ON CHAIN MAPS INDUCING ISOMORPHISMS IN HOMOLOGY

by

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ABSTRACT

Let $A$ be an abelian category, $I$ the full subcategory of $A$ consisting of injective objects of $A$, and $K(A)$ the category whose objects are cochain complexes of elements of $A$, and whose morphisms are homotopy classes of cochain maps.

In (5), lemma 4.6., p. 42, R. Hartshorne has proved that, under certain conditions, a cochain complex $X' \in |K(A)|$ can be embedded in a complex $I' \in |K(I)|$ in such a way that $I'$ has the same cohomology as $X'$.

In Chapter I we show that the construction given in the two first parts of Hartshorne's Lemma is natural i.e. there exists a functor $J : K(A) \longrightarrow K(I)$ and a natural transformation $i : \text{Id}_{K(A)} \longrightarrow EJ$ (where $E : K(I) \longrightarrow K(A)$ is the embedding functor) such that $i_X$, is injective and induces isomorphism in cohomology. The question whether the construction given in the third part of the lemma is functorial is still open.

We also prove that $J$ is left adjoint to $E$, so that $K(I)$ is a reflective subcategory of $K(A)$.

In the special case where $A$ is a category $\underline{\text{M}}$ of left $A$-modules, and $C(\underline{\text{M}})$ the category of cochain complexes in $\underline{\text{M}}$ and cochain maps (not homotopy classes), we prove the existence of a functor $C(\underline{\text{M}}) \longrightarrow C(I)$.

In Chapter II we study the natural homomorphism $\nu : \text{Hom}_A(M,L) \otimes_B N \longrightarrow \text{Hom}_A(M,L \otimes_B N)$ where $A$, $B$ are rings, and $M$, $L$, $N$ modules or chain complexes. In particular we give several sufficient conditions under which $\nu$ is an isomorphism, or
induces isomorphism in homology.

In the appendix we give a detailed proof of Hartshorne's Lemma. We think that this is useful, as no complete proof is, to our knowledge, to be found in the literature.
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CHAPTER I. COCHAIN COMPLEXES CONSISTING OF INJECTIVES.

We first introduce some notations which will be used in this chapter.

If $A$ denotes an abelian category, we denote $K(A)$ the category whose objects are cochain complexes of objects of $A$, and whose morphisms are homotopy classes of cochain maps.

A cochain complex $X^*$ is said to be bounded below if there exists an integer $N$ such $X^n = 0$ if $n > N$.

We denote by $K^-(A)$ the full subcategory of $K(A)$ consisting of the complexes bounded below.

According to (1) and (4), a subcategory $A'$ of an abelian category $A$ is thick (or is a Serre subcategory of $A$) \(^{(1)}\) if

(i) $A'$ is a full subcategory of $A$.

(ii) $A'$ is non empty.

(iii) For any short exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ in $A$, $B \in |A'|$ $\iff$ $B', B'' \in |A'|$ \(^{(2)}\)

\((\text{iii}) \text{ is equivalent to } (\text{iii})': \text{ if } X \rightarrow Y \rightarrow Z \text{ is exact in } A, \text{ then } X, Z \in |A'| \implies Y \in |A'|\).

If $A'$ is a thick subcategory of $A$ we define $K^+_{A'}(A)$ to be the full subcategory of $K(A)$ consisting of those complexes $X'$ whose cohomology objects $H^i(X')$ are in $A'$. We define $K^+_{A'}(A)$ by taking complexes bounded below.

\(^{(1)}\) Note that the definition of a thick subcategory is not absolutely clear in (5)

\(^{(2)}\) We denote by $|A'|$ the class of objects of the category $A'$. 

If $A'$ is a full subcategory of $A$, $I$ is an $A'$-injective object of $A'$ if $I \in |A'|$ and if $I$ is injective, considered as an object of $A$.

A cochain map $f : X' \rightarrow Y'$ is called a quasi-isomorphism (abbreviated quiso) if $H^i(f) : H^i(X') \rightarrow H^i(Y')$ are all isomorphisms. Similarly, we call a homotopy class a quasi-isomorphism if one (hence all) of its representatives induces isomorphisms in cohomology.

If $A$ is isomorphic to $B$, we write $A \cong B$, and if $A$ is homotopic to $B$, we write $A \sim B$.

We now state the lemma we are going to discuss ((5), Lemma 4.6, p.42):

**Lemma 1.**

Let $A$ be an abelian category.

a) Let $\mathcal{P}$ be a subclass of $|A|$, and assume

(i) every object of $A$ admits a monomorphism into an element of $\mathcal{P}$.

Then every $X' \in |K^+(A)|$ admits a monic quasi-isomorphism $i$ into a bounded below complex $I'$ of objects of $\mathcal{P}$.

b) Assume furthermore that $\mathcal{P}$ satisfies

(ii) If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence with $X \in \mathcal{P}$, then $Y \in \mathcal{P} \iff Z \in \mathcal{P}$.

(iii) There exists a positive integer $n$ such that, if $X^0 \rightarrow X^1 \rightarrow \ldots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0$ is an exact sequence, and $X^0, X^1, \ldots, X^{n-1} \in \mathcal{P}$, then $X^n \in \mathcal{P}$.

Then every $X' \in |K(A)|$ admits a quasi-isomorphism into a complex $I'$ of objects of $\mathcal{P}$.

c) Let $A'$ be a thick subcategory of $A$, and assume that $A'$ has
enough $A$-injectives. Then every $X \in \mathcal{K}_A^+(A)$ admits a quasi-isomorphism into a bounded below complex $I'$ of $A$-injective objects of $A'$.

We give the construction of the quasi $i : X' \to I'$ of Lemma 1 a). (For a complete proof of Lemma 1, see the appendix).

We may assume $X^p = 0$ for $p < 0$. Embed $X^0 \to I^0$, with $I^0$ in $\mathcal{I}$. Having defined $I^0 \to I^1 \to \ldots \to I^p$ and $i^k : X^k \to I^k$ ($k=0,\ldots, p$), we choose $I^{p+1}$ to be an element of $\mathcal{I}$ containing the pushout of

$\xymatrix{ X^p \ar[r]^{d^p} \ar[d]_{i^p} & X^{p+1} \ar[d]_{i^{p+1}} \\
 I^p \ar[r]_{Cok d^{p-1}} & I^{p+1}}$

Then, $d^p$ and $i^{p+1}$ are defined by the following commutative diagram:

$\xymatrix{ X^p \ar[rr]^{d^p} \ar[dd]_{i^p} & & X^{p+1} \ar[dd]_{i^{p+1}} \\
 & Cok d^{p-1} \ar[rr] & & K^{p+1} \\
 I^p \ar[rrrr]^{d^p} & & & & I^{p+1}}$

Now we prove that, if $\mathcal{I}$ is the full subcategory of $A$ consisting of all injective objects of $A$, then the construction given in Lemma 1 a) is functorial.

More precisely: let $A$ be an abelian category with enough injectives. Let $I$ be the full subcategory of $A$ consisting of all injective objects of $A$,
and let \( E^+ : K^+(I) \to K^+(A) \) be the embedding functor. Then

**Proposition 2.**

There exists a functor \( J^+ : K^+(A) \to K^+(I) \) and a natural transformation

\[
i : \text{Id}_{K^+(A)} \to E^+ J^+
\]

such that, for each \( X^* \in |K^+(A)| \):

(i) \( i_X \) is a quasi

(ii) \( i_X \) is monic.

\( (K^+(I)) \) denotes the full subcategory of \( K^+(A) \) whose objects \( I^* \) are such that \( I^n \in |I| \) for all \( n \).

If \( A \) has enough injectives, Lemma 1 a) clearly holds for \( IP = |I| \).

We prove that the construction is functorial. It suffices to prove that if \( X^*, Y^* \in |K^+(A)| \) and if \( I^* \) (resp. \( J^* \)) is a complex constructed from \( X^* \) (resp. \( Y^* \)), then, for any \( K^+(A) \)-morphism \( f : X^* \to Y^* \), there exists a unique \( \phi_{IJ} : I^* \to J^* \) such that the diagram

\[
\begin{array}{ccc}
X^* & \xrightarrow{i} & I^* \\
\downarrow f & & \downarrow \phi_{IJ} \\
Y^* & \xrightarrow{j} & J^*
\end{array}
\]

commutes. Indeed, let \( L^* \) be a complex constructed from \( Z^* \in |K^+(A)| \), and let \( g : Y^* \to Z^* \) be a morphism. One checks that

\[
(1) \quad \phi_{JL} \circ \phi_{IJ} = \phi_{IL}
\]
and that \( \text{Id}_X : X' \rightarrow X' \) yields

\[
\tag{2} \phi_{II} = \text{Id}_I.
\]

Moreover, if \( I_1' \) and \( I_2' \) are two complexes constructed from \( X' \), there exists a unique \( \eta_I = \phi_{I_1I_2} : I_1' \rightarrow I_2' \) such that the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{i_1} & I_1' \\
\downarrow & & \downarrow \eta_I \\
X' & \xrightarrow{i_2} & I_2'
\end{array}
\]

commutes. But \( i_1 \) and \( i_2 \) are quasi's, so that \( \eta_I \) is a quasi from a bounded below complex consisting of injectives into \( I_2' \). Hence \( \eta_I \) is an isomorphism \((5), \text{p.} \ 41\). Thus, we are allowed to identify \( I_1' \) and \( I_2' \) via the isomorphism \( \eta_I \), and we write \( J^+X' \) for \( I' \).

Finally, we have to define, for a given \( f : X' \rightarrow Y' \), an induced morphism \( J^+f : J^+X' \rightarrow J^+Y' \). Of course, we define \( J^+f = \phi_{IJ} : I' \rightarrow J' \).

If we do so, then the formulas (1) and (2) will ensure that \( J^+ \) is a functor. The only thing left to check is that the definition of \( J^+f \) is compatible with the identification made under \( \eta \), i.e. that, if \( I_1', I_2' \) (resp. \( J_1', J_2' \)) are constructed from \( X' \) (resp. \( Y' \)), then \( \phi_{I_2J_2} \eta_I = \eta_J \phi_{I_1J_1} \), which is clear since both make the following diagram commutative:

\[
\begin{array}{ccc}
X' & \xrightarrow{i_1} & I_1' \\
\downarrow f & & \downarrow j_2 \\
Y' & \xrightarrow{j_2} & J_2'
\end{array}
\]
a) Existence of $\phi_{IJ}$

We construct $\phi^p$ by induction: as $J^0$ is injective, there exists a map $\phi^0 : I^0 \to J^0$ such that

![Diagram]

commutes. Suppose now that $\phi^0, \ldots, \phi^p$ are defined, and consider

![Diagram with $\phi$ and $\bar{\phi}$]

where $\bar{I}^p = \text{Cok} d^{p-1}_I$, $\bar{J}^p = \text{Cok} d^{p-1}_J$, $K^{p+1} = \text{pushout}(d^p_X; \bar{I}^p)$, $L^{p+1} = \text{pushout}(d^p_Y; \bar{J}^p)$. Then, $\phi^{p-1}, \bar{\phi}^p$ induce $\bar{\phi}^p$ on cokernels. By universal property of the pushout, we get a map $\psi : K^{p+1} \to L^{p+1}$ such that the upper cube commutes. As $J^{p+1}$ is injective, $\psi$ induces a map $\phi^{p+1} : I^{p+1} \to J^{p+1}$ such that the square

![Diagram with $\phi$ and $\bar{\phi}$]
commutes. It is then easy to check that the bottom square of the diagram on p.6 commutes, so that $\phi$ is the required map.

b) Uniqueness.

