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SEMI-METRICS ON THE NORMAL STATES
OF A W^* -ALGEBRA

by

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ABSTRACT

In this thesis we investigate certain semi-metrics defined on the normal states of a W^* -algebra and their applications to infinite tensor products.

This extends the work of Bures, who defined a metric d on the set of normal states by taking $d(\mu, \nu) = \inf \{ \|x-y\| \}$, where the infimum is taken over all vectors x and y which induce the states μ and ν respectively relative to any representation of the algebra as a von-Neumann algebra. He then made use of this metric in obtaining a classification of the various incomplete tensor products of a family of semi-finite W^* -algebras, up to a natural type of equivalence known as product isomorphism.

By removing the semi-finiteness restriction from Bures' "product formula", which relates the distance under d between two finite product states to the distances between their components, we obtain this tensor product classification for families of arbitrary W^* -algebras. Moreover we extend the product formula to apply to the case of infinite product states.

For any subgroup G of the * -automorphism group of a W^* -algebra, we define the semi-metric d^G on the set of normal states by: $d^G(\mu, \nu) = \inf \{ d(\mu^\alpha, \nu^\beta) : \alpha, \beta \in G \}$; where μ^α is defined by $\mu^\alpha(A) = \mu(\alpha(A))$. We show the significance

of d^G in classifying incomplete tensor products up to weak product isomorphism, a natural weakening of the concept of product isomorphism.

In the case of tensor products of semi-finite factors, we obtain explicit criteria for such a classification by calculating $d^G(u,v)$ in terms of the Radon-Nikodym derivatives of the states.

In the course of this calculation we introduce a concept of compatibility which yields some other results about d and d^G . Two self-adjoint operators S and T are said to be compatible, if given any real numbers α and β , either $E(\alpha) \leq F(\beta)$ or $F(\beta) \leq E(\alpha)$; where $\{E(\lambda)\}$, $\{F(\lambda)\}$, are the spectral resolutions of S, T , respectively. We obtain some miscellaneous results concerning this concept.

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INTRODUCTION

In [3] Bures defined a metric d on the set of all normal states of a W^* -algebra G by taking $d(\mu, \nu) = \inf \{ \|x-y\| \}$, where the infimum is taken over all vectors x and y inducing the states μ and ν respectively relative to any representation of G as a von Neumann algebra. In this thesis we generalize this definition by defining, for any subgroup G of the group of $*$ -automorphisms of G , a semi-metric d^G , given by $d^G(\mu, \nu) = \inf \{ d(\mu^\alpha, \nu^\beta) : \alpha, \beta \in G \}$, where $\mu^\alpha(A) = \mu(\alpha(A))$. In particular we are interested in the case where G is either the group of all $*$ -automorphisms, or the group of inner automorphisms, in which case d^G is denoted by $\tilde{d}$ or $\tilde{d}$ respectively. We then investigate certain aspects of d and d^G with a view towards applications to infinite tensor products.

We first prove the "product formula",

$$\rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) = \rho(\mu_1, \nu_1) \cdot \rho(\mu_2, \nu_2)$$

for normal states μ_1 and ν_1 of G_1 and μ_2 and ν_2 of G_2 where G_1 and G_2 are arbitrary W^* -algebras. Here $\rho(\mu, \nu) = \frac{1}{2}[d(\mu, \nu)]^2$. (See Theorem 2.4 below). This extends the result of [3] where this formula is proved under the assumption that G_1 and G_2 are semi-finite.

As a major application of this result we can complete the classification of the incomplete tensor products of von Neumann [12] up to product isomorphism [4]. In fact in the notation of [4] we have that $\bigotimes_{i \in I} (G_i, \mu_i)$ is product isomorphic to $\bigotimes_{i \in I} (G_i, \nu_i)$ if and only if $\sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty$. In the notation of [12] the equivalent statement is that $\bigotimes_{i \in I} (G_i, x_i)$ is product isomorphic to $\bigotimes_{i \in I} (G_i, y_i)$ if and only if the states μ_i and ν_i induced by the vectors x_i and y_i respectively satisfy the above condition.

In connection with this result we extend the product formula to apply to the case of infinite product states.

In Theorem 10.8 below we indicate the significance of $\tilde{d}$ in the classification of incomplete tensor products up to weak product isomorphism. This type of isomorphism (see Definition 9.1 below) is a natural weakening of the concept of product isomorphism. For a countable family $(G_i)_{i \in I}$ of W^* -algebras, $\bigotimes_{i \in I} (G_i, \mu_i)$ and $\bigotimes_{i \in I} (G_i, \nu_i)$ are weakly product isomorphic if and only if $\sum_{i \in I} [\tilde{d}(\mu_i, \nu_i)]^2 < \infty$.

In the case where each G_i is a semi-finite factor we make this a more explicit criteria by calculating $\tilde{d}$ in terms of the Radon-Nikodym derivatives of the states (see Theorem 8.6 below).

In the course of this calculation we introduce a concept of compatibility which yields some other results concerning d^G . Suppose that S is an arbitrary set and that f and g are any real valued functions on S . We can call f and g compatible if for any pair $x, y \in S$,

$f(x) \leq f(y)$ implies $g(x) \leq g(y)$. We extend this idea to define compatibility between two self-adjoint operators on a Hilbert space. We then define two states on a semi-finite factor to be compatible if their Radon-Nikodym derivatives are compatible. We show that for a factor of type I or II_1 , compatibility minimizes the distance between two states, in the sense that $\tilde{d}(\mu, \nu) = d(\mu, \nu)$ whenever μ and ν are compatible. Moreover we show that for this class of factors $\tilde{d} = d$.

Following this introduction is a historical note on infinite tensor products. We indicate there in more detail the connection between some of the problems considered in this thesis and the previous work in this subject.

In Chapter 0 we review some basic results on von Neumann and W^* -algebras and introduce some notation and terminology. The main references for this material are [5] and [9]. It should be noted that reference [5] is for the second edition of Dixmier's book.

HISTORICAL NOTE ON INFINITE TENSOR PRODUCTS

The theory of tensor products of von Neumann algebras was originated by von Neumann, first for the case of a finite family of algebras in [9], and then for the case of an infinite family in [12].

Consider first the case where we have von Neumann algebras G_1 and G_2 acting on Hilbert spaces H_1 and H_2 respectively. The usual vector space tensor product of H_1 and H_2 can be completed, by means of the inner products on the respective spaces, to another Hilbert space. This space is known as the Hilbert space tensor product of H_1 and H_2 and usually denoted by $H_1 \otimes H_2$. Then we can form a $*$ -algebra G_0 , generated by elements $A_1 \otimes A_2$ where $A_1 \in G_1$ and $A_2 \in G_2$, with the usual bilinear relations, and with multiplication and involution defined by

$$(A_1 \otimes A_2) (B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$$

$$(A_1 \otimes A_2)^* = A_1^* \otimes A_2^* .$$

G_0 is then a $*$ -algebra tensor product of the $*$ -algebras G_1 and G_2 . The representations of G_1 on H_1 and G_2 on H_2 induce in a natural way a representation of G_0 on $H_1 \otimes H_2$, and the von Neumann algebra generated by this representation of G_0 is known as the tensor product of G_1 and G_2 .

Next, suppose we have an infinite family of von Neumann algebras $(G_i)_{i \in I}$ acting on Hilbert spaces $(H_i)_{i \in I}$. In [12], von Neumann extends the construction in the finite case to produce a Hilbert space known as the total tensor product of the family (H_i) and denoted by $\bigotimes_{i \in I} (H_i)$. Again we can construct G_0 , a $*$ -algebra tensor product of the family (G_i) , and represent G_0 on $\bigotimes_{i \in I} (H_i)$ to obtain a von Neumann algebra, known as the total tensor product of the family (G_i) . This total tensor product however fails to possess many of the properties which are enjoyed by the finite tensor product. For example: the total tensor product of factors is not itself a factor; the total tensor product of a family of full operator rings is not itself a full operator ring; the usual associativity laws for tensor products fail to hold; the algebra does not act on a separable space even if each H_i is separable and I is countable.

A remedy for this situation is provided by the decomposition of von Neumann. He showed that $\bigotimes_{i \in I} (H_i)$ splits up into a direct sum of orthogonal subspaces each of which is invariant for the algebra G_0 . These subspaces are known as incomplete tensor products for the family (H_i) and the von Neumann algebras generated by G_0 on each of these subspaces are known as incomplete tensor products of the family (G_i) . It is the incomplete tensor products which appear to be the proper generalization of the finite tensor product.

Each incomplete space is completely determined by choosing a family of unit vectors (x_i) with $x_i \in H_i$. If (x_i) and (y_i) are two such families satisfying $\sum_{i \in I} |1 - (x_i | y_i)| < \infty$ they determine the same incomplete space and therefore the same algebra (More details are given in Chapter 0). Consequently, in the finite case there is only one incomplete tensor product algebra which coincides with the tensor product mentioned above. In the infinite case however there are many such algebras, and a basic problem is to determine the relationships between them.

Much of the theory can be simplified by removing consideration of the underlying Hilbert spaces. Misonou showed in [7] that for a finite family of von Neumann algebras the resulting tensor product was independent, as an algebra, of the particular representations of the given family. One could then speak of a W^* -tensor product of a finite family of W^* -algebras. Nakamura in [11] and Takesaki in [19] gave space-free definitions of the finite tensor product in certain special cases. Takeda in [17] constructed a W^* -tensor product of an infinite family of W^* -algebras by using inductive limits. Bures in [4] gave a general definition for the W^* -tensor product of an arbitrary family of W^* -algebras and showed how to obtain the W^* -analogue of von Neumann's incomplete tensor products, called local tensor products, by using families of states in place of families of vectors.

Let $(G_i)_{i \in I}$ be a family of W^* -algebras. For

every family $(\mu_i)_{i \in I}$ where μ_i is a normal state on G_i one obtains a local tensor product of the family (G_i) , denoted by $\bigotimes_{i \in I} (G_i, \mu_i)$. A fundamental problem is then to determine when two such families of states produce tensor products which are essentially the same (i.e. product isomorphic). The solution, obtained partially in [3] and completed in this thesis, shows that this happens if and only if the respective components of the families are sufficiently "close to each other". The metric d is used to make this idea precise.

When we apply this result to von Neumann algebras we see that an incomplete tensor product does not really depend on the family of vectors determining it, but rather on the family of states induced by these vectors. This explains von Neumann's consideration of weak equivalence ([12] def. 6.1.1), since the vectors $e^{i\theta}x$ and x always induce the same state.

The criteria which we develop for weak product isomorphism can be illustrated by looking at countable tensor products of type I factors. This is a situation which has been considered by many authors, [1], [2], [8], [13], [16] and [18].

In particular we can extend the results discussed in ([12] Chapt. VII) concerning tensor products of I_2 factors. Let G be a fixed I_2 factor represented on a two dimensional Hilbert space H . Fix any orthonormal basis (x, y) in H . Let ω_x and ω_y denote the states on G

induced by the vectors x and y respectively, and for each $\frac{1}{2} \leq \alpha \leq 1$ let $w_\alpha = \alpha w_x + (1-\alpha) w_y$. Let F be the set of all sequences of numbers belonging to the interval $[\frac{1}{2}, 1]$, and for $f = (\alpha_1, \alpha_2, \dots) \in F$ let $\mathfrak{J}_F$ denote the tensor product $\bigotimes_{i=1,2,\dots} (G_i, w_{\alpha_i})$, where $G_i = G$ for all i . Then every countable tensor product of I_2 factors is weakly product isomorphic to $\mathfrak{J}_f$ for some $f \in F$. If $f = (\alpha_i)$ and $g = (\beta_i) \in F$, $\mathfrak{J}_f$ and $\mathfrak{J}_g$ are weakly product isomorphic if and only if they are product isomorphic, which occurs if and only if $\sum_{i=1}^{\infty} [(\sqrt{\alpha_i} - \sqrt{\beta_i})^2 + (\sqrt{1-\alpha_i} - \sqrt{1-\beta_i})^2] < \infty$.

Next, consider the more general case. Suppose that G is a factor of type I_n , $n < \infty$, with trace τ , normalized so that its value on a minimal projection = 1. It is well known that any state μ on G is given by $\mu(A) = \tau(SAS)$ where S is a unique positive element of G . If S and T are two positive elements of G we let $\sigma(S, T) = \sum_{i=1}^n a_i b_i$ where $a_1 \geq a_2 \geq \dots \geq a_n$ are the eigenvalues of S , and $b_1 \geq b_2 \geq \dots \geq b_n$ are the eigenvalues of T (in both cases counted with respective multiplicities).

Now suppose we have a family $(G_i)_{i=1,2,\dots}$ where G_i is a factor of type I_{n_i} , $1 < n_i < \infty$. Let X = the set of all sequences $(S_1, S_2, \dots)$ where S_i is a positive operator in G_i such that the sum of the squares of its eigenvalues (counted with respective multiplicities) = 1. Then, by the remark above there is a 1-1 correspondence between families of states (μ_i) where μ_i is a

normal state on G_1 , and the elements of X . We define an equivalence relation $\sim$ on X by setting $S = (S_1, S_2, \dots)$ $\sim$ $T = (T_1, T_2, \dots)$ if and only if $\sum_{i=1}^{\infty} [1 - \sigma(S_i, T_i)] < \infty$. Then it turns out that the local tensor products determined by S and T are weakly product isomorphic if and only if $S \sim T$.

Using this approach we can rephrase the results of [2], [8], [16] and [18], concerning the type of the factor which arises from the tensor product of finite type I factors, in the following form.

The $\sim$ equivalence classes of X which produce a semi-finite factor are precisely those which contain a sequence consisting of scalar multiples of projections. The unique equivalence class for which these projections are all equal to the identity produces the hyperfinite II_1 factor $\mathcal{P}$. The unique equivalence class for which these projections are all minimal produces the I_{∞} factor. All the rest produce $\mathcal{P} \otimes I_{\infty}$, a factor of type II_{∞} .

It follows then that each of the $\sim$ equivalent classes which does not contain a sequence of scalar multiples of projections produces a type III factor, and no two of these are weakly product isomorphic. However there may be some that are algebraically isomorphic. (Definition 9.1 below). The problem of determining criteria for algebraic isomorphism is a much more difficult one. Powers in [13] considered essentially those tensor products of I_2 factors given in our notation above by $\mathcal{J}_f$ where f is a constant sequence.

He showed that for different choices of the constant the tensor products are all non-algebraically isomorphic and thereby produced an uncountable family of non-isomorphic type III factors. His results have been extensively generalized by Araki and Woods in [1].

If we take infinite tensor products of Abelian algebras the problem reduces to the study of infinite product measure spaces. Suppose we have a family (G_i) where each G_i is an abelian von Neumann algebra and therefore can be considered as $L^\infty(\Omega_i)$ for a suitable measure space Ω_i . The total tensor product of this family is just $L^\infty(\Omega)$ where Ω is the infinite product measure space of the Ω_i . Suppose $\Omega = (X, \mathcal{B}, \mu)$. Then the set X splits up into a union of subsets (X_α) which are essentially disjoint (i.e. $\mu(X_\alpha \cap X_\beta) = 0$ for $\alpha \neq \beta$), and each of which corresponds to a product isomorphism class of incomplete tensor products. In this case, the criteria for product isomorphism in terms of the metric d was shown in [3] to yield the Kakutani theorem on infinite product measures [6].

CHAPTER 0

PRELIMINARIES ON W^* -Algebrasvon-Neumann Algebras.

Let H be a Hilbert space over the complex numbers $\mathbb{C}$ with norm, $\| \cdot \|$, and inner product, $(\cdot | \cdot)$. In addition to the usual norm topology on H we have the weak topology, whereby a net of vectors (x_α) converges to a vector x if $(x_\alpha | y)$ converges to $(x | y)$ for all $y \in H$. This induces on the set of all bounded linear operators on H the weak operator topology whereby a net of operators (A_α) converges to the operator A if $(A_\alpha x)$ converges weakly to Ax for all $x \in H$.

An algebra G of bounded linear operators on H is called a von Neumann algebra on H if

- (a) The identity operator $1 \in G$
- (b) G is self adjoint (i.e. $A \in G$ implies that its adjoint $A^* \in G$.)
- (c) G is closed in the weak operator topology.

Particular examples of von Neumann algebras are $\mathcal{L}(H)$, the set of all bounded linear operators on H and $\mathbb{C}(H)$, the set of all scalar multiples of the identity operator.

For an arbitrary subset S of $\mathcal{L}(H)$, the commu-

tant of $\mathcal{S}$, denoted by $\mathcal{S}'$, is defined as $\{T \in \mathcal{L}(H) : TS = ST \text{ for all } S \in \mathcal{S}\}$. The fundamental double commutant theorem of von Neumann states that a self-adjoint algebra G with identity is a von Neumann algebra if and only if $G = G''$.

