HOMOGENEITY OF COMBINATORIAL SPHERES

by

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B.Sc., The University of British Columbia, 1966

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

November 1968

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Date October 24th 1968

Abstract

The object of this thesis is to cover the results of [1] from a piecewise linear point of view. The principal result of [1] is the theorem on the homogeneity of spheres, i.e. the complement of a combinatorial n-cell in a combinatorial n-sphere is a combinatorial n-cell. A piecewise linear proof of this theorem by a "long induction" using regular neighbourhoods and collapsing was given in [4]. A direct piecewise linear proof appeared recently in [2]; it is based on the existence of a "collar" for the boundary of a combinatorial manifold with boundary. Our proof is similar to the proof in [2]. We proceed by induction on dimensions, proving simultaneously the existence of a collar for the boundary of a combinatorial manifold with boundary and the homogeneity theorem. From [2] we adopted an argument which eliminates a certain combinatorial technique applied in [1] and involving induction on the length of stellar subdivisions.

The results of [1] were previously interpreted in piecewise linear topology by use of a theorem in [3] stating that piecewise linearly homeomorphic simplicial complexes have subdivisions which are combinatorially equivalent in the sense of [1].

The thesis is divided into three parts. The first gives definitions and basic properties relating to simplicial

complexes. The second concerns combinatorial manifolds, and in the third we present our proof of the piecewise linear homogeneity of spheres.

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Acknowledgements

The thesis was written under the supervision of Dr. E. Luft whose constant help and encouragement was much appreciated.

I should also like to thank the National Research Council of Canada for giving me financial assistance.

1. Simplicial Complexes in $\mathbb{R}^n$

<u>Definition 1.1</u> Simplex in $\mathbb{R}^n$; Δ^k :

$\Delta^k = [v_o \ldots v_k]$ the barycentric hull of linearly independent points $v_i \in \mathbb{R}^n$ called vertices of Δ^k .

i.e. $\Delta^k = \{x : x \in \mathbb{R}^n, x = \sum_{i=0}^{k} \xi_i v_i \ , \ \xi_i \geq 0 \text{ and } \sum_{i=0}^{k} \xi_i = 1\}$

Dimension of Δ^k ; dim (Δ^k) :

dim (Δ^k) = k-1 = one less than the number of vertices of Δ^k .

Face of Δ^k ; $\Delta^m < \Delta^k$:

If $\{v_o' \ldots v_m'\} \subseteq \{v_o \ldots v_k\}$, then $\Delta^m = [v_o' \ldots v_m']$ is a face of Δ^k of appropriate dimension.

Boundary of Δ^k ; $\partial\Delta_k$:

$\partial\Delta_k = \{\Delta^m : \Delta^m < \Delta^k \ , \ \Delta^m \neq \Delta^k\}$

= the set of proper faces of Δ^k .

Interior of Δ^k , int (Δ^k) :

Int $(\Delta^k) = \{x : x \in \mathbb{R}^n \ , \ x = \sum_{i=0}^{k} \xi_i v_i \ , \ \xi_i > 0 \text{ and } \sum_{i=0}^{k} \xi_i = 1\}$

Barycentre of Δ^k , $b\Delta^k$:

$$b\Delta^k = \frac{1}{k+1} \{v_o + \ldots + v_k\} .$$

<u>Definition 1.2</u> Simplicial Complex in $\mathbb{R}^n$; K :

A countable set of simplices $\{\Delta_i\}_{i \in L}$, $\Delta_i \in \mathbb{R}^n \ni$

(i) $\Delta_1 \in K$ and $\Delta_2 < \Delta_1 \Rightarrow \Delta_2 \in K$.

(ii) $\Delta_1, \Delta_2 \in K \Rightarrow \Delta_1 \cap \Delta_2 < \Delta_1$ and $\Delta_1 \cap \Delta_2 < \Delta_2$.

Dimension of K ; dim (K) :

$\dim(K) = \sup\{\dim(\Delta) : \Delta \in K\}$.

Realization of K ; |K| :

$|K| = \{x : x \in \Delta , \Delta \in K\}$

= the point set spanned by all simplices of K

$\subseteq \mathbb{R}^n$

Note: (1) If Δ is a simplex, the simplicial complex consisting of all faces of Δ including Δ itself is a complex and will also be denoted by Δ . The boundary of a simplex is always a complex.

(2) $|\Delta| = \Delta$ by definition.

Subcomplex L of K ; $L \subseteq K$: $L \subseteq K$ is a subcomplex of K if it is also a simplicial complex.

Full Subcomplex L of K :

$L \subseteq K$ is a full subcomplex of K if:

(i) L is a subcomplex of K

(ii) For $\Delta = [v_o \ldots v_k] \in K$, if $\{v_o, \ldots, v_k\} \subseteq L$ then $\Delta \in L$

Principal Simplex of K :

$\Delta \in K$ is a principal simplex of K if Δ is not a face of a higher dimensional simplex of K .

Note: $\Delta \in K$ is principal $\Leftrightarrow$ $K-\{\Delta\}$ is a complex.

<u>Definition 1.3</u> Linear Map, $f : \Delta \to \mathbb{R}^n$:

$$\Delta = [v_o \ldots v_k]$$

f is linear if $f(x) = f(\sum_{i=0}^{k} \xi_i v_i) = \sum_{i=0}^{k} \xi_i f(v_i) \ \forall x \in \Delta$

Simplicial Map, $f : K \to L$:

A map $f : |K| \to |L|$, K, L simplicial complexes $\ni$

(1) $\forall \Delta \in K$, $f(\Delta)$ is a simplex of L

(2) $\forall \Delta \in K$, $f|_\Delta : \Delta \to |L| \subseteq \mathbb{R}^n$ is linear.

Note: A simplicial map $f : K \to L$ maps vertices of K to vertices of L .

Simplicial Isomorphism, $f : K \to L$:

$f : K \to L$ is a simplicial isomorphism if it is a simplicial map which is 1-1 and onto on vertices.

Note: If $\exists$ a simplicial isomorphism $f : K \to L$ we say that K and L are simplicially isomorphic and write $K \approx L$.

<u>Definition 1.4</u> Convex cell in $\mathbb{R}^n$; C :

$$C = \left\{ x : x \in \mathbb{R}^n \text{ , } x \text{ satisfies } \begin{array}{l} L_i(x) \geq b_i \text{ , } i \in I \\ l_j(x) = d_j \text{ , } j \in J \end{array} \right\}$$

where $L_i(x) = \sum_{k=0}^{n} a_{i_k} x_k \geq b_i$ is a linear inequality $\forall\, i \in I$

$l_j(x) = \sum_{k=0}^{n} c_{j_k} x_k = d_j$ is a linear equation $\forall j \in J$.

Note: (1) C is convex, i.e. $x,y \in C \Rightarrow$

$$\overline{xy} =_{df} \{(1-t)x + ty : 0 \leq t \leq 1\} \subseteq C .$$

(2) A simplex in $\mathbb{R}^n$ is a convex cell in $\mathbb{R}^n$.

<u>Definition 1.5</u> Operations on Complexes:

K a simplicial complex

L_1, L_2 subcomplexes of K .

Sum; $L_1 + L_2$:

$L_1+L_2 = \{\Delta : \Delta \in L_1 \text{ or } \Delta \in L_2\}$.

Intersection; $L_1 \cap L_2$:

$L_1 \cap L_2 = \{\Delta : \Delta \in L_1 \text{ and } \Delta \in L_2\}$.

Note: More generally, if L_1 and L_2 are simplicial complexes in $\mathfrak{R}^n \ni \{\Delta : \Delta \in L_1 \text{ or } \Delta \in L_2\}$ is a simplicial complex then L_1+L_2 and $L_1 \cap L_2$ are defined as above.

Join of Simplices, $\Delta^k * \Delta^l$:

$$\Delta^k = [v_o \ldots v_k] \ , \quad \Delta^l = [v'_o \ldots v'_l]$$

where $v_o, \ldots, v_k, v'_o, \ldots, v'_l$ are linearly independent then $\Delta^k * \Delta^l = [v_o \ldots v_k v'_o \ldots v'_l]$

$= $ a simplex of dimension $(k+l+1)$.

Note: A simplex may be considered as the join of any two of its faces provided these faces have between them as vertices all the vertices of the simplex.

K is the join of subcomplexes L_1 and L_2 ; $K = L_1 * L_2$:

$$\Delta \in K \iff \Delta = \Delta_1 * \Delta_2 \ , \ \Delta_1 \in L_1 \ , \ \Delta_2 \in L_2$$

Join of Complexes K and L ; $K*L$:

$$|K| \subseteq \mathbb{R}^{n_1} \quad , \quad |L| \subseteq \mathbb{R}^{n_2}$$

consider $|K|$, $|L| \subseteq \mathbb{R}^{n_1+n_2+1}$ where $|K|$ and $|L|$ are contained in disjoint planes of $\mathbb{R}^{n_1+n_2+1}$ which together span $\mathbb{R}^{n_1+n_2+1}$.

$$K*L = \{\Delta_1 * \Delta_2 : \Delta_1 \in K \ , \ \Delta_2 \in L\}$$

$$|K*L| \subseteq \mathbb{R}^{n_1+n_2+1}$$

note that this join is defined only up to simplicial isomorphism.

The following elementary properties hold:

(1) $L_1*(L_2+L_3) \approx L_1*L_2 + L_1*L_3$ where L_1, L_2, L_3 are subcomplexes of some complex K and the joins are defined.

(2) $\partial(\Delta_1*\Delta_2) = \partial\Delta_1*\Delta_2 + \Delta_1*\partial\Delta_2$.

(3) If $f : K_1 \to K_2$ and $g : L_1 \to L_2$ are simplicial maps

then $f*g : K_1*L_1 \to K_2*L_2$ is a well-defined simplicial map.

<u>Definition 1.6</u> K a simplicial complex, $\Delta \in K$

Star of Δ in K ; $St(\Delta,K)$:

$$St(\Delta,K) = \{\Delta' : \Delta' < \Delta'', \Delta'' \in K, \Delta < \Delta''\} .$$

Link of Δ in K ; $Lk(\Delta,K)$:

$$Lk(\Delta,K) = \{\Delta' : \Delta' \in St(\Delta,K), \Delta' \cap \Delta = \emptyset\} .$$

Residue of Δ in K ; $Res(\Delta,K)$:

$$Res(\Delta,K) = \{\Delta' : \Delta' \in K \text{ and } \Delta \not< \Delta'\} .$$

Note:

(1) $\Delta' \in Lk(\Delta,K) \iff \Delta*\Delta' \in K$.

(2) $St(\Delta,K) = \Delta*Lk(\Delta,K)$.

(3) $K = St(\Delta,K) + Res(\Delta,K)$.

Elementary Properties:

(1) $v \in K \Rightarrow Lk(v,K*L) = Lk(v,K)*L$.

(2) $Lk(\Delta_1*\Delta_2,K) = Lk(\Delta_1,Lk(\Delta_2,K))$.

