SUMMABILITY AND INVARIANT MEANS ON SEMIGROUPS

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

March, 1970

Supervisor: Professor E. Granirer

ABSTRACT

This thesis consists of two parts. In the first part, we study summability in left amenable semigroups. More explicitly, various summability methods defined by matrices are considered. Necessary and (or) sufficient conditions are given for matrices to be regular, almost regular, Schur, almost Schur, strongly regular and almost strongly regular, generalizing those of O. Toeplitz, J. P. King, J. Schur, G. G. Lorentz and P. Schaefer for the semigroup of additive positive integers. The theorems are of interest even for the semigroup of multiplicative positive integers.

Let S be a topological semigroup which is amenable as a discrete semigroup. Denote by $LUC(S)$ the set of bounded real-valued left uniformly continuous functions on S. It is shown by E. Granirer that if S is a separable topological group which is amenable as a discrete group and has a certain property (B) then $LUC(S)$ has "many" left invariant means. In the second part of this thesis, we extend this result to certain topological subsemigroups of a topological group. In particular, we show that if S is a separable closed non-compact subsemigroup of a locally compact group which is amenable as a discrete semigroup then $LUC(S)$ has "many" left invariant means. Finally, an example is given to show that this result cannot be extended to every topological semigroup.

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ACKNOWLEDGEMENTS

I wish to thank Professor E. Granirer for his guidance, encouragements and valuable suggestions during the preparation of this thesis.

I would also like to thank Professor J. Fournier for his careful reading of the draft form of this thesis, and to Mrs. Monlisa Wang for typing this thesis with patience and care.

Finally the financial support given me by the University of British Columbia and the National Research Council of Canada is gratefully acknowledged.

INTRODUCTION

In [11] G. G. Lorentz introduced a new method of summation which assigns a general limit to certain bounded sequences $x = \{x(n)\}$. He called the sequences which are summable by this method almost convergent. In various summability methods defined by matrices, an infinite matrix A is strongly regular if the sequence Ax is convergent to k whenever x is almost convergent to k , where Ax is the sequence $\{\sum_n A(m,n)x(n)\}$ whenever the sum is convergent for each m . It is almost regular if Ax is almost convergent to k whenever x is convergent to k , and it is almost strongly regular if Ax is almost convergent to k whenever x is almost convergent to k . Necessary and (or) sufficient conditions for matrices to be strongly regular, almost regular and almost strongly regular have been obtained by G. G. Lorentz, J. P. King and P. Schaefer respectively in [11], [10] and [14].

Now the method of G. G. Lorentz is connected with the theory of amenable semigroups. One of the purposes of this thesis is to extend the various summability methods described above to a class of amenable semigroups. This will be our concern in Chapter I. Our setting then is as follows: S is a left amenable semigroup without any finite left ideals. It is shown that only in such semigroups is almost convergence a generalization of convergence in the sense defined in section 1 (theorem 2.1). If A is an infinite matrix on S , we have the following cases:

(1) Af (see section 1 for the definition of Af) is convergent for every bounded real-valued function f on S . (Schur matrices)

(2) Af is left almost convergent for every bounded real-valued function f on S . (Almost Schur matrices)

(3) Af is convergent to k whenever f is convergent to k . (Regular matrices)

(4) Af is left almost convergent to k whenever f is convergent to k . (Almost regular matrices)

(5) Af is convergent to k whenever f is left almost convergent to k . (Strongly regular matrices)

(6) Af is left almost convergent to k whenever f is left almost convergent to k . (Almost strongly regular matrices)

We give necessary and (or) sufficient conditions for matrices to satisfy (1) to (6). It should be pointed out here that the results for (1) and (3) do not in any way depend on the algebraic structure of S , so that the results of O. Toeplitz [17] and J. Schur [15] can easily be extended. The only additional argument needed is given in lemma 3.1. Examples are given to illustrate our results.

The second purpose of this thesis is the study of the set of left invariant means of a topological semigroup. This will be our concern in Chapter II. Our goal is to extend to topological semigroup a theorem of E. Granirer which states that the set of left invariant means on the left uniformly continuous functions of an amenable separable topological group is "huge" if it has an unbounded left uniformly continuous function. However, we are only able to extend the above result to certain topological subsemigroups of a topological group. Examples are then given to show that this extension is non-trivial. Also an example is given to show that the extension to any topological semigroup is not possible.

SUMMABILITY IN AMENABLE SEMIGROUPS

1. DEFINITION AND NOTATIONS.

Let S be a set. A function f on S with values in a linear topological space L is called unconditionally summable to g in L if $\lim_{\sigma \varepsilon \Sigma} \sum_{s \varepsilon \sigma} f(s) = g$, where Σ is the family of all finite subsets of S directed by inclusion. We shall denote this by $g = \sum_{s \varepsilon S} f(s)$ and say the sum $\sum_{s \varepsilon S} f(s)$ converges to g [2]. In particular, we may take L to be the reals. Then $\sum_{s \varepsilon S} f(s) = g$ if for every $\varepsilon > 0$ there is a finite subset σ_1 such that if $\sigma \supseteq \sigma_1$ then $|\sum_{s \varepsilon \sigma} f(s) - g| < \varepsilon$. It is well known that the above definition implies only countably many $f(s)$ are different from 0 [8, p. 19, theorem 1].

Let S be a set and $S \cup \{\infty\}$ be the one-point compactification of S when S has the discrete topology. Let $m(S)$ be the linear space of all bounded real-valued function on S with the sup norm, and let C_∞ be the closed linear subspace of all those f in $m(S)$ such that $\lim_{s \to \infty} f(s)$ exists. From now on, we shall write $\lim_s f(s)$ for $\lim_{s \to \infty} f(s)$, so that $\lim_s f(s) = k$ means that for every $\varepsilon > 0$ there is a finite subset $\sigma \subset S$ such that $|f(s) - k| < \varepsilon$ if $s \varepsilon S \sim \sigma$. If, in addition, S is a semigroup, then, for $f \varepsilon m(S)$, $a \varepsilon S$, $p_a(f) = f(a)$, and $\ell_a [r_a]$ is the left [right] translation operator on $m(S)$ defined by $\ell_a f(s) = f(as)$ $[r_a f(s) = f(sa)]$. The conjugate mapping of ℓ_a will be denoted by L_a. If $\mathrm{Co}A$ denotes the convex hull of A then elements in $\mathrm{Co}\{p_a : a \varepsilon S\}$ are called finite means. A linear functional ϕ on $m(S)$

is a left invariant mean (LIM) if $\phi(f) \geq 0$ for $f \geq 0$, $\phi(1) = 1$ and $\phi(\ell_a f) = \phi(f)$ for all $f \in m(S)$ and all $a \in S$, where 1 is the constant one function on S, and $f \geq 0$ means $f(s) \geq 0$ for all $s \in S$. We denote the set of all left invariant means by $M\ell(S)$. If $M\ell(S) \neq \emptyset$, where $\emptyset$ denotes the empty set, then the semigroup S is said to be left amenable (LA). If, in addition, ϕ is multiplicative, i.e., $\phi(fg) = \phi(f)\phi(g)$ for all $f,g \in m(S)$ then S is said to be extremely left amenable (ELA). Examples of left amenable semigroups are: commutative semigroups, solvable groups and locally finite groups. For details and an excellent reference see [1]. Extremely left amenable semigroups are precisely those semigroups in which every two elements have a common right zero. For details and other interesting results see [3], [4], [5] and [13].

If S is LA, then a function $f \in m(S)$ is said to be left almost convergent to k if $\phi(f) = \psi(f) = k$ for every $\phi, \psi \in M\ell(S)$. We shall denote the set of all almost convergent functions by F, and write f is ℓac to k to mean f is left almost convergent to k.

If $A = (A(s,t))$ is an infinite matrix on S and $f \in m(S)$, let Af be the function defined on S by $Af(s) = \sum_t A(s,t)f(t)$, whenever the sum on the right hand side converges for each $s \in S$. We say f is F_A-summable to k iff $\lim_s \sum_t A(s,t)f(tb) = k$ uniformly in b, where $b \in S$. This generalizes the definition by G. G. Lorentz [11, p.171].

2. CONVERGENCE AND LEFT ALMOST CONVERGENCE.

We show in this section that left almost convergence is a generalization of convergence in a LA semigroup without finite left ideals.

2.1. <u>THEOREM</u>. Let S be a LA semigroup. Then f is ℓac to k whenever f is convergent to k iff S does not contain any finite left ideals.

PROOF. Suppose S does not contain any finite left ideals. We first show $\phi(1_a) = 0$ for any LIM ϕ and any $a \in S$, where 1_A, here and elsewhere, denotes the characteristic function of A. We shall always write $\phi(A)$ for $\phi(1_A)$. If $\phi(a) > 0$ then since ϕ is left invariant, $\phi(sa) \geq \phi(a) > 0$ for all $s \in S$. Since $\phi(S) = 1$, Sa has to be a finite left ideal, which cannot be.

Suppose now $f \in C_\infty$ and $\lim_s f(s) = 0$. For $\varepsilon > 0$ let H be the finite subset of S for which $|f(s)| < \varepsilon$ whenever $s \in S \sim H$. Let $M = \max_{s \in H} |f(s)|$. Then $|f(s)| \leq \sum_{a \in H} M1_a + \varepsilon 1_{S \sim H}$. Hence if ϕ is any LIM then $|\phi(f)| < \varepsilon$. And since ε is arbitrary, we see that $\phi(f) = 0$. If now $\lim_s f(s) = k$ then by considering $f-k$, we see that f is ℓac to k.

Conversely, suppose S has a finite left ideal A. Let G be a minimal left ideal of S contained in A. We now show that G is a group. Let $G = \{g_1, \ldots, g_n\}$. Then by the minimality of G, $Gg = G$ for each $g \in G$. Hence for $1 \leq i,j \leq n$, there is some $1 \leq k \leq n$

such that $g_k g_i = g_j$. Then for any LIM ϕ , we have $\phi(g_j) = \phi(g_k g_i) \geq \phi(g_i)$. And since this argument is symmetric we conclude that $\phi(g_j) = \phi(g_i)$ for each $g_i, g_j \in G$. If now $g \in G$ is arbitrary then $\phi(gG) \geq \phi(G) = n\phi(gg_1) \geq m\phi(gg_1)$, where m is the cardinality of gG . Since $gG \subset G$ we see that $n = m$ and so $gG = G$. This shows that G is a group.

Let now ϕ be defined on $m(S)$ by $\phi(f) = \frac{1}{n} \sum_{a \in G} f(a)$, $f \in m(S)$. Then it is easy to see that ϕ is indeed a LIM . Clearly the function 1_G is convergent to 0 , while 1_G is not ℓac to 0 since $\phi(1_G) = 1$. This completes the proof.

In view of the above theorem, whenever we consider LA semigroups <u>we will always assume the semigroup to be without any finite left ideals</u>, even though we might not explicitly mention so.

3. ALMOST REGULAR MATRICES.

We first prove the following useful lemma. We point out here that unless S is a countable set the usual proof does not work.

3.1. <u>LEMMA</u>. Let A be an infinite matrix on S . A necessary and sufficient condition for $Af \in m(S)$ whenever $f \in C_\infty$ is that there exists an $M > 0$ such that $\sup_s \sum_t |A(s,t)| \leq M$.

PROOF. Suppose $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$. Then clearly for each $s \in S$ the sum $\sum_t A(s,t)f(t)$ converges and $\|Af\| \leq M\|f\|$

for each $f \in C_\infty$.