For any subset $\mathcal{S}$ of $\mathcal{L}(H)$ and any $x \in H$, $[Sx]$ will denote the norm closed linear span of $\{Sx : S \in \mathcal{S}\}$. If G is a von Neumann algebra on H , then $[G'x] \in G'' = G$. (We will follow throughout the convention employed in the sentence above of using the same symbol for a subspace and for the orthogonal projection onto that subspace whenever the meaning is clear from the context.)

The following is a well known and important result.

Lemma 0.1. Let G and H be Hilbert spaces and let G be a von Neumann algebra on H . Suppose ϕ is a 1-1 algebra homomorphism of G into $\mathcal{L}(G)$ satisfying $\phi(A^*) = (\phi(A))^*$ for all $A \in G$, and $\phi(1) = 1$. Then $\phi(G)$ is a von Neumann algebra on G .

Proof. See e.g. ([5] Chapt. 1, §4, Cor. 2).

W^* -algebras

By a W^* -algebra we mean a linear associative algebra G over $\mathbb{C}$ with identity (always denoted by 1), which possesses an involution; that is a map from G to itself, sending A into A^* , which satisfies for all $A, B \in G$ and $\lambda \in \mathbb{C}$,

- a. $(\lambda A + B)^* = \bar{\lambda} A^* + B^*$, ($\bar{\lambda}$ denotes complex conjugation).
- b. $A^{**} = A$,
- c. $(AB)^* = B^* A^*$.

By a homomorphism from G to $\mathfrak{B}$ where G and $\mathfrak{B}$ are $*$ -algebras we will always mean an algebra homomorphism α , satisfying $\alpha(1) = 1$, and $\alpha(A^*) = (\alpha(A))^*$ for all $A \in G$.

A $*$ -algebra G is called a W^* -algebra if for some Hilbert space H there is a 1-1 homomorphism ϕ from G into the $*$ -algebra $\mathfrak{L}(H)$ so that $\phi(G)$ is a von Neumann algebra on H . By a representation of a W^* -algebra G we will always mean a homomorphism ϕ satisfying the above conditions.

Consider now a fixed W^* -algebra G .

If $\mathfrak{B}$ and $\mathfrak{B}'$ are sub $*$ -algebras of G which are themselves W^* -algebras, Lemma 0.1 shows that $\mathfrak{B} \cap \mathfrak{B}'$ is a W^* -algebra. Consequently, for any subset S of G we may speak of the W^* -algebra generated by S .

An element $A \in G$ is called

- a. self-adjoint if $A = A^*$,
- b. a projection if A is self adjoint and $A^2 = A$;
- c. positive if $A = B^* B$ for some $B \in G$,
- d. a partial isometry if $A^* A$ is a projection (necessarily AA^* is also a projection),

e. a unitary if $A^*A = AA^* = 1$.

Two projections E and F are said to be orthogonal if $EF = 0$.

It is easily verified that any representation of G takes elements of the various types mentioned into operators of the same type in the usual sense.

We will denote the set of all positive elements of G by G^+ . We will denote by $\leq$ the usual partial order defined on the set of self adjoint elements of G whereby $A \leq B$ if $B - A \in G^+$. Any element A of G^+ has a unique square root $\in G^+$ which we denote by $A^{\frac{1}{2}}$.

From the well known polar decomposition theorem any $A \in G$ can be written as $U|A|$, where $|A| \in G^+$ and U is a partial isometry $\in G$ satisfying $AU^*U = A$.

We will denote by $\text{Aut}(G)$, the group of all automorphisms of G and by $\text{Int}(G)$ the subgroup of all $\alpha \in \text{Aut}(G)$ for which there exists some unitary $U \in G$ satisfying $\alpha(A) = UAU^*$ for all $A \in G$.

Any representation of G induces a norm on G (the usual operator norm) which makes it into a Banach algebra. It is easy to verify that this norm is independent of the particular representation and we may speak uniquely of the norm of an element A , which we denote by $\|A\|$.

For any projection $E \in G$ we let $G_E = \{A \in G : A = EAE\}$. Then G_E is a $*$ -algebra with identity E . Moreover G_E is a W^* -algebra since any representation ϕ of G on a Hilbert space H induces a representation of

G_E as a von-Neumann algebra on the subspace $\phi(E)$ ([5] Chapt. 1, §2, Prop. 1).

States and Traces

A linear functional μ on a W^* -algebra G is called a state if $\mu(A) \geq 0$ for all $A \in G^+$ and $\mu(1) = 1$. It follows that $\mu(A)$ is real for any self adjoint A . Moreover μ is automatically bounded (considering G as a Banach space) with $\|\mu\| = 1$.

A state μ is said to be faithful if $\mu(A) > 0$ for all non-zero $A \in G^+$. It is said to be normal if for any upwardly directed set (A_α) of self adjoint elements with least upper bound A , $\mu(A) = \sup\{\mu(A_\alpha)\}$.

We will deal exclusively with normal states and will denote by Σ_G the set of all normal states of G .

A trace on G is a mapping τ from G^+ to the closed interval $[0, \infty]$ satisfying

- (a) $\tau(kA + B) = k\tau(A) + \tau(B)$, for all $k \geq 0$ and $A, B \in G^+$ (we define $0 \cdot \infty = 0$).
- (b) $\tau(UAU^*) = \tau(A)$, for all unitary $U \in G$ and all $A \in G^+$.

We define a trace to be faithful or normal in an analogous way to the definition for states.

We say that a trace τ is finite if $\tau(A) < \infty$ for all $A \in G^+$, semi-finite if for every non-zero $S \in G^+$ there exists $T \leq S$ such that $0 < \tau(T) < \infty$.

We say that the algebra G is finite (semi-finite) if for every non-zero $T \in G^+$ there is a normal finite (semi-finite) trace τ on G such that $0 < \tau(T)$.

Definition 0.2 Let μ be a linear functional on G . For any $\alpha \in \text{Aut}(G)$ we define a functional μ^α by

$$\mu^\alpha(A) = \mu(\alpha(A)) \quad , \text{ for all } A \in G .$$

For any $T \in G$ we define a functional μ_T by

$$\mu_T(A) = \mu(TAT^*) \quad , \text{ for all } A \in G .$$

Lemma 0.3 Let $\mu \in \Sigma_G$, $\alpha \in \text{Aut}(G)$, and $T \in G$ with $\mu(TT^*) = 1$. Then μ^α and $\mu_T \in \Sigma_G$ and

$$(\mu_T)^\alpha = (\mu^\alpha)_{\alpha^{-1}(T)}$$

Proof. Obvious calculations.

For any trace τ on G we define in an analogous way the mappings τ^α and τ_T from G^+ to $[0, \infty]$. If τ is normal then τ^α is also a normal trace for all $\alpha \in \text{Aut}(G)$. If τ is normal and if $S \in G$ is such that $\tau(SS^*) = 1$ then τ_S is finite valued and may be extended by linearity to a normal state of G , which we will denote also by τ_S . In particular for a finite trace τ with $\tau(1) = 1$ we can take $S = 1$. The resulting state (denoted simply by τ) has the property that

$$\tau(AB) = \tau(BA) \quad , \text{ for all } A, B \in G \quad .$$

For any state μ on G we have the inequality

$$(0.1) \quad |\mu(AB)| \leq \mu(AA^*)^{\frac{1}{2}} \mu(B^*B)^{\frac{1}{2}} \quad , \text{ for all } A, B \in G \quad ,$$

which we obtain by applying the Cauchy-Schwartz inequality to the positive conjugate bilinear form, $(A, B) \rightarrow \mu(B^*A)$.

We use this to prove some facts about a finite trace.

Lemma 0.4. Let G be a W^* -algebra with a finite trace

τ . Let $A \in G$ and let E be a projection in G . Then

$$(a) \quad \tau(|A|) = \max \{ |\tau(AV)| : V \text{ a partial isometry} \\ \in G \text{ such that } AVV^* = A \} \quad .$$

$$(b) \quad |\tau(EA)| \leq \|A\| \tau(E) \quad .$$

Proof.

(a) Let $A = U|A|$ be the polar decomposition of A . Then for any V satisfying the stated conditions we obtain from (0.1)

$$\begin{aligned} |\tau(AV)| &= |\tau(U|A|^{\frac{1}{2}} |A|^{\frac{1}{2}} V)| \leq [\tau(U|A|U^*)]^{\frac{1}{2}} [\tau(V^*|A|V)]^{\frac{1}{2}} \\ &= [\tau(|A|U^*U)]^{\frac{1}{2}} [\tau(|A|VV^*)]^{\frac{1}{2}} = \tau|A| \quad . \end{aligned}$$

Then taking $V = U^*$ we obtain

$$|\tau(AU^*)| = \tau(U|A|U^*) = \tau(|A|)$$

(b) Using (0.1) and the fact that $EA^*AE \leq \|A^*A\| E$ we have

$$\begin{aligned}
|\tau(EA)| &= |\tau(E^2A)| = |\tau(EAE)| \\
&\leq [\tau(EA^*AE)]^{\frac{1}{2}} [\tau(E)]^{\frac{1}{2}} \\
&\leq \|A^*A\|^{\frac{1}{2}} [\tau(E)]^{\frac{1}{2}} [\tau(E)]^{\frac{1}{2}} = \|A\| \tau(E) .
\end{aligned}$$

We next consider relationships between states and vectors. If $\mu \in \Sigma_G$ and ϕ is a representation of G on a Hilbert space H , we say that a vector $x \in H$ induces μ relative to ϕ if $\mu(A) = (\phi(A)x|x)$, for all $A \in G$. Definition 0.5. Let $(\mu_i)_{i \in I}$ be any nonvoid family of elements of Σ_G . We define

$$Q((\mu_i)_{i \in I}) = \{[\phi, (x_i)_{i \in I}] : \phi \text{ is a representation of } G \text{ on a Hilbert space } H \text{ and } (x_i)_{i \in I} \text{ is a family of vectors in } H \text{ such that } x_i \text{ induces } \mu_i \text{ relative to } \phi \text{ for all } i \in I\}$$

This set is never empty (see e.g. the proof in [5] of Th. 1, Chapt. 1, §4).

Lemma 0.6. Let $\mu \in \Sigma_G$ and let ϕ be any representation of G . Suppose $[\phi, x] \in Q(\mu)$. Then $[\phi, y] \in Q(\mu)$ if and only if $y = U'x$ for some partial isometry U' in $(\phi(G))'$ such that $U'^*U'x = x$.

Proof. Suppose $y = U'x$ for such a U' . Then for any $A \in G$

$$(\phi(A)y|y) = (\phi(A)U'x|U'x) = (\phi(A)U'^*U'x|x) = (\phi(A)x|x) = \mu(A) .$$

Therefore $[\phi, y] \in Q(\mu)$.

Conversely, if both x and y induce μ it is a well known result that we can define an isometry U'_1 from the subspace $[\phi(G)x]$ to the subspace $[\phi(G)y]$ by taking $U'_1(\phi(A)x) = \phi(A)y$ and extending to the closure (see [5], Chapt. 1, §4, Lemma 3). Extending U'_1 to an operator U' on the whole space by setting $U' = 0$ on the orthogonal complement of $[\phi(G)x]$ gives us the desired partial isometry.

If $\mu \in \Sigma_G$ the support of μ is defined to be the smallest projection E of G for which $\mu(1 - E) = 0$. By the normality of μ the support always exists. The following is a well known result (see [5] p. 58).

Lemma 0.7. Let $\mu \in \Sigma_G$. For any $[\phi, x] \in Q(\mu)$, the support of $\mu = \phi^{-1} [(\phi(G))'x]$.

Factors

A W^* -algebra G is called a factor if its center (i.e. $\{T \in G : TA = AT \text{ for all } A \in G\}$) consists only of scalar multiples of the identity.

A factor G can be effectively analyzed by looking at its set of projections. Two projections E and F are said to be equivalent, written $E \sim F$, if for some partial isometry $U \in G$, $E = U^*U$ and $F = UU^*$. We say that $E \preceq F$ if $E \sim G$ for some $G \leq F$. While these definitions can be made for any W^* -algebra, factors have the important property that for any two projections E and F either $E \preceq F$ or $F \preceq E$. A projection E is said to be finite

if $E \sim F$ and $F \leq E$ imply that $F = E$. A projection which is not finite is said to be infinite.

A factor is said to be of type I if it contains a minimal non-zero projection. Any such factor is isomorphic to the algebra of all bounded linear operators on some Hilbert space and we then say it is of type I_n where n is the dimension of the space.

A factor with no minimal projection is said to be of type;

- a. II_1 , if the identity is finite,
- b. II_∞ , if the identity is infinite but some non-zero projection is finite,
- c. III , if all non-zero projections are infinite.

Factors of type I , II_1 and II_∞ are semi-finite.

In fact they each have a semi-finite trace, which is unique up to a positive scalar multiple, and which is normal and faithful. The only trace on a type III factor is that which takes value ∞ on all non-zero positive elements.

If E is a projection in a semi-finite factor G , then G_E is a semi-finite factor. G_E is finite if and only if E is a finite projection and this occurs if and only if the trace of E is finite. It follows that the finite factors are those of type I_n , n a finite cardinal, and of type II_1 .

A factor of type II_1 is said to be hyperfinite if it is generated by an increasing sequence of subalgebras $(G_i)_{i=1,2,\dots}$ where G_i is a factor of type I_{2^i} . Any

two hyperfinite II_1 factors are isomorphic. ([5], Chapt. 3, §7, Th. 2).

Tensor Products of von Neumann algebras

Let I be an arbitrary indexing set and let $(G_i)_{i \in I}$ be a family of von Neumann algebras acting on the Hilbert spaces $(H_i)_{i \in I}$.

A family of vectors (x_i) with $x_i \in H_i$ is called a C_0 -sequence if $\sum_{i \in I} \|x_i\|^2 < \infty$. An equivalence relation is defined on the set of all C_0 -sequences by taking (x_i) equivalent to (y_i) iff $\sum_{i \in I} \|x_i - y_i\|^2 < \infty$.

Let (x_i) be a C_0 -sequence and let Γ be its equivalence class. For each $(y_i) \in \Gamma$ we let $\otimes_{i \in I} (y_i)$ denote the multi-anti linear mapping from Γ to $\mathbb{C}$ defined by

$$\otimes_{i \in I} (y_i)[(z_i)] = \prod_{i \in I} (y_i | z_i), \text{ for all } (z_i) \in \Gamma.$$

These mappings generate a linear subset H_0 of the space of all multi-anti linear functionals on Γ , and H_0 becomes a pre-Hilbert space by defining $(\otimes(y_i) | \otimes(z_i)) = \prod_{i \in I} (y_i | z_i)$, and extending this inner product to all of H_0 by linearity. The completion of H_0 is known as the tensor product of the family (H_i) with respect to the C_0 -sequence (x_i) and denoted by $\otimes_{i \in I} (H_i, x_i)$.

An orthonormal basis for $\bigotimes_{i \in I} (H_i, x_i)$ can be chosen in the following manner. First note that we can assume that $\|x_i\| = 1$ for all i since replacing each x_i by $x_i/\|x_i\|$ yields an equivalent C_0 -sequence. For each $i \in I$, let $(x_i^j)_{j \in J(i)}$ be an orthonormal basis for H_i which includes x_i . Then the set of all elements of the form $\bigotimes_{i \in I} (y_i)$ where for each $i \in I$ $y_i = x_i^j$ for some $j \in J(i)$, and for all but a finite number of $i \in I$ y_i in particular $= x_i$, constitutes an orthonormal basis for $\bigotimes_{i \in I} (H_i, x_i)$. This shows that $\bigotimes_{i \in I} (H_i, x_i)$ is separable if each H_i is separable and I is countable.

Now, for any $k \in I$ and $A_k \in G_k$ we define the operator $\bar{A}_k$ on $\bigotimes (H_i, x_i)$ by taking

$$\bar{A}_k(\bigotimes (y_i)) = \bigotimes (y'_i) \quad \text{where} \quad y'_i = y_i \quad \text{if} \quad i \neq k$$

$$y'_k = A_k y_k.$$

$\bar{A}_k$ is then extended to the whole space by linearity and continuity.

Let G be the von Neumann algebra on $\bigotimes (H_i, x_i)$ generated by $\{\bar{A}_i : i \in I \text{ and } A_i \in G_i\}$. Then the tensor product of the family (G_i) with respect to the C_0 -sequence (x_i) , denoted by $\bigotimes_{i \in I} (G_i, x_i)$, is defined to be the von-Neumann algebra G , together with the injections of each G_i into G given by $A_i \rightarrow \bar{A}_i$.

Note that if (x_i) and (y_i) are equivalent C_0 -sequences it follows immediately from the definitions that $\otimes (H_i, x_i) = \otimes (H_i, y_i)$ and $\otimes (G_i, x_i) = \otimes (G_i, y_i)$.

Complete details and proofs of this material can be found in [12] or [2].

Products and Tensor Products of W^* -algebras

Let I be an arbitrary indexing set and let $(G_i)_{i \in I}$ be a family of W^* -algebras.

