HOMOGENEITY OF COMBINATORIAL SPHERES

by

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Abstract

The object of this thesis is to cover the results of [1] from a piecewise linear point of view. The principal result of [1] is the theorem on the homogeneity of spheres, i.e. the complement of a combinatorial n-cell in a combinatorial n-sphere is a combinatorial n-cell. A piecewise linear proof of this theorem by a "long induction" using regular neighbourhoods and collapsing was given in [4]. A direct piecewise linear proof appeared recently in [2]; it is based on the existence of a "collar" for the boundary of a combinatorial manifold with boundary. Our proof is similar to the proof in [2]. We proceed by induction on dimensions, proving simultaneously the existence of a collar for the boundary of a combinatorial manifold with boundary and the homogeneity theorem. From [2] we adopted an argument which eliminates a certain combinatorial technique applied in [1] and involving induction on the length of stellar subdivisions.

The results of [1] were previously interpreted in piecewise linear topology by use of a theorem in [3] stating that piecewise linearly homeomorphic simplicial complexes have subdivisions which are combinatorially equivalent in the sense of [1].

The thesis is divided into three parts. The first gives definitions and basic properties relating to simplicial
complexes. The second concerns combinatorial manifolds, and in the third we present our proof of the piecewise linear homogeneity of spheres.
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1. Simplicial Complexes in $\mathbb{R}^n$

**Definition 1.1** Simplex in $\mathbb{R}^n$; $\Delta^k$:

$\Delta^k = [v_0 \ldots v_k]$ the barycentric hull of linearly independent points $v_i \in \mathbb{R}^n$ called vertices of $\Delta^k$.

i.e. $\Delta^k = \{ x : x \in \mathbb{R}^n, x = \sum_{i=0}^{k} \xi_i v_i, \xi_i \geq 0 \text{ and } \sum_{i=0}^{k} \xi_i = 1 \}$

Dimension of $\Delta^k$; $\dim(\Delta^k)$:

$\dim(\Delta^k) = k - 1 = \text{one less than the number of vertices of } \Delta^k$.

Face of $\Delta^k$; $\Delta^m < \Delta^k$:

If $\{ v'_0 \ldots v'_m \} \subseteq \{ v_0 \ldots v_k \}$, then $\Delta^m = [v'_0 \ldots v'_m]$ is a face of $\Delta^k$ of appropriate dimension.

Boundary of $\Delta^k$; $\partial \Delta^k$:

$\partial \Delta^k = \{ \Delta^m : \Delta^m < \Delta^k, \Delta^m \neq \Delta^k \}$

= the set of proper faces of $\Delta^k$.

Interior of $\Delta^k$, $\text{int}(\Delta^k)$:

$\text{Int}(\Delta^k) = \{ x : x \in \mathbb{R}^n, x = \sum_{i=0}^{k} \xi_i v_i, \xi_i > 0 \text{ and } \sum_{i=0}^{k} \xi_i = 1 \}$

Barycentre of $\Delta^k$, $b\Delta^k$:

$b\Delta^k = \frac{1}{k+1} \{ v_0 + \ldots + v_k \}$.
Definition 1.2  Simplicial Complex in \( \mathbb{R}^n \); \( K \):
A countable set of simplices \( \{ \Delta_i \}_{i \in I}, \Delta_i \in \mathbb{R}^n \) \( \in \)

(1) \( \Delta \in K \) and \( \Delta_2 \subset \Delta_1 \Rightarrow \Delta_2 \in K \).

(ii) \( \Delta_1, \Delta_2 \in K \Rightarrow \Delta_1 \cap \Delta_2 \subset \Delta_1 \) and \( \Delta_1 \cap \Delta_2 \subset \Delta_2 \).

Dimension of \( K \); \( \dim(K) \):
\[ \dim(K) = \sup \{ \dim(\Delta) : \Delta \in K \} \]

Realization of \( K \); \( |K| \):
\[ |K| = \{ x : x \in \Delta, \Delta \in K \} \]
= the point set spanned by all simplices of \( K \)
\( \subseteq \mathbb{R}^n \)

Note: (1) If \( \Delta \) is a simplex, the simplicial complex consisting of all faces of \( \Delta \) including \( \Delta \) itself is a complex and will also be denoted by \( \Delta \).

The boundary of a simplex is always a complex.
(2) \( |\Delta| = \Delta \) by definition.

Subcomplex \( L \) of \( K \); \( L \subseteq K \): \( L \subseteq K \) is a subcomplex of \( K \) if it is also a simplicial complex.

Full Subcomplex \( L \) of \( K \):
L ⊆ K is a full subcomplex of K if:

(i) L is a subcomplex of K

(ii) For Δ = [v₀, ..., vₖ] ∈ K, if {v₀, ..., vₖ} ⊆ L

then Δ ∈ L

Principal Simplex of K:

Δ ∈ K is a principal simplex of K if Δ is not a face of a higher dimensional simplex of K.

Note: Δ ∈ K is principal ⇐⇒ K-(Δ) is a complex.

Definition 1.3 Linear Map, f : Δ → ℝⁿ:

Δ = [v₀, ..., vₖ]

f is linear if f(x) = f(Σᵢ₌₀^k gᵢv₁) = Σᵢ₌₀^k gᵢf(v₁) ∀ x ∈ Δ

Simplicial Map, f : K → L:

A map f : |K| → |L|, K,L simplicial complexes ⇒

(1) ∀Δ ∈ K, f(Δ) is a simplex of L

(2) ∀Δ ∈ K, f|Δ : Δ → |L| ⊆ ℝⁿ is linear.

Note: A simplicial map f : K → L maps vertices of K to vertices of L.
Simplicial Isomorphism, \( f : K \to L \):

\( f : K \to L \) is a simplicial isomorphism if it is a simplicial map which is 1-1 and onto on vertices.

Note: If \( \exists \) a simplicial isomorphism \( f : K \to L \) we say that \( K \) and \( L \) are simplicially isomorphic and write \( K \approx L \).

**Definition 1.4**

Convex cell in \( \mathbb{R}^n \); \( C \):

\[
C = \{ x : x \in \mathbb{R}^n , x \text{ satisfies } L_i(x) \geq b_i , i \in I \}
\]

\[
l_j(x) = d_j , j \in J
\]

where \( L_i(x) = \sum_{k=0}^{n} a_{ik} x_k \geq b_i \) is a linear inequality \( \forall i \in I \)

\( l_j(x) = \sum_{k=0}^{n} c_{jk} x_k = d_j \) is a linear equation \( \forall j \in J \).

Note: (1) \( C \) is convex, i.e. \( x, y \in C \Rightarrow \overline{xy} = df \{ (1-t)x + ty : 0 \leq t \leq 1 \} \subseteq C \).

(2) A simplex in \( \mathbb{R}^n \) is a convex cell in \( \mathbb{R}^n \).

**Definition 1.5**

Operations on Complexes:

\( K \) a simplicial complex

\( L_1, L_2 \) subcomplexes of \( K \).
Sum; \( L_1 + L_2 \):

\[ L_1 + L_2 = \{ \Delta : \Delta \in L_1 \text{ or } \Delta \in L_2 \} \]

Intersection; \( L_1 \cap L_2 \):

\[ L_1 \cap L_2 = \{ \Delta : \Delta \in L_1 \text{ and } \Delta \in L_2 \} \]

Note: More generally, if \( L_1 \) and \( L_2 \) are simplicial complexes in \( \mathbb{R}^n \) \( \{ \Delta : \Delta \in L_1 \text{ or } \Delta \in L_2 \} \) is a simplicial complex then \( L_1 + L_2 \) and \( L_1 \cap L_2 \) are defined as above.

Join of Simplices, \( \Delta^k \times \Delta^l \):

\[ \Delta^k = [v_0 \ldots v_k], \quad \Delta^l = [v'_0 \ldots v'_l] \]

where \( v_0, \ldots, v_k, v'_0, \ldots, v'_l \) are linearly independent then \( \Delta^k \times \Delta^l = [v_0 \ldots v_k v'_0 \ldots v'_l] \)

= a simplex of dimension \((k+1+l)\).

Note: A simplex may be considered as the join of any two of its faces provided these faces have between them as vertices all the vertices of the simplex.
$K$ is the join of subcomplexes $L_1$ and $L_2$; $K = L_1 \ast L_2$.

$\Delta \in K \iff \Delta = \Delta_1 \ast \Delta_2$, $\Delta_1 \in L_1$, $\Delta_2 \in L_2$

Join of Complexes $K$ and $L$; $K \ast L$:

$$|K| \subseteq \mathbb{R}^{n_1}, \ |L| \subseteq \mathbb{R}^{n_2}$$

consider $|K|, \ |L| \subseteq \mathbb{R}^{n_1+n_2+1}$ where $|K|$ and $|L|$ are contained in disjoint planes of $\mathbb{R}^{n_1+n_2+1}$ which together span $\mathbb{R}^{n_1+n_2+1}$.

$$K \ast L = \{ \Delta_1 \ast \Delta_2 : \Delta_1 \in K, \Delta_2 \in L \}$$

$$|K \ast L| \subseteq \mathbb{R}^{n_1+n_2+1}$$

note that this join is defined only up to simplicial isomorphism.

The following elementary properties hold:

(1) $L_1 \ast (L_2 + L_3) \simeq L_1 \ast L_2 + L_1 \ast L_3$ where $L_1$, $L_2$, $L_3$ are subcomplexes of some complex $K$ and the joins are defined.

(2) $\partial (\Delta_1 \ast \Delta_2) = \partial \Delta_1 \ast \Delta_2 + \Delta_1 \ast \partial \Delta_2$.

(3) If $f : K_1 \rightarrow K_2$ and $g : L_1 \rightarrow L_2$ are simplicial maps
then \( f^*g : K_1 \ast L_1 \to K_2 \ast L_2 \) is a well-defined simplicial map.

**Definition 1.6**  
\( K \) a simplicial complex, \( \Delta \in K \)

- **Star of \( \Delta \) in \( K \);**  
  \( \text{St} (\Delta, K) : \)  
  \( \text{St} (\Delta, K) = \{ \Delta' : \Delta' \prec \Delta'', \Delta'' \in K, \Delta \prec \Delta'' \} \).

- **Link of \( \Delta \) in \( K \);**  
  \( \text{Lk}(\Delta, K) : \)  
  \( \text{Lk}(\Delta, K) = \{ \Delta' : \Delta' \in \text{St}(\Delta, K), \Delta' \cap \Delta = \emptyset \} \).

- **Residue of \( \Delta \) in \( K \);**  
  \( \text{Res}(\Delta, K) : \)  
  \( \text{Res}(\Delta, K) = \{ \Delta' : \Delta' \in K \text{ and } \Delta \notin \Delta' \} \).

**Note:**  
(1) \( \Delta' \in \text{Lk}(\Delta, K) \iff \Delta \ast \Delta' \in K \).  
(2) \( \text{St}(\Delta, K) = \Delta \ast \text{Lk}(\Delta, K) \).  
(3) \( K = \text{St}(\Delta, K) \ast \text{Res}(\Delta, K) \).

