otimes_B \bigoplus_I B) \arrow[d, tail, two heads] \\ \bigoplus_I \mathrm{Hom}_A(M;L) & \mathrm{Hom}_A(M: \bigoplus_I L) \arrow[l, tail, two heads, "\phi"'] \end{tikzcd}$$

If N is projective, we express it as a direct summand in a free module, and we proceed as in Proposition 2.

We would like now to interchange the two conditions on M and N. But it seems that the condition "finitely generated" is not strong enough to prove that ν is an isomorphism; we substitute to it the stronger condition "finitely presented" (these conditions are equivqlent if the ring is Noetherian). But, first, we need a

Lemma :

If $E \in |{}_\Lambda\mathfrak{M}|$ is finitely presented, and if $M_i \in |\mathfrak{M}_\Lambda|$, $i \in I$, then the natural homomorphism

$$\phi : (\prod_I M_i) \otimes_\Lambda E \longrightarrow \prod_I (M_i \otimes_\Lambda E)$$

is an isomorphism.

Proof.

We first suppose that E is finitely generated free : $E = \bigoplus_1^n \Lambda$. Then

$$(\prod_I M_i) \otimes_\Lambda \bigoplus_1^n \Lambda \cong \bigoplus_1^n \left[(\prod_I M_i) \otimes_\Lambda \Lambda \right] \cong \bigoplus_1^n (\prod_I M_i) \cong \prod_I (\bigoplus_1^n M_i) \cong$$

$$\cong \prod_I \left[\bigoplus_1^n (M_i \otimes_\Lambda \Lambda) \right] \cong \prod_I (M_i \otimes_\Lambda \bigoplus_1^n \Lambda).$$

Consider now the general case, and let $L_1 \longrightarrow L_o \twoheadrightarrow E \longrightarrow 0$ be a finite presentation of E ($L_1 \longrightarrow L_o \longrightarrow E \longrightarrow 0$ is an exact sequence with L_o and L_1 finitely generated free).

As the functor $X \otimes_\Lambda -$ is right exact, and as, in ${}_\Lambda\mathfrak{M}$, the product of exact sequences is still an exact sequence ((2), p.A II 10), we get the following commutative diagram with exact rows :

$$(6) \qquad \begin{array}{ccccccc}
(\prod_I M_i) \otimes_\Lambda L_1 & \longrightarrow & (\prod_I M_i) \otimes_\Lambda L_o & \twoheadrightarrow & (\prod_I M_i) \otimes_\Lambda E & \longrightarrow & 0 \\
\downarrow \phi_{L_1} & & \downarrow \phi_{L_o} & & \downarrow \phi_E & & \\
\prod_I (M_i \otimes_\Lambda L_1) & \longrightarrow & \prod_I (M_i \otimes_\Lambda L_o) & \twoheadrightarrow & \prod_I (M_i \otimes_\Lambda E) & \longrightarrow & 0
\end{array}$$

As ϕ_{L_o} and ϕ_{L_1} are isomorphisms, ϕ_E is also an isomorphism.

Proposition 5.

If M is projective and N finitely presented, then ν is an isomorphism.

Proof.

As in Proposition 3, it is sufficient to give a proof for the case where M

is free : $M = \bigoplus_I A$. But in this case, the result is immediate if we consider the commutative diagram

$$
(7) \qquad
\begin{tikzcd}
\mathrm{Hom}_A(\bigoplus_I A\,;\,L)\otimes_B N \arrow[r, "\nu"] \arrow[d, two heads] & \mathrm{Hom}_A(\bigoplus_I A\,;\,L\otimes_B N) \arrow[d, two heads] \\
(\prod_I L)\otimes_B N \arrow[r, tail, "\phi"] & \prod_I (L\otimes_B N)
\end{tikzcd}
$$

Remark.

Over a Noetherian ring, "finitely presented" is equivalent to "finitely generated" ((3), p.36). Hence, if B is Noetherian, by Propositions 4 and 5, ν is an isomorphism as soon as the hypotheses "finitely generated" and "projective" are distributed in any of the four possible ways on M and N.

We state now a last proposition without assumptions on the bimodule L :

Proposition 6.

If M is finitely generated (resp. finitely presented), and if N is flat, then ν is a monomorphism (resp. an isomorphism).

Proof.

Let $F_1 \longrightarrow F_o \longrightarrow M \longrightarrow 0$ be a free presentation of M, with F_o, F_1 free and F_o finitely generated, and consider the commutative diagram

$$(8)\qquad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(M;L)\otimes_B N & \rightarrowtail & \mathrm{Hom}_A(F_o;L)\otimes_B N & \longrightarrow & \mathrm{Hom}_A(F_1;L)\otimes_B N \\ & & \downarrow \nu_M & & \downarrow \nu_{F_o} & & \downarrow \nu_{F_1} \\ 0 & \longrightarrow & \mathrm{Hom}_A(M;L\otimes_B N) & \rightarrowtail & \mathrm{Hom}_A(F_o;L\otimes_B N) & \longrightarrow & \mathrm{Hom}_A(F_1;L\otimes_B N) \end{array}$$

(The top row is exact since $-\otimes_B N$ is an exact functor, and the bottom row is exact since $\mathrm{Hom}_A(-\,;L\otimes_B N)$ is left exact).

By Proposition 3, ν_{F_o} is an isomorphism, hence ν_M is a monomorphism. If M is finitely presented, F_1 can be chosen finitely generated, so that ν_{F_1} is also an isomorphism. In this case, the map ν_M induced on the kernels is an isomorphism.

Remarks.

1. As ν is natural in M, L and N, the propositions 1 to 6 admit an immediate generalization to the case where M, L or N are (co)-chain complexes.

2. We obtain interesting particular cases for Propositions 1 to 6 by taking $A = B = L = \Lambda$. Then ν becomes

$$\psi : \mathrm{Hom}_\Lambda(M;\Lambda)\otimes_\Lambda N \longrightarrow \mathrm{Hom}_\Lambda(M;N)$$

and is defined by $\psi(f\otimes y)(x) = f(x)y$.

If, as usual, we denote $M^* = \mathrm{Hom}_\Lambda(M;\Lambda)$ (M^* = dual of M), the map ψ takes the very simple form

$$\psi : M^*\otimes_\Lambda N \longrightarrow \mathrm{Hom}_\Lambda(M;N)$$

(If Λ is a field, and if M is a finite-dimensional vector space over Λ, then ψ is an isomorphism which is well known in tensor calculus).

We shall now use the previous propositions to prove a few statements about the behaviour of an injective module with respect to the tensor product :

Proposition 7.

Let A be a left Noetherian ring, $L \in |{}_A\mathfrak{M}_B|$ injective as A-module, and $N \in |{}_B\mathfrak{M}|$ flat. Then, $L \otimes_B N$ is an injective A-module.

Proof.

We recall that a left A-module M is injective if and only if, for each left ideal I of A, the morphism

$$i^* : \mathrm{Hom}_A(A;M) \longrightarrow \mathrm{Hom}_A(I;M)$$

induced by the embedding $i : I \rightarrowtail A$, is an epimorphism.

Consider the following commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_A(A;L) \otimes_B N & \xrightarrow{\psi} & \mathrm{Hom}_A(I;L) \otimes_B N \\ \downarrow \nu_A & & \downarrow \nu_I \\ \mathrm{Hom}_A(A;L \otimes_B N) & \xrightarrow{i^*} & \mathrm{Hom}_A(I;L \otimes_B N) \end{array}$$

(ν_A is an isomorphism since $\mathrm{Hom}_A(A;L \otimes_B N) \cong L \otimes_B N \cong \mathrm{Hom}_A(A;L) \otimes_B N$). As L is injective, and as the functor $- \otimes_B N$ is right exact, ψ is an epimorphism. But, since A is Noetherian, I is finitely presented. By Proposition 6, as N is flat, ν_I is an isomorphism. Thus, i^* is an epimorphism, and $L \ {}_B N$ is injective.

If A is a PID, the proposition is true without the assumption N flat :

Proposition 8.

Let A be a PID, $L \in |{}_A\mathfrak{M}_B|$ injective as A-module, and let $N \in |{}_B\mathfrak{M}|$. Then $L \otimes_B N$ is an injective A-module.

Proof.

If A is a PID, then I is free and finitely generated and, by Proposition 3, ν_I is an isomorphism (even if N is not flat).

Remark.

If we take the particular case $A = \mathbb{Z}$, we get the following result :

If $L \in |\mathfrak{M}_B|$ is injective as abelian group, and if $N \in |{}_B\mathfrak{M}|$, then $L \otimes_B N$ is an injective abelian group.

We also have a sort of sual situation :

Proposition 9.

