SOME FIXED POINT THEOREMS FOR
NONEXPANSIVE MAPPINGS IN HAUSDORFF LOCALLY CONVEX SPACES

by

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ABSTRACT

Let $X$ be a Hausdorff locally convex space, $\mathcal{U}$ be a base for closed absolutely convex $0$-neighborhoods in $X$, $K \subseteq X$ be nonempty. For each $U \in \mathcal{U}$, we denote by $P_U$ the gauge of $U$. Then $T : K \to K$ is said to be nonexpansive w.r.t. $\mathcal{U}$ if and only if for each $U \in \mathcal{U}$, $P_u(T(x) - T(y)) \leq P_u(x - y)$ for all $x, y \in K$; $T : K \to K$ is said to be strictly contractive w.r.t. $\mathcal{U}$ if and only if for each $U \in \mathcal{U}$, there is a constant $0 < \lambda_u < 1$ such that $P_u(T(x) - T(y)) \leq \lambda_u P_u(x - y)$ for all $x, y \in K$. The concept of nonexpansive (respectively strictly contractive) mappings is originally defined on a metric space. The above definitions are generalizations if the topology on $X$ is induced by a translation invariant metric, and in particular if $X$ is a normed space.

An analogue of the Banach contraction mapping principle is proved and some examples together with an implicit function theorem are shown as applications. Moreover several fixed point theorems for various kinds of nonexpansive mappings are obtained. The convergence of nets of nonexpansive mappings and that of fixed points are also studied.

Finally on sets with 'complete normal structure', a common fixed point theorem is obtained for an arbitrary family of 'commuting' nonexpansive mappings while on sets with 'normal structure', a common fixed point theorem for an arbitrary family of (not necessarily commuting) 'weakly periodic' nonexpansive mappings is obtained.
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INTRODUCTION

Nonexpansive mappings are mappings $T : K \rightarrow K$, where $K$ is a nonempty subset of a metric space $X$ with metric $d$, such that $d(T(x), T(y)) \leq d(x, y)$, for all $x, y \in K$. Let $H$ be a Hilbert space, \{A(t) : T \geq 0\} be a family of closed linear operators on $H$, $f$ be a mapping of $\mathbb{R}^+ \times H$ into $H$ and $A(t), f(t, u(t))$ be periodic in $t$ with a common period $\xi > 0$. Suppose

$$\frac{du(t)}{dt} + A(t)u(t) = f(t, u(t)) \quad (t \geq 0) \quad \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

is a time-dependent non-linear equation of evolution in $H$ satisfying some conditions. If $R$ is a suitably chosen nonnegative number, for each $v \in B = \{v \in H : \|v\| \leq R\}$, define $T(v) = u(\xi)$, where $u(t)$ is the "mild solution" of equation (1) such that $u(0) = v$. Then F. E. Browder proved in [5] that $T$ is a nonexpansive mapping from $B$ into $B$, and moreover, each fixed point of $T$ corresponds to a periodic solution of equation (1) with period $\xi$.

Since nonexpansive mappings originated with and as noted above, have applications to differential and functional equations, the existence of fixed points for such mappings has been widely studied. However most of the research so far has been done in the Banach and metric space setting.

W. A. Kirk proved in [12] that if $T : K \rightarrow K$ is nonexpansive, where $K$ is a nonempty weakly compact convex subset of a
Banach space $X$ and $K$ has "normal structure", then $T$ has a fixed point in $K$. If we renorm the space $X$ by an equivalent norm, then $T$ needs not be nonexpansive and $K$ needs not have "normal structure" with respect to the new norm yet $T$ still has a fixed point in $K$. This observation leads us to consider the relationship of a given mapping $T$ and the topology of the space (independent of the norm structure) and to work on neighborhoods instead of norms. This is the main purpose of this work.

In the first chapter, the notion of a nonexpansive (respectively contractive, strictly contractive) mapping is defined in a Hausdorff locally convex space such that it will be a generalization if the topology of the space is induced by a translation invariant metric. Next an analogue of the Banach contraction mapping principle is proved and some examples together with an implicit function theorem are shown as applications. Furthermore, convergence of (subsequence of) iterations of contractive mappings are studied and a result of M. Edelstein in [9] is generalized. Finally we discuss the convergence of a net of contractive mappings and the convergence of a net of fixed points of nonexpansive mappings.

In the second chapter, we first study the "center" of a set and then we generalize the notion of "normal structure" introduced by M. S. Brodskii and D. P. Milman in [4] and the notion of complete normal structure introduced by L. P. Belluce and W. A. Kirk in [2]. It is proved that each compact convex subset of a Hausdorff locally convex space has
complete normal structure (and so has normal structure) in our sense.
Some results of R. DeMarr in [6], of L. P. Belluce and W. A. Kirk in [2]
and of L. P. Belluce, W. A. Kirk and E. F. Steiner in [3] are extended in
our sense.

In the third and final chapter, we study the fixed point
theorems for various kinds of nonexpansive mappings. The notions of
nonexpansive mappings with diminishing orbital diameters and bounded
mappings were introduced by L. P. Belluce and W. A. Kirk in [1] and by
W. A. Kirk in [13] respectively. These notions have been generalized into
Hausdorff locally convex spaces and much of their results have been
extended in our general setting. The notion of convex mappings is also
generalized and some fixed point theorems for convex nonexpansive mappings
are obtained. Finally we prove a common fixed point theorem for an
arbitrary family of (not necessarily "commuting") "weakly periodic" (and
also periodic) nonexpansive mappings.
1-1. Definitions and notations.

In this chapter, $X$ denotes a real or complex Hausdorff locally convex space ($T_2$-l.c.s.) and $U$ denotes a base for the closed absolutely convex neighborhoods of zero (0-nbhd). If $U$ is any closed absolutely convex 0-nbhd, we denote by $P_u$ the gauge of $U$ defined by $P_u(x) = \inf\{\lambda > 0 : x \in \lambda U\}$ for all $x \in X$. If $K \subseteq X$, we denote by $\overline{K}$ the closure of $K$, $\text{Co}(K)$ the convex hull of $K$, $K^i$ the interior of $K$. We shall refer to Robertson [14] and Kelley and Naimioka [11] for properties of gauge function and further notations used thereafter.

Definition 1.1.

If $K \subseteq X$ is non-empty, then a mapping $T : K \rightarrow K$ is said to be nonexpansive with respect to (w.r.t.) $U$ if and only if for each $U \in U$, $P_u(T(x) - T(y)) \leq P_u(x - y)$ for all $x, y \in K$.

Definition 1.2.

If $K \subseteq X$ is non-empty, then a mapping $T : K \rightarrow K$ is said to be contractive w.r.t. $U$ if and only if $T$ is nonexpansive w.r.t. $U$ and for each $U \in U$ and for any $x, y \in K$, if $P_u(x - y) > 0$, then $P_u(T(x) - T(y)) < P_u(x - y)$.
Definition 1.3.

If $K \subseteq X$ is non-empty, then a mapping $T : K \rightarrow K$ is said to be strictly contractive w.r.t. $\mathcal{U}$ if and only if for any $U \in \mathcal{U}$, there is a constant $\lambda_U$ such that $0 < \lambda_U < 1$ and

$$P_U(T(x) - T(y)) \leq \lambda_U P_U(x - y) \text{ for all } x, y \in K.$$ 

If $K \subseteq X$ is non-empty and $T : K \rightarrow K$, it is clear that $T$ is strictly contractive w.r.t. $\mathcal{U}$ implies $T$ is contractive w.r.t. $\mathcal{U}$ which in turn implies $T$ is nonexpansive w.r.t. $\mathcal{U}$ and which in turn implies $T$ is continuous.

Remark 1.4.

If the topology on $X$ is induced by a translation invariant metric $d$ and $K \subseteq X$ is non-empty, let $B_r = \{x \in X : d(x, 0) \leq r\}$ and $\mathcal{U} = \{B_r : r \text{ runs through a net of positive numbers tending to } 0\}$, then a mapping $T : K \rightarrow K$ is nonexpansive (respectively contractive and strictly contractive) w.r.t. $\mathcal{U}$ if and only if $d(T(x), T(y)) \leq d(x, y)$ (respectively $x \neq y$ implies $d(T(x), T(y)) < d(x, y)$ and there is a constant $\lambda$ such that $0 < \lambda < 1$ and $d(T(x), T(y)) \leq \lambda d(x, y)$ for all $x, y \in K$).

Remark 1.5.

Suppose $K \subseteq X$ is nonempty and $T : K \rightarrow K$ is nonexpansive w.r.t. $\mathcal{U}$. If $\mathcal{U}$ is the set of all closed absolutely convex 0-nbhd in $X$, it is clear that $T$ is weakly continuous.
Remark 1.6.

Let \( K \subseteq X \) be nonempty and \( T : K \rightarrow K \) be nonexpansive w.r.t. \( U \). If \( X \) is normable, it is not known whether there exists a norm \( \| \cdot \| \) on \( X \) inducing the same topology on \( X \) such that 
\[
\| T(x) - T(y) \| \leq \| x - y \| , \text{ for all } x, y \in K .
\]

I-2. The Banach contraction mapping principle.

Definition 2.1.

Let \( K \subseteq X \) be nonempty, \( T : K \rightarrow K \) and \( x \in K \). Then

(i) \( x \) is said to be a fixed point of \( T \) if and only if \( T(x) = x \),
and

(ii) \( x \) is said to be a periodic point of \( T \) if and only if there is a positive integer \( N \) such that \( T^N(x) = x \), where
\[
T^N(x) = T(T^{N-1}(x)) \quad \text{and} \quad T^0 = I , \text{ the identity mapping}.
\]

Proposition 2.2.

If \( K \subseteq X \) is nonempty and \( T : K \rightarrow K \) is contractive w.r.t. \( U \), then a fixed point of \( T \), whenever it exists, is unique.

Proof: Suppose there were \( \xi \) and \( \eta \) in \( K \) such that \( \xi \neq \eta \) and \( T(\xi) = \xi \) and \( T(\eta) = \eta \). Since \( X \) is Hausdorff, there is a \( U \in U \) with \( \xi - \eta \notin U \), and so \( P_U(\xi - \eta) > 1 > 0 \). Since \( T \) is contractive w.r.t. \( U \), and \( P_U(\xi - \eta) > 0 \), we see that
\[ P_u(\xi - \eta) = P_u(T(\xi) - T(\eta)) < P_u(\xi - \eta), \] which is impossible. Hence a fixed point of \( T \), whenever it exists, is unique.

**Proposition 2.3.**

Let \( K \subseteq X \) be nonempty and \( T : K \rightarrow K \) be contractive w.r.t. \( U \). Then for any \( x \in K \), \( x \) is a fixed point of \( T \) if and only if \( x \) is a periodic point of \( T \).

**Proof:** Suppose \( x \in K \) is a periodic point of \( T \). Let \( N \) be a positive integer such that \( T^N(x) = x \). Suppose \( x \neq T(x) \). Then there is a \( U \in U \) such that \( P_u(x - T(x)) > 0 \). Since \( T \) is contractive w.r.t. \( U \), we have
\[
P_u(x - T(x)) > P_u(T(x) - T^2(x)) \geq \ldots \geq P_u(T^N(x) - T^{N+1}(x)) = P_u(x - T(x)),
\]
which is impossible. Hence we must have \( x = T(x) \). The converse is obvious.

The following theorem generalizes the Banach contraction mapping principle to \( T_2 \)-l.c.s.

**Theorem 2.4.**

If \( K \subseteq X \) is nonempty and sequentially complete and \( T : K \rightarrow K \) is strictly contractive w.r.t. \( U \), then \( T \) has a unique fixed point, say \( \xi \), in \( K \). Moreover \( \lim_{n \to \infty} T^n(x) = \xi \) for all \( x \in K \).

**Proof:** Suppose \( U \in U \). Then there is a constant \( \lambda_u \) such that \( 0 < \lambda_u < 1 \) and \( P_u(T(x) - T(y)) \leq \lambda_u P_u(x - y) \) for all \( x, y \in K \).
If \( x \in K \), then
\[
P_u(T^{n+1}(x) - T^n(x)) \leq \lambda \, P_u(T^n(x) - T^{n-1}(x))
\]
\[
\leq \ldots
\]
\[
\leq \lambda_u \, n \, P_u(T(x) - x)
\]
for all \( n = 1, 2, \ldots \), and so
\[
P_u(T^{n+k}(x) - T^n(x)) \leq \lambda_n \, P_u(T^{n+k-1}(x)) + \ldots + \lambda_u \, P_u(T(x) - x)
\]
\[
= \lambda_u \, \left( \lambda_u^{n+k-1} + \ldots + \lambda_u^n \right)
\]
\[
\leq \lambda_u \frac{n}{1-\lambda_u} \, P_u(T(x) - x)
\]
for all \( n, k = 1, 2, \ldots \). Choose a positive integer \( N \) such that,
\[
\lambda_u \frac{N}{1-\lambda_u} \, P_u(T(x) - x) < 1
\]
then for \( m > n > N \), say \( m = n+k \), we have
\[
P_u(T^m(x) - T^n(x)) = \lambda_u \frac{n}{1-\lambda_u} \, P_u(T(x) - x)
\]
\[
< 1
\]
so that \( T^m(x) - T^n(x) \in U \). Thus \( \{T^n(x) : n = 0, 1, 2, \ldots \} \) is a Cauchy sequence in \( K \) and so there is an \( \xi \in K \) such that
\[
\lim_{n \to \infty} T^n(x) = \xi
\]
Since \( T \) is continuous, we see that \( T(\xi) = T(\lim_{n \to \infty} T^n(x)) = \lim_{n \to \infty} T^{n+1}(x) = \xi \).
Hence $\xi$ is a fixed point of $T$, and so must be unique, by Proposition 2.1.

It is clear that if $K \subseteq X$ is nonempty, $T : K \to K$ is nonexpansive w.r.t. $U$ and if $x \in K$ and there is a subsequence $\{T^n_i(x)\}_{i=1}^{\infty}$ of $\{T^n(x)\}_{n=1}^{\infty}$ such that $\xi = \lim_{i \to \infty} T^n_i(x)$ exists and is a fixed point of $T$, then $\lim_{n \to \infty} T^n(x)$ exists and is $\xi$.

Corollary 2.5.

Let $K \subseteq X$ be nonempty and sequentially complete, and $T : K \to K$ be continuous. If there is a positive integer $N$ such that $T^N$ is strictly contractive w.r.t. $U$, then $T$ has a unique fixed point in $K$. Moreover, $\lim_{n \to \infty} T^n(x) = \xi$, for each $x \in K$.

Proof: Since $T^N : K \to K$ is strictly contractive w.r.t. $U$ and $K$ is sequentially complete, $T^N$ has a unique fixed point $\xi$ in $K$, and $\lim_{n \to \infty} (T^N)_n(x) = \xi$ for all $x \in K$, by Theorem 2.4. Thus $T$ is continuous implies

$$T(\xi) = T(\lim_{n \to \infty} (T^N)_n(x))$$

$$= \lim_{n \to \infty} (T^N)_n(Tx)$$

$$= \xi,$$

and so $\xi$ is a fixed point of $T$. Since a fixed point of $T$ is also a fixed point of $T^N$, we see that $\xi$ must be the unique fixed point of $T$. Thus it follows that $\xi = \lim_{n \to \infty} T^n(x)$, for each $x \in K$. 
I-3. Iterations of contractive mappings.

The purpose of this section is to study the iterations of nonexpansive mappings and contractive mappings and to discuss when the iterations will converge to a fixed point.

Definition 3.1.

Suppose \( K \subseteq X \) is nonempty. Then \( T : K \to K \) is an isometry w.r.t. \( U \) if and only if for any \( U \subseteq U \),
\[
P_u(T(x) - T(y)) = P_u(x - y) \quad \text{for all} \quad x, y \in K.
\]

Proposition 3.2.

Let \( K \subseteq X \) be nonempty, \( T : K \to K \) be nonexpansive w.r.t. \( U \) and \( K^T = \{x \in K : \text{there is an} \ x^o \ \text{in} \ K \ \text{such that} \ x \ \text{is a limit point of} \ \{T^n(x^o) : n = 0, 1, 2, \ldots\} \} \). Then for any \( x \in K \), \( x \) is in \( K^T \) if and only if \( x \) is a limit point of \( \{T^n(x) : n = 0, 1, 2, \ldots\} \). Also \( T(K^T) \subseteq K^T \).

Proof: Suppose \( x \in K \). If \( x \) is a limit point of \( \{T^n(x) : n = 0, 1, 2, \ldots\} \), then \( x \in K^T \), by definition of \( K^T \).

Conversely let us assume that \( x \in K^T \). Then there is an \( x^o \in K \) such that \( x \) is a limit point of \( \{T^n(x^o) : n = 0, 1, 2, \ldots\} \). Thus for arbitrarily fixed \( U \subseteq U \) and for any positive integer \( N \), there are positive integers \( m \) and \( n \) such that \( n > N \), \( m > N + n \),
\[
x - T^n(x^o) \in \frac{1}{2} U \quad \text{and} \quad x - T^m(x^o) \in \frac{1}{2} U,
\]
and so there exists a positive integer \( k = m - n > N \) such that
\[ P_u(x - T^k(x)) \leq P_u(x - T^m(x)) + P_u(T^m(x) - T^{m-n}(x)) \]

\[ \leq \frac{1}{2} + P_u(T^m(x) - x) \]

\[ \leq \frac{1}{2} + \frac{1}{2} \]

\[ = 1 \]

so that \( x - T^k(x) \in U \). Hence \( x \) is a limit point of \( \{T^n(x) : n = 0, 1, 2, \ldots \} \).

Next since \( T \) is nonexpansive w.r.t. \( U \), it is clear that \( T(K^T) \subseteq K^T \).

Proposition 3.3.

Let \( K \subseteq X \) be nonempty and \( T : K \to K \) be nonexpansive w.r.t. \( U \). Suppose \( 0 < m_1 < m_2 < \ldots \), where \( m_i \)'s are positive integers, and \( S = \{ x \in K : \lim_{i \to \infty} T^{m_i}(x) = x \} \). Then \( T(S) \subseteq S \) and \( T \) is an isometry on \( S \) w.r.t. \( U \).

Proof: If \( x \in S \), then \( \lim_{i \to \infty} T^{m_i}(x) = x \) implies \( T(x) = \lim_{i \to \infty} T^{m_i}(T(x)) \) so that \( T(x) \in S \). Thus \( T(S) \subseteq S \).

Next suppose there is a \( U \in U \) and there are \( x_1, x_2 \in S \) such that \( P_u(T(x_1) - T(x_2)) \neq P_u(x_1 - x_2) \). Since \( T \) is nonexpansive w.r.t. \( U \), we must have \( P_u(T(x_1) - T(x_2)) < P_u(x_1 - x_2) \). Let \( \delta = P_u(x_1 - x_2) - P_u(T(x_1) - T(x_2)) \). Choose positive integer \( i_0 \) such that \( i > i_0 \) implies \( P_u(T^{m_i}(x_j) - x_j) < \frac{1}{2} \delta \) for each \( j = 1, 2 \). Since
12.

\[ P_u(T^i(x_1) - T^i(x_2)) \leq P_u(T(x_1) - T(x_2)) \]

we have for \( i > i_0 \),

\[
\delta = P_u(x_1 - x_2) - P_u(T(x_1) - T(x_2)) \\
\leq P_u(x_1 - x_2) - P_u(T^i(x_1) - T^i(x_2)) \\
\leq P_u((x_1 - x_2) - (T^i(x_1) - T^i(x_2))) \\
\leq P_u(x_1 - T^i(x_1)) + P_u(x_2 - T^i(x_2)) \\
< \frac{1}{2} \delta + \frac{1}{2} \delta \\
= \delta,
\]

which is a contradiction. Therefore \( T \) is an isometry on \( S \) w.r.t. \( U \).

Theorem 3.4.

Let \( K \subseteq X \) be nonempty and \( T : K \to K \) be nonexpansive w.r.t. \( U \). If \( x \in K^T \), then \( T \) is an isometry on \( \{T^n(x) : n = 0, 1, 2, \ldots \} \) w.r.t. \( U \).

Proof: Let \( x \in K^T \). Suppose there is a \( U \in U \) and there exist nonnegative integers \( m \) and \( n \) such that

\[ P_u(T^{m+1}(x) - T^{n+1}(x)) \neq P_u(T^m(x) - T^n(x)) \].

Then \( T \) is nonexpansive w.r.t. \( U \) implies \( \delta = P_u(T^m(x) - T^n(x)) - P_u(T^{m+1}(x) - T^{n+1}(x)) > 0 \). By Proposition 3.2, \( x \) is a limit point of \( \{T^n(x) : n = 0, 1, 2, \ldots \} \).

Choose a positive integer \( k \) such that \( P_u(x - T^k(x)) < \frac{1}{2} \delta \). Then
\[ P_u(T^m(x) - T^n(x)) \leq P_u(T^m(x) - T^{m+k}(x)) + P_u(T^{m+k}(x) - T^{n+k}(x)) + P_u(T^{n+k}(x) - T^n(x)) \]
\[ \leq P_u(x - T^k(x)) + P_u(T^{m+1}(x) - T^{n+1}(x)) + P_u(T^k(x) - x) \]
\[ \leq \frac{1}{2} \delta + P_u(T^{m+1}(x) - T^{n+1}(x)) + \frac{1}{2} \delta \]
\[ = \delta + P_u(T^{m+1}(x) - T^{n+1}(x)) \]
\[ = P_u(T^m(x) - T^n(x)) , \]

which is impossible. Hence we must have for each \( U \in \mathcal{U} \) and for any nonnegative integers \( m \) and \( n \), \( P_u(T^{m+1}(x) - T^{n+1}(x)) = P_u(T^m(x) - T^n(x)) \), i.e. \( T \) is an isometry on \( \{ T^n(x) : n = 0, 1, 2, \ldots \} \) w.r.t. \( U \).

Theorem 3.5.

Let \( K \subseteq X \) be nonempty and \( T : K \to K \) be contractive w.r.t. \( U \). Then \( \text{Card}(K_T) \leq 1 \). In case \( \text{Card}(K_T) = 1 \), \( K_T \) contains only the unique fixed point of \( T \).

Proof: Suppose \( K_T \neq \emptyset \) and let \( x \in K_T \). If \( T(x) \neq x \), then there exists \( U \in \mathcal{U} \) with \( P_u(x - T(x)) > 0 \). Thus \( T \) is contractive w.r.t. \( U \) implies \( P_u(T^2(x) - T(x)) < P_u(T(x) - x) \). By Theorem 3.4., \( T \) is an isometry on \( \{ T^n(x) : n = 0, 1, 2, \ldots \} \) w.r.t. \( U \), and so we must have \( P_u(T^2(x) - T(x)) = P_u(T(x) - x) \), which is a contradiction. Thus \( T(x) = x \). By Proposition 2.2, \( K_T = \{ x \} \).

Corollary 3.6.