Suppose that there exist $\phi_1$ and $\phi_2$ such that $\phi_1 i = \phi_2 i$. Consider the short exact sequence $(\ast)$: $0 \rightarrow X' \xrightarrow{i} I' \xrightarrow{p} Y' \rightarrow 0$. From the long exact sequence of $(\ast)$ in cohomology, one sees that $Y'$ is acyclic. In the diagram

we have $(\phi_1 - \phi_2)i = 0$, so that $\phi_1 - \phi_2$ factors through $Y'$: there exists a map $f : Y' \rightarrow J'$ such that $fp = \phi_1 - \phi_2$. As $Y'$ is acyclic, and $J'$ consists of injectives and is bounded below, we have $f = 0$ ((5), p.40).

Thus, $\phi_1 - \phi_2 \simeq fp \simeq 0$, and this concludes the proof of Prop. 2.

The functor $J^+$ has the other nice property to send quasi-isomorphisms into isomorphisms. More precisely:

**Proposition 3.**

If $f : X' \rightarrow Y'$ is a quasi-isomorphism in $K^+(A)$, then $J^+f : J^+X' \rightarrow J^+Y'$ is an isomorphism.
Proof.

From Prop. 2, we have a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{i_X} & E^+J^+X' \\
f \downarrow & & \downarrow E^+J^+f \\
Y' & \xrightarrow{i_Y} & E^+J^+Y'
\end{array}
\]

where \(i_X\), \(i_Y\) and \(f\) are quiso's. Thus \(J^+f\) is a quiso. But \(J^+X'\) is a bounded below complex consisting of injectives, so that \(J^+f\) is an isomorphism ((5), p.41).

We are now able to state a first universal property for \(J^+\):

Corollary 4.

The functor \(J^+: K^+(A) \rightarrow K^+(I)\), satisfies the following universal property:

For each quiso \(f: X' \rightarrow Y'\) there exists a unique homotopy class \(\phi: Y' \rightarrow E^+J^+X'\) such that the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{i_X} & E^+J^+X' \\
f \downarrow & \nearrow \phi \\
Y'
\end{array}
\]

commutes.

Proof.

a) Existence.

As \(J^+f\) is an isomorphism by previous proposition, it is sufficient to take \(\phi = (E^+J^+f)^{-1}i_Y\).
b) Uniqueness.

Suppose \( \phi' : Y' \to E^+J^+X' \) is such that \( \phi'f = i_X \), and consider the diagram

\[
\begin{array}{c}
X' \xleftarrow{i_X} E^+J^+X' \xrightarrow{i_{E^+J^+X'}} E^+J^+E^+J^+X' \\
\downarrow{f} \quad \downarrow{\phi} \quad (1) \quad \downarrow{E^+J^+\phi} \\
Y' \xrightarrow{i_Y} E^+J^+Y'.
\end{array}
\]

As \( \phi f = \phi'f \), we have \( (J^+\phi)(J^+f) = (J^+\phi')(J^+f) \). But \( f \) is a quasi, so that \( J^+f \) is an isomorphism; thus \( J^+\phi = J^+\phi' \). By commutativity of (1), we get \( i_{E^+J^+X'} = i_{E^+J^+X'} \). \( i_{E^+J^+X'} \) being monic, we have \( \phi = \phi' \).

**Proposition 5.**

Notations being as in Prop.2, the functor \( J^+ \) is left adjoint to the embedding functor \( E^+ : K^+(I) \to K^+(A) \). In other words, \( K^+(I) \) is a reflective subcategory of \( K^+(A) \).

**Proof.**

By Prop.2, we have a natural transformation

\[
i : \text{Id}_{K^+(A)} \to E^+J^+
\]

Furthermore, for each \( Y' \in [K^+(I)] \), we have \( J^+E^+Y' = Y' \) by definition of \( J^+ \). Hence, we have a natural transformation

\[
\delta : J^+E^+ \to \text{Id}_{K^+(I)}
\]

which is the identity. Thus, it is sufficient to prove that

(i) \( \delta_{J^+} \circ J^+i = \text{Id}_{J^+} \)

(ii) \( E^+\delta \circ i_{E^+} = \text{Id}_{E^+} \)
(i) Apply Corollary 4 for $Y' = E^+J^+X'$:

\[
\begin{array}{ccc}
X' & \xrightarrow{i_X} & E^+J^+X' \\
\downarrow{\phi} & & \downarrow{\phi}
\end{array}
\]

The unique $\phi$ such that the diagram commutes is defined by

\[
\phi = (E^+J^+i_X)^{-1}i_{E^+J^+X'}.
\]

But, also $\phi = \text{Id}_{E^+J^+X'}$ makes the diagram commutative. Thus

\[
E^+J^+i_X. = i_{E^+J^+X'}, \quad \text{and, by construction of } J^+, i_{E^+J^+X.} = \text{Id}_{E^+J^+X'}, \quad \text{so that}
\]

\[
E^+J^+i_X. = \text{Id}_{E^+J^+X.} = E^+\text{Id}_{J^+X.}. \quad \text{Thus : } J^+i_X. = \text{Id}_{J^+X.}. \quad \text{As } \delta_{J^+X.} = \text{Id}_{J^+X'},
\]

we get $\delta_{J^+X.} \circ J^+i_X. = \text{Id}_{J^+X'}$, and (i) is proved. Similarly, one proves (ii).

Remark.

Corollary 4 and Proposition 5 give two different universal properties for $i_X$:

\[
\begin{array}{ccc}
X' & \xrightarrow{i_X} & E^+J^+X' \\
\downarrow{\phi} & & \downarrow{\phi}
\end{array}
\]

Corollary 4

\[
\begin{array}{ccc}
X' & \xrightarrow{i_X} & E^+J^+X' \\
\downarrow{\phi} & & \downarrow{\phi}
\end{array}
\]

Proposition 5

(Note that, in general, Prop.5 $\Rightarrow$ Cor.4).

Up to now, we have only discussed applications of the first part of Lemma 1. Looking at the second part, we get the following result:

Let $A$ be an abelian category of finite injective homological dimension, let $I$ be the full subcategory of $A$ consisting of injective objects of $A$, and let $E : K(I) \longrightarrow K(A)$ be the embedding functor. Then:
Proposition 6.

There exists a functor $J : K(A) \rightarrow K(I)$ and a natural transformation $i : \text{Id}_{K(A)} \rightarrow EJ$ such that for each $X' \in |K(A)|$

(i) $i_X$ is a quiso

(ii) $i_X$ is monic.

The proof of this proposition depends heavily on the construction of the chain complex $I'$ of Lemma 1 b), so that it will be done in the appendix. The question whether $J$ is left adjoint to the embedding $E$ is still open.

Now we restrict our attention to the case where $A$ is the category $\mathcal{M}_L$ of left $A$-modules (we could get similar results for the category $\mathcal{M}_R$ of right $A$-modules). In this category, we prove the existence of a natural embedding of any module into an injective module. As a consequence of this fact, we shall see that, in $\mathcal{M}_L$, the functor $J^+$ of Prop. 2 may be considered as a functor $C^+(\mathcal{M}) \rightarrow C^+(I)$, where $C^+(A)$ denotes the category whose objects are cochain complexes of objects of $A$ and whose morphisms are cochain maps. Thus here we are working with cochain maps instead of homotopy classes of cochain maps.

Lemma 7.

For any $A \in |\mathcal{M}_L|$, let $A^*$ be the right $A$-module

$A^* = \text{Hom}_{\mathbb{Z}}(A \ ; \ \mathbb{Q}/\mathbb{Z})$

Then the map $\psi : A \rightarrow A^{**}$ defined, for $a \in A$, $f \in A^*$, by

$\psi(a)(f) = f(a)$

is a natural embedding.
Proof.

1. It is clear that \( \psi \in \text{Hom}_A(A;A^{**}) \). (Note that \( A \in |\mathcal{M}_A| \) implies \( A^* \in |\mathcal{M}_A| \)
which implies \( A^{**} \in |\mathcal{M}_A| \).

2. \( \psi \) is a monomorphism.

Let \( a \neq 0 \) be an element of \( A \), and let \( (a) \) be the submodule of \( A \) generated by \( a \). We define \( \alpha : (a) \rightarrow \mathbb{Q}/\mathbb{Z} \) as follows:

- If \( a \) is of infinite order, \( \alpha(a) \) is any non-zero element of \( \mathbb{Q}/\mathbb{Z} \).
- If the order of \( a \) is \( n \), \( \alpha(a) \) is an element of order dividing \( n \).

Clearly, \( \alpha \) is a well defined morphism.

Considering now the embedding \( (a) \hookrightarrow A \) we get, since \( \mathbb{Q}/\mathbb{Z} \) is injective, a non-zero map \( f : A \rightarrow \mathbb{Q}/\mathbb{Z} \) such that

\[
\begin{array}{ccc}
(a) & \longrightarrow & A \\
\downarrow \alpha & & \downarrow f \\
\mathbb{Q}/\mathbb{Z} & & \\
\end{array}
\]

commutes. Hence, for this particular morphism \( f \), we have \( \psi(a)(f) = f(a) = \alpha(a) \neq 0 \), so that \( a \neq 0 \) implies \( \psi(a) \neq 0 \).

3. \( \psi \) is natural.

For any \( f : A \rightarrow B \), we define \( f^{**} : A^{**} \rightarrow B^{**} \) for \( a \in A^{**} \) and \( \phi \in B^{**} \) by \( f^{**}(a)(\phi) = \alpha(\phi f) \). It is then easy to see that

\[
\begin{array}{ccc}
A & \longrightarrow & A^{**} \\
\downarrow f & & \downarrow f^{**} \\
B & \longrightarrow & B^{**} \\
\end{array}
\]

commutes.
Theorem 8.

Every $\Lambda$-module can be embedded in a natural way into a cofree (hence injective) module. In other words, there exists a functor $L: {}_{\Lambda}\mathcal{M} \rightarrow {}_{\Lambda}\mathcal{M}$ and a natural transformation $\mu : \text{Id} {}_{\Lambda}\mathcal{M} \rightarrow L$ such that, for each $\Lambda$-module $M$,

(i) $LM$ is a cofree module,

(ii) $\mu_M$ is a monomorphism.

Proof.

We know that every $\Lambda$-module $A$ is quotient of the free module generated by the underlying set of $A$. Hence, there exists a natural epimorphism $\phi : \bigoplus_{x \in M} A(x) \rightarrow A$. Thus the composition

$$M \rightarrow M^{**} = \text{Hom}_{\mathbb{Z}}(M^*; \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\phi; \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\bigoplus_{x \in M} A(x); \mathbb{Q}/\mathbb{Z}) \Rightarrow$$

$$\cong \rightarrow \prod_{x \in A^*} \text{Hom}_{\mathbb{Z}}(A(x); \mathbb{Q}/\mathbb{Z}) = LM$$

is a natural embedding and, by definition, $LM$ is a cofree module.