Suppose that B is a left Noetherian ring, $M \in |{}_A\mathfrak{M}|$ is projective, and $L \in |{}_A\mathfrak{M}_B|$ is flat as right B-module. Then $\mathrm{Hom}_A(M;L)$ is a flat right B-module.

Proof.

By (3), Proposition 1, p.26, $\mathrm{Hom}_A(M;L)$ is flat if and only if, for each left ideal I of B, the canonical morphism

$$\phi : \mathrm{Hom}_A(M;L) \otimes_B I \longrightarrow \mathrm{Hom}_A(M;L) \otimes_B B$$

is a monomorphism. Again by (3), p.36, if B is left Noetherian, then I is finitely presented.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M;L) \otimes_B I \arrow[r, "\phi"] \arrow[d, "\nu_M"] & \mathrm{Hom}_A(M;L) \otimes_B B \arrow[d, tail, "\beta"] \\
\mathrm{Hom}_A(M;L \otimes_B I) \arrow[r, "\psi"] & \mathrm{Hom}_A(M;L \otimes_B B)
\end{tikzcd}$$

(β is an isomorphism since $\mathrm{Hom}_A(M;L) \otimes_B B \cong \mathrm{Hom}_A(M;L) \cong \mathrm{Hom}_A(M;L \otimes_B B)$). By Proposition 5, ν_M is an isomorphism. Furthermore, L flat entails that $L \otimes_B I \longrightarrow L \otimes_B B$ is a monomorphism. As $\mathrm{Hom}_A(M; -)$ is left exact, then ψ is monic. Hence ϕ is a monomorphism, and $\mathrm{Hom}_A(M;L)$ is flat.

As before, if B is a PID, then Proposition 9 holds without the assumption that M be projective since, by Proposition 2, I projective implies that ν_M is a monomorphism. Thus :

<u>Proposition 10.</u>

Suppose B is a PID, and $L \in |{}_A\mathfrak{M}_B|$ is flat as right B-module. Then $\mathrm{Hom}_A(M;L)$ is a flat right B-module.

We conclude this chapter by a brief discussion of the case where M or N are complexes. We give some sufficient conditions for ν to induce isomorphism in homology.

<u>Proposition 11.</u>

Let A be a left Noetherian ring, M. a bounded below chain complex of objects in $|{}_A\mathfrak{M}|$ such $H_n(M.)$ is finitely generated for all n. If L is injective as A-module, and if N is flat, then ν is a quasi-isomorphism, i.e. $H_n(\nu)$ is an isomorphism for all n.

Proof.

By Remark 1, p.16, we know that for every bounded below chain complex M. such that $H_n(M.)$ is finitely generated for all n, there exists a quasi-isomorphism $\pi : P. \longrightarrow M.$ where P. is a bounded below chain complex with P_n projective and finitely generated for all n.

By Proposition 7, $L \otimes_B N$ is an injective A-module. Hence the functors $\mathrm{Hom}_A(-;L)$ and $\mathrm{Hom}_A(-;L \otimes_B N)$ as well as $- \otimes_B N$ are exact. Moreover, we know that if a functor T is exact, and if $f : X. \longrightarrow Y.$ is a chain map, then $T(H_n(f)) = H_n(Tf)$ so that exact functors preserve the quasi-isomorphisms. Proposition 11 results now from Proposition 3 and from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_A(M.;L) \otimes_B N & \xrightarrow{\nu_{M.}} & \mathrm{Hom}_A(M.;L \otimes_B N) \\
\Big\downarrow \text{quiso} & & \Big\downarrow \text{quiso} \\
\mathrm{Hom}_A(P.;L) \otimes_B N & \xrightarrow[\cong]{\nu_{P.}} & \mathrm{Hom}_A(P.;L \otimes_B N)
\end{array}$$

By similar arguments, using Proposition 9, one proves :

Proposition 12.

Let B be a left Noetherian ring, N. a bounded below chain complex of objects in $|{}_B\mathfrak{M}|$ such that $H_n(N.)$ is finitely generated for all n. If L is flat as B-module, and if M is projective, then ν is a quasi-isomorphism.

Now, we summarize our results in the following list :

$$\nu : \mathrm{Hom}_A({}_AM \; ; \; {}_AL_B) \otimes_B {}_BN \longrightarrow \mathrm{Hom}_A({}_AM \; ; \; {}_AL_B \otimes_B {}_BN)$$

A	B	M	L	N	ν
				projective	mono
				projective + finitely generated	iso
		projective + finitely generated			iso
		finitely generated		projective	iso
		projective		finitely presented	iso
		finitely generated		flat	mono
		finitely presented		flat	iso
left Noetherian		$M. \in K^+_{A'}({}_A\mathfrak{M})$	injective	flat	quiso
	left Noetherian	projective	flat	$N. \quad K^+_{A'}({}_B\mathfrak{M})$	quiso

APPENDIX.

1. Proof of Lemma I.1.

For any co-chain complex

$$X^{\cdot} : \quad \dots \longrightarrow X^{p-1} \xrightarrow{d^{p-1}} X^{p} \xrightarrow{d^{p}} X^{p+1} \longrightarrow \dots$$

we use the usual notations

$$B^{p}(X^{\cdot}) = \operatorname{Im} d^{p-1} \qquad B'^{p}(X^{\cdot}) = \operatorname{Coim} d^{p}$$

$$Z^{p}(X^{\cdot}) = \operatorname{Ker} d^{p} \qquad Z'^{p}(X^{\cdot}) = \operatorname{Cok} d^{p-1}$$

Thus, as objects, we have $B'^{p}(X^{\cdot}) \cong B^{p+1}(X^{\cdot})$.

We recall Lemma I.1 a) :

Let $\mathcal{A}$ be an abelian category, $\mathbb{P}$ a subclass of $|\mathcal{A}|$ and assume that every object of $\mathcal{A}$ admits an injection into an object of $\mathbb{P}$. Then every $X^{\cdot} \in |K^{+}(\mathcal{A})|$ admits a quasi-isomorphism into a bounded below complex $I^{\cdot}$ of objects of $\mathbb{P}$.

Proof.

Given a bounded below co-chain complex $X^{\cdot}$, we can assume $X^{p} = 0$ for $p < 0$. Then, there exist $I^{o} \in \mathbb{P}$ and a monomorphism $i^{o} : X^{o} \rightarrowtail I^{o}$. Suppose

$$0 \longrightarrow I^{o} \xrightarrow{d_{I}^{o}} I^{1} \longrightarrow \dots \xrightarrow{d_{I}^{p-1}} I^{p}$$

and $i^{k} : X^{k} \longrightarrow I^{k}$ $(k = 0,1,\dots,p)$ are defined. Then I^{p+1}, d_{I}^{p} and i^{p+1} are defined by the following commutative diagram :

1.

$$\begin{tikzcd}
 & & X^p \arrow[r, "d_X^p"] \arrow[d, "\bar{i}^p"] \arrow[dl, "i^p"'] \arrow[dr, phantom, "(1)"] & X^{p+1} \arrow[d, "\alpha^{p+1}"'] \arrow[ddr, "i^{p+1}"] & \\
 & & \operatorname{Cok} d_I^{p-1} = \bar{I}^p \arrow[r, "\beta^p"] & K^{p+1} \arrow[dr, tail, "j^{p+1}"] & \\
I^{p-1} \arrow[r, "d_I^{p-1}"] & I^p \arrow[ur, two heads, "s^p"] \arrow[rrr, "d_I^p"'] & & & I^{p+1}
\end{tikzcd}$$

where (1) is a pushout diagram, s^p the canonical projection, and $j^{p+1} : K^{p+1} \rightarrowtail I^{p+1}$ is an embedding of K^{p+1} into an element of $\mathbb{P}$.

It is clear from this diagram that $d_I^p\, d_I^{p-1} = 0$ and that $i^{p+1}\, d_X^p = d_I^p\, i^p$. Thus, we obtain a complex $I^\bullet$ and a chain map $i : X^\bullet \longrightarrow I^\bullet$.