Let \( K \subseteq X \) be nonempty and \( T : K \to K \) be contractive
w.r.t. \( U \). Suppose there is an \( x_0 \in K \) and a sequence of strictly increasing positive integers \( 1 < n_1 < n_2 < \ldots \) such that \( \xi = \lim_{i \to \infty} T^{n_i}(x_0) \) exists in \( K \), then \( \xi \) is the unique fixed point of \( T \). Moreover, \( \lim_{n \to \infty} T^n(x_0) \) exists in \( K \) and \( \xi = \lim_{n \to \infty} T^n(x_0) \).

**Proof:** Since \( \xi = \lim_{i \to \infty} T^{n_i}(x_0) \) exists in \( K \), \( \xi \in K^T \). Hence by Theorem 3.5, \( \xi \) must be the unique fixed point of \( T \) in \( K \).

Next suppose \( U \in U \). Then there is a positive integer \( N \) such that \( T^{n_i}(x_0) \in \xi + U \) for all \( i \geq N \). Thus, for \( \lambda = 0, 1, 2, \ldots \),

\[
P_u(\xi - T^{n_i+N}(x_0)) = P_u(T^\lambda(\xi) - T^{n_i+N}(x_0))
\]

\[
\leq P_u(\xi - T^{n_i}(x_0)) < 1
\]

and so \( T^{n_i+N}(x_0) \in \xi + U \) for all \( \lambda = 0, 1, 2, \ldots \) and thus \( T^m(x_0) \in \xi + U \) for all \( m \geq n_N \). Hence \( \xi = \lim_{n \to \infty} T^n(x_0) \).


**Corollary 3.7.**

Let \( K \subseteq X \) be nonempty sequentially compact and \( T : K \to K \) be contractive w.r.t. \( U \). Then \( T \) has a unique fixed point, say \( \xi \), in \( K \) and \( \xi = \lim_{n \to \infty} T^n(x) \) for any \( x \in K \).
Corollary 3.7 can be slightly generalized to a kind of mappings, called "asymptotically regular mappings".

**Definition 3.8.**

Let \( K \subseteq X \) be nonempty. Then, \( T : K \to K \) is said to be **asymptotically regular** if and only if

\[
\lim_{n \to \infty} (T^n(x) - T^{n+1}(x)) = 0 ,
\]

for each \( x \in K \).

It is clear from Corollary 3.7 that if \( K \subseteq X \) is nonempty sequentially compact and \( T : K \to K \) is contractive w.r.t. \( \mathcal{U} \), then \( T \) is asymptotically regular. Next we have the following

**Remark 3.9.**

Let \( K \subseteq X \) be nonempty and \( T : K \to K \) be strictly contractive w.r.t. \( \mathcal{U} \). Then for any \( U \in \mathcal{U} \) and any \( x \in K \), we have

\[
\lim_{n \to \infty} \sup_{i,j \geq n} \{ P_u(T^i(x) - T^j(x)) \} = 0 .
\]

In particular, \( T \) is asymptotically regular.

**Proof:** If \( U \in \mathcal{U} \), let \( \lambda_u \) be a constant such that \( 0 < \lambda_u < 1 \) and

\[
P_u(T(x) - T(y)) \leq \lambda_u P_u(x - y)
\]

for all \( x, y \in K \). Suppose \( x \in K \) and \( n \) is a positive integer, then

\[
\sup_{i,j \geq n} \{ P_u(T^i(x) - T^j(x)) \} \leq \left( \frac{\lambda_u}{1 - \lambda_u} \right) P_u(T(x) - x) ,
\]

so that

\[
\lim_{n \to \infty} \sup_{i,j \geq n} \{ P_u(T^i(x) - T^j(x)) \} = 0 .
\]
Proposition 3.10.

Let $K \subseteq X$ be nonempty sequentially compact and $T : K \to K$ be continuous and asymptotically regular. Then there is an $x \in K$ such that $T(x) = x$.

Proof: Suppose $x \in K$. Then there are a subsequence $\{T^i(x)\}_{i=1}^{\infty}$ of $\{T^n(x)\}_{n=1}^{\infty}$ and an $z \in K$ such that $\lim_{i \to \infty} T^i(x) = z$.

Since $T$ is asymptotically regular, $\lim_{n \to \infty} (T^n(x) - T^{n+1}(x)) = 0$ and so

$$\lim_{i \to \infty} (T^i(x) - T^{i+1}(x)) = 0.$$ Since $T$ is continuous, we see that

$$T(z) = T(\lim_{i \to \infty} T^i(x)) = \lim_{i \to \infty} T^i(x) = \lim_{i \to \infty} T^i(x) = z,$$

and so $z$ is a fixed point of $T$.

Proposition 3.11.

Let $K \subseteq X$ be nonempty sequentially compact and $T : K \to K$ be nonexpansive w.r.t. $U$. If $T$ is asymptotically regular, then for each $x \in K$, $\lim_{n \to \infty} T^n(x)$ exists in $K$ and is a fixed point of $T$.

Proof: For each $x \in K$, there is a subsequence $\{T^i(x)\}_{i=1}^{\infty}$ of $\{T^n(x)\}_{n=1}^{\infty}$ such that $\lim_{i \to \infty} T^i(x)$ exists in $K$ and is a fixed point of $T$.

Since $T$ is nonexpansive w.r.t. $U$, $\lim_{n \to \infty} T^n(x)$ exists and

$$\lim_{i \to \infty} T^i(x) = \lim_{n \to \infty} T^n(x).$$ Thus $\lim_{n \to \infty} T^n(x)$ is a fixed point of $T$. 

Some examples and applications.

In this section, some examples of contractive mappings and strictly contractive mappings are shown and some applications of the Banach contraction mapping principle are obtained.

Example 4.1.

Suppose \( T : X \to X \) is nonexpansive w.r.t. \( \mathcal{U} \), \( x_0 \in X \), \( n \) is a positive integer and \( \lambda \) is any scalar. Define \( S_\lambda : X \to X \) by
\[
S_\lambda(x) = \lambda T^n(x) + x_0 \quad \text{for all} \quad x \in X .
\]
Then for any \( U \in \mathcal{U} \), and for all \( x, y \in X \),
\[
P_u(S_\lambda(x) - S_\lambda(y)) = P_u(\lambda T^n(x) - \lambda T^n(y)) \leq |\lambda| P_u(x - y) ,
\]
so that \( S \) is nonexpansive w.r.t. \( \mathcal{U} \) if \( |\lambda| \leq 1 \) and is strictly contractive w.r.t. \( \mathcal{U} \) if \( |\lambda| < 1 \).

Example 4.2.

Let \( K \subseteq X \) be nonempty convex and \( T : K \to K \). Suppose \( a_0, a_1, \ldots, a_n \geq 0 \), \( n > 1 \), and \( a_n \neq 0 \) and \( \sum_{i=1}^{n} a_i = 1 \). Define \( S : K \to K \) by
\[
S(x) = \sum_{i=0}^{n} a_i T^i(x) \quad \text{for all} \quad x \in K \quad \text{where} \quad T^0 = I ,
\]
the identity mapping on \( K \). Then \( T \) is nonexpansive (respectively contractive, strictly contractive) w.r.t. \( \mathcal{U} \) implies \( S \) is nonexpansive (respectively contractive, strictly contractive) w.r.t. \( \mathcal{U} \).

Definition 4.3.

If \( \mathcal{U} \) is any closed absolutely convex 0-nbhd in \( X \), and \( K \subseteq X \) is nonempty and bounded, we define
\[ \delta_u(K) = \inf \{ r > 0 : K - K \subseteq rU \}. \]

\( \delta_u(K) \) is called the diameter of \( K \) w.r.t. \( U \).

**Remark 4.4.**

If \( U \) is any closed absolutely convex 0-nbhd in \( X \), and \( K \subseteq X \) is nonempty and bounded, we see that

\[ \delta_u(K) = \sup \{ \mu(x - y) : x, y \in K \} \]

\[ = \delta_u(Co(K)) \]

It follows that if \( K \subseteq H \subseteq Co(K) \), then \( \delta_u(K) = \delta_u(H) \).

**Proposition 4.5.**

Let \( K \subseteq X \) be nonempty convex, and \( T : K \to K \) be nonexpansive w.r.t. \( U \). Suppose \( a_0, a_1, \ldots, a_n \geq 0 \), \( n \geq 1 \), \( a_1 > 0 \) and \( \sum_{i=0}^{n} a_i = 1 \). Define \( S : K \to K \) by \( S(x) = \sum_{i=0}^{n} a_i T^i(x) \) for all \( x \in K \).

Then for any \( x \in K \), \( S(x) = x \) if and only if \( T(x) = x \).

**Proof:**

If \( x \in K \) and \( T(x) = x \), then
\[ S(x) = (a_1 T + \ldots + a_n T^n)(x) \]
\[ = a_0 x + a_1 T(x) + \ldots + a_n T^n(x) = a_0 x + a_1 x + \ldots + a_n x = x. \]

Conversely, suppose \( S(x) = x \). Then
\[ x = a_0 x + a_1 T(x) + \ldots + a_n T^n(x) \]
implies
\[ (1 - a_0)x = a_1 T(x) + \ldots + a_n T^n(x), \]
and so
\[ x = \frac{a_1}{1-a_0} T(x) + \ldots + \frac{a_n}{1-a_0} T^n(x). \]

Let \( \beta_1 = \frac{a_1}{1-a_0} \). If \( \beta_1 = 1 \), then \( x = T(x) \). Thus we may assume
\( \beta_1 \neq 1 \), and so \( 0 < \beta_1 < 1 \). It follows that
\[ x = \beta_1 T(x) + (1 - \beta_1) z, \]
where
\[ z = \frac{a_2}{(1-\beta_1)(1-a_0)} T^2(x) + \ldots + \frac{a_n}{(1-\beta_1)(1-a_0)} T^n(x). \]
Since
\[ \sum_{i=2}^{n} \frac{a_i}{(1-\beta_1)(1-a_0)} = 1, \]
we see that \( z \in \text{Co}(\{T^2(x), \ldots, T^n(x)\}) \).

Suppose \( x \neq T(x) \), then \( x \in \text{Co}(\{T(x), \ldots, T^n(x)\}) \) implies
\( \{T(x), \ldots, T^n(x)\} \) contains more than one point, and so there is a \( U \in \mathcal{U} \)
such that
\[ d = \delta_U(\text{Co}(\{T(x), \ldots, T^n(x)\})) = \delta_U(\{T(x), \ldots, T^n(x)\}) > 0. \]

Since \( \text{P}_U(T^m(x) - T^\ell(x)) \leq \text{P}_U(T^{m-1}(x) - T^{\ell-1}(x)) \) whenever
\( m, \ell \in \{1, 2, \ldots, n\} \), we see that there must be a \( p \in \{1, 2, \ldots, n\} \)
such that \( d = \text{P}_U(T^p(x) - x) \). Let \( p_0 \) be the smallest positive integer \( p \) such that \( d = \text{P}_U(T^p(x) - x) \). Then
\[ d = P_u(x - T^o(x)) = P_u(\beta_1 T(x) - (1 - \beta_1)z - T^o(x)) \]
\[ \leq \beta_1 P_u(T(x) - T^o(x)) + (1 - \beta_1)P_u(z - T^o(x)) \]
\[ \leq \beta_1 P_u(x - T^o(x)) + (1 - \beta_1) \cdot d \]
\[ \leq \beta_1 \cdot d + (1 - \beta_1) \cdot d \]
\[ = d \]

and it follows that
\[ P_u(x - T^o(x)) = d , \]

which contradicts our choice of \( p_o \) if \( p_o > 1 \); but for \( p_o = 1 \),
\[ d = P_u(x - T^o(x)) = P_u(x - x) = 0 \]
which is again a contradiction.

Therefore we must have \( x = T(x) \).

Example 4.6.

Let \( S \) be a nonempty Hausdorff locally compact topological space and let \( C(S) \) be the set of all complex-(resp. real-) valued continuous functions on \( S \). For each nonempty compact subset \( C \) of \( S \), we define \( q_c \) on \( C(S) \) by \( q_c(f) = \sup_{x \in C} |f(x)| \), for all \( f \in C(S) \). Then \( q_c \) is a semi-norm on \( C(S) \) for each nonempty compact subset \( C \) of \( S \).

Let \( F \) be the collection of all such semi-norms, and let \( C(S) \) have the topology generated by \( F \) (Robertson [14]). Then \( C(S) \) is a Hausdorff locally convex space and a base for closed absolutely convex 0-nbhd in \( C(S) \) is given by
U = \{ f \in C(S) : \forall q_1(f) \leq \varepsilon \} : \varepsilon > 0, q_1, \ldots, q_n \in F, n=1,2, \ldots \}.

Firstly we observe that if \( C_1, \ldots, C_n \) are nonempty compact subsets of \( S \) and \( U = \{ f \in C(S) \ : q_{c_1}(f) \vee \ldots \vee q_{c_n}(f) \leq 1 \} \), then the gauge function \( P_u \) of \( U \) is just \( q_{c_1} \vee \ldots \vee q_{c_n} \). Thus for \( K \subseteq C(S) \), \( T : K \to K \) is nonexpansive w.r.t. \( U \) if and only if
\[
q(T(f) - T(g)) \leq q(f - q)
\]
for all \( f, g \in K \) and all \( q \in F \); \( T : K \to K \) is contractive w.r.t. \( U \) if and only if \( T \) is nonexpansive w.r.t. \( U \) and for any \( q \in F \) and \( f, g \in K \), \( q(f - g) > 0 \) implies \( q(T(f) - T(g)) < q(f - g) \); \( T : K \rightarrow K \) is strictly contractive w.r.t. \( U \) if and only if for any \( q \in F \), there is a constant \( \lambda \) with \( 0 \leq \lambda < 1 \) such that \( q(T(f) - T(g)) \leq \lambda q(f - g) \) for all \( f, g \in K \).

Secondly we observe that if \( S \) is not compact, then \( C(S) \) is not normable. Indeed, if \( C(S) \) were normable, say by \( \| \cdot \| \), then there is \( \varepsilon > 0 \) and there are nonempty compact subsets \( C_1, \ldots, C_n \) of \( S \) such that \( \{ f \in C(S) : q_{c_1} \vee \ldots \vee q_{c_n} (f) \leq \varepsilon \} \subseteq \{ f \in C(S) : \| f \| \leq 1 \} \).

Since \( \bigcup_{i=1}^{n} C_i \) is compact and \( S \) is not compact, there is an \( x_0 \in S \) such that \( x_0 \notin \bigcup_{i=1}^{n} C_i \). Since \( S \) is completely regular, there is a continuous function \( f \) on \( S \) satisfying \( f(y) = 0 \) for all \( y \in \bigcup_{i=1}^{n} C_i \), \( f(x_0) = 1 \) and \( 0 \leq f(x) \leq 1 \) for all \( x \in S \). It follows that \( q_{c_i}(f) = 0 \) for all \( i = 1, 2, \ldots, n \) and so \( q_{c_i}(\lambda f) = 0 \) for all \( i = 1, 2, \ldots, n \) and all real number \( \lambda \) and so \( \| \lambda f \| \leq 1 \) for all real numbers \( \lambda \) which implies
\( f \equiv 0 \) on \( S \). This is a contradiction. Hence \( C(S) \) is not normable.

Thirdly it is clear that \( C(S) \) is complete.

Now let \( K = \{ f \in C(S) : \| f \| = \sup_{x \in S} |f(x)| \leq \frac{1}{2} \} \), then it is clear that \( K \) is nonempty closed and convex.

(1) For each scalar \( \lambda \) such that \( |\lambda| = 1 \), and each \( g \in C(S) \) such that \( \| g \| \leq \frac{1}{4} \), we define \( T_{\lambda, g} : K \to K \) by

\[
T_{\lambda, g}(f) = \lambda f + g, \quad \forall f \in K.
\]

Then we shall show that \( T_{\lambda, g} \) is contractive w.r.t. \( U \) with a unique fixed point in \( K \) but is not strictly contractive w.r.t. \( U \).

Indeed, if \( f_1, f_2 \in K \) and \( C \) is any nonempty compact subset of \( S \), then

\[
q_c(T_{\lambda, g}(f_1) - T_{\lambda, g}(f_2)) = q_c(f_1^2 - f_2^2)
\]

\[
= \sup_{x \in C} |f_1^2(x) - f_2^2(x)|
\]

\[
= \sup_{x \in C} |f_1(x) - f_2(x)| \cdot |f_1(x) + f_2(x)|
\]

\[
\leq \sup_{x \in C} |f_1(x) - f_2(x)|
\]

\[
= q_c(f_1 - f_2),
\]

and so \( T_{\lambda, g} \) is nonexpansive w.r.t. \( U \). Next suppose \( q_c(f_1 - f_2) > 0 \). Then there exists an \( x_0 \in C \) such that

\[
|f_1^2(x_0) - f_2^2(x_0)| = \sup_{x \in C} |f_1^2(x) - f_2^2(x)|
\]

\[
= q_c(T_{\lambda, g}(f_1) - T_{\lambda, g}(f_2)).
\]
If \(|f_1(x_0) + f_2(x_0)| < 1\), then

\[ q_c(T_{\lambda, g}(f_1) - T_{\lambda, g}(f_2)) = |f_1^2(x_0) - f_2^2(x)| < |f_1(x_0) - f_2(x_0)| \]

\[ \leq q_c(f_1 - f_2) \]

if \(|f_1(x_0) + f_2(x_0)| = 1\), then since \(|f_1(x_0)| \leq \frac{1}{2}\), we see that

\[ f_1(x_0) = f_2(x_0) \]

and so \(q_c(T_{\lambda, g}(f_1) - T_{\lambda, g}(f_2)) = 0 < q_c(f_1 - f_2)\). Thus

\(T_{\lambda, g}\) is contractive w.r.t. \(U\). Now if \(\mu\) is any constant with

\[ 0 < \mu < 1, \]

choose any constant \(a > 0\) with \(\mu - \frac{1}{2} < a < \frac{1}{2}\). Define

\[ h_1 \equiv \frac{1}{2} \quad \text{and} \quad h_2 \equiv a, \]

then \(h_1, h_2 \in K\). Since \(\mu - \frac{1}{2} < a < \frac{1}{2}\), we have

\[ (\frac{1}{2})^2 - a^2 > \mu(\frac{1}{2} - a) \]

and so \(\mu q_c(h_1 - h_2) < q_c(T_{\lambda, g}(h_1) - T_{\lambda, g}(h_2))\), and hence \(T_{\lambda, g}\) is not strictly contractive w.r.t. \(U\). Finally it can be shown that for each \(U \in U\),

\[ P_u(T_{\lambda, g}^4(f_1) - T_{\lambda, g}^4(f_2)) \leq \frac{63}{64} P_u(f_1 - f_2) \]

for all \(f_1, f_2 \in K\) so that \(T_{\lambda, g}^4\) is strictly contractive w.r.t. \(U\) and so

\(T_{\lambda, g}\) has a unique fixed point in \(K\) by Corollary 2.5.

(2) Suppose \(T: K \rightarrow K\) is nonexpansive w.r.t. \(U\). For each scalar \(\lambda\) with \(|\lambda| \leq 1\) and each \(g \in C(S)\) with \(\|g\|_\infty \leq \frac{1}{4}\), and each positive integer \(n \geq 3\), we define \(T_{\lambda, n, g}: K \rightarrow K\) by

\[ T_{\lambda, n, g}(f) = \lambda(Tf)^n + g, \quad \forall f \in K. \]

Suppose \(C\) is any nonempty compact subset of \(S\) and \(f_1, f_2 \in K\). Then
\[ q_c(T_{\lambda,n,g}(f_1) - T_{\lambda,n,g}(f_2)) \]
\[ = q_c(\lambda(Tf_1)^n - \lambda(Tf_2)^n) \]
\[ = \sup_{x \in C} |((Tf_1)(x))^n - ((Tf_2)(x))^n| \]
\[ = \sup_{x \in C} |(Tf_1)(x) - (Tf_2)(x)| |((Tf_1)(x))^{n-1} + \ldots + (Tf_2)(x))^{n-1}| \text{ (n-terms)} \]
\[ \leq \sup_{x \in C} |(Tf_1)(x) - (Tf_2)(x)| \left\{ \left( \frac{1}{2} \right)^{n-1} + \ldots + \left( \frac{1}{2} \right)^{n-1} \right\} \text{ (n-terms)} \]
\[ = \frac{n}{2^{n-1}} q_c(T(f_1) - T(f_2)) \]
\[ \leq \frac{n}{2^{n-1}} q_c(f_1 - f_2) ; \]

since \( 0 < \frac{n}{2^{n-1}} < 1 \), we see that \( T_{\lambda,n,g} \) is strictly contractive w.r.t. \( U \). Hence by Theorem 2.4., \( T_{\lambda,n,g} \) has a unique fixed point in \( K \).

(3) Suppose \( T : K \to K \) is nonexpansive w.r.t. \( U \). For each scalar \( \lambda \) with \( 0 < \lambda < 1 \), for each \( g \in C(S) \) with \( \|g\|_\infty \leq \frac{1}{4} \) and for each positive integer \( n \geq 2 \), we define \( V_{\lambda,n,g} : K \to K \) by

\[ V_{\lambda,n,g}(f) = \lambda(Tf)^n + (1 - \lambda)g \text{, for all } f \in K \text{.} \]

Then for any nonempty compact subset \( C \) of \( S \), we have

\[ q_c(V_{\lambda,n,g}(f_1) - V_{\lambda,n,g}(f_2)) \leq \lambda \frac{n}{2^{n-1}} q_c(f_1 - f_2) \text{ for all } f_1, f_2 \in K \text{, and} \]

so \( V_{\lambda,n,g} \) is also strictly contractive w.r.t. \( U \) and hence by Theorem 2.4. \( V_{\lambda,n,g} \) has a unique fixed point in \( K \).

Remark 4.7.

Suppose \( X \) is sequentially complete and \( T : X \to X \) be
strictly contractive w.r.t. \( U \). Then for arbitrarily fixed \( y \in X \), the equation \( x - Tx = y \) has a unique solution in \( X \).

**Proof**: Define \( S : X \to X \) by \( S(x) = T(x) + y \). Then \( S \) is strictly contractive w.r.t. \( U \) and so \( S \) has a unique fixed point in \( X \), by Theorem 2.4., and so the equation \( x - Tx = y \) has a unique solution in \( X \).