$L \subseteq K$ a subcomplex of K

Neighbourhood of L in K ; $N(L,K)$:

$N(L,K) = \{\Delta : \Delta \in K, \Delta < \Delta', \Delta' \cap L \neq \emptyset\}$.

$\mathring{N}(L,K) = \{\Delta : \Delta \in N(L,K), \Delta \cap L = \emptyset\}$.

Note: L full in $K \Leftrightarrow \Delta \in N(L,K) \Leftrightarrow \Delta = \Delta_1 * \Delta_2, \Delta_1 \in L$
$\Delta_2 \in \mathring{N}(L,K)$.

Complement of L in K ; $C(L,K)$:

K a simplicial complex, $L \subseteq K$ a subcomplex

$C(L,K) = \{\Delta : \Delta \in K, \Delta < \Delta', \Delta' \in K, \Delta' \notin L\}$.

Note: (1) If K is a simplicial complex and $v \in K$ a vertex
$C(v,K) = \mathrm{Res}\ (v,K)$.

(2) $|C(L,K)| = CL(|K| - |L|)$
where $CL(X)$ = the topological closure of X.

<u>Definition 1.7</u> Subdivision of a simplicial complex K ; $\alpha(K)$:

$\alpha(K)$ is a subdivision of K if:

(i) $\alpha(K)$ is a simplicial complex,

(ii) $\Delta \in \alpha(K) \Rightarrow \exists\, \Delta' \in K \ni \Delta \subseteq \Delta'$.

(iii) $|\alpha(K)| = |K|$.

Elementary Subdivision $\mathcal{E}_{(x,\Delta)}(K)$:

$\Delta \in K$, $x \in \text{int}\ \Delta$

$\mathcal{E}_{(x,\Delta)}(K) = x * \partial\Delta * \text{Lk}(\Delta,K) + \text{Res}\ (\Delta,K)$

Stellar Subdivision; $\sigma(K)$:

A composition of elementary subdivisions

Barycentric Subdivision of K relative to a subcomplex L of K ; $B_L(K)$:

$$B_L(K) = \left\{\Delta' : \Delta' = \Delta_L * b\Delta_1 * \ldots * b\Delta_k,\ \Delta_L < \Delta_1 < \ldots < \Delta_k \in K,\ \Delta_L \in L,\ \Delta_1 \notin L \right\}$$

Note: (1) If $\alpha(K)$ is a subdivision of K and L is a subcomplex of K then α defines a subdivision $\alpha(L)$ of L .

(2) If $\alpha(K)$ is a subdivision of K and $\gamma(L)$ is a subdivision of L , then $\alpha(K) * \gamma(L)$ defines a subdivision of $K * L$.

(3) If L , $K_\circ$ are subcomplexes of K , $L \subseteq K_\circ \subseteq K$, then $B_L(K)$ defines the barycentric subdivision of $K_\circ$ relative to L on $K_\circ$.

<u>Lemma 1.1</u> Δ a simplex

C a convex cell $\ni C \cap \Delta \subseteq \partial\Delta$

Then $C \cap \Delta \subseteq \Delta_1$, $\Delta_1 \in \partial\Delta$

<u>Proof</u>: Choose $\Delta_1 \in \partial\Delta \ni C \cap \text{int}\, \Delta_1 \neq \emptyset$ and Δ_1 is of maximum dimension in $\partial\Delta$ with this property.

Suppose that $C \cap \Delta \not\subseteq \Delta_1$:

i.e. $\exists\, x \in C \cap \Delta \ni x \notin \Delta_1$

now $x \in \text{int}\, \Delta_2$, $\Delta_2 \in \partial\Delta$

consider the smallest simplex

$\Delta_3 \ni \Delta_1 < \Delta_3$ and $\Delta_2 < \Delta_3$

choose any $y \in C \cap \text{int}\, \Delta_1$

$\overline{xy} \in C$ --- C is convex

$\overline{xy} \in \Delta_3$ --- Δ_3 is convex

and minimality of $\Delta_3 \Rightarrow \overline{xy} \cap \text{int}\, \Delta_3 \neq \emptyset$

$\Rightarrow C \cap \text{int}\, \Delta_3 \neq \emptyset$

contradicting the minimality of Δ_1

$\therefore C \cap \Delta \subseteq \Delta_1$

<u>Theorem 1.1</u> K, L finite simplicial complexes

$|L| \subseteq |K|$. Then $\exists$ a stellar subdivision $\sigma(K)$ and a subdivision $\alpha(L) \ni \alpha(L) \subseteq \sigma(K)$.

<u>Proof</u>: By induction on N, the number of simplices of L.

$N = 1$: L is a vertex, say v.

$|L| \subseteq |K| \Rightarrow v \in \text{int}\,\Delta$, $\Delta \in K$

$\therefore \mathcal{E}_{(v,\Delta)}(K)$ is the required subdivision of K. Assume the hypothesis to be true if L has $\leq (N-1)$ simplices

N : Consider Δ, a principal simplex of L. $L-\{\Delta\}$ is a simplicial complex with $<N$ simplices and $|L-\{\Delta\}| \subseteq |K|$. Thus by the induction assumption $\exists$ a stellar subdivision $\sigma_1(K)$ and a subdivision $\alpha(L-\{\Delta\}) \ni \alpha(L-\{\Delta\}) \subseteq \sigma_1(K)$

We order the simplices $\tilde{\Delta} \in \sigma_1(K)$ with $\text{int}\,\tilde{\Delta} \cap \text{int}\,\Delta \neq \emptyset$ by decreasing dimensions:

$$\tilde{\Delta}_1, \tilde{\Delta}_2 \ldots \tilde{\Delta}_m, \quad \dim \tilde{\Delta}_1 \geq \dim \tilde{\Delta}_2 \geq \ldots \geq \dim \tilde{\Delta}_m$$

Choose $x_i \in \text{int}\,\tilde{\Delta}_i \cap \text{int}\,\Delta$, $i = 1,2,\ldots m$

set $\sigma_2(K) = \mathcal{E}_{(x_1,\tilde{\Delta}_1)}(\sigma_1(K))$

$\vdots$

$\sigma_i(K) = \mathcal{E}_{(x_{i-1},\tilde{\Delta}_{i-1})}(\sigma_{i-1}(K))$

and define $\sigma(K) = \sigma_{m+1}(K)$, the resultant of these elementary subdivisions of $\sigma_1(K)$.

set $\alpha(L) = \{\Delta' : \Delta' \in \sigma(K),\ \text{int}\ \Delta' \cap |L| \neq \emptyset\}$

Clearly $\sigma(K)$ is a stellar subdivision of K . $\alpha(L) \subseteq \sigma(K)$ by its definition.

$|\alpha(L)| = |L|$: (i) $|L| \subseteq |\alpha(L)|$:

say $x \in |L| \Rightarrow x \in \text{int}\ \tilde{\Delta},\ \tilde{\Delta} \in L$

if $\tilde{\Delta} \neq \Delta \Rightarrow x \in \text{int}\ \tilde{\tilde{\Delta}},\ \tilde{\tilde{\Delta}} \in \alpha(L-\{\Delta\})$

$\Rightarrow \tilde{\tilde{\Delta}} \in \sigma(K) - \tilde{\Delta}$ unaffected by

$\mathcal{E}(x_i, \Delta_i)(\sigma_i(K))$, $i=1 \ldots m$

$\Rightarrow x \in |\alpha(L)| - \text{int}\ \tilde{\tilde{\Delta}} \cap |L| \neq \emptyset$.

if $\tilde{\Delta} = \Delta$, now

$x \in |\sigma(K)| \Rightarrow \exists\ \tilde{\tilde{\Delta}} \in \sigma(K) \ni x \in \text{int}\ \tilde{\tilde{\Delta}}$

$\Rightarrow x \in \text{int}\ \tilde{\tilde{\Delta}} \cap \text{int}\ \Delta \Rightarrow$

$\text{int}\ \tilde{\tilde{\Delta}} \cap |L| \neq \emptyset$

$\Rightarrow \tilde{\tilde{\Delta}} \in \alpha(L)$

$\Rightarrow x \in |\alpha(L)|$.

(ii) $|\alpha(L)| \subseteq |L|$:

say $x \in |\alpha(L)| \Rightarrow x \in \text{int}\ \Delta', \Delta' \in \alpha(L)$

$\Delta' \in \alpha(L) \Rightarrow \Delta' \in \sigma(K)$ and

$\text{int}\ \Delta' \cap |L| \neq \emptyset$.

(a) if $\text{int}\,\Delta' \cap \text{int}\,\Delta = \emptyset$

$\Rightarrow \text{int}\,\Delta' \cap |L-\{\Delta\}| \neq \emptyset$ — $\text{int}\,\Delta' \cap |L| \neq \emptyset$

now $\Delta' \in \sigma(K) \Rightarrow \exists$ a smallest

$\Delta_1 \in \sigma_1(K) \ni |\Delta'| \subseteq |\Delta_1|$

$\Rightarrow \text{int}\,\Delta_1 \cap |L-\{\Delta\}| \neq \emptyset$

$\Rightarrow \Delta_1 \in \alpha(L-\{\Delta\})$ —

$\alpha(L-\{\Delta\}) \subseteq \sigma_1(K)$

$\Rightarrow \Delta_1 \subseteq |L|$

$\Rightarrow \Delta' \subseteq |L|$

$\Rightarrow x \in |L|$

(b) if $\text{int}\,\Delta' \cap \text{int}\,\Delta \neq \emptyset$

as $\sigma(K)$ is a subdivision of

$\sigma_1(K)$ $\exists$ $\Delta_1 \in \sigma_1(K) \ni \Delta' \subseteq \Delta_1$ and

$\text{int}\,\Delta_1 \cap \text{int}\,\Delta \neq \emptyset$

thus $\Delta_1 = \tilde{\Delta}_{i_1}$ and

$\sigma_{i_1+1}(K) = \mathcal{E}_{(x_{i_1}, \tilde{\Delta}_{i_1})}(\sigma_{i_1}(K))$

i.e. $\Delta' \subseteq x_{i_1} * \Delta_2$, $\Delta_2 \in \partial\tilde{\Delta}_{i_1}$

if $\Delta_2 \subseteq \Delta$ then

$$x_{i_1} * \Delta_2 \subseteq \Delta \Rightarrow \Delta' \subseteq \Delta \Rightarrow x \in \Delta \Rightarrow x \in |L|$$

if $\Delta_2 \not\subseteq \Delta$:

(1) $\Delta_2 \cap \text{int}\, \Delta = \emptyset \Rightarrow x_{i_1} * \Delta_2 \cap \text{int}\, \Delta = x_{i_1}$

$\therefore$ $\text{int}\, \Delta' \cap \text{int}\, \Delta \neq \emptyset \Rightarrow \Delta' = x_{i_1} \subseteq \Delta$

$\Rightarrow x = \Delta' = x_{i_1} \in |L|$

(2) $\text{int}\, \Delta_2 \cap \text{int}\, \Delta \neq \emptyset \Rightarrow \Delta_2 = \tilde{\Delta}_{i_2}$ and