<u>i^p is a monomorphism for all p.</u>

Consider the diagram

2.

$$\begin{tikzcd}
 & & B^p(X) \arrow[dl, tail, "\kappa"] \arrow[d, "\tau"] & \operatorname{Ker} \bar{i}^p \arrow[l, dashed, "\psi"'] \arrow[d, tail, "\varepsilon"] \arrow[ddl, dashed, "\tilde{i}"] \arrow[ddl, dashed, "\tilde{i}"] & \\
 & X^{p-1} \arrow[ur, two heads] \arrow[r, "d_X"] \arrow[d, "\bar{i}^{p-1}"'] & \operatorname{Im} \beta \arrow[dr, tail, "\beta''"] \arrow[d, equal] & X^p \arrow[d, "\alpha"'] \arrow[ddr, "i^p"] \arrow[ddrr, "\bar{i}^p"] & \\
 & \bar{I}^{p-1} \arrow[ur, two heads, "\beta'"] \arrow[r, "\beta"] & \operatorname{Ker} s \arrow[dr, tail, "\nu''"] & K^p \arrow[dr, tail, "j"] & \\
I^{p-1} \arrow[uur, two heads, "s^{p-1}"] \arrow[ur, two heads, "\nu'"] \arrow[rr, "d_I = j\beta s"'] & & & I^p \arrow[r, two heads, "s"] & \bar{I}^p
\end{tikzcd}$$

where τ is the map induced on the images, and ν' the factorization of d_I through Ker s. $i^p : X^p \longrightarrow I^p$ induces a map $\tilde{i} : \text{Ker } \bar{i}^p \longrightarrow \text{Ker } s$ such that

3.
$$\begin{tikzcd}
\text{Ker } \bar{i}^p \arrow[r, tail, "\varepsilon"] \arrow[d, "\tilde{i}"'] & X^p \arrow[r, "\bar{i}^p"] \arrow[d, "i^p"] & \bar{I}^p \arrow[d, equal] \\
\text{Ker } s \arrow[r, tail, "\nu''"] & I^p \arrow[r, "s"] & \bar{I}^p
\end{tikzcd}$$

commutes.

Furthermore, as $\nu''\ \nu' = (j\ \beta'')(\beta'\ s^{p-1})$ are two epi-mono factorizations of $j\beta s$, we can identify Im β to Ker s, so that $\tilde{i}$ may be considered as a map Ker $\bar{i}^p \longrightarrow$ Im β. Hence : $j\beta''\tilde{i} = \nu''\tilde{i} = i^p\varepsilon = j\alpha\varepsilon$. But j is monic, so that $\beta''\tilde{i} = \alpha\varepsilon$.

In the diagram

4.
$$\begin{tikzcd}
\text{Ker } \bar{i}^p \arrow[dr, "\psi"] \arrow[drr, "\varepsilon"] \arrow[ddr, "\tilde{i}"'] & & \\
& B^p(X^\cdot) = \text{Im } d_X \arrow[r, tail, "\kappa"'] \arrow[d, "\tau"] & X^p \arrow[d, "\alpha"] \\
& \text{Im } \beta \arrow[r, tail, "\beta''"] & K^p
\end{tikzcd}$$

the square is bicartesian (see (10), Theorem 13.4.8), hence there exists $\psi : \text{Ker } \bar{i}^p \longrightarrow B^p(X^\cdot)$ such that $\kappa\psi = \varepsilon$.

Finally, consider

5.
$$\begin{tikzcd}
\operatorname{Ker} \bar{i}^p \arrow[r, dashed, "\gamma"] \arrow[d, tail, "\varepsilon"'] & \operatorname{Ker} \alpha \arrow[d, tail, "\mu"] \\
X^p \arrow[r, "d_X"] \arrow[d, "\bar{i}^p"'] \arrow[dr, phantom, "(1)"] & X^{p+1} \arrow[d, "\alpha"] \\
\bar{I}^p \arrow[r] & K^{p+1}
\end{tikzcd}$$

As (1) is pushout, the map γ induced on the kernels is epic ((10), theorem 13.4.8). But $d_X\varepsilon = (d_X\kappa)\psi = 0 = \mu\gamma$. Then : γ epic implies $\mu = 0$, i.e. $\operatorname{Ker} \alpha = 0$, so that α is monic. As j is also monic, $i^p = j\alpha$ is monic.

The following step is to prove that

<u>$\operatorname{Ker} \beta^p \cong H^p(I^\cdot)$</u>.

Recall the definition of $H^p(I^\cdot)$:

6.
$$\begin{tikzcd}
I^{p-1} \arrow[r, "d_I^{p-1}"] & I^p \arrow[r, "d_I^p"] \arrow[d, two heads, "s^p"'] \arrow[dr, two heads, "\mu^p"] & I^{p+1} \\
H^p(I^\cdot) \arrow[r, tail] & \operatorname{Cok} d_I^{p-1} \arrow[r, two heads, "h'^p"] & \operatorname{Coim} d_I^p \arrow[u, tail, "\nu^p"']
\end{tikzcd}$$

We have $H^p(I^\cdot) = \operatorname{Ker} h'^p$, and $h'^p s^p = \mu^p$.

Then :

7.
$$\begin{tikzcd}
\operatorname{Cok} d_I^{p-1} \arrow[r, "h'^p"] \arrow[d, "\beta^{p+1}"] & \operatorname{Coim} d_I^p \cong \operatorname{Im} d_I^p \arrow[d, tail, "\nu^p"] \\
K^{p+1} \arrow[r, tail, "j^{p+1}"] & I^{p+1}
\end{tikzcd}$$

commutes : $j^{p+1}\beta^{p+1}s^p = d_I^p = \nu^p\mu^p = \nu^p h'^p s^p$. As s^p is epic : $j^{p+1}\beta^{p+1} = \nu^p h'^p$.

Thus : $\operatorname{Ker} \beta^{p+1} = \operatorname{Ker} j^{p+1}\beta^{p+1} = \operatorname{Ker} \nu^p h'^p = \operatorname{Ker} h'^p = H^p(I^\cdot)$.

<u>$H^p(i)$ is epic</u>.

Since in the diagram 1., the square (1) is pushout, $\bar{i}^p$ induces an epimorphism η^p: $\operatorname{Ker} d_X^p = Z^p(X^\cdot) \twoheadrightarrow \operatorname{Ker} \beta^p = H^p(I^\cdot)$ such that

8.
$$\begin{tikzcd} Z^p(X^\cdot) \arrow[r, tail] \arrow[d, "\eta^p"'] & X^p \arrow[d] \\ H^p(I^\cdot) \arrow[r, tail] & \bar{I}^p \end{tikzcd}$$

commutes.

Chasing the diagram

9.
$$\begin{tikzcd} X^{p-1} \arrow[rrrr] \arrow[dr] \arrow[ddd] & & & & X^p \arrow[ddd] \arrow[ddl] \\ & B^p(X^\cdot) \arrow[rr, tail] \arrow[urrr] & & Z^p(X^\cdot) \arrow[r, two heads, "\pi"] \arrow[urr] \arrow[d, "\eta"'] & H^p(X^\cdot) \arrow[dl, dashed, "\theta"] \\ & & & H^p(I^\cdot) \arrow[r, tail] & \bar{I}^p \\ I^{p-1} \arrow[rrrr] & & & & I^p \arrow[ul] \end{tikzcd}$$

one sees that $X^{p-1} \twoheadrightarrow B^p(X^\cdot) \longrightarrow Z^p(X^\cdot) \longrightarrow H^p(I^\cdot) \rightarrowtail \bar{I}^p$ is the zero map. Hence, $B^p(X^\cdot) \longrightarrow Z^p(X^\cdot) \xrightarrow{\eta} H^p(I^\cdot)$ is the zero map, and η factors through $\operatorname{Cok}(B^p(X^\cdot) \longrightarrow Z^p(X^\cdot)) = H^p(X^\cdot)$; thus, there exists an epimorphism θ such that $\theta\pi = \eta$.

We claim that $\theta = H^p(i)$. For this, by definition of $H^p(i)$, we have to prove that

10.
$$\begin{tikzcd} H^p(X^\cdot) \arrow[r] \arrow[d, two heads, "\theta"'] & Z'^p(X^\cdot) = \operatorname{Cok} d_X^{p-1} \arrow[d, "\psi"] \\ H^p(I^\cdot) \arrow[r] & Z'^p(I^\cdot) = \operatorname{Cok} d_I^{p-1} \end{tikzcd}$$

commutes, where ψ is defined by the commutative diagram

11.
$$\begin{tikzcd} X^{p-1} \arrow[r] \arrow[d] & X^p \arrow[r, two heads] \arrow[d, "i^p"'] \arrow[dr, "\bar{i}^p"] & Z'^p(X^\cdot) \arrow[d, "\psi"] \\ I^{p-1} \arrow[r] & I^p \arrow[r, two heads, "s^p"'] & Z'^p(I^\cdot) \end{tikzcd}$$

Consider the diagram

12.
$$\begin{tikzcd} & Z^p(X^\cdot) \arrow[r, tail] \arrow[d, two heads, "\pi"] \arrow[ddl, "\eta"'] & X^p \arrow[r, "i^p"] \arrow[d, two heads] \arrow[dd, phantom, "(1)"] & I^p \arrow[dd, "s^p"] \\ & H^p(X^\cdot) \arrow[r, tail] \arrow[dl, "\theta"] \arrow[dl, phantom, shift right=4ex, "(3)"] & Z'^p(X^\cdot) \arrow[dr, "\psi"] \arrow[r, phantom, "(2)"] & \\ H^p(I^\cdot) \arrow[rrr] \arrow[rr, phantom, shift right=4ex, "(4)"] & & & \bar{I}^p = \operatorname{Cok} d_I^{p-1} \end{tikzcd}$$

From previous computations, we know that (2), (3) and the large square of diagram 12. commute. By definition of Z, Z' and H, the square (1) also commutes. Then (4) commutes, so that $H^p(i) = \theta$.