A product for this family is an object $(G, (\alpha_i)_{i \in I})$ where G is a W^* -algebra, and for each $i \in I$ α_i is a 1-1 homomorphism from G_i into G satisfying

(a) $\alpha_i(G_i)$ and $\alpha_j(G_j)$ commute pointwise for $i \neq j$.

(b) $\{\alpha_i(G_i) : i \in I\}$ generates G as a W^* -algebra.

We say that the products $(G, (\alpha_i))$ and $(B, (\beta_i))$ are product isomorphic if there exists an isomorphism ϕ from G onto B such that $\phi \alpha_i = \beta_i$ for all $i \in I$.

We let $\Lambda = \Lambda((G_i))$ denote the set of all families (μ_i) where $\mu_i \in \Sigma_{G_i}$ for all $i \in I$.

We say that μ is a product state of the product $(G, (\alpha_i))$ if μ is a normal state of G such that for some $(\mu_i) \in \Lambda$

$$\mu \left(\prod_{i \in F} \alpha_i(A_i) \right) = \prod_{i \in F} \mu_i(A_i) ,$$

for all finite subsets F of I and all $A_i \in G_i$. We will denote such a μ by $\bigotimes_{i \in I} (\mu_i)$. It follows from the normality of μ that if $\bigotimes_{i \in I} (\mu_i)$ exists it is unique.

Definition 0.8. For any $(\mu_i) \in \Lambda$ we define a product,

$\bigotimes_{i \in I} (G_i, \mu_i)$, as follows. For each $i \in I$ choose $[\phi_i, x_i] \in Q(\mu_i)$. Then $\bigotimes_{i \in I} (G_i, \mu_i)$ is defined to be the algebra $\bigotimes_{i \in I} (\phi_i(G_i), x_i)$ with injections α_i given by

$$\alpha_i(A_i) = \overline{\phi_i(A_i)} \text{ for all } A_i \in G_i.$$

It is shown in [3] or [4] that $\bigotimes_{i \in I} (G_i, \mu_i)$ is unique up to product isomorphism.

In [4] a purely algebraic definition of $\bigotimes_{i \in I} (G_i, \mu_i)$ is given.

It is easy to see that the product state $\bigotimes_{i \in I} (\mu_i)$ exists on $\bigotimes_{i \in I} (G_i, \mu_i)$. In fact in the construction above it is induced by the vector $\bigotimes_{i \in I} (x_i)$.

Following the usage in [4] we will refer to these products, constructed from an element of Λ , as local tensor products.

Suppose that the indexing set I is finite. For ease in notation let $I = \{1, 2\}$. Then all local tensor products of a given family are product isomorphic and we may speak of a unique tensor product $(G, (\alpha_1, \alpha_2))$ of the family (G_1, G_2) . It is usual to denote G by $G_1 \otimes G_2$ and the element $\alpha_1(A_1) \alpha_2(A_2)$ by $A_1 \otimes A_2$. Every element (μ_1, μ_2) of Λ determines a product state, denoted by $\mu_1 \otimes \mu_2$.

Lemma 0.9. Let $\mathcal{G}$ and $\mathcal{B}$ be W^* -algebras, and let $\mu \in \Sigma_{\mathcal{G}}$ and $\nu \in \Sigma_{\mathcal{B}}$. Then for all $S \in \mathcal{G}^+$ and $T \in \mathcal{B}^+$ with $\mu(SS^*) = \nu(TT^*) = 1$

$$(\mu \otimes \nu)_{(S \otimes T)} = \mu_S \otimes \nu_T$$

Proof. Choose any $A \in \mathcal{G}$ and $B \in \mathcal{B}$. Then

$$\begin{aligned} (\mu \otimes \nu)_{(S \otimes T)} [A \otimes B] &= (\mu \otimes \nu)[(S \otimes T) (A \otimes B) (S \otimes T)^*] \\ &= (\mu \otimes \nu)[(SAS^*) \otimes (TAT^*)] \\ &= \mu(SAS^*) \nu(TAT^*) = \mu_S(A) \nu_T(B) \\ &= \mu_S \otimes \nu_T [A \otimes B] . \end{aligned}$$

The lemma now follows from the uniqueness property of product states.

CHAPTER I

THE METRIC d 1. Basic properties of d and ρ

The quantities d and ρ were introduced in [3].

In this section we review some basic properties.

Definition 1.1. Let G be a W^* -algebra and let μ and $\nu \in \Sigma_G$. We define.

$$d(\mu, \nu) = \inf \{ \|x - y\| : [\phi, (x, y)] \in Q(\mu, \nu) \} ,$$

$$\rho(\mu, \nu) = \sup \{ |(x|y)| : [\phi, (x, y)] \in Q(\mu, \nu) \} .$$

It is proved in [3] that d is a metric on Σ_G and that d and ρ are connected by the formula

$$(1.1) \quad [d(\mu, \nu)]^2 = 2[1 - \rho(\mu, \nu)] .$$

The number $d(\mu, \nu)$ can vary from 0 to $\sqrt{2}$. To gain some insight into the definition we can consider the extreme cases.

Lemma 1.2.

- (a) $d(\mu, \nu) = 0$ if and only if $\mu = \nu$.
- (b) $d(\mu, \nu) = \sqrt{2}$ if and only if the support of μ and the support of ν are orthogonal.

Proof.

(a) This follows from the fact that d is a metric.

(b) Let E = the support of μ , F = the support of ν and suppose that $FE = 0$. Choose any $[\phi, (x, y)] \in Q(\mu, \nu)$.

Since $\|\phi(E)x\|^2 = \mu(1) = \|x\|^2$, we have that $\phi(E)x = x$ and similarly $\phi(F)y = y$. Then

$$(x|y) = (\phi(E)x|\phi(F)y) = (\phi(FE)x|y) = 0.$$

Therefore $\rho(\mu, \nu) = 0$ and from (1.1) we have that $d(\mu, \nu) = \sqrt{2}$.

Conversely suppose that $d(\mu, \nu) = \sqrt{2}$. Then $\rho(\mu, \nu) = 0$. Choose any $[\phi, (x, y)] \in Q(\mu, \nu)$. For any unitary U' in $(\phi(G))'$ Lemma 0.6 shows that $[\phi, (U'x, y)] \in Q(\mu, \nu)$ and therefore $0 \leq |(U'x|y)| \leq \rho(\mu, \nu) = 0$. We have then that $(A'x|y) = 0$ for all $A' \in (\phi(G))'$ by writing each such element as a linear combination of unitaries. By Lemma 0.7, y is orthogonal to a dense subset of the subspace $\phi(E)$ and therefore $\phi(E)y = 0$. Then, $\phi(E)A'y = A'\phi(E)y = 0$, for all $A' \in (\phi(G))'$, and by Lemma 0.7 again $\phi(E)\phi(F) = 0$. Since ϕ is a 1-1 mapping we have that $EF = 0$.

Lemma 1.3 Let G be a W^* -algebra and let μ, ν and $\omega \in \Sigma_G$. Then

(a) For any $0 \leq \alpha \leq 1$, $d(\mu, (1-\alpha)\mu + \alpha\nu) \leq \sqrt{2\alpha}$

(b) $|\rho(\omega, \mu) - \rho(\omega, \nu)| \leq \sqrt{2} d(\mu, \nu)$

Proof.

(a) Let μ' denote the state $(1-\alpha)\mu + \alpha\nu$.

Choose any $[\phi, (x, y)] \in Q(\mu, \nu)$. A direct calculation shows

that $[\phi \oplus \phi, (\sqrt{1-\alpha}x \oplus \sqrt{\alpha}x, \sqrt{1-\alpha}x \oplus \sqrt{\alpha}y)] \in Q(\mu, \mu')$.

Here $\phi \oplus \phi$ denotes the direct sum representation of ϕ with itself on the space $H \oplus H$ where H is the underlying space of ϕ . We have then

$$\begin{aligned} d(\mu, \mu') &\leq \|\sqrt{1-\alpha}x \oplus \sqrt{\alpha}x - \sqrt{1-\alpha}x \oplus \sqrt{\alpha}y\| \\ &= \sqrt{\alpha} \|0 \oplus (x-y)\| \leq \sqrt{2\alpha} \end{aligned}$$

(b) Using (1.1) and the backwards triangle inequality for d we obtain,

$$\begin{aligned} |\rho(\omega, \mu) - \rho(\omega, \nu)| &= \frac{1}{2} |[d(\omega, \mu)]^2 - [d(\omega, \nu)]^2| \\ &= \frac{1}{2} (d(\omega, \mu) + d(\omega, \nu)) |d(\omega, \mu) - d(\omega, \nu)| \\ &\leq \sqrt{2} d(\mu, \nu) . \end{aligned}$$

2. The product formula for ρ

In this section we will prove in general the product formula for ρ which was obtained in ([3], Theorem 2.5) for semi-finite algebras. We do this by means of Lemma 2.1 which is similar in statement and proof to ([3], Lemma 1.6). However by dealing with only one element of G we are able to avoid the use of a trace.

Lemma 2.1. Let G be a W^* -algebra, $\mu \in \Sigma_G$ and let $T \in G^+$ be such that $\mu(T^2) = 1$. Then,

$$\rho(\mu, \mu_T) = \mu(T)$$

Proof. Choose any $[\phi, (x, y)] \in Q(\mu, \mu_T)$. A direct calculation shows that $[\phi, (x, \phi(T)x)] \in Q(\mu, \mu_T)$. Therefore,

$$(2.1) \quad \rho(\mu, \mu_T) \geq |(x | \phi(T)x)| = \mu(T).$$

On the other hand, since y and $\phi(T)x$ induce the same state relative to ϕ , Lemma 0.6 shows that $y = U'\phi(T)x$ for some partial isometry U' in $(\phi(G))'$. Therefore

$$\begin{aligned} |(x | y)| &= |(x | U'\phi(T)x)| = |(U'\phi(T)^{\frac{1}{2}}x | \phi(T)^{\frac{1}{2}}x)| \\ &\leq \|\phi(T)^{\frac{1}{2}}x\|^2 = \mu(T). \end{aligned}$$

Taking the supremum over all $[\phi, (x, y)] \in Q(\mu, \nu)$ we obtain that

$$\rho(\mu, \mu_T) \leq \mu(T) .$$

This formula and (2.1) complete the proof.

In the remainder of this section we consider two W^* -algebras G_1 and G_2 and their tensor product $G_1 \otimes G_2$. We also consider states μ_1 and $\nu_1 \in \Sigma_{G_1}$ and μ_2 and $\nu_2 \in \Sigma_{G_2}$. We want to prove the following product formula for ρ .

$$(2.2) \quad \rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) = \rho(\mu_1, \nu_1) \rho(\mu_2, \nu_2) .$$

Lemma 2.2. Let $j = 1$ or 2 . Suppose that $\mu_j = (\nu_j)_{T_j}$ for some $T_j \in G_j^+$. Then (2.2) holds.

Proof. From Lemma 0.9, $\mu_1 \otimes \mu_2 = (\nu_1)_{T_1} \otimes (\nu_2)_{T_2} = (\nu_1 \otimes \nu_2)_{T_1 \otimes T_2}$ and by successive applications of Lemma 2.1

$$\begin{aligned} \rho(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) &= \nu_1 \otimes \nu_2 (T_1 \otimes T_2) \\ &= \nu_1(T_1) \nu_2(T_2) = \rho(\mu_1, \nu_1) \rho(\mu_2, \nu_2) . \end{aligned}$$

Lemma 2.3. For any δ such that $0 \leq \delta \leq 1$ let $\nu'_j = (1-\delta)\nu_j + \delta\mu_j$, where $j = 1$ or 2 . Then

$$d(\nu_1 \otimes \nu_2, \nu'_1 \otimes \nu'_2) \leq 2\sqrt{2\delta}$$

Proof. By the triangle inequality,

$$d(v_1 \otimes v_2, v'_1 \otimes v'_2) \leq d(v_1 \otimes v_2, v_1 \otimes v'_2) \\ + d(v_1 \otimes v'_2, v'_1 \otimes v'_2) .$$

From Lemma 1.3 (a),

$$d(v_1 \otimes v_2, v_1 \otimes v'_2) \\ = d(v_1 \otimes v_2, [(1-\delta)v_1 \otimes v_2 + \delta(v_1 \otimes \mu_2)]) \\ \leq \sqrt{2\delta}$$

Similarly, $d(v_1 \otimes v'_2, v'_1 \otimes v'_2) \leq \sqrt{2\delta}$, which completes the proof.

We now remove the restrictions of Lemma 2.3

Theorem 2.4. The product formula (2.2) holds in general.

Proof. Given any ϵ such that $0 < \epsilon < 1$, let $\delta = \frac{\epsilon^2}{144}$. Let $v'_j = (1-\delta)v_j + \delta\mu_j$, where $j = 1$ or 2 . Then for $n > \frac{1}{\delta}$, $\mu_j(A) \leq n v'_j(A)$ for all $A \in G_j^+$. From Sakai's Radon-Nikodym Theorem ([5], Chapt. 1, §4, Th. 5),

$$(2.3) \quad \mu_j = (v'_j)_{T_j}, \text{ for some } T_j \in G_j^+ .$$

For ease in notation let

$$a = \rho(u_1 \otimes u_2, v_1 \otimes v_2),$$

$$b = \rho(u_1, v_1),$$

$$c = \rho(u_2, v_2),$$

and let a' , b' and c' be defined similarly with v'_j replacing v_j . We want to show that $(a-bc) = 0$. From Lemmas 1.3, 2.1, 2.3 and formula (2.3) we have

$$(a'-b'c') = 0,$$

$$|a-a'| \leq 4\delta^{\frac{1}{2}},$$

$$|b-b'| \leq 2\delta^{\frac{1}{2}} \quad \text{and} \quad |c-c'| \leq 2\delta^{\frac{1}{2}}.$$

Then, since δ , b and c are all ≤ 1 we have

$$|a-bc| = |(a-bc) - (a'-b'c')|$$

$$\leq |a-a'| + |bc-b'c'|$$

$$\leq |a-a'| + b|c-c'| + c|b-b'| + |b-b'||c-c'|$$

$$\leq 2[\delta^{\frac{1}{2}}(2+b+c) + 2\delta] \leq 2\delta^{\frac{1}{2}}(4+b+c)$$

$$\leq 12\delta^{\frac{1}{2}} \leq \epsilon.$$

Since ϵ was chosen arbitrarily we conclude that $(a-bc) = 0$.

Remark 2.5 We can of course define d and ρ for any positive normal linear functionals as was done in [3], instead of restricting the definition to states. Since a vector x induces a functional μ if and only if the vector $k^{\frac{1}{2}}x$ induces $k\mu$, we have that $\rho(k\mu, \nu) = k^{\frac{1}{2}} \rho(\mu, \nu)$ for all $k > 0$.

Moreover $\mu \otimes \nu$ is defined for functionals and the mapping (μ, ν) to $\mu \otimes \nu$ is bilinear. It follows from these remarks that the product formula will still hold for positive normal linear functionals.

CHAPTER II

THE SEMI-METRICS d^G 3. Definitions of d^G and ρ^G

Definition 3.1. Let G be a W^* -algebra and let G be any subgroup of $\text{Aut}(G)$. For any μ and $\nu \in \Sigma_G$ we define

$$d^G(\mu, \nu) = \inf \{d(\mu^\alpha, \nu^\beta) : \alpha, \beta \in G\},$$

$$\rho^G(\mu, \nu) = \sup \{\rho(\mu^\alpha, \nu^\beta) : \alpha, \beta \in G\}.$$

From (1.1) we obtain

$$(3.1) \quad [d^G(\mu, \nu)]^2 = 2[1 - \rho^G(\mu, \nu)].$$

Lemma 3.2.

- (a) $d^G(\mu, \nu) = \inf \{d(\mu, \nu^\alpha) : \alpha \in G\}$
- (b) $\rho^G(\mu, \nu) = \sup \{\rho(\mu, \nu^\alpha) : \alpha \in G\}$

Proof. Let $\alpha \in G$. For any $[\phi, (x, y)] \in Q(\mu, \nu)$ $\phi\alpha$ is also a representation of G and a direct calculation shows $[\phi\alpha, (x, y)] \in Q(\mu^\alpha, \nu^\alpha)$. Therefore $d(\mu^\alpha, \nu^\alpha) \leq \|x - y\|$ and taking the infimum over all elements of $Q(\mu, \nu)$ we obtain

$$d(\mu^\alpha, \nu^\alpha) \leq d(\mu, \nu) .$$

Then replacing μ and ν by μ^α and ν^α respectively and replacing α by α^{-1} we have

$$d(\mu^\alpha, \nu^\alpha) = d(\mu, \nu) , \text{ for all } \alpha \in G .$$

Therefore $d(\mu^\alpha, \nu^\beta) = d(\mu, \nu^{\beta\alpha^{-1}})$ which proves the lemma.

It follows easily that for all G , d^G is a semi-metric on Σ_G .

Notation. If $G = \text{all of } \text{Aut}(G)$ we will let $\tilde{\rho}$ and $\tilde{d}$ denote ρ^G and d^G respectively.