Corresponding to Theorem 2.4., we have the following implicit function theorem which is analogous to a result of E. Dubinsky in [7]:

**Theorem 4.8.**

Let \( K \subseteq X \) be nonempty bounded and sequentially complete, \( S \) be a topological space and \( f : K \times S \to K \) be continuous. Suppose for each \( U \in \mathcal{U} \), there is a constant \( \lambda_U \) with \( 0 < \lambda_U < 1 \) such that

\[
P_u(f(x, s) - f(y, s)) \leq \lambda_u P_u(x - y) \quad \text{for all } x, y \in K \text{ and all } s \in S.
\]

Then there is a unique continuous mapping \( T : S \to K \) such that

\[
f(T(s), s) = T(s) \quad \text{for all } s \in S.
\]

**Proof**: For each \( s \in S \), define \( g_s : K \to K \) by \( g_s(x) = f(x, s) \) for all \( x \in K \). Then \( g_s \) is strictly contractive w.r.t. \( U \). Thus by Theorem 2.4., there is a unique \( T(s) \in K \) such that \( g_s(T(s)) = T(s) \). Hence there is a unique mapping \( T : S \to K \) such that \( f(T(s), s) = T(s) \), for all \( s \in S \). It remains to show that \( T \) is continuous.
Fix any \( x_0 \in K \), we define \( T_n : S \to K \) for all positive integers \( n \) as follows: \( T_1(s) = f(x_0, s) \) and \( T_{n+1}(s) = f(T_n(s), s) \) for all \( s \in S \) and all \( n = 1, 2, \ldots \). Suppose \( s \to s \) in \( S \), then

\[
(x_0, s) \to (x_1, s) \quad \text{and so} \quad T_1(s) = f(x_0, s) \to f(x_1, s) = T_1(s), \quad \text{and hence} \quad T_1 \quad \text{is continuous. Suppose} \quad T_k \quad \text{is continuous and} \quad s \to s \quad \text{in} \quad S, \quad \text{then}
\]

\[
T_k(s) \to T_k(s) \implies (T_k(s), s) \to (T_k(s), s) \quad \text{and so}
\]

\[
T_{k+1}(s) = f(T_k(s), s) \to f(T_k(s), s) = T_{k+1}(s), \quad \text{and therefore} \quad T_{k+1} \quad \text{is also continuous. Thus by induction,} \quad T_n : S \to K \quad \text{is continuous for all} \quad n = 1, 2, \ldots .
\]

Next we want to show that for any \( U \in \mathcal{U} \), \( \exists \) a positive integer \( N(U) \) such that \( P_u(T_n(s) - T(s)) \leq \frac{1}{3} \), for all \( n > N \) and all \( s \in S \).

Indeed, since \( K \) is bounded, \( \delta_u(K) = \sup_{x,y \in K} P_u(x - y) < \infty \), and so we may choose a positive integer \( N(U) \) with \( \lambda_u \delta_u(K) < \frac{1}{3} \). Thus for \( n > N \),

\[
P_u(T_n(s) - T(s)) = P_u(f(T_{n-1}(s), s) - f(T(s), s))
\]

\[
\leq \lambda_u P_u(T_{n-1}(s) - T(s))
\]

\[
\leq \ldots
\]

\[
\leq \lambda_u^{n-1} P_u(T_1(s) - T(s))
\]

\[
= \lambda_u^{n-1} P_u(f(x_0, s) - f(T(s), s))
\]

\[
\leq \lambda_u^n P_u(x_0 - T(s))
\]

\[
\leq \lambda_u^n \delta_u(K)
\]

\[
\leq \lambda_u^{N} \delta_u(K)
\]

\[
\leq \frac{1}{3}.
\]
Finally suppose \( s_\mu \rightarrow s \) in \( S \). Let \( U \in \mathcal{U} \), then there is a positive integer \( N(U) \) with \( P_n(T(s) - T(s)) \leq \frac{1}{3} \), for all \( n > N \) and all \( s \in S \). Since \( T_{N+1} \) is continuous, there is a \( \mu_0 \) with

\[
P_n(T_{N+1}(s_\mu) - T_{N+1}(s)) \leq \frac{1}{3} \quad \text{for all} \quad \mu > \mu_0.
\]

Thus

\[
P_n(T(s_\mu) - T(s))
\]

\[
\leq P_n(T(s_\mu) - T_{N+1}(s_\mu)) + P_n(T_{N+1}(s_\mu) - T_{N+1}(s)) + P_n(T_{N+1}(s) - T(s))
\]

\[
\leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1,
\]

for all \( \mu > \mu_0 \) and so \( T(s_\mu) - T(s) \in U \), for all \( U \in \mathcal{U} \). Therefore \( T(s_\mu) \rightarrow T(s) \) and hence \( T \) is continuous. This completes the proof.

**I-5. Nets of contractive mappings.**

The purpose of this section is to discuss the convergence of nets of contractive mapping and nets of fixed points.

**Definition 5.1.**

Let \( K \subseteq X \) be nonempty. Then \( T : K \rightarrow K \) is said to be **uniformly continuous w.r.t. \( \mathcal{U} \)** if and only if for each \( \varepsilon > 0 \) and for each \( U \in \mathcal{U} \), there is a \( \delta(\varepsilon, U) > 0 \) such that for all \( x, y \in K \), if

\[
P_n(x - y) < \delta,
\]

then \( P_n(T(x) - T(y)) < \varepsilon \).

If \( K \subseteq X \) is nonempty and \( T : K \rightarrow K \) is uniformly continuous w.r.t. \( \mathcal{U} \), then it is clear that \( T \) is continuous. Also if
T: K → K is nonexpansive w.r.t. U then T is uniformly continuous w.r.t. U.

**Definition 5.2.**

Let K ⊆ X be nonempty and B ⊆ K be nonempty. Let F be a family of mappings from K into itself. Then F is said to be **equicontinuous w.r.t.** U on B if and only if for each ε > 0 and each U ∈ U, there is a δ(ε, U) > 0 such that for all x, y ∈ B, if P_u(x - y) < δ, then P_u(T(x) - T(y)) < ε for each T ∈ F.

It is clear that if F consists of nonexpansive mappings w.r.t. U, then F is equicontinuous w.r.t. U.

**Definition 5.3.**

Let B, K ⊆ X be nonempty with B ⊆ K, T: K → K and \{T_α\}_α∈Γ be a net of mappings on K (into itself), then

(i) \(T_α → T\) pointwise on B if and only if \(T_α(x) → T(x)\) for each \(x ∈ B\),

(ii) \(T_α → T\) uniformly w.r.t. U on B if and only if for each ε > 0 and each \(U ∈ U\), there is a \(α_0 ∈ Γ\) such that if \(α ∈ Γ\) and \(α ≥ α_0\), then \(P_u(T_α(x) - T(x)) < ε\), for all \(x ∈ B\).

**Proposition 5.4.**

Let B ⊆ K ⊆ X be nonempty and B is compact. Suppose T: K → K and \{T_α\}_α∈Γ is a net of mappings on K such that (i) \(T_α → T\) pointwise on B and (ii) \{T_α : α ∈ Γ\} ∪ \{T\} is equicontinuous w.r.t. U on B. Then \(T_α → T\) uniformly w.r.t. U on B.
Proof: Suppose $\varepsilon > 0$ and $U \in \mathcal{U}$. Then there is a $\delta(\varepsilon, U) > 0$ such that $P_u(T_\alpha(x) - T_\alpha(y)) < \frac{\varepsilon}{3}$ for each $\alpha \in \Gamma$ and $P_u(T(x) - T(y)) < \frac{\varepsilon}{3}$ for all $x, y \in K$ with $P_u(x - y) < \delta$. Since $B \subseteq \bigcup_{x \in B} x + \delta U^i$ and $B$ is compact, there exists $\{x_1, \ldots, x_n\} \subseteq B$ with $B \subseteq \bigcup_{j=1}^n x_j + \delta U^i$. Since $T_\alpha \to T$ pointwise on $B$, there are $\beta_j \in \Gamma$ such that $T_\alpha(x_j) - T(x_j) \leq \frac{\varepsilon}{3} U^i$ for all $\alpha > \beta_j$ and all $i = 1, 2, \ldots, n$.

Take $\beta \in \Gamma$ such that $\beta_j \leq \beta$ for all $j = 1, 2, \ldots, n$. Then for any $x \in B$, $x \in x_j + \delta U^i$ for some $j \in \{1, 2, \ldots, n\}$, so that $P_u(x - x_j) < \delta$ and it follows that $P_u(T_\alpha(x) - T_\alpha(x_j)) < \frac{\varepsilon}{3}$ for all $\alpha \in \Gamma$ and $P_u(T(x) - T(x_j)) < \frac{\varepsilon}{3}$. Hence for $\alpha > \beta$,

$$P_u(T_\alpha(x) - T(x))$$

$$\leq P_u(T_\alpha(x) - T_\alpha(x_j)) + P_u(T_\alpha(x_j) - T(x_j)) + P_u(T(x_j) - T(x))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon .$$

Hence $P_u(T_\alpha(x) - T(x)) < \varepsilon$ for all $x \in B$, for all $\alpha > \beta$. Therefore $T_\alpha \to T$ uniformly w.r.t. $U$ on $B$.

Proposition 5.5.

Let $K \subseteq X$ be nonempty, $T : K \to K$ be strictly contractive w.r.t. $U$ and $\{T_\alpha\}_{\alpha \in \Gamma}$ be a net of mappings on $K$. Suppose (i) there are $a, a_\alpha \in K$ such that $T(a) = a$ and $T_\alpha(a_\alpha) = a_\alpha$ for all $\alpha \in \Gamma$, and (ii) $T_\alpha \to T$ uniformly w.r.t. $U$ on $K$. Then $a_\alpha \to a$. 
Proof: Suppose $0 < \varepsilon < 1$ and $U \in \mathcal{U}$. Then there is a constant $0 < \lambda_U < 1$ such that $P_u(T(x) - T(y)) \leq \lambda_u P_u(x - y)$ for all $x, y \in K$. Since $T_\alpha \to T$ uniformly w.r.t. $U$ on $K$, there is an $\beta \in \Gamma$ such that if $\alpha > \beta$, then $P_u(T_\alpha(x) - T(x)) < \varepsilon(1 - \lambda_u)$, for all $x \in K$. Thus for $\alpha > \beta$,

$$P_u(a_\alpha - a) = P_u(T_\alpha(a_\alpha) - T(a))$$

$$\leq P_u(T_\alpha(a_\alpha) - T(a_\alpha)) + P_u(T(a_\alpha) - T(a))$$

$$< \varepsilon(1 - \lambda_u) + \lambda_u P_u(a_\alpha - a),$$

so that $P_u(a_\alpha - a) < \varepsilon$ for all $\alpha > \beta$, and hence $a_\alpha - a \in U$, for all $\alpha > \beta$. Therefore $a_\alpha \to a$.

Theorem 5.6.

Let $K \subseteq X$ be nonempty, $T : K \to K$ be strictly contractive w.r.t. $U$, and $\{T_\alpha : \alpha \in \Gamma\}$ be a net of contractive mappings w.r.t. $U$ on $K$. Suppose there are $a, a_\alpha \in K$ such that $T(a) = a$ and $T_\alpha(a_\alpha) = a_\alpha$ for all $\alpha \in \Gamma$. If $T_\alpha \to T$ pointwise on $K$ and $a$ has a compact nbhd in $K$, then $a_\alpha \to a$.

Proof: Let $U \in \mathcal{U}$ such that $B = a + U \cap K$ is compact. Since $\{T_\alpha : \alpha \in \Gamma\} \cup \{T\}$ is equicontinuous w.r.t. $U$ on $K$ and hence on $B$ and $T_\alpha \to T$ pointwise on $K$, and hence on $B$ and $B$ is compact, we see that $T_\alpha \to T$ uniformly w.r.t. $U$ on $B$, by Proposition 5.4. Let $\lambda_U$ be a constant such that $0 < \lambda_U < 1$ and $P_u(T(x) - T(y)) \leq \lambda_u P_u(x - y)$ for all
31.

Let $K \subseteq X$ be nonempty, $T : K \to K$ and $\{T_\alpha\}_{\alpha \in \Gamma}$ be a net of nonexpansive mappings w.r.t. $U$ on $K$. Suppose (i) $T_\alpha \to T$ pointwise on $K$, (ii) there are $a_\alpha \in K$ such that $T_\alpha(a_\alpha) = a_\alpha$ for all $\alpha \in \Gamma$ and (iii) there is a subnet $\{a_{\alpha'}\}_{\alpha'}$ of $\{a_\alpha\}_{\alpha \in \Gamma}$ and there is an $x_0 \in K$ such that $a_{\alpha'} \to x_0$. Then $T(x_0) = x_0$. 

\textbf{Proposition 5.7.}

Choose $\beta \in \Gamma$ such that $P_u(T_\beta(x) - T(x)) < 1 - \lambda_u$ for all $x \in B$ and all $\alpha > \beta$. If $x \in B$, then $x - a \in U$ so that $P_u(x-a) \leq 1$, and so for $\alpha > \beta$, we have

$$P_u(T_\alpha(x) - a) \leq P_u(T_\alpha(x) - T(x)) + P_u(T(x) - a)$$

$$< (1 - \lambda_u) + \lambda_u P_u(x - a)$$

$$\leq (1 - \lambda_u) + \lambda_u \cdot 1$$

$$= 1,$$

and so $T_\alpha(x) - a \in U$ for all $\alpha > \beta$ and all $x \in B$. Thus $T_\alpha(x) \in a + U \cap K = B$ for all $x \in B$ and all $\alpha > \beta$. Thus $T_\alpha : B \to B$ is contractive w.r.t. $U$ for all $\alpha > \beta$ and $B$ is compact and so by Corollary 3.7. $T_\alpha$ has a unique fixed point in $B$ for each $\alpha > \beta$. But $T_\alpha$ has a fixed point $a_\alpha \in K$ which must also be unique, we see that $a_\alpha \in B$ for all $\alpha > \beta$. Also for $x \in B$, we see that

$$P_u(T(x) - a) = P_u(T(x) - T(a)) \leq \lambda_u P_u(x - a) \leq \lambda_u \cdot 1 < 1,$$

so that $T(x) - a \in U$ for all $x \in B$ and so $T(x) \in a + U \cap K = B$ for all $x \in B$. Thus $T : B \to B$. Hence by Proposition 5.5., $a_\alpha \to a$.
Proof: Suppose $U \in U$. Then there is an $\beta'$ such that 

$$P_u(a', - x_0) < \frac{1}{2} \text{ for all } a' > \beta'. \quad (*)$$

Since $T_{\beta}(x_0) \to T(x_0)$ and so

$$T_{\beta}(x_0) \to T(x_0),$$

there is an $\gamma'$ such that 

$$P_u(T_{\beta}(x_0) - T(x_0)) < \frac{1}{2}$$

for all $a' > \gamma'$. Fix any $\delta' > \beta'$ and $\gamma'$, then for $a' > \delta'$,

$$P_u(a', - T(x_0)) = P_u(T_{\beta}^{-1}(a), - T(x_0))$$

$$< P_u(T_{\beta}^{-1}(a), - T_{\beta}(x_0)) + P_u(T_{\beta}(x_0) - T(x_0))$$

$$< P_u(a', - x_0) + \frac{1}{2}$$

$$< \frac{1}{2} + \frac{1}{2}$$

$$= 1,$$

so that $a', - T(x_0) \in U$ for all $a' > \delta'$. Hence $a_{\beta} \to T(x_0)$. Since $X$ is Hausdorff, $a_{\beta} \to T(x_0)$ and $a_{\beta} \to x_0$ imply $T(x_0) = x_0$.

Corollary 5.8.

Let $K \subseteq X$ be nonempty sequentially compact, $T : K \to K$ and $T_n : K \to K$ be nonexpansive w.r.t. $U$ for each $n = 1, 2, \ldots$.

Suppose $T_n \to T$ pointwise on $K$ and there are $a_n \in K$ with $T_n(a_n) = a_n$ for all $n = 1, 2, \ldots$. Then $T$ has a fixed point in $K$.

Definition 5.9.

Let $X_i$ be a $T_2$-l.c.s., $i = 1, 2$ and $U_1$ be a base for closed absolutely convex 0-nbhd's in $X_1$. Let $\pi_i : X_1 \times X_2 \to X_i$ be the natural projection for each $i = 1, 2$, $K \subseteq X_1 \times X_2$ be nonempty and
T : K \to K. Then T is said to be strictly contractive in the 1st variable w.r.t. \( U_1 \) if and only if for each \( y \in \pi_2(K) \), for each \( U \in U_1 \), there is a constant \( \lambda_u \) such that \( 0 < \lambda_u < 1 \) and
\[
P_u(\pi_1(T(x_1,y)) - \pi_1(T(x_2,y))) \leq \lambda_u P_u(x_1 - x_2),
\]
for all \( x_1, x_2 \in X_1 \) with \( (x_1, y), (x_2, y) \in K \).

**Lemma 5.10.**

Let \( X_i \) be \( T_2 \)-l.c.s. and \( U_i \) is a closed absolutely convex 0-nbhd in \( X_i \), \( i = 1, 2 \). Then
\[
P_{u_1}((x_1, x_2)) = \max\{P_{u_1}(x_i) : i = 1, 2\},
\]
for all \( x_i \in X_i \), \( i = 1, 2 \).

**Proof:**
\[
P_{u_1}((x_1, x_2)) = \inf\{\lambda > 0 : (x_1, x_2) \in \lambda(U_1 \times U_2)\}
\leq \max\{P_{u_1}(x_i) : i = 1, 2\},
\]
since \( x_i \in P_{u_1}(x_i)U_i \) for all \( i = 1, 2 \). But if
\[
P_{u_1}((x_1, x_2)) < \max\{P_{u_1}(x_i) : i = 1, 2\} = P_{u_1}(x_i),
\]
say, then
\[
x_1 \notin P_{u_1}((x_1, x_2)) \text{ and so } (x_1, x_2) \notin P_{u_1}((x_1, x_2))U_1 \times U_2 \text{ which is impossible. Hence } P_{u_1}((x_1, x_2)) = \max\{P_{u_1}(x_i) : i = 1, 2\}.
\]

**Theorem 5.11.**

Let \( X_i \) be \( T_2 \)-l.c.s., \( U_i \) be a base for closed absolutely convex 0-nbhds in \( X_i \) for all \( i = 1, 2 \). Suppose \( M \subseteq X_i \) is
nonempty such that every strictly contractive mapping w.r.t. \( U_1 \) on \( M \) has a fixed point (e.g. \( M \) is sequentially complete) and \( N \subseteq X_2 \) is nonempty such that every continuous mapping on \( N \) has a fixed point. (e.g. \( N \) is compact convex). If (i) \( T : M \times N \rightarrow M \times N \) is uniformly continuous w.r.t. \( U_1 \times U_2 \) and (ii) \( T \) is strictly contractive in the 1st variable w.r.t. \( U_1 \), then \( T \) has a fixed point in \( M \times N \).

**Proof:** For any \( y \in N \), we define \( T_y : M \rightarrow M \) by

\[
T_y(x) = \pi_1 \circ T(x, y), \text{ for all } x \in M.
\]

Then by (ii), \( T_y \) is strictly contractive w.r.t. \( U_1 \) and so by assumption, \( T_y \) has a (unique) fixed point in \( M \) for each \( y \in N \). Define \( F : N \rightarrow M \) by \( F(y) = \) the unique fixed point of \( T_y \) in \( M \), for all \( y \in N \). Thus \( T_y(F(y)) = F(y) \) for all \( y \in N \). We shall show that \( F \) is continuous. Indeed, suppose \( y_\alpha \rightarrow y \) in \( N \). If \( U_1 \in U_1 \) and \( U_2 \in U_2 \), then since \( T \) is uniformly continuous w.r.t. \( U_1 \times U_2 \), for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[
P_{u_1 \times u_2}((x_1, y_1) - (x_2, y_2)) < \delta \text{ implies } P_{u_1 \times u_2}(T(x_1, y_1) - T(x_2, y_2)) < \varepsilon
\]

for all \( (x_1, y_1), (x_2, y_2) \in M \times N \). Since \( y_\alpha \rightarrow y \), there is an \( \alpha_0 \) such that \( \alpha > \alpha_0 \) implies \( P_{u_2}(y_\alpha - y) < \delta \). Hence for \( x_1, x_2 \in M \) such that \( P_{u_1}(x_1 - x_2) < \delta \), we have

\[
P_{u_1 \times u_2}((x_1, y_\alpha) - (x_2, y)) = \max \{ P_{u_1}(x_1 - x_2), P_{u_2}(y_\alpha - y) \} < \delta,
\]

for all \( \alpha > \alpha_0 \), by Lemma 5.10, and it follows that

\[
P_{u_1 \times u_2}(T(x_1, y_\alpha) - T(x_2, y)) < \varepsilon \text{ for all } \alpha > \alpha_0.
\]

Hence for any \( x \in M \),
and $\alpha > \alpha_0$, we have $P_{\gamma_1}(\pi_0 \circ T(x, \gamma) - \pi_1 \circ T(x, y)) < P_{\gamma_1 \times \gamma_2}(T(x, \gamma) - T(x, y)) < \varepsilon$.

Thus $P_{\gamma_1}(T_1(x) - T_1(y)) < \varepsilon$, for all $x \in M$ and for all $\alpha > \alpha_0$. It follows that $T_1 \rightarrow T_1$ uniformly w.r.t. $\gamma_1$ on $M$. Thus by Proposition 5.6., $F(y) = F(y)$, and so $F$ is continuous.

Define $G : N \rightarrow N$ by $G(y) = \pi_2 \circ T(F(y), y)$ for all $y \in N$. Then clearly $G$ is a continuous mapping on $N$ and so by assumption, there exists an $p \in N$ such that $G(p) = p$. Since $p = G(p) = \pi_2 \circ T(F(y), y)$ and $F(p) = T_p(F(y)) = \pi_1 \circ T(F(p), p)$, we see that

$$T(F(p), p) = (\pi_1 \circ T(F(p), p), \pi_2 \circ T(F(p), p))$$

$$= (F(p), p),$$

so that $(F(p), p)$ is a fixed point of $T$ in $M \times N$. 
CHAPTER II

NORMAL STRUCTURE

II-1. Centre of a set.

Suppose \( X \) is a \( T_2\)-l.c.s., \( H \) and \( K \) are subsets of \( X \) such that \( H \) is bounded. If \( U \) is any closed absolutely convex 0-nbhd, we denote

\[
\begin{align*}
    r_x(U; H) &= \inf \{ r > 0 : x - H \subseteq rU \} \quad \text{for any } x \in X; \\
    r(U; H, K) &= \inf \{ r_x(U; H) : x \in K \} ; \\
    C(U; H, K) &= \{ x \in K : r_x(U; H) = r(U; H, K) \} .
\end{align*}
\]

\( C(U; H, H) \) is called the centre of \( H \) relative to \( U \) (or w.r.t. \( U \)).

If \( H \neq \emptyset \), we see that \( r_x(U; H) = \sup \{ P_u(x - y) : y \in H \} \). The following two lemmas are obvious.

Lemma 1.1.

If \( X \) is a \( T_2\)-l.c.s., \( H \subseteq X \) is a nonempty bounded subset, \( U \) is a closed absolutely convex 0-nbhd, define \( f : X \to \mathbb{R} \) by \( f(x) = r_x(U; H) \) for all \( x \in X \). Then \( f \) is continuous and

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) , \quad \text{for all } x, y \in X \text{ and } 0 \leq \lambda \leq 1.
\]

Lemma 1.2.

Let \( X \) be a \( T_2\)-l.c.s., \( K \subseteq X \) be nonempty, and \( H \subseteq X \).
be nonempty bounded. If \( U \) is any closed absolutely convex \( 0 \)-nbhd, then for any subsets \( H_1, K_1 \) of \( X \) such that \( H \subseteq H_1 \subseteq \overline{\text{co}}(H) \) and 
\( K \subseteq K_1 \subseteq \overline{\text{co}}(K) \), we have 
(i) \( r(U; H, K) = r(U; H_1, K_1) \),
and (ii) \( C(U; H, K) = C(U; H_1, K) \subseteq C(U; H_1, K_1) \).