Conversely, suppose $Af \in m(S)$ for every $f \in C_\infty$. For each fixed $s \in S$ let σ be the countable subset such that $A(s,t) = 0$ for all $t \notin \sigma$. Let σ_n be an increasing sequence of finite subsets such that $\bigcup_n \sigma_n = \sigma$. For each finite subset σ_n define the continuous linear functional A_n on C_∞ by $A_n(f) = \sum_{t\in\sigma_n} A(s,t)f(t)$. Then $A_n(f)$ converges to $A_\sigma(f)$ for each $f \in C_\infty$, where $A_\sigma(f) = \sum_{t\in\sigma} A(s,t)f(t)$. Hence for all n , $\|A_n\| \leq M(s)$ for some constant $M(s)$. This implies $\|A_\sigma\| \leq M(s)$. Let now A_s be the linear functional on C_∞ defined by $A_s(f) = Af(s)$. Then A_s is continuous since $\|A_s\| = \|A_\sigma\| \leq M(s)$.

Now the set $\{A_s : s \in S\}$ is a pointwise bounded set of continuous linear functional on C_∞ since for each $f \in C_\infty$ $\sup_s |A_s(f)| = \sup_s |Af(s)| < \infty$ by hypothesis. By the principle of uniform boundedness there is an $M > 0$ such that $|A_s| \leq M$ for all $s \in S$.

Once again let $s \in S$ be fixed, and for each finite subset σ_n define the function $f_n \in C_\infty$ by

$$f_n(t) = \begin{cases} \operatorname{sgn} A(s,t) & \text{if } t \in \sigma_n \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f_n \in C_\infty$ and $\|f_n\| \leq 1$. Therefore

$$\sum_{t\in\sigma_n} |A(s,t)| = \left|\sum_t A(s,t)f_n(t)\right| = |A_s(f_n)| \leq \|A_s\| \, \|f_n\| \leq M .$$

Consequently, $\sum_t |A(s,t)| = \sum_{t\in\sigma} |A(s,t)| \leq M$. And since this is true for each $s \in S$, it follows that $\sup_s \sum_t |A(s,t)| \leq M$.

3.2. THEOREM. Let S be a LA semigroup. Then a matrix A is almost regular iff the following conditions are satisfied:

(3.2.1) $\sup_s \sum_t A(s,t) \leq M$ for some $M > 0$.

(3.2.2) $A(s,t)$,as a function of s , is ℓac to 0 for each $t \varepsilon S$.

(3.2.3) $\sum_t A(s,t)$, as a function of s , is ℓac to 1 .

PROOF. Suppose A is almost regular. Then (3.2.1) follows from lemma 3.1. Conditions (3.2.2) and (3.2.3) follows if we note that $A1_t(s) = A(s,t)$ and $A1(s) = \sum_t A(s,t)$.

Conversely, suppose (3.2.1) and (3.2.2) and (3.2.3) hold. Then (3.2.1) together with 3.1 implies $Af \varepsilon m(S)$ for every $f \varepsilon C_\infty$. Using (3.2.2) and (3.2.3) Af is ℓac to $\lim_s f(s)$ whenever f is in the set $B = \{1, 1_t : t \varepsilon S\}$. The proof is then completed by noting that B is fundamental in C_∞ , i.e., the uniform closure of the linear span of B is C_∞ .

3.3. REMARK.

(a) J. P. King was the first to consider almost regular matrices, and for the semigroup of additive positive integers, 3.2 yields King's theorem 3.2 in [10].

(b) Let $(N,+)$ and $(N,\cdot)$ denote the semigroup of additive positive integers and the semigroup of multiplicative positive integers respectively. Define the matrix A by

$$A(m,n) = \begin{cases} \delta(1,n) & \text{if } m \text{ is odd} \\ \delta(m,n) & \text{if } m \text{ is even ,} \end{cases}$$

where $\delta(m,n) = 1$ iff $m = n$ and 0 otherwise. Then with $(N,+)$, A is not almost regular because the sequence $(1,0,1,0,\ldots)$ which appears in the first column of the matrix is ℓac to $1/2$. However, with $(N,\cdot)$, the sequence $(1,0,1,0,\ldots)$ restricted to the ideal $2N$ is the identically 0 sequence and hence by proposition 3.5 below, is ℓac to 0 . It can easily be checked that A satisfies the conditions of 3.2, so that A is almost regular when the semigroup is $(N,\cdot)$. We feel that this example together with those that will follow justify the study of summability in LA semigroups.

In the following let S be a LA semigroup, C be the constant functions on S , and

$$H = \left\{ \sum_{i=1}^{n} f_i(g_i - \ell_{a_i} g_i) : a_i \in S \text{ , } f_i, g_i \in m(S) \text{ , } n = 1,2,\ldots \right\}$$

$$K = \left\{ \sum_{i=1}^{n} (f_i - \ell_{a_i} f_i) : a_i \in S \text{ , } f_i \in m(S) \text{ , } n = 1,2,\ldots \right\} .$$

We denote the uniform closure of any set A in $m(S)$ by $Cl(A)$. For $\phi \in m(S)^*$ define $T_\phi : m(S) \to m(S)$ by $(T_\phi f)(s) = \phi(r_s f)$. In particular, if ϕ is a finite mean, i.e. $\phi = \sum_{i=1}^{n} \phi(t_i) p_{t_i}$, then $(T_\phi f)(s) = \sum_{i=1}^{n} \phi(t_i) f(t_i s)$. Therefore, if $LO(f) = \{\ell_s f : s \in S\}$ then $T_\phi f \in CoLO(f)$.

We now bring in a result of E. Granirer [5, p. 71, theorem 7] which will be used throughout this chapter. For completeness, we will give its proof.

3.4. <u>THEOREM</u>. Let S be a LA semigroup. Then $F = C \oplus Cl(K)$, and f is ℓac to k iff $f \in k1 + Cl(K)$. Furthermore,

(i) If f is ℓac to k and $\{\phi_\alpha\}$ is a net of means such that $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$ then $\lim_\alpha \|T_{\phi_\alpha} f - k1\| = 0$.

(ii) If $k1 \in ClCoLO(f)$ then f is ℓac to k.

PROOF. If f is ℓac to 0 then $f \in Cl(K)$. Otherwise, there would be some $\phi \in m(S)^*$ such that $\phi(f) \neq 0$ and $\phi(Cl(K)) = 0$ by the Hahn-Banach theorem. But then ϕ is left invariant and hence $\phi = \alpha\phi_1 - \beta\phi_2$, where ϕ_1, ϕ_2 are LIM on $m(S)$ and $\alpha,\beta \geq 0$ [6, p. 55]. Since $\phi_1(f) = \phi_2(f) = 0$ it follows that $\phi(f) = 0$, which cannot be. Thus $f \in Cl(K)$. And since $\phi(Cl(K)) = 0$ for any LIM ϕ we get $Cl(K)$ coincides with the set of functions ℓac to 0. If f is ℓac to k then $f-k1$ is ℓac to 0, so that $f \in k1 + Cl(K)$. If $k1 \in Cl(K)$ then $k = \phi(k1) = 0$. Hence $F = C \oplus Cl(K)$.

If ϕ is a LIM then we note $\phi(f) = \phi(g)$ for every $g \in CoLO(f)$. Hence if $k1 \in ClCoLO(f)$ then it follows from the fact that $k1$ can be uniformly approximated by a sequence $g_n \in CoLO(f)$ we have $\phi(f) = k$. This proves (ii).

Suppose now $f = g - \ell_a g$ for some $g \in m(S)$, $a \in S$. If now $\{\phi_\alpha\}$ is any net of means such that $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$, then for $t \in S$,

$$\begin{aligned}|(T_{\phi_\alpha} f)(t)| &= |\phi_\alpha(r_t g - r_t \ell_a g)| \\ &= |\phi_\alpha(r_t g - \ell_a r_t g)| \\ &= |(\phi_\alpha - L_a\phi_\alpha) r_t g| \\ &\le \|L_a\phi_\alpha - \phi_\alpha\| \|g\| \quad .\end{aligned}$$

Thus $\lim_\alpha \|T_{\phi_\alpha} f\| = 0$. Since each α , $\|T_{\phi_\alpha}\| \le \|\phi_\alpha\| = 1$, and T_{ϕ_α} is linear it is easy to see that $\lim_\alpha \|T_{\phi_\alpha} f\| = 0$ whenever $f \in Cl(K)$. If f is ℓac to k then $f = k1 + g$ for some $g \in Cl(K)$. Thus $\lim_\alpha \|T_{\phi_\alpha} f - k1\| = \lim_\alpha \|T_{\phi_\alpha}(f - k1)\| = \lim_\alpha \|T_{\phi_\alpha} g\| = 0$. This proves (i).

3.5. PROPOSITION. Let $f \in m(S)$ and A be any right ideal of S . If $\pi f \in m(A)$ is the restriction of f to A then πf is ℓac to k iff f is ℓac to k .

PROOF. The map $\pi : m(S) \to m(A)$ is defined by $\pi f(s) = f(s)$ for $s \in A$, $f \in m(S)$. Let ϕ_α be a net of finite means converging to left invariance in norm, i.e., $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$ [1, p. 524, theorem 1]. We may assume that each ϕ_α have their support in A, since otherwise, we replace ϕ_α by $L_a\phi_\alpha$ for some fixed $a \in A$. Let $\phi_\alpha(f) = \sum_{i=1}^{n} \phi_\alpha(t_i) f(t_i)$. If f is ℓac to k then by theorem 3.4(i), it follows for all $s \in S$,

$$\begin{aligned}|(T_{\phi_\alpha} \pi f)(s) - k| &= \left|\sum_{i=1}^{n} \phi_\alpha(t_i) \pi f(t_i s) - k\right| \\ &= \left|\sum_{i=1}^{n} \phi_\alpha(t_i) f(t_i s) - k\right| \\ &\le \|T_{\phi_\alpha} f - k1\| \to 0 \ .\end{aligned}$$

Hence f is ℓac to k by theorem 3.4(ii).

Similarly by taking ϕ_α to be a net of finite means on $m(A)$ such that $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$, we can show f is ℓac to k whenever πf is ℓac to k.

4. REGULAR MATRICES.

The following is the extension of the well known Toeplitz theorem for regular matrices. However, this extension does not depend on the algebraic structure of S. The only additional argument needed in the usual proof is given in lemma 3.1.

4.1. <u>THEOREM</u>. Let S be a LA semigroup. The following conditions are both necessary and sufficient for an infinite matrix A to be regular:

(4.1.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(4.1.2) $\lim_s A(s,t) = 0$ for each $t \in S$.

(4.1.3) $\lim_s \sum A(s,t) = 1$.

PROOF. Suppose A is regular. Then (4.1.1) follows from lemma 3.1. Conditions (4.1.2) and (4.1.3) follow if we note that $A1_t(s) = A(s,t)$ and $A1(s) = \sum_{t\in S} A(s,t)$.

Conversely, suppose (4.1.1), (4.1.2) and (4.1.3) hold. Then (4.1.1) and lemma 3.1 implies $Af \in m(S)$ for every $f \in C_\infty$. Using (4.1.2) and (4.1.3) Af is convergent to $\lim_s f(s)$ whenever f is in the set $B = \{1, 1_t : t \in S\}$. We then complete the proof by noting that B is fundamental in C_∞ .

4.2. REMARK. It is interesting to compare theorem 3.2 and 4.1. More specifically, we draw to the reader's attention the similarities in the conditions of the theorems as well as the proofs.

5. STRONGLY REGULAR MATRICES.

We will frequently need the following: If S is a left cancellative semigroup, $b \in S$, $\sup_s \sum_t |A(s,t)| \leq M$ then

(i) $$\sum_{t \in bS} A(s,t) = \sum_{t \in S} A(s,bt)$$

(ii) $$\sum_{t \in S \sim bS} |A(s,t)| = \sum_{t \in S} |A(s,t)| - \sum_{t \in bS} |A(s,t)|$$

$$= \sum_{t \in S} (|A(s,t)| - |A(s,bt)|)$$

(5.0.1)

$$\leq \sum_{t \in S} |A(s,t) - A(s,bt)|$$

The following theorem 5.1 contains one of the main results of this chapter. When S is the semigroup of additive positive integers, theorem 5.1 yields G. G. Lorentz's theorem 8 in [11, p. 181].