$\sigma_{i_2+1}(K) = \varepsilon_{(x_{i_2}, \tilde{\Delta}_{i_2})}(\sigma_{i_2}(K))$

i.e. $\Delta' \subseteq x_{i_1} * x_{i_2} * \Delta_3$, $\Delta_3 \in \partial\tilde{\Delta}_{i_2}$

if $\Delta_3 \subseteq \Delta$ then $x \in |L|$ as above

if $\Delta_3 \not\subseteq \Delta$ continue as in case $\Delta_2 \not\subseteq \Delta$

(3) $\Delta_2 \cap \text{int}\, \Delta \subseteq \partial\Delta_2 \Rightarrow \Delta_2 \cap \text{int}\, \Delta \subseteq \Delta_3$, $\Delta_3 \in \partial\Delta_2$ by lemma 1.2

$\Rightarrow x_{i_1} * \Delta_2 \cap \text{int}\, \Delta \subseteq x_{i_1} * \Delta_3$

$\Rightarrow \text{int}\, \Delta' \cap x_{i_1} * \Delta_3 \neq \emptyset$

$\Rightarrow \Delta' \subseteq x_{i_1} * \Delta_3$

again, if $\Delta_3 \subseteq \Delta$ then $x \in L$

if $\Delta_3 \not\subseteq \Delta$ continue as in case $\Delta_2 \not\subseteq \Delta$

Continuing this inspection of the subdivision of Δ_1 in forming $\sigma(K)$ from $\sigma_1(K)$ we will eventually arrive at

$$\Delta' \subseteq x_{i_1} * x_{i_2} * \ldots * x_{i_p} * \Delta_s \ , \quad \Delta_s \subseteq \Delta$$

$$\text{or} \quad \Delta' \subseteq x_{i_1} * x_{i_2} * \ldots * x_{i_s} * \Delta_s \ , \quad \Delta_s \cap \operatorname{int} \Delta = \emptyset$$

which, in either case $\Rightarrow \Delta' \subseteq \Delta \Rightarrow x \in \Delta$

$\Rightarrow x \in |L|$

<u>Corollary 1.1</u> K a finite Simplicial Complex

$\alpha_1(K)$, $\alpha_2(K)$ subdivisions of K.

Then $\exists$ a common subdivision $\alpha(K)$ of $\alpha_1(K)$ and $\alpha_2(K)$.

<u>Proof</u>: Set $\alpha_1(K) = K$ and $\alpha_2(K) = L$ in Thm. 1.1.

<u>Corollary 1.2</u> K a finite Simplicial Complex

$\alpha(K)$ a subdivision of K.

Then $\exists$ a stellar subdivision $\sigma(K)$ of K which is a subdivision of $\alpha(K)$.

<u>Proof</u>: Set $K = K$, $\alpha(K) = L$ in Thm. 1.1.

<u>Definition 1.8</u> Piecewise linear Homeomorphism:

K_1, K_2 simplicial complexes

A piecewise linear homeomorphism, denoted PL homeomorphism, $f : K_1 \to K_2$ is a map $f : |K_1| \to |K_2| \ni \exists$ subdivisions $\alpha_1(K_1)$, $\alpha_2(K_2)$ of K_1 and K_2 for which $f : \alpha_1(K_1) \to \alpha_2(K_2)$ is a simplicial isomorphism. K_1 and K_2 are said to be PL homeomorphic.

Note: (1) If $f : K_1 \to K_2$ and $g : K_2 \to K_3$ are PL homeomorphisms, then $g \circ f : K_1 \to K_3$ is a PL homeomorphism by Corol. 1.1.

(2) If $f : K_1 \to K_2$ and $g : L_1 \to L_2$ are PL homeomorphisms, then $g * f : K_1 * L_1 \to K_2 * L_2$ is well defined and a P-L homeomorphism.

<u>Definition 1.9</u> The Cartesian Product of a Complex with the Unit interval:

Let $I = [0,1]$ and K be a simplicial complex. If $|K| \subseteq \mathbb{R}^n$ then $K \times I$ $|K \times I| \subseteq \mathbb{R}^n \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ is given by:

$K \times I = \{\Delta : \Delta = (\Delta_0, 0) * (b\Delta_1, 1) * \ldots * (b\Delta_k, 1),\ \Delta_0 < \Delta_1 < \ldots < \Delta_k \in K\}$.

Set $K \times 0 = \{\Delta : \Delta = (\Delta_0, 0),\ \Delta_0 \in K\} \subseteq K \times I$.

$K \times 1 = \{\Delta : = (b\Delta_1, 1) * \ldots * (b\Delta_k, 1),\ \Delta_1 < \ldots < \Delta_k \in K\} \subseteq K \times I$.

Clearly $K \times 0 \approx K$ and $K \times 1 \approx B(K)$.

Note: $|K \times I| = |K| \times I$.

<u>Theorem 1.2</u> K a simplicial complex

$\alpha_1(K)$, $\alpha_2(K)$ subdivisions of K

$v \in \alpha_1(K)$ and $v \in \alpha_2(K)$

Then $Lk(v, \alpha_1(K))$ is PL homeomorphic to $Lk(v, \alpha_2(K))$.

<u>Proof</u>: Since by Corol. 1.1 we may take a common subdivision $\alpha(K)$ of $\alpha_1(K)$ and $\alpha_2(K)$ it suffices to prove the theorem for $K = \alpha_1(K)$ and $\alpha(K) = \alpha_2(K)$.

We produce a subdivision $\gamma(Lk(v,K)) \ni \gamma(Lk(v,K)) \approx Lk(v, \alpha(K))$ by "radial projection" from v.

Define $r : |Lk(v, \alpha(K))| \to |Lk(v,K)|$ on vertices

of $Lk(v,\alpha(K))$ by, $v_1 \in Lk(v,\alpha(K))$.

Choose a simplex Δ^k of $St\ (v,K) \ni v_1 \in \Delta^k$

$$\Delta^k = v*\Delta^{k-1}\ ,\ \Delta^{k-1} \in Lk(v,K)$$

Set $r(v_1) = w_1 \in \Delta^{k-1}$ such that $\overline{vv_1} \subseteq \overline{vw_1}$

extend r to $|Lk(v,\alpha(K))|$ by defining, for $\Delta \in Lk(v,\alpha(K))$

$$\Delta = [v_o \ldots v_k]\ ,\ \Delta \subseteq \text{some } \Delta^k\ ,\ \Delta^k \in St(v,K)$$

$$r(\Delta) = [r(v_o) \ldots r(v_k)]\ .$$

$v_o \ldots v_k$ linearly independent $\Rightarrow r(v_o) \ldots r(v_k)$ linearly independent

$\Rightarrow r(\Delta)$ is a simplex

and $r(\Delta) \subseteq |Lk(v,K)|$.

Set $\gamma(Lk(v,K)) = \{r(\Delta) : \Delta \in Lk(v,\alpha(K))\}$.

$\gamma(Lk(v,K))$ is a subdivision of $Lk(v,K)$:

(a) $\{r(\Delta) : \Delta \in Lk(v,\alpha(K))\}$ is a simp. comp. since

$$\Delta_1,\Delta_2 \in Lk(v,\alpha(K)) \Rightarrow \Delta_1 \cap \Delta_2 \in Lk(v,\alpha(K))$$

thus $r(\Delta_1) \cap r(\Delta_2) = r(\Delta_1 \cap \Delta_2) \in \{r(\Delta) : \Delta \in Lk(v,\alpha(K))\}$

(b) $|Lk(v,K)| = |\{r(\Delta) : \Delta \in Lk(v,\alpha(K))\}|$:

by defn $|\{r(\Delta) : \Delta \in Lk(v,\alpha(K))\}| \subseteq |Lk(v,K)|$

say $x \in |Lk(v,K)| \Rightarrow x \in |\Delta^{k-1}|$, $\Delta^k = v*\Delta^{k-1}$,

$\Delta^k \in St\ (v,K)$

thus $x_1 = \overline{v \cdot x} \cap |Lk(v,\alpha(K))|$ has $r(x_1) = x$

$\Rightarrow x \in |r(\Delta_1)|$, $x_1 \in \text{int}\ \Delta$, $\Delta_1 \in Lk(v,\alpha(K))$

By construction $Lk(v,\alpha(K)) \approx \gamma(Lk(v,K))$ through the radial projection map r .

<u>Corollary 1.3</u> K a simplicial complex

$\alpha(K)$ a subdivision of K

$v \in K \ni Lk(v,\alpha(K)) \cap \alpha(Lk(v,K)) = \emptyset$

Then $C(St\ (v,\alpha(K))),\alpha(St(v,K))$ is PL homeomorphic to $Lk(v,K) \times I$.

2. Combinatorial Manifolds

<u>Definition 2.1</u> Combinatorial Cell and Sphere:

$\mathcal{L}^n$ is a combinatorial cell of dimension n if $\mathcal{L}^n$ is PL homeomorphic to Δ^n.

γ^n is a combinatorial sphere of dimension n if γ^n is PL homeomorphic to $\partial\Delta^{n+1}$.

<u>Definition 2.2</u> Combinatorial Manifold of dimension n:

$\mathcal{M}^n$ is a combinatorial manifold of dimension n if $\forall$ vertex v of $\mathcal{M}^n$, $Lk(v, \mathcal{M}^n)$ is a combinatorial cell or sphere of dimension $(n-1)$.

<u>Theorem 2.1</u> $\mathcal{L}^n$ a comb. cell of dim n, $\mathcal{L}^m$ a comb. cell of dim m, γ^n a comb. sphere of dim n, γ^m a comb. sphere of dim m. Then,

(a) $\mathcal{L}^n * \mathcal{L}^m$ is a comb. cell of dim $(n+m+1)$

(b) $\mathcal{L}^n * \gamma^m$ is a comb. cell of dim $(n+m+1)$

(c) $\gamma^n * \gamma^m$ is a comb. sphere of dim $(n+m+1)$

<u>Proof</u>: (a) We have PL homeomorphisms $f : \mathcal{L}^n \to \Delta^n$

$g : \mathcal{L}^m \to \Delta^m$

$\therefore$ $\exists$ PL homeomorphism

$f*g : \mathcal{L}^n * \mathcal{L}^m \to \Delta^n * \Delta^m = \Delta^{n+m+1}$

(b) We have PL homeomorphisms $f : \mathcal{L}^n \to \Delta^n$

$g : \gamma^m \to \partial\Delta^{m+1}$

thus $f*g : \mathcal{L}^n * \gamma^m \to \Delta^n * \partial\Delta^{m+1}$ is a PL homeomorphism. Now $\Delta^o * \partial\Delta^{m+1} \approx \varepsilon_{(x,\Delta^{m+1})}(\Delta^{m+1})$, let $h : \Delta^o * \partial\Delta^{m+1} \to \Delta^{m+1}$ be the corresponding PL homeomorphism. Then $id_{\Delta^{n-1}} * h : \Delta^{n-1} * (\Delta^o * \partial\Delta^{m+1}) = \Delta^n * \partial\Delta^{m+1} \to \Delta^{n-1} * \Delta^{m+1} = \Delta^{n+m+1}$ is a PL homeomorphism

$\therefore (id_{\Delta^{n-1}} * h) \circ (f*g) : \mathcal{L}^n * \gamma^m \to \Delta^{n+m+1}$ is a PL homeomorphism.