**Proposition 1.3.**

Let \( X \) be a \( T_2 \)-l.c.s., \( H \subseteq K \) be nonempty bounded and convex, and \( U \) be any closed absolutely convex \( 0 \)-nbhd. If \( x \in H \) is such that \( r_x(U; H) < \delta_u(H) \), then there is an \( x_o \in H \) with 
\[ 0 < r_{x_o}(U; H) < \delta_u(H) \, . \]

**Proof:** We may assume \( r_x(U; H) = 0 \). Then \( \delta_u(H) > 0 \) implies \( r_y(U; H) > 0 \) for some \( y \in H \). Since \( \lambda x + (1 - \lambda)y \to y \) as \( \lambda \to 0^+ \), we have \( r_{\lambda x+(1-\lambda)y}(U; H) \to r_y(U; H) \) as \( \lambda \to 0^+ \), by Lemma 1.1. Thus there is \( 0 < \lambda_o < 1 \) with 
\[ |r_y(U; H) - r_{\lambda_o x+(1-\lambda_o)y}(U; H)| < \frac{1}{2} r_y(U; H) \, , \]
and so 
\[ r_{\lambda_o x+(1-\lambda_o)y}(U; H) > r_y(U; H) - \frac{1}{2} r_y(U; H) = \frac{1}{2} r_y(U; H) > 0 \, . \] Take \( x_o = \lambda_o x + (1 - \lambda_o)y \), then \( x_o \in H \) since \( H \) is convex, and \( r_{x_o}(U; H) > 0 \).

Also by Lemma 1.1,
\[ r_{x_o}(U; H) = r_{\lambda_o x+(1-\lambda_o)y}(U; H) \] 
\[ \leq \lambda_o r_x(U; H) + (1 - \lambda_o)r_y(U; H) \] 
\[ = (1 - \lambda_o)r_y(U; H) \] 
\[ < r_y(U; H) \leq \delta_u(H) \, . \]
Proposition 1.4.

Let $X$ be a $T_2$-l.c.s., $K \subseteq X$ be nonempty weakly compact convex, $H \subseteq K$ be nonempty convex. If $U$ is any closed absolutely convex $0$-nbhd in $X$, then $C(U; H, K)$ is nonempty closed and convex.

Proof:

For each positive integer $n$, we denote

$$F(y; n) = \{x \in K : x - y \in (r(U; H, K) + \frac{1}{n})U\},$$

for each $y \in H$ and

$$C_n(U) = \bigcap_{y \in H} F(y; n) = \{x \in K : x - H \subseteq (r(U; H, K) + \frac{1}{n})U\}.$$  

Then it is clear that $C(U; H, K) = \bigcap_{n=1}^{\infty} C_n(U)$. For each positive integer $n$, since $r(U; H, K) + \frac{1}{n} > r(U; H, K)$, there is an $x \in K$ such that $r_x(U; H) < r(U; H, K) + \frac{1}{n}$, and so $x - H \subseteq (r(U; H, K) + \frac{1}{n})U$ and so $x \in C_n(U)$; thus $C_n(U) \neq \emptyset$. It is also clear that $F(y, n)$ is closed and convex for each positive integer $n$ and therefore $C_n(U)$ is nonempty closed and convex. Moreover, since $K$ is weakly compact and

$K \supseteq C_n(U) \supseteq C_{n+1}(U) \neq \emptyset$, we see that $\bigcap_{n=1}^{\infty} C_n(U) \neq \emptyset$. Hence $C(U; H, K)$ is nonempty closed and convex.

The concept of normal structure was introduced by M. S. Brodskii and D. P. Milman in [4]. The following definition is its generalization.

Definition 2.1.

Let $X$ be a $T_2$-l.c.s., $U$ be a base for closed absolutely convex $0$-nbhds, and $K \subseteq X$. Then $K$ is said to have normal structure w.r.t. $U$ if and only if for any bounded convex subset $H$ of $K$ containing more than one point, there is an $U \in U$ and $x_0 \in H$ such that $r_{x_0} (U; H) < \delta_u (H)$. In this case $x_0$ is called a non-diametral point of $H$ w.r.t. $U$.

Theorem 2.2.

Let $X$ be a $T_2$-l.c.s., $U$ be a base for closed absolutely convex $0$-nbhds and $K \subseteq X$ be nonempty weakly compact convex. Then $K$ has normal structure w.r.t. $U$ if and only if for any convex subset $H$ of $K$ containing more than one point there is an $U \in U$ such that $\delta_u (C(U; H, H)) < \delta_u (H)$.

Proof: Suppose $K$ has normal structure w.r.t. $U$. If $H$ is a nonempty subset of $K$ containing more than one point, then there is a $U \in U$ and an $x \in H$ such that $r_x (U; \overline{H}) = r_x (U; H) < \delta_u (H) = \delta_u (\overline{H})$. If $z, w \in C(U; \overline{H}, \overline{H})$, then $z - w \in r_z (U; \overline{H})U = r(U; \overline{H}, \overline{H})U$ implies
\[ \delta_u(C(U; H, H)) \leq r(U; H, H) \leq r_x(U; \overline{H}) \] and so \( \delta_u(C(U; H, H)) \leq \delta_u(H) \).

But \( C(U; H, H) \subseteq C(U; \overline{H}, \overline{H}) \), we see that
\[ \delta_u(C(U; H, H)) \leq \delta_u(C(U; \overline{H}, \overline{H})) \leq \delta_u(H) \).

Conversely, suppose \( H \) is any convex subset of \( K \) containing more than one point and \( U \in \mathcal{U} \) is such that
\[ \delta_u(C(U; H, H)) < \delta_u(H) \). By Proposition 1.4. \( C(U; \overline{H}, \overline{H}) \neq \emptyset \). Choose \( x \in C(U; \overline{H}, \overline{H}) \). Then \( r_x(U; H) = r_x(U; \overline{H}) = r(U; \overline{H}, \overline{H}) \). Since there is a \( y \in \overline{H} \) such that \( r_y(U; H) > r(U; \overline{H}, \overline{H}) \), we see that
\[ r(U; \overline{H}, \overline{H}) < \delta_u(H) = \delta_u(H) \). By Lemma 1.1., there is an \( x_0 \in H \) such that
\[ |r_x(U; H) - r_{x_0}(U; H)| < \frac{1}{2}(\delta_u(H) - r(U; H, H)) \).

Thus \( r_{x_0}(U; H) < \frac{1}{2}(\delta_u(H) + r(U; H, H)) < \delta_u(H) \). Hence \( K \) has normal structure w.r.t. \( U \).

The above theorem generalizes a result due to W. A. Kirk in [13].

**Theorem 2.3.**

Let \( X \) be a \( T_2 \)-l.c.s., \( K \subseteq X \) be compact convex. If \( H \) is any convex subset of \( K \) containing more than one point, then for any closed absolutely convex 0-nbhd \( U \) with \( \delta_u(H) > 0 \), there exists an \( x_o \in H \) such that \( x_o \) is a non-diametral point of \( H \) w.r.t. \( U \). In particular \( K \) has normal structure w.r.t. any base for closed absolutely convex 0-nbhds.
Proof: Choose \( x_n, y_n \in H \) such that \( P_u(x_n - y_n) \neq \delta_u(H) = r \).

Since \( H \) is compact, there are \( x, y \in \overline{H} \) and subsequences \( \{x_{n_i}\}_{i=1}^{\infty} \) of \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_{n_i}\}_{i=1}^{\infty} \) of \( \{y_n\}_{n=1}^{\infty} \) with \( x_{n_i} \to x \) and \( y_{n_i} \to y \). It follows that

\[
P_u(x - y) = P_u\left(\lim_{i \to \infty} (x_{n_i} - y_{n_i})\right) = \lim_{i \to \infty} P_u(x_{n_i} - y_{n_i}) = r.
\]

Let \( F = \{M \subseteq \overline{H} : \{x, y\} \subseteq M \text{ and } P_u(a - b) = r \text{ for all } a, b \in M \text{ with } a \neq b\} \). Partially order \( F \) by \( \subseteq \). Then \( F \neq \emptyset \) since \( \{x, y\} \in F \). If \( \{M_\lambda\}_{\lambda \in \Gamma} \) is any chain in \( F \), then clearly

\[
\bigcup_{\lambda \in \Gamma} M_\lambda \in F.
\]

Therefore by Zorn's Lemma, \( F \) has a maximal element, say \( M_0 \). If \( M_0 \) were infinite, let \( a_1, a_2, \ldots \) be countably infinite distinct elements of \( M_0 \). Then there is a subsequence \( \{a_{n_i}\}_{i=1}^{\infty} \) of \( \{a_n\}_{n=1}^{\infty} \) and \( a \in \overline{H} \) with \( a_{n_i} \to a \). Thus there is an \( i_0 \) with \( a_{n_i} \in a + \frac{1}{3}rU \) for all \( i > i_0 \). It follows that \( a_{n_{i_0}+1}, a_{n_{i_0}+2} \in a + \frac{1}{3}rU \) and so

\[
a_{n_{i_0}+1} - a_{n_{i_0}+2} \in \frac{2}{3}rU, \text{ and hence } P_u(a_{n_{i_0}+1} - a_{n_{i_0}+2}) \leq \frac{2}{3}r, \text{ which is impossible. Thus } M_0 \text{ must be finite, say } M_0 = \{x_1, \ldots, x_n\} \text{ where } x_i \neq x_j \text{ if } i \neq j. \text{ Denote } w = \frac{1}{n} \sum_{i=1}^{n} x_i. \text{ Then } w \in \overline{H} \text{ since } \overline{H} \text{ is convex. We shall show that } w \text{ is a non-diametral point of } \overline{H} \text{ w.r.t. } U. \]

Indeed we may assume \( s_0 = r_w(U; \overline{H}) > 0 \). Since \( \overline{H} \) is compact, there is
an $z \in \overline{H}$ such that $P_{u}(w - z) = s_o$ by the same proof as above. If $s_o = r$, then $P_{u}(z - x_i) = r$ for all $i = 1, 2, \ldots, n$, for if

$$t_o = P_{u}(z - x_{i_o}) < r$$

for some $i_o \in \{1, 2, \ldots, n\}$, then

$$z - w = \sum_{i=1}^{n} \frac{1}{n}(z - x_i) \in \bigoplus_{i \neq i_o} \frac{1}{n} rU + \frac{t_o}{n} U = \frac{n-1}{n} r + \frac{t_o}{n} U,$$

so that $P_{u}(z - w) \leq \frac{n-1}{n} r + \frac{t_o}{n} < r$, which is impossible. Thus $M_o \cup \{z\} \in F$ and $M_o$ is maximal in $F$ imply $M_o = M_o \cup \{z\}$ and hence $z = x_j$ for some $j \in \{1, 2, \ldots, n\}$. But then

$$r = P_{u}(x_j - z) = P_{u}(x_j - x_j) = 0,$$

again contradicts $r > 0$. Hence

$$s_o < r$$

and so $w$ is a non-diametral point of $H$ w.r.t. $U$.

Finally choose $w_o \in H$ with $w - w_o \in \frac{r - s_o}{2} U$. Then for any $h \in H$, $w_o - h = w_o - w + w - h \in \frac{r - s_o}{2} U + s_o U = \frac{r + s_o}{2} U$. It follows that $r_{w_o}(U; H) \leq \frac{r + s_o}{2}$. Therefore $w_o$ is a non-diametral point of $H$ w.r.t. $U$.

The above theorem generalizes a result due to R. DeMarr in [6].

Theorem 2.4.

Let $X_i$ be $T_2$-l.c.s. and $U$ be a base for closed absolutely convex 0-nbhd in $X_i$ where $i = 1, 2$. If $X_i$ has normal structure w.r.t. $U_i$ for all $i = 1, 2$, then $X = X_1 \times X_2$ has normal structure w.r.t. $U = U_1 \times U_2 = \{U_1 \times U_2 : U_i \in U_i, i = 1, 2\}$.
Proof: Let $K$ be a bounded subset of $X$ containing more than one point. Let $\pi_i$ be the natural projection of $X$ onto $X_i$ and $K_i = \pi_i(K)$ for $i = 1, 2$. Then $K_i$ is bounded and convex for $i = 1, 2$.

If either $K_1$ or $K_2$, say $K_1$, contains only one point, say $x_1$, then $K_2$ contains more than one point; since $X_2$ has normal structure w.r.t. $U_2$, there is an $x_2 \in K_2$ and a $U_2 \in U_2$ with $\delta_{u_2}(K_2) > r_{x_2}(U_2; K_2)$. Thus for any $U_1 \in U_1$, we have

$$\delta_{u_1 \times u_2}(K) = \inf\{r > 0 : K \subseteq r(U_1 \times U_2)\}$$

$$= \inf\{r > 0 : (x_1, y_1) - (x_1, y_2) \in r(U_1 \times U_2) \text{ for all } y_1, y_2 \in K_2\}$$

$$= \inf\{r > 0 : y_1 - y_2 \in rU_2 \text{ for all } y_1, y_2 \in K_2\}$$

$$= \inf\{r > 0 : K_2 - K_2 \subseteq rU_2\}$$

$$= \delta_{u_2}(K_2) > r_{x_2}(U_2; K_2)$$

$$= \inf\{r > 0 : x_2 - K_2 \subseteq rU_2\}$$

$$= \inf\{r > 0 : (x_1, x_2) - K \subseteq r(U_1 \times U_2)\}$$

$$= r(x_1, x_2)(U_1 \times U_2; K),$$

so that $(x_1, x_2)$ is a non-diametral point of $K$ w.r.t. $U_1 \times U_2$ for any $U_1 \in U_1$. Thus we may assume that for each $i = 1, 2$, $K_i$ contains more than one point. It follows that there are $U_1 \in U_1$ and $x_i \in K_i$ such that $r_i = \delta_{u_1}(K_1) > r_{x_i}(U_i; K_i) = s_i$, for all $i = 1, 2$. Choose any $u_i \in \pi_i^{-1}(\{x_i\}) \cap K$, $i = 1, 2$. Then $u_1 = (x_1, v)$ and $u_2 = (w, x_2)$ for
some \( v \in K_2 \) and \( w \in K_1 \). Denote \( m = \frac{1}{2}(u_1 + u_2) \), then \( m \in K \) since \( K \) is convex. If \( z \in K \), then \( z = (z_1, z_2) \) where \( z_i \in K_i, i = 1, 2 \).

Thus

\[
m - z = \frac{1}{2}(u_1 + u_2) - z
\]

\[
= \frac{1}{2}((x_1, v) + (w, x_2)) - (z_1, z_2)
\]

\[
= (\frac{1}{2}(x_1 + w) - z_1, \frac{1}{2}(v + x_2) - z_2)
\]

\[
= \frac{1}{2}(x_1 - z_1, x_2 - z_2) + \frac{1}{2}(w - z_1, v - z_2)
\]

\[
\in \frac{1}{2}(s_1 U_1 \times s_2 U_2) + \frac{1}{2}(r_1 U_1 \times r_2 U_2)
\]

\[
\subseteq \frac{1}{2} s(U_1 \times U_2) + \frac{1}{2} r(U_1 \times U_2)
\]

\[
= \frac{1}{2}(r + s)(U_1 \times U_2)
\]

where \( s = \max \{s_1, s_2\} \) and \( r = \max \{r_1, r_2\} \). Thus

\[
r_m(U_1 \times U_2; K) \leq \frac{1}{2}(r + s) \quad \text{Suppose} \quad r = r_1 \quad \text{then since} \quad \frac{1}{2}(r + s) < r,
\]

\( K_1 - K_1 \subseteq \frac{1}{2}(r + s)U \) and so there are \( y_1, y_2 \in K_1 \) with \( y_1 - y_2 \notin \frac{1}{2}(r + s)U_1 \).

Choose any \( z_1, z_2 \in K \) with \( (y_i, z_i) \in K \) for \( i = 1, 2 \), then

\[
(y_1, z_1) - (y_2, z_2) \notin \frac{1}{2}(r + s)(U_1 \times U_2) \quad \text{implies} \quad \delta_u(K) > \frac{1}{2}(r + s).
\]

Hence \( m \) is a non-diametral point of \( K \) w.r.t. \( U_1 \times U_2 \). Therefore \( X_1 \times X_2 \) has normal structure w.r.t. \( U_1 \times U_2 \).

The above theorem generalizes Theorem 2.1. proved by L. P. Belluce, W. A. Kirk and E. F. Steiner in [3].
II-3. Some fixed point theorems.

Throughout the remaining of this chapter, \( X \) will denote a \( T_2 \)-l.c.s., and \( U \) will denote a base for closed absolutely convex 0-nbhd's in \( X \).

Definition 3.1.

Suppose \( K \subseteq X \) is nonempty and \( T : K \to K \). For each nonnegative integer \( n \), we denote
\[
0(T, n, x) = \{T^{n+i}(x) : i = 0, 1, 2, \ldots \}
\]
for all \( x \in K \).

Theorem 3.2.

Let \( K \subseteq X \) be nonempty bounded closed convex, \( T : K \to K \) be nonexpansive w.r.t. \( U \) and \( M \subseteq K \) be nonempty weakly compact such that

(i) \( \text{Co}(0(T, n, x)) \cap M \neq \emptyset \), for all \( x \in K \) and

(ii) \( \text{Co}(0(T, n, x)) \) has normal structure w.r.t. \( U \) for all \( x \in K \). Then there is an \( x \in M \) with \( T(x) = x \).

Proof: Let \( G = \{ A \subseteq K : A \) is nonempty closed convex and \( T(A) \subseteq A \} \). Partially order \( G \) by \( \supseteq \). Then by weak compactness of \( M \) and Zorn's Lemma, \( G \) has a minimal element, say \( K_1 \). Suppose there were an \( x \in K_1 \) with \( T(x) \neq x \). Then \( 0(T, o, x) \) and thus \( \text{Co}(0(T, o, x)) \) contains more than one point, and so by (ii), there is a \( U \in U \) and \( y \in \text{Co}(0(T, o, x)) \) such that \( r = r_y(U; \text{Co}(0(T, o, x))) \) < \( \delta_u(\text{Co}(0(T, o, x))) \).

By Proposition 1.3., we may assume that \( r > 0 \). Define \( A = \{ z \in K_1 : z - 0(T, n, x) \subseteq rU \) for some nonnegative integer \( n \} \), then
A ≠ ∅, since y ∈ A. Clearly A is convex and T(A) ⊆ A. Thus T(A) ⊆ A since T is continuous. Since A is also closed and convex A ∈ G and so A = K₀ by the minimality of K₀ in G.

Define S = \{z ∈ K₀ : z - K₀ ⊆ rU\}. Then clearly S is closed and convex. Now if z ∈ A, then z - O(T, n, x) ⊆ rU for some nonnegative integer n, so that O(T, n, x) ⊆ z + rU implies \(\overline{\text{co}}(O(T, n, x)) \subseteq z + rU\) and so \(\bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, n, x)) \subseteq \overline{\text{co}}(O(T, n, x)) \subseteq z + rU\) for all z ∈ A. By (i) \(\overline{\text{co}}(O(T, m, x)) \cap M ≠ ∅\) for all m = 0, 1, 2,...

Hence \(\bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, x)) \cap M ≠ ∅\) by weak compactness of M; and so \(\bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, x)) \cap M ≠ ∅\) implies \(\bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, x)) \neq ∅\). If \(t ∈ \bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, x))\), then \(t ∈ z + rU\) for all \(z ∈ A\) implies \(z ∈ t + rU\) for all \(z ∈ A\) and thus \(K₀ = A ⊆ t + rU\), and hence \(t - K₀ ⊆ rU\) implies \(t ∈ S\) for all \(t ∈ \bigcap_{m=0}^{\infty} \overline{\text{co}}(O(t, m, x))\). Thus S ≠ ∅. Moreover, if T(S) ⊆ S, take any \(z ∈ S\) with T(z) ∈ S. Let H = (T(z) + rU) ∩ K₀. If \(y ∈ H\), then \(y ∈ K₀ ⊆ z + rU\) implies \(y - z ∈ rU\) and so \(T(y) - T(z) ∈ rU\) since T is nonexpansive w.r.t. U.

It follows that \(T(y) ∈ T(z) + rU\); but \(T(y) ∈ K₀\), and so \(T(y) ∈ (T(z) + rU) ∩ K₀ = H\). Thus \(T(H) ⊆ H\). Since \(T(z) ∈ H\) so that \(H ≠ ∅\) and clearly H is closed and convex, \(H ∈ G\). Hence \(H = K₀\) by the minimality of K₀ in G. But \(T(z) ∈ S\) implies \(K₀ ⊆ T(z) + rU\) and so there exists an \(y ∈ K₀\) with \(y ∈ (T(z) + rU) \cap K₀ = H\), so that
H \subseteq K$, which is a contradiction. Hence $T(S) \subseteq S$, and so $S \in G$.

Therefore $S = K_1$ again by the minimality of $K_1$ in $G$. But

$\delta_u(K_1) = \delta_u(S) \leq r$, while $\delta_u(K_1) = \delta_u(\text{Co}(O(T, n, x))) > r$, which is impossible. This leads to the conclusion that $T(x) = x$ for all $x \in K_1$.

Since $K_1 \neq \emptyset$, there is an $x \in K_1 \subseteq K$ with $T(x) = x$. By (i), $x \in M$.

**Corollary 3.3.**

Let $K \subseteq X$ be nonempty bounded closed convex with normal structure w.r.t. $U$, $M \subseteq K$ be weakly compact and $T : K \rightarrow K$ be nonexpansive w.r.t. $U$ such that $\text{Co}(O(T, o, x)) \cap M \neq \emptyset$ for all $x \in K$.

Then there is an $x \in M$ with $T(x) = x$.

**Corollary 3.4.**

Let $K \subseteq X$ be nonempty weakly compact convex with normal structure w.r.t. $U$ and $T : K \rightarrow K$ be nonexpansive w.r.t. $U$.

Then there is an $x \in K$ with $T(x) = x$.

Next we shall see that the above corollary holds for a finite number of "commuting" nonexpansive mappings.

**Definition 3.5.**

Let $K \subseteq X$ be nonempty and $F$ be a family of mappings on $K$. Then (i) for $T_1, T_2 \in F$, $T_1$ commutes with $T_2$ if and only if $T_1T_2 = T_2T_1$, i.e. $T_1(T_2(x)) = T_2(T_1(x))$ for all $x \in K$, (ii) $F$ is a commuting family if and only if $T_1$ commutes with $T_2$ for all $T_1, T_2 \in F$. 

Theorem 3.6.

Let $K \subseteq X$ be nonempty weakly compact convex with normal structure w.r.t. $U$ and $F$ be a finite commuting family of nonexpansive mappings w.r.t. $U$ on $K$. Then there is an $x \in K$ such that $T(x) = x$ for all $T \in F$.