5.1. THEOREM. Let S be a left cancellative LA semigroup generated by $B \subset S$. The following conditions are necessary and sufficient for an infinite matrix A to be strongly regular:

(5.1.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(5.1.2) $\lim_s \sum_t A(s,t) = 1$.

(5.1.3) $\lim_s \sum_t |A(s,t) - A(s,at)| = 0$ for each $a \in B$.

PROOF. We first show (5.1.3) implies that $\lim_s \sum_t |A(s,t) - A(s,bt)| = 0$ for every $b \in S$. Let $b = a_n a_{n-1} \dots a_1$, where $a_i \in B$ for $i = 1, 2, \dots, n$. Let $b_j = a_j a_{j-1} \dots a_1$, $j = 2, 3, \dots, n-1$. Then the desired result follows from the following:

$$\begin{aligned} \sum_t |A(s,t) - A(s,bt)| &\le \sum_t |A(s,t) - A(s,a_1 t)| \\ &+ \sum_{t \in a_1 S} |A(s,t) - A(s,a_2 t)| \\ &+ \sum_{t \in b_2 S} |A(s,t) - A(s,a_3 t)| + \dots \\ &+ \sum_{t \in b_{n-1} S} |A(s,t) - A(s,a_n t)| . \end{aligned} \tag{5.1.4}$$

Now suppose f is ℓac to k . Then $Af \in m(S)$ by 3.1. Let ϕ_α be a net of finite means converging in norm to left invariance, i.e., $\lim_\alpha \|L_s \phi_\alpha - \phi_\alpha\| = 0$ [1, p. 524, theorem 1]. Let $\phi_\alpha(f) = \sum_{i=1}^n \phi_\alpha(t_i) f(t_i)$. Let $\varepsilon > 0$. Then there is an α_0 such that if $\alpha \ge \alpha_0$ then $|\sum_{i=1}^n \phi_\alpha(t_i) f(t_i t) - k| < \varepsilon$ for all $t \in S$ by theorem 3.4(i). Then for all $t,s \in S$, and all $\alpha \ge \alpha_0$,

(5.1.5) $$|\sum_t A(s,t) \sum_{i=1}^n \phi_\alpha(t_i) f(t_i t) - \sum_t A(s,t) k| < M\varepsilon \quad .$$

From (5.1.2) there is a finite subset H_1 such that for $s \notin H_1$ then

$|\sum_t A(s,t) - 1| < \varepsilon$. Hence for $s \notin H_1$,

(5.1.6) $$|\sum_t A(s,t)k - k| < \varepsilon|k| \quad .$$

Fix an $\alpha \geq \alpha_0$. From (5.1.4) there is a finite subset H_2 such that if $s \notin H_2$, then $\sum_t |A(s,t_it) - A(s,t)| < \varepsilon$ for $i = 1, 2, \ldots, n$. Hence for $s \notin H_2$,

$$|Af(s) - \sum_t A(s,t) \sum_{i=1}^n \phi_\alpha(t_i)f(t_it)|$$

$$= |\sum_{i=1}^n \phi_\alpha(t_i)Af(s) - \sum_{i=1}^n \phi_\alpha(t_i)\sum_t A(s,t)f(t_it)|$$

$$= |\sum_{i=1}^n \phi_\alpha(t_i)[\sum_t A(s,t)f(t) - \sum_t A(s,t)f(t_it)]|$$

(5.1.7) $$= |\sum_{i=1}^n \phi_\alpha(t_i)[\sum_{t\varepsilon t_iS} A(s,t)f(t) + \sum_{t\varepsilon S\sim t_iS} A(s,t)f(t) - \sum_t A(s,t)f(t_it)]|$$

$$\leq \|f\| \sum_{i=1}^n \phi_\alpha(t_i)\sum_t |A(s,t_it) - A(s,t)| + \|f\| \sum_{i=1}^n \phi_\alpha(t_i) \sum_{t\varepsilon S\sim t_iS} |A(s,t)|$$

$$\leq 2\|f\| \sum_{i=1}^n \phi_\alpha(t_i)\sum_t |A(s,t_it) - A(s,t)| \qquad \text{(see (5.0.1))}$$

$$< 2\,\|f\|\,\varepsilon \quad .$$

Then for the fixed $\alpha \geq \alpha_0$ and $s \notin H_0 = H_1 \cup H_2$, it follows from (5.1.5), (5.1.6) and (5.1.7) that

$$|Af(s) - k| \leq |Af(s) - \sum_t A(s,t) \sum_{i=1}^n \phi_\alpha(t_i)f(t_it)|$$

$$+ |\sum_t A(s,t) \sum_{i=1}^n \phi_\alpha(t_i)f(t_it) - \sum_t A(s,t)k|$$

$$+ \left|\sum_t A(s,t)k - k\right|$$

$$< (2\|f\| + M + |k|)\varepsilon \ .$$

Conversely, suppose A is strongly regular. Then A is regular and hence, (5.1.1) and (5.1.2) follows. If (5.1.3) does not hold for some a, then there is an $\varepsilon > 0$ such that $\sum_t |A(s,t) - A(s,at)| > 5\varepsilon$ for infinitely many $s \in S$. Using this and the fact that $\lim_s [A(s,t) - A(s,at)] = 0$ for each $t \in S$ we now choose an increasing sequence $\sigma(k)$ of finite subsets of S and an infinite subset $\{s_k\}$ of S as follows: For convenience, denote $A(s,t) - A(s,at)$ by $B(s,t)$. In general, for $k = 1, 2, \ldots$, if $\sigma(k) \supset \sigma(k-1)$ (where $\sigma(0) = \emptyset$) let $s_k \in S$ be such that

(5.1.8) $$\sum_{t\in S} |B(s_k,t)| > 5\varepsilon \quad \text{and}$$

(5.1.9) $$\sum_{t\in\sigma(k)} |B(s_k,t)| < \varepsilon \ .$$

And since the sum $\sum_{t\in S} |B(s_k,t)|$ is convergent there is a finite subset $\sigma(k+1) \supset \sigma(k)$ such that

(5.1.10) $$\sum_{t\in S\sim\sigma(k+1)} |B(s_k,t)| < \varepsilon \ .$$

Then from (5.1.8), (5.1.9) and (5.1.10) we have

$$\sum_{t\in\sigma(k+1)\sim\sigma(k)} |B(s_k,t)|$$

(5.1.11) $$= \Big(\sum_{t\in S} - \sum_{t\in\sigma(k)} - \sum_{t\in S\sim\sigma(k+1)}\Big)|B(s_k,t)|$$

$$> 5\varepsilon - \varepsilon - \varepsilon = 3\varepsilon$$

We now define $f \in m(S)$ by

$$f(t) = \begin{cases} \operatorname{sgn} B(s_k,t) & \text{if} \quad t \in a(\sigma(k+1)\sim\sigma(k)) \\ 0 & \text{otherwise.} \end{cases}$$

Since S is left cancellative, f is well-defined. Moreover, $\|f\| \leq 1$ and $\ell_a f - f$ is ℓac to 0. But for $k = 1, 2, \ldots,$ it follows from (5.1.9), (5.1.10) and (5.1.11) that

$$|A(\ell_a f - f)(s_k)| = |\sum_{t\varepsilon S} A(s,t)f(at) - \sum_{t\varepsilon S} A(s,t)f(t)|$$

$$= |\sum_{t\varepsilon S} A(s_k,t)f(at) - \sum_{t\varepsilon aS} A(s_k,t)f(t) - \sum_{t\varepsilon S\sim aS} A(s_k,t)f(t)|$$

$$= |\sum_{t\varepsilon S} [A(s_k,t) - A(s_k,at)]f(at)|$$

$$= |\sum_{t\varepsilon S} B(s_k,t)f(at)|$$

$$= |(\sum_{t\varepsilon\sigma(k)} + \sum_{t\varepsilon\sigma(k+1)\sim\sigma(k)} + \sum_{t\varepsilon S\sim\sigma(k+1)})B(s_k,t)f(at)|$$

$$\geq (\sum_{t\varepsilon\sigma(k+1)\sim\sigma(k)} - \sum_{t\varepsilon\sigma(k)} - \sum_{t\varepsilon S\sim\sigma(k+1)})|B(s_k,t)|$$

$$> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon \quad .$$

But this cannot be since $A(\ell_a f - f)$ converges to 0. Thus (5.1.3) holds.

5.2. <u>REMARK</u>. If A is a strongly regular matrix we cannot hope that (5.1.3) hold in general as the following example shows: Let S be the set of positive integers with multiplication $*$ defined by $i*j = k$, where

k is the smallest odd integer greater than or equal to $i \vee j = \max(i,j)$. Then $*$ is associative since for each $i,j,k \in S$, both $(i*j)*k$ and $i*(j*k)$ are equal to the smallest odd integer greater than or equal to $\max(i,j,k)$. Moreover, for every $i,j \in S$ then either $i \vee j$ or $(i \vee j) + 1$ is a right zero for i and j. Hence S is an ELA semigroup. Let now A be a matrix defined on S by

(i) $A(m,n) = 0$ whenever n is even, or $n < 2m-1$.

(ii) $A(m,2n-1) \geq A(m,2n+1) > 0$ whenever $2n-1 \geq 2m-1$.

(iii) $\sum_n A(m,n) = 1$ for each m.

Then A does not satisfy (5.1.3) since for example, $\lim_m \sum_n |A(m,n) - A(m,3*n)| = 1$. However, A is strongly regular as theorem 5.4 below shows. We leave the details for the reader to check.

5.3. <u>REMARK</u>. If S is a cancellative LA semigroup without any finite left ideals then C_∞ is a proper subset of F, since otherwise, the identity matrix would have to satisfy (5.1.3). Then there exist finite subsets σ_1, σ_2, a, $b \in S$, $a \neq b$, such that $at = t$ for $t \in S \sim \sigma_1$ and $bt = t$ for $t \in S \sim \sigma_2$. Hence if $t \in S \sim (\sigma_1 \cup \sigma_2)$ then $at = bt = t$. Since S is right cancellative, $a = b$, which cannot be.

5.4. <u>THEOREM</u>. If S is ELA semigroup then the following conditions are both necessary and sufficient for an infinite matrix on S to be strongly regular:

(5.4.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(5.4.2) $$\lim_s \sum_t A(s,t) = 1 .$$

(5.4.3) $$\lim_s \sum_{t\varepsilon S\sim aS} |A(s,t)| = 0 \quad \text{for every } a \varepsilon S \text{ such that } a \varepsilon Sa.$$

PROOF. Let $f = g - \ell_b g$, $g \varepsilon m(S)$, $g \neq 0$, and $b \varepsilon S$. Let $a \varepsilon S$ be such that $ba = a$. For $\varepsilon > 0$, let H_0 be a finite subset such that if $s \notin H_0$ then $\sum_{t\varepsilon S\sim aS} |A(s,t)| < \varepsilon/2\|g\|$. Then for $s \notin H_0$, we have

$$|Af(s)| = |\sum_{t\varepsilon S} A(s,t)g(t) - \sum_{t\varepsilon S} A(s,t)g(bt)|$$

$$\leq |\sum_{t\varepsilon aS} A(s,t)g(t) - \sum_{t\varepsilon aS} A(s,t)g(bt)| + 2\|g\| \sum_{t\varepsilon S\sim aS} |A(s,t)|$$

$$< \varepsilon .$$

Thus Af is convergent to 0 . And since A is linear, Af is convergent to 0 whenever $f \varepsilon K$. Suppose now $f \varepsilon Cl(K)$, and let $g_n \varepsilon K$ be such that $\lim_n \|g_n - f\| = 0$. Then $\lim_n \|Ag_n - Af\| < \lim_n M\|g_n - f\| = 0$. Since lim is a continuous linear functional on C_∞ this implies $\lim_s Af(s) = 0$. If now f is ℓac to k then $f-k$ is ℓac to 0 and hence $\lim_s A(f - k)(s) = 0$, i.e., $\lim_s Af(s) = k$.