The last step to prove the first part of the lemma is to show that <u>$H^p(i)$ is monic</u>.

For this, consider the diagram

13.
$$\begin{tikzcd}
X^p \arrow[r, "d_X"] \arrow[d, "\bar{i}"'] \arrow[dd, bend right=60, "i"'] \arrow[dr, phantom, "(1)"] & X^{p+1} \arrow[r, two heads] \arrow[d, "\alpha"] \arrow[dd, bend left=60, "i"] & \mathrm{Cok}\ d_X \arrow[d, dashed, "\phi_1"] \\
\bar{I}^p \arrow[r, "\beta"] & K^{p+1} \arrow[r, two heads] \arrow[d, equal] \arrow[dr, phantom, "(2)"] & \mathrm{Cok}\ \beta \arrow[d, dashed, "\phi_2^{-1}"] \\
I^p \arrow[u, two heads, "s"] \arrow[r, "\beta s"] \arrow[d, equal] \arrow[dr, phantom, "(3)"] & K^{p+1} \arrow[r, two heads] \arrow[d, tail, "j"] & \mathrm{Cok}\ \beta s \arrow[d, dashed, "\phi_3"] \\
I^p \arrow[r, "d_I = j\beta s"'] & I^{p+1} \arrow[r, two heads] & \mathrm{Cok}\ d_I
\end{tikzcd}$$

where (1) is pushout by construction, (2) is pushout since s is epic, and (3) is pullback since j is monic; thus, by (10), theorem 13.4.8 : ϕ_1 and ϕ_2 are isomorphisms, and ϕ_3 is monic. Hence,

$$\begin{tikzcd}
X^p \arrow[r, "d_X"] \arrow[d, "i"'] & X^{p+1} \arrow[d, "i"] \\
I^p \arrow[r, "d_I"] & I^{p+1}
\end{tikzcd}$$

induces a monomorphism ϕ : Cok $d_X \rightarrowtail$ Cok d_I.

Finally, look at the commutative diagram

14.
$$\begin{tikzcd}
0 \arrow[r] & H^p(X^\cdot) \arrow[r, tail] \arrow[d, "H^p(i)"'] \arrow[dr, phantom, "(1)"] & \mathrm{Cok}\ d_X^{p-1} \arrow[r, two heads] \arrow[d, tail, "\phi"] & \mathrm{Coim}\ d_X^p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^p(I^\cdot) \arrow[r, tail] & \mathrm{Cok}\ d_I^{p-1} \arrow[r, two heads] & \mathrm{Coim}\ d_I^p \arrow[r] & 0
\end{tikzcd}$$

As (1) commutes and ϕ is monic, so is $H^p(i)$. Hence, $H^p(i)$ is an isomorphism and i is a quiso. This ends the proof of Lemma 1 a).

Now, we recall Lemma I. 1 b) :

Let $\mathcal{A}$ be an abelian category, let $\mathbb{P}$ be a subclass of $|\mathcal{A}|$, and assume

(i) Every object of $\mathcal{A}$ admits an injection into an element of $\mathbb{P}$.

(ii) If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a short exact sequence with $X \in \mathbb{P}$, then $Y \in \mathbb{P} \iff Z \in \mathbb{P}$.

(iii) There exists a positive integer n such that if $X^o \longrightarrow X^1 \longrightarrow \dots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0$ is an exact sequence, and $X^o, X^1, \dots, X^{n-1} \in \mathbb{P}$, then $X^n \in \mathbb{P}$.

Then, every $X^\cdot \in |K(\mathcal{A})|$ admits a quasi-isomorphism into a complex $I^\cdot$ of objects of $\mathbb{P}$.

<u>Proof</u>.

<u>First step</u>.

Let $X^\cdot \in |K(\mathcal{A})|$, and $i_o \in \mathbb{Z}$. Then, by Lemma 1 a), we can find a quasi-isomorphism i of the truncated complex

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^{i_o} \longrightarrow X^{i_o+1} \longrightarrow \dots.$$

into a complex

$$I^\cdot : \dots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^{i_o} \longrightarrow I^{i_o+1} \longrightarrow \dots$$

with each $i^p : X^p \longrightarrow I^p$ monic. Define $X_o^\cdot$ to be the complex

$$\dots \longrightarrow X^{i_o-2} \xrightarrow{d_X^{i_o-2}} X^{i_o-1} \xrightarrow{i \circ d_X^{i_o-1}} I^{i_o} \xrightarrow{d_I} I^{i_o+1} \longrightarrow \dots$$

and consider the diagram

$$15. \quad \begin{tikzcd}
X^\cdot \arrow[d, "i"'] & \dots \arrow[r] & X^{i_o-2} \arrow[r, "d_X"] \arrow[d, equal] & X^{i_o-1} \arrow[r, "d_X"] \arrow[d, equal] & X^{i_o} \arrow[r, "d_X"] \arrow[d, "i"'] & X^{i_o+1} \arrow[r] \arrow[d, "i"'] & \dots \\
X_o^\cdot & \dots \arrow[r] & X^{i_o-2} \arrow[r, "d_X"] & X^{i_o-1} \arrow[r, "i \circ d_X"] & I^{i_o} \arrow[r, "d_I"] & I^{i_o+1} \arrow[r] & \dots
\end{tikzcd}$$

Then, $X_o^i \in \mathbb{P}$ for $i \geqslant i_o$, and each $i^p : X^p \longrightarrow X_o^p$ is monic.

<u>$i : X^{\cdot} \longrightarrow X_o^{\cdot}$ is a quasi-isomorphism.</u>

It is clear from the construction of $X_o^{\cdot}$ that $H^n(X^{\cdot}) \cong H^n(X_o^{\cdot})$ for $n \neq i_o - 1$ and for $n \neq i_o$.

(i) $H^{i_o-1}(X^{\cdot}) \cong H^{i_o-1}(X_o^{\cdot})$:

Clearly, $B^{i_o-1}(X^{\cdot}) = B^{i_o-1}(X_o^{\cdot}) = \operatorname{Im} d_X^{i_o-2}$.

Furthermore, $Z^{i_o-1}(X^{\cdot}) = \operatorname{Ker} d_X^{i_o-1} \cong \operatorname{Ker} i d_X^{i_o-1} = Z^{i_o-1}(X_o^{\cdot})$, since i is monic.

Hence, $H^{i_o-1}(X^{\cdot}) \cong H^{i_o-1}(X_o^{\cdot})$.

(ii) $H^{i_o}(X^{\cdot}) \cong H^{i_o}(X_o^{\cdot})$:

Consider the diagram

16.
$$\begin{tikzcd}
X^{i_o-1} \arrow[rr] \arrow[dr, two heads] \arrow[dd, equal] & & X^{i_o} \arrow[dd, tail, "i"] \\
& B(X^{\cdot}) \arrow[ur, tail] \arrow[dd, dashed, "\phi"] & \\
X^{i_o-1} \arrow[rr] \arrow[dr, two heads] & & I^{i_o} \\
& B(X_o^{\cdot}) \arrow[ur, tail] &
\end{tikzcd}$$

which shows readily that ϕ is an isomorphism : $B^{i_o}(X^{\cdot}) \cong B^{i_o}(X_o^{\cdot})$

We now look at

17.
$$\begin{tikzcd}
Z^{i_o}(X^{\cdot}) = \operatorname{Ker} d_X \arrow[r, tail] \arrow[d, dashed, "\psi"] \arrow[dr, phantom, "(1)"] & X^{i_o} \arrow[r, "d_X"] \arrow[d, tail] \arrow[dr, phantom, "(2)"] & X^{i_o+1} \arrow[d, tail, "\alpha"] \\
Z^{i_o}(X_o^{\cdot}) = \operatorname{Ker} d_I \arrow[r, tail] & I^{i_o} \arrow[r, "\beta"] \arrow[dr, "d_I"'] & K^{i_o+1} \arrow[d, tail, "j"] \\
& & I^{i_o+1}
\end{tikzcd}$$

(By construction, $I^{i_o} = \mathrm{Cok}\, d_I^{i_o-1} = \bar{I}^{i_o}$, since $d_I^{i_o-1} = 0$, and j is monic. Thus, $\mathrm{Ker}\, d_I = \mathrm{Ker}\, \beta$).