Proof:

Let $F = \{T_1, \ldots, T_n\}$ and we may assume that $n \geq 2$. By weak compactness of $K$ and by Zorn's Lemma, let $K_1 \subseteq K$ be minimal w.r.t. being nonempty closed convex invariant under $T_i$ for all $i = 1, 2, \ldots, n$.

Since $T_1 \ldots T_n : K_1 \to K_1$ is nonexpansive w.r.t. $U$ and $K_1$ is weakly compact convex with normal structure w.r.t. $U$, we see that by Corollary 3.4., there is an $x_0 \in K_1$ such that $T_1 \ldots T_n(x_0) = x_0$. Thus the set $M = \{x \in K_1 : T_1 \ldots T_n(x) = x\}$ is nonempty.

If $i \in \{1, 2, \ldots, n\}$ and $x \in M$, then $T_1 \ldots T_n(x) = x$ so that $T_i(x) = T_i(T_1 \ldots T_n(x)) = T_1 \ldots T_n(T_i(x))$ and so $T_i(x) \in M$. Thus $T_i(M) \subseteq M$ for each $i = 1, 2, \ldots, n$. Also for each $i \in \{1, 2, \ldots, n\}$ and each $x \in M$, let $y_i = T_1 \ldots \hat{T}_i \ldots T_n(x)$, where $T_1 \ldots \hat{T}_i \ldots T_n = T_2 \ldots T_n$ if $i = 1$,

$T_1 \ldots \hat{T}_i \ldots T_n = T_{n-1} \ldots T_n$ if $i = n$ and $T_1 \ldots \hat{T}_i \ldots T_n = T_1 \ldots T_{i-1} T_{i+1} \ldots T_n$ if $1 < i < n$. Since $T_j(M) \subseteq M$ for all $j = 1, 2, \ldots, n$, we see that $y_i \in M$. But
\( T_i(y_i) = T_i(T_1 \ldots T_i \ldots T_n(x)) = T_1 \ldots T_n(x) = x \). Hence \( T_i(M) = M \) for each \( i \in \{1, 2, \ldots, n\} \).

If \( M \) contains more than one point, then \( \overline{\text{Co}(M)} \) contains more than one point; since \( K \) has normal structure w.r.t. \( U \) there exists a \( U \in \mathcal{U} \) and an \( x \in \overline{\text{Co}(M)} \) such that \( 0 < r = r_x(U; \overline{\text{Co}(M)}) < \delta_u(\overline{\text{Co}(M)}) \).

Define \( C = \{ y \in K_1 : r_y(U; \overline{\text{Co}(M)}) < r \} \), then \( C \) is nonempty closed and convex. For each \( i \in \{1, 2, \ldots, n\} \), if \( y \in C \), then
\[
y - \overline{\text{Co}(M)} \subseteq rU \quad \text{and so} \quad y - M \subseteq rU ;
\]
since \( T_i \) is nonexpansive w.r.t. \( U \), it follows that \( T_i(y) - T_i(M) \subseteq rU \), and so \( T_i(y) - M \subseteq rU \) since \( T_i(M) = M \), and hence \( r_y(U; \overline{\text{Co}(M)}) = r_y(U; M) < r \), so that \( T_i(y) \in C \).

Thus \( T_i(C) \subseteq C \) for each \( i = 1, 2, \ldots, n \). Thus \( C = K_1 \), by the minimality of \( K_1 \). But then \( K_1 - \overline{\text{Co}(M)} = C - \overline{\text{Co}(M)} \subseteq rU \) and so
\[
\delta_u(\overline{\text{Co}(M)}) < r = r_x(U; \overline{\text{Co}(M)}) < \delta_u(M),
\]
which is impossible. Hence \( M \) contains exactly one point, say \( x \), then \( T_i(x) = x \) for all \( i = 1, 2, \ldots, n \) since \( T_i(M) = M \) for all \( i = 1, 2, \ldots, n \).

One will ask whether the above theorem holds for an infinite commuting family \( F \) of nonexpansive mappings. This needs a concept so called "complete normal structure", and we shall answer this question in the next section. Finally we conclude this section with some structure of the set of fixed points of a nonexpansive mapping.

**Theorem 3.7.**

Let \( K \subseteq X \) be nonempty weakly compact convex with normal
structure w.r.t. \( U \), \( T : K \to K \) be nonexpansive w.r.t. \( U \) and
\[
M = \{ x \in K : T(x) = x \} .
\]
Then for any \( x, y \in M \) with \( x \neq y \) and
\( 0 < \lambda < 1 \), there is an \( z \in M \) with \( P_u(x - z) = (1 - \lambda)P_u(x - y) \) and
\( P_u(y - z) = \lambda P_u(x - y) \), for all \( U \in \mathcal{U} \). In particular, \( \text{Card} \ M > 2 \)
implies \( \text{Card} \ M > 2 \).

Proof: Define
\[
A_\lambda = \{ z \in K : P_u(x - z) = (1 - \lambda)P_u(x - y) \text{ and } P_u(y - z) = \lambda P_u(x - y) \}
\]
for all \( U \in \mathcal{U} \).

If \( U \in \mathcal{U} \), then
\[
P_u(x - (\lambda x + (1 - \lambda)y)) = P_u((1 - \lambda)x - (1 - \lambda)y) = (1 - \lambda)P_u(x - y)
\]
and
\[
P_u(y - (\lambda x + (1 - \lambda)y)) = P_u(\lambda y - \lambda x) = \lambda P_u(x - y),
\]
so that \( \lambda x + (1 - \lambda)y \in A_\lambda \) and hence \( A_\lambda \neq \emptyset \). Now if \( z_1, z_2 \in A \) and
\( 0 < \mu \leq 1 \), then
\[
P_u(x - (\mu z_1 + (1 - \mu)z_2)) \leq \mu P_u(x - z_1) + (1 - \mu)P_u(x - z_2)
\]
\[
= \mu(1 - \lambda)P_u(x - y) + (1 - \mu)(1 - \lambda)P_u(x - y)
\]
\[
= (1 - \lambda)P_u(x - y),
\]
and
\[
P_u(y - (\mu z_1 + (1 - \mu)z_2)) \leq \mu P_u(y - z_1) + (1 - \mu)P_u(y - z_2)
\]
\[
= \mu \lambda P_u(x - y) + (1 - \mu)\lambda P_u(x - y)
\]
\[
= \lambda P_u(x - y),
\]
and
\[
P_u(x - y) \leq P_u(x - (\mu z_1 + (1 - \mu)z_2)) + P_u(y - (\mu z_1 + (1 - \mu)z_2))
\]
\[
\leq P_u(x - y),
\]
for all \( u \in U \) imply \( P_u(x - (\mu z_1 + (1 - \mu)z_2)) = (1 - \lambda)P_u(x - y) \) and 
\( P_u(y - (\mu z_1 + (1 - \mu)z_2)) = \lambda P_u(x - y) \) for all \( u \in U \). Thus 
\[
\mu z_1 + (1 - \mu)z_2 \in A_\lambda \text{ and so } A_\lambda \text{ is convex. Clearly } A_\lambda \text{ is closed.}
\]
Next if \( z \in A_\lambda \), then
\[
P_u(x - T(z)) = P_u(T(x) - T(z)) \leq P_u(x - z) = (1 - \lambda)P_u(x - y),
\]
\[
P_u(y - T(z)) = P_u(T(y) - T(z)) \leq P_u(y - z) = \lambda P_u(x - y),
\]
\[
P_u(x - y) \leq P_u(x - T(z)) + P_u(y - T(z)) \leq P_u(x - y)
\]
for all \( u \in U \), and so \( P_u(x - T(z)) = (1 - \lambda)P_u(x - y) \) and
\[
P_u(y - T(z)) = \lambda P_u(x - y) \text{ for all } u \in U. \text{ Thus } T(z) \in A_\lambda \text{ and so}
\]
\( T(A_\lambda) \subseteq A_\lambda \). Since \( A_\lambda \) is weakly compact convex with normal structure w.r.t. \( U \), there is an \( z_\lambda \in A_\lambda \) with \( T(z_\lambda) = z_\lambda \), by Corollary 3.4. Thus \( z_\lambda \in M \). Finally we shall show that \( z_{\lambda_1} \neq z_{\lambda_2} \) if \( \lambda_1 \neq \lambda_2 \).

Indeed, choose any \( u \in U \) with \( P_u(x - y) > 0 \). If \( z_{\lambda_1} = z_{\lambda_2} \), then
\[
(1 - \lambda_1)P_u(x - y) = P_u(x - z_{\lambda_1}) = P_u(x - z_{\lambda_2}) = (1 - \lambda_2)P_u(x - y)
\]
so that \( \lambda_1 = \lambda_2 \). Thus \( \text{Card } M \geq 2 \) implies \( \text{Card } M \geq N_0 \).

II-4. Complete normal structure.

The concept of complete normal structure was first introduced by L. P. Belluce and W. A. Kirk in [2]. We shall now generalize this notion to \( T_2 \)-l.c.s.
Definition 4.1.
Let \( K \subseteq X \). Then \( K \) is said to have complete normal structure w.r.t. \( \mathcal{U} \) if and only if for each bounded closed convex subset \( H \) of \( K \), if \( H \) contains more than one point, then \( H \) satisfies the following property: (*) For any descending net \( \{W_\alpha\}_{\alpha \in \Gamma} \) of nonempty subsets of \( H \) and for any \( U \in \mathcal{U} \), if \( r(U; W_\alpha, H) = r(U; H, H) > 0 \), for all \( \alpha \in \Gamma \), then \( \bigcup_{\alpha \in \Gamma} C(U; W_\alpha, H) \) is a nonempty proper subset of \( H \).

Lemma 4.2.
Let \( K \subseteq X \) be bounded. Then for each \( U \in \mathcal{U} \),
\[
\delta_u(K) \geq r(U; K, K) \geq \frac{1}{2} \delta_u(K).
\]
In particular, if \( \text{Card } K > 1 \), then there is a \( U \in \mathcal{U} \) such that \( r(U; K, K) > 0 \).

Proof: Let \( U \in \mathcal{U} \). If \( r(U; K, K) < \frac{1}{2} \delta_u(K) \), let \( x \in K \) be such that \( r_x(U; K) < \frac{1}{2} \delta_u(K) \). Suppose \( r_x(U; K) < r < \frac{1}{2} \delta_u(K) \). Since \( 2r < \delta_u(K) \), there are \( k_1, k_2 \in K \) such that \( P_u(k_1 - k_2) > 2r \). But \( x - K \subseteq rU \), so that \( x - k_i \in rU \) for each \( i = 1, 2 \). Hence
\[
P_u(x - k_1) \leq r, \quad \text{for all } i = 1, 2, \quad \text{and so } \quad P_u(k_1 - k_2) \leq P_u(k_1 - x) + P_u(x - k_2) \leq r + 2r = 3r, \]
which contradicts our choices of \( k_1 \) and \( k_2 \) in \( K \). Thus \( r(U; K, K) \geq \frac{1}{2} \delta_u(K) \). Also it is clear that \( r(U; H, H) \leq \delta_u(K) \). Therefore if \( \text{Card } K > 1 \), let \( k_1, k_2 \in K \) be such that \( k_1 \neq k_2 \), then there is a \( U \in \mathcal{U} \) with \( k_1 - k_2 \notin U \), so that \( \delta_u(K) > P_u(k_1 - k_2) > 1 \) and so \( r(U; K, K) > 0 \).
Proposition 4.3.

Let \( K \subset X \) be closed. If \( K \) has complete normal structure, w.r.t. \( U \), then \( K \) has normal structure w.r.t. \( U \).

Proof: Let \( H \) be a bounded convex subset of \( K \) containing more than one point, then \( \overline{H} \) is a bounded closed convex subset of \( K \) containing more than one point. Thus there is a \( U \in U \) with \( \delta_u(\overline{H}) > 0 \).

By Lemma 4.2., \( r(U; \overline{H}, \overline{H}) = r(U; H, H) > 0 \). Since \( K \) has complete normal structure w.r.t. \( U \), \( C(U; \overline{H}, \overline{H}) = C(U; H, H) \) is a nonempty proper subset of \( \overline{H} \). Take \( h \in C(U; \overline{H}, \overline{H}) \), then \( r_h(U; \overline{H}) = r(U; \overline{H}, \overline{H}) \).

Choose any \( x \) in \( \overline{H} \) but not in \( C(U; H, H) \), then \( r_x(U; \overline{H}) > r(U; \overline{H}, \overline{H}) \).

By Lemma 1.1., there is an \( x_0 \in H \) such that
\[
|r_{x_0}(U; \overline{H}) - r_x(U; \overline{H})| < \frac{1}{2}|r_x(U; \overline{H})|.
\]
It follows that
\[
r_{x_0}(U; \overline{H}) = r_x(U; \overline{H}) < \frac{1}{2}(r_x(U; \overline{H}) + r_{x_0}(U; \overline{H})) < r_x(U; \overline{H}) \leq \delta_u(H) = \delta_u(H),
\]
so that \( x_0 \) is a non-diametral point of \( H \) w.r.t. \( U \). Hence \( K \) has normal structure w.r.t. \( U \).

Theorem 4.4.

Let \( K \subset X \) be compact convex. If \( H \) is any closed convex subset of \( K \) containing more than one point, \( U \) is any closed absolutely convex 0-nbhd, and \( \{W_\alpha\}_{\alpha \in \Gamma} \) is a descending net of subsets of \( H \) with \( r(U; W_\alpha, H) = r(U; H, H) > 0 \) for all \( \alpha \in \Gamma \), then \( \overline{C(U; W_\alpha, H)} \) is a nonempty proper subset of \( H \). In particular, \( K \) has complete normal structure w.r.t. \( U \).
Proof: Partially order $\Gamma$ as follows: if $\alpha, \beta \in \Gamma$, then $\alpha \leq \beta$ if and only if $W_\alpha \supset W_\beta$.

Case 1: Assume that $W_\alpha$ is closed and convex for each $\alpha \in \Gamma$. First we shall show that for each $\epsilon > 0$, there is an $\alpha_0 \in \Gamma$ such that for all $\beta \in \Gamma$ with $\beta \geq \alpha_0$, $\sup \{\inf\{P_u(y-x) : x \in W_\alpha \} : y \in W_\beta \} < \epsilon$.

Indeed, since each $W_\alpha$ is compact, $\bigcap_{\alpha \in \Gamma} W_\alpha$ is nonempty. Then for any $\epsilon > 0$, $\bigcap_{\alpha \in \Gamma} W_\alpha + \frac{1}{2} \epsilon U^i$ is an open set and $\bigcap_{\alpha \in \Gamma} W_\alpha \subseteq \bigcap_{\alpha \in \Gamma} W_\alpha + \frac{1}{2} \epsilon U^i$ imply there is an $\alpha_0 \in \Gamma$ such that for all $\beta \in \Gamma$, with $\beta \geq \alpha_0$, we have

\[ W_\beta \subseteq \bigcap_{\alpha \in \Gamma} W_\alpha + \frac{1}{2} \epsilon U^i. \]

Thus for $\beta \in \Gamma$ with $\beta \geq \alpha_0$, if $y \in W_\beta$, then $y \in \bigcap_{\alpha \in \Gamma} W_\alpha + \frac{1}{2} \epsilon U$ and so there is an $x_0 \in \bigcap_{\alpha \in \Gamma} W_\alpha$ such that $P_u(x_0 - y) \leq \frac{1}{2} \epsilon$, so that $\inf\{P_u(x - y) : x \in \bigcap_{\alpha \in \Gamma} W_\alpha \} \leq \frac{1}{2} \epsilon$. Hence

\[ \sup \{\inf\{P_u(y-x) : x \in \bigcap_{\alpha \in \Gamma} W_\alpha \} : y \in W_\beta \} \leq \frac{1}{2} \epsilon < \epsilon, \]

for all $\beta \in \Gamma$ with $\beta \geq \alpha_0$.

Next we shall show that for each $\beta \in \Gamma$,

\[ r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) = r(U; W_\beta, H) = r(U; H, H). \]

Indeed, let $r_o = r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H)$. Since $H$ is compact convex and $\bigcap_{\alpha \in \Gamma} W_\alpha$ is nonempty convex, $C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H)$ is nonempty closed and convex. Choose any $x \in C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H)$. Then

\[ r_o = r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) = r_x(U; \bigcap_{\alpha \in \Gamma} W_\alpha). \]

For each $\epsilon > 0$,

\[ r_x(U; \bigcap_{\alpha \in \Gamma} W_\alpha) < r_o + \frac{1}{2} \epsilon, \]

and so $\bigcap_{\alpha \in \Gamma} W_\alpha \subseteq x + (r_o + \frac{1}{2} \epsilon)U$; also there is an $\alpha_\epsilon \in \Gamma$ such that $W_\alpha \subseteq x + (r_o + \frac{1}{2} \epsilon)U$, and so

\[ W_\alpha \subseteq x + (r_o + \frac{1}{2} \epsilon)U + \frac{1}{2} \epsilon U = x + (r_o + \epsilon)U, \]

so that $r_x(U; W_\alpha) < x + \epsilon$. 
Hence for each \( \varepsilon > 0 \), \( r(U; H, H) = r(U; W_\varepsilon, H) < r_{x(U; W_\varepsilon)} < r + \varepsilon \),
and thus \( r(U; H, H) < r \). For each \( \beta \in \Gamma \), since \( \bigcap_{\alpha \in \Gamma} W_\alpha \subseteq W_\beta \), we see that \( r_\beta = r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) \leq r(U; W_\beta, H) = r(U; H, H) \leq r \), so that
\[
\begin{align*}
r_\beta &= r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) = r(U; W_\beta, H) = r(U; H, H).
\end{align*}
\]

Third we shall show that
\[
\emptyset \neq \overline{co}(\bigcup_{\alpha \in \Gamma} C(U; W_\alpha, H)) \subseteq C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H).
\]
Note that by Proposition 1.4., each \( C(U; W_\alpha, H) \) is nonempty. If \( x \in \bigcup_{\alpha \in \Gamma} C(U; W_\alpha, H) \), then
\[
x \in C(U; W_\alpha, H) \text{ for some } \alpha \in \Gamma,
\]
so that \( r_{x(U; W_\alpha)} \leq r_{x(U; W_\alpha)} < r \). But
\[
r_{x(U; \bigcap_{\alpha \in \Gamma} W_\alpha)} > r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) = r \text{, and so } r_{x(U; \bigcap_{\alpha \in \Gamma} W_\alpha)} = r_\beta.
\]
Thus
\[
x \in C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) \text{ for each } x \in \bigcup_{\alpha \in \Gamma} C(U; W_\alpha, H) \text{ so that } \bigcup_{\alpha \in \Gamma} C(U; W_\alpha, H) \subseteq C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H).
\]

Finally it suffices to show that
\[
C(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) \subseteq H.
\]
Indeed, since \( \delta_u(\bigcap_{\alpha \in \Gamma} W_\alpha) \geq r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, \bigcap_{\alpha \in \Gamma} W_\alpha) > r(U; \bigcap_{\alpha \in \Gamma} W_\alpha, H) = r(U; H, H) \)
> 0, and \( \bigcap_{\alpha \in \Gamma} W_\alpha \) is a compact convex set containing more than one point,
by Theorem 2.3., there is an \( x_0 \in \bigcap_{\alpha \in \Gamma} W_\alpha \) such that
\[
r_{x_0(U; \bigcap_{\alpha \in \Gamma} W_\alpha)} < \delta_u(\bigcap_{\alpha \in \Gamma} W_\alpha).
\]
But then
\[ \delta_u(C(U; \bigcap_{\alpha \in \Gamma} W_{\alpha}, H)) \leq r_0 - r(U; \bigcap_{\alpha \in \Gamma} W_{\alpha}, H) \leq r_x(U; \bigcap_{\alpha \in \Gamma} W_{\alpha}) < \delta_u(\bigcap_{\alpha \in \Gamma} W_{\alpha}) \leq \delta_u(H), \]

so that \( C(U; \bigcap_{\alpha \in \Gamma} W_{\alpha}, H) \notin H \).

**Case 2:** In general, by Case 1, Proposition 1.4., and Lemma 1.2., \( \emptyset \neq C_0\left( \bigcup_{\alpha \in \Gamma} C(U; W_{\alpha}, H) \right) = C_0\left( \bigcup_{\alpha \in \Gamma} C(U; C_0(W_{\alpha}), H) \right) \)

\[ \subseteq C(U; \bigcap_{\alpha \in \Gamma} C_0(W_{\alpha}), H) \notin H. \] This completes the proof.

We are now ready to answer the question raised in Section II-3.

**Theorem 4.5.**

Let \( K \subseteq X \) be nonempty weakly compact convex with complete normal structure w.r.t. \( U \). If \( F \) is any commuting family of nonexpansive mappings w.r.t. \( U \) on \( K \), then there is an \( x \in K \) with \( T(x) = x \) for all \( T \in F \).

**Proof:** By weak compactness of \( K \) and by Zorn's Lemma, let \( K_1 \) be minimal w.r.t. being a nonempty closed convex subset of \( K \) invariant under each \( T \) in \( F \). Suppose \( K_1 \) contains more than one point, then there is a \( U \in \mathcal{U} \) with \( \delta_u(K) > 0 \). By Lemma 4.2., \( r(U; K_1, K_1) > 0 \).

Let \( \mathcal{A} \) be the family of all nonempty finite subsets of \( F \). For each \( A \in \mathcal{A} \), let \( M_A = \{x \in K_1 : T(x) = x \text{ for all } T \in A\} \).