Conversely, if A is strongly regular then (5.4.1) and (5.4.2) hold. If (5.4.3) does not hold there is an $\varepsilon > 0$ and an $a \varepsilon S$ such that $\sum_{t\varepsilon S\sim aS} |A(s,t)| > 5\varepsilon$ for an infinite number of $s \varepsilon S$. Using this together with the fact that $\lim_s A(s,t) = 0$ for each $t \varepsilon S$, we can choose, as in the proof of 5.1, an increasing sequence $\sigma(k)$ of finite subsets of $S \sim aS$ and an infinite subset $\{s_k\}$ of S so that the following conditions holds:

(5.4.4) $\sum_{t\in\sigma(k)} |A(s_k,t)| < \varepsilon$.

(5.4.5) $\sum_{t\in S\sim aS\sim\sigma(k+1)} |A(s_k,t)| < \varepsilon$.

(5.4.6) $\sum_{t\in\sigma(k+1)\sim\sigma(k)} |A(s_k,t)| > 3\varepsilon$.

We can choose the sets $\sigma(k)$ to be subsets of $S \sim aS$ since $S \sim aS$ is infinite (otherwise $\sum_{t\in S\sim aS} |A(s,t)|$ would be a finite sum of convergent functions) and the sum $\sum_{t\in S\sim aS} |A(s,t)|$ is finite. Define

$$f(t) = \begin{cases} \operatorname{sgn} A(s_k,t) & \text{if } t \in \sigma(k+1)\sim\sigma(k) \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that $\|f\| \leq 1$, $\ell_a f = 0$, the support of f is contained in $S \sim aS$, and that $f - \ell_a f$ is ℓac to 0 . Using (5.4.4), (5.4.5) and (5.4.6) we have for all k ,

$$|A(f - \ell_a f)(s_k)| = |\sum_{t\in S} A(s_k,t)f(t) - \sum_{t\in S} A(s_k,t)f(at)| = |\sum_{t\in S\sim aS} A(s_k,t)f(t)|$$

$$= |(\sum_{t\in\sigma(k)} + \sum_{t\in\sigma(k+1)\sim\sigma(k)} + \sum_{t\in S\sim aS\sim\sigma(k+1)})A(s_k,t)f(t)|$$

$$\geq (\sum_{t\in\sigma(k+1)\sim\sigma(k)} - \sum_{t\in\sigma(k)} - \sum_{t\in S\sim aS\sim\sigma(k+1)})|A(s_k,t)|$$

$$> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon .$$

But this cannot be since $A(f - \ell_a f)$ is convergent to 0. Thus (5.4.3) hold.

5.5. REMARK.

(a) In the proof of the necessity in 5.4, we did not use the fact that a is a right zero of some element in S, so that in any LA semigroup, (5.4.1), (5.4.2) and (5.4.3) are necessary conditions whenever A is strongly regular.

(b) We note that in the proof of 5.1, if f is replaced by $r_b f$ for any $b \in S$, the same α_0 is obtained since $\|T_{\phi_\alpha}(r_b f)\| \leq \|T_{\phi_\alpha}(f)\|$. It follows from this that the same finite subset H_0 there is obtained if f is replaced by $r_b f$ for any $b \in S$. Also in the proof of 5.4, if f is replaced by $r_b f$ for any $b \in S$, the same finite subset H_0 is obtained. In his proof, for the semigroup of additive positive integers, G. G. Lorentz made the same observation (it should be pointed out that our proof differs in many ways from his) and used it in the proof of the following theorem 5.6 for the additive positive integers [11, p. 181]. We now use this observation in the following theorem.

5.6. THEOREM. Let S be a left cancellative LA [ELA, not necessarily left cancellative] semigroup and A be an infinite matrix on S satisfying the conditions of theorem 5.1 [theorem 5.4]. Then f is F_A-summable to k iff f is ℓac to k.

PROOF. If f is ℓac to k then, as known, $r_t f$ is ℓac to k for every $t \in S$. This can easily be seen from the fact that the left translation operator commutes with the right translation operator. By remark 5.5(b) $\lim_s A(r_t f)(s) = \lim_s \sum_{t'} A(s,t')f(t't) = k$ uniformly in t, i.e., f is F_A-summable.

The converse follows easily from corollary 5.8 to the following theorem, proved first for the semigroup of additive positive integers by P. Schaefer [14, p. 51].

5.7. THEOREM. Let S be a LA semigroup. If A is almost regular and f is F_A-summable to k then f is ℓac to k.

PROOF. We basically adapt Schaefer's proof to the general semigroup case. Suppose f is F_A-summable to k. Then $\lim_s \sum_t A(s,t)f(tb) = k$ uniformly in b. Let g be a function of s and b defined by $g(s,b) = \sum_t A(s,t)f(tb)$. Then $g(s,b) = k + h(s,b)$, where h is a function of s and t such that h, as a function of s, is convergent to 0 uniformly in $b \in S$. Now for each finite subset σ of S, define g_σ as a function of s and b by $g_\sigma(s,b) = \sum_{t\in\sigma} A(s,t)f(tb)$. Then g_σ converges uniformly to g for each fixed $s \in S$ since $\|g_\sigma - g\| = \sup_b |g_\sigma(b) - g(b)| \leq \sum_{t\in S\sim\sigma} |A(s,t)|\ \|f\|$ and this can be made as small as we please.

If now ϕ is any LIM then for each fixed $s \in S$,

$$\phi(g) = \phi(\lim_\sigma g_\sigma) = \lim_\sigma \sum_{t\varepsilon\sigma} A(s,t)\phi(\ell_t f) = \sum_{t\varepsilon S} A(s,t)\phi(f) = \phi(k+h) = k + \phi(h).$$

Thus $\phi(f)\sum_t A(s,t) = k + \phi(h)$. Since h , as a function of s converges to 0 uniformly in $b \varepsilon S$, for every $\varepsilon > 0$ there is a finite subset H such that if $s \notin H$ then $|\phi(h)| < \varepsilon$, i.e., $\phi(h)$, as a function of s , is convergent to 0 . If now ψ is any LIM then

$$\psi[\phi(f)\sum_t A(s,t)] = \phi(f)\psi[\sum_t A(s,t)] = \psi(k) + \psi(\phi(h)) \quad . \tag{5.7.1}$$

Since A is almost regular, $\psi[\sum_t A(s,t)] = 1$ and $\psi(\phi(h)) = 0$. Therefore, we see from (5.7.1) that $\phi(f) = \psi(k) = k$, i.e., f is ℓac to k . This completes the proof.

The following corollary, which is due to G. G. Lorentz for the additive positive integers [11, p. 171], is an immediate consequence of 5.7 since every regular matrix is almost regular.

5.8. COROLLARY. Let S be a LA semigroup. If A is regular and f is F_A-summable to k then f is ℓac to k .

6. ALMOST SCHUR MATRICES.

The following theorem gives sufficient conditions for a matrix to be almost Schur.

6.1. THEOREM. Let S be a LA semigroup. Let A be an infinite matrix on S satisfying the following conditions:

(6.1.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(6.1.2) The sum $\sum_t |A(s,t)|$ converges uniformly in s .

(6.1.3) $A(s,t)$, as a function of s , is ℓac to α_t for each $t \varepsilon S$.

Then Af is ℓac to $\sum_t \alpha_t f(t)$ for each $f \varepsilon m(S)$.

PROOF. Let Σ be the family of all finite subsets of S directed by inclusion. Let $f \varepsilon m(S)$. For each $\sigma \varepsilon \Sigma$ define g_σ by $g_\sigma(s) = \sum_{t\varepsilon\sigma} A(s,t)f(t)$. Then clearly g_σ is ℓac to $\sum_{t\varepsilon\sigma} \alpha_t g(t)$ by (6.1.3). Now (6.1.1) implies $Af \varepsilon m(S)$. And using (6.1.2), one can readily show that Af is the uniform limit of g_σ . Hence Af is ℓac and if ϕ is any LIM then

$$\phi(Af) = \phi(\lim_\sigma g_\sigma) = \lim_\sigma \phi(g_\sigma) = \lim_\sigma \sum_{t\varepsilon\sigma} \alpha_t f(t) = \sum_t \alpha_t f(t) .$$

6.2. <u>COROLLARY</u>. If A is an almost regular matrix then A cannot be an almost Schur matrix.

PROOF. If A is an almost regular matrix $A(s,t)$ is ℓac to 0 and $\sum_t A(s,t)$ is ℓac to 1 . If A is also an almost Schur matrix then Af is ℓac to 0 by the theorem. In particular, $\sum_t A(s,t)$ is ℓac to 0 , which cannot be.

6.3. <u>REMARK</u>. It is easy to see that if A is an almost Schur matrix then (6.1.1) and (6.1.3) are necessary. However, (6.1.2) is not necessary as the following example shows: Let S be the semigroup of ordinals less than the first uncountable ordinal Ω with the usual addition of order types. Then S is a non-commutative, left cancellative, ELA semigroup [5, p. 73]. Define A on S by

$$A(s,t) = \begin{cases} \delta(s,t) & \text{if } 1 \leq s < \omega \text{ , } t \in S \\ 0 & \text{otherwise,} \end{cases}$$

where ω is the first countable infinite ordinal. Then for any $f \in m(S)$, it is easy to see that $Af(s) = 0$ for $s \in \alpha + S$ for any $\alpha > \omega$. By 3.5 Af is ℓac to 0. But clearly (6.1.2) is not satisfied.

6.4. EXAMPLE. Let $S = \{(m,n) : m = 1, 2, \ldots, n = 1, 2, \ldots\}$. Define the operation $*$ on S by

(a) $(m_1,n_1)*(m_2,n_2) = (m_1+n_2, n_1+n_2)$ if $m_1 \neq 1$ and $m_2 \neq 1$.

(b) $(m_1,n_1)*(1,n_2) = (1,n_2)*(m_1,n_1) = (1,n_2)$ if $m_1 \neq 1$.

(c) $(1,n_1)*(1,n_2) = (1,n_1 \vee n_2)$, where $n_1 \vee n_2 = \max(n_1,n_2)$.

That S is an ELA semigroup actually follows from the following general construction: Let $S = S_1 \cup S_2$, where S_1 is any semigroup and S_2 is any ELA semigroup. For $a,b \in S$, define the product $a*b$ to be the product of a and b in S_i , $i = 1, 2$, if both $a,b \in S_i$. If $a \in S_1$, $b \in S_2$ then $a*b = b*a = b$.

Now for each k and each ℓ fixed, define

$$f(m,n) = \begin{cases} (1/2)^{k+\ell} & \text{if } m = 1 \\ 0 & \text{otherwise .} \end{cases}$$

Then $g(m,n) = \ell_{(1,1)}f(m,n) = f[(1,1)*(m,n)] = (1/2)^{k+\ell}$ for all m and n. Then $g(m,n) - f(m,n)$ is ℓac to 0. Define the matrix A on S by $A(m,n;k,\ell) = g(m,n) - f(m,n)$. Then $A(m,n;k,\ell)$, as a function of (m,n), is ℓac to 0 for each k and ℓ; and $\sum_{k,\ell} |A(m,n;k,\ell)| = \sum_{k,\ell} (1/2)^{k+\ell}$ converges uniformly in (m,n) to 1. By 6.1 A is an almost Schur matrix.

7. SCHUR MATRICES.

The following is the extension of Schur's theorem. However, this extension does not depend in any way on the algebraic structure of S.

7.1. THEOREM. The following conditions are both necessary and sufficient for an infinite matrix A to be a Schur matrix:

(7.1.1) $\lim_s A(s,t)$ exists for every $t \in S$.

(7.1.2) The sum $\sum_t |A(s,t)|$ converges uniformly in S.

Moreover, if $\lim_s A(s,t) = \alpha_t$ then Af converges to $\sum_t \alpha_t f(t)$ for every $f \in m(S)$.