(c) Note that $\partial\Delta^1 * \partial\Delta^k \approx \varepsilon_{(x,\Delta^{k+1})}(\partial\Delta^{k+2})$.

By induction on n :

$n = 0$: We have PL homeomorphisms $f : \gamma^o \to \partial\Delta^1$ and $g : \gamma^m \to \partial\Delta^{m+1}$

$\therefore$ $f*g : \gamma^o * \gamma^m \to \partial\Delta^1 * \partial\Delta^{m+1}$ is a PL homeomorphism. As noted $\exists$ a PL homeomorphim $h : \partial\Delta^1 * \partial\Delta^{m+1} \to \partial\Delta^{m+2}$

$\therefore$ $h \circ (f*g) : \gamma^o * \gamma^m \to \partial\Delta^{m+2}$ is a PL homeomorphism.

Assume the hypothesis in dimensions $\leq n$

$n+1$: We have PL homeomorphisms

$f : \gamma^{n+1} \to \partial\Delta^{n+2}$ and $g : \gamma^m \to \partial\Delta^{m+1}$.

Consider the PL homeomorphism

$f*g : \gamma^{n+1}*\gamma^m \to \partial\Delta^{n+2}*\partial\Delta^{m+1}$

now $h*1_{\partial\Delta^{m+1}} : \partial\Delta^{n+2}*\partial\Delta^{m+1} \to (\partial\Delta^1*\partial\Delta^{n+1})*\partial\Delta^{m+1}$

$= \partial\Delta^1*(\partial\Delta^{n+1}*\partial\Delta^{m+1})$ is a PL homeomorphism.

By the induction hypothesis, $\partial\Delta^{n+1}*\partial\Delta^{m+1}$ is PL homeomorphic to $\partial\Delta^{n+m+1}$, thus $\partial\Delta^1*(\partial\Delta^{n+1}*\partial\Delta^{m+1})$ is PL homeomorphic to $\partial\Delta^{n+m+2}$ by again applying the above note.

<u>Theorem 2.2</u> $\mathcal{M}^n$ a comb. man. of dim n.

$\Delta^k \in \mathcal{M}^n$ a k-simplex

K a simp. comp., $\alpha(K)$ a subdivision of K

$\mathcal{E}^n$ a comb. n-cell, γ^n a comb. n-sphere.

Then

(i) $Lk(\Delta^k, \mathcal{M}^n)$ is a comb. (n-k-1) cell or sphere.

(ii) K is a comb. man. of dim n $\Leftrightarrow$ $\alpha(K)$ is a comb. man. of dim n.

(iii) $\mathcal{E}^n$ and γ^n are comb. mans. of dim n.

<u>Proof</u>: By induction on n :

n = 0 trivial

assume the theorem is true for dimensions $\leq (n-1)$.

n : (i) By induction on $k = \dim(\Delta^k)$

$k=0$: $\Delta^k = v \in \mathcal{M}^n$

$\therefore$ $Lk(\Delta^k, \mathcal{M}^n)$ is a comb $(n-1)$ cell or sphere since $\mathcal{M}^n$ is a comb. man. of dim n .

Assume (i) is true for $\dim(\Delta) < k$, $\Delta \in \mathcal{M}^n$.

k : $\Delta^k = v*\Delta^{k-1}$, $\dim \Delta^{k-1} = k-1$

$$\therefore \quad Lk(\Delta^k, \mathcal{M}^n) = Lk(v*\Delta^{k-1}, \mathcal{M}^n)$$
$$= Lk(v, Lk(\Delta^{k-1}, \mathcal{M}^n))$$

but $Lk(\Delta^{k-1}, \mathcal{M}^n)$ is a comb. $(n-k)$ cell or sphere by the induction hypothesis on k $\Rightarrow$ $Lk(\Delta^{k-1}, \mathcal{M}^n)$ is a comb. man. of dim $(n-k)$ by the induction hypothesis on n in dimension $(n-k) < n$

$\Rightarrow$ $Lk(\Delta^k, \mathcal{M}^n) = Lk(v, Lk(\Delta^{k-1}, \mathcal{M}^n))$ is a comb. $(n-k-1)$ cell or sphere.

(ii) $\Leftarrow$ Say $\alpha(K)$ is a comb. man. of dim n . Choose any $v \in K$. Since $v \in \alpha(K)$, $Lk(v, \alpha(K))$ is a comb. $(n-1)$ cell or sphere and $Lk(v, \alpha(K))$ is PL homeomorphic to $Lk(v,K)$ — Thm. 1.2

$\therefore$ $Lk(v,K)$ is a comb. $(n-1)$ cell or sphere.

=> Say K is a comb. man. of dim n . Choose any $v \in \alpha(K)$. Now $v \in \text{int}\,\Delta$, $\Delta \in K$. Consider $\mathcal{E}_{(v,\Delta)}(K) = v*\partial\Delta*Lk(\Delta,K) + \text{Res}\,(\Delta,K)$. $Lk(v,\alpha(K))$ is PL homeomorphic to $Lk(v,\mathcal{E}_{(v,\Delta)}(K))$ - Thm. 1.2 and $Lk(v,\mathcal{E}_{(v,\Delta)}(K)) = \partial\Delta*Lk(\Delta,K)$. But $\partial\Delta$ is a comb. sphere of dimension $(\dim\Delta-1)$ and $Lk(\Delta,K)$ is a comb. cell or sphere of dimension $(n-\dim\Delta-1)$ by (i) in dimension n since K is a comb. man. of dim n .

$\therefore$ $Lk(v,\mathcal{E}_{(v,\Delta)}(K))$ is a comb. cell or sphere of dim (n-1) — Thm. 2.1

=> $Lk(v,\alpha(K))$ is a comb (n-1) cell or sphere.

(iii) $\mathcal{E}^n \approx \alpha_1(\Delta^n)$ and Δ^n is a comb. man. of dim n — $Lk(v,\Delta^n) = \Delta^{n-1}$

$\therefore$ $\mathcal{E}^n$ is a comb. man. of dim n — (ii) in dim n .

$\gamma^n \approx \alpha_2(\partial\Delta^{n+1})$ and $\partial\Delta^{n+1}$ is a comb. man. of dim n — $Lk(v,\partial\Delta^{n+1}) = \partial\Delta^n$

$\therefore$ γ^n is a comb. man. of dim n — (ii) in dim n .

<u>Corollary 2.1</u> $\mathcal{M}^n$ a comb. man. of dim n

K a simp. comp. $\ni$ K is PL homeomorphic to $\mathcal{M}^n$. Then K is a comb. man. of dim n .

<u>Proof</u>: Thm. 2.2 (ii).

<u>Theorem 2.3</u> K , L simplicial complexes . Then $K*L$ is a comb. man. $\Leftrightarrow$ K and L are comb. cells or spheres.

<u>Proof</u>: $\Leftarrow$ Thm. 2.1

$\Rightarrow$ By induction on $\dim K*L$, say n :

$n=1$: Choose $v \in K$, $Lk(v,K*L) = Lk(v,K)*L$

$L \neq \emptyset \Rightarrow \dim K = 0 \Rightarrow Lk(v,K) = \emptyset$

$\therefore$ $Lk(v,K*L) = L$

= comb 0-cell or sphere.

Choose $v \in L$ to show K is a comb. 0 cell or sphere.

Assume the theorem is true for dimensions $\leq (n-1)$

n : Choose $v \in K$, $Lk(v,K*L) = Lk(v,K)*L$

but $Lk(v,K*L)$ = comb $(n-1)$ cell or sphere

= comb man. of dim $(n-1)$

— Thm. 2.2 (iii)

$\therefore$ by the ind. assumption $Lk(v,K)$ and L are comb cells or spheres.

Choose $v \in L$ to show K is a comb. cell or sphere.

<u>Definition 2.3</u> Boundary of a Combinatorial Manifold:

$\partial\mathcal{M}^n = \{\Delta : \Delta \in \mathcal{M}^n$ and $Lk(\Delta, \mathcal{M}^n)$ is a combinatorial cell$\}$.

<u>Theorem 2.4</u> $\mathcal{M}^n$ a comb. man. of dim n

$\alpha(\mathcal{M}^n)$ a subdivision of $\mathcal{M}^n$

$\mathcal{E}^n$ a comb. n-cell, γ^n a comb. n-sphere.

Then (i) $\partial\mathcal{M}^n$ is a subcomplex of $\mathcal{M}^n$

(ii) $\partial(\alpha(\mathcal{M}^n)) = \alpha(\partial\mathcal{M}^n)$

(iii) $\partial\mathcal{E}^n = \gamma^{n-1}$, $n \geq 1$,

γ^{n-1} a comb. (n-1) sphere

(iv) $\partial\gamma^n = \emptyset$.

<u>Proof</u>: By induction on n:

n = 0 and 1 : trivial

Assume the theorem is true in dimensions $\leq (n-1)$

n : (i) $\partial\mathcal{M}^n$ is a subcomplex of $\mathcal{M}^n$:

say $\Delta \in \partial\mathcal{M}^n$ and $\Delta' < \Delta$ i.e. $\Delta = \Delta' * \Delta''$

now $Lk(\Delta, \mathcal{M}^n) = Lk(\Delta'', Lk(\Delta', \mathcal{M}^n))$. If

$Lk(\Delta', \mathcal{M}^n) \neq$ a comb. cell

$\Rightarrow$ $Lk(\Delta', \mathcal{M}^n)$ = a comb sphere of dim $\leq (n-1)$

$\Rightarrow$ $Lk(\Delta'', Lk(\Delta', \mathcal{M}^n))$ = a comb. sphere

— (iv) in dim $Lk(\Delta', \mathcal{M}^n)$.