By Theorem 3.6., \( M_A \neq \emptyset \) for each \( A \in \mathcal{A} \). Let \( A_0 \in \mathcal{A} \) be arbitrarily...
fixed, and let \( r_o = r(U; M_{A_o}, K) \). For each \( A \in \mathcal{A}_\mathcal{T} \), let
\[
H_A = \{ x \in K_1 : r_x(U; M_A) \leq r_o \}.
\]
Since for each \( x \in K_1 \),
\[
r_x(U; M_A) \leq r(U; M_{A_o}, K) \quad \text{if and only if} \quad r_x(U; M_A) = r(U; M_{A_o}, K),
\]
we see that \( H_{A_o} = C(U; M_A, K) \), so that \( H_{A_o} \) is nonempty closed convex, by Lemma 1.1 and Proposition 1.4. It is clear that \( H_A \) is closed and convex for each \( A \in \mathcal{A}_\mathcal{T} \). Denote \( H_0 = \bigcup_{A \in \mathcal{A}_\mathcal{T}} H_A \). Since for any \( A, B \in \mathcal{A}_\mathcal{T} \) if \( A \subseteq B \), then \( M_A \geq M_B \) and so \( H_A \subseteq H_B \), it follows that \( H_0 \) is also convex. Thus \( \overline{H_0} \) is convex. Note that for each \( T \in \mathcal{F} \) and each \( A \in \mathcal{A}_\mathcal{T} \),
\[
T(H_A) \subseteq H_{A \cup \{T\}}.
\]
Indeed, if \( x \in H_A \), then \( x \in H_{A \cap \{T\}} \), and so
\[
r_x(U; M_{A \cap \{T\}}) \leq r_o.
\]
If \( y \in M_{A \cup \{T\}} \), then \( T(y) = y \) so that
\[
P_u(T(x) - y) \leq P_u(x - y) \leq r_o \implies r_{T(x)}(U; M_{A \cup \{T\}}) \leq r_o.
\]
Thus \( T(x) \in H_{A \cup \{T\}} \) for each \( x \in H_A \) so that \( T(H_A) \subseteq H_{A \cup \{T\}} \). Hence for each \( T \in \mathcal{F} \), \( T(H) = T(\bigcup_{A \in \mathcal{A}_\mathcal{T}} H_A) = \bigcup_{A \in \mathcal{A}_\mathcal{T}} T(H_A) \subseteq \bigcup_{A \in \mathcal{A}_\mathcal{T}} H_{A \cup \{T\}} \subseteq \bigcup_{A \in \mathcal{A}_\mathcal{T}} H_A = H_0 \), and so
\[
T(H) \subseteq H_0; \quad \text{since } T \text{ is continuous, } T(\overline{H}) \subseteq \overline{H}. \quad \text{Hence } \overline{H} \text{ is nonempty closed convex invariant under each } T \in \mathcal{F}. \quad \text{Thus } \overline{H} = K_1, \text{ by the minimality of } K_1. \quad \text{Suppose } \varepsilon > 0. \text{ If } x \in K_1 = \overline{H}, \text{ then there is an } y \in H \text{ with } y - x \leq \frac{1}{2} \varepsilon U. \text{ But then } y \in H_{A_1} \text{ for some } A_1 \in \mathcal{A}_\mathcal{T} \text{ with } A_1 \supseteq A_o, \text{ and so }
\[
r_y(U; M_{A_1}) \leq r_o < r_o + \frac{1}{2} \varepsilon \quad \text{so that}
\]
Thus

\[ M_{A_1} \subseteq y + (r_0 + \frac{1}{2}\varepsilon)U \subseteq (x + \frac{1}{2}\varepsilon U) + (r_0 + \frac{1}{2}\varepsilon)U = x + (r_0 + \varepsilon)U. \]

\[ \overline{\text{Co}(M_{A_1})} \subseteq x + (r_0 + \varepsilon)U, \] and hence

\[ \bigcap_{A \in \Theta_{0}} \overline{\text{Co}(M_{A})} = \bigcap_{x \in K_{1}} (x + (r_0 + \varepsilon)U). \]

It follows that

\[ \bigcap_{A \in \Theta_{0}} \overline{\text{Co}(M_{A})} \subseteq \bigcap_{x \in K_{1}} (x + (r_0 + \varepsilon)U). \]

if \( A \supseteq B \) and each \( \overline{\text{Co}(M_{A})} \) is a nonempty weakly closed subset of \( K_{1} \),

and so \( \bigcap_{A \in \Theta_{0}} \overline{\text{Co}(M_{A})} \) is nonempty, since \( K_{1} \) is weakly compact. Choose any \( z \in \bigcap_{A \in \Theta_{0}} \overline{\text{Co}(M_{A})} \), then \( z \in x + (r_0 + \varepsilon)U \) for all \( x \in K_{1} \) so that

\[ K_{1} \subseteq z + (r_0 + \varepsilon)U. \]

Thus \( r_z(U; K_{1}) \leq r_0 + \varepsilon \) so that

\[ r(U; K_{1}, K_{1}) \leq r_z(U; K_{1}) \leq r_0 + \varepsilon. \]

Thus \( r(U; K_{1}, K_{1}) \leq r_0 + \varepsilon \) for all \( \varepsilon > 0 \) and so \( r(U; K_{1}, K_{1}) \leq r_0 \). On the other hand,

\[ r_0 = r(U; M_{A_0}, K_{1}) \leq r(U; K_{1}, K_{1}) \leq r_0. \]

Hence \( r_0 = r(U; K_{1}, K_{1}) > 0 \).

Since for each \( A \in \Theta_{0}, r(U; M_{A_0}, K_{1}) = r_0 = r(U; K_{1}, K_{1}) > 0 \),

\( \{M_{A} : A \in \Theta_{0}\} \) forms a decreasing net of nonempty subsets of \( K_{1} \subseteq K \) and \( K \) has complete normal structure w.r.t. \( U \), \( \bigcup_{A \in \Theta_{0}} \overline{C(U; M_{A}, K_{1})} \) is a nonempty proper subset of \( K_{1} \). Since \( H_{A} = C(U; M_{A}, K_{1}) \) for each \( A \in \Theta_{0} \),

\( \bigcup_{A \in \Theta_{0}} \overline{H_{A}} \) is a nonempty proper subset of \( K_{1} \). Since \( \{H_{A} : A \in \Theta_{0}\} \) is a net of convex sets, \( \bigcup_{A \in \Theta_{0}} \overline{H_{A}} \) and so \( \bigcup_{A \in \Theta_{0}} \overline{H_{A}} \) is also convex. Thus for each \( T \in F \), we have
\[
T(\bigcup_{A \in H} H_A) \subseteq T(\bigcup_{A \in H} H_A) = \bigcup_{A \in H} T(H_A) \subseteq \bigcup_{A \in H} H_A \cup \{T\} \subseteq \bigcup_{A \in H} H_A.
\]
This contradicts the minimality of \( K \). Hence \( K \) contains exactly one point, say \( x_0 \). Then since \( T(K) \subseteq K \) for each \( T \in F \), \( T(x_0) = x_0 \) for each \( T \in F \).

**Corollary 4.6.**

Let \( K \subseteq X \) be nonempty compact convex. If \( F \) is any commuting family of nonexpansive mappings w.r.t. \( U \) on \( K \), then there is an \( x \in K \) with \( T(x) = x \) for each \( T \in F \).
CHAPTER III

FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS

III-1. Relative nonexpansive mappings.

In this chapter we shall investigate some fixed point theorems for various kinds of nonexpansive mappings.

Lemma 1.1.

Let $X$ be a topological vector space and $\{x_1, \ldots, x_n\}$ be a finite subset of $X$. Then $\text{Co}(\{x_1, \ldots, x_n\})$ is compact.

Proof: Define

$$ T : [0, 1] \times \ldots \times [0, 1] \times \{x_1\} \times \ldots \times \{x_n\} \to \text{Co}(\{x_1, \ldots, x_n\}) $$

by

$$ T(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i x_i. $$

Then clearly $T$ is continuous and onto. Since $[0, 1] \times \ldots \times [0, 1] \times \{x_1\} \times \ldots \times \{x_n\}$ is compact, by Tichonov, we see that $\text{Co}(\{x_1, \ldots, x_n\})$ is also compact.

Next we shall show that Mazur's theorem [6, p. 416-417] holds in $T_2$-l.c.s.

Theorem 1.2.

Let $X$ be a complete $T_2$-l.c.s. If $K \subseteq X$ is compact, then $\overline{\text{Co}(K)}$ is also compact.
Proof: Since $\text{Co}(K)$ is closed and $X$ is complete, $\text{Co}(K)$ is complete. Thus by Hausdorff's Theorem on total boundedness in [9, p. 61], it remains to show that $\text{Co}(K)$ is also totally bounded.

Let $U$ be any closed absolutely convex 0-nbhd in $X$.

Since $K$ is compact, there are distinct $x_1, \ldots, x_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} x_i + \frac{1}{4} U$. If $y \in \text{Co}(K)$, then $y + \frac{1}{4} U \cap \text{Co}(K) \neq \emptyset$. Choose any $z \in y + \frac{1}{4} U \cap \text{Co}(K)$, then $y - z \in \frac{1}{4} U$ and so

$y = z - (z - y) \in \text{Co}(K) - \frac{1}{4} U = \text{Co}(K) + \frac{1}{4} U$, for all $y \in \text{Co}(K)$. Thus $\text{Co}(K) \subseteq \text{Co}(K) + \frac{1}{4} U$. Since $K \subseteq \bigcup_{i=1}^{n} x_i + \frac{1}{4} U$, define $\nu : K \to \{1, 2, \ldots, n\}$ such that $y - x_{\nu(y)} \in \frac{1}{4} U$ for all $y \in K$. If $y \in \text{Co}(K)$, then $y = \sum_{i=1}^{m} \lambda_i y_i$ for some $\{y_1, \ldots, y_m\} \subseteq K$ and $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{m} \lambda_i = 1$; thus

$y - \sum_{i=1}^{m} \lambda_i x_{\nu(y_i)} = \sum_{i=1}^{m} \lambda_i y_i - \sum_{i=1}^{m} \lambda_i x_{\nu(y_i)}$

$= \sum_{i=1}^{m} \lambda_i (y_i - x_{\nu(y_i)})$

$\in \sum_{i=1}^{m} \lambda_i \left( \frac{1}{4} U \right)$

$= \frac{1}{4} U$, and so $y \in \sum_{i=1}^{m} \lambda_i x_{\nu(y_i)} + \frac{1}{4} U \subseteq \text{Co}({x_1, \ldots, x_n}) + \frac{1}{4} U$. Hence

$\text{Co}(K) \subseteq \text{Co}({x_1, \ldots, x_n}) + \frac{1}{4} U$ and so
\[ \overline{\text{Co}}(K) \subseteq \text{Co}(K) + \frac{1}{4} U \subseteq \text{Co}(\{x_1, \ldots, x_n\}) + \frac{1}{2} U. \] By Lemma 1.1, \(\text{Co}(\{x_1, \ldots, x_n\})\) is compact. Thus there are \(y_1, \ldots, y_\ell\) in \(\text{Co}(\{x_1, \ldots, x_n\}) \subseteq \text{Co}(K)\) such that \(\text{Co}(\{x_1, \ldots, x_n\}) \subseteq \bigcup_{j=1}^{\ell} y_j + \frac{1}{2} U\). It follows

\[ \overline{\text{Co}}(K) \subseteq \text{Co}(\{x_1, \ldots, x_n\}) + \frac{1}{2} U \subseteq (\bigcup_{j=1}^{\ell} (y_j + \frac{1}{2} U)) + \frac{1}{2} U = \bigcup_{j=1}^{\ell} y_j + U. \]

Hence \(\overline{\text{Co}}(K)\) is also totally bounded. Therefore \(\overline{\text{Co}}(K)\) is compact.

**Definition 1.3.**

Let \(X\) be a \(T_2\)-l.c.s., \(U\) be a base for closed absolutely convex 0-nbhd in \(X\) and \(M, K \subseteq X\) be nonempty with \(M \subseteq K\). Then \(T: K \to K\) is said to be nonexpansive relative to \(M\) w.r.t. \(U\) if and only if for any \(U \in U\), \(P_u(T(x) - T(y)) \leq P_u(x - y)\), for all \(x \in M\) and all \(y \in K\).

**Theorem 1.4.**

Let \(X\) be a complete \(T_2\)-l.c.s., \(U\) be a base for closed absolutely convex 0-nbhd in \(X\), \(K \subseteq X\) be nonempty bounded closed convex, \(M \subseteq K\) be nonempty compact and \(T: K \to K\) be nonexpansive relative to \(M\) w.r.t. \(U\) such that (i) \(\overline{\text{Co}}(O(T, o, x)) \cap M \neq \emptyset\) for all \(x \in K\) and (ii) \(T(M) \subseteq M\). Then there is an \(x \in M\) with \(T(x) = x\).

**Proof:** Let \(F = \{A \subseteq K : A\) is nonempty closed convex and \(T(A) \subseteq A\}\). Partially order \(F\) by \(\supseteq\). Then by (i), compactness of \(M\)
and Zorn's Lemma, \( F \) has a minimal element, say \( K_1 \). Let \( M_1 = K_1 \cap M \). Note that \( M_1 \) is nonempty compact and \( T(M_1) \subseteq M_1 \).

Let \( G = \{ B \subseteq M_1 : B \) is nonempty compact and \( T(B) \subseteq B \} \). Then \( G \neq \emptyset \) for \( M_1 \in G \). Partially order \( G \) by \( \supseteq \). Then by compactness of \( M_1 \) and Zorn's Lemma, \( G \) has a minimal element, say \( M_2 \). Since \( T \) is continuous on \( M \) and so on \( M_2 \) and \( M_2 \) is compact, \( T(M_2) \) is compact. Thus \( T(M_2) = M_2 \) by minimality of \( M_2 \) in \( G \).

Suppose \( M_2 \) contains more than one point, then there is a \( U \in U \) with \( \delta_u(M_2) > 0 \). Since \( X \) is complete and \( M_2 \) is compact, it follows that \( \overline{\text{Co}(M_2)} \) is also compact, by Theorem 1.2. Since \( \delta_u(\overline{\text{Co}(M_2)}) = \delta_u(M_2) > 0 \), by Proposition II-1.3.

and by Theorem II-2.3., there is an \( x_0 \in \overline{\text{Co}(M_2)} \) such that
\[
0 < r = r_{x_0}(U; \overline{\text{Co}(M_2)}) < \delta_u(\overline{\text{Co}(M_2)}).
\]

Let \( S = \{ x \in K_1 : r_x(U; M_2) \leq r = r_{x_0}(U; \overline{\text{Co}(M_2)}) \} \).

Then \( S \neq \emptyset \) since \( x_0 \in S \). By Lemma II-1.1. \( S \) is closed and convex.

We shall show that \( T(S) \subseteq S \). Indeed, suppose \( x \in S \) and \( z \in M_2 = T(M_2) \), then there is an \( w \in M_2 \subseteq M \) with \( T(w) = z \). Thus
\[
P_u(T(x) - z) = P_u(T(x) - T(w)) \leq P_u(x - w) \leq r, \quad \text{and so} \quad r_{T(x)}(U; M_2) \leq r\]
implies \( T(x) \in S \) for all \( x \in S \). Hence \( T(s) \subseteq S \). Thus \( S \in F \) and so \( S = K_1 \) by the minimality of \( K_1 \) in \( F \). But \( S - M_2 \subseteq rU \), and so \( M_2 \subseteq K_1 = S \) implies \( M_2 - M_2 \subseteq rU \). It follows that
\[
\delta_u(M_2) \leq r = r_{x_0}(U; M_2) = r_{x_0}(U; \overline{\text{Co}(M_2)}) < \delta_u(\overline{\text{Co}(M_2)}) = \delta_u(M_2), \quad \text{which is}
\]
impossible. Hence $M_2$ must contain only a single point, say $x_o$, and so $T(x_o) = x_o$ since $T(M_2) = M_2$. Moreover $x_o \in M_2 \subseteq M$.

**III-2. Affine mappings and convex mappings.**

Throughout the remaining of this chapter, $X$ will denote a $T_2$-l.c.s. and $U$ a base for closed absolutely convex 0-nbhd in $X$. If $K \subseteq X$ is nonempty, $T : K \to K$ is said to be **affine** if and only if for any $x, y \in K$ and $0 \leq \lambda \leq 1$, $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1-\lambda)T(y)$.

Note that if $T$ is affine, it can be shown by induction that if $x_1, \ldots, x_n \in K$ and $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{n} \lambda_i = 1$, then

$$T\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i T(x_i)$$

A mapping $T : K \to K$ is said to be **convex** w.r.t. $U$ if and only if for any $U \in U$, we have

$$P_u(T(\lambda x + (1 - \lambda)y)) \leq \lambda P_u(T(x)) + (1 - \lambda)P_u(T(y))$$

for all $x, y \in K$ and $0 \leq \lambda \leq 1$. In this case, one can show by induction that for $x_1, \ldots, x_n \in K$, $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{n} \lambda_i = 1$,

$$P_u(T(\sum_{i=1}^{n} \lambda_i x_i)) \leq \sum_{i=1}^{n} \lambda_i P_u(T(x_i))$$

for all $U \in U$. Note that $T$ is affine implies $T$ is convex w.r.t. $U$.
Theorem 2.1.

Let $K \subseteq X$ be nonempty weakly compact convex, $T : K \to K$ be continuous such that $I - T$ is convex w.r.t. $\mathcal{U}$ on $K$. Suppose for any $U \in \mathcal{U}$, $\inf\{P_{\mathcal{U}}(T(x) - x) : x \in K\} = 0$. Then there is an $x \in K$ with $T(x) = x$.

Proof: For each $U \in \mathcal{U}$ and each $r > 0$, let $H_r(U) = \{z \in K : P_{\mathcal{U}}(T(z) - z) < r\}$. Since $\inf\{P_{\mathcal{U}}(T(x) - x) : x \in K\} = 0$, we see that $H_r(U) \neq \emptyset$ for any $r > 0$. If $z_\lambda \in H_r(U)$ and $z_\lambda \to z$, then $T(z_\lambda) - z_\lambda \to T(z) - z$ for $T$ is continuous, and so $P_{\mathcal{U}}(T(z) - z) \leq r$ since $P_{\mathcal{U}}(T(z_\lambda) - z_\lambda) \leq r$ for all $\lambda$. Thus $z \in H_r(U)$ and so $H_r(U)$ is closed. Next suppose $z_1, z_2 \in H_r(U)$ and $0 \leq \lambda \leq 1$. Then

$$P_{\mathcal{U}}(T(\lambda z_1 + (1 - \lambda)z_2) - (\lambda z_1 + (1 - \lambda)z_2))$$

$$= P_{\mathcal{U}}((I - T)(\lambda z_1 + (1 - \lambda)z_2))$$

$$\leq \lambda P_{\mathcal{U}}((I - T)(z_1)) + (1 - \lambda)P_{\mathcal{U}}((I - T)(z_2))$$

$$\leq \lambda r + (1 - \lambda)r$$

$$= r,$$

since $I - T$ is convex w.r.t. $\mathcal{U}$. It follows that $\lambda z_1 + (1 - \lambda)z_2 \in H_r(U)$, and consequently $H_r(U)$ is convex. Note that $H_r(U)$ is a weakly closed nonempty subset of $K$, for all $r > 0$. Now if $r_1, \ldots, r_n > 0$, let $r = \min\{r_1, \ldots, r_n\}$, then $H_r(U) \subseteq H_{r_i}(U)$ for
all $i = 1, 2, \ldots, n$ so that \[ \bigcap_{i=1}^{n} H_r(U) = H_r(U) \neq \emptyset. \] Since \( \{H_r(U) : r > 0\} \) has finite intersection property and $K$ is weakly compact, $H(U) = \bigcap_{r>0} H_r(U) \neq \emptyset$. Thus for each $U \in \mathcal{U}$, $H(U)$ is a nonempty closed convex and so weakly closed subset of $K$. If $U_1, U_2 \in \mathcal{U}$ and $U_1 \subseteq U_2$, then $P_{U_1}(x) \geq P_{U_2}(x)$, for all $x \in K$, and so $H_{r}(U_1) \subseteq H_{r}(U_2)$ for all $r > 0$ and so $H(U_1) = \bigcap_{r>0} H_r(U_1) \subseteq \bigcap_{r>0} H_r(U_2) = H(U_2)$. If $U_1, \ldots, U_m \in \mathcal{U}$, then there is a $U \in \mathcal{U}$ with $U \subseteq \bigcap_{i=1}^{m} U_i$; since $U \subseteq U_i$ for each $i = 1, 2, \ldots, m$, it follows $H(U) \subseteq H(U_1)$ for each $i = 1, 2, \ldots, m$, and so $\bigcap_{i=1}^{m} H(U_i) \supseteq H(U) \neq \emptyset$. Therefore $\{H(U) : U \in \mathcal{U}\}$ has finite intersection property. Thus $H = \bigcap_{U \in \mathcal{U}} H(U) \neq \emptyset$ by weak compactness of $K$.

Finally we shall show that $x \in H$ implies $T(x) = x$.

Indeed if $x \in H$, then $x \in H(U)$ for all $U \in \mathcal{U}$ and so $x \in H_r(U)$ for all $r > 0$ and all $U \in \mathcal{U}$ and so $x \in H_1(U)$ for all $U \in \mathcal{U}$. Thus $P_U(T(x) - x) \leq 1$ for all $U \in \mathcal{U}$ implies $T(x) - x \in U$ for all $U \in \mathcal{U}$, and so $T(x) - x \in \bigcap_{U \in \mathcal{U}} U = \{0\}$. Consequently $T(x) = x$.

**Theorem 2.2.**

Let $K \subseteq X$ be nonempty closed convex, $T : K \to K$ and $M \subseteq K$ be nonempty weakly compact with $M \cap \text{Co}(0(T, 0, x)) \neq \emptyset$ for all $x \in K$. Suppose (i) $T$ is nonexpansive w.r.t. $U$ on $O(T, 1, x)$ for
all \( x \in K \); (ii) \( x \neq Tx \) implies \( T \) is not an isometry w.r.t. \( \mathcal{U} \) on \( 0(T, l, x) \) and (iii) \( I - T^n \) is convex w.r.t. \( \mathcal{U} \) for each \( n = 1, 2, \ldots \).

Then there exists an \( x \in M \) with \( T(x) = x \).

**Proof:** By weak compactness of \( M \) and by Zorn's Lemma, let \( K_1 \) be minimal w.r.t. being a nonempty closed convex subset of \( K \) invariant under \( T \). Suppose there were an \( x \in K_1 \) with \( T(x) \neq x \). Then by (ii), there is a \( u \in \mathcal{U} \) and there are positive integers \( k \) and \( j \) with

\[
P_u(T^{k+1}(x) - T^{j+1}(x)) \neq P_u(T^k(x) - T^j(x)).
\]

By (i), it follows that

\[
P_u(T^{k+1}(x) - T^{j+1}(x)) < P_u(T^k(x) - T^j(x)).
\]

Assume \( k > j \) and \( k = n + j \), then for \( w = T^n(x) \) and \( y = T^j(x) \), we see that

\[
P_u(w - T^n(y)) < P_u(y - T^n(y)).
\]

Define \( H = \{ z \in K_1 : P_u(z - T^n(z)) \leq P_u(w - T^n(w)) \} \).

Then \( H \) is nonempty since \( w \in H \). By (iii), \( H \) is convex. Clearly \( H \) is also closed. Hence \( H = K_1 \) by the minimality of \( K_1 \). But \( y \in K \) while \( y \notin H \) which is a contradiction. Hence \( T(x) = x \) for all \( x \in K_1 \).

Since \( K_1 \neq \emptyset \), there is an \( x \in K_1 \) with \( T(x) = x \). But then \( x \) must be in \( M \) since \( M \cap \text{cl} \{0(T, 0, x)\} \neq \emptyset \).

**Theorem 2.3.**

Let \( K \subseteq X \) be nonempty bounded closed convex and sequentially complete, \( M \subseteq K \) be nonempty compact and \( T : K \to K \) be nonexpansive w.r.t. \( \mathcal{U} \). Suppose (i) \( M \cap \text{cl} \{0(T, l, x)\} \neq \emptyset \) for all \( x \in K \) and (ii) for each \( x \in K \) and \( u \in \mathcal{U} \), \( P_u(y - T(y)) \leq P_u(x - T(x)) \) for all \( y \in \text{cl} \{0(T, l, x)\} \). Then there is an \( x_o \in M \) with \( T(x_o) = x_o \).
Proof: Suppose \( T(x) \neq x \) for each \( x \in M \). Define 
\[ A = \{ T(x) - x : x \in M \} \]. Suppose \( x, y \in M \) and \( T(x) - x, y \). Then \( M \) is compact implies there is a subnet \( \{ x, \} \) with \( x, y \). Thus \( T(x) \to T(x) \), and so \( T(x) - x, y \to T(x) - x, y \). Since \( X \) is \( T_{2} \), \( y = T(x) - x \in A \). Hence \( A \) is closed. Since \( 0 \notin A \), there is a \( U \in U \) with \( A \cap U = \emptyset \), and it follows that: (\( * \)) \( P_{u}(T(m) - m) > 1 \) for all \( m \in M \).