PROOF. Suppose (7.1.1) and (7.1.2) are satisfied. If Σ denote the family of all finite subsets of S directed by inclusion, for each $\sigma \in \Sigma$ and $f \in m(S)$ define $g_\sigma(s) = \sum_{t\in\sigma} A(s,t)f(t)$. Then (7.1.1) and

(7.1.2) imply that each $g_\sigma \in C_\infty$ and Af is the uniform limit of g_σ ; and thus $Af \in C_\infty$. Since lim is a continuous linear functional on C_∞, $\lim_s Af(s) = \sum_t \alpha_t f(t)$.

Conversely, suppose A is a Schur matrix. Then $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$ and $\lim_s A(s,t) = \alpha_t$ exists for each $t \in S$. Define g_σ as above. Then g_σ is a norm-bounded net converging pointwise to Af in C_∞ . By [12, p. 249, lemma 3] g_σ converges w* to Af i.e., for every $\phi \in \ell_1(S)$, $\lim_\sigma \phi(g_\sigma) = \phi(Af)$. In particular, $\lim_s Af(s) = \lim_\sigma \lim_s g_\sigma(s) = \sum_t \alpha_t f(t)$. Hence the matrix B defined on S by $B(s,t) = A(s,t) - \alpha_t$ is a Schur matrix such that Bf converges to 0 for every $f \in m(S)$. If now (7.1.2) is not satisfied then there is an $\varepsilon > 0$ such that for all finite subsets σ there is an infinite subset $S(\sigma)$ such that $\sum_{t \in S \sim \sigma} |B(s,t)| > 5\varepsilon$ for all $s \in S(\sigma)$. Using this, together with $\lim_s B(s,t) = 0$ for all $t \in S$, choose an increasing sequence $\sigma(k)$ of finite subsets of S and an infinite subset $\{s_k\}$ of S as follows: In general, for $k = 1, 2, \ldots,$ if $\sigma(k) \supset \sigma(k-1)$ (where $\sigma(0) = \emptyset$) , let $s_k \in S$ be such that

(7.1.3) $$\sum_{t \in S \sim \sigma(k)} |B(s_k,t)| > 5\varepsilon \qquad \text{and}$$

(7.1.4) $$\sum_{t \in \sigma(k)} |B(s_k,t)| < \varepsilon \quad .$$

And since the sum $\sum_{t \in S} |B(s_k,t)|$ is convergent, there is a finite subset $\sigma(k+1) \supset \sigma(k)$ such that

(7.1.5) $$\sum_{t \in S \sim \sigma(k+1)} |B(s_k,t)| < \varepsilon \; .$$

Then from (7.1.3), (7.1.4) and (7.1.5) it follows that

$$(7.1.6)\qquad \begin{aligned}\sum_{t\varepsilon\sigma(k+1)\sim\sigma(k)} |B(s_k,t)| &= \Big(\sum_{t\varepsilon S} - \sum_{t\varepsilon\sigma(k)} - \sum_{t\varepsilon S\sim\sigma(k+1)}\Big)|B(s_k,t)| \\ &> 5\varepsilon - \varepsilon - \varepsilon = 3\varepsilon \quad .\end{aligned}$$

Define now $f \varepsilon m(S)$ by

$$f(t) = \begin{cases} \operatorname{sgn} B(s_k,t) & \text{if} \quad t \varepsilon \quad \sigma(k+1)\sim\sigma(k) \\ 0 & \text{otherwise} .\end{cases}$$

Using (7.1.4), (7.1.5), (7.1.6) and $\|f\| \leq 1$, we have, for $k = 1,2, \ldots,$

$$\begin{aligned}|Bf(s_k)| &= |\sum_{t\varepsilon S} B(s_k,t)f(t)| \\ &\geq \Big(\sum_{t\varepsilon\sigma(k+1)\sim\sigma(k)} - \sum_{t\varepsilon\sigma(k)} - \sum_{t\varepsilon S\sim\sigma(k+1)}\Big)|B(s_k,t)| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon \quad .\end{aligned}$$

But this cannot be since Bf is converging to 0 .

8. ALMOST STRONGLY REGULAR MATRICES.

The following theorem gives sufficient conditions for a matrix to be almost strongly regular. We strengthen the theorem when the semigroup S is ELA .

8.1. <u>THEOREM</u>. Let S be a left cancellative LA semigroup generated by $B \subset S$. Let A be an infinite matrix on S such that the following conditions hold:

(8.1.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(8.1.2) $\sum_t A(s,t)$, as a function of s , is ℓac to 1 .

(8.1.3) $\sum_t |A(s,t) - A(s,at)|$, as a function of s , is ℓac to 0 for every $a \in B$.

Then Af is ℓac to k whenever f is ℓac to k .

PROOF. Condition (8.1.3) together with (5.1.4) implies $\sum_t |A(s,t) - A(s,at)|$, as a function of s , is ℓac to 0 for all $a \in S$.

If $f \in F$ then (8.1.1) implies $Af \in m(S)$, while (8.1.2) implies Af is ℓac to $f(s)$ whenever f is a constant function. Suppose $f \in K$ and $f = g - \ell_a g$ for some $g \in m(S)$, $a \in S$. Let ϕ_α be a net of finite means converging in norm to left invariance, i.e., $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$ [1, p. 524, theorem 1]. Let $\phi_\alpha(f) = \sum_{i=1}^n \phi_\alpha(t_i)f(t_i)$. Then (8.1.3) together with theorem 3.4 shows that for every $\varepsilon > 0$, there is an α_0 such that if $\alpha \geq \alpha_0$ then for all $s \in S$, $\sum_{i=1}^n \phi_\alpha(t_i)\sum_t |A(t_is,t) - A(t_is,at)| < \varepsilon$. Then for all $s \in S$, $\alpha \geq \alpha_0$,

$$\left|\sum_{i=1}^n \phi_\alpha(t_i)[Ag - A(\ell_a g)](t_is)\right|$$

$$= \left|\sum_{i=1}^n \phi_\alpha(t_i)Ag(t_is) - \sum_{i=1}^n \phi_\alpha(t_i)A(\ell_a g(t_is))\right|$$

$$= \left| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S} A(t_i s,t) g(t) - \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S} A(t_i s,t) g(at) \right|$$

$$\leq \left| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S} [A(t_i s,at) - A(t_i s,t)] g(at) \right|$$

$$+ \left| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S\sim aS} A(t_i s,t) g(t) \right|$$

$$\leq \|g\| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S} |A(t_i s,at) - A(t_i s,t)| + \|g\| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S\sim aS} |A(t_i s,t)|$$

$$\leq 2\|g\| \sum_{i=1}^{n} \phi_\alpha(t_i) \sum_{t\varepsilon S} |A(t_i s,t) - A(t_i s,at)| \qquad \text{(see (5.0.1))}$$

$$< 2\,\|g\|\,\varepsilon \quad .$$

It follows from theorem 3.4 that Af is ℓac to 0 . Since A is linear, Af is ℓac to 0 whenever $f \varepsilon K$. Suppose now $f \varepsilon Cl(K)$, and let $g_n \varepsilon K$ be such that $\lim_n \|g_n - f\| = 0$. Then $\lim_n \|Ag_n - Af\| \leq \lim_n M \|g_n - f\| = 0$. Hence if ϕ is any LIM then $\lim_n |\phi(Ag_n) - \phi(Af)| = 0$. Thus Af is ℓac to 0 since $\phi(Ag_n) = 0$ for each n . This completes the proof.

8.2. REMARK.

(a) If S is the additive positive integers theorem 8.1 yields P. Schaefer's theorem 2 [14, p. 52]. Our proof is entirely different from his. His proof does not seem to carry over to the general case.

(b) It is clear that if A is an almost strongly regular matrix then (8.1.1) and (8.1.2) are both necessary conditions. However, (8.1.3) does not always hold, since the identity matrix A is almost strongly regular but for the additive positive integers, $\lim_m \sum_n |A(m,n) - A(m,n+1)| = 2$. When S is ELA (not necessarily left cancellative) we have the following stronger result.

8.3. THEOREM. Let S be ELA, and A be an infinite matrix on S satisfying the following conditions:

(8.3.1) $\sup_s \sum_t |A(s,t)| \leq M$ for some $M > 0$.

(8.3.2) $\sum_t A(s,t)$, as a function of s, is ℓac to 1.

(8.3.3) $\sum_{t\varepsilon S\sim aS} |A(s,t)|$, as a function of s, is ℓac to 0 for every $a \varepsilon S$ such that $a \varepsilon Sa$.

Then Af is ℓac to k whenever f is ℓac to k.

PROOF. Let f ℓac to k'. By [5, p. 72, theorem 8] for every $\varepsilon > 0$ there is a $b \varepsilon S$ such that if $t \varepsilon bS$ then $|f(t) - k| < \varepsilon$. Let $a \varepsilon S$ be such that $ba = a$. Then if $t \varepsilon aS \subset bS$, $|f(t) - k| < \varepsilon$. By (8.3.2) and (8.3.3) let $c,d \varepsilon S$ be such that if $s \varepsilon cS$ then $|\sum_t A(s,t) - 1| < \varepsilon$, and if $s \varepsilon dS$ then $\sum_{t\varepsilon S\sim aS} |A(s,t)| < \varepsilon$. Now $Af \varepsilon m(S)$ by (8.3.1) and if $s \varepsilon cS \cap dS \neq \emptyset$ (since S is ELA), then

$$|Af(s) - k| \leq |\sum_t A(s,t)(f(t) - k)| + |\sum_t A(s,t)k - k|$$

$$\leq (\sum_{t\varepsilon S\sim aS} + \sum_{t\varepsilon aS})|A(s,t)||f(t) - k| + |k| \; |\sum_t A(s,t) - 1|$$

$$< (\|f\| + |k|)\varepsilon + M\varepsilon + |k|\varepsilon$$

$$= (2|k| + \|f\| + M)\varepsilon \quad .$$

By [5, p. 72, theorem 8], Af is ℓac to k .

8.4. <u>REMARK</u>. If A is non-negative, i.e., $A(s,t) \geq 0$ for all $s,t \varepsilon S$, then (8.3.1), (8.3.2) and (8.3.3) are necessary also. For, $\sum_{t\varepsilon S\sim aS} |A(s,t)| = \sum_{t\varepsilon S\sim aS} A(s,t) = A1_{S\sim aS}(s)$. Since $1_{S\sim aS}$ is ℓac to 0 it follows that (8.3.3) holds.

8.5. <u>EXAMPLE</u>. Let S be the semigroup described in 5.2. Let A be defined for each m,n by

(i) $A(2m-1,2n-1) \geq A(2m-1,2n+1) > 0$ whenever $2n-1 \geq 2m-1$ and 0 otherwise.

(ii) $A(2m,n) = 1$ only if $n = 1$ and 0 otherwise.

(iii) $A(m,n) = 0$ whenever n is even.

(iv) $\sum_n A(m,n) = 1$ for each m .

By 8.3 A is almost strongly regular. However, if we replace the operation * by the ordinary addition, then A is not almost strongly regular since the sequence f = (1,0,1,0,...) is ℓac to 1/2 while Af is the sequence (1,1,1,...) , which is ℓac to 1 .

CHAPTER II

THE INVARIANT MEAN ON A TOPOLOGICAL SEMIGROUP

9. DEFINITIONS AND NOTATIONS.

For general topological terms we will follow Kelley [9]. We recall if $(S,\mathcal{U})$ is a uniform space, then for each $V_1, V_2, V \in \mathcal{U}$, $x \in S$, $A \subset S$,

(i) $V[x] = \{y \in S : (x,y) \in V\}$.

(ii) $V[A] = \bigcup_{x\in A} V[x]$.

(iii) $V_1 \circ V_2 = \{(x,y) \in S\times S : \text{for some } z \in S,\ (x,z) \in V_2 \text{ and } (z,y) \in V_1\}$.

(iv) $V^{-1} = \{(x,y) \in S\times S : (y,x) \in V\}$.

(v) $V^n = V \circ V \circ \ldots \circ V$ (n terms)

(vi) $(V_1 \circ V_2)[A] = V_1[V_2[A]]$.