Then : (1) commutes implies ψ is monic

(2) is pushout implies ψ is epic (see (10), theorem 13.4.8).

Hence, $H^{i_o}(X^\cdot) \cong H^{i_o}(X_o^\cdot)$, and $i : X^\cdot \longrightarrow X_o^\cdot$ is a quasi-isomorphism.

<u>Second step</u>.

Suppose given $X_1^\cdot \in |K(\mathcal{A})|$ with $X_1^i \in \mathbb{P}$ for $i \geqslant i_1$, and let $i_2 < i_1$. We want to construct a quiso $X_1^\cdot \longrightarrow X_2^\cdot$ such that $X_2^i \in \mathbb{P}$ for $i \geqslant i_2$, and $X_1^i = X_2^i$ for $i > i_1+n$ (n=integer of condition iii)).

By the first step, we can find a quiso $X_1^\cdot \longrightarrow X'^\cdot$ such that $X'^i \in \mathbb{P}$ for $i \geqslant i_2$, and $X_1^i \longrightarrow X'^i$ monic. Define $Y^i = \mathrm{Cok}(X_1^i \longrightarrow X'^i)$. The long exact cohomology sequence of

$$0 \longrightarrow X_1^\cdot \longrightarrow X'^\cdot \longrightarrow Y^\cdot \longrightarrow 0$$

then shows that $Y^\cdot$ is acyclic, i.e. $H^p(Y^\cdot) = 0$ for all p. Furthermore, for $i \geqslant i_1 > i_2$, $X_1^i \in \mathbb{P}$ and $X'^i \in \mathbb{P}$. Hence, by condition (ii), $Y^i \in \mathbb{P}$ for $i \geqslant i_1$. The exact sequence

$$Y^i \longrightarrow Y^{i+1} \longrightarrow \dots \longrightarrow Y^{i+n-1} \twoheadrightarrow B^{i+n}(Y^\cdot) \longrightarrow 0$$

together with condition (iii) entails that $B^i(Y^\cdot) \in \mathbb{P}$ for $i \geqslant i_1+n$.

We are now ready to define $X_2^\cdot$: let K_i be defined by the pushout square

18.
$$\begin{tikzcd}
 & X_1^{i-1} \arrow[r, "d_{X_1}"] \arrow[dl, "\text{quiso}"'] \arrow[d] & X_1^i \arrow[d, "\alpha"] \\
X'^{i-1} \arrow[dr, two heads, "\mu"'] & & \\
 & B^i(X'^\cdot) \arrow[r, "\beta"] & K^i
\end{tikzcd}$$

and let

$$X_2^i = \begin{cases} X'^i & \text{for } i < i_1 + n \\ K^i & \text{for } i = i_1 + n \\ X_1^i & \text{for } i > i_1 + n \end{cases}$$

<u>Convention</u> : To simplify the notations, we set $\underline{i_1 + n = 0}$ for the proof that $X_2^\bullet$ is quasi-isomorphic to $X_1^\bullet$.

We define the complex $X_2^\bullet$ as follows :

$$\cdots \longrightarrow X'^{-2} \xrightarrow{d_{X'}^{-2}} X'^{-1} \xrightarrow{d_{X_2}^{-1}} K_o \xrightarrow{d_{X_2}^{o}} X_1^1 \xrightarrow{d_{X_1}^{1}} X_1^2 \longrightarrow \cdots$$

where $d_{X_2}^{-1}$ and $d_{X_2}^{-1}$ are defined by the following diagram :

19.

$$\begin{tikzcd}
 & & X_1^{-1} \arrow[r, "d_{X_1}"] \arrow[d] \arrow[ddl, "i^{-1}"'] & X_1^o \arrow[d, "\alpha^o"'] \arrow[ddr, "d_{X_1}"] & \\
 & & B^o(X'^\bullet) \arrow[r, "\beta^o"] \arrow[dr, "o"] & K^o \arrow[dr, dashed, "d_{X_2}^o"] & \\
X'^{-2} \arrow[r, "d_{X'}^{-2}"] & X'^{-1} \arrow[uur, "\mu^o"] \arrow[urr, dashed, "d_{X_2}^{-1}"'] & & & X_1^1
\end{tikzcd}$$

i.e. $d_{X_2}^{-1} = \beta^o \mu^o$ and $d_{X_2}^{o}$ is the unique map making the diagram commutative (universal property of the pushout).

It is immediate from this diagram that $d_{X_2}^{-1}\, d_{X'}^{-2} = 0$, and that $d_{X_2}^{o}\, d_{X_2}^{-1} = 0$.

To prove that $d_{X_1}^{1}\, d_{X_2}^{o} = 0$, we consider the diagram

20.
$$\begin{tikzcd}
X_1^{-1} \arrow[r] \arrow[d] & X_1^{o} \arrow[d, "\alpha^{o}"'] \arrow[ddr, "0"] & & \\
B^{o}(X'^{\cdot}) \arrow[r] \arrow[drr, "0"] & K^{o} \arrow[dr, "d^{o}_{X_2}"'] & & \\
& & X_1^{1} \arrow[dr, "d^{1}_{X_1}"'] & \\
& & & X_1^{2}
\end{tikzcd}$$

This diagram commutes, but $0 : K^{o} \longrightarrow X_1^2$ makes also the diagram commutative, hence $d^1_{X_1}\, d^o_{X_2} = 0$; hence $X_2^{\cdot}$ is a co-chain complex.

We now construct a chain map $X_1^{\cdot} \longrightarrow X_2^{\cdot}$:

21.
$$\begin{tikzcd}
X_1^{\cdot} \arrow[d, tail, "i"] & \cdots \arrow[r] & X_1^{-2} \arrow[r] \arrow[d, tail, "i^{-2}"'] & X_1^{-1} \arrow[r] \arrow[d, tail, "i^{-1}"'] & X_1^{o} \arrow[r] \arrow[d, "\alpha^{o}"] & X_1^{1} \arrow[r] \arrow[d, equal] & X_1^{2} \arrow[r] \arrow[d, equal] & \cdots \\
X_2^{\cdot} & \cdots \arrow[r] & X'^{-2} \arrow[r, "d_{X'}"'] & X'^{-1} \arrow[r, "\beta^{o}\mu^{o}"'] & K^{o} \arrow[r, "d^{o}_{X_2}"'] & X_1^{1} \arrow[r, "d_{X_1}"'] & X_1^{2} \arrow[r] & \cdots
\end{tikzcd}$$

where i^{-2} and i^{-1} are parts of the quiso $i : X_1^{\cdot} \rightarrowtail X'^{\cdot}$.

We claim that $i : X_1^{\cdot} \longrightarrow X_2^{\cdot}$ is a quiso. To prove this, it suffices to show that $H^p(i)$ is an isomorphism for $p = -1,0,1$. First, we show that

<u>α^o and β^o are monomorphisms</u>.

Consider the diagram 22., where $\nu\mu$ is the epi-mono factorization of $d^{-1}_{X'}$:

22.
$$\begin{tikzcd}
 & X_1^{-1} \arrow[r, "d_{X_1}^{-1}"] \arrow[d] & X_1^{o} \arrow[d, "\alpha^{o}"'] \arrow[ddr, "i^{o}"] & \\
 & B^{o}(X') \arrow[r, "\beta^{o}"] \arrow[ddr, "\nu"] & K^{o} \arrow[dr, dashed, "j^{o}"] & \\
X'^{-1} \arrow[uur, "i^{-1}"] \arrow[ur, "\mu"'] \arrow[rrr, "d_{X'}^{-1}"] & & & X'^{o}
\end{tikzcd}$$

As $d_{X'}^{-1}\, i^{-1} = i^{o} d_{X_1}$, the universal property of pushouts implies the existence of a map $j^{o} : K^{o} \longrightarrow X'^{o}$ such that $j^{o}\alpha^{o} = i^{o}$ and $j^{o}\beta^{o} = \nu$. But i^{o} and ν are monomorphisms, thus α^{o} and β^{o} are also monomorphisms.