**Elementary Properties:**  
(1) \( v \in K \Rightarrow \text{Lk}(v, K \ast L) = \text{Lk}(v, K) \ast L \).  
(2) \( \text{Lk}(\Delta_1 \ast \Delta_2, K) = \text{Lk}(\Delta_1, \text{Lk}(\Delta_2, K)) \).  

\( L \subseteq K \) a subcomplex of \( K \).
Neighbourhood of \( L \) in \( K \); \( N(L,K) \):

\[
N(L,K) = \{ \Delta : \Delta \in K, \Delta < \Delta', \Delta' \cap L \neq \emptyset \}.
\]

\( \hat{N}(L,K) = \{ \Delta : \Delta \in N(L,K), \Delta \cap L = \emptyset \} \).

Note: \( L \) full in \( K \) \( \iff \Delta \in N(L,K) \iff \Delta = \Delta_1 \cup \Delta_2, \Delta_1 \in L \)
\( \Delta_2 \in \hat{N}(L,K) \).

Complement of \( L \) in \( K \); \( C(L,K) \):

\( K \) a simplicial complex, \( L \subseteq K \) a subcomplex

\[
C(L,K) = \{ \Delta : \Delta \in K, \Delta < \Delta', \Delta' \in K, \Delta' \not\subseteq L \}.
\]

Note: (1) If \( K \) is a simplicial complex and \( v \in K \) a vertex

\( C(v,K) = \text{Res}(v,K) \).

(2) \(|C(L,K)| = \text{CL}(|K| - |L|)\)

where \( \text{CL}(X) = \) the topological closure of \( X \).

Definition 1.7

Subdivision of a simplicial complex \( K \); \( \alpha(K) \):

\( \alpha(K) \) is a subdivision of \( K \) if:

(i) \( \alpha(K) \) is a simplicial complex,

(ii) \( \Delta \in \alpha(K) \Rightarrow \exists \Delta' \in K : \Delta \subseteq \Delta' \).

(iii) \(|\alpha(K)| = |K|\).
Elementary Subdivision $\varepsilon(x,\Delta)(K)$:

$\Delta \in K, \ x \in \text{int} \ \Delta$

$\varepsilon(x,\Delta)(K) = x \cdot \partial \Delta \star \text{lk}(\Delta, K) + \text{Res}(\Delta, K)$

Stellar Subdivision; $\sigma(K)$:
A composition of elementary subdivisions

Barycentric Subdivision of $K$ relative to a subcomplex $L$ of $K$; $B_L(K)$:

$B_L(K) = \{ \Delta' : \Delta' = \Delta_L \star b\Delta_1 \star \cdots \star b\Delta_k, \ \Delta_L < \Delta_1 < \cdots < \Delta_k \in K \}$

$\Delta_L \in L, \ \Delta_1 \notin L$

Note:

(1) If $\alpha(K)$ is a subdivision of $K$ and $L$ is a subcomplex of $K$ then $\alpha$ defines a subdivision $\alpha(L)$ of $L$.

(2) If $\alpha(K)$ is a subdivision of $K$ and $\gamma(L)$ is a subdivision of $L$, then $\alpha(K) \star \gamma(L)$ defines a subdivision of $K \star L$.

(3) If $L, K_0$ are subcomplexes of $K$, $L \subset K_0 \subset K$, then $B_L(K)$ defines the barycentric subdivision of $K_0$ relative to $L$ on $K_0$. 
Lemma 1.1  
\[ \Delta \text{ a simplex} \]
\[ C \text{ a convex cell } \ni C \cap \Delta \subseteq \partial \Delta \]
Then \[ C \cap \Delta \subseteq \Delta_1, \Delta_1 \in \partial \Delta \]

Proof: Choose \( \Delta_1 \in \partial \Delta \ni C \cap \text{int} \Delta_1 \neq \emptyset \) and \( \Delta_1 \) is of maximum dimension in \( \partial \Delta \) with this property.

Suppose that \( C \cap \Delta \notin \Delta_1 \):

i.e. \[ \exists x \in C \cap \Delta \ni x \notin \Delta_1 \]

now \( x \in \text{int} \Delta_2, \Delta_2 \in \partial \Delta \)

consider the smallest simplex \( \Delta_3 \ni \Delta_1 \subset \Delta_3 \) and \( \Delta_2 \subset \Delta_3 \)

choose any \( y \in C \cap \text{int} \Delta_1 \)

\[ xy \in C \quad \text{---} \quad C \text{ is convex} \]
\[ xy \in \Delta_3 \quad \text{---} \quad \Delta_3 \text{ is convex} \]

and minimality of \( \Delta_3 \Rightarrow xy \cap \text{int} \Delta_3 \neq \emptyset \)

\[ \Rightarrow C \cap \text{int} \Delta_3 \neq \emptyset \]
contradicting the minimality of \( \Delta_1 \)

\[ \therefore C \cap \Delta \subseteq \Delta_1 \]

Theorem 1.1  
\[ K, L \text{ finite simplicial complexes} \]
\[ |L| \leq |K| \]
Then \( \exists \) a stellar subdivision \( \sigma(K) \) and a subdivision \( \alpha(L) \ni \alpha(L) \subseteq \sigma(K) \).
Proof: By induction on $N$, the number of simplices of $L$.

$N = 1$: $L$ is a vertex, say $v$.

$|L| \leq |K| \Rightarrow v \in \text{int } \Delta$, $\Delta \in K$

$\therefore \mathcal{E}(v, \Delta)(K)$ is the required subdivision of $K$. Assume the hypothesis to be true if $L$ has $\leq (N-1)$ simplices.

$N$: Consider $\Delta$, a principal simplex of $L$. $L\{\Delta\}$ is a simplicial complex with $\leq N$ simplices and $|L\{\Delta\}| \leq |K|$. Thus by the induction assumption $\exists$ a stellar subdivision $\sigma_1(K)$ and a subdivision $\alpha(L\{\Delta\}) \ni \alpha(L\{\Delta\}) \subseteq \sigma_1(K)$.

We order the simplices $\tilde{\Delta} \in \sigma_1(K)$ with $\text{int } \tilde{\Delta} \cap \text{int } \Delta \neq \emptyset$ by decreasing dimensions:

$$\tilde{\Delta}_1, \tilde{\Delta}_2 \ldots \tilde{\Delta}_m, \quad \dim \tilde{\Delta}_1 \geq \dim \tilde{\Delta}_2 \geq \ldots \geq \dim \tilde{\Delta}_m$$

Choose $x_i \in \text{int } \tilde{\Delta}_i \cap \text{int } \Delta$, $i = 1, 2, \ldots, m$

set $\sigma_2(K) = \mathcal{E}(x_1, \tilde{\Delta}_1)(\sigma_1(K))$

$\therefore \sigma_1(K) = \mathcal{E}(x_{i-1}, \tilde{\Delta}_{i-1})(\sigma_{i-1}(K))$
and define \( \sigma(K) = \sigma_{m+1}(K) \), the resultant of these elementary subdivisions of \( \sigma_1(K) \).

set \( \alpha(L) = \{ \Delta' : \Delta' \in \sigma(K), \text{ int } \Delta' \cap |L| \neq \emptyset \} \)

Clearly \( \sigma(K) \) is a stellar subdivision of \( K \). \( \alpha(L) \subseteq \sigma(K) \) by its definition.

\[ |\alpha(L)| = |L|: \quad (i) \quad |L| \subseteq |\alpha(L)|; \]

say \( x \in |L| \Rightarrow x \in \text{ int } \Delta, \Delta \in L \)

if \( \tilde{\Delta} \neq \Delta \Rightarrow x \in \text{ int } \tilde{\Delta}, \tilde{\Delta} \in \alpha(L-\{L\}) \)

\[ \Rightarrow \tilde{\Delta} \in \sigma(K) - \tilde{\Delta} \quad \text{unaffected by} \]

\( \varepsilon(x_1, \tilde{\Delta}_1)(\sigma_1(K)), i=1 \ldots m \)

\[ \Rightarrow x \in |\alpha(L)| - \text{ int } \tilde{\Delta} \cap |L| \neq \emptyset. \]

if \( \tilde{\Delta} = \Delta \), now

\[ x \in |\sigma(K)| \Rightarrow \exists \tilde{\Delta} \in \sigma(K) \ni x \in \text{ int } \tilde{\Delta} \]

\[ \Rightarrow x \in \text{ int } \tilde{\Delta} \cap \text{ int } \Delta \Rightarrow \]

\[ \text{ int } \tilde{\Delta} \cap |L| \neq \emptyset \]

\[ \Rightarrow \tilde{\Delta} \in \alpha(L) \]

\[ \Rightarrow x \in |\alpha(L)|. \]

(\( ii \)) \( |\alpha(L)| \subseteq |L|: \)

say \( x \in |\alpha(L)| \Rightarrow x \in \text{ int } \Delta', \Delta' \in \alpha(L) \)

\( \Delta' \in \alpha(L) \Rightarrow \Delta' \in \sigma(K) \) and

\[ \text{ int } \Delta' \cap |L| \neq \emptyset. \]
(a) if int $\Delta' \cap \text{int } \Delta = \emptyset$

$\Rightarrow$ int $\Delta' \cap |L-(\Delta)| \neq \emptyset$ — int $\Delta' \cap |L|$ \neq \emptyset

Now $\Delta' \in \sigma(K)$ $\Rightarrow \exists$ a smallest

$\Delta_1 \in \sigma_1(K) \ni |\Delta'| \subseteq |\Delta_1|$

$\Rightarrow$ int $\Delta_1 \cap |L-(\Delta)| \neq \emptyset$

$\Rightarrow \Delta_1 \in \alpha(L-(\Delta))$

$\alpha(L-(\Delta)) \subseteq \sigma_1(K)$

$\Rightarrow \Delta_1 \subseteq |L|$

$\Rightarrow \Delta' \subseteq |L|$

$\Rightarrow x \in |L|$

(b) if int $\Delta' \cap \text{int } \Delta \neq \emptyset$

as $\sigma(K)$ is a subdivision of

$\sigma_1(K) \exists \Delta_1 \in \sigma_1(K) \ni \Delta' \subseteq \Delta_1$ and

int $\Delta_1 \cap \text{int } \Delta \neq \emptyset$

thus $\Delta_1 = \bar{\Delta}_{1_1}$ and

$\sigma_{1_1+1}(K) = \varepsilon(x_{1_1}, \bar{\Delta}_{1_1})(\sigma_{1_1}(K))$

i.e. $\Delta' \subseteq x_{1_1} \ast \Delta_2$, $\Delta_2 \in \partial \bar{\Delta}_{1_1}$

if $\Delta_2 \subseteq \Delta$ then
\[
x_{1 \cdot} \Delta_2 \subseteq \Delta \Rightarrow \Delta' \subseteq \Delta \Rightarrow x \in \Delta \Rightarrow x \in |L|
\]

if \( \Delta_2 \not\subseteq \Delta \):

(1) \( \Delta_2 \cap \text{int} \Delta = \emptyset \Rightarrow x_{1 \cdot} \Delta_2 \cap \text{int} \Delta = x_{1 \cdot} \Delta_1 \)

\[
\therefore \text{int} \Delta' \cap \text{int} \Delta \neq \emptyset \Rightarrow \Delta' = x_{1 \cdot} \Delta_1 \subseteq \Delta
\]

\[
\Rightarrow x = \Delta' = x_{1 \cdot} \Delta_1 \in |L|
\]