We can now apply theorem 8 to Hartshorne's construction to get the following result:

Let $C^+(\mathcal{M})$ be the category whose objects are bounded below cochain complexes of elements in $\mathcal{M}$, and whose morphisms are cochain maps. If $I$ denotes the full subcategory of $\mathcal{M}$ consisting of injective modules, and if $E^+ : C^+(I) \rightarrow C^+(\Lambda \mathcal{M})$ is the embedding functor, then
Proposition 9.

There exists a functor $J^+ : C^+(\mathcal{M}) \rightarrow C^+(I)$ together with a natural transformation

$$i : \text{Id}_{C^+(\mathcal{M})} \rightarrow E^+J^+$$

such that, for each $X' \in |C^+(\mathcal{M})|$, 

(i) $i_X$ is a quasi

(ii) $i_X$ is monic.

Remark.

Note that, if we work in the category $C^+(\mathcal{M})$ (and not in $K^+(\mathcal{M})$), the functor $J^+$ is no more left adjoint to the embedding functor: if $J^+$ were left adjoint to $E^+$, then the counit

$$\delta : J^+E^+ \rightarrow \text{Id}_{C^+(I)}$$

of the adjunction would be an isomorphism since $E^+$ is full and faithful ((8), p. 88). As $E^+\delta \circ i_{E^+} = \text{Id}_{E^+}$, $i_{E^+}$ would be an isomorphism; whence, to prove that $J^+$ is not left adjoint to $E^+$, it is sufficient to show that $i_{E^+}$ is not an isomorphism.

First, note that if $\text{Hom}(\beta; \mathbb{Q}/\mathbb{Z}) : \text{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(B; \mathbb{Q}/\mathbb{Z})$ is epic, then $\beta : B \rightarrow A$ is monic. Indeed, suppose that $x$ is a non-zero element of $B$ such that $\beta(x) = 0$. By an argument similar to the one used in the proof of Lemma 7, one shows that there exists $f : B \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $f(x) \neq 0$. As $\text{Hom}(\beta; \mathbb{Q}/\mathbb{Z})$ is epic, there exists $g : A \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $g\beta = f$. Then, $f(x) = g(\beta(x)) = 0$ since $\beta(x) = 0$, contradicting the fact that $f(x) \neq 0$.

Also, if $A$ is a non-zero $\Lambda$-module, the canonical epimorphism
is not monic, so that, for a non-zero $\Lambda$-module $M$,
\[
\text{Hom}_{\mathbb{Z}}(\phi; \mathbb{Q}/\mathbb{Z}) : \text{Hom}_{\mathbb{Z}}(M^*; \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}\left( \bigoplus_{x \in M^*} \Lambda(x); \mathbb{Q}/\mathbb{Z} \right)
\]
is not epic, and the map $\nu_M : M \longrightarrow \text{IM}$ of theorem 8 is not an isomorphism.

Thus, the map $i_{E^+_X} : E^+_X \longrightarrow E^+_X$ is not an isomorphism.

In Chapter II, we shall use the dual of Lemma 1; so let's make some remarks about the dual situation.

As the proof of Lemma 1 invokes only properties of abelian categories and as the dual of any abelian category is abelian, the dual of lemma 1 holds. We rewrite it in details:

**Lemma 1**

Let $A$ be an abelian category.

a) Let $\mathcal{P}$ be a subclass of $|A|$, and assume

(i) For each object $X$ of $A$, there exists an element $P$ of $\mathcal{P}$ and an epimorphism $P \longrightarrow A$.

Then, for every $X \in |X(A)^{op}|$ (where $C^{op}$ represents the dual of $C$), there exists a bounded below chain complex $P$. of elements of $\mathcal{P}$ together with a quasi-isomorphism $\pi : P. \longrightarrow A$, with each $\pi_n$ epic.

b) Assume furthermore that $\mathcal{P}$ satisfies

(ii) If $0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$ is a short exact sequence with $X \in \mathcal{P}$, then $Y \in \mathcal{P} \iff Z \in \mathcal{P}$, and

(iii) There exists a positive integer $n$ such that, if $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_0$ is an exact sequence, and
X_0, \ldots, X_{n-1} \in \mathcal{P}, \text{ then } X_n \in \mathcal{P}.

Then, for every \( X \in |K(A)^{\text{op}}| \) there exists a complex \( P \) of elements of \( \mathcal{P} \) and a quasi \( P \rightarrow X \).

c) Let \( A' \) be a thick subcategory of \( A \), and assume that \( A' \) has enough \( A \)-projectives. Then, for every \( X \in |K^+_A(A)^{\text{op}}| \) there exists a bounded below complex \( P \) of \( A \)-projective objects of \( A' \) and a quasi \( P \rightarrow X \).

Remarks.

1. Consider the particular case \( A = \mathcal{M} \). If \( \mathcal{P} \) is the class of projective modules of \( \mathcal{M} \), then the condition (i) holds, since \( \mathcal{M} \) has enough projectives. Furthermore, if the sequence \( 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \) is exact and if \( X \) is projective, then the sequence splits, so that \( Y = Z \oplus X \). Thus, \( Y \) is projective if and only if \( Z \) is projective, and (ii) holds. Also, by definition, (iii) holds if the left homological dimension of \( \Lambda \) is finite.

Finally, the category \( A' \) of finitely generated \( \Lambda \)-modules is a thick subcategory of \( \mathcal{M} \) since if \( 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \) is exact, then \( A' \) and \( A'' \) are finitely generated if and only if \( A \) is finitely generated.

Also, \( A' \) has enough \( A \)-projectives since every finitely generated module is a quotient of a finitely generated projective module. Hence, we may write Lemma 1\(^d\) c) as follows for the case of modules:

For every bounded below chain complex \( X \) such that all \( H_n(X) \) are finitely generated, there exists a quasi-isomorphism \( P \rightarrow X \), where \( P \) is a bounded below complex, with \( P_n \) projective and finitely generated.

2. The dual of this statement is false since in general a finitely generated
module cannot be embedded in a finitely generated injective module (take for instance $\Lambda = \mathbb{Z}$).

3. Even in $\mathcal{M}_\Lambda$, the quiso of lemma 1d a) is not a homotopy equivalence in general, as we see in the following example: take

$$X : \cdots \to 0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Then, Hartshorne's construction yields a chain complex $P.$ and a quiso $\pi$ as follows:

$$P. : \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

$$\pi \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad p$$

$$X. : \cdots \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Then, $\text{Hom}(P.;\mathbb{Z}) \cong P.$ and $\text{Hom}(X.;\mathbb{Z}) = 0$, since $\mathbb{Z}/2\mathbb{Z}$ is a torsion group. Hence, $H^n(\text{Hom}(P.;\mathbb{Z}))$ is not isomorphic to $H^n(\text{Hom}(X.;\mathbb{Z}))$, which proves that $P.$ and $X.$ are not homotopy equivalent.

As a final remark, note that the dual of Propositions 2, 3 and Corollary 4 hold. There is also a dual construction for theorem 8, since every $\Lambda$-module $M$ is a quotient of $\bigoplus_{x \in M} \Lambda(x)$. Hence, the dual of Proposition 9 also holds.
CHAPTER II. THE GROUP HOMOMORPHISM \( \nu : \text{Hom}_A(M;L) \otimes_B N \to \text{Hom}_A(M;L \otimes_B N) \).

In this chapter, we denote \( \mathcal{M}_A \) the category of left \( A \)-modules, and \( \mathcal{M}_{A,B} \) the category of \((A,B)\)-bimodules. \((M \in \mathcal{M}_A)\) means that \( M \) is a left \( A \)-module, a right \( B \)-module, and that \((ax)b = a(xb)\) for all \( a \in A, x \in M, b \in B\).

We first prove the existence of a group homomorphism

\[
\nu : \text{Hom}_A(M;L) \otimes_B N \to \text{Hom}_A(M;L \otimes_B N) :
\]

Let \( A, B \) be two rings, \( M \in \mathcal{M}_A, L \in \mathcal{M}_{A,B}, N \in \mathcal{M}_B \). Then

**Proposition 1.**

The map \( \nu : \text{Hom}_A(M;L) \otimes_B N \to \text{Hom}_A(M;L \otimes_B N) \) defined by

\[
\nu(u \otimes y)(x) = u(x) \otimes y \quad (u \in \text{Hom}_A(M;L), y \in N, x \in M)
\]

is a group homomorphism, natural in \( M, L, N \).

**Proof.**

\( \nu \) is well defined: for \( y \in N \) and \( u \in \text{Hom}_A(M;L) \), define

\[
\nu'(u,y) : M \to L \otimes_B N
\]

by \( \nu'(u,y)(x) = u(x) \otimes y \quad (x \in M) \). Then, by definition of the left \( A \)-module structure of \( L \otimes_B N \), \( \nu'(u,y) \in \text{Hom}_A(M;L \otimes_B N) \). Furthermore, one checks directly that

\[
\nu' : \text{Hom}_A(M;L) \times N \to \text{Hom}_A(M;L \otimes_B N)
\]

\[
(u,y) \mapsto \nu'(u,y)
\]

is bilinear and \( B \)-balanced. By the universal property of \( \otimes_B \), \( \nu' \) induces the group homomorphism \( \nu \).
\( \nu \) is natural in \( M \). Indeed, for each \( f: M' \to M \), the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Hom}_A(M; L \otimes B N) & \xrightarrow{\nu_M} & \text{Hom}_A(M; L \otimes B N) \\
\downarrow & & \downarrow \\
\text{Hom}_A(f; L \otimes B N) & \xrightarrow{\nu_{M'}} & \text{Hom}_A(f; L \otimes B N)
\end{array}
\]

results immediately of the definitions.

Similarly, one verifies the naturality in \( L \) and \( N \).

**Convention.**

To the end of this chapter, \( M, L, N, A, B, \nu \) will have the same meaning as in Proposition 1.

In general, \( \nu \) is neither an epimorphism, nor a monomorphism. We give a series of sufficient conditions under which \( \nu \) is an isomorphism. These conditions will be summarized at the end of the chapter.

**Proposition 2.**

If \( N \) is projective (resp. finitely generated projective), then \( \nu \) is a monomorphism (resp. \( \nu \) is an isomorphism).

