Inequivalence and equivalence of certain kinds of non-normal operators

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Abstract

This thesis is concerned with the problem of unitary equivalence of certain kinds of non-normal operators. Suppose $[m, K, G, g \mapsto U_g]$ is an ergodic and free $C$-system, with $G$ abelian. Let $m = m \otimes 1,$ $n = R(U_g \otimes V_g : g \in G),$ and let $a = R(m, n) = R[m, K, G, g \mapsto U_g]$ be the von Neumann algebra constructed from $[m, K, G, g \mapsto U_g]$ according to von Neumann. We compute: (1) the group $A(a; m, n)$ of all automorphisms of $a$ which keep $m$ pointwise fixed and keep $n$ invariant, and (2) the group $A(m, a; n)$ (resp. $G(m, a; n)$) of all automorphisms of $m$ which extend to automorphisms (resp. inner automorphisms) of $a$ keeping $n$ pointwise fixed. These calculations lead us to compute $G' \cap [G]$ and $G'$ (where $[G]$ is the full group generated by $G$). We show that for an abelian and ergodic $G$ on an abelian $m$, $G' \cap [G] = G$. In the course of this investigation we obtain several interesting results. For example we see that such $G$ is automatically free on $m$. For a large class of tensor algebras (and in particular for a large class of multiplication algebras) we succeed in determining $G'$. (For the particular cases of multiplication algebras we only use measure-theoretical arguments.) These results are applied to solve the problem of unitary equivalence of certain kinds of non-normal operators. Finally for most of the interesting thick subalgebras $E$ in the literature, we construct numerous unitarily non-equivalent operators $A$ with $R(Re A) = E$. 
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Table of contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. The basic set up</td>
<td>6</td>
</tr>
<tr>
<td>3. The calculation of $A(a; m,n)$</td>
<td>9</td>
</tr>
<tr>
<td>4. Operators distinguishable by means of $A(a; m,n)$</td>
<td>13</td>
</tr>
<tr>
<td>5. The calculation of $A(m,a; n)$ and $G(m,a; n)$</td>
<td>15</td>
</tr>
<tr>
<td>6. The calculation of $G' \cap [G]$</td>
<td>20</td>
</tr>
<tr>
<td>7. The calculation of $G'$</td>
<td>24</td>
</tr>
<tr>
<td>8. Operators distinguishable by means of $G' \cap [G]$ and $G'$</td>
<td>44</td>
</tr>
<tr>
<td>9. Examples of non-equivalent operators of various type with $R(Re A)$</td>
<td>49</td>
</tr>
<tr>
<td>thick and of various type in $R(A)$</td>
<td></td>
</tr>
</tbody>
</table>

Appendix                                                                | 79   |

References                                                              | 86   |
1. Introduction

The following (so-called unitary equivalence) problem is of paramount importance in the theory of operators: given two (bounded linear) operators \( A_1, A_2 \) on a (complex) Hilbert space \( H \), to determine whether or not they are unitarily equivalent, i.e. whether or not there is a unitary operator \( U \) on \( H \) such that \( U^*A_1U = A_2 \). For normal operators this question is completely answered by the classical multiplicity theory [20 or 10]. Many authors, in particular Brown [3], Pearcy [15], Deckard [8], Radjavi [19], and Arveson [1, 2], considered the problem for non-normal operators and have obtained various significant results. However most of their results (cf. [22]) deal only with operators which are of type I in the following sense [21]: an operator is of type I (II, II\(_1\), III) if the von Neumann algebra generated by \( A \) is of type I (resp. II, II\(_1\), III). For non-normal operators of type I the problem is already known to be difficult, and the known results are far from exhaustive; the author is therefore very pleased if he has obtained (as he so believes) some interesting results for some kinds of operators of more general type.

The problem of unitary equivalence is closely connected with the following problem of algebraic equivalence: given two operators \( A_1, A_2 \) on the Hilbert space \( H \), and denoting by \( a_1, a_2 \) the von Neumann algebras generated by \( A_1, A_2 \) respectively, to determine whether or not they are algebraically equivalent, i.e. whether or not there is an (algebraic \( \ast \)-) isomorphism \( \phi \) of \( a_1 \) onto \( a_2 \) such that \( \phi(A_1) = A_2 \). In fact, if
\( a_1 = a_2 \) is a factor, it is well-known that the two concepts coincide. So we concentrate on the algebraic equivalence.