(2) \( \text{int} \Delta_2 \cap \text{int} \Delta \neq \emptyset \Rightarrow \Delta_2 = \tilde{\Delta}_{1 \cdot} \Delta_1 \) and

\[
\sigma_{1 \cdot}^2 \alpha(K) = \varepsilon(x_{1 \cdot}, \tilde{\Delta}_{1 \cdot} \Delta_1) \sigma_{1 \cdot}^2(K)
\]

i.e. \( \Delta' \subseteq x_{1 \cdot} \Delta_1 \Delta_3 \), \( \Delta_3 \in \tilde{\Delta}_{1 \cdot} \Delta_1 \)

if \( \Delta_3 \subseteq \Delta \) then \( x \in |L| \) as above

if \( \Delta_3 \not\subseteq \Delta \) continue as in case \( \Delta_2 \not\subseteq \Delta \)

(3) \( \Delta_2 \cap \text{int} \Delta \subseteq \tilde{\Delta}_{1 \cdot} \Delta_2 \Rightarrow \Delta_2 \cap \text{int} \Delta \subseteq \Delta_3 \), \( \Delta_3 \in \tilde{\Delta}_{1 \cdot} \Delta_2 \) by

lemma 1.2

\[
\Rightarrow x_{1 \cdot} \Delta_2 \cap \text{int} \Delta \subseteq x_{1 \cdot} \Delta_3
\]

\[
\Rightarrow \text{int} \Delta' \cap x_{1 \cdot} \Delta_3 \neq \emptyset
\]

\[
\Rightarrow \Delta' \subseteq x_{1 \cdot} \Delta_3
\]
again, if $\Delta_3 \subseteq \Delta$ then $x \in L$

if $\Delta_3 \not\subseteq \Delta$ continue as in case $\Delta_2 \not\subseteq \Delta$

Continuing this inspection of the subdivision of $\Delta_1$ in forming $\sigma(K)$ from $\sigma_1(K)$ we will eventually arrive at

$$\Delta' \subseteq x_{i_1} \times x_{i_2} \times \cdots \times x_{i_p} \times \Delta_s, \Delta_s \subseteq \Delta$$

or $$\Delta' \subseteq x_{i_1} \times x_{i_2} \times \cdots \times x_{i_s} \times \Delta_s, \Delta_s \cap \text{int} \Delta = \emptyset$$

which, in either case $\Rightarrow \Delta' \subseteq \Delta \Rightarrow x \in \Delta$

$$\Rightarrow x \in |L|$$

**Corollary 1.1**

K a finite Simplicial Complex

$\alpha_1(K), \alpha_2(K)$ subdivisions of $K$.

Then $\exists$ a common subdivision $\alpha(K)$ of $\alpha_1(K)$ and $\alpha_2(K)$.

**Proof:** Set $\alpha_1(K) = K$ and $\alpha_2(K) = L$ in Thm. 1.1.

**Corollary 1.2**

K a finite Simplicial Complex

$\sigma(K)$ a subdivision of $K$.

Then $\exists$ a stellar subdivision $\sigma(K)$ of $K$ which is a subdivision of $\alpha(K)$. 
Proof: Set $K = K$, $\alpha(K) = L$ in Thm. 1.1.

Definition 1.8 Piecewise linear Homeomorphism:

Let $K_1, K_2$ simplicial complexes

A piecewise linear homeomorphism, denoted $PL$ homeomorphism, $f : K_1 \to K_2$ is a map

$f : |K_1| \to |K_2| \ni \exists$ subdivisions $\alpha_1(K_1), \alpha_2(K_2)$ of $K_1$ and $K_2$ for which $f : \alpha_1(K_1) \to \alpha_2(K_2)$ is a simplicial isomorphism. $K_1$ and $K_2$ are said to be PL homeomorphic.

Note: (1) If $f : K_1 \to K_2$ and $g : K_2 \to K_3$ are PL homeomorphisms, then $g \circ f : K_1 \to K_3$ is a PL homeomorphism by Corol. 1.1.

(2) If $f : K_1 \to K_2$ and $g : L_1 \to L_2$ are PL homeomorphisms, then $g \circ f : K_1 \times L_1 \to K_2 \times L_2$ is well defined and a P-L homeomorphism.

Definition 1.9 The Cartesian Product of a Complex with the Unit interval:

Let $I = [0,1]$ and $K$ be a simplicial complex. If $|K| \subseteq \mathbb{R}^n$ then $K \times I$, $|K \times I| \subseteq \mathbb{R}^n \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ is given by:
\[ K \times I = \{ \Delta : \Delta = (\Delta_0, 0) \star (b\Delta_1, 1) \star \cdots \star (b\Delta_k, 1), \Delta_0 < \Delta_1 < \cdots < \Delta_k \in K \}. \]

Set \( K \times 0 = \{ \Delta : \Delta = (\Delta_0, 0), \Delta_0 \in K \} \subseteq K \times I. \)

\[ K \times 1 = \{ \Delta : \Delta = (b\Delta_1, 1) \star \cdots \star (b\Delta_k, 1), \Delta_1 < \cdots < \Delta_k \in K \} \subseteq K \times I. \]

Clearly \( K \times 0 \approx K \) and \( K \times 1 \approx B(K) \).

Note: \(|K \times I| = |K| \times I|.

**Theorem 1.2**

\( K \) a simplicial complex

\( \alpha_1(K), \alpha_2(K) \) subdivisions of \( K \)

\( v \in \alpha_1(K) \) and \( v \in \alpha_2(K) \).

Then \( Lk(v, \alpha_1(K)) \) is PL homeomorphic to \( Lk(v, \alpha_2(K)) \).

**Proof:** Since by Corol. 1.1 we may take a common subdivision \( \alpha(K) \) of \( \alpha_1(K) \) and \( \alpha_2(K) \) it suffices to prove the theorem for \( K = \alpha_1(K) \) and \( \alpha(K) = \alpha_2(K) \).

We produce a subdivision \( \gamma(Lk(v,K)) \ni \gamma(Lk(v,K)) = Lk(v,\alpha(K)) \) by "radial projection" from \( v \).

Define \( r : |Lk(v,\alpha(K))| \to |Lk(v,K)| \) on vertices.
of \( \text{Lk}(v,\alpha(K)) \) by, \( v_1 \in \text{Lk}(v,\alpha(K)) \).

Choose a simplex \( \Delta^k \) of \( \text{St}(v,K) \) s.t. \( v_1 \in \Delta^k \)
\[
\Delta^k = v \ast \Delta^{k-1}, \Delta^{k-1} \in \text{Lk}(v,K)
\]
Set \( r(v_1) = w_1 \in \Delta^{k-1} \) such that \( \overline{vv_1} \subseteq \overline{vw_1} \)
extend \( r \) to \( |\text{Lk}(v,\alpha(K))| \) by defining, for \( \Delta \in \text{Lk}(v,\alpha(K)) \)
\[
\Delta = [v_o \ldots v_k], \Delta \subseteq \text{some } \Delta^k, \Delta^k \in \text{St}(v,K)
\]
\[
r(\Delta) = [r(v_o) \ldots r(v_k)].
\]

\( v_o \ldots v_k \) linearly independent \( \Rightarrow \) \( r(v_o) \ldots r(v_k) \) linearly independent
\[
\Rightarrow r(\Delta) \text{ is a simplex}
\]
and \( r(\Delta) \subseteq |\text{Lk}(v,K)| \).

Set \( \gamma(\text{Lk}(v,K)) = \{r(\Delta) : \Delta \in \text{Lk}(v,\alpha(K))\} \).

\( \gamma(\text{Lk}(v,K)) \) is a subdivision of \( \text{Lk}(v,K) \):

(a) \( \{r(\Delta) : \Delta \in \text{Lk}(v,\alpha(K))\} \) is a simp. comp. since
\[
\Delta_1, \Delta_2 \in \text{Lk}(v,\alpha(K)) \Rightarrow \Delta_1 \cap \Delta_2 \in \text{Lk}(v,\alpha(K))
\]
thus \( r(\Delta_1) \cap r(\Delta_2) = r(\Delta_1 \cap \Delta_2) \in \{r(\Delta) : \Delta \in \text{Lk}(v,\alpha(K))\} \).
(b) \(|\text{Lk}(v, K)| = |\{r(\Delta) : \Delta \in \text{Lk}(v, \alpha(K))\}| :\)

by defn \(|\{r(\Delta) : \Delta \in \text{Lk}(v, \alpha(K))\}| \leq |\text{Lk}(v, K)|\)
say \(x \in |\text{Lk}(v, K)| \Rightarrow x \in |\Delta_{k-1}^k|, \Delta_{k}^k = v_x \Delta_{k-1}^k,\)
\(\Delta_{k}^k \in \text{St}(v, K)\)
thus \(x_\bot = v \cdot x \cap |\text{Lk}(v, \alpha(K))|\) has \(r(x_\bot) = x\)
\(\Rightarrow x \in |r(\Delta_{\bot})|, x_\bot \in \text{int} \Delta, \Delta_{\bot} \in \text{Lk}(v, \alpha(K))\)

By construction \(\text{Lk}(v, \alpha(K)) \approx \gamma(\text{Lk}(v, K))\) through the radial projection map \(r\).

**Corollary 1.3**

\(K\) a simplicial complex
\(\alpha(K)\) a subdivision of \(K\)
\(v \in K \ni \text{Lk}(v, \alpha(K)) \cap \alpha(\text{Lk}(v, K)) = \emptyset\)
Then \(C(\text{St}(v, \alpha(K))), \alpha(\text{St}(v, K))\) is PL homeomorphic to \(\text{Lk}(v, K) \times I\).
2. Combinatorial Manifolds

**Definition 2.1** Combinatorial Cell and Sphere:

$n^c$ is a combinatorial cell of dimension $n$ if $n^c$ is PL homeomorphic to $\Delta^n$.

$\gamma^n$ is a combinatorial sphere of dimension $n$ if $\gamma^n$ is PL homeomorphic to $\Delta^{n+1}$.

**Definition 2.2** Combinatorial Manifold of dimension $n$:

$\mathcal{M}^n$ is a combinatorial manifold of dimension $n$ if \( \forall \) vertex $v$ of $\mathcal{M}^n$, $\text{Lk}(v, \mathcal{M}^n)$ is a combinatorial cell or sphere of dimension $(n-1)$.

**Theorem 2.1** $n^c$ a comb. cell of dim $n$, $m^c$ a comb. cell of dim $m$, $\gamma^n$ a comb. sphere of dim $n$, $\gamma^m$ a comb. sphere of dim $m$. Then,

(a) $n^c \ast m^c$ is a comb. cell of dim $(n+m+1)$

(b) $n^c \ast \gamma^m$ is a comb. cell of dim $(n+m+1)$

(c) $\gamma^n \ast \gamma^m$ is a comb. sphere of dim $(n+m+1)$

**Proof:** (a) We have PL homeomorphisms $f: n^c \rightarrow \Delta^n$

$g: m^c \rightarrow \Delta^m$
\[ \exists \text{ PL homeomorphism} \]
\[ f \ast g : \Delta^n \ast \Delta^m \rightarrow \Delta^n \ast \Delta^m = \Delta^{n+m+1} \]

(b) We have PL homeomorphisms \( f : \Delta^n \rightarrow \Delta^n \)
\( g : \Delta^m \rightarrow \Delta^{m+1} \)

thus \( f \ast g : \Delta^n \ast \Delta^m \rightarrow \Delta^n \ast \Delta^{m+1} \) is a PL homeomorphism. Now \( \Delta^n \ast \Delta^{m+1} = \varepsilon(x_1, \Delta^{m+1}) \), let
\( h : \Delta^n \ast \Delta^{m+1} \rightarrow \Delta^{m+1} \) be the corresponding PL homeomorphism.