If $G = \text{Int}(G)$ we will let $\tilde{\rho}$ and $\tilde{d}$ denote ρ^G and d^G respectively.

4. An Inequality.

In this section we prove a matrix inequality which is a fundamental result for the calculation of d^G .

Notation. For any integer n let R_n denote the real vector space consisting of all sequences $a = (a_1, a_2, \dots, a_n)$ of real numbers. We let

$$D_n^+ = \{a \in R_n : a_i \geq a_j \text{ for } i \leq j \text{ and } a_i \geq 0 \text{ for all } i\}$$

Note that D_n^+ is closed under addition, and multiplication by a non-negative scalar.

For a positive integer $r \leq n$, let δ_r denote the element a of D_n^+ with $a_i = 1$, $1 \leq i \leq r$; $a_i = 0$, $r < i \leq n$. Then any element of D_n^+ can be written as a linear combination of δ_r 's with non-negative coefficients.

In fact for $a \in D_n^+$

$$a = \sum_{i=1}^n (a_i - a_{i+1}) \delta_i + a_n \delta_n.$$

Finally we let $M_n(\mathbb{C})$ denote the set of all n by n matrices $k = (k_{ij})$ over the complex numbers.

Definition 4.1. For $a, b \in D_n^+$ and $k \in M_n(\mathbb{C})$ we define

$$T_k(a, b) = \sum_{i,j=1}^n a_i b_j k_{ij}.$$

Lemma 4.2. Let c be an element of R_n such that $c_i \geq 0$ for $1 \leq i \leq n$, and let $k \in M_n(\mathbb{C})$ be such that for all

r with $1 \leq r \leq n$

$$\left| \sum_{j=1}^r k_{1j} \right| \leq c_1, \text{ and } \left| \sum_{i=1}^r k_{ij} \right| \leq c_j.$$

Then for all $a, b, \in D_n^+$

$$(4.1) \quad |T_k(a, b)| \leq \sum_{i=1}^n a_i b_i c_i.$$

Proof. Let $a, \bar{a}, b, \bar{b} \in D_n^+$ and let λ be a positive number.

Suppose that (4.1) holds for the pairs (a, b) and $(\bar{a}, b)$.

Then

$$\begin{aligned} |T_k(\lambda \bar{a} + a, b)| &= \left| \sum_{i,j=1}^n (\lambda \bar{a}_i + a_i) b_j k_{ij} \right| \\ &= \left| \lambda \sum_{i,j=1}^n \bar{a}_i b_j k_{ij} + \sum_{i,j=1}^n a_i b_j k_{ij} \right| \\ &\leq \lambda |T_k(\bar{a}, b)| + |T_k(a, b)| \\ &\leq \lambda \sum_{j=1}^n \bar{a}_i b_i c_i + \sum_{i=1}^n a_i b_i c_i \\ &= \sum_{i=1}^n (\lambda \bar{a}_i + a_i) b_i c_i. \end{aligned}$$

Therefore (4.1) holds for $((\lambda \bar{a} + a), b)$. Similarly, if (4.1) holds for (a, b) and $(a, \bar{b})$ it will hold for $(a, (b + \lambda \bar{b}))$.

It follows then, from the remarks above concerning D_n^+ that it is sufficient to prove (4.1) for $a = \delta_r$ and $b = \delta_s$.

Moreover by the symmetry of the conditions on k we may assume that $r \leq s$. We have in this case.

$$\begin{aligned}
|T_k(a,b)| &= \left| \sum_{i=1}^r \sum_{j=1}^s k_{ij} \right| \leq \sum_{i=1}^r \left| \sum_{j=1}^s k_{ij} \right| \\
&\leq \sum_{i=1}^r c_i = \sum_{i=1}^n a_i b_i c_i,
\end{aligned}$$

and the lemma is proved.

Theorem 4.3. Let G be a W^* -algebra with a finite trace

τ . Let S and T be elements of G^+ such that:

$S = \sum_{i=1}^n a_i E_i$, where $a = (a_1, a_2, \dots, a_n) \in D_n^+$ and the E_i 's are mutually orthogonal projections in G ; $T = \sum_{i=1}^n b_i F_i$, with similar conditions; and $\tau(E_i) = \tau(F_i)$ for $i = 1, 2, \dots, n$.

We will denote this common value by c_i .

Then, for any $V \in G$ with $\|V\| \leq 1$

$$|\tau(STV)| \leq \sum_{i=1}^n a_i b_i c_i.$$

Proof.

$$STV = \sum_{i,j=1}^n a_i b_j E_i F_j V$$

Therefore

$$\tau(STV) = \sum_{i,j=1}^n a_i b_j k_{ij}, \text{ where } k_{ij} = \tau(E_i F_j V).$$

The result will now follow from Lemma 4.2 after we verify the appropriate conditions on (k_{ij}) . Choose any $1 \leq r \leq n$.

Since $\sum_{j=1}^r F_j$ is a projection, $\left\| \sum_{j=1}^r F_j V \right\| \leq 1$. Then, using Lemma 0.4(b)

$$\begin{aligned}
\left| \sum_{j=1}^r k_{ij} \right| &= \left| \tau \left[E_i \left(\sum_{j=1}^r F_j V \right) \right] \right| \\
&\leq \tau(E_i) \left\| \sum_{j=1}^r F_j V \right\| \leq c_i.
\end{aligned}$$

Similarly, since $\tau(E_i F_j V) = \tau(F_j V E_i)$ we have

$$\left| \sum_{i=1}^r k_{ij} \right| = \left| \tau \left[F_j \left(\sum_{i=1}^r V E_i \right) \right] \right| \leq c_j .$$

5. Monotone Functions

Notation

Let R^+ denote the non-negative real numbers and let $(R^+)' = (R^+ \cup +\infty)$.

Let M = the set of all functions f from $(R^+)'$ to $(R^+)'$ such that $x \leq y$ implies $f(x) \geq f(y)$.

Definition 5.1. For any $f \in M$ we define a function $\bar{f}$ on $(R^+)'$ by

$$\bar{f}(a) = \inf \{x \in (R^+)' : f(x) \leq a\}$$

(Note that $\inf\{\emptyset\} = +\infty$, $\sup\{\emptyset\} = 0$.)

The following facts about $\bar{f}$ are either well known or easily verified.

Lemma 5.2

(a) $\bar{f} \in M$ and moreover it is right continuous (i.e. $f(\inf(S)) = \sup(f(S))$, for all $S \subset (R^+)'$.)

(b) $\bar{f}(a) = x$, if and only if

(i) for every $y < x$, $f(y) > a$

(ii) for every $y > x$, $f(y) \leq a$

(c) Let $k \in R^+$. Then

$g(x) = kf(x)$ for all x implies $\bar{g}(x) = \bar{f}(\frac{x}{k})$ for all x .

$f(x) = f(\frac{x}{k})$ for all x implies $\bar{g}(x) = k\bar{f}(x)$ for all x .

(d) Let $(f_n)_{n=1,2,\dots}$ be a sequence of functions in M such that $f_i \leq f_j$ for $i \leq j$. Define f on $(R^+)'$ by $f(x) = \sup_n (f_n(x))$. Then,

$$f \in M \text{ and } (\bar{f}_n) \uparrow \bar{f}.$$

(e) Let ϕ be a continuous function defined on the interval $[a, b]$ and let $f \in M$ be finite valued on this interval. Then the Stieltjes integral $\int_a^b \phi(x) df(x) = \int_{f(a)}^{f(b)} \phi(\bar{F}(\alpha)) d\alpha$.

We need an extension of (e) to improper integrals.

Lemma 5.3 Let ϕ be a continuous function on R^+ such that $\phi(x) \geq 0$ for all x and $\phi(0) = 0$. Let $f \in M$ be such that $f(x) < \infty$ for $x > 0$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Then

$$-\int_0^\infty \phi(x) df(x) = \int_0^\infty \phi(\bar{F}(\alpha)) d\alpha.$$

($\int_0^\infty$ denotes $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left\{ \int_{\frac{1}{n}}^m \right\}$ and is allowed to take the value $+\infty$).

Proof. Let $f^+(0) = \sup \{f(x) : x \in (R^+)'\}$ and note first that $y \in f^+(0)$ implies by Lemma 5.2(b) that $\bar{F}(y) = 0$, and therefore that $\phi(\bar{F}(y)) = \phi(0) = 0$. Then using Lemma 5.2(d)

$$\begin{aligned} -\int_0^\infty \phi(x) df(x) &= -\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left(\int_{\frac{1}{n}}^m \phi(x) df(x) \right) \\ &= \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \left(\int_{f(m)}^{f(\frac{1}{n})} \phi(\bar{F}(\alpha)) d\alpha \right) = \int_0^{f^+(0)} \phi(\bar{F}(\alpha)) d\alpha \\ &= \int_0^\infty \phi(\bar{F}(\alpha)) d\alpha. \end{aligned}$$

The main result of this section is the following.

Theorem 5.4. Let ϕ be any continuous function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$. Let f, g and $h \in M$ be such that for all $x, y \in \mathbb{R}^+$.

$$h(\phi(x, y)) \in [f(x), g(y)] .$$

(We let $[a, b]$ denote the closed interval with end-points a and b even if $a > b$). Then for all $a \in (\mathbb{R}^+)'$ such that $\bar{f}(a)$ and $\bar{g}(a) \in \mathbb{R}^+$

$$\bar{h}(a) = \phi(\bar{f}(a), \bar{g}(a)) .$$

Proof. Let $\bar{f}(a) = x$ and $\bar{g}(a) = y$. Given any $\epsilon > 0$ choose, by the continuity of ϕ , $\delta > 0$ so that

$$(5.1) \quad \phi(x-\delta, y-\delta) \geq \phi(x, y) - \epsilon$$

$$(5.2) \quad \phi(x+\delta, y+\delta) \leq \phi(x, y) + \epsilon$$

By our hypothesis

$$h(\phi(x-\delta, y-\delta)) \in [f(x-\delta), g(y-\delta)]$$

and from Lemma 5.2(b) both $f(x-\delta)$ and $g(y-\delta)$ are $> a$.

Using (5.1) we obtain

$$(5.3) \quad h(\phi(x, y) - \epsilon) \geq h(\phi(x-\delta, y-\delta)) > a$$

From Lemma 5.2(b) again both $f(x+\delta)$ and $g(y+\delta)$ are $\leq a$ and using (5.2) we obtain similarly that

$$(5.4) \quad h(\phi(x,y)+\epsilon) \leq a$$

From (5.3), (5.4) and the other direction of Lemma 5.2(b) we have

$$\bar{h}(a) = \phi(x,y) = \phi(\bar{f}(a), \bar{g}(a)) .$$

6. Distribution Functions and σ .

Spectral Resolutions.

Let A be a self-adjoint operator (not necessarily bounded) on a Hilbert space H . Let $\mathcal{D}_A$ denote the domain of A . (For terminology concerning unbounded operators refer to [20].) By a spectral resolution of A we will always mean the unique family of projections in H , $\{E(\lambda)\}$, λ a real number, which satisfies the following properties:

1. $\lambda \leq \mu$ implies $E(\lambda) \leq E(\mu)$;
2. $\bigcup_{\lambda} E(\lambda) = 1$, $\bigcap_{\lambda} E(\lambda) = 0$;
3. $E(\lambda)$ commutes with any operator in $\mathfrak{L}(H)$ which commutes with A ;
4. $\bigcap_{\lambda > \lambda_0} E(\lambda) = E(\lambda_0)$, for all λ_0 ;
5. For $-\infty < \lambda \leq \mu < +\infty$, $x \in E(\mu) - E(\lambda)$ implies that $x \in \mathcal{D}_A$ and if $\|x\| = 1$, $\lambda < (Ax|x) \leq \mu$.

Integrable Operators and Radon-Nikodym Derivatives.

Suppose that G is a von Neumann algebra on a Hilbert space H . An operator S on H (not necessarily bounded) is said to be affiliated with G , written $S \eta G$, if $U S U^* = S$ for every unitary $U \in G'$. If S is in fact bounded and everywhere defined, the double commutant theorem shows that $S \eta G$ if and only if $S \in G$. For any self-adjoint

S with spectral resolution $\{E(\lambda)\}$, it is easy to see that $S \in G$ if and only if $E(\lambda) \in G$ for all real λ .

Now suppose that τ is a semi-finite, faithful, normal trace on G . Let $\mathcal{N} = \{A \in G : \tau(A^*A) < \infty\}$. It is shown in ([5] Chapt. 1, §6, Prop. 1) that $\mathcal{N}$ is a two-sided ideal of G and that the ideal $\mathcal{M} = \mathcal{N}^2$ consists precisely of those elements of G which are finite linear combinations of positive elements of finite trace. Then τ can be extended by linearity to a linear functional τ' on $\mathcal{M}$. It satisfies

$$(6.1) \quad \tau'(ST) = \tau'(TS), \text{ for all } S, T \in \mathcal{N}.$$

We can define an inner product on $\mathcal{N}$ by taking $(S|T) = \tau'(T^*S)$, and this produces a norm on $\mathcal{N}$ given by

$$\|S\|_2 = [\tau'(S^*S)]^{\frac{1}{2}}.$$

It is shown in ([5], Chapt. 1, §6, Th. 8) that we can also define a norm on $\mathcal{M}$ by

$$\|S\|_1 = \tau'(|S|).$$

In [15], Segal defines two subclasses of the operators affiliated with G which are known as square integrable and integrable, with respect to τ , and which contain $\mathcal{N}$ and $\mathcal{M}$ respectively. Moreover he extends τ' to a functional on the class of integrable operators. These classes can be considered as completions of $\mathcal{N}$ and $\mathcal{M}$ with respect to the above mentioned norms. We summarize a few basic facts that we need in the following lemma. Proofs can be found in [15].

Lemma 6.1

(a) Let S be a positive self-adjoint integrable operator with spectral resolution $\{E(\lambda)\}$. Then

$$(i) \quad \tau(1-E(\lambda)) < \infty, \text{ for } \lambda > 0,$$

$$(ii) \quad \tau'(S) = -\int_0^\infty \lambda \, d\tau(1-E(\lambda)) < \infty.$$

(b) Let S be a positive self-adjoint square-integrable operator with spectral resolution $\{E(\lambda)\}$. Then

$$(i) \quad \tau(1-E(\lambda)) < \infty, \text{ for } \lambda > 0,$$

$$(ii) \quad S^2 \text{ is integrable and } \tau'(S^2) = -\int_0^\infty \lambda^2 \, d\tau(1-E(\lambda)) < \infty.$$

The key result which we need is the following theorem of Segal ([15], Theorem 14.1). It is a generalization of both the Radon-Nikodym theorem in measure theory and of a well known result concerning $\mathfrak{L}(H)$.

Lemma 6.2. Let $\mathfrak{A}$ be a von Neumann algebra with a semi-finite, faithful, normal trace τ , and let $\mu \in \Sigma_{\mathfrak{A}}$. Then there exists a unique square integrable, positive self-adjoint operator S such that

$$\mu(A) = \tau'(S^2 A), \text{ for all } A \in \mathfrak{A}.$$

The operator S^2 is commonly known as the Radon-Nikodym derivative of μ with respect to τ . We will denote this by writing

$$\mu = \tau_S$$

Note that if $S \in \mathfrak{A}$, formula (6.1) shows that this agrees with our notation of Chapter 0.

Remark. In the above definitions and results we considered a von Neumann algebra G on a Hilbert space. However a square integrable, positive self-adjoint operator can be viewed as simply an abstract spectral resolution; that is a family of projections in G satisfying the appropriate conditions, all of which relate only to the algebra G itself and the trace. Accordingly we can apply these results to a W^* -algebra G without regard to any particular representation of G .

Distribution Functions

Definition 6.3. Let G be a von Neumann algebra and let τ be a faithful, normal semi-finite trace on G . For any positive self-adjoint $A \in G$ we define the function f_A on $(\mathbb{R}^+)$ by

$$f_A(\lambda) = \tau(1 - E(\lambda)) \quad , \text{ for } 0 \leq \lambda < \infty$$

$$f_A(\infty) = 0 .$$

where $\{E(\lambda)\}$ is the spectral resolution of A .

f_A of course depends on the trace τ .

We call f_A the distribution function of A with respect to τ , by analogy to the probability definition.

Note that the definition above is preferable to taking the increasing function, $\lambda \rightarrow \tau(E(\lambda))$, since if A is square

integrable f_A is finite valued, except perhaps at 0.

Obviously, in the notation of Section 5, f_A is a right continuous function $\in M$ and we can form the function $\bar{f}_A$.

Lemma 6.4. Let A be a positive self-adjoint operator in G and let $(A_n)_{n=1,2,\dots}$ be a sequence of mutually commuting elements of G^+ such that

- (i) $A_1 \leq A_j$ for $i \leq j$
- (ii) $A = \sup_n (A_n)$.