For each $V \in \mathcal{U}$ we say V totally covers S if for some finite subset $\{a_1, \ldots, a_k\} \subset S$, $S \subset \bigcup_{i=1}^{k} V[a_i]$. We say S has property (B) if there is some $V \in \mathcal{U}$ such that for each n and each finite subset $\{a_1, \ldots, a_k\} \subset S$, $S \sim \bigcup_{i=1}^{k} V[a_i] \neq \emptyset$. Examples of uniform spaces having property (B) are non-compact locally compact group and any uniform space which has an unbounded real-valued uniformly continuous function on S [7, p. 118]. To see the later, let $V = \{(x,y) \in S\times S : |f(x) - f(y)| < 1\}$. Then $(x,y) \in V^2$ implies $y \in V^2[x] = V[V[x]]$ and hence $y \in V[s]$ for some $s \in V[x]$. It follows that $|f(x) - f(y)| \leq |f(x) - f(s)| + |f(s) - f(y)| < 2$. By induction, if $(x,y) \in V^n$ then $|f(x) - f(y)| < n$. Therefore, if S does not have property (B) then $S = \bigcup_{i=1}^{n} V^n[a_i]$ for some n and some finite subset $\{a_1, \ldots, a_n\} \subset S$. Hence for $s \in S$, $|f(s)| \leq n + \max_{1 \leq i \leq n} |f(a_i)|$, which cannot be.

Throughout this chapter we are mainly interested in a topological subsemigroup S of a Hausdorff topological group G. For such a semigroup S we shall adopt the following notations throughout: If $\mathcal{U}$ is the left uniformity for a topological group G then for each $V \in \mathcal{U}$, the set $V \cap S \times S$ will be denoted by V^*, so that $\mathcal{U}^* = \{V^* : V^* = V \cap S \times S , V \in \mathcal{U}\}$ is the relative left uniformity for the topological subsemigroup S of G. The space of bounded real-valued left uniformily continuous functions with respect to $\mathcal{U}^*$ will be denoted by LUC(S). It is well known that LUC(S) is a closed left translation invariant (i.e., $\ell_a f \in LUC(S)$ for each $f \in LUC(S)$, $a \in S$) subspace of $m(S)$ containing the constants. If $L \subset m(S)$ is a left invariant subspace let $\ell_s' : L \to L$ be the restriction of ℓ_s to L. We will use the symbol ℓ_s instead of ℓ_s' so that L_s will be the conjugate mapping $\ell_s'^*$ as well as ℓ_s^*. It will be clear from the context which mapping we have in mind. Let $J\ell(S) = \{\phi \in m(S)^* : L_s\phi = \phi$ for each $s \in S\}$ and $J_u\ell(S) = \{\phi \in LUC(S)^* : L_s\phi = \phi$ for each $s \in S\}$. Then $J\ell(S)$ or $J_u\ell(S)$ is infinite dimensional if considered as vector spaces, it is not finite dimensional.

10. TECHNICAL LEMMAS.

We will need the following lemmas. The crucial lemma is

10.4, and the proof follows more or less the proof of lemma III-1 in [7, p. 119].

10.1. LEMMA. If W is a neighbourhood of the identity e in G and $V = \{(x,y) \in G \times G : y \in Wx\}$ then for every $a \in S$, $V^*[a] = Wa \cap S$.

PROOF.
$$\begin{aligned} V^*[a] &= \{y \in S : (a,y) \in V\} \\ &= V[a] \cap S \\ &= Wa \cap S . \end{aligned}$$

10.2. LEMMA. If W is a symmetric neighbourhood of the identity e in G and $V = \{(x,y) \in G \times G : y \in Wx\}$ then $V^{*n} \subset V^{n*}$ for every n.

PROOF. Suppose $(p,q) \in V^{*2}$. Then for some $r \in S$, $(p,r) \in V \cap S \times S$ and $(r,q) \in V \cap S \times S$. Hence $(p,q) \in V^2 \cap S \times S = V^{2*}$. The lemma now follows by induction.

10.3. LEMMA. If W is a symmetric neighbourhood of the identity e in G, $V = \{(x,y) \in G \times G : y \in Wx\}$, and $a_2 \in S \sim V^{2*}[a_1]$ then $V^*[a_1] \cap V^*[a_2] = \emptyset$.

PROOF. Suppose $q \in V^*[a_1] \cap V^*[a_2]$. Then $q \in Wa_1 \cap Wa_2 \cap S$. Hence $q = w_1 a_1 = w_2 a_2$, which implies $a_2 = w_2^{-1} w_1 a_1 \in W^{-1} W a_1 = W^2 a_1$. Thus $a_2 \in W^2 a_1 \cap S = V^{2*}[a_1]$, which cannot be.

10.4. LEMMA. Let S be a topological subsemigroup of a topological group G, and $\{p_n\}_{n=1}^{\infty}$ be a countable dense subset of S in the relative topology. Let $V^* \in \mathcal{U}^*$ be symmetric and such that V^{*n} does not totally cover S for any positive integer n. Then there is an unbounded non-negative left uniformly continuous function F on S such that $F^{-1}[0,k] \subset \bigcup_{i=1}^{k+2} V^{*2(k+2)}[p_i]$.

PROOF. We choose a sequence of increasing subsets of S as follows: In general, if $A_1, \ldots, A_{n-1}$ has been chosen such that for $1 \leq j \leq n-1$,

(1) $V^*[p_1] \cup \ldots \cup V^*[p_j] \subset A_j$,

(2) $V^*[\overline{A}_{j-1}] \subset A_j$,

(3) $\overline{A}_j \subset V^{*2j}[p_1] \cup \ldots \cup V^{*2j}[p_j]$,

we let $A_n = V^*[\overline{A}_{n-1} \cup V^*[p_n]]$, where $A_0 = \emptyset$, $n = 1, 2, \ldots$. (Here $\overline{A}$ denotes the closure of A in S.) Then we have

(4) $V^*[p_1] \cup \ldots \cup V^*[p_n] \subset A_n$.

(5) $V^*[\overline{A}_{n-1}] \subset A_n$.

(6) $$\begin{aligned}\overline{A}_n &\subset V^*[A_n] \subset V^{*2}[\overline{A}_{n-1} \cup V^*[p_n]] \\ &\subset V^{*2n}[p_1] \cup \ldots \cup V^{*2n-1}[p_n] \\ &\subset V^{*2n}[p_1] \cup \ldots \cup V^{*2n}[p_n] .\end{aligned}$$

Continuing this way we obtain the sequence $\{A_n\}$ with the following properties:

(7) $\bigcup_{n=1}^{\infty} A_n = S$.

(8) $V^*[\overline{A}_n] \subset A_{n+1}$.

(9) $A_{n+1} \subset \bigcup_{i=1}^{n+1} V^{*2(n+1)}[p_i]$.

Conditions (8) and (9) are clear from (5) and (6), while (7) follows because $\{p_n\}_{n=1}^{\infty}$ is dense in S and so $S = \bigcup_{i=1}^{\infty} V^*[p_i] \subset \bigcup_{n=1}^{\infty} A_n \subset S$. We may even assume $A_n \sim A_{n-1} \neq \emptyset$, since otherwise, we could choose a subsequence with this property. That this is possible follows from (9) and the fact that V^{*n} does not totally cover S .

It is proved by A. Weil in [18, p. 13] that if (E,U) is a uniform space, $p \in E$, $V \in U$, V_n is a sequence of symmetric members in U such that $V_{n+1} \circ V_{n+1} \subset V_n \subset V$ then there is a uniformly continuous function $f : E \to [0,1]$ such that $f(p) = 0$, $f(E \sim V[p]) = 1$ and $|f(q) - f(r)| < 1/2^{n-1}$ whenever $(q,r) \in V_n$. Moreover, as noted by E. Granirer [6, p.121] if $P \subset S$ then the function f can even be chosen in such a way that $f(P) = 0$ and $f(E \sim V[P]) = 1$. Applying this to our situation, if V_n^* , $n = 0, 1, 2, \ldots$ is any sequence of symmetric member of U^* such that $V_0^* = V^*$ and $V_{n+1}^* \circ V_{n+1}^* \subset V_n^*$, then there is, for each $k = 1, 2, \ldots$, a left uniformly continuous function $f_k : S \to [0,1]$ such that

(10) $f_k(\overline{A}_k) = 0$.

(11) $f_k(S \sim V^*[\overline{A}_k]) = 1$ and hence $f_k(S \sim A_{k+1}) = 1$.

(12) $|f_k(p) - f_k(q)| < 1/2^{n-1}$ if $(p,q) \in V_n^*$.

Define now the sequence of functions $\{h_k\}$ by $h_k(s) = f_k(s) + k-1$ for $k = 1, 2, \ldots$. Then

$$h_k(s) = \begin{cases} k-1 & \text{if } s \in \overline{A}_k \\ k & \text{if } s \in S \sim A_{k+1} \end{cases},$$

and if $(p,q) \in V_n^*$ then for every k,

$$|h_k(p) - h_k(q)| = |f_k(p) - f_k(q)| < 1/2^{n-1} .$$

Define the required function F on S by

$$F(s) = \begin{cases} h_1(s) & \text{if } s \in A_2 \\ h_k(s) & \text{if } s \in A_{k+1} \sim A_k \text{ for } k \geq 2 . \end{cases}$$

Then F is unbounded since $A_k \sim A_{k-1} \neq \emptyset$. Suppose now $s \notin A_{k+2}$. Then $s \in A_{n+1} \sim A_n$ for $n > k+1$, and so $F(s) = h_n(s) \geq n-1 > k+1-1 = k$. Thus $F^{-1}[0,1] \subset A_{k+2} \subset \bigcup_{i=1}^{k+2} V^{*2(k+2)}[p_i]$.

We now prove F is left uniformly continuous. Let $\varepsilon > 0$ and choose n so large so that $1/2^{n-2} < \varepsilon$. Let $(p,q) \in V_n^*$. We have three cases to consider:

(i) If both p and q are in $A_{k+1} \sim A_k$ for some $k \geq 2$, then $|F(p) - F(q)| = |h_k(p) - h_k(q)| < 1/2^{n-1} < \varepsilon$.

(ii) If both p and q are in A_2 then $|F(p) - F(q)| = |h_1(p) - h_1(q)| < \varepsilon$.

(iii) Assume that i is the first index for which $p \in A_i$ and j is the first index for which $q \in A_j$. Assume also $i < j$. Since

$q \in V_n^*[p] \subset V^*[p] \subset V^*[\overline{A}_i] \subset V^*[\overline{A}_{i+1}] \subset A_{i+1}$, we have $j = i + 1$.

Also, we may assume $i \geq 2$. Then since $p \in A_i \sim A_{i+1}$ and $q \in A_{i+1} \sim A_i$

we have
$$\begin{aligned} |F(q) - F(p)| &= |h_i(q) - h_{i-1}(p)| \\ &= |h_i(q) - (i-1) + (i-1) - h_{i-1}(p)| \\ &= |h_i(q) - h_i(p) + h_{i-1}(q) - h_{i-1}(p)| \\ &\leq |h_i(q) - h_i(p)| + |h_{i-1}(q) - h_{i-1}(p)| \\ &< 1/2^{n-1} + 1/2^{n-1} = 1/2^{n-2} < \varepsilon \quad . \end{aligned}$$

This completes the proof of the lemma.

11. THE MAIN THEOREM.

The following theorem is due to E. Granirer [7, p. 124, theorem 1]:

11.1. <u>THEOREM</u>. Let G be a separable Hausdorff topological group which is amenable as a discrete group and satisfied property (B) . Then $J_u\ell(G)$ is infinite dimensional.

It is our aim to extend the above theorem to certain topological semigroups (theorem 11.3). Our method is more or less of that used in the proof of the above theorem. We will give an example to show that the above theorem cannot be extended to <u>any semigroup</u>. First, we need the following theorem of E. Granirer [7, p. 112, theorem 1]. For the sake of completeness, we will give its proof below.