Then \( id_{\Delta^n \ast h} : \Delta^{n-1} \ast (\Delta^n \ast \Delta^{m+1}) = \Delta^n \ast \Delta^{m+1} \rightarrow \Delta^{n-1} \ast \Delta^{m+1} = \Delta^{n+m+1} \)
is a PL homeomorphism

\[ (id_{\Delta^n \ast h}) \ast (f \ast g) : \Delta^n \ast \Delta^m \rightarrow \Delta^{n+m+1} \] is a PL homeomorphism.

(c) Note that \( \Delta^n \ast \Delta^{k+1} = \varepsilon(x_1, \Delta^{k+1}) \).

By induction on \( n \):

\( n = 0 \): We have PL homeomorphisms \( f : \Delta^0 \rightarrow \Delta^1 \)
and \( g : \Delta^m \rightarrow \Delta^{m+1} \)

\[ \therefore f \ast g : \Delta^0 \ast \Delta^m \rightarrow \Delta^1 \ast \Delta^{m+1} \] is a PL homeomorphism.

As noted \( \exists \) a PL homeomorphism \( h : \Delta^1 \ast \Delta^{m+1} \rightarrow \Delta^{m+2} \)

\[ \therefore h \ast (f \ast g) : \Delta^0 \ast \Delta^m \rightarrow \Delta^{m+2} \] is a PL homeomorphism.

Assume the hypothesis in dimensions \( \leq n \)

\( n+1 \): We have PL homeomorphisms
\( f : \Delta^{n+1} \rightarrow \Delta^{n+2} \)
\( g : \Delta^m \rightarrow \Delta^{m+1} \).

Consider the PL homeomorphism
\[ f \circ g : \gamma^{n+1} \ast \gamma^m \rightarrow \partial \Delta^{n+2} \ast \partial \Delta^{m+1} \]

Now \[ h+1 : \partial \Delta^{n+2} \ast \partial \Delta^{m+1} \rightarrow (\partial \Delta^1 \ast \partial \Delta^{n+1}) \ast \partial \Delta^{m+1} \]

\[ = \partial \Delta^1 \ast (\partial \Delta^{n+1} \ast \partial \Delta^{m+1}) \]

is a PL homeomorphism.

By the induction hypothesis, \( \partial \Delta^{n+1} \ast \partial \Delta^{m+1} \)

is PL homeomorphic to \( \partial \Delta^{n+m+1} \),

thus \( \partial \Delta^1 \ast (\partial \Delta^{n+1} \ast \partial \Delta^{m+1}) \) is PL homeomorphic to \( \partial \Delta^{n+m+2} \) by again applying the above note.

**Theorem 2.2**

\( \mathcal{M}^n \) a comb. man. of dim \( n \).

\( \Delta^k \in \mathcal{M}^n \) a \( k \)-simplex

\( K \) a simp. comp., \( \alpha(K) \) a subdivision of \( K \)

\( \mathcal{S}^n \) a comb. \( n \)-cell, \( \gamma^n \) a comb. \( n \)-sphere.

Then

(i) \( \text{lk}(\Delta^k, \mathcal{M}^n) \) is a comb. \((n-k-1)\) cell or sphere.

(ii) \( K \) is a comb. man. of dim \( n \) \iff \( \alpha(K) \) is a comb. man. of dim \( n \).

(iii) \( \mathcal{S}^n \) and \( \gamma^n \) are comb. mans. of dim \( n \).

**Proof:** By induction on \( n \):

\( n = 0 \) trivial

assume the theorem is true for dimensions

\( \leq (n-1) \).

\( n \) : (i) By induction on \( k = \text{dim } (\Delta^k) \).
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k=0 : \( \Delta^k = v \in \mathcal{U}^n \)  

\( \therefore \text{Lk}(\Delta^k, \mathcal{U}^n) \) is a comb \((n-1)\) cell or sphere since \( \mathcal{U}^n \) is a comb. man. of dim \( n \).

Assume (i) is true for \( \dim(\Delta) < k \),  
\( \Delta \in \mathcal{U}^n \).

\( k : \Delta^k = v \times \Delta^{k-1}, \dim \Delta^{k-1} = k-1 \)  

\( \therefore \text{Lk}(\Delta^k, \mathcal{U}^n) = \text{Lk}(v \times \Delta^{k-1}, \mathcal{U}^n) = \text{Lk}(v, \text{Lk}(\Delta^{k-1}, \mathcal{U}^n)) \)

but \( \text{Lk}(\Delta^{k-1}, \mathcal{U}^n) \) is a comb. \((n-k)\) cell or sphere by the induction hypothesis on \( k \)  

\( \Rightarrow \text{Lk}(\Delta^{k-1}, \mathcal{U}^n) \) is a comb. man. of \( \dim (n-k) \) by the induction hypothesis on \( n \) in dimension \((n-k) < n \)  

\( \Rightarrow \text{Lk}(\Delta^k, \mathcal{U}^n) = \text{Lk}(v, \text{Lk}(\Delta^{k-1}, \mathcal{U}^n)) \) is a comb. \((n-k-1)\) cell or sphere.

(ii) \( \Leftarrow \) Say \( \alpha(K) \) is a comb. man. of dim \( n \). Choose any \( v \in K \). Since \( v \in \alpha(K) \), \( \text{Lk}(v, \alpha(K)) \) is a comb. \((n-1)\) cell or sphere and \( \text{Lk}(v, \alpha(K)) \) is PL homeomorphic to \( \text{Lk}(v, K) \) — Thm. 1.2  

\( \therefore \text{Lk}(v, K) \) is a comb. \((n-1)\) cell or sphere.
Say $K$ is a comb. man. of dim $n$. Choose any $v \in \alpha(K)$. Now $v \in \text{int } \Delta$, $\Delta \in K$. Consider $e(v, \Delta)(K) = v \cdot \partial \Delta * Lk(\Delta, K) + \text{Res}(\Delta, K)$. $Lk(v, \alpha(K))$ is PL homeomorphic to $Lk(v, e(v, \Delta)(K))$ - Thm. 1.2 and $Lk(v, e(v, \Delta)(K)) = \partial \Delta * Lk(\Delta, K)$. But $\partial \Delta$ is a comb. sphere of dimension $(\text{dim } \Delta - 1)$ and $Lk(\Delta, K)$ is a comb. cell or sphere of dimension $(n - \text{dim } \Delta - 1)$ by (i) in dimension $n$ since $K$ is a comb. man. of dim $n$.

$\therefore Lk(v, e(v, \Delta)(K))$ is a comb. cell or sphere of dim $(n-1)$ — Thm. 2.1

$\Rightarrow Lk(v, \alpha(K))$ is a comb. $(n-1)$ cell or sphere.

(iii) $\Delta^n \cong \alpha_1(\Delta^n)$ and $\Delta^n$ is a comb. man. of dim $n$ — $\Rightarrow Lk(v, \Delta^n) = \Delta^{n-1}$

$\therefore \Delta^n$ is a comb. man. of dim $n$ — (ii) in dim $n$.

$\gamma^n \sim \alpha_2(\partial \Delta^{n+1})$ and $\partial \Delta^{n+1}$ is a comb. man. of dim $n$ — $\Rightarrow Lk(v, \partial \Delta^{n+1}) = \partial \Delta^n$

$\therefore \gamma^n$ is a comb. man. of dim $n$ — (ii) in dim $n$.

**Corollary 2.1**

$\mathcal{M}^n$ a comb. man. of dim $n$

$K$ a simp. comp. $\vDash K$ is PL homeomorphic to $\mathcal{M}^n$. Then $K$ is a comb. man. of dim $n$. 
Theorem 2.3 \( K, L \) simplicial complexes. Then \( K*L \) is a comb. man. \( \iff \) \( K \) and \( L \) are comb. cells or spheres.

Proof: \( \implies \) Thm. 2.1

\( \implies \) By induction on \( \dim K*L \), say \( n \):

\( n=1 \): Choose \( v \in K \), \( Lk(v,K*L) = Lk(v,K)*L \)

\( L \neq \emptyset \implies \dim K = 0 \implies Lk(v,K) = \emptyset \)

\( \therefore \) \( Lk(v,K*L) = L \)

\( = \) comb 0-cell or sphere.

Choose \( v \in L \) to show \( K \) is a comb. 0-cell or sphere.

Assume the theorem is true for dimensions \( \leq (n-1) \)

\( n \): Choose \( v \in K \) , \( Lk(v,K*L) = Lk(v,K)*L \)

but \( Lk(v,K*L) = \) comb \( (n-1) \) cell or sphere

\( = \) comb man. of dim \( (n-1) \)

\( \therefore \) Thm. 2.2 (iii)

\( \therefore \) by the ind. assumption \( Lk(v,K) \) and \( L \) are comb cells or spheres.

Choose \( v \in L \) to show \( K \) is a comb. cell or sphere.
Definition 2.3  
Boundary of a Combinatorial Manifold:
\[ \partial \mathcal{M}^n = \{ \Delta : \Delta \in \mathcal{M}^n \text{ and } \text{Lk}(\Delta, \mathcal{M}^n) \text{ is a combinatorial cell} \} \].

Theorem 2.4  
\[ \mathcal{M}^n \text{ a comb. man. of dim } n \]
\[ \alpha(\mathcal{M}^n) \text{ a subdivision of } \mathcal{M}^n \]
\[ \mathcal{S}^n \text{ a comb. n-cell, } \gamma^n \text{ a comb. n-sphere} \].

Then (i) \( \partial \mathcal{M}^n \) is a subcomplex of \( \mathcal{M}^n \)

(ii) \( \partial(\alpha(\mathcal{M}^n)) = \alpha(\partial \mathcal{M}^n) \)

(iii) \( \partial \mathcal{S}^n = \gamma^{n-1}, \ n \geq 1, \)
\[ \gamma^{n-1} \text{ a comb. (n-1) sphere} \]

(iv) \( \partial \gamma^n = \emptyset \).