Then,

$$f_{A_n} \uparrow f_A.$$

Proof. Let $\{E_n(\lambda)\}$ be the spectral resolution of A_n and let $\{E(\lambda)\}$ be the spectral resolution of A . It is not hard to verify that $E_n(\lambda) \downarrow E(\lambda)$ for all λ . Therefore $(1-E_n(\lambda)) \uparrow (1-E(\lambda))$. By the normality of τ , we have that for all $\lambda \geq 0$

$$f_{A_n}(\lambda) = \tau(1-E_n(\lambda)) \uparrow \tau(1-E(\lambda)) = f_A(\lambda).$$

Lemma 6.5 Let $T \in G^+$ and let P be a projection in G such that $TP = T$. Let T' and τ' denote the restrictions of T and τ respectively to the algebra G_P . Then $f_{T'}$, computed with respect to the algebra G_P and trace τ' is equal to f_T computed with respect to τ .

Proof. Let $\{E(\lambda)\}$ be the spectral resolution of T .

Then the spectral resolution of T' as an operator on the Hilbert space P is $\{E'(\lambda)\}$ where $E'(\lambda) = E(\lambda)P$ for all λ . Now for any unit vector x in $1-E(\lambda)$, $(Tx|x) > \lambda$. For any unit vector x in $1-P$, $(Tx|x) = (TPx|x) = 0$. This shows that $(1-E(\lambda)) \cap (1-P) = 0$, or $1-E(\lambda) \leq P$ for all $\lambda > 0$. Then for any $\lambda > 0$,

$$\begin{aligned} f_{T'}(\lambda) &= \tau'(P-E(\lambda)P) = \tau'(P(1-E(\lambda))) \\ &= \tau(P(1-E(\lambda))) = \tau(1-E(\lambda)) = f_T(\lambda), \end{aligned}$$

and by right continuity $f_{T'}(0) = f_T(0)$.

Definitions of σ

Definition 6.6. Let G be a W^* -algebra and τ a normal semi-finite trace on G . For any positive self adjoint operators S and T affiliated with G and square integrable with respect to τ we define

$$\sigma(S, T) = \int_0^\infty \bar{F}_S(x) \bar{F}_T(x) dx.$$

By Lemma 5.3 and Lemma 6.1(b) we have that $\bar{F}_S$ and $\bar{F}_T$ are square integrable functions on $[0, \infty)$. Therefore $\sigma(S, T)$ is always $< \infty$.

Note that σ depends on the trace τ . When G is a factor we can eliminate this dependence by defining σ for states.

Definition 6.7. Let G be a factor and let μ and $\nu \in \Sigma_G$. Suppose that for some semi-finite trace τ on G , $\mu = \tau_S$ and $\nu = \tau_T$. We then define

$$\sigma(\mu, \nu) = \sigma(S, T) \text{ computed with respect to } \tau.$$

To justify this definition we need to show that it is independent of the particular trace chosen. Let τ' be another semi-finite trace on G . Since G is a factor, $\tau' = k\tau$ for some $k > 0$, and it follows that $\mu = \tau'_S / \sqrt{k}$. If the spectral resolution of S is $\{E(\lambda)\}$ the spectral resolution of $S/\sqrt{k}$ is easily seen to be $\{E'(\lambda)\}$ where $E'(\lambda) = E(\sqrt{k}\lambda)$ for all λ . Letting primed symbols denote calculation with respect to τ' we have

$$f'_{S/\sqrt{k}}(\lambda) = \tau'(1 - E'(\lambda)) = kf_S(\sqrt{k}\lambda)$$

From Lemma 5.2(c)

$$F_{S/\sqrt{k}}(\lambda) = \frac{1}{\sqrt{k}} F_S\left(\frac{\lambda}{k}\right)$$

After a similar calculation for ν we obtain

$$\begin{aligned} \sigma'(\mu, \nu) &= \sigma'(S/\sqrt{k}, T/\sqrt{k}) \\ &= \frac{1}{k} \int_0^\infty F_S\left(\frac{x}{k}\right) F_T\left(\frac{x}{k}\right) dx \\ &= \int_0^\infty F_S(y) F_T(y) dy = \sigma(S, T) = \sigma(\mu, \nu). \end{aligned}$$

Definition 6.8. Let G be a factor with a semi-finite trace τ . We let

$$P(G) = \{\alpha \in \text{Aut}(G) : \tau^\alpha = \tau\}$$

(Note that $P(G)$ is obviously independent of any particular trace). Now let G be any subgroup of $\text{Aut}(G)$. Let ψ be the homomorphism from G onto the multiplicative group of positive real numbers defined by $\psi(\alpha) = k$ if $\tau^\alpha = k\tau$. The kernel of ψ is obviously $G \cap P(G)$. Then $G/G \cap P(G)$ can be identified as a subgroup of the multiplicative group of reals which we denote by $r(G)$.

Definition 6.9. Let G be a factor with a semi-finite trace τ , let G be a subgroup of $\text{Aut}(G)$, and let μ and $\nu \in \Sigma_G$. Suppose that $\mu = \tau_T$. We define,

$$\sigma_k(\mu, \nu) = \sqrt{k} \int_0^\infty \bar{F}_S(x) \bar{F}_T(kx) dx, \text{ for all } k > 0,$$

$$\sigma^G(\mu, \nu) = \sup \{\sigma_k(\mu, \nu) : k \in r(G)\}.$$

As above we can show that σ^G is independent of the particular trace.

The motivation for these definitions is shown by the following.

Lemma 6.10. Let $\nu \in \Sigma_G = \tau_T$ for some $T \in G^+$ and some semi-finite trace τ . Let $\alpha \in \text{Aut}(G)$ with $\tau^\alpha = k\tau$.

Then

$$\sigma(\mu, \nu^\alpha) = \sigma_k(\mu, \nu) .$$

Proof. By lemma 0.3, $\nu^\alpha = (\tau_T)^\alpha = \tau^\alpha_{(\alpha^{-1}(T))} = k\tau_{(\alpha^{-1}(T))}$
 $= \tau_{\sqrt{k}(\alpha^{-1}(T))}$. Let $T' = \sqrt{k}(\alpha^{-1}(T))$. If T has spectral
 resolution $\{E(\lambda)\}$, T' has spectral resolution $\{E'(\lambda)\}$
 where $E'(\lambda) = \alpha^{-1}(\lambda/\sqrt{k})$ for all λ .

Then

$$\begin{aligned} f_{T'}(\lambda) &= \tau(1 - \alpha^{-1}(E(\lambda/\sqrt{k}))) = \frac{1}{k}\tau^\alpha(1 - \alpha^{-1}(E(\lambda/\sqrt{k}))) \\ &= \frac{1}{k} f_T(\lambda/\sqrt{k}) \end{aligned}$$

By lemma 5.2(c), $\bar{f}_{T'}(\lambda) = \sqrt{k} \bar{f}_T(k\lambda)$, for all λ and the lemma follows from the definitions of σ and σ_k .

Remark 6.11. If G is a factor of type I or II_1 ,
 $P(G) = \text{Aut}(G)$. So for any $G \subseteq \text{Aut}(G)$, $r(G) = \{1\}$ and
 $\sigma^G = \sigma$. The purpose of defining σ^G is therefore to deal
 with factors of type II_∞ where this is not the case. An
 example of Suzuki ([14] p. 188) shows that if $G = \mathcal{P} \otimes I_\infty$
 where $\mathcal{P}$ is the hyperfinite II_1 factor then $r(\text{Aut}(G)) =$
 the entire group of multiplicative real numbers. The same
 reasoning employed in this example shows that if $G = \mathcal{B} \otimes I_\infty$
 for a II_1 factor $\mathcal{B}$ then $r(\text{Aut}(G)) =$ the fundamental group
 of $\mathcal{B}$. (See [10] Theorem VIII for the definition of funda-
 mental group).

In the rest of this section we will consider a fi-
 nite factor G with normalized trace τ (i.e. $\tau(1) = 1$).

We want to develop some continuity properties of σ .

The following is a well known result. (See [9]

lemma 14.22).

Lemma 6.12. Let E and F be projections in G with $\tau(E) < \tau(F)$. Then $(1-E) \cap F \neq 0$.

The next lemma is essentially proved in ([9] Lemma 15.21).

Lemma 6.13. Consider any representation of G . For any $A \in G^+$ and $0 \leq a \leq \infty$,

$$(6.2) \quad \bar{f}_A(a) = \inf_E (\sup \{ (Ax|x) : \|x\| = 1, Ex = x \})$$

as E runs over the projections of G with $\tau(1-E) \leq a$.

Proof. If $a = \infty$ both sides of (6.2) are easily seen to be 0, so assume $a < \infty$. Let $\bar{f}_A(a) = b$ and let the right side of (6.2) = b_0 . Let $\{E(\lambda)\}$ be the spectral resolution of A . Choose any b' and b'' with $b' < b < b''$. Then $\tau(1-E(b')) = f_A(b') > a$ by Lemma 5.2(b). So for any projection E with $\tau(1-E) \leq a$, $(1-E(b')) \cap E \neq 0$ by Lemma 6.12. Therefore E contains a unit vector x such that $(Ax|x) > b'$. It follows that $b_0 > b'$.

On the other hand, $\tau(1-E(b'')) = f_A(b'') \leq a$ by Lemma 5.2(b). So for any unit vector in $E(b'')$, $(Ax|x) \leq b''$. It follows that $b_0 \leq b''$.

Since b' and b'' were chosen arbitrarily we have that $b_0 = b$.

Lemma 6.14. Let G be a finite factor. Suppose that

S, S', T and $T' \in G^+$ satisfy

$$\|S - S'\| \leq \delta, \quad \|T - T'\| \leq \delta, \quad \text{where } 0 < \delta < 1.$$

Then

$$|\sigma(S, T) - \sigma(S', T')| \leq \delta(\|S\| + \|T\| + 1),$$

where σ is calculated with respect to the normalized trace τ .

Proof. Note first that since $\tau(1) = 1$, for any $A \in G^+$ $f_A \leq 1$, and therefore $\bar{f}_A(a) = 0$ for all $a > 1$. Therefore the integral in the definition of σ can be taken over the interval $[0, 1]$.

Since $f_S(\|S\|) = f_T(\|T\|) = 0$ we have

$$(6.3) \quad \bar{f}_S(a) \leq \|S\|, \quad \bar{f}_T(a) \leq \|T\|, \quad \text{for } 0 \leq a \leq \infty.$$

Now consider any representation of G and any unit vector x .

$$\text{Then } |(Sx|x) - (S'x|x)| = |((S - S')x|x)| \leq \|S - S'\| \leq \delta.$$

From Lemma 6.13 we can conclude

$$(6.4) \quad |\bar{f}_S(a) - \bar{f}_{S'}(a)| \leq \delta \quad \text{and} \quad |\bar{f}_T(a) - \bar{f}_{T'}(a)| \leq \delta, \quad \text{for } 0 \leq a \leq \infty.$$

Using (6.3) and (6.4) we have for all $a \geq 0$

$$\begin{aligned}
& |\bar{f}_S(a)\bar{f}_T(a) - \bar{f}_{S'}(a)\bar{f}_{T'}(a)| \\
& \leq \bar{f}_S(a)|\bar{f}_T(a) - \bar{f}_{T'}(a)| + \bar{f}_T(a)|\bar{f}_S(a) - \bar{f}_{S'}(a)| \\
& \quad + |f_S(a) - f_{S'}(a)||f_T(a) - f_{T'}(a)| \\
& \leq \delta(\|S\| + \|T\| + \delta) \leq \delta(\|S\| + \|T\| + 1) .
\end{aligned}$$

Therefore

$$\begin{aligned}
|\sigma(S, T) - \sigma(S', T')| & \leq \int_0^1 |\bar{f}_S(a)\bar{f}_T(a) - \bar{f}_{S'}(a)\bar{f}_{T'}(a)| da \\
& \leq \delta(\|S\| + \|T\| + 1) .
\end{aligned}$$

7. Compatibility.

Definition 7.1. Let S and T be self-adjoint operators (not necessarily bounded) on a Hilbert space H . Suppose that $\{E(\lambda)\}$ is the spectral resolution of S , and $\{F(\lambda)\}$ is the spectral resolution of T . We say that S and T are compatible, if given any ordered pair (α, β) of real numbers either (1) or (2) holds:

$$(1) \quad E(\alpha) \leq F(\beta) \quad ,$$

$$(2) \quad F(\beta) \leq E(\alpha) \quad .$$

Remarks. It is not hard to verify that the above definition remains unchanged if we use left continuous spectral resolutions in place of right continuous ones.

Note that if S and T are compatible they obviously commute with each other.

The main result of compatibility which we need is the following.

Theorem 7.2. Let G be a von Neumann algebra with a normal, semi-finite trace τ . Let S and $T \in G^+$ be compatible, and satisfy $\tau(S^2) < \infty$, $\tau(T^2) < \infty$. Then for $0 < \alpha \leq \infty$,

$$\bar{F}_{ST}(\alpha) = \bar{F}_S(\alpha) \bar{F}_T(\alpha) \quad .$$

Proof. We apply Theorem 5.4, taking $\phi(x, y) = xy$. It re-

mains to verify the conditions of that theorem. Let $\{E(\lambda)\}$, $\{F(\lambda)\}$, and $\{G(\lambda)\}$ be the spectral resolutions of S , T , and ST respectively. Since the operators all commute with each other any two of these spectral projections will commute. Let (α, β) be any ordered pair of non-negative real numbers.

Case 1. Assume that $E(\alpha) \leq F(\beta)$.

For any unit vector x in $E(\alpha)$, $S^{\frac{1}{2}}x \in E(\alpha) \leq F(\beta)$, which shows that

$$(STx|x) = (TS^{\frac{1}{2}}x|S^{\frac{1}{2}}x) \leq \beta(Sx|x) \leq \alpha\beta .$$

For any unit vector y in $1 - G(\alpha\beta)$

$$(STy|y) > \alpha\beta .$$

This shows that $E(\alpha)(1 - G(\alpha\beta)) = 0$, so

$$(7.1) \quad (1 - G(\alpha\beta)) \leq (1 - E(\alpha)) .$$

Similarly, we see that for any unit vector $x \in (1 - F(\beta))$ $\leq (1 - E(\alpha))$, we have $(STx|x) > \alpha\beta$, and we obtain

$$(7.2) \quad (1 - G(\alpha\beta)) \geq (1 - F(\beta)) .$$

From (7.1) and (7.2) we obtain by the positivity of τ that

$$f_T(\beta) \leq f_{ST}(\alpha\beta) \leq f_S(\alpha) .$$

Case 2. Assume that $E(\alpha) \nmid F(\beta)$.

By compatibility we must have $F(\beta) \leq E(\alpha)$, and arguing as above we obtain

$$f_S(\alpha) \leq f_{ST}(\alpha\beta) \leq f_T(\beta) .$$

We have then that in any event, for all non-negative α and β

$$f_{ST}(\alpha\beta) \in [f_S(\alpha) , f_T(\beta)] .$$

Now we note from Lemma 6.1 that since $\tau(S^2)$ and $\tau(T^2)$ are finite, $\bar{F}_S$ and $\bar{F}_T$ are finite valued except perhaps at 0 . We apply Theorem 5.4 to complete the proof.

Corollary 7.3. If S and $T \in G^+$ are compatible, and satisfy $\tau(S^2) < \infty$, $\tau(T^2) < \infty$, then

$$\tau(ST) = \sigma(S,T) .$$

Proof. From Lemma 6.1(a), Lemma 5.3, and Theorem 7.2,

$$\tau(ST) = \int_0^\infty \bar{F}_{ST}(\alpha) d\alpha = \sigma(S,T) .$$

Remark. As a further application of Theorem 5.4 we may

obtain that $\bar{F}_{(S+T)} = \bar{F}_S + \bar{F}_T$ for compatible S and T , by taking $\phi(x,y) = x + y$ and proceeding as in Theorem 7.2. Similarly, taking $\phi(x,y) = kx$ yields $\bar{F}_{kS} = k\bar{F}_S$ for $k \geq 0$ (a result which we established previously in the remark following Definition 6.7).

Definition 7.4. Let G be a W^* -algebra. An element $A \in G$ will be called simple, if it can be written as a finite linear combination of mutually orthogonal projections $e \in G$.

Remark. Let A be simple and self-adjoint. Then $A = \sum_{i=1}^n c_i E_i$, for some real numbers (c_i) and mutually orthogonal, non-zero projections (E_i) with $\sum_{i=1}^n E_i = 1$. (We can always obtain the latter condition by inserting if necessary an appropriate projection with 0 coefficient.) It is easy to see that the spectral resolution $\{E(\lambda)\}$ of A is given by

$$(7.3) \quad E(\lambda) = \sum E_j, \text{ the sum taken over all } j \text{ such that } c_j \leq \lambda,$$

and it follows that

$$(7.4) \quad 1 - E(\lambda) = \sum E_j, \text{ the sum taken over all } j \text{ such that } c_j > \lambda.$$

Theorem 7.5. Let S and T be simple, self-adjoint elements of a W^* -algebra G . Then S and T are compatible if and only if there exist:

(i) a set of non-zero, mutually orthogonal projections in G , $(G_i)_{i=1,2,\dots,m}$ with $\sum_{i=1}^m G_i = 1$;

(ii) non-increasing sequences of real numbers, $(c_1, c_2, \dots, c_m)$ and $(d_1, d_2, \dots, d_m)$; such that $S = \sum_{i=1}^m c_i G_i$ and $T = \sum_{i=1}^m d_i G_i$.