11.2. THEOREM. Let S be a LA semigroup and $L \subset m(S)$ be a left invariant subspace containing the constants. If $J_L\ell(S) = \{\phi \in L^* : L_s\phi = \phi$ for each $s \in S\}$, assume that there is a sequence $\{p_n\}_{n=1}^{\infty} \subset S$ such that

$$\{\phi \in L^* : L_{p_n}\phi = \phi , n = 1, 2, \ldots\} = J_L\ell(S) .$$

If $J_L\ell(S)$ is n dimensional for some $n < \infty$ then each LIM $\phi \in L^*$ is a w*-sequential limit of finite means on L .

PROOF. Let $\phi_o \in L^*$ be a LIM and let $\psi \in m(S)^*$ be the norm preserving extension of ϕ_o . Then $1 = \|\phi_o\| = \phi_o(1) = \psi(1)$. This implies ψ is a mean on $m(S)$. Let ν be any LIM on $m(S)$. Let $\phi_o' = \nu \odot \psi$, where $\nu \odot \psi$ is the functional defined on $m(S)$ by $\nu \odot \psi(f) = \nu(h)$, where h is defined by $h(s) = \psi(\ell_s f)$, $f \in m(S)$, $s \in S$. Then ϕ_o' is a LIM on $m(S)$ which is an extension of ϕ_o [1, p. 526-527 and p. 529, corollary 2]. In fact, if $f \in L$, then $h(s) = \psi(\ell_s f) = \phi_o(\ell_s f) = \phi_o(f)$, i.e., h is a constant on S . Therefore $\nu \odot \psi(f) = \nu(h) = \nu(\phi_o(f)1) = \phi_o(f)\nu(1) = \phi_o(f)$, i.e., $\nu \odot \mu$ is an extension of ϕ_o .

Let now $\{\phi_\alpha'\}$ be a net of finite means in $m(S)^*$ such that $\lim_\alpha \phi_\alpha'(f) = \phi_o'(f)$ for every $f \in m(S)$ and $\lim_s \|L_s\phi_\alpha' - \phi_\alpha'\| = 0$ for each $s \in S$ [6, p. 44, (5.8)*]. If ϕ_α is the restriction of ϕ_α' to L then it can easily be checked that $\lim_\alpha \phi_\alpha(f) = \phi_o(f)$ for each $f \in L$ and $\lim_\alpha \|L_s\phi_\alpha - \phi_\alpha\| = 0$ for each $s \in S$ (where the norm now is that of L^*).

Let $S(\phi_o, 1/n) = \{\phi \in L^* : \|\phi - \phi_o\| < 1/n\}$. Since $J_L \ell(S)$ is finite dimensional, there is a decreasing sequence V_n of w*-closed convex neighbourhoods of ϕ_o such that

$$\phi_o \in V_n \cap J_L \ell(S) \subset S(\phi_o, 1/n) \cap J_L \ell(S) .$$

Now for each n there is α_n' such that if $\alpha \geq \alpha_n'$ then $\|L_{p_i} \phi_\alpha - \phi_\alpha\| < 1/n$ for $1 \leq i \leq n$. Since ϕ_o is a w*-limit of $\{\phi_\alpha\}$ there is $\alpha_n \geq \alpha_n'$ such that $\phi_{\alpha_n} \in V_n$. For convenience, write ϕ_n for ϕ_{α_n} . Since $\{\phi_n\}$ is a net contained in the w*-compact set of means, let ψ_o be any w*-cluster point of $\{\phi_n\}$. Let $f \in L$, $f \neq 0$, and p_j be fixed. If $\varepsilon > 0$ is given, there is n_o such that $1/n_o < \varepsilon/3\|f\|$ and for $n \geq n_o$, $\|L_{p_j} \phi_n - \phi_n\| < \varepsilon/3\|f\|$. Since ψ_o is a w*-cluster point of $\{\phi_n\}$ there is an $n_1 \geq n_o$ such that $|(\psi_o - \phi_{n_1})\ell_{p_j} f| < \varepsilon/3$ and $|(\phi_{n_1} - \psi_o)f| < \varepsilon/3$. Thus

$$\begin{aligned}
|(L_{p_j}\psi_o - \psi_o)f| &\leq |L_{p_j}(\psi_o - \phi_{n_1})f| + |(L_{p_j}\phi_{n_1} - \phi_{n_1})f| \\
&\quad + |(\phi_{n_1} - \psi_o)f| \\
&\leq |(\psi_o - \phi_{n_1})\ell_{p_j} f| + \|L_{p_j}\phi_{n_1} - \phi_{n_1}\| \, \|f\| \\
&\quad + |(\phi_{n_1} - \psi_o)f| \\
&< \varepsilon/3 + \varepsilon/3\|f\| \cdot \|f\| + \varepsilon/3 = \varepsilon \quad .
\end{aligned}$$

Therefore, for each p_j , $L_{p_j}\psi_o = \psi_o$ and hence by hypothesis, $\psi_o \in J_L\ell(S)$. Also, since ψ_o is a w*-cluster point of $\{\phi_n\}_{n=k}^{\infty} \subset V_k$, we have $\psi_o \in V_k \cap J_L\ell(S) \subset S(\phi_o, 1/k) \cap J_L\ell(S)$. This shows $\|\psi_o - \phi_o\| < 1/k$ for each k and thus $\psi_o = \phi_o$. Summarizing, we have shown that the sequence $\{\phi_n\}$ has a unique w*-cluster point ϕ_o .

We now show $\lim_n \phi_n(f) = \phi_o(f)$ for every $f \in L$. If not, there would be a $f_o \in L$ and a subsequence n_i such that $|(\phi_{n_i} - \phi_o)f_o| \geq \varepsilon$ for some $\varepsilon > 0$. But $\{\phi_{n_i}\}$, as net in the w*-compact set of means, has a w*-cluster point μ . But μ , being a w*-cluster point of ϕ_{n_i} , has to be a w*-cluster point of $\{\phi_n\}$. By uniqueness $\mu = \phi_o$, which cannot be, since $\mu \in \{\psi : |(\psi - \phi_o)f| \geq \varepsilon\}$. This completes the proof.

The following is the main thoerem in this chapter:

11.3. THEOREM. Let S be a discrete amenable semigroup which is a separable topological subsemigroup of a topological group G . Let W be a symmetric neighbourhood of the identity e in G and $V = \{(x,y) \in G \times G : y \in Wx\}$ such that for every n , V^{n*} does not totally cover S . Then $J_u\ell(S)$ is infinite dimensional.

PROOF. Let $\{p_n\}_{n=1}^{\infty}$ be the countable dense subset of S and let $\phi \in LUC(S)^*$ be such that $L_{p_n}\phi = \phi$ for every n . Let $s \in S$ be arbitrary, $f \in LUC(S)$. Then since

$$(L_s\phi - \phi)(f) = \phi(\ell_s f - f)$$
$$= \phi(\ell_s f - \ell_{p_n} f + \ell_{p_n} f - f)$$
$$= \phi(\ell_s f - \ell_{p_n} f) \quad ,$$

it follows that $|(L_s\phi - \phi)(f)| \le \|\phi\| \|\ell_s f - \ell_{p_n} f\| \to 0$ as $p_n \to s$. Hence $\{\phi \in LUC(S)^* : L_{p_n}\phi = \phi \ , n = 1, 2, \ldots\} = J_u\ell(S)$.

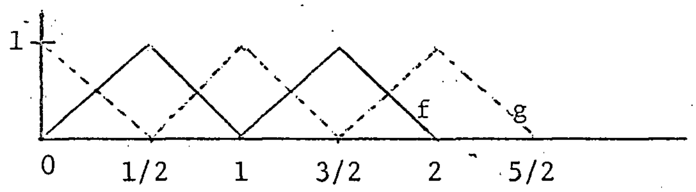
Suppose $J_u\ell(S)$ is n dimensional for some $n < \infty$. Let ϕ be a <u>fixed</u> two sided invariant mean on $m(S)$. Since the restriction of ϕ to $LUC(S)$ is a LIM on $LUC(S)$, by 11.2 there is a sequence $\{\phi_k\}$ of finite means on $LUC(S)$ such that $\phi(f) = \lim_k \phi_k(f)$ for every $f \in LUC(S)$.

Define now the following bounded sequence of uniformly continuous function on the real line: For $n = 1, 2, \ldots$

$$f_n(x) = \begin{cases} 1 - 2|x - (n - 1/2)| & \text{if } n-1 \le x \le n \\ 0 & \text{otherwise} \end{cases}$$

$$g_n(x) = \begin{cases} 1 - 2|x - n+1| & \text{if } n - \frac{3}{2} \le x \le n - \frac{1}{2} \\ 0 & \text{otherwise} \ . \end{cases}$$

Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then f is well defined since for $i \ne j$, f_i and f_j have disjoint carriers. Similarly, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ is well defined. The graphs of f and g are shown below.



And if $\{a_n\}$ is a bounded sequence of reals then $\sum_n a_n f_n$ and $\sum_n a_n g_n$ are bounded uniformly continuous functions on the real line.

Let W be the symmetric neighbourhood of the identity e in G and $V = \{(x,y) : y \in Wx\}$ such that for every n $V^{n}*$ does not totally cover S. By lemma 10.2, $V*^{n}$ does not totally cover S for every n. Let F be the unbounded left uniformly continuous function obtained in lemma 10.4. Then for each $s \in S$, $[(f + g)\circ F](s) = 1$ and so $\phi((f + g)\circ F) = 1$. Hence $\phi(f\circ F) > 0$ or $\phi(g\circ F) > 0$.

Without loss of generality assume $\phi(f\circ F) > 0$. For the sequence ϕ_k of finite means which converges $w*$ to the LIM ϕ in $LUC(S)*$ define the following linear functionals ϕ_k', ϕ' on the space m of bounded real sequences $\{a_n\}$ by

$$\phi_k'(\{a_n\}) = \phi_k(\sum_n a_n f_n \circ F)$$

$$\phi'(\{a_n\}) = \phi(\sum_n a_n f_n \circ F) \quad .$$

Let $\phi_k(f) = \sum_{i=1}^{m} \phi_k(t_i) f(t_i)$. Then

$$\begin{aligned}
\phi_k'(\{1\}) &= \phi_k(\sum_n f_n \circ F) \\
&= \sum_{i=1}^{m} \phi_k(t_i)(\sum_n f_n \circ F)(t_i) \\
&= \sum_n \sum_i \phi_k(t_i) f_n \circ F(t_i) \\
&= \sum_n \phi_k(f_n \circ F) \\
&= \sum_n \phi_k'(\{1_n\}) \quad ,
\end{aligned}$$

where $\{1_n\}$ is the sequence which is 1 at the n-th place and 0 everywhere else. This shows $\phi'_k \varepsilon Q[\ell_1]$, where ℓ_1 is the space of absolutely convergent sequences and Q is the natural mapping of ℓ_1 into its second conjugate ℓ_1^{**} . Since ℓ_1 is weakly sequentially complete [1, p. 33, corollary 3], $Q[\ell_1]$ is w*-sequentially complete in m* . Thus for any sequence $\{a_n\}$,

$$\phi'(\{a_n\}) = \phi(\sum_n a_n f_n \circ F)$$

$$= \lim_k \phi_k(\sum_n a_n f_n \circ F)$$

$$= \lim_k \phi'_k(\{a_n\}) \quad .$$

Thus $\phi' \varepsilon \ell_1$ and it follows that $\phi'(\{1\}) = \sum_n \phi'(\{1_n\}) = \sum_n \phi(f_n F)$. This means $\phi(f_n \circ F) > 0$ for at least one n since

$$0 < \phi(\sum_n f_n \circ F) = \phi'(\{1\}) = \sum_n \phi(f_n \circ F) \quad .$$

Since $\{s \varepsilon S : f_n \circ F(s) > 0\} \subset \{s \varepsilon S : F(s) \subset [0,n]\}$

$$\subset \bigcup_{i=1}^{n+2} V^{*2(n+2)}[p_i] \quad ,$$

it follows $\phi(V^{*2(n+2)}[p_j]) > 0$ for some p_j . By lemma 10.2 $\phi(V^{2(n+2)}*[p_j]) > 0$.