Proof: By induction on \( n \):
\( n = 0 \) and \( 1 \) : trivial

Assume the theorem is true in dimensions \( \leq (n-1) \)

\( n \) :

(i) \( \partial \mathcal{M}^n \) is a subcomplex of \( \mathcal{M}^n \):

say \( \Delta \in \partial \mathcal{M}^n \) and \( \Delta' < \Delta \) i.e. \( \Delta = \Delta' \Delta'' \)

now \( \text{Lk}(\Delta, \mathcal{M}^n) = \text{Lk}(\Delta'', \text{Lk}(\Delta', \mathcal{M}^n)) \). If
\( \text{Lk}(\Delta', \mathcal{M}^n) \neq \text{a comb. cell} \)

\( \Rightarrow \text{Lk}(\Delta', \mathcal{M}^n) = \text{a comb sphere of dim } \leq (n-1) \)

\( \Rightarrow \text{Lk}(\Delta'', \text{Lk}(\Delta', \mathcal{M}^n)) = \text{a comb. sphere} \)

(iv) in dim \( \text{Lk}(\Delta', \mathcal{M}^n) \).
Contradicting \( \text{Lk}(\Delta, \mathcal{M}^n) = \text{a comb. cell} \)

\[ \Delta \in \partial \mathcal{M}^n \]

\[ \therefore \text{Lk}(\Delta', \mathcal{M}^n) = \text{a comb. cell} \]

\[ \Rightarrow \Delta' \in \partial \mathcal{M}^n \]

(11) \( \partial \alpha(\mathcal{M}^n) \subseteq \alpha(\partial \mathcal{M}^n) : \)

Say \( \Delta' \in \partial \alpha(\mathcal{M}^n) \) i.e. \( \text{Lk}(\bar{\Delta}', \alpha(\mathcal{M}^n)) = \text{a comb. cell} \)

Choose \( x \in \text{int} \Delta' \)

\[ \varepsilon(x, \Delta')(\alpha(\mathcal{M}^n)) = x \cdot \partial \Delta' \cdot \text{Lk}(\Delta', \alpha(\mathcal{M}^n)) + \text{Res}(\Delta', \alpha(\mathcal{M}^n)) \]

\[ \therefore \text{Lk}(x, \varepsilon(x, \Delta')(\alpha(\mathcal{M}^n))) = \partial \Delta' \cdot \text{Lk}(\Delta', \alpha(\mathcal{M}^n)) \]

\[ = \text{a comb. cell} \quad \text{--- Thm. 2.1} \]

Now \( \Delta' \in \alpha(\mathcal{M}^n) => \Delta' \subseteq \Delta, \Delta \in \mathcal{M}^n \ni \Delta' \cap \text{int} \Delta \neq \emptyset \)

\[ \therefore x \in \text{int} \Delta \]

\[ \varepsilon(x, \Delta)(\mathcal{M}^n) = x \cdot \partial \Delta \cdot \text{Lk}(\Delta, \mathcal{M}^n) + \text{Res}(\Delta, \mathcal{M}^n) \]

\[ \therefore \text{Lk}(x, \varepsilon(x, \Delta)(\mathcal{M}^n)) = \partial \Delta \cdot \text{Lk}(\Delta, \mathcal{M}^n) \]

but \( \text{Lk}(x, \varepsilon(x, \Delta)(\mathcal{M}^n)) \) is PL homeomorphic to \( \text{Lk}(x, \varepsilon(x, \Delta')(\alpha(\mathcal{M}^n))) \) --- Thm. 1.2

\[ \therefore \partial \Delta \cdot \text{Lk}(\Delta, \mathcal{M}^n) \text{ is a comb. cell} \]

and \( \partial \Delta \text{ a comb. sphere} => \text{Lk}(\Delta, \mathcal{M}^n) \text{ is a comb. cell} \quad \text{--- Thm. 2.1} \)

\[ \Rightarrow \Delta \in \partial \mathcal{M}^n \]

\[ \Rightarrow \Delta' \in \alpha(\partial \mathcal{M}^n) \]
\( \alpha(\partial \mathcal{U}^n) \subseteq \partial \alpha(\mathcal{U}^n) : \)

Say \( \Delta' \in \alpha(\partial \mathcal{U}^n) \Rightarrow \Delta' \subseteq \Delta \), \( \Delta \in \partial \mathcal{U}^n \) and \( \text{int } \Delta \cap \Delta' \neq \emptyset \), choose \( x \in \text{int } \Delta' \cap \text{int } \Delta \)

\( e(x, \Delta)(\mathcal{U}^n) = x \ast \partial \ast \text{Lk}(\Delta, \mathcal{U}^n) + \text{Res } (\Delta, \mathcal{U}^n) \)

\( \therefore \text{Lk}(x, e(x, \Delta)(\mathcal{U}^n)) = \partial \ast \text{Lk}(\Delta, \mathcal{U}^n) \)

is a comb. cell --- \( \Delta \in \partial \mathcal{U}^n \)

and Thm. 2.1

Also \( e(x, \Delta')(\alpha(\mathcal{U}^n)) = x \ast \partial \ast \text{Lk}(\Delta', \alpha(\mathcal{U}^n)) + \text{Res } (\Delta', \alpha(\mathcal{U}^n)) \)

\( \therefore \text{Lk}(x, e(x, \Delta')(\alpha(\mathcal{U}^n))) = \partial \ast \text{Lk}(\Delta', \alpha(\mathcal{U}^n)) \)

but \( \text{Lk}(x, e(x, \Delta')(\alpha(\mathcal{U}^n))) \) is PL homeomorphic to \( \text{Lk}(x, e(x, \Delta)(\mathcal{U}^n)) \) — Thm. 1.2

\( \therefore \partial \ast \text{Lk}(\Delta', \alpha(\mathcal{U}^n)) \) is a comb. cell.

and \( \partial \Delta' \) a comb. sphere \( \Rightarrow \text{Lk}(\Delta', K(\mathcal{U}^n)) \) is a comb. cell --- Thm. 2.1

\( \Rightarrow \Delta' \in \partial \alpha(\mathcal{U}^n) \)

(iii) \( \partial \mathcal{A}^n = \gamma^{n-1} : \mathcal{A}^n \) PL homeomorphic to \( \Delta^n \Rightarrow \exists \text{ subdivisions } \alpha(\mathcal{A}^n) \approx \gamma(\Delta^n) \)

\( \therefore \alpha(\partial \mathcal{A}^n) = \partial(\alpha(\mathcal{A}^n)) \approx \partial(\gamma(\Delta^n)) = \gamma(\partial \Delta^n) \)

\( \Rightarrow \partial \mathcal{A}^n \) is a comb. sphere of dim \((n-1)\)

(iv) \( \partial \gamma^n = \emptyset : \gamma^n \) PL homeomorphic to \( \partial \Delta^{n+1} \Rightarrow \)
∃ subdivisions \( \gamma^n \Rightarrow \gamma(\Delta^{n+1}) \)

\[ \Rightarrow \gamma(\Delta^{n+1}) = \gamma(\Delta^{n+1}) = \gamma(\Delta^{n+1}) \]

\[ \Rightarrow \gamma^n = \emptyset \]

**Corollary 2.2**

\( \mathcal{M}^n \) a comb. man. of dim \( n \)

\[ \partial \mathcal{M}^n = \emptyset \iff \forall \text{ vertex } v \in \mathcal{M}^n, \]

\( \text{Lk}(v, \mathcal{M}^n) \) is a comb. sphere.

**Proof:**

\( \Rightarrow \) by defn of \( \partial \mathcal{M}^n \)

\( \Leftarrow \) by Thm. 2.4 (iii) and (iv) a combinatorial cell cannot be a sphere. Hence if \( \forall v \in \mathcal{M}^n \text{ Lk}(v, \mathcal{M}^n) \)

is a sphere, \( \partial \mathcal{M}^n \) cannot contain vertices. But \( \partial \mathcal{M}^n \) is a subcomplex by Thm. 2.4 (i) so it must be empty.

**Theorem 2.5**

\( \mathcal{M}^n \) a comb. man. of dim \( n \) \( \exists \partial \mathcal{M}^n \neq \emptyset \)

Then \( \partial \mathcal{M}^n \) is a comb. man. of

\( \text{dim} (n-1) \exists \partial(\partial \mathcal{M}^n) = \emptyset . \)

**Proof:**

(a) \( \text{Lk}(v, \partial \mathcal{M}^n) = \partial \text{Lk}(v, \mathcal{M}^n) : \)

\[ \Delta \in \text{Lk}(v, \partial \mathcal{M}^n) \iff v \neq \Delta \in \partial \mathcal{M}^n \]

\[ \iff \text{Lk}(v, \Delta, \mathcal{M}^n) = \text{Lk}(\Delta, \text{Lk}(v, \mathcal{M}^n)) \]

is a comb. cell

\[ \iff \Delta \in \partial \text{Lk}(v, \mathcal{M}^n) \]
(b) \( \partial \mathcal{U}^n \) is a comb. man. of dim \((n-1) \Rightarrow \partial (\partial \mathcal{U}^n) = \emptyset \):

\[ v \in \partial \mathcal{U}^n \Rightarrow \text{Lk}(v, \partial \mathcal{U}^n) = \partial \text{Lk}(v, \mathcal{U}^n) \]

Lk\((v, \mathcal{U}^n)\) is a comb. cell of dim \((n-1)\)

\[ \therefore \partial \text{Lk}(v, \mathcal{U}^n) \text{ is a comb. sphere of dim } (n-2) \]

\[ \text{Thm. 2.4 (iii)} \]

\[ \Rightarrow \partial \mathcal{U}^n \text{ is a comb. man. of dim } (n-1) \]

\[ \exists \partial (\partial \mathcal{U}^n) = \emptyset . \]
3. Homogeneity of Combinatorial Spheres

**Lemma 3.1**  
$s^1_n, s^2_n$ combinatorial cells  
$h : \partial s^1_n \rightarrow \partial s^2_n$ a PL homeomorphism  

Then $\exists$ a PL homeomorphism  
$h : s^1_n \rightarrow s^2_n \ni h|_{s^1_n} = h$

**Proof:**  
$\exists$ PL homeomorphisms $g_1 : s^1_n \rightarrow \Delta^n$,  
$g_2 : s^2_n \rightarrow \Delta^n$, set $h' = g_2|_{\partial s^2_n} \circ h^{-1}|_{\partial \Delta^2_n}$  

$h' : \partial \Delta^n \rightarrow \partial \Delta^n$ is a PL homeomorphism.  

Consider its underlying simplicial isomorphism  
$h' : a_1(\partial \Delta^n) \rightarrow a_2(\partial \Delta^n)$  
choose $v \in \text{int} \Delta^n$, set  
$h' = h'|_v = \alpha_1(\Delta^n)*v = \alpha_1(\Delta^n) \rightarrow a_2(\partial \Delta^n)*v = \alpha_2(\Delta^n)$  

$h = g_2^{-1} \circ h' \circ g_1 : s^1_n \rightarrow s^2_n$ is a PL homeomorphism  

$\exists h|_{s^1_n} = h$.

**Lemma 3.2**  
$s^n$ a combinatorial cell of dim $n$  
$c^n$ PL homeomorphic to $s^n \times I$ with  
$c^n \cap s^n = \partial s^n$. Then $s^n + c^n$ is a combinatorial cell of dimension $n$. 
Proof: \exists PL homeomorphisms \( h_1 : S^n \to \Delta^n \),
\( h_2 : C^n \to \partial S^n \times I \) take \( h_3 = (h_1|_{\partial S^n} \times \{1\}) \circ h_2 : C^n \to \partial \Delta^n \times I \).