Proof. Suppose that S and T are of the above form.

From (7.4), any non-zero, spectral projection of either S or T is $= \sum_{i=k}^m G_i$ for some k such that $1 \leq k \leq m$. Obviously S and T are compatible.

Conversely, suppose that S and T are compatible. Let $S = \sum_{i=1}^n a_i E_i$, $T = \sum_{j=1}^k b_j F_j$, where (E_i) and (F_j) are sets of non-zero, mutually orthogonal projections with sum $= 1$. For any i, j , $G_{ij} = E_i F_j$ is a projection and,

$$S = \sum_{i,j} a_i G_{ij} \quad \text{and} \quad T = \sum_{i,j} b_j G_{ij}.$$

Deleting those $G_{ij} = 0$ and suitably renumbering the remaining ones, we obtain

$$S = \sum_{i=1}^m c_i G_i \quad \text{and} \quad T = \sum_{i=1}^m d_i G_i,$$

where;

(i) $(G_i)_{i=1,2,\dots,m}$ is a set of non-zero mutually

orthogonal projections with $\text{sum} = 1$,

$$(ii) \quad c_1 \geq c_2 \dots \geq c_m ,$$

$$(iii) \quad \text{If } c_i = c_{i+1} , \text{ then } d_i \geq d_{i+1} .$$

This of course is a familiar procedure which can be applied whenever S and T commute. We want to show now that the compatibility implies that the d_i 's are non-decreasing.

Suppose this is not the case. Then for some i , $d_i < d_{i+1}$ and necessarily $c_i > c_{i+1}$. Choose numbers c and d such that $d_i < d < d_{i+1}$ and $c_i > c > c_{i+1}$. Let $\{E(\lambda)\}$ and $\{F(\lambda)\}$ be the spectral resolutions of S and T respectively. From (7.3) and (7.4), we obtain

$$G_i \leq F(d) \cap (1 - E(c)) ,$$

$$G_{i+1} \leq E(c) \cap (1 - F(d)) .$$

Since G_i and G_{i+1} are both non-zero, we necessarily have

$$E(c) \not\perp F(d) \quad \text{and} \quad F(d) \not\perp E(c) ,$$

a contradiction to the compatibility. Therefore, we must have

$d_i \geq d_{i+1}$ for $i = 1, 2, \dots, m-1$ and the theorem is proved.

We now need some lemmas concerning the comparison of projections in a factor. In the proofs we use several standard results which can be found in ([9] Chapters VI and VII).

Lemma 7.6. Let E and H be projections in a factor such that H is infinite and $E \leq H$. Then either $E \sim H$ or $H-E \sim H$.

Proof. Suppose $E \not\sim H-E$. Then $H-E$ must be infinite, for if not, E would be finite and then H would be finite. Therefore $H-E = P + Q$ where $H-E \sim P \sim Q$. Then $H = E + H-E \lesssim P + Q = H-E$, and it follows that $H \sim H-E$.

If $E \not\sim H-E$, then $H-E \cong E$ and we obtain similarly that $H \sim E$.

Definition 7.7. Let H be a projection in a W^* -algebra G .

If E and F are projections of G contained in H we say that $E \underset{(H)}{\sim} F$, if $E \sim F$ and $H-E \sim H-F$. We say that $E \underset{(H)}{\lesssim} F$, if $E \leq P \underset{(H)}{\sim} Q \leq F$, for some P and Q .

Lemma 7.8. Let E , F , and H be projections in a factor G such that $E \leq H$ and $F \leq H$. Then either

$$E \underset{(H)}{\lesssim} F, \text{ or } F \underset{(H)}{\lesssim} E.$$

Proof. If H is finite, it is a well known result that $\underset{(H)}{\sim}$ is the same as $\sim$, and we are done. So assume H is infinite. By symmetry we may assume that for some F_1 , $E \sim F_1 \leq F$. If $H-E \sim H-F_1$ we are done. If not, $H-E$ and $H-F_1$ cannot both be $\sim H$. From Lemma 7.6, either E or F_1 is $\sim H$ and necessarily $E \sim F \sim H$. By symmetry

again we may assume that for some G , $H-E \sim G \leq H-F$.

Then $H-G$ contains F and must be equivalent to H . Therefore, $F \leq H-G \underset{(H)}{\sim} E$, which shows that $F \underset{(H)}{\leq} E$.

Corollary 7.9. Assuming the conditions of Lemma 7.8, there exists a unitary $U \in G$ such that: (a) $U^*GU = G$ for all projections $G \leq 1-H$; (b) Either (i) $U^*FU \leq E$ or (ii) $U^*FU \geq E$ holds. (cf. [5], Chapt. III, §2, Prop. 6.)

Proof. We can suppose by Lemma 7.8 that $E \underset{(H)}{\leq} F$. Then,

$E \leq P \sim Q \leq F$ for some P and Q . Let V and W be partial isometries in G satisfying $V^*V = P$, $VV^* = Q$, $W^*W = H-P$, $WW^* = H-Q$. Then, $U = V + W + 1-H$ is a unitary in G which satisfies (a) and (b) (ii).

Theorem 7.10. Let S and T be self-adjoint, simple elements of a factor G . Then, there exists a unitary $U \in G$ such that S and U^*TU are compatible.

Proof. Let $S = \sum_{k=1}^m a_k E_k$ and $T = \sum_{l=1}^p b_l F_l$, where (E_k) and (F_l) are sets of non-zero, mutually orthogonal projections with sum = 1, and $a_1 < a_2 < \dots < a_m$, and $b_1 < b_2 < \dots < b_p$.

We define the projections

$$E(i) = \begin{cases} \sum_{k=1}^i E_k, & \text{for } i = 1, 2, \dots, m \\ 0, & \text{for } i = 0 \end{cases}$$

Then any spectral projection of $S = E^{(i)}$ for some $i = 0, 1, \dots, m$. We define in a similar way the projections $F^{(j)}$, $j = 0, 1, \dots, p$, corresponding to T . Now, let $A = \{0, 1, \dots, m\}$, $B = \{0, 1, \dots, p\}$, and order the set $A \times B$ lexicographically. If S and T are not already compatible there must exist some (i, j) in $A \times B$ such that

$$(*) \quad E^{(i)} \not\leq F^{(j)} \text{ and } F^{(j)} \not\leq E^{(i)}.$$

Let (i_0, j_0) be the least such element. Obviously $i_0 > 0$ and $j_0 > 0$. Then $E^{(i_0)} \not\leq F^{(j_0)}$, so obviously $E^{(i_0)} \leq F^{(j_0-1)}$. This implies by the minimality of (i_0, j_0) that

$$(7.5) \quad F^{(j_0-1)} \leq E^{(i_0)}$$

Similarly we obtain

$$(7.6) \quad E^{(i_0-1)} \leq F^{(j_0)}.$$

Case 1. Suppose $E^{(i_0-1)} \leq F^{(j_0-1)}$.

Let $E'_{i_0} = E^{(i_0)} - F^{(j_0-1)}$, and let $H = 1 - F^{(j_0-1)}$.

From (7.5), E'_{i_0} is a projection contained in H . Ob-

viously $F_{j_0} \leq H$. Choose a unitary $U \in G$ which satisfies

the conditions of Corollary 7.9 for E'_{i_0} , F_{j_0} and H .

Consider the element $T' = U^*TU$, and the corresponding projections, $F'_k = U^*F_kU$, and $F'(j) = \sum_{k=1}^j F'_k$. Let

$(*)'$ denote the statement $(*)$ above with $F^{(j)}$ replaced by $F'(j)$.

From the conclusions of Corollary 7.9 we have that $F(j) = F'(j)$ for $j < j_0$; and that either

(a) $F'_{j_0} \leq E'_{i_0}$, in which case

$$F'(j_0) = (F'(j_0-1) + F'_{j_0}) \leq (F^{(j_0-1)} + E'_{i_0}) = E^{(i_0)};$$

or

(b) $E'_{i_0} \leq F'_{j_0}$, in which case

$$E^{(i_0)} = F^{(j_0-1)} + E'_{i_0} \leq F'(j_0).$$

So $(*)'$ does not hold for (i_0, j_0) .

Now consider any $(i, j) < (i_0, j_0)$.

(a) If $j < j_0$, $F'(j) = F(j)$, and by the minimality of (i_0, j_0) , $(*)'$ does not hold for (i, j) .

(b) If $j = j_0$, we must have $i \leq i_0 - 1$, and

$$E^{(i)} \leq E^{(i_0-1)} \leq F^{(j_0-1)} = F'(j_0-1) \leq F'(j_0) = F'(j).$$

So again, $(*)'$ does not hold for (i, j) .

Case 2. Suppose $E^{(i_0-1)} \not\leq F^{(j_0-1)}$,

By the minimality of (i_0, j_0) , we have $F^{(j_0-1)} \leq E^{(i_0-1)}$.

We can then argue in a similar manner to Case 1, using (7.6) in place of (7.5).

In any event we can find a unitary $U \in G$ such that when we replace T by U^*TU , the number of pairs (i, j) for which (*) holds is strictly decreased. The theorem then follows easily by induction.

Example 7.11. Theorem 7.10 will not hold in general if S and T are not simple. Let M be the von Neumann algebra consisting of $L^\infty[0,1]$ acting on $L^2[0,1]$ by multiplication. Let G be any free, ergodic, measure preserving group of transformations of $[0,1]$ and let $\mathcal{G}$ be the factor constructed from M and G in the standard manner (see [9], Chapter XII).

For any bounded measurable function h on $[0,1]$ we let T_h denote the corresponding element of $\mathcal{G}$. There is a faithful trace τ on $\mathcal{G}$ which satisfies

$$(7.7) \quad \tau(T_h) = \int_0^1 h(x) dx.$$

In particular consider the functions f and g defined by

$$f(x) = x, \quad 0 \leq x \leq 1,$$

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2-2x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Letting χ denote the characteristic function of a set we have for $0 \leq \lambda \leq 1$,

the spectral projection $E(\lambda)$ of $T_f = T_{\chi_{[0,\lambda]}}$,

the spectral projection $F(\lambda)$ of $T_g = T_{\chi([0,\lambda/2] \cup [1-\lambda/2,1])}$,

and $E(\lambda) = F(\lambda) = 1$ for $\lambda > 1$; $E(\lambda) = F(\lambda) = 0$ for $\lambda < 0$. Applying (7.7), we see that for any unitary $U \in G$

$$(7.8) \quad \tau(U^* F(\lambda) U) = \tau(E(\lambda)), \quad -\infty < \lambda < +\infty.$$

Consequently, if there existed a unitary U such that $U^* T_g U$ and T_f were compatible, we would have from (7.8) and the fact that τ is faithful,

$$U^* F(\lambda) U = E(\lambda), \quad \text{for } 0 < \lambda < +\infty.$$

By the uniqueness of the spectral resolution we would have that $U^* T_g U = T_f$ and therefore that the von Neumann algebras generated by T_f and T_g are unitarily equivalent. But this is a contradiction, since T_f generates $\{T_h : h \in L^\infty[0,1]\}$ which is maximal abelian, while T_g generates $\{T_h : h \in L^\infty[0,1], \text{ and } h(x) = h(1-x)\}$ which is not maximal abelian.

8. Calculation of ρ^G for semi-finite factors.

The calculations for ρ^G made in this section will be based on the following result for ρ proved in [3].

Lemma 8.1 ([3], Prop. 2.3). Let G be a W^* -algebra with a normal finite trace τ . Let μ and $\nu \in \Sigma_G$ be such that $\mu = \tau_S$ and $\nu = \tau_T$ for some S and $T \in G^+$. Then

$$\rho(\mu, \nu) = \tau(|ST|) .$$

We want to talk about the type of states used in the above lemma, so we make the following definition.

Definition 8.2. Let G be a W^* -algebra with a normal, semi-finite trace τ . An element μ of Σ_G will be called elementary with respect to τ , if $\mu = \tau_S$ for some $S \in G^+$ such that $\tau(E) < \infty$, where E is the range projection of S .

In the remainder of this section G will always be a factor and we can speak of elementary states without regard to τ .

Lemma 8.3. Let G be a finite factor with normalized trace τ and let S and T be simple elements of G^+ . Then, for any $V \in G$ with $\|V\| \leq 1$,

$$|\tau(STV)| \leq \sigma(S, T) .$$

Proof. By Theorem 7.10, choose a unitary $U \in G$ so that S and U^*TU are compatible. Let $S = \sum_{i=1}^m c_i G_i$ and $U^*TU = \sum_{i=1}^m d_i G_i$ be the representations given by Theorem 7.5. Let $g_i = \tau(G_i)$. By the unitary invariance of τ and Corollary 7.3,

$$\begin{aligned}\sigma(S, T) &= \sigma(S, U^*TU) = \tau(SU^*TU) \\ &= \sum_{i=1}^m c_i d_i g_i.\end{aligned}$$

The lemma now follows easily from Theorem 4.3.

Lemma 8.4. Let G be a factor with a finite trace τ . Let μ and ν be elements of Σ_G such that $\mu = \tau_S$ and $\nu = \tau_T$ for some S and $T \in G^+$. Then:

(a) $\rho(\mu, \nu) \leq d(\mu, \nu)$, and equality holds if S and T are compatible;

(b) Given any $\epsilon > 0$, there exists $\alpha \in \text{Int}(G)$ such that

$$\rho(\mu, \nu^\alpha) > \sigma(\mu, \nu) - \epsilon.$$

Proof. We can assume that $\tau(1) = 1$. Choose any ϵ with $0 < \epsilon < 1$. Let $r = (\|S\| + \|T\| + 1)$. It follows immediately from the spectral theorem that the simple elements of G^+ are norm dense in G^+ . Therefore, we can choose simple elements S' and T' of G^+ such that

$$\|S-S'\| \leq \frac{\epsilon}{2r}, \quad \|T-T'\| \leq \frac{\epsilon}{2r}.$$

From Lemma 6.14,

$$(8.1) \quad |\sigma(S,T) - \sigma(S',T')| \leq \epsilon/2,$$

and a similar type of calculation shows that

$$\|ST - S'T'\| \leq \epsilon/2.$$

For any $V \in G$ with $\|V\| \leq 1$ we have

$$\|STV - S'T'V\| \leq \|ST - S'T'\| \|V\| \leq \epsilon/2,$$

and since the state τ has norm 1,

$$(8.2) \quad \begin{aligned} | |\tau(STV)| - |\tau(S'T'V)| | &\leq |\tau(STV) - \tau(S'T'V)| \\ &= |\tau(STV - S'T'V)| \\ &\leq \epsilon/2. \end{aligned}$$

From (8.1), (8.2), and Lemma 8.3,

$$\begin{aligned} |\tau(STV)| &\leq |\tau(S'T'V)| + \epsilon/2 \\ &\leq \sigma(S',T') + \epsilon/2 \leq \sigma(S,T) + \epsilon \end{aligned}$$

for all V with $\|V\| \leq 1$.

Then from Lemma 0.4(a) and Lemma 8.1,

$$\rho(\mu, \nu) = \tau|ST| \leq \sigma(S, T) + \epsilon = \sigma(\mu, \nu) + \epsilon.$$

Since ϵ was chosen arbitrarily,

$$\rho(\mu, \nu) \leq \sigma(\mu, \nu).$$

If S and T are compatible the equality follows directly from Corollary 7.3. This proves (a).

By Theorem 7.10, choose a unitary $U \in G$ such that S' and $U^*T'U$ are compatible.

Calculating as in (8.2) we can show

$$(8.3) \quad |\tau(SU^*TU)| \geq |\tau(S'U^*T'U)| - \epsilon/2.$$

Let α be the inner automorphism induced by U . Then $\nu^\alpha = \tau_{U^*TU}$. From (8.1), (8.2), Lemma 0.4(a), Corollary 7.3, and Lemma 8.1,

$$\begin{aligned} \rho(\mu, \nu^\alpha) &= \tau(|SU^*TU|) \geq |\tau(SU^*TU)| \\ &\geq |\tau(S'U^*T'U)| - \epsilon/2 \\ &= \sigma(S', T') - \epsilon/2 \geq \sigma(S, T) - \epsilon \\ &= \sigma(\mu, \nu) - \epsilon, \end{aligned}$$

which proves (b).

Lemma 8.5. Let G be a semi-finite factor and let μ and ν be elementary states of τ_G . Then, (a) and (b) of Lemma 8.4 hold.