Contradicting $Lk(\Delta,\mathcal{M}^n)$ = a comb. cell

— $\Delta \in \partial\mathcal{M}^n$

$\therefore$ $Lk(\Delta',\mathcal{M}^n)$ = a comb. cell

$\Rightarrow$ $\Delta' \in \partial\mathcal{M}^n$

(ii) $\partial\alpha(\mathcal{M}^n) \subseteq \alpha(\partial\mathcal{M}^n)$:

Say $\Delta' \in \partial\alpha(\mathcal{M}^n)$ i.e. $Lk(\Delta',\alpha(\mathcal{M}^n))$ = a comb. cell

Choose $x \in \text{int}\,\Delta'$

$\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n)) = x*\partial\Delta'*Lk(\Delta',\alpha(\mathcal{M}^n)) + \text{Res}\ (\Delta',\alpha(\mathcal{M}^n))$

$\therefore$ $Lk(x,\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n)) = \partial\Delta'*Lk(\Delta',\alpha(\mathcal{M}^n))$

= a comb. cell — Thm. 2.1

Now $\Delta' \in \alpha(\mathcal{M}^n) \Rightarrow \Delta' \subseteq \Delta$, $\Delta \in \mathcal{M}^n \ni \Delta' \cap \text{int}\,\Delta \neq \emptyset$

$\therefore$ $x \in \text{int}\,\Delta$

$\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n) = x*\partial\Delta*Lk(\Delta,\mathcal{M}^n) + \text{Res}\ (\Delta,\mathcal{M}^n)$

$\therefore$ $Lk(x,\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n)) = \partial\Delta*Lk(\Delta,\mathcal{M}^n)$

but $Lk(x,\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n))$ is PL homeomorphic to

$Lk(x,\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n)))$ — Thm. 1.2

$\therefore$ $\partial\Delta*Lk(\Delta,\mathcal{M}^n)$ is a comb. cell

and $\partial\Delta$ a comb. sphere $\Rightarrow$ $Lk(\Delta,\mathcal{M}^n)$ is a comb.

cell — Thm. 2.1

$\Rightarrow$ $\Delta \in \partial\mathcal{M}^n$

$\Rightarrow$ $\Delta' \in \alpha(\partial\mathcal{M}^n)$

$\alpha(\partial\mathcal{M}^n) \subseteq \partial\alpha(\mathcal{M}^n)$:

Say $\Delta' \in \alpha(\partial\mathcal{M}^n) \Rightarrow \Delta' \subseteq \Delta$, $\Delta \in \partial\mathcal{M}^n$ and

int $\Delta \cap \Delta' \neq \emptyset$, choose $x \in \text{int}\,\Delta' \cap \text{int}\,\Delta$

$\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n) = x*\partial\Delta*Lk(\Delta,\mathcal{M}^n) + \text{Res}\ (\Delta,\mathcal{M}^n)$

$\therefore\ Lk(x,\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n) = \partial\Delta*Lk(\Delta,\mathcal{M}^n)$

= a comb. cell --- $\Delta \in \partial\mathcal{M}^n$

and Thm. 2.1

Also $\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n)) = x*\partial\Delta'*Lk(\Delta',\alpha(\mathcal{M}^n))$

$+ \text{Res}\ (\Delta',\alpha(\mathcal{M}^n))$

$\therefore\ Lk(x,\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n))) = \partial\Delta'*Lk(\Delta',\alpha(\mathcal{M}^n))$

but $Lk(x,\mathcal{E}_{(x,\Delta')}(\alpha(\mathcal{M}^n)))$ is PL homeomorphic

to $Lk(x,\mathcal{E}_{(x,\Delta)}(\mathcal{M}^n))$ — Thm. 1.2

$\therefore\ \partial\Delta'*Lk(\Delta',\alpha(\mathcal{M}^n))$ is a comb. cell.

and $\partial\Delta'$ a comb. sphere $\Rightarrow Lk(\Delta',K(\mathcal{M}^n))$ is a

comb. cell --- Thm. 2.1

$\Rightarrow \Delta' \in \partial\alpha(\mathcal{M}^n)$

(iii) $\partial\mathcal{L}^n = \gamma^{n-1}$: $\mathcal{L}^n$ PL homeomorphic to

$\Delta^n \Rightarrow \exists$ subdivisions $\alpha(\mathcal{L}^n) \approx \gamma(\Delta^n)$

$\therefore\ \alpha(\partial\mathcal{L}^n) = \partial(\alpha(\mathcal{L}^n)) \approx \partial(\gamma(\Delta^n)) = \gamma(\partial\Delta^n)$

$\Rightarrow \partial\mathcal{L}^n$ is a comb. sphere of dim (n-1)

(iv) $\partial\gamma^n = \emptyset$: γ^n PL homeomorphic to $\partial\Delta^{n+1} \Rightarrow$

$$\exists \text{ subdivisions } \alpha(\gamma^n) \approx \gamma(\partial\Delta^{n+1})$$

$$\therefore \alpha(\partial\gamma^n) = \partial(\alpha(\gamma^n)) \approx \partial(\gamma(\partial\Delta^{n+1})) = \gamma(\partial(\partial\Delta^{n+1}))$$

$$= \emptyset$$

$$\Rightarrow \partial\gamma^n = \emptyset$$

<u>Corollary 2.2</u> $\mathcal{M}^n$ a comb. man. of dim n

$\partial\mathcal{M}^n = \emptyset \Leftrightarrow \forall$ vertex $v \in \mathcal{M}^n$,

$Lk(v, \mathcal{M}^n)$ is a comb. sphere.

<u>Proof:</u> $\Rightarrow$ by defn of $\partial\mathcal{M}^n$

$\Leftarrow$ by Thm. 2.4 (iii) and (iv) a combinatorial cell cannot be a sphere. Hence if $\forall v \in \mathcal{M}^n$ $Lk(v, \mathcal{M}^n)$ is a sphere, $\partial\mathcal{M}^n$ cannot contain vertices. But $\partial\mathcal{M}^n$ is a subcomplex by Thm. 2.4 (i) so it must be empty.

<u>Theorem 2.5</u> $\mathcal{M}^n$ a comb. man. of dim n $\ni \partial\mathcal{M}^n \neq \emptyset$

Then $\partial\mathcal{M}^n$ is a comb. man. of

dim $(n-1) \ni \partial(\partial\mathcal{M}^n) = \emptyset$.

<u>Proof:</u> (a) $Lk(v, \partial\mathcal{M}^n) = \partial Lk(v, \mathcal{M}^n)$:

$$\Delta \in Lk(v, \partial\mathcal{M}^n) \Leftrightarrow v*\Delta \in \partial\mathcal{M}^n$$

$$\Leftrightarrow Lk(v\,\Delta, \mathcal{M}^n) = Lk(\Delta, Lk(v, \mathcal{M}^n))$$

is a comb. cell

$$\Leftrightarrow \Delta \in \partial Lk(v, \mathcal{M}^n)$$

(b) $\partial\mathcal{M}^n$ is a comb. man. of dim $(n-1) \ni \partial(\partial\mathcal{M}^n) = \emptyset$:

$v \in \partial\mathcal{M}^n \Rightarrow Lk(v, \partial\mathcal{M}^n) = \partial Lk(v, \mathcal{M}^n)$

$Lk(v, \mathcal{M}^n)$ is a comb. cell of dim $(n-1)$

$\therefore$ $\partial Lk(v, \mathcal{M}^n)$ is a comb. sphere of dim $(n-2)$

— Thm. 2.4 (iii)

$\Rightarrow \partial\mathcal{M}^n$ is a comb. man. of dim $(n-1)$

$\ni \partial(\partial\mathcal{M}^n) = \emptyset$.

3. Homogeneity of Combinatorial Spheres

<u>Lemma 3.1</u> $\mathfrak{L}_1^n$ $\mathfrak{L}_2^n$ combinatorial cells

$h : \partial\mathfrak{L}_1^n \to \partial\mathfrak{L}_2^n$ a PL homeomorphism

Then $\exists$ a PL homeomorphism

$\hat{h} : \mathfrak{L}_1^n \to \mathfrak{L}_2^n \ni \hat{h}|_{\partial\mathfrak{L}_1^n} = h$

<u>Proof</u>: $\exists$ PL homeomorphisms $g_1 : \mathfrak{L}_1^n \to \Delta^n$, $g_2 : \mathfrak{L}_2^n \to \Delta^n$, set $h' = g_2|_{\partial\mathfrak{L}_2^n} \circ h \circ g_1^{-1}|_{\partial\Delta_2^n}$

$h' : \partial\Delta^n \to \partial\Delta^n$ is a PL homeomorphism.

Consider its underlying simplicial isomorphism

$h' : \alpha_1(\partial\Delta^n) \to \alpha_2(\partial\Delta^n)$

choose $v \in \operatorname{int}\Delta^n$, set

$\hat{h}' = h' * 1_v : \alpha_1(\partial\Delta^n)*v = \hat{\alpha}_1(\Delta^n) \to \alpha_2(\partial\Delta^n)*v = \hat{\alpha}_2(\Delta^n)$

$\hat{h} = g_2^{-1} \circ \hat{h}' \circ g_1 : \mathfrak{L}_1^n \to \mathfrak{L}_2^n$ is a PL homeomorphism

$\ni \hat{h}|_{\partial\mathfrak{L}_1^n} = h$.

<u>Lemma 3.2</u> $\mathfrak{L}^n$ a combinatorial cell of dim n

C^n PL homeomorphic to $\partial\mathfrak{L}^n \times I$ with $C^n \cap \mathfrak{L}^n = \partial\mathfrak{L}^n$. Then $\mathfrak{L}^n + C^n$ is a combinatorial cell of dimension n .

<u>Proof</u>: $\exists$ PL homeomorphisms $h_1 : \mathcal{E}^n \to \Delta^n$, $h_2 : C^n \to \partial\mathcal{E}^n \times I$ take $h_3 = (h_1|_{\partial\mathcal{E}^n} \times 1_I) \circ h_2 : C^n \to \partial\Delta^n \times I$ consider $h_3|_{\partial\mathcal{E}^n \subseteq C^n} : \partial\mathcal{E}^n \to \partial\Delta^n$ by lemma 3.1 it may be extended to $\hat{h}_3 : \mathcal{E}^n \to \Delta^n$ thus h_3 and $\hat{h}_3$ define a PL homeomorphism

$$h : C^n + \mathcal{E}^n \to \partial\Delta^n \times I + \Delta^n \text{ which is PL homeomorphic to } \Delta^n.$$

<u>Lemma 3.3</u> $\mathcal{E}_1^n, \mathcal{E}_2^n$ combinatorial n-cells $\ni \partial\mathcal{E}_1^n = \partial\mathcal{E}_2^n = \mathcal{E}_1^n \cap \mathcal{E}_2^n$. Then $\mathcal{E}_1^n + \mathcal{E}_2^n$ is a combinatorial n-sphere.

<u>Proof</u>: By lemma 3.1 $\exists$ PL homeomorphisms

$h_1 : \mathcal{E}_1^n \to \partial\mathcal{E}_1^n * v_1$, $v_1 \notin \mathcal{E}_1^n + \mathcal{E}_2^n \ni h_1|_{\partial\mathcal{E}_1^n} = 1_{\partial\mathcal{E}_1^n}$

$h_2 : \mathcal{E}_2^n \to \partial\mathcal{E}_2^n * v_2$, $v_2 \neq v_1$ $v_2 \notin \mathcal{E}_1^n + \mathcal{E}_2^n \ni h_2|_{\partial\mathcal{E}_2^n} = 1_{\partial\mathcal{E}_2^n}$

since $\mathcal{E}_1^n \cap \mathcal{E}_2^n = \partial\mathcal{E}_1^n = \partial\mathcal{E}_2^n$, $\exists$ a PL homeomorphism

$$h : \mathcal{E}_1^n + \mathcal{E}_2^n \to \partial\mathcal{E}_1^n * v_1 + \partial\mathcal{E}_2^n * v_2$$

but $\partial\mathcal{E}_1^n * v_1 + \partial\mathcal{E}_2^n * v_2 = \partial\mathcal{E}_1^n * (v_1 + v_2)$ which is a comb. n-sphere by Thm. 2.1.