**Proof.**

We first prove the proposition when \( N \) is free (resp. finitely generated free). In this case, \( N \) can be identified to a direct sum of copies of the ring \( B \):

\[ N = \bigoplus_{I} B \] (if \( N \) is finitely generated, the set \( I \) of indices can be chosen finite).
Consider the following diagram:

\[
\begin{array}{cccccc}
\text{Hom}_A(M; L) \oplus_B I & \xrightarrow{\nu} & \text{Hom}_A(M; L \otimes_B I) \\
\uparrow{\alpha} & & \downarrow{\beta} \\
\bigoplus_I \text{Hom}_A(M; L) & \xrightarrow{i} & \prod_I \text{Hom}_A(M; L)
\end{array}
\]

where \(\alpha\) is the canonical isomorphism, defined by \(\alpha(u \otimes (y_i)_I) = (uy_i)_I\), \(j\) the embedding, and \(\beta\) is defined, for \(f \in \text{Hom}_A(M; L \otimes_B I)\) such that \(f(x) = u(x) \otimes (y_i)_I\), by \(\beta(f) = (f_i)_I\), with \(f_i(x) = u(x)y_i\). \(\beta\) is a monomorphism, as composition of the following natural maps:

\[
\begin{array}{c}
\text{Hom}_A(M; L \otimes_B I) \xrightarrow{\text{embedding}} \text{Hom}_A(M; \bigoplus_I L) \\
\xrightarrow{\text{embedding}} \prod_I \text{Hom}_A(M; L)
\end{array}
\]

One checks that (2) commutes, so that \(\beta \nu = j \alpha\) is a monomorphism. Hence, \(\nu\) is a monomorphism.

If \(N\) is finitely generated, the set \(I\) is finite, thus since \(\bigoplus X \cong \prod X\), \(j\) and \(\beta\) are isomorphisms, and \(\nu\) is also an isomorphism, and the proposition is proved when \(N\) is free.

Suppose now \(N\) projective. Then \(N\) is a direct summand in a free module, i.e. there exist a module \(R\) and a free module \(F\) such that \(F = N \oplus R\). But \(\nu_F\) is an isomorphism if and only if \(\nu_N\) and \(\nu_R\) are isomorphisms. Indeed, consider the following diagram:
where $\phi$ and $\phi'$ are the canonical isomorphisms.

One checks that (3) commutes, hence $\nu_F$ is an isomorphism if and only if $\nu_N \oplus \nu_R$ is an isomorphism. But $\nu_N \oplus \nu_R$ is an isomorphism if and only if $\nu_N$ and $\nu_R$ are both isomorphisms. This concludes the proof of Proposition 2.

We have an analogous result when we put the assumptions on the module $M$:

Proposition 3.

If $M$ is finitely generated and projective, then $\nu$ is an isomorphism.

Proof.

As for Proposition 2, we first suppose that $M$ is finitely generated free, so that we can write $M = \bigoplus_{i=1}^{n} A$. Then, if $\alpha$, $\beta$ and $\gamma$ are the canonical isomorphisms, the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}_A((\bigoplus_{i=1}^{n} A) \otimes_B N) & \xrightarrow{\nu} & \text{Hom}_A((\bigoplus_{i=1}^{n} A ; L \otimes_B N)\\
\bigoplus_{i=1}^{n} L & \xrightarrow{\gamma} & \bigoplus_{i=1}^{n} (L \otimes_B N)
\end{array}
\]
Hence, $v$ is an isomorphism if $M$ is free.

We pass from the case $M$ free to the case $M$ projective by the same argument as in the proof of Proposition 2.

Now, we "spread" the two conditions onto $M$ and $N$. In order to prove our next result, we need a

**Lemma:**

If $M$ is finitely generated, then there exists a natural isomorphism

$$
\phi : \text{Hom}_A(M \oplus I \ L) \rightarrow \bigoplus_I \text{Hom}_A(M;L)
$$

defined, for each $f \in \text{Hom}_A(M \oplus I \ L)$ such that $f(x) = (y_i)_I$

defined by $\phi(f) = (f_i)_I$, where $f_i$ is defined by $f_i(x) = y_i$.

**Proof.**

$\phi$ is well defined: if $M$ is generated by $x_1, \ldots, x_n$, and if $f : M \rightarrow \bigoplus_I L$ is such that $f(x) = (y_i)_I$, then $f$ is entirely determined by $f(x_i) = (y_{ij})_{i \in I}$. But, by definition of $\bigoplus_I L$, $y_{ij}$ is nonzero for only finitely many $i \in I$. This implies that $f_i$ is nonzero for only finitely many indices $i \in I$, thus $(f_i)_I \in \bigoplus_I \text{Hom}_A(M;L)$.

One checks directly that $\phi$ is a monomorphism and an epimorphism.

**Proposition 4.**

If $M$ is finitely generated and if $N$ is projective, then $v$ is an isomorphism.
Proof.

If $N$ is free: $N = \bigoplus_I B$, the proposition follows from the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Hom}_A(M;L) \otimes_B \bigoplus_I B & \xrightarrow{\psi} & \text{Hom}_A(M;L \otimes_B \bigoplus_I B) \\
\downarrow & & \downarrow \\
\bigoplus_I \text{Hom}_A(M;L) & \xleftarrow{\phi} & \text{Hom}_A(M; \bigoplus_I L)
\end{array}
\]

(5)

If $N$ is projective, we express it as a direct summand in a free module, and we proceed as in Proposition 2.

We would like now to interchange the two conditions on $M$ and $N$. But it seems that the condition "finitely generated" is not strong enough to prove that $\psi$ is an isomorphism; we substitute to it the stronger condition "finitely presented" (these conditions are equivalent if the ring is Noetherian). But, first, we need a

Lemma:

If $E \in \bigcup_{\Lambda} M_i$ is finitely presented, and if $M_i \in \bigcup_{\Lambda} M_i$, $i \in I$, then the natural homomorphism

$$\phi : \bigcap_I M_i \otimes_{\Lambda} E \longrightarrow \bigcap_I (M_i \otimes_{\Lambda} E)$$

is an isomorphism.
Proof.

We first suppose that $E$ is finitely generated free: $E = \bigoplus I \Lambda$. Then

\[
\left( \prod_I M_i \right) \otimes \Lambda \approx \bigoplus_I \left( \prod_I M_i \right) \otimes \Lambda \approx \bigoplus_I \left( \prod_I M_i \right) \otimes \sum_I \Lambda \approx \bigoplus_I \left( \prod_I M_i \right) \otimes \Lambda.
\]

Consider now the general case, and let $L_1 \longrightarrow L_0 \longrightarrow E \longrightarrow 0$ be a finite presentation of $E$ ($L_1 \longrightarrow L_0 \longrightarrow E \longrightarrow 0$ is an exact sequence with $L_0$ and $L_1$ finitely generated free).

As the functor $X \otimes \Lambda$ is right exact, and as, in $M$, the product of exact sequences is still an exact sequence (2, p.A II 10), we get the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
( \prod_I M_i ) \otimes \Lambda L_1 & \longrightarrow & ( \prod_I M_i ) \otimes \Lambda L_0 & \longrightarrow & ( \prod_I M_i ) \otimes \Lambda E & \longrightarrow & 0 \\
\downarrow \phi_{L_1} & & \downarrow \phi_{L_0} & & \downarrow \phi_E & & \\
\prod_I (M_i \otimes \Lambda L_1) & \longrightarrow & \prod_I (M_i \otimes \Lambda L_0) & \longrightarrow & \prod_I (M_i \otimes \Lambda E) & \longrightarrow & 0
\end{array}
\]

As $\phi_{L_0}$ and $\phi_{L_1}$ are isomorphisms, $\phi_E$ is also an isomorphism.

Proposition 5.

If $M$ is projective and $N$ finitely presented, then $\nu$ is an isomorphism.

Proof.

As in Proposition 3, it is sufficient to give a proof for the case where $M$
is free: \( M = \bigoplus_I A \). But in this case, the result is immediate if we consider the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_A \left( \bigoplus_I A \otimes_B N \right) & \xrightarrow{\nu} & \text{Hom}_A \left( \bigoplus_I A \otimes_B N \right) \\
\downarrow & & \downarrow \\
\left( \bigoplus_I L \right) \otimes_B N & \xrightarrow{\phi} & \left( \bigoplus_I \left( L \otimes_B N \right) \right)
\end{array}
\]

(7)

**Remark.**

Over a Noetherian ring, "finitely presented" is equivalent to "finitely generated" ([3], p.36). Hence, if \( B \) is Noetherian, by Propositions 4 and 5, \( \nu \) is an isomorphism as soon as the hypotheses "finitely generated" and "projective" are distributed in any of the four possible ways on \( M \) and \( N \).

We state now a last proposition without assumptions on the bimodule \( L \):

**Proposition 6.**

If \( M \) is finitely generated (resp. finitely presented), and if \( N \) is flat, then \( \nu \) is a monomorphism (resp. an isomorphism).

**Proof.**

Let \( F_1 \twoheadrightarrow F_0 \twoheadrightarrow M \rightarrow 0 \) be a free presentation of \( M \), with \( F_0, F_1 \) free and \( F_0 \) finitely generated, and consider the commutative diagram
\[ 0 \longrightarrow \text{Hom}_A(M;L) \otimes_B N \longrightarrow \text{Hom}_A(F_0;L) \otimes_B N \longrightarrow \text{Hom}_A(F_1;L) \otimes_B N \]

(8)

\[ 0 \longrightarrow \text{Hom}_A(M;L \otimes_B N) \longrightarrow \text{Hom}_A(F_0;L \otimes_B N) \longrightarrow \text{Hom}_A(F_1;L \otimes_B N) \]

(The top row is exact since \(- \otimes_B N\) is an exact functor, and the bottom row is exact since \(\text{Hom}_A(-; L \otimes_B N)\) is left exact).

By Proposition 3, \(\nu_{F_0}\) is an isomorphism, hence \(\nu_M\) is a monomorphism.

If \(M\) is finitely presented, \(F_1\) can be chosen finitely generated, so that \(\nu_{F_1}\) is also an isomorphism. In this case, the map \(\nu_M\) induced on the kernels is an isomorphism.

Remarks.

1. As \(\nu\) is natural in \(M, L\) and \(N\), the propositions 1 to 6 admit an immediate generalization to the case where \(M, L\) or \(N\) are (co)-chain complexes.

2. We obtain interesting particular cases for Propositions 1 to 6 by taking \(A = B = L = \Lambda\). Then \(\nu\) becomes

\[ \psi : \text{Hom}_\Lambda(M;\Lambda) \otimes_\Lambda N \longrightarrow \text{Hom}_\Lambda(M;N) \]

and is defined by \(\psi(f \otimes y)(x) = f(x)y\).

If, as usual, we denote \(M^* = \text{Hom}_\Lambda(M;\Lambda)\) (\(M^*\) = dual of \(M\)), the map \(\psi\) takes the very simple form

\[ \psi : M^* \otimes_\Lambda N \longrightarrow \text{Hom}_\Lambda(M;N) \]

(If \(\Lambda\) is a field, and if \(M\) is a finite-dimensional vector space over \(\Lambda\), then \(\psi\) is an isomorphism which is well known in tensor calculus).
We shall now use the previous propositions to prove a few statements about the behaviour of an injective module with respect to the tensor product:

**Proposition 7.**

Let $A$ be a left Noetherian ring, $L \in \mathcal{M}_A$ injective as $A$-module, and $N \in \mathcal{M}_B$ flat. Then, $L \otimes_B N$ is an injective $A$-module.