Now let $\pi : m(G) \to m(S)$ be defined by $\pi f(s) = f(s)$ for $f \varepsilon m(G)$, $s \varepsilon S$. Then $\pi^* : m(S)^* \to m(G)^*$ and if ψ is a mean on m(S) then $\pi^*\psi$ is a mean on m(G) . In particular, for our invariant mean ϕ , $\pi^*\phi$ is a mean on m(G) . We now show that $\pi^*\phi(r_s f) = \pi^*\phi(f)$

for all $s \in S$ and $f \in m(G)$. First $\pi r_s f = r_s \pi f$ since for $t \in S$, $\pi r_s f(t) = r_s f(t) = f(ts) = \pi f(ts) = r_s \pi f(t)$. Hence $\pi^*\phi(r_s f) = \phi(\pi r_s f) = \phi(r_s \pi f) = \phi(\pi f) = \pi^*\phi(f)$.

Let $U = V^{2(n+2)}$. Since U does not totally cover S, let $a_1 \in S \sim U^2*[p_j]$. By lemma 10.3, $U*[p_j] \cap U*[a_1] = \emptyset$. In general, if $U*[p_j], U*[a_1], \ldots, U*[a_{n-1}]$ has been chosen to be pairwise disjoint sets let

$$a_n \in S \sim (U^2*[p_j] \cup \ldots \cup U^2*[a_{n-1}]) .$$

By lemma 10.3, $U*[p_j], \ldots, U*[a_n]$ are pairwise disjoint. Thus for any n,

$$\begin{aligned} 1 = \phi(S) &\geq \phi(U*[p_j]) + \ldots + \phi(U*[a_n]) \\ &= \phi(W^{2(n+2)}p_j \cap S) + \ldots + \phi(W^{2(n+2)}a_n \cap S) \\ &= \phi(\pi W^{2(n+2)}p_j) + \ldots + \phi(\pi W^{2(n+2)}a_n) \\ &= \pi^*\phi(W^{2(n+2)}p_j) + \ldots + \pi^*\phi(W^{2(n+2)}a_n) \\ &= (n+1)\pi^*\phi(W^{2(n+2)}) \quad . \end{aligned}$$

This implies $\pi^*\phi(W^{2(n+2)}) = 0$, which cannot be since $\phi(V^{2(n+2)}*[p_j]) > 0$ implies $\pi^*\phi(W^{2(n+2)}) > 0$. Hence $J_u\ell(S)$ cannot be finite dimensional.

11.4. REMARK.

(a) It is known that if S is LA and L is a left invariant subspace of $m(S)$ then each LIM on L can be extended to a LIM on $m(S)$. Such an extension can be obtained by an extension theorem of

R. J. Silverman [16] or by a direct argument as given in the proof of theorem 11.2.

(b) If S is LA and L is a left invariant subspace of $m(S)$ let $J_L\ell(S) = \{\phi \in L^* : L_s\phi = \phi$ for each $s \in S\}$. It is shown in [7, p. 114, remark 2] that $J_L\ell(S)$ coincides with the linear subspace spanned by the left invariant means in $J_L\ell(S)$.

In the following corollary $C(S)$ denotes the space of bounded real-valued continuous function on S and $J_c\ell(S) = \{\phi \in C(S)^* : L_s\phi = \phi$ for each $s \in S\}$.

11.5. <u>COROLLARY</u>. If S is the semigroup in the theorem 11.3 then $J_c\ell(S)$ is infinite dimensional.

PROOF. Each LIM on $LUC(S)$ can be extended to a LIM on $C(S)$. For, by remark 11.4(a) each LIM ϕ on $LUC(S)$ can be extended to a LIM ϕ'' on $m(S)$ and the required extension is obtained by restricting ϕ'' to $C(S)$. Suppose $J_c\ell(S)$ has dimension n for some $n < \infty$. Let $\{\phi_1, \ldots, \phi_{n+1}\}$ be $n+1$ linearly independent set of LIM on $LUC(S)$, and let $\{\phi_1', \ldots, \phi_{n+1}'\}$ be the respective extensions to LIM on $C(S)$. Then $\sum_{i=1}^{n+1} \alpha_i\phi_i' = 0$ for some $\alpha_i \neq 0$. But for every $f \in LUC(S)$, $\sum_{i=1}^{n+1} \alpha_i\phi_i'(f) = \sum_{i=1}^{n+1} \alpha_i\phi(f) = 0$ implies $\alpha_i = 0$ for each i, which cannot be. Hence $J_c\ell(S)$ is infinite dimensional.

12. EXAMPLES.

12.1. Let $S = \{0,1,2,\ldots\}$ with ordinary multiplication and the discrete topology. This topology is generated by the metric d defined by $d(x,y) = 1$ iff $x \neq y$ and 0 otherwise. If $LUC(S)$ is the bounded real-valued uniformly continuous functions on S then $LUC(S) = m(S)$. By [7, p. 34, theorem 3.1] $J\ell(S)$ has dimension 1. Clearly, S is separable and has property (B). This example shows theorem 11.3 cannot be extended to every topological semigroup.

The following lemma is essentially known:

12.2. LEMMA. If S_1 is a LA subsemigroup of a topological group and S is a dense subsemigroup of S_1 then there is a positive linear isometry from $J_u(S)$ onto $J_u\ell(S_1)$.

PROOF. Let $\pi : LUC(S_1) \to LUC(S)$ be defined by $\pi f(s) = f(s)$ for $f \in LUC(S_1)$ and $s \in S$. If $f \in LUC(S)$ then f has a unique extension f' in $LUC(S_1)$ by [9, p. 195, theorem 26]. Moreover, this extension preserves norm since if $s \in S_1$ and $s_\alpha \to s$, $s_\alpha \in S$, then for every $\varepsilon > 0$ there is an α_0 such that $|f'(s) - f(s_\alpha)| < \varepsilon$ if $\alpha \geq \alpha_0$. Thus $|f'(s)| \leq |f'(s) - f(s_\alpha)| + |f(s_\alpha)| < \varepsilon + \|f\|$ and so $\|f'\| \leq \|f\|$. On the other hand, it is clear that $\|f\| \leq \|f'\|$. Hence π is a map which sends the unit ball in $LUC(S_1)$ onto the unit ball in $LUC(S)$. It follows from this that $\pi^* : LUC(S)^* \to LUC(S_1)^*$ is a linear isometry. That it is positive is easy to see.

Since S_1 is LA, $J_u\ell(S_1)$ is non-empty. If $\phi' \in J_u\ell(S_1)$ then define $\phi \in LUC(S)^*$ by $\phi(f) = \phi'(f')$, where f' is the unique extension of f to S_1. Then $\phi \in J_u\ell(S)$ since if $s \in S$ and $f \in LUC(S)$ then $\phi(\ell_s f) = \phi'((\ell_s f)') = \phi'(\ell_s f') = \phi'(f') = \phi(f)$. To see the second equality above, we note that if $s \in S$, $t \in S_1$ and $s_\alpha \in S$ is such that $s_\alpha \to t$ then $(\ell_s f)'(t) = (\ell_s f)'(\lim_\alpha s_\alpha)$ $= \lim_\alpha (\ell_s f)'(s_\alpha) = \lim_\alpha \ell_s f(s_\alpha) = \lim_\alpha f(ss_\alpha) = f(\lim_\alpha ss_\alpha) = f'(\lim_\alpha ss_\alpha)$ $= f'(st) = \ell_s f'(t)$. Thus $J_u\ell(S) \neq \emptyset$ and $\pi^*\phi = \phi'$.

Now let $f \in LUC(S_1)$, $s \in S_1$, $s_\alpha \in S$ such that $s_\alpha \to s$. Then

$$\begin{aligned}\|\pi\ell_s f - \ell_{s_\alpha}\pi f\| &= \sup_{t\in S} |\pi\ell_s f(t) - \ell_{s_\alpha}\pi f(t)| \\ &= \sup_{t\in S} |\ell_s f(t) - f(s_\alpha t)| \\ &\leq \sup_{t\in S_1} |\ell_s f(t) - \ell_{s_\alpha} f(t)| \\ &= \|\ell_s f - \ell_{s_\alpha} f\| \to 0 \text{ as } s_\alpha \to s .\end{aligned}$$

Hence if $\phi \in J_u\ell(S)$ then

$$\begin{aligned}\pi^*\phi(\ell_s f) &= \phi(\pi\ell_s f) = \phi(\lim_\alpha \ell_{s_\alpha}\pi f) \\ &= \lim_\alpha \phi(\ell_{s_\alpha}\pi f) = \lim_\alpha \phi(\pi f) \\ &= \pi^*\phi(f) .\end{aligned}$$

Consequently $\pi^*[J_u\ell(S)] = J_u\ell(S_1)$.

12.3. <u>THEOREM</u>. Let S be a separable subsemigroup of a locally compact group G such that S_1, the closure of S in G, is amenable and non-compact. Then $J_u\ell(S)$ is infinite dimensional.

PROOF. Let W be a compact symmetric neighbourhood of the identity e in G. If $V = \{(x,y) \in G \times G : y \in Wx\}$ then for every n, V^n* does not totally cover S_1, since otherwise, there is a positive integer n and a finite subset $\{a_1, \ldots, a_k\} \subset S_1$ such that $S_1 \subset \bigcup_{i=1}^{k} V^n*[a_i] = \bigcup_{i=1}^{k} (W^n a_i \cap S_1)$, which is a compact set. But this cannot be since S_1 is non-compact. By theorem 11.3, $J_u\ell(S_1)$ is infinite dimensional and hence by lemma 12.2, $J_u\ell(S)$ is infinite dimensional.

12.4. <u>EXAMPLES</u>. Using theorem 12.3 we can see $J_u\ell(S)$ is infinite dimensional if S is the following topological semigroups:

(i) $S = [0,\infty)$ with ordinary addition and the induced topology from the usual topology on the real line R.

(ii) $S = [1,\infty)$ with ordinary multiplication and the induced topology from R.

(iii) $S = (0,1]$ with ordinary multiplication and the induced topology from R.

(iv) S is any positive cone in a Euclidean vector space E with the usual addition of vectors and the induced topology from E. (S is a positive cone if $S + S \quad S$ and $\lambda S \quad S$ for any non-negative scalar λ.)

(v) $S = P \cup [1,\infty)$, where P is the set of negative irrationals, with ordinary multiplication and the induced topology from R .

(vi) Let S be the set of all real 3×3 diagonal matrices whose determinant is greater than or equal to 1. Then S with the usual multiplication of two matrices is a commutative (thus amenable) subsemigroup of the group $G_3(R)$, the full linear group. It is well known that $G_3(R)$ can be considered as a subset of R^9 , and $G_3(R)$ becomes a topological group with the induced topology from R^9 . The semigroup S is a separable closed non-compact subset in $G_3(R)$. That S is closed because the determinant function D is continuous and $S = D^{-1}([1,\infty))$. And since for each $A = (a_{ij}) \in S$, the norm of A is $\|A\| = (\sum_{i=1}^{3} |a_{ii}|^2)^{1/2}$, S is an unbounded set and hence cannot be a compact set.

(vii) Let A be a real $n\times n$ matrix of the form

$$\begin{pmatrix} \lambda I_p & 0 \\ B & \lambda I_q \end{pmatrix} ,$$

where λ is any scalar greater than or equal to 1 , I_p and I_q are identity matrices of fixed orders p and q respectively, $p + q = n$, and B is <u>any</u> $q\times p$ matrix. Let S be the set of all such matrices A . Then S is a commutative (thus amenable) separable closed non-compact subsemigroup of $G_n(R)$, the full linear group. That S is closed because the determinant function D is continuous and $S = D^{-1}([1,\infty))$. Also, since for each matrix $A = (a_{ij})$, its norm is $||A|| = (\sum_{i,j=1}^{n} |a_{ij}|^2)^{1/2}$ we see that S is unbounded and hence cannot be compact.

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