Consider \( h_3|_{\partial S^n} \subseteq C^n \) : \( \partial S^n \to \partial \Delta^n \) by lemma 3.1 it may be extended to \( h_3 : S^n \to \Delta^n \) thus \( h_3 \) and \( \hat{h}_3 \) define a PL homeomorphism.

\( h : C^n + S^n \to \partial \Delta^n \times I + \Delta^n \) which is PL homeomorphic to \( \Delta^n \).

Lemma 3.3 
\( S^n_1, S^n_2 \) combinatorial n-cells \( \ni \partial S^n_1 \)
\( = \partial S^n_2 = S^n_1 \cap S^n_2 \). Then \( S^n_1 + S^n_2 \) is a combinatorial n-sphere.

Proof: By lemma 3.1 \( \exists PL homeomorphisms \)
\( h_1 : S^n_1 \to \partial S^n_1 \times V_1 \), \( V_1 \not\in S^n_1 + S^n_2 \ni h_1|_{\partial S^n_1} = 1_{\partial S^n_1} \)
\( h_2 : S^n_2 \to \partial S^n_2 \times V_2 \), \( V_2 \not\in V_1 \) \( V_2 \not\in S^n_1 + S^n_2 \ni h_2|_{\partial S^n_2} = 1_{\partial S^n_2} \)
since \( S^n_1 \cap S^n_2 = \partial S^n_1 = \partial S^n_2 \), \( \exists \) a PL homeomorphism
\( h : S^n_1 + S^n_2 \to \partial S^n_1 \times V_1 + \partial S^n_2 \times V_2 \)
but \( \partial S^n_1 \times V_1 + \partial S^n_2 \times V_2 = \partial S^n_1 \times (V_1 + V_2) \) which is a comb. n-sphere by Thm. 2.1.
Theorem 3.1

\[ \gamma^n \text{ a comb. } n\text{-sphere} \]
\[ \mathcal{E}^n \subseteq \gamma^n \text{ a comb. } n\text{-cell} \]

Then \( C(\mathcal{E}^n, \gamma^n) \) is a comb. \( n\)-cell.

Proof: By induction on \( n \):

- **Case \( n = 0 \):** \( \gamma^0 = 2 \) 0-simplices, \( \mathcal{E}^0 = 1 \) 0-simplex
  \[ \therefore C(\mathcal{E}^0, \gamma^0) = 1 \text{ 0-simplex} \]
  = a comb. 0-cell.

Assume the theorem is true in dimensions \( \leq (n-1) \).

This assumption implies result (A), the existence of a collar of the boundary of a combinatorial manifold of dimension \( n \) which will be used to prove the main theorem in that dimension.

(A). \( \mathcal{M}^n \) a comb. manifold of \( \dim n \gg \partial \mathcal{M}^n \) is full in \( \mathcal{M}^n \). Then \( N(\partial \mathcal{M}^n, \gamma(\mathcal{M}^n)) \) is PL homeomorphic to \( \partial \mathcal{M}^n \times I \) where \( \gamma(\mathcal{M}^n) = \mathcal{B}_0 \mathcal{M}^n(\mathcal{M}^n) = 1\text{-st} \)

barycentric subdivision of \( \mathcal{M}^n \) relative to \( \partial \mathcal{M}^n \).

Proof: By induction on \( n \):

Case \( n = 0 \) is trivial. We assume (A) is true for dimension \( \leq (n-1) \) as in the main theorem.

Case \( n \): (1) Defn of \( O(\Delta) \) for \( \Delta \in \partial \mathcal{M}^n \):

\[ O(\Delta) = \{ \Delta_1 : \Delta_1 \in \text{Lk}(\Delta, \gamma(\mathcal{M})) \} \] and
\( \Delta \cap \delta \mathcal{M} = \emptyset \)

since \( \delta \mathcal{M} \) is full in \( \gamma(\mathcal{M}) \) we have that

\[
N(\delta \mathcal{M}, \gamma(\mathcal{M})) = \sum_{\Delta \in \delta \mathcal{M}} \Delta \ast O(\Delta)
\]

also \( \text{Lk}(\Delta, \gamma(\mathcal{M})) = N(\text{Lk}(\Delta, \delta \mathcal{M}), \text{Lk}(\Delta, \gamma(\mathcal{M}))) \ast O(\Delta) \)

since \( \text{Lk}(\Delta, \delta \mathcal{M}) = \delta \text{Lk}(\Delta, \gamma(\mathcal{M})) = \text{Lk}(\Delta, \gamma(\mathcal{M})) \cap \delta \mathcal{M} \)

(2) Properties of \( O(\Delta) \):

I. If \( \Delta \in \delta \mathcal{M} \) is an \( l \)-simplex, then \( O(\Delta) \) is a comb.

\( (n-l-1) \) cell:

(i) Choose a vertex \( v \) not considered so far.

Since \( \text{Lk}(\Delta, \delta \mathcal{M}) \) is a comb. \( (n-l-2) \) sphere,

\( v \ast \text{Lk}(\Delta, \delta \mathcal{M}) \) is a comb. \( (n-l-1) \) cell.

Also \( \text{Lk}(\Delta, \gamma(\mathcal{M})) \) is a comb. \( (n-l-1) \) cell \( \Delta \)

\( \text{Lk}(\Delta, \gamma(\mathcal{M})) \cap v \ast \text{Lk}(\Delta, \delta \mathcal{M}) \) is

\( \text{Lk}(\Delta, \delta \mathcal{M}) = \delta \text{Lk}(\Delta, \gamma(\mathcal{M})) = \delta (v \ast \text{Lk}(\Delta, \delta \mathcal{M})) \) is full in

\( \gamma(\mathcal{M}). \) Thus \( v \ast \text{Lk}(\Delta, \delta \mathcal{M}) + \text{Lk}(\Delta, \gamma(\mathcal{M})) \) is a comb. \( (n-l-1) \) sphere — Lemma 3.3.

(ii) \( N(\text{Lk}(\Delta, \delta \mathcal{M}), \text{Lk}(\Delta, \gamma(\mathcal{M}))) \approx N(\text{Lk}(\Delta, \delta \mathcal{M}), \hat{\gamma}(\text{Lk}(\Delta, \mathcal{M}))) \)

where \( \hat{\gamma}(\text{Lk}(\Delta, \mathcal{M})) = \beta_{\text{Lk}(\Delta, \delta \mathcal{M})} \text{Lk}(\Delta, \mathcal{M}) \)

but \( N(\text{Lk}(\Delta, \delta \mathcal{M}), \hat{\gamma}(\text{Lk}(\Delta, \mathcal{M}))) \) is PL homeomorphic to \( \text{Lk}(\Delta, \delta \mathcal{M}) \times I \) by (A) in dimension \( (n-l-1) \)
(iii) \( N(Lk(\Delta, \partial M), Lk(\Delta, \gamma(M))) \) is a "collar" of \( v \cdot Lk(\Delta, \partial M) \) in the sense of Lemma 3.2, hence 
\[ v \cdot Lk(\Delta, \partial M) + N(Lk(\Delta, \partial M), Lk(\Delta, \gamma(M))) \] 
is a comb. (\( n-l-1 \)) cell, say \( \gamma^{n-l-1} \).

(iv) Now 
\[ O(\Delta) + N(Lk(\Delta, \partial M), Lk(\Delta, \gamma(M))) + v \cdot Lk(\Delta, \partial M) \]
\[ = Lk(\Delta, \gamma(M)) + v \cdot Lk(\Delta, \partial M) \]
\[ = \text{a comb. (} n-l-1 \text{) sphere} \]
\[ \text{—— (i) say } \gamma^{n-l-1} \]

And so, by the induction hypothesis to the main theorem \( \text{dim } (n-l-1) \), \( C(\gamma^{n-l-1}, \gamma^{n-l-1}) \) is a comb. (\( n-l-1 \)) cell.

(v) But 
\[ O(\Delta) \cap [N(Lk(\Delta, \partial M), Lk(\Delta, \gamma(M))) + v \cdot Lk(\Delta, \partial M)] = N(Lk(\Delta, \partial M), Lk(\Delta, \gamma(M))) \]
which can only contain simplices of dimension < (\( n-l-1 \))

Thus 
\[ C(\gamma^{n-l-1}, \gamma^{n-l-1}) = O(\Delta) \]. And \( O(\Delta) \) is a comb. (\( n-l-1 \)) cell.

II. for \( \Delta_1, \Delta_2 \in \partial M, \Delta_1 < \Delta_2 \Rightarrow O(\Delta_2) \subseteq O(\Delta_1) \)
say \( \Delta \in O(\Delta_2) \Rightarrow \Delta \in Lk(\Delta_2, \gamma(M)) \) and \( \Delta \cap \partial M = \emptyset \)
\[ \Rightarrow \Delta \cap \Delta_2 \in \gamma(M) \Rightarrow \Delta \cap \Delta_1 \in \gamma(M) \]
\[ \Rightarrow \Delta \in Lk(\Delta_1, \gamma(M)) \) and \( \Delta \cap \partial M = \emptyset \)
\[ \Rightarrow \Delta \in O(\Delta_1) \]
III. for $\Delta \in \partial \mathcal{M}$, $\partial \mathcal{O}(\Delta) = \bigcup_{\Delta < \Delta'} \mathcal{O}(\Delta')$

(i) say $\Delta_1 \in \partial \mathcal{O}(\Delta) \Rightarrow \Delta_1 \in \text{Lk}(\Delta, \gamma(\mathcal{M}))$, $\Delta_1 \cap \partial \mathcal{M} = \emptyset$, and $\text{Lk}(\Delta_1, \mathcal{O}(\Delta))$ is a comb. cell.

now $\text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M}))) = \text{a comb-sphere} --$

$\Delta \in \partial \mathcal{M} \Rightarrow \text{Lk}(\Delta, \gamma(\mathcal{M}))$ is a comb. sphere,
thus $\text{Lk}(\Delta_1, \mathcal{O}(\Delta)) \subseteq \text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M}))) \subseteq \mathcal{O}(\Delta)$
$\subseteq \text{Lk}(\Delta, \gamma(\mathcal{M}))$. But $\text{Lk}(\Delta_1, \mathcal{O}(\Delta)) \neq \text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M})))$

$\therefore \exists \Delta_2 \in \text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M}))), \Delta_2 \notin \text{Lk}(\Delta_1, \mathcal{O}(\Delta))$

$\Rightarrow \Delta_2 \ast \Delta_1 \in \text{Lk}(\Delta, \gamma(\mathcal{M}))$

as $\Delta_2 \notin \text{Lk}(\Delta_1, \mathcal{O}(\Delta))$ we must have $\Delta_2 \ast \Delta_1 \cap \partial \mathcal{M}^2 \neq \emptyset$
but $\Delta_1 \cap \partial \mathcal{M}^2 = \emptyset$. $\therefore \Delta_3 = \Delta_2 \ast \Delta_1 \cap \partial \mathcal{M}^2 < \Delta_2$

Consider $\Delta_3 \ast \Delta$, clearly $\Delta < \Delta_3 \ast \Delta$

we show $\Delta_3 \in \mathcal{O}(\Delta_3 \ast \Delta)$:

$\Delta_2 \ast \Delta_1 \in \text{Lk}(\Delta, \gamma(\mathcal{M})) \Rightarrow \Delta_2 \ast \Delta_1 \ast \Delta \in \gamma(\mathcal{M})$

$\Rightarrow \Delta_3 \ast \Delta_1 \ast \Delta \in \gamma(\mathcal{M}) \Rightarrow \Delta_3 < \Delta_2$

$\Rightarrow \Delta_1 \in \text{Lk}(\Delta_3 \ast \Delta, \gamma(\mathcal{M}))$ and

$\text{Lk}(\Delta, \gamma(\mathcal{M}))$

$\Rightarrow \Delta_1 \in \mathcal{O}(\Delta_3 \ast \Delta)$

$\Rightarrow \Delta_1 \in \bigcup_{\Delta < \Delta_1} \mathcal{O}(\Delta') \Rightarrow \Delta < \Delta_3 \ast \Delta$. 