<u>$H^{-1}(i) : H^{-1}(X_1^{\cdot}) \longrightarrow H^{-1}(X_2^{\cdot})$ is an isomorphism.</u>

As we already know that $H^{-1}(X_1^{\cdot}) \cong H^{-1}(X'^{\cdot})$. we only have to prove that $H^{-1}(X'^{\cdot}) \cong H^{-1}(X_2^{\cdot})$. For this, consider

23.
$$\begin{tikzcd}
X_2^{\cdot} & \cdots \arrow[r] & X'^{-2} \arrow[r, "d_{X'}"] \arrow[d, equal] & X'^{-1} \arrow[r, "\beta\mu"] \arrow[d, equal] & K^{o} \arrow[r] \arrow[d, "j^{o}"] & \cdots \\
X'^{\cdot} & \cdots \arrow[r] & X'^{-2} \arrow[r, "d_{X'}"] & X'^{-1} \arrow[r, "d_{X'}"] \arrow[dr, "\mu"'] & X'^{o} \arrow[r] & \cdots \\
 & & & & B^{-1}(X'^{\cdot}) \arrow[u, tail, "\nu"'] &
\end{tikzcd}$$

Since $B^{-1}(X_2^{\cdot}) = B^{-1}(X'^{\cdot})$, we have to prove that $Z^{-1}(X_2^{\cdot}) \cong Z^{-1}(X'^{\cdot})$.

β and ν are monomorphisms, so that

$$Z^{-1}(X_2^{\cdot}) = \operatorname{Ker} \beta\mu \cong \operatorname{Ker} \mu \cong \operatorname{Ker} \nu\mu = \operatorname{Ker} d_{X'}^{-1} = Z^{-1}(X'^{\cdot}).$$

Hence : $H^{-1}(i) : H^{-1}(X_1^{\cdot}) \longrightarrow H^{-1}(X_2^{\cdot})$ is an isomorphism.

$H^o(i) : H^o(X_1^\cdot) \longrightarrow H^o(X_2^\cdot)$ is an isomorphism.

In the diagram

24.
$$\begin{tikzcd}
X_1^{-1} \arrow[r, "d_{X_1}"] \arrow[d] \arrow[dr, phantom, "(1)"] & X_1^o \arrow[r, two heads] \arrow[d, "\alpha"] & \operatorname{Cok} d_{X_1} = Z'^o(X_1^\cdot) \arrow[d, dashed, "\bar{\alpha}"] \\
B^o(X') \arrow[r, "\beta"] & K^o \arrow[r, two heads] & \operatorname{Cok} \beta = \operatorname{Cok} d_{X_2}^{-1} = Z'^o(X_2^\cdot) \\
X'^{-1} \arrow[u] \arrow[ur, "d_{X_2}^{-1}"'] & &
\end{tikzcd}$$

(1) is pushout entails that $\bar{\alpha} : Z'^o(X_1^\cdot) \longrightarrow Z'^o(X_2^\cdot)$ is an isomorphism.

Consider the epi-mono factorization of $d_{X_1}^o$ and $d_{X_2}^o$:

$$\begin{tikzcd}
X_1^o \arrow[rr, "d_{X_1}^o"] \arrow[dr, two heads, "\theta"'] & & X_1^1 \\
& B^1(X_1^\cdot) \arrow[ur, tail, "\psi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K^o \arrow[rr, "d_{X_2}^o"] \arrow[dr, two heads, "\mu"'] & & X_1^1 \\
& B^1(X_2^\cdot) \arrow[ur, tail, "\nu"'] &
\end{tikzcd}$$

In the diagram

25.
$$\begin{tikzcd}
X_1^{-1} \arrow[r, "d_{X_1}^{-1}"] \arrow[d, "\bar{i}"'] \arrow[dr, phantom, "(1)"] & X_1^o \arrow[r, two heads, "\theta"] \arrow[d, tail, "\alpha"] & B^1(X_1^\cdot) \arrow[r, tail, "\psi"] \arrow[d, tail, "\phi"] \arrow[dr, phantom, "(2)"] & X_1^1 \arrow[d, equal] \\
B^o(X') \arrow[r, tail, "\beta"] \arrow[d, two heads, "\tilde{i}"'] & K^o \arrow[r, two heads, "\mu"] \arrow[d, two heads, "\tilde{\alpha}"] & B^1(X_2^\cdot) \arrow[r, tail, "\nu"] \arrow[d, two heads, "\tilde{\phi}"] & X_1^1 \\
\operatorname{Cok} \bar{i} \arrow[r, tail, two heads, "\bar{\beta}"] & \operatorname{Cok} \alpha \arrow[r, "\bar{\mu}"] & \operatorname{Cok} \phi &
\end{tikzcd}$$

(2) commutes implies that the induced map ϕ is monic

(1) is pushout implies that $\bar{\beta}$ is an isomorphism.

Then : $d_{X_2}\ \beta = 0 \implies \mu\ \beta = 0 \implies \tilde{\phi}\ \mu\ \beta = \bar{\mu}\ \bar{\beta}\ \tilde{\bar{i}} = 0.$

But $\bar{\beta}\ \tilde{\bar{i}}$ epic $\implies \bar{\mu} = 0 \implies \bar{\mu}\ \tilde{\alpha} = \tilde{\phi}\ \mu = 0 \implies \tilde{\phi} = 0.$

Hence : $\tilde{\phi}$ epic $\implies$ Cok $\phi = 0 \implies \phi$ is an isomorphism:

$B'^{o}(X_1^{\cdot}) \cong B^1(X_1^{\cdot}) \overset{\phi}{\cong} B^1(X_2^{\cdot}) \cong B'^{o}(X_2^{\cdot})$, and $H^o(i) : H^o(X_1^{\cdot}) \longrightarrow H^o(X_2^{\cdot})$ is an isomorphism.

<u>$H^1(i) : H^1(X_1^{\cdot}) \longrightarrow H^1(X_2^{\cdot})$ is an isomorphism.</u>

Consider the diagram

26.
$$\begin{tikzcd}
X_1^o \arrow[r] \arrow[d, tail, "\alpha"] & X_1^1 \arrow[r, "d^1_{X_1}"] \arrow[d, equal] & X_1^2 \arrow[d, equal] \\
K^o \arrow[r] & X_1^1 \arrow[r, "d^1_{X_1}"] & X_1^2.
\end{tikzcd}$$

We have $Z^1(X_1^{\cdot}) = \text{Ker } d^1_{X_1} = Z^1(X_2^{\cdot})$. As we already know that $B^1(X_1^{\cdot}) \cong B^1(X_2^{\cdot})$, we conclude that $H^1(i)$ is an isomorphism.

Hence : $i : X_1^{\cdot} \longrightarrow X_2^{\cdot}$ is a quasi-isomorphism.

In order to complete the second step, we have to show that $X_2^i \in \mathbb{P}$ for $i \geqslant i_2$, i.e. that $K^o \in \mathbb{P}$. For this, it suffices to prove that $B^o(Y^{\cdot}) = \text{Cok } \alpha^o$, since then, as $0 \longrightarrow X_1^o \overset{\alpha^o}{\rightarrowtail} K^o \twoheadrightarrow B^o(Y^{\cdot}) \longrightarrow 0$ is exact and as X_1^o, $B^o(Y^{\cdot}) \in \mathbb{P}$, we have, by assumption (ii), $K^o \in \mathbb{P}$.

In the diagram

27.
$$\begin{tikzcd}
 & X^{-1} \arrow[r] \arrow[dl, tail, "i"'] \arrow[d] \arrow[dr, phantom, "(1)"] & X_1^o \arrow[d, "\alpha^o"] \\
X'^{-1} \arrow[r, two heads, "\mu'"'] & B^o(X') \arrow[r] & K^o
\end{tikzcd}$$

(1) is pushout implies that $\operatorname{Cok} \alpha^{o} \cong \operatorname{Cok} \mu' i$. Hence, it suffices to prove that $\operatorname{Cok} \mu' i \cong B^{o}(Y^{\cdot})$.

Consider

28.
$$\begin{tikzcd}
X^{-2} \arrow[r, "d_1"] \arrow[d, tail, "i"'] & X_1^{-1} \arrow[r, two heads, "\mu"] \arrow[d, tail, "i"'] & B^{o}(X_1^{\cdot}) \arrow[r, tail] \arrow[d, tail, "\eta"] & X_1^{o} \arrow[d, tail, "i"] \\
X'^{-2} \arrow[r, "d'"] \arrow[dd, two heads, "\tilde{i}"'] & X'^{-1} \arrow[r, two heads, "\mu'"] \arrow[dr, "f\mu'"] \arrow[dd, two heads, "\tilde{i}"'] & B^{o}(X'^{\cdot}) \arrow[r, tail] \arrow[dl, "f"'] \arrow[dd, two heads, "\tilde{\eta}"] & X'^{o} \arrow[dd, two heads, "\tilde{i}"] \\
 & & M & \\
Y^{-2} \arrow[r, "\bar{d}"] & Y^{-1} \arrow[r, two heads, "\bar{\mu}"] \arrow[ur, dashed, "g"] & B^{o}(Y^{\cdot}) \arrow[r, tail] \arrow[ul, dashed, "\phi"'] & Y^{o}
\end{tikzcd}$$

where η and $\tilde{\eta}$ are induced on the images.