**Proof.**

We recall that a left $A$-module $M$ is injective if and only if, for each left ideal $I$ of $A$, the morphism

$$i^* : \text{Hom}_A(A;M) \rightarrow \text{Hom}_A(I;M)$$

induced by the embedding $i : I \rightarrow A$, is an epimorphism.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_A(A;L) \otimes_B N & \rightarrow & \text{Hom}_A(I;L) \otimes_B N \\
\downarrow \psi & & \downarrow \psi \\
\text{Hom}_A(A;L \otimes_B N) & \rightarrow & \text{Hom}_A(I;L \otimes_B N)
\end{array}
\]

($\psi$ is an isomorphism since $\text{Hom}_A(A;L \otimes_B N) \cong L \otimes_B N \cong \text{Hom}_A(A;L) \otimes_B N$).

As $L$ is injective, and as the functor $- \otimes_B N$ is right exact, $\psi$ is an epimorphism. But, since $A$ is Noetherian, $I$ is finitely presented. By Proposition 6, as $N$ is flat, $\nu_i$ is an isomorphism. Thus, $i^*$ is an epimorphism, and $L \otimes_B N$ is injective.

If $A$ is a PID, the proposition is true without the assumption $N$ flat:
Proposition 8.

Let $A$ be a PID, $L \in \text{M}_A^B$ injective as $A$-module, and let $N \in \text{M}_B$. Then $L \otimes_B N$ is an injective $A$-module.

Proof.

If $A$ is a PID, then $I$ is free and finitely generated and, by Proposition 3, $\nu_I$ is an isomorphism (even if $N$ is not flat).

Remark.

If we take the particular case $A = \mathbb{Z}$, we get the following result:

If $L \in \text{M}_B$ is injective as abelian group, and if $N \in \text{M}_B$, then $L \otimes_B N$ is an injective abelian group.

We also have a sort of usual situation:

Proposition 9.

Suppose that $B$ is a left Noetherian ring, $M \in \text{M}_A$ is projective, and $L \in \text{M}_A^B$ is flat as right $B$-module. Then $\text{Hom}_A(M;L)$ is a flat right $B$-module.

Proof.

By (3), Proposition 1, p.26, $\text{Hom}_A(M;L)$ is flat if and only if, for each left ideal $I$ of $B$, the canonical morphism

$$\phi : \text{Hom}_A(M;L) \otimes_B I \rightarrow \text{Hom}_A(M;L) \otimes_B B$$

is a monomorphism. Again by (3), p.36, if $B$ is left Noetherian, then $I$ is finitely presented.

Consider the commutative diagram
By Proposition 5, $\psi_M$ is an isomorphism. Furthermore, $L$ flat entails that $L \otimes_B I \rightarrow L \otimes_B B$ is a monomorphism. As $\text{Hom}_A(M; -)$ is left exact, then $\psi$ is monic. Hence $\phi$ is a monomorphism, and $\text{Hom}_A(M; L)$ is flat.

As before, if $B$ is a PID, then Proposition 9 holds without the assumption that $M$ be projective since, by Proposition 2, $I$ projective implies that $\nu_M$ is a monomorphism. Thus:

**Proposition 10.**

Suppose $B$ is a PID, and $L \in A_B$ is flat as right $B$-module.

Then $\text{Hom}_A(M; L)$ is a flat right $B$-module.

We conclude this chapter by a brief discussion of the case where $M$ or $N$ are complexes. We give some sufficient conditions for $\nu$ to induce isomorphism in homology.

**Proposition 11.**

Let $A$ be a left Noetherian ring, $M$ a bounded below chain complex of objects in $A_B$ such $H_n(M)$ is finitely generated for all $n$.

If $L$ is injective as $A$-module, and if $N$ is flat, then $\nu$ is a quasi-isomorphism, i.e. $H_n(\nu)$ is an isomorphism for all $n$. 

\[
\begin{array}{ccc}
\text{Hom}_A(M; L) \otimes_B I & \xrightarrow{\phi} & \text{Hom}_A(M; L) \otimes_B B \\
\downarrow{\nu_M} & & \downarrow{\beta} \\
\text{Hom}_A(M; L \otimes_B I) & \xrightarrow{\psi} & \text{Hom}_A(M; L \otimes_B B)
\end{array}
\]

($\beta$ is an isomorphism since $\text{Hom}_A(M; L) \otimes_B B \cong \text{Hom}_A(M; L) \cong \text{Hom}_A(M; L \otimes_B B)$).
Proof.

By Remark 1, p.16, we know that for every bounded below chain complex $M$ such that $H_n(M)$ is finitely generated for all $n$, there exists a quasi-isomorphism $\pi : P \rightarrow M$ where $P$ is a bounded below chain complex with $P_n$ projective and finitely generated for all $n$.

By Proposition 7, $L \otimes_B N$ is an injective $A$-module. Hence the functors $\text{Hom}_A(-;L)$ and $\text{Hom}_A(-;L \otimes_B N)$ as well as $- \otimes_B N$ are exact. Moreover, we know that if a functor $T$ is exact, and if $f : X \rightarrow Y$ is a chain map, then $T(H_n(f)) = H_n(Tf)$ so that exact functors preserve the quasi-isomorphisms. Proposition 11 results now from Proposition 3 and from the commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_A(M;L) \otimes_B N & \xrightarrow{\nu_M} & \text{Hom}_A(M;L \otimes_B N) \\
\downarrow \text{quiso} & & \downarrow \text{quiso} \\
\text{Hom}_A(P;L) \otimes_B N & \xrightarrow{\nu_P} & \text{Hom}_A(P;L \otimes_B N)
\end{array}
$$

By similar arguments, using Proposition 9, one proves:

Proposition 12.

Let $B$ be a left Noetherian ring, $N$ a bounded below chain complex of objects in $\mathcal{M}$ such that $H_n(N)$ is finitely generated for all $n$. If $L$ is flat as $B$-module, and if $M$ is projective, then $\nu$ is a quasi-isomorphism.

Now, we summarize our results in the following list:
\[ \nu : \text{Hom}_A(\mathcal{A}; \mathcal{A}_B) \otimes_B B_\mathcal{N} \rightarrow \text{Hom}_A(\mathcal{A}; \mathcal{A}_B \otimes_B B_\mathcal{N}) \]

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APPENDIX.

1. Proof of Lemma 1.1.

For any co-chain complex
\[ X' : \ldots \rightarrow x^{p-1} \xrightarrow{d^{p-1}} x^p \xrightarrow{d^p} x^{p+1} \rightarrow \ldots \]
we use the usual notations
\[ B^p(X') = \text{Im } d^{p-1} \quad B^p(X') = \text{Coim } d^p \]
\[ Z^p(X') = \text{Ker } d^p \quad Z^p(X') = \text{Cok } d^{p-1} \]
Thus, as objects, we have \( B^p(X') \cong B^{p+1}(X') \).

We recall Lemma 1.1 a):

Let \( A \) be an abelian category, \( \mathcal{P} \) a subclass of \( |A| \) and assume that every object of \( A \) admits an injection into an object of \( \mathcal{P} \). Then every \( X' \in \mathcal{K}^+(A) \) admits a quasi-isomorphism into a bounded below complex \( I' \) of objects of \( \mathcal{P} \).

Proof.

Given a bounded below co-chain complex \( X' \), we can assume \( x^p = 0 \) for \( p < 0 \).

Then, there exist \( I^0 \in \mathcal{P} \) and a monomorphism \( i^0 : X^0 \rightarrow I^0 \). Suppose

\[ 0 \rightarrow I^0 \xrightarrow{d^0} I^1 \rightarrow \ldots \xrightarrow{d^{p-1}} I^p \]

and \( i^k : x^k \rightarrow I^k \) (\( k = 0,1,\ldots,p \)) are defined. Then \( I^{p+1} \), \( d^p_I \) and \( i^{p+1} \) are defined by the following commutative diagram:
where (1) is a pushout diagram, $s^p$ the canonical projection, and 

$$j^{p+1} : K^{p+1} \to I^{p+1}$$

is an embedding of $K^{p+1}$ into an element of $I^p$.

It is clear from this diagram that $d^p_I d^{p-1}_I = 0$ and that

$$i^{p+1} d^p_X = d^p_I i^p.$$ Thus, we obtain a complex $I'$ and a chain map $i : X' \to I'$.

$i^p$ is a monomorphism for all $p$.

Consider the diagram
where \( \tau \) is the map induced on the images, and \( v' \) the factorization of \( d_i \) through \( \ker s \). \( i^p : x^p \to \bar{i}^p \) induces a map \( \tilde{i} : \ker \bar{i}^p \to \ker s \) such that

\[
\begin{array}{ccc}
kern \bar{i}^p & \xrightarrow{\epsilon} & x^p & \xrightarrow{\bar{i}^p} & \bar{i}^p \\
\tilde{i} & \downarrow & \downarrow & \downarrow & \downarrow \\
kern s & \xrightarrow{v''} & \bar{i}^p & \xrightarrow{s} & \bar{i}^p
\end{array}
\]

commutes.

Furthermore, as \( v'' v' = (j_\beta')(\beta' s^{p-1}) \) are two epi-mono factorizations of \( j \beta s \), we can identify \( \Im \beta \) to \( \ker s \), so that \( \tilde{i} \) may be considered as a map \( \ker \bar{i}^p \to \Im \beta \). Hence: \( j \beta'' \tilde{i} = v'' \tilde{i} = \bar{i}^p \epsilon = j \alpha \epsilon \). But \( j \) is monic, so that \( \beta'' \tilde{i} = \alpha \epsilon \).

In the diagram

\[
\begin{array}{c}
\ker \bar{i}^p \\
\xrightarrow{\psi} \\
\xrightarrow{\beta''} \\
\xrightarrow{\alpha} \\
\bar{i}^p \\
\xrightarrow{\tau} \\
\xrightarrow{\kappa} \xrightarrow{\epsilon} x^p
\end{array}
\]

the square is bicartesian (see (10), Theorem 13.4.8), hence there exists \( \psi : \ker \bar{i}^p \to \beta''(x^p) \) such that \( \kappa \psi = \epsilon \).

Finally, consider
As (1) is pushout, the map $\gamma$ induced on the kernels is epic ((10), theorem 13.4.8). But $d_x \varepsilon = (d_x \psi) \psi = 0 = \mu \gamma$. Then : $\gamma$ epic implies $\mu = 0$, i.e. $\text{Ker } \alpha = 0$, so that $\alpha$ is monic. As $j$ is also monic, $i^p = ja$ is monic.

The following step is to prove that

$$\text{Ker } \beta^p \cong H^p(I').$$

Recall the definition of $H^p(I')$:

$$\begin{array}{c}
i^p-1 \rightarrow d^p-1_I \rightarrow i^p \rightarrow d^p_I \rightarrow i^{p+1} \\
\downarrow s^p \quad \downarrow \mu^p \quad \downarrow \nu^p \\
H^p(I') \rightarrow \text{Cok } d^p-1_I \rightarrow h'^p_I \rightarrow \text{Coim } d^p_I 
\end{array}$$

We have $H^p(I') = \text{Ker } h'^p$, and $h'^ps^p = \mu^p$.