Proof. Suppose $\mu = \tau_S$, $\nu = \tau_T$ where S and $T \in G^+$. Let E and F be the range projections of S and T respectively. Since E and F are finite it is a well known result that the projection $G = E \cup F$ is finite (see [9] Lemma 7.35). Let μ' and ν' denote the restrictions of μ and ν respectively to the finite factor G_G . Let α' be any inner automorphism of G_G . We extend α' to an inner automorphism α of G as follows. If the unitary U in G_G induces α' , we take α as the inner automorphism induced by the unitary $U + (1-G)$. Then $(\nu')^{\alpha'}$ is the restriction of ν^α to G_G . Moreover,

$$\nu^\alpha(G) = \nu(\alpha(G)) = \nu(G) = \tau(SGS) = \tau(S^2) = \mu(1) = 1,$$

and similarly $\mu(G) = 1$.

From ([3] Prop. 1.10),

$$\rho(\mu, \nu^\alpha) = \rho(\mu', (\nu')^{\alpha'}) ,$$

and taking $\alpha' =$ the identity automorphism,

$$\rho(\mu, \nu) = \rho(\mu', \nu') .$$

Finally, we see from Lemma 6.5 that

$$\sigma(\mu, \nu) = \sigma(\mu', \nu') .$$

The lemma now follows directly from Lemma 8.4.

Theorem 8.6. Let G be a semi-finite factor and let G be a subgroup of $\text{Aut}(G)$ which contains $\text{Int}(G)$. Then, for any $\mu, \nu \in \Sigma_G$

$$\rho^G(\mu, \nu) = \sigma^G(\mu, \nu) .$$

We first prove the result in a special case.

Lemma 8.7. Theorem 8.6 holds if μ and ν are elementary.

Proof. Choose any $\alpha \in G$. Then obviously ν^α is also elementary. From Lemmas 8.5 and 6.10 we see that

$$\rho(\mu, \nu^\alpha) \leq \sigma(\mu, \nu^\alpha) \leq \sigma^G(\mu, \nu) .$$

Then from Lemma 3.2.,

$$(8.4) \quad \rho^G(\mu, \nu) \leq \sigma^G(\mu, \nu) .$$

Now choose any $\epsilon > 0$. Let $k \in r(G)$ be such that $\sigma_k(\mu, \nu) > \sigma^G(\mu, \nu) - \epsilon/2$, and choose $\beta \in G$ so that $\tau^\beta = k\tau$ for any trace τ on G . Applying Lemma 8.5, there exists $\gamma \in \text{Int}(G)$ such that

$$\begin{aligned}
 (8.5) \quad \rho(\mu, \nu^{\beta\gamma}) &\geq \sigma(\mu, \nu^{\beta}) - \epsilon/2 = \sigma_k(\mu, \nu) - \epsilon/2 \\
 &\geq \sigma^G(\mu, \nu) - \epsilon.
 \end{aligned}$$

Since $\text{Int}(G) \subseteq G$, we have that $\beta\gamma \in G$, and the lemma follows from (8.4) and (8.5).

Remark 8.8. The definition for $\sigma(\mu, \nu)$ can obviously be made for any positive, normal, linear functionals as well as for states. If a state $\mu = \tau_S$, the functional $k\mu = \tau_{\sqrt{k}S}$ for any positive k , and we have seen that $\bar{F}_{\sqrt{k}S} = \sqrt{k} \bar{F}_S$.

Consequently, for any positive k and r , $\sigma(k\mu, r\nu) = \sqrt{kr} \sigma(\mu, \nu)$. In Remark 2.5 we noted that $\rho(k\mu, r\nu) = \sqrt{kr} \rho(\mu, \nu)$. It follows that Lemma 8.7 holds for positive, normal, linear functionals, and we will apply it in this form in the following proof.

Proof of Theorem 8.6.

Let τ be a semi-finite trace on G which we keep fixed throughout the proof. Suppose that $\mu = \tau_S$ and $\nu = \tau_T$. Let $\{E(\lambda)\}$ be the spectral resolution of S . For any integer $n > 1$ let $P_n = (E(n) - E(\frac{1}{n}))$, and let $S_n = SP_n$. Since S is square integrable, P_n is a finite projection and S_n is a bounded positive operator with finite range. We define the functional $\mu_n = \mu_{P_n}$. Then $\mu_n = \tau_{S_n}$,

so it is elementary. Since $S_n \uparrow S$, we obtain by Lemmas 5.2(d) and 6.4 that

$$(8.6) \quad \bar{F}_{S_n} \uparrow \bar{F}_S .$$

Analogously, we define projections Q_n , operators T_n , and elementary functionals v_n corresponding to v . We obtain

$$(8.7) \quad \bar{F}_{T_n} \uparrow \bar{F}_T .$$

From Lemmas 6.1 and 5.3 we have

$$(8.8) \quad \tau(S_n^2) = \int_0^\infty (\bar{F}_{S_n}(\alpha))^2 d\alpha, \text{ and } \tau(S^2) = \int_0^\infty (\bar{F}_S(\alpha))^2 d\alpha$$

Now using (8.6) - (8.8) and the monotone convergence theorem

$$\mu(P_n) = \tau(S_n^2) \uparrow \tau(S^2) = \mu(1) = 1$$

$$v(Q_n) \uparrow 1 .$$

$$\sigma(\mu_n, v_n) = \sigma(S_n, T_n) \uparrow \sigma(S, T) = \sigma(\mu, v) .$$

Similarly $\sigma_k(\mu_n, v_n) \uparrow \sigma_k(\mu, v)$ for all $k > 0$ and therefore

$$\sigma^G(\mu_n, v_n) \uparrow \sigma^G(\mu, v) .$$

Then given any $\epsilon > 0$ we can choose an integer m so that

$$\mu(P_m) > 1 - (\frac{\epsilon}{10})^2, \text{ and } \nu(Q_m) > 1 - (\frac{\epsilon}{10})^2, \text{ and}$$

$$(8.9) \quad \sigma^G(\mu_m, \nu_m) > \sigma(\mu, \nu) - \frac{\epsilon}{2}.$$

Let α be any element of G . Then

$$(\nu_m)^\alpha = (\nu(Q_m))^\alpha = \nu_{(\alpha^{-1}(Q_m))} \quad \text{by Lemma 0.3,}$$

and

$$\nu^\alpha(\alpha^{-1}(Q_m)) = \nu(Q_m) > 1 - (\frac{\epsilon}{10})^2.$$

We can then apply ([3] Prop. 1.9(c)) to obtain

$$|\rho(\mu, \nu^\alpha) - \rho(\mu_m, \nu_m)| \leq \epsilon/2, \text{ for all } \alpha \in G,$$

and therefore,

$$(8.10) \quad |\rho^G(\mu, \nu) - \rho^G(\mu_m, \nu_m)| \leq \epsilon/2.$$

Lemma 8.7, applied to the elementary functionals μ_m and ν_m gives

$$(8.11) \quad \rho^G(\mu_m, \nu_m) = \sigma^G(\mu_m, \nu_m).$$

From formulas (8.9) - (8.11) we have $|\rho^G(\mu, \nu) - \sigma^G(\mu, \nu)| < \epsilon$, and since ϵ was chosen arbitrarily the proof is completed.

Definition 8.9. Let G be a semi-finite factor and let μ and $\nu \in \Sigma_G$. We say that μ and ν are compatible,

if for some semi-finite trace τ on G , $\mu = \tau_S$ and $\nu = \tau_T$ where S and T are compatible operators.

Lemma 8.10. Suppose that μ and $\nu \in \Sigma_G$ are compatible. Then $\rho(\mu, \nu) = \sigma(\mu, \nu)$.

Proof. This follows directly from Lemma 8.4(a) if the Radon-Nikodym derivatives of μ and $\nu \in G$. For the general case we can employ the same type of approximation procedure used in proving Theorem 8.6.

Theorem 8.11. Let G be a factor of type I or of type II_1 .

Then, for any μ and $\nu \in \Sigma_G$:

$$(a) \quad \tilde{d}(\mu, \nu) = \sqrt{2[1 - \sigma(\mu, \nu)]} ;$$

$$(b) \quad \text{If } \mu \text{ and } \nu \text{ are compatible, } d(\mu, \nu) = \tilde{d}(\mu, \nu) .$$

Proof. Immediate from Theorem 8.6, Lemma 8.10 and formula (3.1).

Example 8.12. Consider the algebra $G = \mathfrak{L}(H)$ where H is separable. Let τ be the trace on G , normalized so that its value = 1 on minimal projections. Let S be a positive, Hilbert-Schmidt operator (i.e. $\tau(S^2) < \infty$). It is well known that

$$S = \sum_{i=1}^{\infty} a_i E_i, \text{ in the sense of strong convergence,}$$

where (a_i) is a non-increasing sequence of positive real

numbers such that $\sum_{i=1}^{\infty} (a_i)^2 < \infty$, and (E_i) is a sequence of non-zero, mutually orthogonal, minimal projections such that $\sum_{i=1}^{\infty} E_i = 1$. Moreover the sequence (a_i) is uniquely determined by S . Let $\{E(\lambda)\}$ be the spectral resolution of S . We can apply (7.4) to see that

$$1 - E(\lambda) = \sum_{i=1}^k E_i, \text{ where } k \text{ is such that } a_k > \lambda \geq a_{k+1}.$$

It follows that

$$f(\lambda) = \text{the number of } a_i \text{'s } > \lambda.$$

Therefore, if $\lambda > a_{n+1}$, $f(\lambda) \leq n$; and if $\lambda \leq a_{n+1}$, $f(\lambda) \geq n+1$. It follows from Lemma 5.2(b) that

$$\bar{F}(n+r) = a_{n+1}, \text{ where } n \text{ is any non-negative integer and } 0 \leq r < 1.$$

Then, if $S = \sum_{i=1}^{\infty} a_i E_i$ and $T = \sum_{i=1}^{\infty} b_i F_i$ are two such operators we have

$$\sigma(S, T) = \sum_{i=1}^{\infty} a_i b_i.$$

A similar result of course holds when H is finite dimensional.

Since any state u on $G = \tau_S$ for some Hilbert-Schmidt $S \in G^+$, this gives us an immediate calculation formula for $\tilde{d}(u, v)$ in the case of a type I factor.

Example 8.13. Let G be a semi-finite factor with trace τ . For any projection $E \in G$ such that $\tau(E) = \alpha < \infty$, we can easily verify that

$$\bar{\tau}_E = \chi_{[0, \alpha)} \quad (\chi \text{ denoting the characteristic function of a set}).$$

Therefore, if E and F are any two projections $\in G$ such that $\alpha = \tau(E) \leq \tau(F) = \beta < \infty$, we have

$$\sigma(E, F) = \alpha.$$

Now if we consider the states $\mu = 1/\alpha(\tau_E)$ and $\nu = 1/\beta(\tau_F)$, we have from Theorem 8.6 and Remark 8.8 that

$$\begin{aligned} \tilde{\rho}(\mu, \nu) &= \sigma(\mu, \nu) = 1/\sqrt{\alpha\beta} (\sigma(E, F)) \\ &= \sqrt{\alpha/\beta} = \sqrt{\tau(E)/\tau(F)}. \end{aligned}$$

Note that if $\alpha/\beta \in r(\text{Aut}(G))$, the same type of calculation shows that $\tilde{\rho}(\mu, \nu) = 1$. Of course this is evident immediately since we can in this case find an automorphism sending F onto E and therefore ν onto μ .

CHAPTER III

APPLICATIONS TO INFINITE TENSOR PRODUCTS

9. Isomorphisms of Products: Definitions.

Definition 9.1. Let $(G, (\alpha_i))$ and $(B, (\beta_i))$ be two products for the family of W^* -algebras $(G_i)_{i \in I}$. We say that they are:

- (a) product isomorphic, if there exists an isomorphism ϕ from G onto B satisfying $\phi \alpha_i = \beta_i$ for all $i \in I$. In such a case we will write $(G, (\alpha_i)) \sim (B, (\beta_i))$;
- (b) weakly product isomorphic, if there exists an isomorphism ϕ from G onto B satisfying $\phi(\alpha_i(G_i)) = \beta_i(G_i)$ for all $i \in I$;
- (c) permutably product isomorphic, if there exists an isomorphism ϕ from G onto B and a permutation π of I , satisfying $\phi(\alpha_i(G_i)) = \beta_{\pi(i)}(G_{\pi(i)})$ for all $i \in I$, (in other words they are weakly product isomorphic after a rearrangement of the G_i 's);
- (d) algebraically isomorphic if there exists an isomorphism from G onto B .

It is obvious that (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d), and it is not hard to show that all implications are strict.

Remark 9.2. Note that if $(G, (\alpha_i))$ and $(B, (\beta_i))$ are weakly product isomorphic, we can define automorphisms ϕ_i of G_i for each $i \in I$ by

$$\phi_i(A_i) = \beta_i^{-1} \alpha_i(A_i), \text{ for all } A_i \in G_i.$$

We then have $\phi \alpha_i = \beta_i$, or in other words the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & B \\ \alpha_i \uparrow & \phi_i & \uparrow \beta_i \\ G_i & \xrightarrow{\quad} & B_i \end{array}$$

is commutative for all $i \in I$.

A similar result of course holds for permutable product isomorphisms.

Remark 9.3. It follows immediately from the definitions that a product isomorphism preserves product states. That is, if $v \in \Sigma_B = \otimes(v_i)$ for the product $(B, (\beta_i))$, and if ϕ is a product isomorphism from $(G, (\alpha_i))$ onto $(B, (\beta_i))$, then $v^\phi = \otimes(v_i)$ for the product $(G, (\alpha_i))$. If ϕ is instead a weak product isomorphism, $v^\phi = \otimes(v_i^{\phi_i})$ for the ϕ_i defined in Remark 9.2.

Lemma 9.4. Suppose that $(G, (\alpha_i)) \sim \bigotimes_{i \in I} (G_i, \mu_i)$. For each $i \in I$ let ϕ_i be an automorphism of G_i . Then

$$(G, (\alpha_i \phi_i)) \sim \bigotimes_{i \in I} (G_i, \mu_i^{\phi_i}) .$$

Proof. For each $i \in I$ choose $[\psi_i, x_i] \in Q(\mu_i^{\phi_i})$. Let $\mathcal{R}$ be the von-Neumann algebra $\mathcal{Q}(\mathcal{R}_i, x_i)$ where $\mathcal{R}_i = \psi_i(G_i)$. Then obviously $[\psi_i \phi_i^{-1}, x_i] \in Q(\mu_i)$, and since ϕ_i is an automorphism, $\mathcal{R}_i = \psi_i \phi_i^{-1}(G_i)$. So by the definition of $\mathcal{Q}(G_i, \mu_i)$ there exists an isomorphism ψ from G onto $\mathcal{R}$ satisfying

$$\psi \alpha_i(A_i) = \psi_i \phi_i^{-1}(A_i) \quad , \quad \text{for all } i \in I \text{ and } A_i \in G_i .$$

Then, $\psi \alpha_i \phi_i(A_i) = \psi_i \phi_i^{-1} \phi_i(A_i) = \psi_i(A_i)$ for all $i \in I$ and $A_i \in G_i$, which by the definition of $\mathcal{Q}(G_i, \mu_i^{\phi_i})$ proves the lemma.

Lemma 9.5. $\bigotimes_{i \in I} (G_i, \mu_i)$ and $\bigotimes_{i \in I} (G_i, \nu_i)$ are weakly product isomorphic, if and only if there exists for each $i \in I$, an automorphism ϕ_i of G_i such that $\bigotimes_{i \in I} (G_i, \mu_i)$ and $\bigotimes_{i \in I} (G_i, \nu_i^{\phi_i})$ are product isomorphic.

Proof. This follows immediately from Lemma 9.4 and Remark 9.2.

10. Main Results on Tensor Products.

In this section we apply the product formula for ρ which we obtained in Section 2 to develop necessary and sufficient conditions for two local tensor products of a given family of W^* -algebras to be product isomorphic. The same result was proved in ([3], Theorem 4.1) under the assumptions that the algebras are semi-finite, and it was shown to be a non-commutative extension of Kakutani's Theorem on infinite product measures [6]. The only need of the semi-finiteness restriction was in invoking the product formula for ρ , and as we have in Theorem 2.4 removed this restriction, we can appeal directly to [3] for a proof of this present result. We will however present a complete proof here, simplifying part of it, and at the same extending the product formula for ρ to the case of an infinite number of factors. Finally we obtain necessary and sufficient conditions of a similar nature for two local tensor products to be weakly product isomorphic and to be permutably product isomorphic.

Definition 10.1. Let $(G, (\alpha_i))$ be a product for the family $(G_i)_{i \in I}$. Let ϕ be any representation of G on a Hilbert space H . Let $\mu \in \Sigma_G$. For any non-void subset J of I we define:

G^J = the W^* -algebra generated by $\{\alpha_i(A_i) : i \in J, A_i \in G_i\}$;

ϕ^J = the restriction ϕ to G^J ;

μ^J = the restriction of μ to G^J .

Lemma 0.1 shows that ϕ^J is in fact a representation of G^J on H .

Lemma 10.2. Suppose $(G, (\alpha_i)) \sim \bigotimes_{i \in I} (G_i, \mu_i)$. Let μ be the element of $\Sigma_G = \bigotimes_{i \in I} (\mu_i)$ and let v be any element of Σ_G . Then there exists some subset J of I with a finite complement such that $\rho(\mu^J, v^J) > 0$.