(ii) say $\Delta_1 \in \bigcup_{\Delta < \Delta'} O(\Delta')$ i.e. $\Delta_1 \in O(\Delta')$ for some $\Delta < \Delta'$. By II $\Delta_1 \in O(\Delta)$, $\therefore$ we must show $\text{Lk}(\Delta_1, O(\Delta))$ is a comb. cell.

Now $\Delta_1 \notin \partial \text{Lk}(\Delta, \gamma(\mathcal{M})) = \partial \text{Lk}(\Delta, \gamma(\mathcal{M}))$ = $\partial \mathcal{M} \cap \text{Lk}(\Delta, \gamma(\mathcal{M}))$

$\therefore \text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M})))$ is a comb. sphere of the same dimension as $\text{Lk}(\Delta_1, O(\Delta))$ but $\text{Lk}(\Delta_1, O(\Delta)) \subset \text{Lk}(\Delta_1, \text{Lk}(\Delta, \gamma(\mathcal{M})))$

And as a comb. k-sphere cannot contain a comb. k-sphere as a proper subcomplex $\text{Lk}(\Delta_1, O(\Delta))$ must be a comb. cell $\Rightarrow O(\Delta)$ is a comb. man.

$\therefore \Delta_1 \in \partial O(\Delta)$

IV. $\Delta_1, \Delta_2 \in \partial \mathcal{M}$

then $O(\Delta_1) \cap O(\Delta_2) = \begin{cases} \emptyset & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ are not faces of a simplex of } \gamma(\mathcal{M}) \\ O(\Delta_3) \text{ where } \Delta_3 \text{ is the smallest simplex of } \gamma(\mathcal{M}) \text{ containing } \Delta_1 \text{ and } \Delta_2 \text{ as faces} \end{cases}$

(i) if $\Delta \in O(\Delta_3)$ then $\Delta \in O(\Delta_1) \cap O(\Delta_2)$, $\Delta_1, \Delta_2 < \Delta_3$ by II

(ii) say $\Delta \in O(\Delta_1) \cap O(\Delta_2)$, now $\Delta = b\Delta'_1 \ldots \ldots b\Delta'_k$, $\Delta'_1 < \ldots < \Delta'_k \in \mathcal{M}$
in the canonical rep. of \( \gamma(M) \)

\[ \Delta \in \text{lk}(\Delta_1, \gamma(M)) \& \Delta \in \text{lk}(\Delta_2, \gamma(M)) \Rightarrow \Delta \ast \Delta_1 \text{ and } \Delta \ast \Delta_2 \in \gamma(M) \]

but \( \Delta_1, \Delta_2 \in \partial M \Rightarrow \Delta \ast \Delta_1 = \Delta_1 \partial M \ast b \Delta_1 \ast \ldots \ast b \Delta_k \)

\[ \Delta \ast \Delta_2 = \Delta_2 \partial M \ast b \Delta_1 \ast \ldots \ast b \Delta_k \text{ in canonical representation} \]

\[ \Rightarrow \Delta_1 \ast \Delta_2 < \Delta_1 \Rightarrow \Delta_1 \ast \Delta_2 \in M \]

\[ \Rightarrow \Delta_1 \ast \Delta_2 \in \partial M \]

\[ \therefore \Delta_3 = \Delta_1 \ast \Delta_2 \text{ has } \Delta \in \partial(\Delta_3) \iff \Delta \in \partial(\Delta_1) \cap \partial(\Delta_2). \]

(3) Consider now the simplicial complex \( \partial M \times I \):

for \( \Delta \in \partial M \times 0 \), set

\[ D(\Delta) = \{ \Delta_1 : \Delta_1 \cap \partial M \times 0 = \emptyset, \Delta_1 \ast \Delta \in \partial M \times I \} \]

\[ \subseteq \partial M \times I \]

Clearly \( \partial M \times 0 \) is full in \( \partial M \times I \)

and \( \partial M \times I = \sum_{\Delta \in \partial M \times 0} \Delta \ast D(\Delta) \)

\[ \partial M \times 1 \approx \beta(\partial M) \]

\[ \partial M \times I \]

\[ \partial M \times 0 \approx \partial M \]

(4) Properties of \( D(\Delta) \):

I. If \( \Delta_1 \in \partial M \times 0 \) is an \( i \)-simplex, then \( D(\Delta_1) \) is a combinatorial \( (n-i-1) \)-cell:
D(\Delta_1) = \{(b\Delta_1,1)\cdots(b\Delta_m,1) : \Delta_1 \prec \cdots \prec \Delta_m \in \partial \mathcal{M}\}
\approx (b\Delta_1,1)*\{(b\Delta_2,1)\cdots(b\Delta_m,1) : \Delta_1 \prec \cdots \prec \Delta_m \in \partial \mathcal{M}\}
= \text{a comb. } (n-l-1) \text{ cell } \text{ and } \text{Lk}(\Delta_1,\partial \mathcal{M}) \text{ is a comb. } \text{(n-l-1) sphere}

II. If \Delta_1, \Delta_2 \in \partial \mathcal{M}^{\times 0}, \Delta_1 \prec \Delta_2 \text{ then } \text{D}(\Delta_2) \subseteq \text{D}(\Delta_1):
\text{ say } \Delta \in \text{D}(\Delta_2) \Rightarrow \Delta * \Delta_2 \in \partial \mathcal{M} \times I, \Delta \cap \partial \mathcal{M}^0 = \emptyset \\
\Rightarrow \Delta * \Delta_1 \in \partial \mathcal{M} \times I, \Delta_1 \prec \Delta_2 \\
\Rightarrow \Delta \in \text{D}(\Delta_1)

III. If \Delta \in \partial \mathcal{M}^{\times 0}, \partial \text{D}(\Delta) = \bigcup_{\Delta \prec \Delta'} \text{D}(\Delta')
\text{(1) say } \Delta_1 \in \partial \text{D}(\Delta).
\Rightarrow \text{Lk}(\Delta_1, \text{D}(\Delta)) = \text{a comb. cell}
\text{ but } \text{Lk}(\Delta_1, \text{Lk}(\Delta, \partial \mathcal{M} \times I)) \supseteq \text{Lk}(\Delta_1, \text{D}(\Delta)) \text{ and }
\text{Lk}(\Delta_1, \text{Lk}(\Delta, \partial \mathcal{M} \times I)) \nprec \text{Lk}(\Delta_1, \text{D}(\Delta))
\therefore \exists \Delta_2 \in \text{Lk}(\Delta_1, \text{Lk}(\Delta, \partial \mathcal{M} \times I)), \Delta_2 \nprec \text{Lk}(\Delta_1, \text{D}(\Delta)),
\text{and } \Delta_1 \in \text{D}(\Delta'), \Delta_3 = \Delta_2 * \Delta_1 \cap \partial \mathcal{M}^{\times 0}
\text{(ii) say } \Delta_1 \in \text{D}(\Delta') \Delta \prec \Delta'

Lk(Δ₁, Lk(Δ, ∂M × I)) is a comb. sphere of the same dim as Lk(Δ₁, D(Δ))
⇒ Lk(Δ₁, D(Δ)) cannot be a combinatorial sphere since Lk(Δ₁, D(Δ)) ⊆ Lk(Δ₁, Lk(Δ, ∂M × I))
⇒ Lk(Δ₁, D(Δ)) is a comb. cell
⇒ Δ₁ ∈ ∂D(Δ).

IV. If Δ₁, Δ₂ ∈ ∂M × 0 then

\[ D(Δ₁) ∩ D(Δ₂) = \begin{cases} \\
\emptyset & \text{if } Δ₁ \text{ and } Δ₂ \text{ are not faces of a simplex of } ∂M \times I \\
D(Δ₃) & \text{where } Δ₃ \text{ is the smallest simplex of } ∂M \times I \text{ containing } Δ₁ \text{ and } Δ₂ \text{ as faces} \\
\end{cases} \]

(1) if Δ ∈ D(Δ₃) then Δ ∈ D(Δ₁) ∩ D(Δ₂) by II

(ii) if Δ ∈ D(Δ₁) ∩ D(Δ₂)
⇒ Δ = (bΔ₁, 1) ∗ (bΔ₂, 1) ∗ ∗ ∗ (bΔₘ, 1) Δ₁ < Δ₂ < ∗ ∗ < Δₘ ∈ ∂M

and Δ = (bΔ₂, 1) ∗ (bΔ₂, 1) ∗ ∗ ∗ (bΔₘ, 1) Δ₂ < Δ₂ < ∗ ∗ < Δₘ ∈ ∂M

but we must have Δₘ ′ = Δₘ

⇒ Δ₁ ∗ Δ₂ < Δₘ ′ ⇒ Δ₁ ∗ Δ₂ = Δ₃ ∈ ∂M

has Δ ∈ D(Δ₃) \iff Δ ∈ D(Δ₁) ∩ D(Δ₂).
(5) We now exhibit a PL homeomorphism

\[ h : N(\partial \mathcal{M}, \gamma(\mathcal{M})) \to \partial \mathcal{M} \times I \]

note that \( N(\partial \mathcal{M}, \gamma(\mathcal{M})) = \sum_{\Delta \in \partial \mathcal{M}} \Delta \ast 0(\Delta) \) and

\[ \partial \mathcal{M} \times I = \sum_{\Delta \in \partial \mathcal{M} \times 0} \Delta \ast D(\Delta) \]

we define \( h \) inductively on simplices of \( N(\partial \mathcal{M}, \gamma(\mathcal{M})) \) in order of decreasing dimension.

Choose any \( \Delta^{n-1} \in \partial \mathcal{M} \), \( 0(\Delta^{n-1}) = v_1 \) and \( D(\Delta^{n-1}) = v_2 \), define \( h|_{\Delta^{n-1} \ast 0(\Delta^{n-1})} : \Delta^{n-1} \ast 0(\Delta^{n-1}) \to \Delta^{n-1} \ast D(\Delta^{n-1}) \) by \( h = 1_{\Delta^{n-1} \ast g} \) where \( g(v_1) = v_2 \).

Assume we have defined \( h \) on \( \Delta^i \ast 0(\Delta^i) \forall i > k \).

Choose \( \Delta^k \in \partial \mathcal{M} \) as \( \partial 0(\Delta^k) = \bigcup_{k < l} 0(\Delta^i) \) and

\[ \partial D(\Delta^k) = \bigcup_{k < l} D(\Delta^l) \]

we have defined \( h|_{\partial 0(\Delta^k)} \) on \( \partial 0(\Delta^k) \) already. \( h|_{\partial 0(\Delta^k)} \) may be extended to \( h|_{0(\Delta^k)} \) by lemma 3.1.