Since \( K \) is bounded, there is \( \delta > 0 \) with \( K \subseteq \delta U \), and so \( K - K \subseteq 2\delta U \) and consequently \( P_{u}(x - y) \leq 2\delta \) for all \( x, y \in K \). Choose \( 0 < q < 1 \) with \( (1 - q) \cdot 2\delta < \frac{1}{3} \). Fix an \( x, y \in K \). Define 
\( f : K \to K \) by \( f(x) = qT(x) + (1 - q)x \), for all \( x \in K \). By Example I-4.2.; \( f \) is strictly contractive w.r.t. \( U \). Since \( K \) is sequentially complete, there is a unique \( y_{o} \in K \) with \( f(y_{o}) = y_{o} \), by Theorem I-2.4.

Thus 
\[ P_{u}(y_{o} - T(y_{o})) = P_{u}((1 - q)(x_{o} - f(y_{o}))) \]
\[ = (1 - q)P_{u}(x_{o} - f(y_{o})) \]
\[ \leq (1 - q) \cdot 2\delta \]

and so (\( ** \)) \( P_{u}(y_{o} - T(y_{o})) < \frac{1}{3} \).

Since \( M \cap \overline{\text{Co}(O(T, 1, y))} \neq \emptyset \), choose any \( m \in M \cap \overline{\text{Co}(O(T, 1, y))} \). Then there exists an \( z \in \text{Co}(O(T, 1, y)) \) with (\( *** \)) \( P_{u}(m - z) < \frac{1}{3} \). Hence
\[ P_u(m - T(m)) = P_u(m - z + z - T(z) + T(z) - T(m)) \]

\[ < P_u(m - z) + P_u(z - T(z)) + P_u(T(z) - T(m)) \]

\[ < P_u(m - z) + P_u(y_o - T(y_o)) + P_u(z - m) \]

\[ \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \]

\[ = 1 , \]

by (ii), (**) and (***) . This contradicts (**) . Hence there is an

\[ x_o \in M \] such that \[ T(x_o) = x_o . \]

Theorem 2.4.

Let \( K \subseteq X \) be nonempty bounded closed convex and sequentially complete, \( T : K \to K \) be affine and nonexpansive w.r.t. \( U \) . Then either (i) \( K \) contains only a single points or (ii) some proper nonempty closed convex subset of \( K \) is invariant under \( T \).

Proof: Suppose no proper nonempty closed convex subsets of \( K \) is invariant under \( T \), then since

\[ T(\overline{Co}(0(T, 1, x))) \subseteq T(\overline{Co}(0(T, 1, x))) \subseteq \overline{Co}(0(T, 1, x)) \]

for each \( x \in K \),

\[ K = \overline{Co}(0(T, 1, x)) , \] for each \( x \in K \). Fix any \( x_o \in K \) and let \( M = \{x_o\} \).

Then \( x_o \in K = \overline{Co}(0(T, 1, x)) \) for each \( x \in K \) implies

\[ M \cap \overline{Co}(0(T, 1, x)) \neq \emptyset \]

for each \( x \in K \) so that the hypothesis (i) of Theorem 2.3. is satisfied. Moreover, for any \( x \in K \), if \( y \in \overline{Co}(0(T, 1, x)) \),

say \( y = \sum_{i=1}^{m} \lambda_i T^i(x) \), where \( 0 \leq \lambda_i \leq 1 \) and \( \sum_{i=1}^{m} \lambda_i = 1 \), then
T(y) = \sum_{i=1}^{m} \lambda_i T_i^{n_i+1}(x) \quad \text{since } T \text{ is affine, and so for } U \in \mathcal{U},

P_u(y - T(y)) = P_u(\sum_{i=1}^{m} \lambda_i T_i^{n_i}(x) - \sum_{i=1}^{m} \lambda_i T_i^{n_i+1}(x))

\leq \sum_{i=1}^{m} \lambda_i P_u(T_i^{n_i}(x) - T_i^{n_i+1}(x))

\leq \sum_{i=1}^{m} \lambda_i P_u(x - T(x))

= P_u(x - T(x))

so that the hypothesis (ii) of Theorem 2.3. is also satisfied. Since

M = \{x_0\} \quad \text{is compact, Theorem 2.3. implies } T(x_0) = x_0. \quad \text{Thus}

K = \overline{Co}(0(T, 1, x_0)) = \{x_0\} \quad \text{and so } K \text{ contains only a single point.}

\textbf{Corollary 2.5.}

Let } K \subseteq X \text{ be nonempty bounded closed convex sequentially complete, } M \subseteq K \text{ be nonempty weakly compact and } T : K \to K \text{ be affine and nonexpansive w.r.t. } U. \text{ If } M \cap \overline{Co}(0(T, 1, x)) \neq \emptyset \text{ for each } x \in K, \text{ then there is an } x \in M \text{ with } T(x) = x.

\textbf{Proof:} \quad \text{By weak compactness of } M \text{ and Zorn's Lemma, let } K_1 \text{ be minimal w.r.t. being a nonempty closed convex subset of } K \text{ invariant under } T. \text{ Since } K_1 \text{, being a subset of } K, \text{ is also bounded and sequentially complete, the minimality of } K_1 \text{ and Theorem 2.4. imply } K_1 \text{ contains only a single point, say } x. \quad \text{Thus } T(K_1) \subseteq K_1 \text{ implies } T(x) = x. \text{ Since}
3. Mappings with diminishing orbital diameters.

The concept of diminishing orbital diameters has been introduced by W. A. Kirk in [1]. We shall generalize this concept into $T_2$-l.c.s.

**Definition 3.1.**

If $K \subseteq X$ is nonempty, $T : K \to K$ and $x \in K$, then $T$ is said to have diminishing orbital diameters w.r.t. $\mathcal{U}$ at $x$ if and only if for any $U \in \mathcal{U}$, $\delta_U(0(T, o, x)) > 0$ implies

$$\lim_{n \to \infty} \delta_U(0(T, n, x)) < \delta_U(0(T, o, x)).$$

$T$ is said to have diminishing orbital diameters (d.o.d.) w.r.t. $\mathcal{U}$ if and only if $T$ has diminishing orbital diameters w.r.t. $\mathcal{U}$ at $y$ for each $y$ in $K$.

Since $0(T, n+1, x) \subseteq 0(T, n, x)$ for each $n = 0, 1, 2, \ldots$ and each $x \in K$, we see that $\delta_U(0(T, n+1, x)) < \delta_U(0(T, n, x))$ for any $U \in \mathcal{U}$ so that $\lim_{n \to \infty} \delta_U(0(T, n, x))$ exists. Thus the above notion is well-defined. By Remark 1-3.9., $T$ is strictly contractive w.r.t. $\mathcal{U}$ implies $T$ has d.o.d. w.r.t. $\mathcal{U}$. 

$$M \cap \overline{Co}(0(T, 1, x)) \neq \emptyset$$ and $$\{x\} = \overline{Co}(0(T, 1, x)),$$ we see that $x \in M$. 

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Theorem 3.2.

Let \( K \subseteq X \) be nonempty compact and \( T : K \to K \) be continuous with diminishing orbital diameters w.r.t. \( U \). Then there is an \( x \in K \) such that \( T(x) = x \).

Proof: By compactness of \( K \) and Zorn's Lemma, let \( K_1 \) be minimal w.r.t. being a nonempty closed subset of \( K \) invariant under \( T \). Suppose there is an \( x \in K_1 \) with \( T(x) \neq x \), then there is a \( U \in U \) with \( \delta_u(0(T, x)) > 0 \), so that \( \lim_{n \to \infty} \delta_u(0(T, n, x)) < \delta_u(0(T, o, x)) \), since \( T \) has diminishing orbital diameters w.r.t. \( U \). Thus there is a positive integer \( N \) with \( r = \delta_u(0(T, N, x)) < \delta_u(0(T, o, x)) \). It follows that

\[
r = \delta_u(O(T, N, x)) < \delta_u(0(T, o, x)) < \delta_u(K_1),
\]

and so \( \emptyset \neq O(T, N, x) \subseteq K \).

But \( O(T, N, x) \) is invariant under \( T \), and thus \( O(T, N, x) \) is invariant under \( T \) as \( T \) is continuous, which contradicts the minimality of \( K_1 \). Thus \( T(x) = x \) for each \( x \in K_1 \). Since \( K_1 \neq \emptyset \), there is an \( x \in K_1 \subseteq K \) with \( T(x) = x \).

Theorem 3.3.

Let \( K \subseteq X \) be nonempty weakly compact, \( T : K \to K \) be nonexpansive w.r.t. \( U \). Suppose there does not exist nonempty proper weakly compact subset of \( K \) which is invariant under \( T \). If for some \( x \in K \), \( T \) has d.o.d. w.r.t. \( U \) at \( x \), then \( T(x) = x \).
Proof: First we note that every weakly compact subset of $X$ is bounded, by Theorem 17.5, p. 155 in Kelley and Namioka [11]. Suppose $T(x) \neq x$, then there is a $U \in \mathcal{U}$ with $\delta_u(0(T, o, x)) > 0$. Since $T$ has d.o.d. w.r.t. $\mathcal{U}$ at $x$, there is a positive integer $N$ with $r = \delta_u(0(T, N, x)) < \delta_u(0(T, o, x))$. Take $x_o = T^N(x)$, then $P_u(x_o - T^i(x)) < r$ for each positive integer $i \geq N$.

For each $\varepsilon > 0$, let $S_\varepsilon = \{y \in X : P_u(y - T^i(x)) \leq r + \varepsilon, \text{ for all but a finite number of positive integers } i\}$. Define $S = \bigcap_{\varepsilon > 0} S_\varepsilon$. Clearly each $S_\varepsilon$ is convex and hence $S$ is convex. If $\{y_\lambda\}_{\lambda \in \Gamma}$ is a net in $S$ and $y_\lambda \to y$ in $X$, then for any $\varepsilon > 0$, there is an $\lambda_0$ with $P_u(y_\lambda - y) \leq \frac{\varepsilon}{2}$, for all $\lambda \geq \lambda_0$. Since $y_\lambda \in S \subseteq S_{\varepsilon/2}$, there is a positive integer $N_0$ with $P_u(y_\lambda - T^i(x)) \leq r + (\varepsilon/2)$, for all $i \geq N_0$. Thus for all $i \geq N_0$,

$$P_u(y - T^i(x)) \leq P_u(y - y_\lambda) + P_u(y_\lambda - T^i(x)) \leq \frac{\varepsilon}{2} + (r + \frac{\varepsilon}{2}) = r + \varepsilon;$$

thus $y \in S_\varepsilon$, for all $\varepsilon > 0$ so that $y \in \bigcap_{\varepsilon > 0} S_\varepsilon = S$. Therefore $S$ is also closed. Since $x_o \in S \cap K$, we see that $M = S \cap K$ is a nonempty weakly compact subset of $K$. It is clear that $T(S \cap K) \subseteq S \cap K$, for each $\varepsilon > 0$ and so $T(S \cap K) \subseteq S \cap K$ and hence $T(M) \subseteq M$. Thus $M = K$, by the hypothesis on $K$.

For each $\varepsilon > 0$, $w \in K$, let $B_\varepsilon(w) = \{y \in X : P_u(w - y) \leq r + \varepsilon\}$. If $\{w_1, \ldots, w_n\} \subseteq K = M = S \cap K$,
then for each $\epsilon > 0$, $\{w_1, \ldots, w_n\} \subseteq S_\epsilon$, and so there are positive integers $N_1, \ldots, N_n$ with $P_u(w_j - T^i(x)) \leq r + \epsilon$ for all $i > N_j$ and for each $j = 1, 2, \ldots, n$. Thus for $N = \max\{N_1, \ldots, N_n\}$ and $i > N$,

$$P_u(w_j - T^i(x)) \leq r + \epsilon$$

for all $j = 1, 2, \ldots, n$, and hence $T^i(x) \in B_\epsilon(w_j)$ for all $j = 1, 2, \ldots, n$, and hence $T^i(x) \in \bigcap_{j=1}^n B_\epsilon(w_j) \cap K$. It is clear that $B_\epsilon(w)$ is closed and convex for each $\epsilon > 0$ and each $w \in K$. Thus for $\epsilon > 0$, $\{B_\epsilon(w) \cap K : w \in K\}$ is a family of weakly closed subsets of $K$ with finite intersection property, and it follows that $C_\epsilon = \bigcap_{w \in K} B_\epsilon(w) \cap K \neq \emptyset$, by weak compactness of $K$. It is also clear that $C_{\epsilon_1} \subseteq C_{\epsilon_2}$ if $0 < \epsilon_1 \leq \epsilon_2$. Thus $\{C_\epsilon : \epsilon > 0\}$ is a family of weakly closed subsets of $K$ with finite intersection property, and hence $C = \bigcap_{\epsilon > 0} C_\epsilon \neq \emptyset$, again by the weak compactness of $K$.

If $v \in C$, then $P_u(w - v) \leq r + \epsilon$, for each $w \in K$ and each $\epsilon > 0$, so that $r_v(U; K) \leq r + \epsilon$ for all $\epsilon > 0$ and thus $r_v(U; K) \leq r$. Define $R = \{y \in X : r_y(U; K) \leq r\}$, then $R \neq \emptyset$ since $\emptyset \neq C \subseteq R \cap K$. Clearly $R$ is closed and convex, and so $R \cap K$ is a nonempty weakly closed subset of $K$. If $y \in R \cap K$, then $r_y(U; K) \leq r$.

Fix any $\epsilon > 0$, then $y - K \subseteq (r + \epsilon)U$ implies $T(y) - T(K) \subseteq (r + \epsilon)U$ since $T$ is nonexpansive w.r.t. $U$. Thus $T(K) \subseteq T(y) + (r + \epsilon)U$, and it follows $T(K \cap T(y) + (r + \epsilon)U) \subseteq T(K) \subseteq T(y) + (r + \epsilon)U$. Since $T(K) \subseteq K$, we see that $T(K \cap T(y) + (r + \epsilon)U) \subseteq K \cap T(y) + (r + \epsilon)U$. 
Hence $K \cap T(y) + (r + \varepsilon)U$ is a nonempty weakly compact subset of $K$ invariant under $T$, and therefore $K \cap T(y) + (r + \varepsilon)U = K$. It follows that $K \subseteq T(y) + (r + \varepsilon)U$ and so $T(y) - K \subseteq (r + \varepsilon)U$. Hence $r_{T(y)}(U; K) \leq r + \varepsilon$ for all $\varepsilon > 0$, and so $r_{T(y)}(U; K) \leq r$. Thus $T(y) \in R \cap K$ for each $y \in R \cap K$, so that $T(R \cap K) \subseteq R \cap K$. By hypothesis on $K$, $R \cap K = K$. But then $\delta_u(K) = \delta_u(K \cap R) \leq r < \delta_u(O(T, o, x)) \leq \delta_u(K)$, which is impossible. Therefore we must have $T(x) = x$.

**Corollary 3.4.**

Let $K \subseteq X$ be nonempty weakly compact and $T : K \to K$ be nonexpansive w.r.t. $U$. If $T$ has d.o.d. w.r.t. $U$, then there is an $x \in K$ with $T(x) = x$.

**Proof:** By weak compactness of $K$ and by Zorn's Lemma, let $K_1$ be minimal w.r.t. being a nonempty weakly compact subset of $K$ invariant under $T$. Since $T$ has d.o.d. w.r.t. $U$ on $K$ and hence on $K_1$, $T(x) = x$, for all $x \in K_1$, by Theorem 3.3. Since $K_1 \neq \emptyset$, there is an $x \in K_1 \subseteq K$ with $T(x) = x$.

**Corollary 3.5.**

Let $K \subseteq X$ be nonempty weakly compact convex and $T : K \to K$ be nonexpansive w.r.t. $U$. Suppose there is a positive integer $N$ such that $T^N : K \to K$ has d.o.d.w.r.t. $U$. Then there is an $x \in K$ with $T(x) = x$. 


Proof: By weak compactness of $K$ and Zorn's Lemma, let $K_1$ be minimal w.r.t. being a nonempty closed convex subset of $K$ invariant under $T$. Since $K_1$ is also weakly compact, $T^N : K_1 \to K_1$ is nonexpansive w.r.t. $U$ and has d.o.d. w.r.t. $U$, there is an $x \in K_1$ with $T^N(x) = x$, by Corollary 3.4.

Suppose $T(x) \neq x$, then $M = \{x, T(x), \ldots, T^{N-1}(x)\}$ contains more than one point, and so $Co(M)$ contains more than one point. By Lemma 1.1., $Co(M)$ is compact, and so by Theorem II-2.3., there are a $U \in U$ and an $x_0 \in Co(M)$ with $0 < r_0 = r_{x_0}(U; Co(M)) < \delta_u(\text{Co}(M))$.

Let $R = \{y \in K_1 : r_y(U; M) \leq r_0\}$. By Lemma II-1.1., $R$ is closed and convex. Also $R \neq \emptyset$ since $x_0 \in R$. If $y \in R$, then $r_y(U; M) \leq r_0$, so that $P_u(T(y) - x) = P_u(T(y) - T^N(x)) \leq P_u(y - T^{N-1}(x)) \leq r_0$ and for $1 \leq m < N$, $P_u(T(y) - T^m(x)) \leq P_u(y - T^{m-1}(x)) \leq r_0$. Thus $T(y) \in R$ for each $y \in R$. Hence $T(R) \subseteq R$, and so $R = K_1$, by the minimality of $K_1$. But $\delta_u(K_1) = \delta_u(R) \leq r_0 < \delta_u(\text{Co}(M)) \leq \delta_u(K)$, which is impossible. Thus $T(x) = x$.

Proposition 3.6.

Let $K \subseteq X$ be nonempty weakly compact. Then (i) for any $U \in U$, there is an $x_U \in K$ with $P_u(x_U) = \inf \{P_u(k) : k \in K\}$ and (ii) $\emptyset \notin K$ implies there is a $U \in U$ with $\inf \{P_u(k) : k \in K\} > 0$. 
Proof:

(i) Suppose $U \in \mathcal{U}$. Then for any $\lambda > 0$ such that $\lambda U \cap K \neq \emptyset$, $\lambda U \cap K$ is a nonempty weakly closed subset of $K$. Thus the intersection of all such nonempty sets $\lambda U \cap K$ is nonempty, by the weak compactness of $K$. Take any $x_u$ in that intersection. Then it is clear that $P_u(x_u) = \inf\{P_u(k) : k \in K\}$.

(ii) If $o \notin K$, then since $K$ is weakly closed, there is a closed absolutely convex 0-nbhd $U_o$ in the weak topology such that $U_o \cap K = \emptyset$. But then $U_o$ is also an 0-nbhd in $X$ so that there is a $U \in \mathcal{U}$ with $U \subseteq U_o$. Hence $P_u(k) \geq P_{u_o}(k) > 1$, for all $k \in K$. Thus

$\inf\{P_u(k) : k \in K\} \geq 1 > 0$.

It is clear that if $K_1, K_2 \subseteq X$ are weakly compact, then $K_1 - K_2$ is weakly compact. Thus we have the following:

Corollary 3.7.

If $K_1, K_2 \subseteq X$ are disjoint nonempty weakly compact sets, then (i) for each $U \in \mathcal{U}$, there are $x_1 \in K_1$ and $x_2 \in K_2$ such that $P_u(x_1 - x_2) = \inf\{P_u(k_1 - k_2) : k_1 \in K_1, k_2 \in K_2\}$; (ii) there is a $U \in \mathcal{U}$ with $\inf\{P_u(k_1 - k_2) : k_1 \in K_1, k_2 \in K_2\} > 0$.

Theorem 3.8.

Let $K \subseteq X$ be nonempty, weakly compact and $T : K \to K$ be
contractive w.r.t. $U$. Suppose for each $x \in K$, there is a positive integer $N(x)$ such that $T^N$ has d.o.d. w.r.t. $U$ at $x$. Then there is an $x \in K$ with $T(x) = x$.

**Proof:** By weak compactness of $K$ and by Zorn's Lemma, let $K_1$ be minimal w.r.t. being a nonempty weakly closed subset of $K$ invariant under $T$. For each $n = 2, 3, \ldots$, let $K_n$ be minimal w.r.t. being a nonempty weakly closed subset of $K_1$ invariant under $T^n$.

**Case 1.** Suppose $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Since each $K_n$ is weakly compact, there is a positive integer $m > 2$ such that $\bigcap_{n=1}^{m-1} K_n \neq \emptyset$. Since $K_m$ and $\bigcap_{n=1}^{m-1} K_n$ are nonempty disjoint weakly compact sets, by Corollary 3.7., there are $U \in U$, $x \in K_m$ and $y \in \bigcap_{n=1}^{m-1} K_n$ such that $P_u(x - y) = \inf\{P_u(w - v) : w \in K_m, v \in \bigcap_{n=1}^{m-1} K_n\} > 0$. Since $T^m(K_m) \subseteq K_n$ for each $n = 1, 2, \ldots, n$ and $T^m$ is contractive w.r.t. $U$, we see that $P_u(T^m(x) - T^m(y)) < P_u(x - y)$. But $T^m(x) \in K_m$ and $T^m(y) \in \bigcap_{n=1}^{m-1} K_n$ imply $P_u(x - y) \leq P_u(T^m(x) - T^m(y))$, which is a contradiction. Hence we must have:

**Case 2.** $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Choose any $z \in \bigcap_{n=1}^{\infty} K_n$. Then there is a positive integer $N(z)$ such that $T^N$ has d.o.d. w.r.t. $U$ at $z$.

Since $f = T^N : K_N \rightarrow K_N$ is nonexpansive w.r.t. $U$ and $f$ has d.o.d.w.r.t.
at \( z \in K \), we must have \( T^N(z) = f(z) = z \), by Theorem 3.3. Since \( T \) is contractive and \( T^N(z) = z \), we must have \( T(z) = z \), by Proposition 1-2.3.

**Theorem 3.9.**

Let \( K \subseteq X \) be nonempty bounded closed convex, \( T : K \to K \) be nonexpansive w.r.t. \( \mathcal{U} \) such that no nonempty proper closed convex subset of \( K \) is invariant under \( T \). Suppose (i) \( M \subseteq K \) is nonempty weakly compact such that \( \overline{\text{co}}(O(T, o, x)) \cap M \neq \emptyset \) for each \( x \in K \), and (ii) for some \( z \in K \), \( T \) has d.o.d. w.r.t. \( \mathcal{U} \) at \( z \). Then \( T(z) = z \).