<u>Theorem 3.1</u> γ^n a comb. n-sphere

$\mathcal{E}^n \subseteq \gamma^n$ a comb. n-cell

Then $C(\mathcal{E}^n, \gamma^n)$ is a comb. n-cell.

<u>Proof</u>: By induction on n :

$n = 0$: $\gamma^0 = 2$ 0-simplices, $\mathcal{E}^0 = 1$ 0-simplex

$\therefore C(\mathcal{E}^0, \gamma^0) = 1$ 0-simplex

$= $ a comb. 0-cell.

Assume the theorem is true in dimensions $\leq (n-1)$.

This assumption implies result (A), the existence of a collar of the boundary of a combinatorial manifold of dimension n which will be used to prove the main theorem in that dimension.

(A). $\mathcal{M}^n$ a comb. manifold of dim $n \ni \partial\mathcal{M}^n$ is full in $\mathcal{M}^n$. Then $N(\partial\mathcal{M}^n, \gamma(\mathcal{M}^n))$ is PL homeomorphic to $\partial\mathcal{M}^n \times I$ where $\gamma(\mathcal{M}^n) = B_{\partial\mathcal{M}^n}(\mathcal{M}^n) =$ 1st barycentric subdivision of $\mathcal{M}^n$ relative to $\partial\mathcal{M}^n$.

<u>Proof</u>: By induction on n :

Case $n = 0$ is trivial. We assume (A) is true for dimension $\leq (n-1)$ as in the main theorem.

case n : (1) Defn of $O(\Delta)$ for $\Delta \in \partial\mathcal{M}$:

$O(\Delta) = \{\Delta_1 : \Delta_1 \in Lk(\Delta, \gamma(\mathcal{M}))$ and

$\Delta_1 \cap \partial\mathcal{M} = \emptyset\}$

since $\partial\mathcal{M}$ is full in $\gamma(\mathcal{M})$ we have that

$$N(\partial\mathcal{M},\gamma(\mathcal{M})) = \sum_{\Delta\in\partial\mathcal{M}} \Delta * O(\Delta)$$

also $Lk(\Delta,\gamma(\mathcal{M})) = N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))+O(\Delta)$

since $Lk(\Delta,\partial\mathcal{M}) = \partial Lk(\Delta,\gamma(\mathcal{M})) = Lk(\Delta,\gamma(\mathcal{M}))\cap\partial\mathcal{M}$

(2) Properties of $O(\Delta)$:

I. If $\Delta\in\partial\mathcal{M}$ is an l-simplex, then $O(\Delta)$ is a comb. $(n-l-1)$ cell:

(i) Choose a vertex v not considered so far.

Since $Lk(\Delta,\partial\mathcal{M})$ is a comb. $(n-l-2)$ sphere, $v*Lk(\Delta,\partial\mathcal{M}))$ is a comb. $(n-l-1)$ cell.

Also $Lk(\Delta,\gamma(\mathcal{M}))$ is a comb. $(n-l-1)$ cell $\ni$ $Lk(\Delta,\gamma(\mathcal{M}))\cap v*Lk(\Delta,\partial\mathcal{M})$ is $Lk(\Delta,\partial\mathcal{M}) = \partial Lk(\Delta,\gamma(\mathcal{M})) = \partial(v*Lk(\Delta,\partial\mathcal{M}))-\partial\mathcal{M}$ is full in $\gamma(\mathcal{M})$. Thus $v*Lk(\Delta,\partial\mathcal{M})+Lk(\Delta,\gamma(\mathcal{M}))$ is a comb. $(n-l-1)$ sphere — Lemma 3.3.

(ii) $N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})) \approx N(Lk(\Delta,\partial\mathcal{M}), \hat{\gamma}(Lk(\Delta,\mathcal{M}))$

where $\hat{\gamma}(Lk(\Delta,\mathcal{M})) = \beta_{Lk(\Delta,\partial\mathcal{M})}Lk(\Delta,\mathcal{M})$

but $N(Lk(\Delta,\partial\mathcal{M}), \hat{\gamma}(Lk(\Delta,\mathcal{M})))$ is PL homeomorphic to $Lk(\Delta,\partial\mathcal{M})\times I$ by (A) in dimension $(n-l-1)$

(iii) $N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))$ is a "collar" of $v*Lk(\Delta,\partial\mathcal{M})$ in the sense of Lemma 3.2, hence $v*Lk(\Delta,\partial\mathcal{M})+N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))$ is a comb. $(n-\ell-1)$ cell, say $\mathcal{E}^{n-\ell-1}$.

(iv) Now $O(\Delta)+N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))+v*Lk(\Delta,\partial\mathcal{M})$

$$= Lk(\Delta,\gamma(\mathcal{M})) + v*Lk(\Delta,\partial\mathcal{M})$$

$$= \text{a comb. } (n-\ell-1) \text{ sphere} \text{ — (i) say } \gamma^{m-\ell-1}$$

And so, by the induction hypothesis to the main theorem in dim $(n-\ell-1)$, $C(\mathcal{E}^{n-\ell-1},\gamma^{n-\ell-1})$ is a comb. $(n-\ell-1)$ cell

(v) But $O(\Delta) \cap [N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))+v*Lk(\Delta,\partial\mathcal{M})]$

$$= N(Lk(\Delta,\partial\mathcal{M}),Lk(\Delta,\gamma(\mathcal{M})))$$

which can only contain simplices of dimension $< (n-\ell-1)$

Thus $C(\mathcal{E}^{n-\ell-1},\gamma^{n-\ell-1}) = O(\Delta)$. And $O(\Delta)$ is a comb. $(n-\ell-1)$ cell.

II. for $\Delta_1, \Delta_2 \in \partial\mathcal{M}$, $\Delta_1 < \Delta_2 \Rightarrow O(\Delta_2) \subseteq O(\Delta_1)$

say $\Delta \in O(\Delta_2) \Rightarrow \Delta \in Lk(\Delta_2,\gamma(\mathcal{M}))$ and $\Delta \cap \partial\mathcal{M} = \emptyset$

$\Rightarrow \Delta * \Delta_2 \in \gamma(\mathcal{M}) \Rightarrow \Delta * \Delta_1 \in \gamma(\mathcal{M})$

$\Rightarrow \Delta \in Lk(\Delta_1,\gamma(\mathcal{M}))$ and $\Delta \cap \partial\mathcal{M} = \emptyset$

$\Rightarrow \Delta \in O(\Delta_1)$

III. for $\Delta \in \partial\mathcal{M}$, $\partial O(\Delta) = \bigcup_{\Delta<\Delta'} O(\Delta')$

(i) say $\Delta_1 \in \partial O(\Delta) \Rightarrow \Delta_1 \in Lk(\Delta,\gamma(\mathcal{M}))$, $\Delta_1 \cap \partial\mathcal{M} = \emptyset$,

and $Lk(\Delta_1,O(\Delta))$ = a comb. cell.

now $Lk(\Delta_1,Lk(\Delta,\gamma(\mathcal{M})))$ = a comb-sphere —

$\Delta \in \partial\mathcal{M} \Rightarrow Lk(\Delta,\gamma(\mathcal{M}))$ is a comb. sphere,

thus $Lk(\Delta_1,O(\Delta)) \subseteq Lk(\Delta_1,Lk(\Delta,\gamma(\mathcal{M}))) - O(\Delta)$

$\subseteq Lk(\Delta,\gamma(\mathcal{M}))$. But $Lk(\Delta_1,O(\Delta)) \neq Lk(\Delta_1,Lk(\Delta,\gamma(\mathcal{M})))$

$\therefore \exists\, \Delta_2 \in Lk(\Delta_1,Lk(\Delta,\gamma(\mathcal{M}))), \Delta_2 \notin Lk(\Delta_1,O(\Delta))$

$\Rightarrow \Delta_2 * \Delta_1 \in Lk(\Delta,\gamma(\mathcal{M}))$

as $\Delta_2 \notin Lk(\Delta_1,O(\Delta))$ we must have $\Delta_2 * \Delta_1 \cap \partial\mathcal{M}^n \neq \emptyset$

but $\Delta_1 \cap \partial\mathcal{M}^n = \emptyset$ $\therefore$ $\Delta_3 = \Delta_2 * \Delta_1 \cap \partial\mathcal{M}^2 < \Delta_2$

Consider $\Delta_3 * \Delta$, clearly $\Delta < \Delta_3 * \Delta$

we show $\Delta_1 \in O(\Delta_3 * \Delta)$:

$\Delta_2 * \Delta_1 \in Lk(\Delta,\gamma(\mathcal{M})) \Rightarrow \Delta_2 * \Delta_1 * \Delta \in \gamma(\mathcal{M})$

$\Rightarrow \Delta_3 * \Delta_1 * \Delta \in \gamma(\mathcal{M}) - \Delta_3 < \Delta_2$

$\Rightarrow \Delta_1 \in Lk(\Delta_3 * \Delta,\gamma(\mathcal{M}))$ and

$\Delta_1 \cap \partial\mathcal{M} = \emptyset$

$\Rightarrow \Delta_1 \in O(\Delta_3 * \Delta)$

$\Rightarrow \Delta_1 \in \bigcup_{\Delta<\Delta_1} O(\Delta') - \Delta < \Delta_3 * \Delta$.

(ii) say $\Delta_1 \in \bigcup_{\Delta < \Delta'} O(\Delta')$ i.e. $\Delta_1 \in O(\Delta')$ for some $\Delta < \Delta'$. By II $\Delta_1 \in O(\Delta)$, $\therefore$ we must show $Lk(\Delta_1, O(\Delta))$ is a comb. cell.

Now $\Delta_1 \notin \partial Lk(\Delta, \gamma(\mathcal{M})) - \partial Lk(\Delta, \gamma(\mathcal{M}))$

$= \partial\mathcal{M} \cap Lk(\Delta, \gamma(\mathcal{M}))$

$\therefore$ $Lk(\Delta_1, Lk(\Delta, \gamma(\mathcal{M})))$ is a comb. sphere of the same dimension as $Lk(\Delta_1, O(\Delta))$ but

$Lk(\Delta_1, O(\Delta)) \subseteq Lk(\Delta_1, Lk(\Delta, \gamma(\mathcal{M})))$

And as a comb. k-sphere cannot contain a comb. k-sphere as a proper subcomplex $Lk(\Delta_1, O(\Delta))$ must be a comb. cell — $O(\Delta)$ is a comb. man.