Then :

$$\begin{array}{c}
\text{Cok } d^p-1_I \rightarrow h'^p_I \\
\downarrow \beta^{p+1} \\
\text{Coim } d^p_I \cong \text{Im } d^p_I 
\end{array}$$

$$\begin{array}{c}
\downarrow \nu^p \\
k^{p+1} \rightarrow i^{p+1} \\
\downarrow \eta \quad \downarrow \nu^p \\
i^{p+1} \rightarrow i^{p+1} 
\end{array}$$

commutes : $j^{p+1} \beta^{p+1}s^p = d^p_I = \nu^p \mu^p = \nu^p h'^p s^p$. As $s^p$ is epic : $j^{p+1} \beta^{p+1} = \nu^p h'^p$. 
Thus: $\text{Ker } \beta^{p+1} = \text{Ker } j^p \beta^{p+1} = \text{Ker } v^p h^p = \text{Ker } h^p = H^p(I')$.

$H^p(i)$ is epic.

Since in the diagram 1., the square (1) is pushout, $\overline{I}^p$ induces an epimorphism $\eta^p: \text{Ker } d^p_X = Z^p(X') \rightarrow \text{Ker } \beta^p = H^p(I')$ such that

\[
\begin{array}{ccc}
Z^p(X') & \xrightarrow{\eta^p} & X^p \\
\downarrow & & \downarrow \\
H^p(I') & \xrightarrow{\overline{I}^p} & \overline{I}^p
\end{array}
\]

commutes.

Chasing the diagram

\[
\begin{array}{ccc}
x^{p-1} & \xrightarrow{B^p(X')} & Z^p(X') & \xrightarrow{\pi} & H^p(X') \\
\downarrow & & \downarrow \eta & & \downarrow \theta \\
I^{p-1} & & H^p(I') & \xrightarrow{\overline{I}^p} & \overline{I}^p
\end{array}
\]

one sees that $x^{p-1} \rightarrow B^p(X') \rightarrow Z^p(X') \rightarrow H^p(I') \rightarrow \overline{I}^p$ is the zero map.

Hence, $B^p(X') \rightarrow Z^p(X') \rightarrow H^p(I')$ is the zero map, and $\eta$ factors through $\text{Cok}(B^p(X') \rightarrow Z^p(X')) = H^p(X')$; thus, there exists an epimorphism $\theta$ such that $\theta \pi = \eta$. 
We claim that $\theta = H^P(i)$. For this, by definition of $H^P(i)$, we have to prove that

$$
H^P(X') \xrightarrow{\theta} Z'^P(X') = \text{Cok } d^{P-1}_X
$$

commutes, where $\theta$ is defined by the commutative diagram

Consider the diagram

$$
\begin{array}{c}
Z^P(X') \xrightarrow{\pi} X^P \xrightarrow{i^P} I^P \\
\downarrow \quad \downarrow \quad \downarrow \\
H^P(X') \xrightarrow{\theta} Z'^P(X') \xrightarrow{(1)} Z'^P(I') \xrightarrow{\psi} I^P = \text{Cok } d^{P-1}_I
\end{array}
$$

From previous computations, we know that (2), (3) and the large square of diagram 12. commute. By definition of $Z$, $Z'$ and $H$, the square (1) also commutes. Then (4) commutes, so that $H^P(i) = \theta$.

The last step to prove the first part of the lemma is to show that $H^P(i)$ is monic.

For this, consider the diagram
(1) is pushout by construction, (2) is pushout since \( s \) is epic, and (3) is pullback since \( j \) is monic; thus, by (10), theorem 13.4.8: \( \phi_1 \) and \( \phi_2 \) are isomorphisms, and \( \phi_3 \) is monic. Hence,

\[
\begin{array}{c}
\xymatrix{
P^X & X^{p+1} & Cok\ d_X \\
\uparrow^i & \downarrow^\alpha & \downarrow^{\phi_1} \\
P^I & X^{p+1} & Cok\ \beta \\
\downarrow^s & \downarrow^i & \downarrow^{\phi_2} \\
P^I & X^{p+1} & Cok\ \beta s \\
\downarrow^{d_I = j\beta s} & \downarrow^j & \downarrow^{\phi_3} \\
P^I & X^{p+1} & Cok\ d_I \\
\end{array}
\]

induces a monomorphism \( \phi: Cok\ d_X \rightarrow Cok\ d_I \).

Finally, look at the commutative diagram

\[
\begin{array}{c}
\xymatrix{
0 & \ar[r] & H^P(X') & \ar[r] & Cok\ d_{X}^{p-1} & \ar[r] & Coim\ d_{X}^{p} & \ar[r] & 0 \\
\downarrow \ar[r] & \ar[r] & \downarrow^\phi \ar[r] & \ar[r] & \downarrow \ar[r] & \ar[r] & \downarrow \ar[r] & \downarrow & \ar[r] & 0 \\
0 & \ar[r] & H^P(I') & \ar[r] & Cok\ d_{I}^{p-1} & \ar[r] & Coim\ d_{I}^{p} & \ar[r] & 0 \\
\end{array}
\]

As (1) commutes and \( \phi \) is monic, so is \( H^P(i) \). Hence, \( H^P(i) \) is an isomorphism and \( i \) is a quiso. This ends the proof of Lemma 1 a).

Now, we recall Lemma 1.1 b):
Let $A$ be an abelian category, let $\mathcal{IP}$ be a subclass of $|A|$, and assume

(i) Every object of $A$ admits an injection into an element of $\mathcal{IP}$.

(ii) If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence with $X \in \mathcal{IP}$, then $Y \in \mathcal{IP} \iff Z \in \mathcal{IP}$.

(iii) There exists a positive integer $n$ such that if $X^0 \rightarrow X^1 \rightarrow \ldots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0$ is an exact sequence, and $X^0, X^1, \ldots, X^{n-1} \in \mathcal{IP}$, then $X^n \in \mathcal{IP}$.

Then, every $X' \in |K(A)|$ admits a quasi-isomorphism into a complex $I'$ of objects of $\mathcal{IP}$.

Proof.

First step.

Let $X' \in |K(A)|$, and $i_0 \in \mathbb{Z}$. Then, by Lemma 1 a), we can find a quasi-isomorphism $i$ of the truncated complex

$$\ldots \rightarrow 0 \rightarrow 0 \rightarrow X^{i_0} \rightarrow X^{i_0+1} \rightarrow \ldots$$

into a complex

$$I' : \ldots \rightarrow 0 \rightarrow 0 \rightarrow I^{i_0} \rightarrow I^{i_0+1} \rightarrow \ldots$$

with each $i^p : X^p \rightarrow I^p$ monic. Define $X'_o$ to be the complex

$$\ldots \rightarrow X^{i_0-2} \xrightarrow{d_{X^{i_0-2}}} X^{i_0-1} \xrightarrow{id_{X^{i_0-1}}} I^{i_0} \xrightarrow{i} I^{i_0+1} \rightarrow \ldots$$

and consider the diagram

$$\begin{array}{cccccccccc}
X' & \ldots & X^{i_0-2} & \xrightarrow{d_X} & X^{i_0-1} & \xrightarrow{d_X} & X^{i_0} & \xrightarrow{d_X} & X^{i_0+1} & \rightarrow & \ldots \\
\downarrow \quad i & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X'_o & \ldots & X^{i_0-2} & \xrightarrow{d_X} & X^{i_0-1} & \xrightarrow{id_{X^{i_0-1}}} & I^{i_0} & \xrightarrow{d_I} & I^{i_0+1} & \rightarrow & \ldots
\end{array}$$

Then, $X^i_o \in \mathcal{IP}$ for $i \geq i_0$, and each $i^p : X^p \rightarrow X^p$ is monic.
\( i : X' \longrightarrow X_0 \) is a quasi-isomorphism.

It is clear from the construction of \( X_0 \) that \( H^n(X') \cong H^n(X'_0) \) for \( n \neq i_0 - 1 \) and for \( n \neq i_0 \).

(i) \( H^{i_0-1}(X') \cong H^{i_0-1}(X'_0) \):

Clearly, \( B^{i_0-1}(X') = B^{i_0-1}(X'_0) = \text{Im } d'^{i_0-2}_X \).

Furthermore, \( Z^{i_0-1}(X') = \text{Ker } d'^{i_0-1}_X = \text{Ker } \text{id}_X d'^{i_0-1}_X = Z^{i_0-1}(X'_0) \), since \( i \) is monic.

Hence, \( H^{i_0-1}(X') \cong H^{i_0-1}(X'_0) \).

(ii) \( H^{i_0}(X') \cong H^{i_0}(X'_0) \):

Consider the diagram

\[
\begin{array}{ccc}
X^{i_0-1} & \xrightarrow{\phi} & X^{i_0} \\
\downarrow & & \downarrow i \\
B(X') & & \text{id}_{X^{i_0}} \\
\downarrow & & \downarrow \\
B(X'_0) & & \\
\end{array}
\]

which shows readily that \( \phi \) is an isomorphism : \( B^{i_0}(X') \cong B^{i_0}(X'_0) \)

We now look at

\[
\begin{array}{cccccc}
Z^{i_0}(X') = \text{Ker } d_X & \longrightarrow & X^{i_0} & \xrightarrow{d_X} & X^{i_0+1} \\
\downarrow & & \downarrow & & \downarrow \alpha \\
Z^{i_0}(X'_0) = \text{Ker } d_I & \longrightarrow & I^{i_0} & \xrightarrow{\beta} & K^{i_0+1} \\
& & \downarrow & & \downarrow j \\
& & \downarrow d_I & & \\
& & I^{i_0+1} & & \\
\end{array}
\]
(By construction, $i_1^0 = \text{Cok } d_1^{i_0-1} = i_0^0$, since $d_1^{i_0-1} = 0$, and $j$ is monic. Thus, $\text{Ker } d_1 = \text{Ker } \beta$).

Then: (1) commutes implies $\psi$ is monic

(2) is pushout implies $\psi$ is epic (see (10), theorem 13.4.8).

Hence, $H^i_0(X') \cong H^i_0(X_0')$, and $i : X' \longrightarrow X_0'$ is a quasi-isomorphism.

Second step.

Suppose given $X_1' \in |K(A)|$ with $X_1' \in \mathbb{P}$ for $i > i_1$, and let $i_2 < i_1$.

We want to construct a quiso $X_1' \longrightarrow X_2'$ such that $X_2' \in \mathbb{P}$ for $i > i_2$, and $X_1' = X_2'$ for $i > i_1 + n$ ($n=\text{integer of condition iii}$).

By the first step, we can find a quiso $X_1' \longrightarrow X_1''$ such that $X_1'' \in \mathbb{P}$ for $i > i_2$, and $X_1' \longrightarrow X_1''$ monic. Define $Y^i = \text{Cok}(X_1' \longrightarrow X_1''$).

The long exact cohomology sequence of

$$0 \longrightarrow X_1' \longrightarrow X_1'' \longrightarrow Y' \longrightarrow 0$$

then shows that $Y'$ is acyclic, i.e. $H^p(Y') = 0$ for all $p$. Furthermore, for $i > i_1 > i_2$, $X_1'' \in \mathbb{P}$ and $X_1'' \in \mathbb{P}$. Hence, by condition (ii), $Y^i \in \mathbb{P}$ for $i > i_1$.