**Proof:** Suppose \( T(z) \neq z \), then \( O(T, o, z) \) contains more than one point. Let \( U \in \mathcal{U} \) with \( \delta_u(O(T, o, z)) > 0 \), and so
\[
\lim_{n \to \infty} \delta_u(O(T, n, z)) < \delta_u(O(T, o, z)).
\]
Thus there is a positive integer \( N \) with \( r = \delta_u(O(T, N, z)) < \delta_u(O(T, o, z)) \). Define
\[
B = \{ x \in K : \delta_x(U ; O(T, n, x) \leq r, \text{ for some positive integer } n \}. \]
Since \( T^N(z) \in B \), \( B \neq \emptyset \). Clearly \( B \) is convex and \( T(B) \subseteq B \), and so \( T(B) \subseteq \overline{T(B)} \subseteq B \). Hence \( B = K \), by hypothesis on \( K \).

Since \( \overline{\text{co}}(O(T, m_1, z)) \supseteq \overline{\text{co}}(O(T, m_2, z)) \) for \( m_1 \leq m_2 \), \( \overline{\text{co}}(O(T, m, z)) \) is closed convex for each \( m = 0, 1, 2, \ldots \) and \( \overline{\text{co}}(O(T, m, z)) \cap M \neq \emptyset \) for each \( m = 0, 1, 2, \ldots \), it follows that
\[
\bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, z)) \cap M \neq \emptyset , \text{ by weak compactness of } M \).

Suppose \( t \in \bigcap_{m=0}^{\infty} \overline{\text{co}}(O(T, m, x)) \cap M \). If \( q \in K = \overline{B} \), then for any \( \varepsilon > 0 \), there is
an \( q_\varepsilon \in B \) with \( P_u(q - q_\varepsilon) \leq \varepsilon \). But then there is a positive integer \( N_\varepsilon \) with \( r(q, (U; 0(T, N_\varepsilon, z)) \leq r \). Thus for \( n > N_\varepsilon \),
\[
P_u(q - T^n(z)) \leq P_u(q - q_\varepsilon) + P_u(q_\varepsilon - T^n(z)) \leq r + \varepsilon,
\]
so that \( r(q, (\overline{Co}(0(T, N_\varepsilon, z))) = r(q, (U; 0(T, N_\varepsilon, z)) \leq r + \varepsilon \). Since \( t \in \overline{Co}(0(T, N_\varepsilon, z)) \), \( P_u(t - q) \leq r + \varepsilon \), for any \( \varepsilon > 0 \) so that \( P_u(t - q) \leq r \) which is true for any \( q \in K \). Thus \( r(t, (U; K) \leq r \),
\[
\forall t \in \bigcap_{m=0}^{\infty} \overline{Co}(0(T, m, z)) \subseteq M .
\]

Define \( S = \{x \in K : r_x(U; K) \leq r\} \), then \( S \neq \emptyset \) since \( \emptyset \neq \bigcap_{m=0}^{\infty} \overline{Co}(0(T, m, z)) \subseteq M \subseteq S \). By Lemma II-1.1., \( S \) is closed and convex.

Suppose there is an \( x \in S \) with \( T(x) \notin S \). Then there is an \( \varepsilon > 0 \) such that \( K \subseteq T(x) + (r + \varepsilon)U \). Let \( H = K \cap T(x) + (r + \varepsilon)U \). Clearly, \( H \) is a nonempty closed convex subset of \( K \). If \( y \in H \), then \( y \in K \subseteq x + (r + \varepsilon)U \) implies \( P_u(T(y) - T(x)) \leq P_u(y - x) \leq r + \varepsilon \), so that \( T(y) \in K \cap T(x) + (r + \varepsilon)U = H \). Thus \( T(H) \subseteq H \). Hence by hypothesis on \( K \), \( K = H = T(x) + (r + \varepsilon)U \cap K \), so that \( K \subseteq T(x) + (r + \varepsilon)U \), which is a contradiction. Hence \( T(S) \subseteq S \). But then \( S = K \), again by the hypothesis on \( K \). Thus \( \delta_u(K) = \delta_u(S) \leq r < \delta_u(0(T, 0, z)) \leq \delta_u(K) \), which is impossible. Therefore we must have \( T(z) = z \).
Corollary 3.10.

Let \( K \subseteq X \) be nonempty bounded closed convex, \( T : K \rightarrow K \) be nonexpansive w.r.t. \( U \) and \( M \subseteq K \) be nonempty weakly compact. Suppose (i) \( \text{Co}(O(T, o, x)) \cap M \neq \emptyset \) for each \( x \in K \) and (ii) \( T \) has d.o.d. w.r.t. \( U \). Then there is an \( x \in M \) with \( T(x) = x \).

Proof: By weak compactness of \( M \) and by Zorn's Lemma, let \( K_1 \) be minimal w.r.t. being a nonempty closed convex subset of \( K \) invariant under \( T \). Thus by Theorem 3.9., \( T(x) = x \) for all \( x \in K_1 \). Since \( K_1 \neq \emptyset \), there is an \( x \in K_1 \subseteq K \) with \( T(x) = x \). By (i), \( x \in M \).

We shall now discuss the fixed point sets of a finite commuting family of nonexpansive mappings with d.o.d. w.r.t. \( U \).

Lemma 3.11.

Let \( K \subseteq X \) be nonempty bounded, \( F = \{T_1, \ldots, T_n\} \) be a finite commuting family of nonexpansive mappings w.r.t. \( U \) on \( K \). If for each \( i = 1, 2, \ldots, n \), \( T_i \) has d.o.d. w.r.t. \( U \), then for any \( x \in K \) with \( T_1 \ldots T_n(x) = x \), we have for each \( U \in U \),

\[ \delta_u(O(T_i, m, x)) = \delta_u(O(T_i, o, x)) \]

for each \( i = 1, 2, \ldots, n \), and \( m = 1, 2, \ldots \).

Proof: Suppose \( U \in U \) and \( m \) is any positive integer. If we denote \( T_1 \ldots T_n = T_2 \ldots T_n \), \( T_1 \ldots T_{n-1} \hat{T}_n = T_1 \ldots T_{n-1} \), and for \( 1 < i < n \), \( T_1 \ldots \hat{T}_i \ldots T_n = T_1 \ldots T_{i-1} \hat{T}_{i+1} \ldots T_n \), then for each
i \in \{1, 2, \ldots, n\}, \quad \delta_u(\{T_1 \ldots T_i \ldots T_n\}0(T_i, m, x)) \leq \delta_u(0(T_i, m, x))$

since $T_j$ is nonexpansive w.r.t. $U$ for each $j = 1, 2, \ldots, n$. But

$(T_1 \ldots T_i \ldots T_n)0(T_i, m, x) = 0(T_i, m-1, x)$ since $T_1 \ldots T_n(x) = x$,

and so $\delta_u(0(T_i, m-1, x)) \leq \delta_u(0(T_i, m, x))$. Also

$0(T_i, m, x) \subseteq 0(T_i, m-1, x)$ implies $\delta_u(0(T_i, m, x)) \leq \delta_u(0(T_i, m-1, x))$.

Hence $\delta_u(0(T_i, m, x)) = \delta_u(0(T_i, m-1, x))$. Therefore

$\delta_u(0(T_i, m, x)) = \delta_u(0(T_i, o, x))$ for each $m = 1, 2, \ldots$, and each $i = 1, 2, \ldots, n$ and each $U \in \mathcal{U}$.

**Theorem 3.12.**

Let $K \subseteq X$ be nonempty bounded and $\mathcal{F} = \{T_1, \ldots, T_n\}$ be a finite commuting family of nonexpansive mappings w.r.t. $U$ on $K$.

If $T_i$ has d.o.d. w.r.t. $U$ on $K$ for each $i \in \{1, 2, \ldots, n\}$, then for each $x \in K$, $T_1 \ldots T_n(x) = x$ if and only if $T_i(x) = x$ for each $i = 1, 2, \ldots, n$. In particular,

$$\{x \in K : T_1 \ldots T_n(x) = x\} = \{x \in K : T_i(x) = x\} \text{ for all } i = 1, 2, \ldots, n.$$ 

**Proof:** Suppose $T_1 \ldots T_n(x) = x$. If there were some $i \in \{1, 2, \ldots, n\}$ with $T_i(x) \neq x$, then $0(T_i, o, x)$ contains more than one point, and so there will be a $U \in \mathcal{U}$ with $\delta_u(0(T_i, o, x)) > 0$.

Thus $\lim_{m \to \infty} \delta_u(0(T_i, m, x)) < \delta_u(0(T_i, o, x))$. By Lemma 3.11,
\[ \delta_u(O(T_1^m, m, x)) = \delta_u(O(T_1^0, 0, x)) \] for all \( m = 1, 2, \ldots \), which is a contradiction. Hence \( T_1^i(x) = x \) for each \( i = 1, 2, \ldots, n \). The converse is obvious.

**Corollary 3.13.**

Let \( K \subseteq X \) be nonempty weakly compact and \( F = \{T_1, \ldots, T_n\} \) be a finite commuting family of nonexpansive mappings w.r.t. \( U \) on \( K \). Suppose \( T_1, \ldots, T_n, T_1 \ldots T_n : K \to K \) have d.o.d. w.r.t. \( U \). Then (i) there is an \( x \in K \) with \( T_1^i(x) = x \) for each \( i = 1, 2, \ldots, n \), and (ii) \( \{x \in K : T_1 \ldots T_n(x) = x\} = \{x \in K : T_1^i(x) = x, \text{ for all } i = 1, 2, \ldots, n\} \).

**Proof:**

By Corollary 3.4., there is an \( x \in K \) with \( T_1 \ldots T_n(x) = x \). By Theorem 3.12., for any \( y \in K \), \( T_1 \ldots T_n(y) = y \) if and only if \( T_1^i(y) = y \) for each \( i = 1, 2, \ldots, n \).

**III-4. Bounded mappings.**

In this section we shall discuss further properties of mappings with d.o.d. and their iterations. If \( K \subseteq X \) is nonempty, we say \( T : K \to K \) is bounded w.r.t. \( U \) if and only if for each \( U \in U \) there is a real number \( c_u > 0 \) with \( P_u(T^k(x) - T^k(y)) \leq c_u P_u(x - y) \), for all \( x, y \in K \), and all \( k = 1, 2, \ldots \). It is clear that \( T \) is bounded
w.r.t. $U$ implies $T$ is continuous.

**Proposition 4.1.**

Let $K \subseteq X$ be nonempty bounded and $T : K \to K$ be bounded w.r.t. $U$. For each $U \in \mathcal{U}$, define $\rho_u(x) = \delta_u(0(T, o, x))$, for each $x \in K$, then $\rho_u$ is continuous from $K$ into $\mathbb{R}$.

**Proof:** Suppose $x, x \in K$ with $x \to x$. Then for any $\varepsilon > 0$, there is an $\lambda_0$ such that $\lambda > \lambda_0$ implies $\rho_u(x, x) < \min\left\{ \frac{\varepsilon}{2c}, \frac{\varepsilon}{2} \right\}$. Thus for $\lambda > \lambda_0$, $\rho_u(x, x) < \varepsilon$ and $\rho_u(T^k(x, x) - T^k(x)) \leq c \rho_u(x, x) < \frac{\varepsilon}{2}$ for all $k = 1, 2, \ldots$, and so

$$\left| \rho_u(T^j(x) - T^k(x)) - \rho_u(T^j(x, x) - T^k(x, x)) \right|$$

$$\leq \rho_u(T^j(x) - T^k(x) - T^j(x, x) + T^k(x, x))$$

$$\leq \rho_u(T^j(x) - T^j(x, x)) + \rho_u(T^k(x) - T^k(x, x))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

for all $j, k = 0, 1, 2, \ldots$. Thus for $\lambda > \lambda_0$, ...
\[ |\rho_u(x) - \rho_u(x_\lambda)| = |\delta_u(0(T, o, x)) - \delta_u(0(T, o, x_\lambda))| \]

\[ = |\sup_{j,k \geq 0} P_u(T_j^k(x)) - \sup_{j,k \geq 0} P_u(T_j^k(x_\lambda))| \]

\[ \leq \sup_{j,k \geq 0} |P_u(T_j^k(x)) - P_u(T_j^k(x_\lambda))| \]

\[ \leq \varepsilon \]

Hence \( \rho_u(x) \to \rho_u(x) \), and so \( \rho_u \) is continuous.

Since every nonexpansive mapping \( T \) w.r.t. \( U \) is bounded w.r.t. \( U \), we have the following:

**Corollary 4.2.**

If \( K \subseteq X \) is nonempty bounded and \( T : K \to K \) is nonexpansive w.r.t. \( U \) , then for any \( U \in U \) , the mapping \( \rho_u : K \to \mathbb{R} \) defined by \( \rho_u(x) = \delta_u(0(T, o, x)) \) for each \( x \in K \) , is continuous.

**Theorem 4.3.**

Let \( K \subseteq X \) be nonempty bounded, \( T : K \to K \) be bounded w.r.t. \( U \) . Suppose (i) \( T \) has d.o.d. w.r.t. \( U \) and (ii) there is an \( x \in K \) such that a subsequence of \( \{T^n(x)\}_{n=1}^{\infty} \) converges to \( z \in K \) . Then

\[ \lim_{n \to \infty} T^n(x) = z \text{ and } T(z) = z . \]

**Proof:** Suppose \( \lim_{k \to \infty} T_k^n(x) = z \) . If \( U \in U \) , then by Proposition 4.1., \( \rho_u : K \to \mathbb{R} \) define by \( \rho_u(x) = \delta_u(0(T, o, x)) \) for each \( x \in K \) , is continuous, and so \( \rho_u(z) = \lim_{k \to \infty} \rho_u(T_k^n(x)) \) . But
\[
\lim p_u(T^n_k(x)) = \lim \delta_u(O(T, n, x)) = \lim \delta_u(O(T, n, x)) \text{, so that}
\]
\[
\lim \delta_u(O(T, n, x)) = p_u(z) \text{. If } p_u(z) > 0 \text{, then}
\]
\[
\delta_u(O(T, o, z)) = p_u(z) > 0 \text{ and } T \text{ has d.o.d. w.r.t. } U \text{ imply}
\]
\[
\lim \delta_u(O(T, n, z)) < \delta_u(O(T, o, z)) \text{. Thus there is a positive integer } N
\]
\[
\text{with } \delta_u(O(T, N, z)) < \delta_u(O(T, o, z)) \text{. On the other hand, since}
\]
\[
\lim T^n_k(x) = z \text{, we have } T^N(z) = T^N(\lim T^n_k(x)) = \lim T^n_k(x) \text{ and so}
\]
\[
\delta_u(O(T, N, z)) = p_u(T^N(z))
\]
\[
= \lim p_u(T^N_k(x))
\]
\[
= \lim \delta_u(O(T, N+n, k, x))
\]
\[
= \lim \delta_u(O(T, n, x))
\]
\[
= \delta_u(O(T, o, z)) \text{,}
\]
which is a contradiction. Hence \( \lim \delta_u(O(T, n, x)) = p_u(z) = \delta_u(O(T, o, z)) = 0 \text{, for each } U \in U \), and it follows that \( T(z) = z \) and \( \{T^n(x)\}_{n=1}^{\infty} \) is a Cauchy sequence; since also \( \lim T^n_k(x) = z \), and \( X \) is Hausdorff, we see that \( \lim T^n(x) \) exists and \( \lim T^n(x) = \lim T^n_k(x) = z \).

**Corollary 4.4.**

Let \( K \subseteq X \) be nonempty bounded and \( T : K \rightarrow K \) be
nonexpansive w.r.t. \( U \). If \( T \) has d.o.d. w.r.t. \( U \) and there is an \( x \in K \) such that \( \lim_{k \to \infty} T^k(x) = z \in K \), then \( \lim_{n \to \infty} T^n(x) = z \) and \( T(z) = z \).

Since \( K \subseteq X \) is sequentially compact implies \( K \) is bounded, we have the following:

**Corollary 4.5.**

Let \( K \subseteq X \) be nonempty sequentially compact and \( T : K \to K \) be bounded w.r.t. \( U \). If \( T \) has d.o.d. w.r.t. \( U \), then for each \( x \in K \), \( \lim_{n \to \infty} T^n(x) \) exists in \( K \) and is a fixed point of \( T \) in \( K \).

Moreover, \( T \) is asymptotically regular.

**Corollary 4.6.**

Let \( K \subseteq X \) be nonempty sequentially compact and \( T : K \to K \) be nonexpansive w.r.t. \( U \). If \( T \) had d.o.d. w.r.t. \( U \), then for any \( x \in K \), \( \lim_{n \to \infty} T^n(x) \) exists and is a fixed point for \( T \) in \( K \).

**III-5. Weakly periodic mappings.**

In this section, the existence of a common fixed point for a family of weakly periodic nonexpansive mapping is proved.

**Theorem 5.1.**

Let \( K \subseteq X \) be nonempty weakly compact convex and \( F \) be
a (not necessarily finite and not necessarily commuting) family of
nonexpansive mappings w.r.t. $U$ on $K$. Suppose for each $x \in K$,
$x \in \bigcap_{T \in F} \text{Co}(O(T, 1, x))$ and if $M = \{x\} \cup \{T_1 \ldots T_n(x) : \{T_1, \ldots, T_n\} \subseteq F\}$,
then $\text{Co}(M)$ has normal structure w.r.t. $U$. Then there is an $x \in K$
with $T(x) = x$ for each $T \in F$.

Proof: By weak compactness of $K$ and by Zorn's Lemma, let $K_1$
be minimal w.r.t. being a nonempty closed convex subset of $K$ invariant
under each $T \in F$. Suppose there exists an $x \in K$ and a $T_o \in F$ such
that $T_o(x) \neq x$. Define $M = \{x\} \cup \{T_1 \ldots T_n(x) : \{T_1, \ldots, T_n\}$ is any
finite subset of $F\}$, then $M$, and so $\text{Co}(M)$, contains more than one
point. Thus there are a $U \in U$ and an $x_o \in \text{Co}(M)$ with

$$0 < r_o = r_{x_o}(U; \text{Co}(M)) < \delta_u(\text{Co}(M)).$$

Define $R = \{y \in K_1 : r_Y(U; M) \leq r_o\}$, then $R \neq \emptyset$ since $x_o \in R$.

Clearly $R$ is closed and convex. Suppose $T \in F$. If $y \in R$, then
$r_Y(U; M) \leq r_o$. If $z \in M$, then $z \in \text{Co}(O(T, 1, z))$ and so for any
$\varepsilon > 0$, there is an $z_1 \in \text{Co}(O(T, 1, z))$ with $P_u(z - z_1) < \varepsilon$. But
then $z_1 = \sum_{i=1}^{m} \lambda_i T_i(z)$ for some $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{m} \lambda_i = 1$ and $n_i > 1$.

Since $T_i(z) \in M$ for each $i = 1, 2, \ldots, m$, we see that
$P_u(y - T_i(z)) \leq r_o$ for each $i = 1, 2, \ldots, m$, and thus
\[ P_u(T(y) - z) = P_u(T(y) - z_1) + P_u(z_1 - z) \]

\[ < P_u(T(y) - \sum_{i=1}^{m} \lambda_i T(z)) + \epsilon \]

\[ \leq \sum_{i=1}^{m} \lambda_i P_u(T(y) - T(z)) + \epsilon \]

\[ \leq (\sum_{i=1}^{m} \lambda_i P_u(y - T(z))) + \epsilon \]

\[ \leq (\sum_{i=1}^{m} \lambda_i r_0) + \epsilon \]

\[ = r_0 + \epsilon \],

so that \( P_u(T(y) - z) \leq r_0 + \epsilon \) for all \( \epsilon > 0 \) implies \( P_u(T(y) - z) \leq r_0 \) for each \( z \in M \). Thus \( r_{T(y)}(U; M) \leq r_o \) and so \( T(y) \in R \). Hence \( R \) is also invariant under each \( T \in F \), and thus \( R = K_1 \) by minimality of \( K_1 \).

But \( \delta_u(K_1) = \delta_u(R) \leq r_0 < \delta_u(which is impossible.

Therefore we must have \( T(x) = x \) for each \( T \in F \) and each \( x \in K_1 \).

Since \( K_1 \neq \emptyset \), \( F \) has a common fixed point.

Corollary 5.2.

Let \( K \subseteq X \) be nonempty weakly compact convex with normal structure w.r.t. \( U \) and \( F \) be a (not necessarily finite and not necessarily commuting) family of nonexpansive mappings w.r.t. \( U \) on \( K \). Suppose for each \( x \in K \), \( x \in \bigcap_{T \in F} \text{Co}(O(T, 1, x)) \). Then there is an \( \text{Co}(O(T, 1, x)) \) such that \( T(x) = x \), for all \( T \in F \).
Definition 5.3.

Let \( K \subseteq X \) be nonempty. Then (i) \( T : K \to K \) is said to be weakly periodic if and only if for each \( x \in K \), there is a subsequence \( \{ T_{1}^{n}(x) \}_{n=1}^{\infty} \) of \( \{ T^{n}(x) \}_{n=1}^{\infty} \) such that \( T_{1}^{n}(x) \to x \) weakly; (ii) \( T : K \to K \) is said to be pointwise periodic if and only if for each \( x \in K \), there is a positive integer \( N(x) \) such that \( T^{N(x)}(x) = x \); (iii) \( T : K \to K \) is said to be periodic if and only if there is a positive integer \( N \) such that \( T^{N}(x) = x \) for all \( x \in K \).

If \( K \subseteq X \) is nonempty and \( T : K \to K \), then it is clear that \( T \) is periodic implies \( T \) is pointwise periodic; \( T \) is pointwise periodic implies \( T \) is weakly periodic which in turn implies \( x \in \overline{O(T, 1, x)} \) for each \( x \in K \). Hence we have

Corollary 5.4.

Let \( K \subseteq X \) be nonempty weakly compact convex with normal structure w.r.t. \( U \) and \( F \) be a (not necessarily finite nor commuting) family of weakly periodic (respectively pointwise periodic and periodic) nonexpansive mappings w.r.t. \( U \) on \( K \). Then there is an \( x \in K \) with \( T(x) = x \) for all \( T \in F \).

Corollary 5.5.

Let \( K \subseteq X \) be nonempty weakly compact convex and \( F \) be a finite commuting family of pointwise periodic (respectively periodic) nonexpansive mappings w.r.t. \( U \) on \( K \). Then there is an \( x \in K \) with \( T(x) = x \) for each \( T \in F \).
Proof: Suppose $F = \{T_1, \ldots, T_n\}$. Since $T_i T_j = T_j T_i$ for all $i, j = 1, 2, \ldots, n$ and $T_i$ is pointwise periodic (respectively periodic) for each $i = 1, 2, \ldots, n$, we see that for each $x \in K$, the set

$$M = \{T_{i_1}^{m_1} \cdots T_{i_n}^{m_n}(x) : m_1, \ldots, m_n \text{ are nonnegative integers}\}$$

is finite. Thus $\overline{\text{Co}(M)} = \text{Co}(M)$ is compact, by Lemma 1.1., and so $\overline{\text{Co}(M)}$ has normal structure w.r.t. $U$, by Theorem II-2.3. Hence by Theorem 5.1., there is an $x \in K$ with $T_i(x) = x$ for each $i = 1, 2, \ldots, n$. 
BIBLIOGRAPHY