Define \( h|_{\Delta^k \ast 0(\Delta^k)} : \Delta^k \ast 0(\Delta^k) \to \Delta^k \ast D(\Delta^k) \) by

\[ h = 1_{\Delta^k \ast h|_{0(\Delta^k)}} \]

noting that \( 0(\Delta^k_1) \cap 0(\Delta^k_2) = 0(\Delta^l) \) \( \forall l > k \),

by induction we define \( h \) on \( \Delta^k \ast 0(\Delta^k) \forall \Delta^k \in \partial \mathcal{M} \),

\( k = 0, 1, \ldots, (n-1) \).

Thus we have a PL homeomorphism
h : N(\mathcal{M}, \gamma(\mathcal{M})) \to \mathcal{M} \times I.

(B) Proof of the Main Theorem in dimension n :

(1) Set \( K = C(\mathcal{M}^n, \gamma^n) \), clearly \( \gamma^n = K + \mathcal{M}^n \).

(a) \( K \) is a comb. man. of dim n :

Choose any \( v \in K \), \( \text{Lk}(v, \gamma^n) = \text{Lk}(v, \mathcal{M}^n) + \text{Lk}(v, K) \)

\( v \notin \mathcal{M}^n \Rightarrow \text{Lk}(v, K) = \text{Lk}(v, \gamma^n) = \) a comb. (n-1) sphere.

\( v \in \mathcal{M}^n \Rightarrow \text{Lk}(v, \gamma^n) = \) a comb. (n-1) sphere.

if \( \text{Lk}(v, \mathcal{M}^n) \) is a sphere,

\( \Rightarrow \text{Lk}(v, \mathcal{M}^n) = \text{Lk}(v, \gamma^n) \)

\( \Rightarrow \text{Lk}(v, K) = \emptyset \)

contradicting \( v \in K \)

\( \therefore \text{Lk}(v, \mathcal{M}^n) \) is a comb. (n-1) cell

thus \( \text{Lk}(v, K) = C(\text{Lk}(v, \mathcal{M}^n), \text{Lk}(v, \gamma^n)) \)

= a comb. (n-1) cell by the induction assumption to the main theorem.

\( \therefore \text{Lk}(v, K) \) is a comb. (n-1) cell or sphere

\( \Rightarrow K \) is a comb. man. of dim n.

(b) \( \partial K = \partial \mathcal{M}^n = K \cap \mathcal{M}^n \):

\( \Delta \in \partial K \iff \text{Lk}(\Delta, K) \) is a comb. cell \( \iff \text{Lk}(\Delta, \mathcal{M}^n) \)

is a comb. cell — (a)

\( \iff \Delta \in \partial \mathcal{M}^n \).
also $\Delta \in K \cap \mathbb{S}^n \Rightarrow \text{Lk}(\Delta, \mathbb{S}^n)$ and $\text{Lk}(\Delta, K)$ are comb. cells — as in (a) 

$\Rightarrow \Delta \in \partial K$ and $\Delta \in \partial \mathbb{S}^n$

$\therefore \partial K = \partial \mathbb{S}^n = K \cap \mathbb{S}^n$

(ii) Given $\mathbb{S}^n \subseteq \gamma^n$ consider $\mathcal{M} = C(\mathbb{S}^n, \gamma^n)$

We show $\mathcal{M}$ is a combinatorial $n$-cell

Now, $M$ is a comb. man. of dim $n \Rightarrow \mathcal{M} = \partial \mathbb{S}^n = \mathcal{M} \cap \mathbb{S}^n$

--- (i) ---

As $\partial \mathcal{M} = \mathcal{M} \cap \mathbb{S}^n$, by Lemma 3.1 we may consider $\gamma^n$ as

$\mathcal{M} + \partial \mathcal{M} * v$.

(iii) Consider a PL homeomorphism

$h : \gamma^n = \mathcal{M} + \partial \mathcal{M} * v \to \partial \Delta^{n+1}$

Let $\Delta^{n+1} = w \Delta^n$. Then $\partial \Delta^{n+1} = w \partial \Delta^n + \Delta^n$. We note that if $x, y \in \partial \Delta^{n+1}$, then there is a PL homeomorphism of $\partial \Delta^{n+1}$ onto itself which maps $x$ onto $y$. Hence we may assume that the PL homeomorphism $h$ satisfies the property

$h(v) = w$.

We choose now subdivisions $\alpha_1(\gamma^n)$ of $\gamma^n$ and $\alpha_2(\partial \Delta^{n+1})$ of $\partial \Delta^{n+1}$ such that $h : \alpha_1(\gamma^n) \to \alpha_2(\partial \Delta^{n+1})$ is a simplicial isomorphism and such that

$\text{Lk}(v, \alpha_1(\gamma^n)) \cap \alpha_1(\text{Lk}(v, \gamma^n)) = \text{Lk}(v, \alpha_1(\gamma^n)) \cap \alpha_1(\partial \mathcal{M}) = \emptyset$
and

\[ \text{Lk}(w, \alpha_2(\partial \Delta^{n+1})) \cap \alpha_2(\text{Lk}(w, \partial \Delta^{n+1})) = \text{Lk}(w, \alpha_2(\partial \Delta^{n+1})) \cap \alpha_2(\partial \Delta^n) = \emptyset. \]

By Corollary 1.3, \( C(\text{St}(w, \alpha(\partial \Delta^{n+1})), \alpha(\text{St}(w, \partial \Delta^{n+1}))) \) is PL homeomorphic to \( \partial \Delta^n \times I \). By Lemma 3.2,

\[ C(\text{St}(w, \alpha_2(\partial \Delta^{n+1})), \alpha_2(\partial \Delta^{n+1})) = C(\text{St}(w, \alpha_2(\partial \Delta^{n+1})), \alpha_2(\text{St}(w, \partial \Delta^{n+1}))) \]

\[ + \alpha_2(\Delta^n) \]

is a combinatorial cell of dimension \( n \). Since

\[ h(\text{St}(v, \alpha_1(\gamma^n))) = \text{St}(w, \alpha_2(\partial \Delta^{n+1})) \]

we conclude that

\[ C(\text{St}(v, \alpha_1(\gamma^n)), \alpha_1(\gamma^n)) \]

is a combinatorial \( n \)-cell.

To satisfy the hypothesis of (A) with respect to \( \alpha_1(\mathcal{U}) \) and \( \alpha_1(\partial \mathcal{U}) \), we define
\[ \alpha(\gamma^n) = B^2 \alpha_1(\partial \mathcal{M} \times \nu)(\alpha_1(\gamma^n)). \]

(iv) By (iii), \( C(\text{St}(v, \alpha(\gamma^n)), \alpha(\gamma^n)) \) is a comb. n-cell.

We will produce a PL homeomorphism of \( \mathcal{M} \) onto
\[ C(\text{St}(v, \alpha(\gamma^n)), \alpha(\gamma^n)). \]

Observe that \( \exists \) PL homeomorphisms
\[ h_1: N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M})) - \mathcal{M} \times I \rightarrow (A) \]
\[ h_2: C(\text{St}(v, \alpha(\gamma^n)), \text{St}(v, \gamma^n)) \]
\[ \rightarrow \mathcal{M} \times I \rightarrow \text{Corol. 1.3} \]

Thus \( \exists \) a PL homeomorphism
\[ g: N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M})) - N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M})) + C(\text{St}(v, \alpha(\gamma^n)), \text{St}(v, \gamma^n)) \]
\[ \exists g|_{N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M}))} = \text{id}_{N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M}))} \]

Using \( g \) and \( \text{id}_{C(N(\partial \alpha(\mathcal{M}), \alpha(\mathcal{M})), \alpha(\mathcal{M}))} \), since they agree on intersections of domains, we may define a PL homeomorphism
\[ h: \mathcal{M} \rightarrow C(\text{St}(v, \alpha(\gamma^n)), \alpha(\gamma^n)). \]

Thus \( \mathcal{M} \) is a combinatorial n-cell.

**Corollary 3.1**  \( \mathcal{M}^n \) a comb. man. of dim \( n \) \( \Rightarrow \mathcal{M}^n \) is
full in $\mathcal{M}^n$. $B(\mathcal{M}^n)$ the 1st barycentric subdivision of $\mathcal{M}^n$ relative to $\partial\mathcal{M}^n$. Then $N(\partial\mathcal{M}^n, B(\mathcal{M}^n))$ is PL homeomorphic to $\partial\mathcal{M}^n \times I$.

**Proof:** c.f. (A) of Thm. 3.1.

**Corollary 3.2** $\mathcal{S}_1^n, \mathcal{S}_2^n$ combinatorial $n$-cells $\exists$

$\mathcal{S}_1^n \cap \mathcal{S}_2^n = \partial\mathcal{S}_1^n \cap \partial\mathcal{S}_2^n = \mathcal{S}^{n-1}$, a comb. $(n-1)$ cell.

Then $\mathcal{S}_1^n + \mathcal{S}_2^n$ is a combinatorial $n$-cell.

**Proof:** $\exists$ a PL homeomorphism $h : \mathcal{S}^{n-1} \to \Delta^{n-1}$.

$C(\mathcal{S}^{n-1}, \partial\mathcal{S}_1^n)$ is a comb. $(n-1)$ cell by theorem 3.1. By lemma 3.1, $\exists$ a PL homeomorphism

$$h_1 : C(\mathcal{S}^{n-1}, \partial\mathcal{S}_1^n) \to v_1 \ast \partial\Delta^{n-1}$$

which extends the PL homeomorphism $h$. Similarly $\exists$ a PL homeomorphism

$$h_2 : C(\mathcal{S}^{n-1}, \partial\mathcal{S}_2^n) \to v_2 \ast \partial\Delta^{n-1}$$

which extends the PL homeomorphism $h$.

Set $\Delta^n_1 = \Delta^{n-1} \ast v_1$, and $\Delta^n_2 = \Delta^{n-1} \ast v_2$. Hence we have PL homeomorphism

$$h'_1 : \partial\mathcal{S}_1^n = \mathcal{S}^{n-1} + C(\mathcal{S}^{n-1}, \partial\mathcal{S}_1^n) \to \partial\Delta^n_1$$
and

\[ h'_2 : \mathbb{A}^n_2 = \mathbb{A}^{n-1} + C(\mathbb{A}^{n-1} \times A_2) \to \Delta^n_2 \]

with \( h'_1 |_{\mathbb{A}^{n-1}} = h'_2 |_{\mathbb{A}^{n-1}} \).

By lemma 3.1 these may be extended to PL homeomorphisms

\[ h_1 : \mathbb{A}^n_1 \to \Delta^n_1 \]
\[ h_2 : \mathbb{A}^n_2 \to \Delta^n_2 \]

Thus giving a PL homeomorphism

\[ h : \mathbb{A}^n_1 + \mathbb{A}^n_2 \to \Delta^n_1 + \Delta^n_2 = \Delta^{n-1} * (v_1 + v_2) \]

and \( \Delta^{n-1} * (v_1 + v_2) \) is a comb. n-cell — Thm. 2.1.
Bibliography