The exact sequence

$$Y^i \longrightarrow Y^{i+1} \longrightarrow \ldots \longrightarrow Y^{i+n-1} \longrightarrow B^{i+n}(Y') \longrightarrow 0$$

together with condition (iii) entails that $B^i(Y') \in \mathbb{P}$ for $i > i_1 + n$.

We are now ready to define $X_2'$: let $K_1$ be defined by the pushout square
and let

\[ X'_2 = \begin{cases} 
  X'_{i} & \text{for } i < i_1 + n \\
  K_{i} & \text{for } i = i_1 + n \\
  X_{i} & \text{for } i > i_1 + n
\end{cases} \]

Convention: To simplify the notations, we set \( i_1 + n = 0 \) for the proof that \( X'_2 \) is quasi-isomorphic to \( X'_1 \).

We define the complex \( X'_2 \) as follows:

\[ \ldots \rightarrow X'_{-2} \xrightarrow{d'_{X'}} X'_{-1} \xrightarrow{d'_{X_2}} K_0 \xrightarrow{d_{X_2}} X_1 \xrightarrow{d_{X_1}} X_2 \rightarrow \ldots \]

where \( d'_{X_2} \) and \( d_{X_2} \) are defined by the following diagram:

\[ \begin{array}{ccc}
  X'_{-2} & \xrightarrow{d'_{X'}} & X'_{-1} \\
  \downarrow{d_{X_2}} & & \downarrow{d_{X_2}} \\
  X'_1 & \xrightarrow{i_1} & X'_1 \\
  \downarrow{d_{X_1}} & & \downarrow{d_{X_1}} \\
  X_1 & \xrightarrow{i_1} & X_1 \\
  \downarrow{\beta} & & \downarrow{\alpha} \\
  K_0 & \xrightarrow{d_{X_1}} & K_0 \\
  \end{array} \]

i.e. \( d'_{X_2} = \beta \circ \mu \) and \( d_{X_2} \) is the unique map making the diagram commutative (universal property of the pushout).

It is immediate from this diagram that \( d'_{X_2} \circ d'_{X'} = 0 \), and that \( d_{X_2} \circ d_{X'} = 0 \).

To prove that \( d_{X_2} \circ d_{X'} = 0 \), we consider the diagram
This diagram commutes, but \( 0 : k^0 \longrightarrow x_1^2 \) makes also the diagram commutative, hence \( d_{x_1}^1 d_{x_2}^0 = 0 \); hence \( x_2^* \) is a co-chain complex.

We now construct a chain map \( x_1^* \longrightarrow x_2^* \):

\[
\begin{array}{cccccccc}
X_1^* & \longrightarrow & x_1^{-2} & \longrightarrow & x_1^{-1} & \longrightarrow & x_1^0 & \longrightarrow & x_1^1 & \longrightarrow & x_1^2 & \longrightarrow & \cdots \\
\downarrow i & & \downarrow i^{-2} & & \downarrow i^{-1} & & \downarrow \alpha^0 & & \downarrow & & \downarrow & & \\
x_2^* & \longrightarrow & x_2^{-2} & \longrightarrow & x_2^{-1} & \longrightarrow & x_2^0 & \longrightarrow & x_2^1 & \longrightarrow & x_2^2 & \longrightarrow & \cdots \\
\end{array}
\]

where \( i^{-2} \) and \( i^{-1} \) are parts of the quiso \( i : X_1^* \longrightarrow x_2^* \).

We claim that \( i : x_1^* \longrightarrow x_2^* \) is a quiso. To prove this, it suffices to show that \( H^p(i) \) is an isomorphism for \( p = -1,0,1 \). First, we show that \( \alpha^0 \) and \( \beta^0 \) are monomorphisms.

Consider the diagram 22., where \( \nu \mu \) is the epi-mono factorization of \( d_{x_2}^0 \):
As $d_{X_1}^{-1}i^{-1} = i^0d_X$, the universal property of pushouts implies the existence of a map $j^0 : K^0 \to X'^0$ such that $j^0a^o = i^0$ and $j^0\beta^0 = \nu$. But $i^0$ and $\nu$ are monomorphisms, thus $a^0$ and $\beta^0$ are also monomorphisms.

$H^{-1}(i) : H^{-1}(X'_1) \to H^{-1}(X'_2)$ is an isomorphism.

As we already know that $H^{-1}(X'_1) \cong H^{-1}(X''_1)$, we only have to prove that $H^{-1}(X''_1) \cong H^{-1}(X'_2)$. For this, consider

$$
\begin{array}{c}
\xymatrix{ X'_2 & \ldots & X'_{i-2} & X'_{i-1} & K^0 & \ldots \\
| & | & | & j^0 \\
X'' & \ldots & X''_{i-2} & X''_{i-1} & X''^0 & \ldots 
}
\end{array}
$$

Since $B^{-1}(X'_2) = B^{-1}(X''_1)$, we have to prove that $Z^{-1}(X'_2) \cong Z^{-1}(X''_1)$.

$\beta$ and $\nu$ are monomorphisms, so that

$$Z^{-1}(X'_2) = \text{Ker } \beta \mu \cong \text{Ker } \mu \cong \text{Ker } \nu \mu = \text{Ker } d_{X'_1}^{-1} = Z^{-1}(X''_1).$$

Hence : $H^{-1}(i) : H^{-1}(X'_1) \to H^{-1}(X'_2)$ is an isomorphism.
\( H^0(i) : H^0(X_1') \to H^0(X_2') \) is an isomorphism.

In the diagram

\[
\begin{array}{ccc}
X_1^{-1} & \xrightarrow{d_{X_1}} & X_1^0 \\
\downarrow & \downarrow & \downarrow \alpha \\
B^0(X') & \xrightarrow{\beta} & K^0 \\
\uparrow d_{X_2}^{-1} & \downarrow & \downarrow \\
X_1^{-1} & \xrightarrow{} & \end{array}
\]

(1) is pushout entails that \( \bar{\alpha} : Z^0(X_1') \to Z^0(X_2') \) is an isomorphism.

Consider the epi-mono factorization of \( d_{X_1}^0 \) and \( d_{X_2}^0 \):

\[
\begin{array}{ccc}
X_1^0 & \xrightarrow{d_{X_1}^0} & X_1^1 \\
\downarrow \theta & \downarrow & \downarrow \psi \\
B^1(X_1') & \xrightarrow{\phi} & X_1^1 \\
\end{array}
\]

In the diagram

\[
\begin{array}{ccc}
X_1^{-1} & \xrightarrow{d_{X_1}^{-1}} & X_1^0 \\
\downarrow & \downarrow & \downarrow \phi \\
B^0(X') & \xrightarrow{\beta} & K^0 \\
\uparrow & \downarrow \bar{\alpha} & \downarrow \phi \\
X_1^{-1} & \xrightarrow{} & \end{array}
\]

(2) commutes implies that the induced map \( \phi \) is monic

(1) is pushout implies that \( \bar{\beta} \) is an isomorphism.
Then: $d_{X_2} \beta = 0 \implies \mu \beta = 0 \implies \tilde{\phi} \mu \beta = \mu \tilde{\beta} \tilde{t} = 0$.

But $\tilde{\beta} \tilde{t}$ epic $\implies \mu = 0 \implies \mu \tilde{a} = \tilde{\phi} \mu = 0 \implies \tilde{\phi} = 0$.

Hence: $\tilde{\phi}$ epic $\implies \text{Cok } \phi = 0 \implies \phi$ is an isomorphism:

$B^0(X_1^*) \cong B^1(X_1^*) \overset{\phi}{\not\cong} B^1(X_2^*) \cong B^0(X_2^*)$, and $H^0(i): H^0(X_1^*) \to H^0(X_2^*)$ is an isomorphism.

$H^1(i): H^1(X_1^*) \to H^1(X_2^*)$ is an isomorphism.

Consider the diagram

\[
\begin{array}{cccccc}
X_0^i & \overset{\alpha}{\longrightarrow} & X_1^i & \overset{d_{X_1}}{\longrightarrow} & X_2^i \\
\downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
K_0^i & \overset{d_{X_1}}{\longrightarrow} & X_1^i & \overset{d_{X_1}}{\longrightarrow} & X_2^i.
\end{array}
\]

We have $Z^1(X_1^i) = \text{Ker } d_{X_1}^1 = Z^1(X_2^i)$. As we already know that $B^1(X_1^i) = B^1(X_2^i)$, we conclude that $H^1(i)$ is an isomorphism.

Hence: $i: X_1^i \to X_2^i$ is a quasi-isomorphism.

In order to complete the second step, we have to show that $X_2^i \in \mathcal{P}$ for $i \geq i_2$, i.e. that $K^0 \in \mathcal{P}$. For this, it suffices to prove that $B^0(Y') = \text{Cok } \alpha^0$, since then, as $0 \to X_1^0 \overset{\alpha^0}{\to} K^0 \to B^0(Y') \to 0$ is exact and as $X_1^0, B^0(Y') \in \mathcal{P}$, we have, by assumption (ii), $K^0 \in \mathcal{P}$.

In the diagram

\[
\begin{array}{ccc}
X^{-1} & \overset{i}{\longrightarrow} & X^0 \\
\downarrow{\alpha} & & \downarrow{\alpha^0} \\
B^0(X') & \longrightarrow & K^0.
\end{array}
\]
(1) is pushout implies that \( \text{Cok } \alpha^0 \cong \text{Cok } \mu' i \). Hence, it suffices to prove that \( \text{Cok } \mu' i = B^0(Y') \).

Consider

\[
\begin{array}{c}
X^{-2} \xrightarrow{d_1} X^{-1} \xrightarrow{\mu} B^0(X^1) \xrightarrow{\eta} X^0 \\
\downarrow i \downarrow i \downarrow \eta \downarrow i \\
X'^{-2} \xrightarrow{d'} X'^{-1} \xrightarrow{\mu'} B^0(X'^1) \xrightarrow{\eta'} X'^0 \\
\downarrow \tilde{i} \downarrow \tilde{i} \downarrow \tilde{\eta} \downarrow \tilde{i} \\
Y^{-2} \xrightarrow{\bar{d}} Y^{-1} \xrightarrow{\bar{\mu}} B^0(Y') \xrightarrow{\tilde{\eta}} Y^0
\end{array}
\]

where \( \eta \) and \( \tilde{\eta} \) are induced on the images.

We have \( \text{Cok } \mu' i = \text{Cok } \eta \mu = \text{Cok } \eta \). Hence, it suffices to prove that \( \tilde{\eta} = \text{Cok } \eta \).

For this, let \( f : B^0(X'^1) \to M \) be a map such that \( f \eta = 0 \). We have to find \( \phi : B^0(Y') \to M \) such that \( \phi \tilde{\eta} = f \) (such a \( \phi \) will necessarily be unique since \( \tilde{\eta} \) is epic).

We have the following implications:

\[
f \eta = 0 \implies f \eta \mu = f \mu' i = 0 \implies f \mu' \text{ factors through } Y^{-1} = \text{Cok } i :
\]

there exists \( g : Y^{-1} \to M \) with \( g i = f \mu' \).

Now, \( g \bar{d} \tilde{i} = g i \bar{d}' = f \mu' \bar{d}' = 0 \). But \( \tilde{i} \) is epic, so that \( g \bar{d} = 0 \).