$\therefore$ $\Delta_1 \in \partial O(\Delta)$

IV. $\Delta_1, \Delta_2 \in \partial\mathcal{M}$

then $O(\Delta_1) \cap O(\Delta_2) = \begin{cases} \emptyset & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ are not faces of a simplex of } \gamma(\mathcal{M}) \\ O(\Delta_3) & \text{where } \Delta_3 \text{ is the smallest simplex of } \gamma(\mathcal{M}) \text{ containing } \Delta_1 \text{ and } \Delta_2 \text{ as faces} \end{cases}$

(i) if $\Delta \in O(\Delta_3)$ then $\Delta \in O(\Delta_1) \cap O(\Delta_2)$,

$\Delta_1, \Delta_2 < \Delta_3$ by II

(ii) say $\Delta \in O(\Delta_1) \cap O(\Delta_2)$,

now $\Delta = b\Delta'_1 * \ldots * b\Delta'_k$, $\Delta'_1 < \ldots < \Delta'_k \in \mathcal{M}$

in the canonical rep. of $\gamma(\mathcal{M})$

$\Delta \in Lk(\Delta_1, \gamma(\mathcal{M}))$ & $\Delta \in Lk(\Delta_2, \gamma(\mathcal{M})) \Rightarrow \Delta * \Delta_1$ and

$\Delta * \Delta_2 \in \gamma(\mathcal{M})$

but $\Delta_1, \Delta_2 \in \partial\mathcal{M} \Rightarrow \Delta * \Delta_1 = \Delta^1_{\partial\mathcal{M}} * b\Delta'_1 * \cdots * b\Delta'_k$

$\Delta * \Delta_2 = \Delta^2_{\partial\mathcal{M}} * b\Delta'_1 * \cdots * b\Delta'_k$ in canonical representation

$\Rightarrow \Delta_1, \Delta_2 < \Delta'_1 \Rightarrow \Delta_1 * \Delta_2 \in \mathcal{M}$

$\Rightarrow \Delta_1 * \Delta_2 \in \partial\mathcal{M}$

— $\partial\mathcal{M}$ full in $\mathcal{M}$

$\therefore \Delta_3 = \Delta_1 * \Delta_2$ has $\Delta \in O(\Delta_3) \Leftrightarrow \Delta \in O(\Delta_1) \cap O(\Delta_2)$.

(3) Consider now the simplicial complex $\partial\mathcal{M} \times I$:

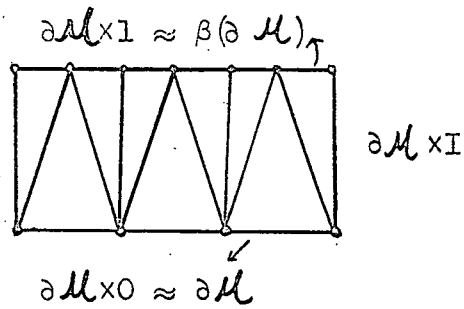
for $\Delta \in \partial\mathcal{M} \times 0$, set

$D(\Delta) = \{\Delta_1 : \Delta_1 \cap \partial\mathcal{M} \times 0 = \emptyset, \Delta_1 * \Delta \in \partial\mathcal{M} \times I\}$

$\subseteq \partial\mathcal{M} \times 1$

Clearly $\partial\mathcal{M} \times 0$ is full in $\partial\mathcal{M} \times I$

and $\partial\mathcal{M} \times I = \sum_{\Delta \in \partial\mathcal{M} \times 0} \Delta * D(\Delta)$



(4) Properties of $D(\Delta)$:

I. If $\Delta_1 \in \partial\mathcal{M} \times 0$ is an i-simplex, then $D(\Delta_1)$ is a combinatorial $(n-i-1)$ cell:

$$D(\Delta_1) = \{(b\Delta_1,1)*\ldots*(b\Delta_m,1) : \Delta_1<\ldots<\Delta_m \in \partial\mathcal{M}\}$$

$$= (b\Delta_1,1)*\{(b\Delta_2,1)*\ldots*(b\Delta_m,1):\Delta_1<\ldots<\Delta_m \in\partial\mathcal{M}\}$$

$$\approx (b\Delta_1,1)*\beta(Lk(\Delta_1,\partial\mathcal{M}))$$

= a comb. $(n-\nu-1)$ cell --- $Lk(\Delta_1,\partial\mathcal{M})$ is a comb. $(n-\nu-1)$ sphere

II. If $\Delta_1,\Delta_2 \in \partial\mathcal{M}\times 0$, $\Delta_1 < \Delta_2$ then $D(\Delta_2) \subseteq D(\Delta_1)$:

say $\Delta \in D(\Delta_2) \Rightarrow \Delta*\Delta_2 \in \partial\mathcal{M}\times I$, $\Delta \cap \partial\mathcal{M}\times 0 = \emptyset$

$\Rightarrow \Delta*\Delta_1 \in \partial\mathcal{M}\times I$, — $\Delta_1 < \Delta_2$

$\Rightarrow \Delta \in D(\Delta_1)$

III. If $\Delta \in \partial\mathcal{M}\times 0$, $\partial D(\Delta) = \bigcup_{\Delta<\Delta'} D(\Delta')$

(i) say $\Delta_1 \in \partial D(\Delta)$

$\Rightarrow Lk(\Delta_1,D(\Delta))$ = a comb. cell

but $Lk(\Delta_1,Lk(\Delta,\partial\mathcal{M}\times I)) \supseteq Lk(\Delta_1,D(\Delta))$ and

$Lk(\Delta_1,Lk(\Delta,\partial\mathcal{M}\times I)) \neq Lk(\Delta_1,D(\Delta))$

$\therefore\ \exists\ \Delta_2 \in Lk(\Delta_1,Lk(\Delta,\partial\mathcal{M}\times I))$, $\Delta_2 \notin Lk(\Delta_1,D(\Delta))$,

and $\Delta_1 \in D(\Delta_3)$, $\Delta_3 = \Delta_2*\Delta_1 \cap \partial\mathcal{M}^n\times 0$

(ii) say $\Delta_1 \in D(\Delta')$ $\Delta < \Delta'$

$Lk(\Delta_1, Lk(\Delta, \partial\mathcal{M}\times I))$ is a comb. sphere of the same dim as $Lk(\Delta_1, D(\Delta))$

$\Rightarrow Lk(\Delta_1, D(\Delta))$ cannot be a combinatorial sphere since $Lk(\Delta_1, D(\Delta)) \subseteq Lk(\Delta_1, Lk(\Delta, \partial\mathcal{M}\times I))$

$\Rightarrow Lk(\Delta_1, D(\Delta))$ is a comb. cell

$\Rightarrow \Delta_1 \in \partial D(\Delta)$.

IV. If $\Delta_1, \Delta_2 \in \partial\mathcal{M}\times 0$ then

$$D(\Delta_1)\cap D(\Delta_2) = \begin{cases} \emptyset & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ are not faces of a simplex of } \partial\mathcal{M}\times I \\ D(\Delta_3) & \text{where } \Delta_3 \text{ is the smallest simplex of } \partial\mathcal{M}\times I \text{ containing } \Delta_1 \text{ and } \Delta_2 \text{ as faces} \end{cases}$$

(i) if $\Delta \in D(\Delta_3)$ then $\Delta \in D(\Delta_1) \cap D(\Delta_2)$ by II

(ii) if $\Delta \in D(\Delta_1) \cap D(\Delta_2)$

$\Rightarrow \Delta = (b\Delta_1, 1)*(b\Delta_2', 1)*\ldots*(b\Delta_m', 1) \quad \Delta_1 < \Delta_2' < \ldots < \Delta_m' \in \partial\mathcal{M}$

and $\Delta = (b\Delta_2, 1)*(b\Delta_2'', 1)*\ldots*(b\Delta_m'', 1) \quad \Delta_2 < \Delta_2'' < \ldots < \Delta_m'' \in \partial\mathcal{M}$

but we must have $\Delta_m' = \Delta_m''$

$\therefore \Delta_1 * \Delta_2 < \Delta_m' \Rightarrow \Delta_1 * \Delta_2 = \Delta_3 \in \partial\mathcal{M}$

has $\Delta \in D(\Delta_3) \Leftrightarrow \Delta \in D(\Delta_1) \cap D(\Delta_2)$.

(5) We now exhibit a PL homeomorphism

$$h : N(\partial(\mathcal{M}),\gamma(\mathcal{M})) \to \partial\mathcal{M}\times I$$

note that $N(\partial\mathcal{M},\gamma(\mathcal{M})) = \sum_{\Delta\in\partial\mathcal{M}} \Delta * O(\Delta)$ and

$$\partial\mathcal{M}\times I = \sum_{\Delta\in\partial\mathcal{M}\times O} \Delta * D(\Delta)$$

we define h inductively on simplices of $N(\partial\mathcal{M},\gamma(\mathcal{M}))$ in order of decreasing dimension.

Choose any $\Delta^{n-1} \in \partial\mathcal{M}$. $O(\Delta^{n-1}) = v_1$ and $D(\Delta^{n-1}) = v_2$ define $h|_{\Delta^{n-1}*O(\Delta^{n-1})} : \Delta^{n-1}*O(\Delta^{n-1}) \to \Delta^{n-1}*D(\Delta^{n-1})$ by $h = 1_{\Delta^{n-1}}*g$ where $g(v_1) = v_2$

Assume we have defined h on $\Delta^{l}*O(\Delta^{l})\ \forall\, l > k$

Choose $\Delta^{k} \in \partial\mathcal{M}$ as $\partial O(\Delta^{k}) = \bigcup_{k<l} O(\Delta^{l})$ and $\partial D(\Delta^{k}) = \bigcup_{k<l} D(\Delta^{l})$ we have defined $h|_{\partial O(\Delta^{k})} : \partial O(\Delta^{k}) \to \partial D(\Delta^{k})$ already. $h|_{\partial O(\Delta^{k})}$ may be extended to $h|_{O(\Delta^{k})}$ by lemma 3.1.

Define $h|_{\Delta^{k}*O(\Delta^{k})} : \Delta^{k}*O(\Delta^{k}) \to \Delta^{k}*D(\Delta^{k})$ by $h = 1_{\Delta^{k}}*h|_{O(\Delta^{k})}$ noting that $O(\Delta_1^{k}) \cap O(\Delta_2^{k}) = O(\Delta^{l})$ $l > k$, by induction we define h on $\Delta^{k}*O(\Delta^{k})\ \forall\, \Delta^{k} \in \partial\mathcal{M}$, $k = 0, 1, \ldots. (n-1)$.

Thus we have a PL homeomorphism

$$h : N(\partial\mathcal{M}, \gamma(\mathcal{M})) \to \partial\mathcal{M} \times I .$$

(B) Proof of the Main Theorem in dimension n :

(i) Set $K = C(\mathcal{L}^n, \gamma^n)$, clearly $\gamma^n = K + \mathcal{L}^n$.