We have $\operatorname{Cok} \mu' i = \operatorname{Cok} \eta\mu = \operatorname{Cok} \eta$. Hence, it suffices to prove that $\tilde{\eta} = \operatorname{Cok} \eta$. For this, let $f : B^{o}(X'^{\cdot}) \longrightarrow M$ be a map such that $f\eta = 0$. We have to find $\phi : B^{o}(Y^{\cdot}) \longrightarrow M$ such that $\phi\,\tilde{\eta} = f$ (such a ϕ will necessary be unique since $\tilde{\eta}$ is epic).

We have the following implications :

$$f\eta = 0 \Longrightarrow f\eta\mu = f\mu' i = 0 \Longrightarrow f\mu' \text{ factors through } Y^{-1} = \operatorname{Cok} i :$$

there exists $g : Y^{-1} \longrightarrow M$ with $g\tilde{i} = f\mu'$.

Now, $g\,\bar{d}\,\tilde{i} = g\,\tilde{i}\,d' = f\,\mu'\,d' = 0$. But $\tilde{i}$ is epic, so that $g\,\bar{d} = 0$.

As $Y^{\cdot}$ is acyclic, $B^{o}(Y^{\cdot}) = \operatorname{Cok} \bar{d}$, and $g\,\bar{d} = 0$ implies that g factors through $B^{o}(Y^{\cdot})$: there exists ϕ such that $\phi\,\bar{\mu} = g$. One checks then directly that $\phi\,\tilde{\eta} = f$.

Third step.

Finally, given a complex $X^{\cdot} \in |K(\mathcal{A})|$, choose a sequence of integers $i_o > i_1 > i_2 > \ldots$ tending to $-\infty$. Choose $X_o^{\cdot}$ for i_o as in the first step, and choose $X_1^{\cdot}, X_2^{\cdot}, \ldots$ for $i_1, i_2, \ldots$ successively as in the second step. Then, we have quasi-isomorphisms $X^{\cdot} \longrightarrow X_o^{\cdot} \longrightarrow X_1^{\cdot} \longrightarrow X_2^{\cdot} \longrightarrow \ldots$ and, for each i, the sequence $X^i \longrightarrow X_o^i \longrightarrow X_1^i \longrightarrow X_2^i \longrightarrow \ldots$ is either constant, or in $\mathbb{P}$. Then $I^{\cdot} = \varinjlim X_r^{\cdot}$ is the required complex of objects in $\mathbb{P}$.

This ends the proof of Lemma I.1, b).

Let's now recall Lemma I. 1 c) :

> Let $\mathcal{A}$ be an abelian category, let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$, and assume that $\mathcal{A}'$ has enough $\mathcal{A}$-injectives. Then every $X^{\cdot} \in |K_{\mathcal{A}}^{+}, (\mathcal{A})|$ admits a quasi-isomorphism into a bounded below complex $I^{\cdot}$ of $\mathcal{A}$-injective objects of $\mathcal{A}'$.

Proof.

We may assume $X^i = 0$ for $i < 0$. Embed $H^o(X^{\cdot})$ into I^o, an $\mathcal{A}$-injective object of $\mathcal{A}'$, which is possible, since $H^o(X^{\cdot}) \in |\mathcal{A}'|$. Then we can extend this embedding to a map $f^o : X^o \longrightarrow I^o$:

29.
$$\begin{tikzcd} 0 \arrow[r] & X^o \arrow[r] \arrow[dr, dashed, "f^o"] & X^1 \\ & H^o(X^{\cdot}) \arrow[u] \arrow[r] & I^o \end{tikzcd}$$

which is possible, since I^o is $\mathcal{A}$-injective.

Having defined $I^o \xrightarrow{d_I^o} I^1 \longrightarrow \ldots \xrightarrow{d_I^{p-1}} I^p$ and $f^i : X^i \longrightarrow I^i$ for $i=0,1,\ldots,p$ we define I^{p+1}, d_I^p and f^{p+1} by the diagram

30.

$$\begin{tikzcd}
 & & & B^{p+1}(X^\cdot) \arrow[dr, tail] & & \\
 & & X^p \arrow[ur] \arrow[rr, "\phi"] \arrow[dl, "f^p"'] \arrow[d, "s^p f^p"] \arrow[dr, phantom, "(1)"] & & Z^{p+1}(X^\cdot) \arrow[r, tail] \arrow[d, "\beta"] & X^{p+1} \arrow[dd, "f^{p+1}"] \\
I^{p-1} \arrow[r, "d_I^{p-1}"] & I^p \arrow[r, two heads, "s^p"] \arrow[drrr, "d_I^p"'] & \operatorname{Cok} d_I^{p-1} = Z'^p(I^\cdot) \arrow[r, "g"] & & K^{p+1} \arrow[dr, tail, "j^{p+1}"] & \\
 & & & & & I^{p+1}
\end{tikzcd}$$

where (1) is a pushout square.

We must check that K^{p+1} is in A' (to be sure that K^{p+1} can be embedded in an A-injective object of A').

Recall that a full subcategory A' of an abelian category A is thick (or is a Serre subcategory of A) if one of the following equivalent conditions hold :

(i) If $X \longrightarrow Y \longrightarrow Z$ is exact, then $X,Z \in |A'| \Longrightarrow Y \in |A'|$.

(ii) If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact, then $B \in |A'|$ if and only if $B' \in |A'|$ and $B'' \in |A'|$.

Thus, it suffices to prove that, in the exact sequence

$$Z'^p(I^\cdot) \xrightarrow{g} K^{p+1} \twoheadrightarrow \operatorname{Cok} g$$

$Z'^p(I) \in |A'|$, and $\operatorname{Cok} g \in |A'|$. But : $0 \longrightarrow B^p(I^\cdot) \rightarrowtail I^p \twoheadrightarrow Z'^p(I^\cdot) \longrightarrow 0$ is a short exact sequence, and $I^p \in |A'|$, so that $Z'^p(I^\cdot) \in |A'|$.

Moreover, as the square (1) of diagram 30. is a pushout, $\operatorname{Cok} g \cong \operatorname{Cok} \phi = H^{p+1}(X^\cdot) \in |A'|$. Hence, $K^{p+1} \in |A'|$, and there is a monomorphism $K^{p+1} \rightarrowtail I^{p+1}$ where I^{p+1} is an A-injective object of A'.

Finally, as I^{p+1} is injective, the map $j^{p+1}\beta$ extends to $f^{p+1} : X^{p+1} \longrightarrow I^{p+1}$ and one defines $d_I^p = j^{p+1}gs^p$. One checks directly that $d_I^p d_I^{p-1} = 0$ and that $f : X^\cdot \longrightarrow I^\cdot$ is a chain map.

<u>$H^p(f)$ is an isomorphism</u>.

1. <u>$H^p(f)$ is epic</u> : consider the diagram

31.
$$\begin{tikzcd}
 & & B^{p+1}(X^\cdot) \arrow[dr, tail] & \\
Z^p(X^\cdot) \arrow[r, tail] \arrow[d, two heads, "\eta"'] & X^p \arrow[ur] \arrow[rr, "\phi"] \arrow[d] \arrow[dr, phantom, "(1)"] & & Z^{p+1}(X^\cdot) \arrow[d] \\
H^p(I^\cdot) \arrow[r, tail] & Z'^p(I^\cdot) \arrow[rr, "g"] \arrow[dr] & & K^{p+1} \\
 & & B^{p+1}(I^\cdot) = B'^p(I^\cdot) \arrow[ur, tail] &
\end{tikzcd}$$

One checks directly that $\operatorname{Im}\phi = B^{p+1}(X^\cdot)$ and that $\operatorname{Im} g = B^{p+1}(I^\cdot) = B'^p(I^\cdot)$. As the square (1) of diagram 31. is a pushout, the map η induced on the kernels is epic. With a same reasoning as in diagram 9. (p.36), one sees that η induces an epimorphism $\theta = H^p(f) : H^p(X^\cdot) \twoheadrightarrow H^p(I^\cdot)$.