Proof. Suppose to the contrary that $\rho(\mu^J, v^J) = 0$ for all $J \subset I$ with finite complements.

For any $J \subset I$ let E^J denote the support of v^J , and let E denote the support of v .

We can consider each G_i as a von Neumann algebra on a Hilbert space H_i with a vector x_i in H_i inducing μ_i , and G as a von Neumann algebra on $H = \bigotimes_{i \in I} (H_i, x_i)$.

Choose an orthonormal basis for H as described in Chapter 0. Let x' be any element of this basis. Recall that there exists a subset $J = J(x')$ of I , with finite complement, such that $x' = \bigotimes_{i \in I} (x'_i)$, where $x'_i = x_i$ for all $i \in J$.

Since $\bigotimes_{i \in I} (x_i)$ induces μ on G it is obvious that x' induces μ^J on G^J . Then, by our assumption and Lemma

1.2 the support of μ^J is orthogonal in G^J to E^J .

Therefore, $\|E^J x'\|^2 = \mu^J(E^J) = 0$. Since $v(E^J) = v^J(E^J)$

$= 1$, $E^J \geq E$ and we must have $E x' = 0$. Since this is

true for all x' in some orthonormal basis of H we have

$E = 0$, and therefore $\mu = 0$, which is a contradiction

to the fact that μ is a state.

Remark 10.3. We next recall some elementary facts about infinite products of numbers. If $(r_i)_{i \in I}$ is a family of positive numbers, we say that $\prod_{i \in I} r_i$ converges if $\lim_F (\prod_{i \in I} r_i)$ as F runs over the finite subsets of I exists as a positive number r . We then define $\prod_{i \in I} r_i = r$. Suppose $(r_i)_{i \in I}$ is a family of non-negative numbers with at least one $r_i = 0$. We then say that $\prod_{i \in I} r_i$ converges if there are only a finite number of zero factors and after deleting these the resulting family of positive numbers converges as above. In this case we define $\prod_{i \in I} r_i = 0$. It follows from these definitions that $\prod_{i \in I} r_i$ converges if and only if $\sum_{i \in I} |1 - r_i| < \infty$. It is easy to verify that if $\prod_{i \in I} r_i = r$ and $\prod_{i \in I} s_i = s$ then $\prod_{i \in I} (r_i s_i) = rs$.

We now consider the following situation; the notation will remain fixed for the remainder of this section. Let I be an arbitrary indexing set and let $(G_i)_{i \in I}$ be a family of W^* -algebras. Let $(u_i)_{i \in I}$ and $(v_i)_{i \in I}$ be two elements of $\wedge[(G_i)]$.

As a notational convenience we will let $\otimes$ denote $\otimes_{i \in I}$ (similarly for $\sum$ and $\prod$).

Lemma 10.4. The following conditions on (u_i) and (v_i) are equivalent:

$$(a) \quad \sum [d^{G_i}(u_i, v_i)]^2 < \infty ;$$

$$(b) \quad \Sigma [1 - \rho^{G_i}(\mu_i, \nu_i)] < \infty ;$$

$$(c) \quad \prod \rho^{G_i}(\mu_i, \nu_i) \text{ converges;}$$

for any subgroups G_i of $\text{Aut}(G_i)$.

Proof: Immediate from (3.1) and Remark 10.3.

Theorem 10.5. Suppose that $\nu = \otimes(\nu_i)$ exists as a product state on $\otimes(G_i, \mu_i)$. Let $\mu = \otimes(\mu_i)$. Then

$$\rho(\mu, \nu) = \prod \rho(\mu_i, \nu_i) \text{ , a convergent product.}$$

Proof. Suppose $(G, (\alpha_i)) \sim \otimes(G_i, \mu_i)$. We consider μ and ν as elements of Σ_G . Choose any $[\phi(x, y)] \in Q(\mu, \nu)$ and let F be any finite subset of I . A direct calculation shows that $[\phi^F(x, y)] \in Q(\mu^F, \nu^F)$ and therefore $\rho(\mu^F, \nu^F) \geq |(x|y)|$. Taking the supremum over all elements of $Q(\mu, \nu)$ we obtain

$$(10.1) \quad \rho(\mu^F, \nu^F) \geq \rho(\mu, \nu) .$$

It is obvious that $\mu^F = \otimes_{i \in F} (\mu_i)$ and $\nu^F = \otimes_{i \in F} (\nu_i)$ on $\otimes_{i \in F} (G_i) = G^F$. Therefore, by Theorem 2.4, which extends from two to any finite number of factors by an obvious induction and standard associativity arguments,

$$(10.2) \quad \rho(\mu^F, \nu^F) = \prod_{i \in F} \rho(\mu_i, \nu_i) .$$

By (10.1) and (10.2),

$$(10.3) \quad \rho(u, v) \leq \inf \left\{ \prod_{i \in F} \rho(u_i, v_i) : F \text{ a finite subset of } I \right\}.$$

Now by Lemma 10.2, choose a subset J of I with finite complement such that $\rho(u^J, v^J) > 0$. By applying the above argument to the algebra G^J , we obtain from (10.3) that

$$(10.4) \quad \inf \left\{ \prod_{i \in F} \rho(u_i, v_i) : F \text{ a finite subset of } J \right\} > 0.$$

Let $J' = \{i \in I : \rho(u_i, v_i) > 0\}$. By (10.4), $J \subseteq J'$, so J' has finite complement. Moreover $J' - J$ is finite so (10.4) holds with J replaced by J' . Then, since the value of ρ is always ≤ 1 ,

$$\lim_F \left(\prod_{i \in F} \rho(u_i, v_i) \right) = \inf_{F'} \left(\prod_{i \in F'} \rho(u_i, v_i) \right),$$

where F, F' runs over the finite subsets of I, J' respectively. Therefore $\prod \rho(u_i, v_i)$ converges and we may write (10.3) as

$$(10.5) \quad \rho(u, v) \leq \prod \rho(u_i, v_i).$$

We now prove the other direction. Let k be any positive number < 1 . Choose any sequence (k_n) of positive numbers < 1 so that $\prod_{n=1}^{\infty} k_n = k$. Let $I_0 = \{i \in I : \rho(u_i, v_i) > 0\}$. The convergence of $\prod \rho(u_i, v_i)$ and Lemma

10.4 show that I_0 is at most countable. Let γ be an injection from I_0 into the positive integers. Define a function g on I by

$$g(i) = \begin{cases} 1, & \text{for } i \in I - I_0 \\ k_{\gamma(i)}, & \text{for } i \in I_0. \end{cases}$$

Then choose, for each $i \in I$, an element $[\phi, (x, y)]$ of $Q(u_i, v_i)$ satisfying $|(x_i | y_i)| \geq g(i)[\rho(u_i, v_i)]$. This is certainly possible by the definition of ρ and the fact that $i \in I - I_0$ implies that $u_i = v_i$. By multiplying the vectors by suitable scalars if necessary we can assume that

$$(10.6) \quad (x_i | y_i) \geq g(i)[\rho(u_i, v_i)], \text{ for all } i \in I.$$

It is easy to see that $\prod_{i=1}^n g(i)$ converges and that its value is $\geq \prod_{n=1}^{\infty} k_n = k$. Therefore, (10.6) shows that $\prod (x_i | y_i)$ converges and that

$$(10.7) \quad \prod (x_i | y_i) \geq k[\prod \rho(u_i, v_i)], \text{ for all } i \in I.$$

We have then that $\Sigma[1 - (x_i | y_i)]$ converges, which shows that for the family of Hilbert spaces (H_i) , where H_i is the underlying space of ϕ_i , (x_i) and (y_i) are equivalent C_0 -sequences. Therefore, the vectors $x = \otimes(x_i)$ and $y = \otimes(y_i)$ are elements of $H = \otimes(H_i, x_i)$ which induce the states μ and ν respectively. Then using (10.7),

$$\rho(u, v) \geq |(\otimes(x_i) | \otimes(y_i))| = \overline{\prod (x_i | y_i)} \geq k[\prod \rho(u_i, v_i)]$$

Since k was chosen arbitrarily we have that

$$\rho(u, v) \geq \prod \rho(u_i, v_i) \quad , \text{ for all } i \in I$$

The above formula and 10.5 complete the proof.

Corollary 10.6 ([3] Lemma 3.6).

Suppose that $\Sigma[d(u_i, v_i)]^2 < \infty$. Then $\otimes(G_i, v_i)$ and $\otimes(G_i, u_i)$ are product isomorphic.

Proof. In the second part of the proof of Theorem 10.5 the convergence of $\Sigma[d(u_i, v_i)]^2$ was all that was necessary to produce the Hilbert space $H = \otimes(H_i, x_i) = \otimes(H_i, y_i)$, and the tensor product $\otimes(\phi_i(G_i), x_i) = \otimes(\phi_i(G_i), y_i)$ which is product isomorphic to both $\otimes(G_i, u_i)$ and $\otimes(G_i, v_i)$. The corollary follows immediately.

We now will state the three main theorems of this section and then give the proofs.

Theorem 10.7. The following conditions on (u_i) and (v_i) are equivalent:

- (a) $\Sigma[d(u_i, v_i)]^2 < \infty$;
- (b) $\otimes(G_i, u_i)$ and $\otimes(G_i, v_i)$ are product isomorphic;

(c) $\otimes(v_i)$ exists as a product state of $\otimes(G_i, u_i)$.
Moreover, if any of these conditions hold we have

$$\rho(\otimes(u_i), \otimes(v_i)) = \prod \rho(u_i, v_i) \quad , \text{ a convergent product.}$$

Theorem 10.8. The following conditions on (u_i) and (v_i) are equivalent:

- (a) $\Sigma[\tilde{d}(u_i, v_i)]^2 < \infty$, and for all but a countable number of $i \in I$ the infimum in the definition of $\tilde{d}$ is attained;
- (b) $\otimes(G_i, u_i)$ and $\otimes(G_i, v_i)$ are weakly product isomorphic;
- (c) There exist automorphisms ϕ_i of G_i such that $\otimes(v_i^{\phi_i})$ exists as a product state of $\otimes(G_i, u_i)$.

Theorem 10.9. The following conditions on (u_i) and (v_i) are equivalent;

- (a) For some permutation π of I , $\Sigma[\tilde{d}(u_i, v_{\pi(i)})]^2 < \infty$, and for all but a countable number of $i \in I$ the infimum in the definition of $\tilde{d}$ is attained;
- (b) $\otimes(G_i, u_i)$ and $\otimes(G_i, v_i)$ are permutably product isomorphic;
- (c) For some permutation π of I , there exist isomorphisms ϕ_i of G_i onto $G_{\pi(i)}$ such that $\otimes(v_{\pi(i)}^{\phi_i})$ exists as a product state of $\otimes(G_i, u_i)$.

Proof of Theorem 10.7. By Theorem 10.5 and Lemma 10.4, (c) implies (a). By Corollary 10.6, (a) implies (b). It is immediate that (b) implies (c) (see Remark 9.3). The last statement is immediate from Theorem 10.5.

Proof of Theorem 10.8. Suppose that (a) holds. Let I_0 be the subset of I for which the infimum is not attained in $\tilde{d}(\mu_1, \nu_1)$, and therefore by (3.1) the supremum is not attained in $\tilde{\rho}(\mu_1, \nu_1)$. By hypothesis, there is an injection γ from I_0 into the positive integers. Define a function g on I by

$$g(i) = \begin{cases} 0 & , \text{ for } i \in I - I_0 \\ 1/2^{\gamma(i)} & , \text{ for } i \in I_0 . \end{cases}$$

By Lemma 3.2, we can choose automorphisms ϕ_i of G_i such that

$$(10.8) \quad \rho(\mu_1, \nu_1^{\phi_i}) \geq \tilde{\rho}(\mu_1, \nu_1) - g(i) .$$

Now since $\sum g(i) < 1$, and since $\sum [1 - \tilde{\rho}(\mu_1, \nu_1)] < \infty$ by the hypothesis and Lemma 10.4, we obtain from (10.8) that

$$(10.9) \quad \sum [1 - \rho(\mu_1, \nu_1^{\phi_i})] \leq \sum [1 - \tilde{\rho}(\mu_1, \nu_1)] + \sum g(i) < \infty .$$

Then from Lemma 10.4 again,

$$(10.10) \quad \sum [d(\mu_1, \nu_1^{\phi_i})]^2 < \infty .$$

Conversely, suppose there exist automorphisms ϕ_i of G_i such that (10.10) holds. Since $\tilde{d}(\mu_1, \nu_1) \leq d(\mu_1, \nu_1^{\phi_i})$ we have

$$\sum [\tilde{d}(\mu_i, \nu_i)]^2 < \infty ,$$

and moreover, for all but a countable number of $i \in I$ we must have $\tilde{d}(\mu_i, \nu_i) = d(\mu_i, \nu_i^{\phi_i}) = 0$. So (a) is equivalent to the statement that (10.10) holds for some automorphisms ϕ_i of G_i . Replacing (b) with the equivalent statement given by Lemma 9.5, we can apply Theorem 10.7 to (μ_i) and $(\nu_i^{\phi_i})$ to prove Theorem 10.8.

Proof of Theorem 10.9. This follows in an obvious manner from Theorem 10.8.

Lemma 10.10. Suppose that for each $i \in I$, G_i is a factor of type I or type II_1 , and that μ_i and ν_i are compatible states. Then $\otimes(G_i, \mu_i)$ and $\otimes(G_i, \nu_i)$ are weakly product isomorphic if and only if they are product isomorphic.

Proof. Immediate from Theorems 10.7, Theorem 10.8, and Theorem 8.11.

Example 10.11. Let G be a I_2 factor with normalized trace τ , and let E and F be two orthogonal projections in G . Suppose that $G_i = G$ for all $i \in I$. Using Example 8.12 and the theorems of this section we can easily obtain the following.

(a) Let $\mu_i = 2\tau_E$ and $\nu_i = 2\tau_F$ for all $i \in I$. Then $\otimes(G_i, \mu_i)$ and $\otimes(G_i, \nu_i)$ are not product isomorphic but they are weakly product isomorphic.

(b) Let I be the positive integers. Suppose that:

$$\mu_i = \begin{cases} 2\tau_E & , \text{ for } i \text{ odd} \\ \tau & , \text{ for } i \text{ even} ; \end{cases}$$

$$v_i = \mu_{(i+1)} \quad , \text{ for all } i \in I ;$$

Then $\otimes(G_i, \mu_i)$ and $\otimes(G_i, v_i)$ are not weakly product isomorphic but they are permutably product isomorphic.

Example 10.12. Let G be a factor of type I_8 with normalized trace τ , and let B be a factor of type I_2 with normalized trace τ' . Let E be a minimal projection in B . Then $G = B \otimes B \otimes B$. Define the states μ and ν on G by

$$\mu = (2\tau'_E) \otimes (2\tau'_E) \otimes \tau' = 4\tau_{(E \otimes E \otimes 1)} ,$$

$$\nu = (2\tau'_E) \otimes \tau' \otimes \tau' = 2\tau_{(E \otimes 1 \otimes 1)} .$$

Let $G_i = G$, $\mu_i = \mu$ and $\nu_i = \nu$ for all $i \in I$. By Example 8.13, $\rho(\mu, \nu) = \sqrt{\frac{1}{2}}$, so $\otimes(G_i, \mu_i)$ and $\otimes(G_i, \nu_i)$ are not weakly (and therefore in this case not permutably) product isomorphic. However there exist "refinements" of each of these tensor products which are permutably product isomorphic. Therefore they are algebraically isomorphic. In fact, suppose I is countable. Using standard associativity arguments,

we can find in each case countable disjoint subsets J and K with $J \cup K = I$ so that the resulting algebra
 $= [\otimes_{J} (\mathcal{B}, \tau')] \otimes [\otimes_{K} (\mathcal{B}, 2\tau'_E)]$, which by standard results on type
 (see [2] e.g.), $= \mathcal{P} \otimes I_{\infty}$ where $\mathcal{P}$ is the hyperfinite II_1
 factor.

Example 10.13. The product formula does not hold for $\tilde{\rho}$.

As a trivial example, let μ and ν be any two states of a
 W^* -algebra G such that $\tilde{\rho}(\mu, \nu) < 1$. Then on $G \otimes G$,
 $\tilde{\rho}(\mu \otimes \nu, \nu \otimes \mu) = 1$. But $\tilde{\rho}(\mu, \nu) \tilde{\rho}(\nu, \mu) < 1$.

It is easy to show however that we always have

$$\tilde{\rho}(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) \geq \tilde{\rho}(\mu_1, \nu_1) \tilde{\rho}(\mu_2, \nu_2).$$

This is a useful fact, since it shows that in certain cases
 we may be able to deduce the algebraic isomorphism of two non-
 weakly product isomorphic tensor products by partitioning I
 into disjoint subsets and computing the distance $\tilde{d}$ between
 the resulting product states on the respective pieces.

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