Let us outline at this point a 'sieving' program for the above problem of algebraic equivalence, and put the present work into perspective. For simplicity let us call two operators equivalent when they are algebraically equivalent in the above sense. First we examine whether \( a_1 \) and \( a_2 \) are isomorphic. If they are not, then \( A_1 \) and \( A_2 \) are non-equivalent; if they are, we can assume that \( a_1 = a_2 \), and proceed to the second stage. Secondly, assuming that \( A_2 \) generates the same von Neumann algebra \( a \) as \( A_1 \) does, and denoting by \( m_1(m_2) \) the von Neumann algebra generated by the real part \( \text{Re} A_1 \) (resp. \( \text{Re} A_2 \)) of \( A_1 \) (resp. \( A_2 \)), we examine whether \([a, m_1]\) and \([a, m_2]\) are equivalent, i.e. whether there is an automorphism \( \phi \) of \( a \) such that \( \phi(m_1) = m_2 \). When \( a \) is a type I factor, the classical multiplicity theory [see 20] provides a complete solution to this question. For type II\(_1\) factor \( a \), this question has been examined and some results have been obtained by Bures [5]. If the answer to this question is negative, then \( A_1 \) and \( A_2 \) are non-equivalent; otherwise we can assume \( m_1 = m_2 \) and proceed to the third stage. Thirdly, assuming that \( A_2 \) generates the same von Neumann algebra \( a \) as \( A_1 \) does, and that \( \text{Re} A_1 \) generates the same von Neumann algebra \( m \) as \( \text{Re} A_2 \) does, we examine whether \( \text{Im} A_2 = \alpha(\text{Im} A_1) \) for some \( \alpha \in A(a; m) \), where \( \text{Im} A \) is the imaginary part of \( A \) and \( A(a; m) \) is the group of all automorphisms of \( a \) which leave \( m \) invariant. The computation of \( A(a; m) \) is in general very difficult and only fragmentary information is available. [see §3 of this thesis.] If the answer to the above question...
is negative, then $A_1$ and $A_2$ are non-equivalent; otherwise, we can assume that $\text{Im} A_1 = \text{Im} A_2$ and proceed to the final stage. Finally, assuming the same conditions as in the third stage, and in addition that $\text{Im} A_1 = \text{Im} A_2 = T$, we examine whether there is an automorphism $\phi$ of $\mathfrak{a}$ such that $\phi(\text{Re} A_1) = \text{Re} A_2$ and $\phi(T) = T$. Obviously $A_1$ and $A_2$ are equivalent if and only if the answer to the above question is affirmative. To settle this question one needs to compute the group $A(m, a; n)$ of all automorphisms of $m$ which extend to automorphisms of $\mathfrak{a}$ keeping $n$ pointwise fixed, where $n$ denotes the von Neumann algebra generated by $T$. For a large class of $[a, m, n]$ this $A(m, a; n)$ is determined in §§5,7 of this thesis.

As we have seen in the last paragraph, we can assume without much loss of generality that $a_1 = a_2$. In that case, the algebraic equivalence is one inside one and the same algebra $\mathfrak{a}$. By analogy to the classical situation, we are also interested in the problem of inner equivalence, i.e. whether or not there is a unitary $U$ in $\mathfrak{a}$ such that $U^* A_1 U = A_2$. In this thesis we consider the special case when $R(\text{Re} A_1) = R(\text{Re} A_2) = m$, $\text{Im} A_1 = \text{Im} A_2 = T$ and $a = R(m, T)$. For the same class of $[a, m, n]$ as in the last paragraph, we have succeeded in calculating the group $G(m, a; n)$ of all automorphisms of $m$ which extend to inner automorphisms of $\mathfrak{a}$ keeping $n$ pointwise fixed, and settled the afore-mentioned question completely in §§5,6.

We have not pretended that this program is an easy one; indeed all questions listed above are very difficult. After all we cannot and do not expect a simple solution to the general problem.
As an application of our results, we construct in the last section numerous examples of non-equivalent operators (of type II\textsubscript{1}, II\textsubscript{\infty} and III). In fact, for a large class of \([E, a]\), where \(E\) is thick in \(a\) (\(a\) can be a factor of type II\textsubscript{1}, II\textsubscript{\infty}, III etc.) [see 5], we construct a family \((A_i)\) of pairwise inequivalent operators such that \(R(A_i) = a\), \(R(\text{Re } A_i) = E\), the \(\text{Im } A_i\)'s are identical, and the \(\text{Re } A_i\)'s can be chosen so that they are unitarily equivalent to each other.