2. <u>$H^p(f)$ ia monic</u>. Consider the diagram

32.
$$\begin{tikzcd}
X^p \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & Z^{p+1}(X^\cdot) \arrow[r, "\gamma"] \arrow[d, "\beta"] & X^{p+1} \arrow[r, "d_X"] \arrow[dd, "f^{p+1}"] & X^{p+2} \arrow[dd, "f^{p+2}"] \\
Z'^p(I^\cdot) \arrow[r, "\alpha"] \arrow[dr] & K^{p+1} \arrow[dl] \arrow[dd, dashed, "\phi"] \arrow[ddr, "j"] & & \\
B^{p+1}(I^\cdot) \arrow[uu, "s"] \arrow[ddr] \arrow[dr, phantom, "(2)"] & & & \\
I^p \arrow[rr, "d_I"] & & I^{p+1} \arrow[r, "d_I"] & I^{p+2} \\
& Z^{p+1}(I^\cdot) \arrow[ur, "\varepsilon"] & &
\end{tikzcd}$$

We have $d_I j\beta = f^{p+2} d_X \gamma = 0$, and $d_I j\alpha s = d_I d_I = 0$, hence, since s is epic, $d_I j\alpha = 0$. By the universal property of the pushout (1), $d_I j = 0$. Thus, there exists a unique monomorphism ϕ such that $\varepsilon\phi = j$. One also checks that the triangle (2) of diagram 32. commutes. Finally, consider

33.
$$\begin{tikzcd}
B^{p+1}(X^\cdot) \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & Z^{p+1}(X^\cdot) \arrow[r, two heads] \arrow[d] & H^{p+1}(X^\cdot) \arrow[d, tail, "\mu"] \\
B^{p+1}(I^\cdot) \arrow[r, tail, "\alpha'"] \arrow[d, equal] & K^{p+1} \arrow[r, two heads] \arrow[d, "\phi"] & \operatorname{Cok} \alpha' \arrow[d, "\mu'"] \\
B^{p+1}(I^\cdot) \arrow[r, tail] & Z^{p+1}(I^\cdot) \arrow[r, two heads] & H^{p+1}(I^\cdot)
\end{tikzcd}$$

As the square (1) of diagram 32. is a pushout, the square (3) of diagram 33 is a pushout, so that μ is an isomorphism. Furthermore, by the short five lemma, as ϕ is monic, μ' is monic. Hence, $H^p(f) = \mu'\mu$ is a monomorphism.

To conclude this appendix, we give a proof of Proposition I.6. We recall the statement of this proposition :

Let $\mathcal{A}$ be an abelian category of finite injective homological dimension, let $\mathcal{I}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects of $\mathcal{A}$, and let $E : K(\mathcal{I}) \longrightarrow K(\mathcal{A})$ be the embedding functor. Then :

> There exists a functor $J : K(\mathcal{A}) \longrightarrow K(\mathcal{I})$ and a natural transformation $i : Id_{K(\mathcal{A})} \longrightarrow EJ$ such that, for each $X^{\bullet} \in |K(\mathcal{A})|$:
>
> (i) i_X is a quasi-isomorphism
>
> (ii) i_X is a monomorphism.

<u>Proof</u>.

If we take $\mathbb{P} = |\mathcal{I}|$, we see that the axioms (i), (ii) and (iii) of Lemma I. 1 b) are fulfilled. Hence, it is sufficient to prove that the association $X^{\bullet} \longrightarrow JX^{\bullet} = I^{\bullet}$ constructed in Lemma I. 1 b) is functorial. (The fact that i_X is monic results from the construction of $I^{\bullet}$). We keep the same notations as in the proof of Lemma I. 1 b).

1. <u>The association $X^{\bullet} \longrightarrow X^{\bullet}_{o}$ is functorial</u>.

Given a map $f : X^{\bullet} \longrightarrow Y^{\bullet}$, let $I^{\bullet}$ (resp. $J^{\bullet}$) be the cochain complex consisting of injectives quasi-isomorphic to the truncated complex $X^{o} \longrightarrow X^{1} \longrightarrow \ldots$ (resp. $Y^{o} \longrightarrow Y^{1} \longrightarrow \ldots$). Then, by the functoriality of the first part of the lemma (Proposition I. 2), f induces a map $f_{*} : I^{\bullet} \longrightarrow J^{\bullet}$. Hence, f induces a map $X^{\bullet} \longrightarrow J^{\bullet}$ as follows :

34.
$$\begin{tikzcd}
\cdots \arrow[r] & X^{i_o-2} \arrow[r] \arrow[d, "f"] & X^{i_o-1} \arrow[r, "i\ d_X"] \arrow[d, "f"] \arrow[dr, phantom, "(1)"] & I^{i_o} \arrow[r] \arrow[d, "f_*"] & I^{i_o+1} \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & Y^{i_o-2} \arrow[r] & Y^{i_o-1} \arrow[r, "j\ d_Y"] & J^{i_o} \arrow[r] & J^{i_o+1} \arrow[r] & \cdots
\end{tikzcd}$$

The square (1) of diagram 34. commutes, since both squares commute in

35.
$$\begin{tikzcd}
X^{i_o-1} \arrow[r, "d_X"] \arrow[d, "f"] & X^{i_o} \arrow[r, tail, "i"] \arrow[d, "f"] & I^{i_o} \arrow[d, "f_*"] \\
Y^{i_o-1} \arrow[r, "d_Y"] & Y^{i_o} \arrow[r, tail, "j"] & J^{i_o}
\end{tikzcd}$$

2. <u>The association $X_1^\bullet \longrightarrow X_2^\bullet$ of the second step is functorial</u>.

By construction of X'', the association $X_1^\bullet \longrightarrow X''$ is functorial, and $X_2^\bullet$ is defined (if we suppose $i_o+n = 0$ as before) by

$$X_2^i = \begin{cases} X'^i & \text{for } i < 0 \\ K^o & \text{for } i = 0 \\ X_1^i & \text{for } i > 0 \end{cases}$$

where K^o is defined by the pushout diagram

36.
$$\begin{tikzcd}
 & X_1^{-1} \arrow[r] \arrow[dl] \arrow[d] & X_1^o \arrow[d] \\
X'^{-1} \arrow[r] & B^o(X'') \arrow[r] & K^o
\end{tikzcd}$$

Then, given a co-chain map $f : X_1^\bullet \longrightarrow Y_1^\bullet$ we have to find an induced map $\tilde{f} : X_2^\bullet \longrightarrow Y_2^\bullet$.

We already know that f induces $f_* : X'' \longrightarrow Y''$. Hence, we have to define $\tilde{f} : K^o \longrightarrow L^o$ such that

37.
$$\begin{tikzcd}
\cdots \arrow[r] & X'^{-2} \arrow[r] \arrow[d, "f_*"'] & X'^{-1} \arrow[r] \arrow[d, "f_*"'] & K^o \arrow[r] \arrow[d, dashed, "\tilde{f}"'] & X_1^1 \arrow[r] \arrow[d, "f"'] & X_1^2 \arrow[r] \arrow[d, "f"'] & \cdots \\
\cdots \arrow[r] & Y'^{-2} \arrow[r] & Y'^{-1} \arrow[r] & L^o \arrow[r] & Y_1^1 \arrow[r] & Y_1^2 \arrow[r] & \cdots
\end{tikzcd}$$

commutes. This map is given by the following diagram (defining K^o and L^o), where we make use of the universal property of the pushouts :

38.
$$\begin{tikzcd}
 & & X_1^{-1} \arrow[rr] \arrow[dl] \arrow[d] \arrow[ddr] & & X_1^o \arrow[d] \arrow[ddr] \arrow[dddr] & & \\
 & X'^{-1} \arrow[r, two heads] \arrow[ddr, "f_*"'] & B^o(X'^{\cdot}) \arrow[rr] \arrow[ddr] & & K^o \arrow[rr] \arrow[ddr, dashed, "\tilde{f}"'] & & X_1^1 \arrow[ddr, "f"] \\
 & & & Y_1^{-1} \arrow[rr] \arrow[dl] \arrow[d] & & Y_1^o \arrow[d] \arrow[ddr] & \\
 & & Y'^{-1} \arrow[r, two heads] & B^o(Y'^{\cdot}) \arrow[rr] & & L^o \arrow[rr] & & Y_1^1
\end{tikzcd}$$

Hence, we have a sequence of natural maps

39.
$$\begin{tikzcd}
X^{\cdot} \arrow[r, tail] \arrow[d, "f"] & X_o^{\cdot} \arrow[r, tail] \arrow[d, "f_{*o}"] & X_1^{\cdot} \arrow[r] \arrow[d, "f_{*1}"] & \cdots \arrow[r] & \varinjlim X_r^{\cdot} = I^{\cdot} \arrow[d, dashed, "If"] \\
Y^{\cdot} \arrow[r, tail] & Y_o^{\cdot} \arrow[r, tail] & Y_1^{\cdot} \arrow[r] & \cdots \arrow[r] & \varinjlim Y_r^{\cdot} = J^{\cdot}
\end{tikzcd}$$

By universal property of $\varinjlim$, f_{*r} induce a unique map

$$If : \varinjlim X_r^{\cdot} \longrightarrow \varinjlim Y_r^{\cdot} .$$

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