(a) K is a comb. man. of dim n :

Choose any $v \in K$, $Lk(v,\gamma^n) = Lk(v,\mathcal{L}^n) + Lk(v,K)$

$v \notin \mathcal{L}^n \Rightarrow Lk(v,K) = Lk(v,\gamma^n)$ = a comb. $(n-1)$ sphere

$v \in \mathcal{L}^n \Rightarrow Lk(v,\gamma^n)$ = a comb. $(n-1)$ sphere

if $Lk(v,\mathcal{L}^n)$ is a sphere,

$\Rightarrow Lk(v,\mathcal{L}^n) = Lk(v,\gamma^n)$

$\Rightarrow Lk(v,K) = \emptyset$

contradicting $v \in K$

$\therefore$ $Lk(v,\mathcal{L}^n)$ is a comb. $(n-1)$ cell

thus $Lk(v,K) = C(Lk(v,\mathcal{L}^n) , Lk(v,\gamma^n))$

= a comb. $(n-1)$ cell by the induction assumption to the main theorem.

$\therefore$ $Lk(v,K)$ is a comb. $(n-1)$ cell or sphere

$\Rightarrow K$ is a comb. man. of dim n .

(b) $\partial K = \partial\mathcal{L}^n = K \cap \mathcal{L}^n$:

$\Delta \in \partial K \Leftrightarrow Lk(\Delta,K)$ is a comb. cell $\Leftrightarrow$ $Lk(\Delta,\mathcal{L}^n)$ is a comb. cell — (a)

$\Leftrightarrow \Delta \in \partial\mathcal{L}^n$

also $\Delta \in K \cap \mathcal{L}^n \Rightarrow Lk(\Delta,\mathcal{L}^n)$ and $Lk(\Delta,K)$ are comb. cells — as in (a)

$\Rightarrow \Delta \in \partial K$ and $\Delta \in \partial\mathcal{L}^n$

$\therefore \partial K = \partial\mathcal{L}^n = K \cap \mathcal{L}^n$

(ii) Given $\mathcal{L}^n \subseteq \gamma^n$ consider $\mathcal{M} = C(\mathcal{L}^n,\gamma^n)$

We show $\mathcal{M}$ is a combinatorial n-cell

Now, $\mathcal{M}$ is a comb. man. of dim n $\ni \partial\mathcal{M} = \partial\mathcal{L}^n = \mathcal{M} \cap \mathcal{L}^n$ — (i)

As $\partial\mathcal{M} = \mathcal{M} \cap \mathcal{L}^n$, by Lemma 3.1 we may consider γ^n as

$$\mathcal{M} + \partial\mathcal{M} * v .$$

(iii) Consider a PL homeomorphism

$$h : \gamma^n = \mathcal{M} + \partial\mathcal{M} * v \to \partial\Delta^{n+1} .$$

Let $\Delta^{n+1} = w*\Delta^n$. Then $\partial\Delta^{n+1} = w*\partial\Delta^n + \Delta^n$. We note that if $x, y \in \partial\Delta^{n+1}$, then there is a PL homeomorphism of $\partial\Delta^{n+1}$ onto itself which maps x onto y. Hence we may assume that the PL homeomorphism h satisfies the property

$$h(v) = w .$$

We choose now subdivisions $\alpha_1(\gamma^n)$ of γ^n and $\alpha_2(\partial\Delta^{n+1})$ of $\partial\Delta^{n+1}$ such that $h : \alpha_1(\gamma^n) \to \alpha_2(\partial\Delta^{n+1})$ is a simplicial isomorphism and such that

$$Lk(v,\alpha_1(\gamma^n)) \cap \alpha_1(Lk(v,\gamma^n)) = Lk(v,\alpha_1(\gamma^n)) \cap \alpha_1(\partial\mathcal{M}) = \emptyset$$

and

$$Lk(w,\alpha_2(\partial\Delta^{n+1})) \cap \alpha_2(Lk(w,\partial\Delta^{n+1})) =$$

$$Lk(w,\alpha_2(\partial\Delta^{n+1})) \cap \alpha_2(\partial\Delta^n) = \emptyset .$$

By Corollary 1.3, $C(St(w,\alpha(\partial\Delta^{n+1})) , \alpha(St(w,\partial\Delta^{n+1})))$ is PL homeomorphic to $\partial\Delta^n \times I$. By Lemma 3.2,

$$C(St(w,\alpha_2(\partial\Delta^{n+1})),\alpha_2(\partial\Delta^{n+1})) = C(St(w,\alpha_2(\partial\Delta^{n+1})),\alpha_2(St(w,\partial\Delta^{n+1})))$$

$$+ \alpha_2(\Delta^n)$$

is a combinatorial cell of dimension n . Since

$$h(St(v,\alpha_1(\gamma^n))) = St(w,\alpha_2(\partial\Delta^{n+1})) ,$$

we conclude that

$$C(St(v,\alpha_1(\gamma^n)),\alpha_1(\gamma^n))$$

is a combinatorial n-cell.

To satisfy the hypothesis of (A) with respect to $\alpha_1(\mathcal{U})$ and $\alpha_1(\partial\mathcal{U})$, we define

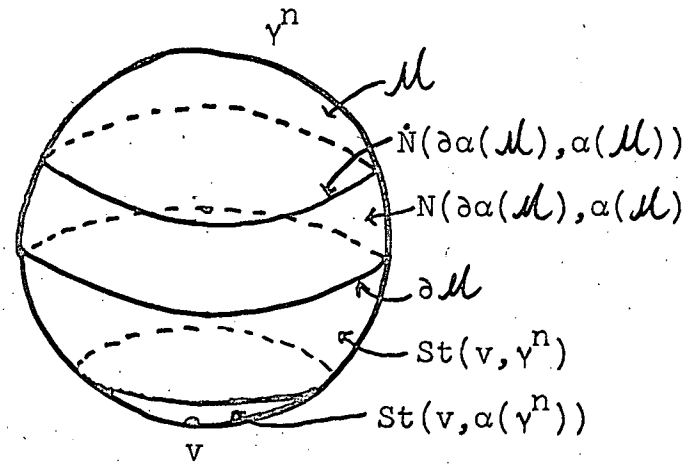
$$\alpha(\gamma^n) = B^2_{\alpha_1(\partial\mathcal{M}*v)}(\alpha_1(\gamma^n)) \ .$$

(iv) By (iii), $C(St(v,\alpha(\gamma^n)) \ , \ \alpha(\gamma^n))$ is a comb. n-cell. We will produce a PL homeomorphism of $\mathcal{M}$ onto $C(St(v,\alpha(\gamma^n)),\alpha(\gamma^n))$.

Observe that $\exists$ PL homeomorphisms

$h_1 : N(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M})) \to \partial\mathcal{M}\times I$ — (A)

$h_2 : C(St(v,\alpha(\gamma^n)),St(v,\gamma^n)) \to \partial\mathcal{M}\times I$ — Corol. 1.3



Thus $\exists$ a PL homeomorphism

$$g : N(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M})) \to N(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M}))+C(St(v,\alpha(\gamma^n)),St(v,\gamma^n))$$

$$\ni \ g|_{\dot{N}(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M}))} = id_{\dot{N}(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M}))}$$

Using g and $id_{C(N(\partial\alpha(\mathcal{M}),\alpha(\mathcal{M})),\alpha(\mathcal{M}))}$, since they agree on intersections of domains, we may define a PL homeomorphism

$$h : \mathcal{M} \to C(St(v,\alpha(\gamma^n)),\alpha(\gamma^n)) \ .$$

Thus $\mathcal{M}$ is a combinatorial n-cell.

<u>Corollary 3.1</u> $\mathcal{M}^n$ a comb. man. of dim n $\ni$ $\partial\mathcal{M}^n$ is

full in $\mathcal{M}^n$. $B(\mathcal{M}^n)$ the 1st barycentric subdivision of $\mathcal{M}^n$ relative to $\partial\mathcal{M}^n$. Then $N(\partial\mathcal{M}^n, B(\mathcal{M}^n))$ is PL homeomorphic to $\partial\mathcal{M}^n \times I$.

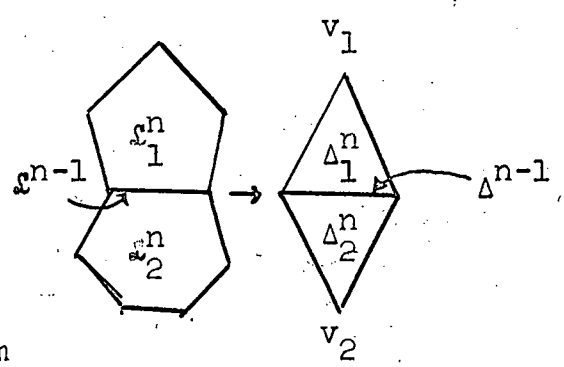
<u>Proof</u>: c.f. (A) of Thm. 3.1.

<u>Corollary 3.2</u> $\mathcal{L}_1^n$, $\mathcal{L}_2^n$ combinatorial n-cells $\ni$ $\mathcal{L}_1^n \cap \mathcal{L}_2^n = \partial\mathcal{L}_1^n \cap \partial\mathcal{L}_2^n = \mathcal{L}^{n-1}$, a comb. (n-1) cell.

Then $\mathcal{L}_1^n + \mathcal{L}_2^n$ is a combinatorial n-cell.

<u>Proof</u>: $\exists$ a PL homeomorphism $h : \mathcal{L}^{n-1} \to \Delta^{n-1}$. $C(\mathcal{L}^{n-1}, \partial\mathcal{L}_1^n)$ is a comb. (n-1) cell by theorem 3.1. By lemma 3.1, $\exists$ a PL homeomorphism

$$h_1 : C(\mathcal{L}^{n-1}, \partial\mathcal{L}_1^n) \to v_1 * \partial\Delta^{n-1}$$



which extends the PL homeomorphism h. Similarly $\exists$ a PL homeomorphism

$$h_2 : C(\mathcal{L}^{n-1}, \partial\mathcal{L}_2^n) \to v_2 * \partial\Delta^{n-1}$$

which extends the PL homeomorphism h.

Set $\Delta_1^n = \Delta^{n-1} * v_1$ and $\Delta_2^n = \Delta^{n-1} * v_2$. Hence we have PL homeomorphism

$$h_1' : \partial\mathcal{L}_1^n = \mathcal{L}^{n-1} + C(\mathcal{L}^{n-1}, \partial\mathcal{L}_1^n) \to \partial\Delta_1^n$$

and

$$h_2' : \partial\mathcal{L}_2^n = \mathcal{L}^{n-1} + C(\mathcal{L}^{n-1}, \partial\mathcal{L}_2^n) \to \partial\Delta_2^n$$

with $h_1'|_{\mathcal{L}^{n-1}} = h_2'|_{\mathcal{L}^{n-1}}$.

By lemma 3.1 these may be extended to PL homeomorphisms

$$\hat{h}_1 : \mathcal{L}_1^n \to \Delta_1^n$$

$$\hat{h}_2 : \mathcal{L}_2^n \to \Delta_2^n$$

Thus giving a PL homeomorphism

$$h : \mathcal{L}_1^n + \mathcal{L}_2^n \to \Delta_1^n + \Delta_2^n = \Delta^{n-1} * (v_1 + v_2)$$

and $\Delta^{n-1} * (v_1 + v_2)$ is a comb. n-cell — Thm. 2.1.

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