As \( Y' \) is acyclic, \( B^0(Y') = \text{Cok } \bar{d} \), and \( g \bar{d} = 0 \) implies that \( g \) factors through \( B^0(Y') \): there exists \( \phi \) such that \( \phi \bar{\mu} = g \). One checks then directly that \( \phi \tilde{\eta} = f \).
Third step.

Finally, given a complex $X \in |K(A)|$, choose a sequence of integers $i_0 > i_1 > i_2 > \ldots$ tending to $-\infty$. Choose $X_{i_0}^*$ for $i_0$ as in the first step, and choose $X_{i_1}^*$, $X_{i_2}^*$, ... for $i_1$, $i_2$, ... successively as in the second step. Then, we have quasi-isomorphisms $X^* \to X_{i_1}^* \to X_{i_2}^* \to \ldots$ and, for each $i$, the sequence $X^i \to X_{i_1}^i \to X_{i_2}^i \to \ldots$ is either constant, or in $\mathcal{P}$. Then $I^* = \lim_{\to} X^*$ is the required complex of objects in $\mathcal{P}$.

This ends the proof of Lemma 1.1, b).

Let's now recall Lemma 1.1 c):

Let $A$ be an abelian category, let $A'$ be a thick subcategory of $A$, and assume that $A'$ has enough $A$-injectives. Then every $X \in |K^+_A(A)|$ admits a quasi-isomorphism into a bounded below complex $I^*$ of $A$-injective objects of $A'$.

Proof.

We may assume $X^i = 0$ for $i < 0$. Embed $H^0(X')$ into $I^0$, an $A$-injective object of $A'$, which is possible, since $H^0(X') \in |A'|$. Then we can extend this embedding to a map $f^0 : X^0 \to I^0$:

$$
\begin{array}{ccc}
0 & \to & X^0 \\
& \searrow & \downarrow f^0 \\
& & X^1 \\
H^0(X') & \to & I^0
\end{array}
$$

which is possible, since $I^0$ is $A$-injective.

Having defined $I^0 \xrightarrow{d^0_I} I^1 \to \ldots \xrightarrow{d^{p-1}_I} I^p$ and $f^i : X^i \to I^i$ for $i=0,1,\ldots,p$ we define $I^{p+1}$, $d^p_I$ and $f^{p+1}$ by the diagram.
where (1) is a pushout square.

We must check that $K^{p+1}$ is in $A'$ (to be sure that $K^{p+1}$ can be embedded in an $A$-injective object of $A'$).

Recall that a full subcategory $A'$ of an abelian category $A$ is thick (or is a Serre subcategory of $A$) if one of the following equivalent conditions hold:

(i) If $X \rightarrow Y \rightarrow Z$ is exact, then $X, Z \in |A'| \implies Y \in |A'|$.

(ii) If $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ is exact, then $B \in |A'|$ if and only if $B' \in |A'|$ and $B'' \in |A'|$.

Thus, it suffices to prove that, in the exact sequence

$$Z^p(I') \xrightarrow{g} K^{p+1} \rightarrow \text{Cok } g$$

$Z^p(I) \in |A'|$, and Cok $g \in |A'|$. But $0 \rightarrow B^p(I') \rightarrow I^p \rightarrow Z^p(I') \rightarrow 0$ is a short exact sequence, and $I^p \in |A'|$, so that $Z^p(I') \in |A'|$.

Moreover, as the square (1) of diagram 30. is a pushout, Cok $g \cong \text{Cok } \phi = H^{p+1}(X') \in |A'|$. Hence, $K^{p+1} \in |A'|$, and there is a monomorphism $K^{p+1} \rightarrow I^{p+1}$ where $I^{p+1}$ is an $A$-injective object of $A'$. 
Finally, as $I_{\mathbb{P}+1}$ is injective, the map $j_{\mathbb{P}+1}$ extends to $f_{\mathbb{P}+1} : X_{\mathbb{P}+1} \rightarrow I_{\mathbb{P}+1}$ and one defines $d_{I}^{\mathbb{P}} = j_{\mathbb{P}+1} g_{\mathbb{P}}$. One checks directly that $d_{I}^{\mathbb{P}}d_{I}^{\mathbb{P}-1} = 0$ and that $f : X \rightarrow I$ is a chain map.

$H_{\mathbb{P}}(f)$ is an isomorphism.

1. $H_{\mathbb{P}}(f)$ is epic: consider the diagram

\[
\begin{diagram}
\node{Z_{\mathbb{P}}(X')} \arrow{e} \node{X_{\mathbb{P}}} \arrow{s}{\eta} \node{Z_{\mathbb{P}}^{+1}(X')} \arrow{n}
\node{H_{\mathbb{P}}(I')} \arrow{e} \node{Z'_{\mathbb{P}}(I')} \arrow{s}{g} \node{K_{\mathbb{P}}^{+1}} \arrow{n}
\node{B_{\mathbb{P}}^{+1}(X')} \arrow{e} \node{B_{\mathbb{P}}^{+1}(I')} = B_{\mathbb{P}}^{+1}(I').
\end{diagram}
\]

One checks directly that $\text{Im} \, \phi = B_{\mathbb{P}}^{+1}(X')$ and that $\text{Im} \, g = B_{\mathbb{P}}^{+1}(I') = B_{\mathbb{P}}^{+1}(I')$.

As the square (1) of diagram 31. is a pushout, the map $\eta$ induced on the kernels is epic. With a same reasoning as in diagram 9. (p.36), one sees that $\eta$ induces an epimorphism $\theta = H_{\mathbb{P}}(f) : H_{\mathbb{P}}(X') \rightarrow H_{\mathbb{P}}(I')$.

2. $H_{\mathbb{P}}(f)$ is monic. Consider the diagram
We have $d_1 j \circ \delta = f^p \circ d_X \gamma = 0$, and $d_1 j \circ s = d_1 \circ d_I = 0$, hence, since $s$ is epic, $d_1 j \circ \alpha = 0$. By the universal property of the pushout (1), $d_1 j = 0$. Thus, there exists a unique monomorphism $\phi$ such that $\epsilon \phi = j$. One also checks that the triangle (2) of diagram 32. commutes. Finally, consider

As the square (1) of diagram 32. is a pushout, the square (3) of diagram 33 is a pushout, so that $\mu$ is an isomorphism. Furthermore, by the short five lemma, as $\phi$ is monic, $\mu'$ is monic. Hence, $H^p(f) = \mu' \mu$ is a monomorphism.
To conclude this appendix, we give a proof of Proposition I.6. We recall the statement of this proposition:

Let $A$ be an abelian category of finite injective homological dimension, let $I$ be the full subcategory of $A$ consisting of injective objects of $A$, and let $E : K(I) \rightarrow K(A)$ be the embedding functor. Then:

There exists a functor $J : K(A) \rightarrow K(I)$ and a natural transformation $i : Id_{K(A)} \rightarrow E J$ such that, for each $X^* \in |K(A)|$:

(i) $i_X$ is a quasi-isomorphism

(ii) $i_X$ is a monomorphism.

Proof.

If we take $P = |I|$, we see that the axioms (i), (ii) and (iii) of Lemma I.1 b) are fulfilled. Hence, it is sufficient to prove that the association $X^* \rightarrow JX^* = I^*$ constructed in Lemma I.1 b) is functorial. (The fact that $i_X$ is monic results from the construction of $I^*$). We keep the same notations as in the proof of Lemma I.1 b).

1. The association $X^* \rightarrow X_0^*$ is functorial.

Given a map $f : X^* \rightarrow Y^*$, let $I^*$ (resp. $J^*$) be the cochain complex consisting of injectives quasi-isomorphic to the truncated complex $X^0 \rightarrow X^1 \rightarrow \ldots$ (resp. $Y^0 \rightarrow Y^1 \rightarrow \ldots$). Then, by the functoriality of the first part of the lemma (Proposition I.2), $f$ induces a map $f_* : I^* \rightarrow J^*$.

Hence, $f$ induces a map $X^* \rightarrow J^*$ as follows:
\[ \cdots \rightarrow x^{i_0-2} \rightarrow x^{i_0-1} \rightarrow x^{i_0} \rightarrow x^{i_0+1} \rightarrow \cdots \]

34. \[ \begin{array}{ccc}
& f & \\
\downarrow & & \downarrow \\
& f & \quad (1) & f_* & \quad f_* \\
\cdots \rightarrow y^{i_0-2} \rightarrow y^{i_0-1} \rightarrow y^{i_0} \rightarrow y^{i_0+1} \rightarrow \cdots
\end{array} \]

The square (1) of diagram 34. commutes, since both squares commute in

\[ \begin{array}{ccc}
x^{i_0-1} & \xrightarrow{d_X} & x^{i_0} \\
\downarrow & f & \downarrow \\
y^{i_0-1} & \xrightarrow{d_Y} & y^{i_0}
\end{array} \]

2. The association \( X'_1 \rightarrow X'_2 \) of the second step is functorial.

By construction of \( X'' \), the association \( X'_1 \rightarrow X'' \) is functorial, and \( X'_2 \) is defined (if we suppose \( i_o + n = 0 \) as before) by

\[
X'_2 = \begin{cases} 
  X'_i & \text{for } i < 0 \\
  K^0 & \text{for } i = 0 \\
  X'_1 & \text{for } i > 0 
\end{cases}
\]

where \( K^0 \) is defined by the pushout diagram

36. \[
\begin{array}{ccc}
X_{1-1} & \xrightarrow{X_{1-1}} & X_1^0 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
B^{0}(X'') & \xrightarrow{B^{0}(X'')} & K^0
\end{array}
\]

Then, given a co-chain map \( f : X'_1 \rightarrow Y'_1 \) we have to find an induced map \( \tilde{f} : X'_2 \rightarrow Y'_2 \).

We already know that \( f \) induces \( f_* : X'' \rightarrow Y'' \). Hence, we have to define \( \tilde{f} : K^0 \rightarrow L^0 \) such that
commutes. This map is given by the following diagram (defining $K^0$ and $L^0$), where we make use of the universal property of the pushouts:

\[
\begin{array}{c}
\cdots \longrightarrow X_{r-2} \longrightarrow X_{r-1} \longrightarrow K^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \\
\downarrow f_x \downarrow f_x \downarrow f \downarrow f \downarrow f \downarrow f \\
\cdots \longrightarrow Y_{r-2} \longrightarrow Y_{r-1} \longrightarrow L^0 \longrightarrow Y^1 \longrightarrow Y^2 \longrightarrow \cdots
\end{array}
\]

Hence, we have a sequence of natural maps

\[
\begin{array}{c}
X' \longrightarrow X'_0 \longrightarrow X'_1 \longrightarrow \cdots \longrightarrow \lim X' = I' \\
\downarrow f \downarrow f_{*0} \downarrow f_{*1} \\
Y' \longrightarrow Y'_0 \longrightarrow Y'_1 \longrightarrow \cdots \longrightarrow \lim Y' = J'
\end{array}
\]

By universal property of $\lim$, $f_{*r}$ induce a unique map

\[
\text{If } : \lim X'_r \longrightarrow \lim Y'_r.
\]
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