In the course of the investigation of \(\mathcal{A}(m, a; n)\) we have determined the commutant \(G'\) of a group \(G\) of automorphisms of \(m\) (see §7 and the appendix) for a large class of \(G\). \([G'\] is the group consisting of automorphisms of \(m\) which commute with each element of \(G\).] Also we have succeeded in showing that an abelian and ergodic group of automorphisms of an abelian von Neumann algebra is necessarily free [cf. §6]. These results are of independent interest.

Finally a few words about the contents. In §2 we have the basic set up: \(a\) is the von Neumann algebra constructed from a free, ergodic C-system \([m; K, G, g \leftrightarrow U_g]\) according to von Neumann and Dixmier [13, 8], \(m = m \Theta 1\), and \(n = R(U_g \Theta V_g : g \in G)\) [For details of notations, cf. §2 below.]. In section 3 we compute the group \(\mathcal{A}(a; m, n)\) of all automorphisms of \(a\) which keep \(m\) pointwise fixed and keep \(n\) invariant. This result indicates that \(\mathcal{A}(a; m)\) can be rather complicated [Note \(\mathcal{A}(a; m, n) \subseteq A(a; m)\)]. In §4 we present simple examples of operators which are distinguishable (up to unitary equivalence) by means of the calculation of \(\mathcal{A}(a; m, n)\) in
§3. In §5 we compute \( A(m, a; n) \) and \( G(m, a; n) \), the importance of which became clear in the preceding paragraphs. This calculation leads us to compute \( G' \cap [G] \) ([G] being the full group generated by \( G \); cf. §5 below for detailed explanation) in §6, and \( G' \) in §7. The results contained in §6 are very general; those in §7 is rather specific. Thus we present in the appendix a result which generalizes those of §7 to a large family of tensor product algebras; this arrangement is due mainly to the tone and the machinery of this thesis. Then in §8 we apply the results of §§6,7 to operators. Finally in §9, for almost all interesting thick subalgebras \( E \) conceived in the literature \([5]\), we construct numerous (unitarily) non-equivalent operators \( A \) with \( R(\text{Re} A) = E \).
2. The basic set up

In this thesis we shall work mostly in the framework outlined in Definition 2.1 (ii) below. Though in general our notation and terminology is that of Dixmier [9], we find it helpful to use the following standard (yet different from Dixmier's) notations and definitions.

Definition 2.1 [cf. 4]

(i) The system $[m, K, G, g \mapsto U_g]$ is called a C-system if $m$ is a maximal abelian von Neumann algebra on the Hilbert space $K$, if $G$ is a group of automorphisms of $m$, and if $g \mapsto U_g$ is a unitary representation of $G$ on $K$ such that $U_g(M)U_g^* = g(M)$ for each $M \in m$.

(ii) Suppose $[m, K, G, g \mapsto U_g]$ is a C-system. Define $K_G$ to be the Hilbert space with an orthonormal basis $(\phi_g)_{g \in G}$ indexed on $G$. Define $H = K \otimes K_G$, $m = m \otimes 1_{K_G}$, $n = R(U_g \otimes V_g : g \in G)$, the von Neumann algebra generated by $\{ U_g \otimes V_g : g \in G \}$ in $L(H)$, where $V_g$ is the unitary operator on $K_G$ which maps $\phi_h$ to $\phi_{gh}$. Define

$$a[m, K, G, g \mapsto U_g] = R(m, n).$$

This construction of $a$ is originally due to von Neumann [15], and further developed by Dixmier [9, pp. 129-137].

(iii) The C-system $[m, K, G, g \mapsto U_g]$ is free if for any $g \in G$, not the identity, there is a family $(E_i)_{i \in I}$ of projections of $m$ such
that \( \sum_{i \in I} E_i = 1 \) and \( g(E_i)E_i = 0 \) for all \( i \in I \). This is not the original definition of freeness due to von Neumann [15] or Dixmier [9]; but it is proved in [4, §4; 5, §7] that these two definitions of freeness are equivalent. We find it more convenient to employ the above definition in this thesis.

(iv) The C-system \([m, K, G, g \mapsto U_g]\) is ergodic if

\[
\mathcal{m} \cap \{ U_g : g \in G \}' = \mathcal{C}.
\]

(v) We shall say that the C-system \([m, K, G, g \mapsto U_g]\) is abelian if \( G \) is abelian.

Throughout this thesis we shall consider only free and ergodic C-systems (or rather the algebras \( a, m, n \) constructed from such C-systems), and we shall stick to the notations set above without further explanation. We remark in passing that if the C-system is free and ergodic, then \( m \) is maximal abelian in \( \mathcal{a} \) [this is a well-known fact; cf. 9, p.133, Lemma 2], that \( (U_g \otimes V_g)m(U_g \otimes V_g)^* = m \) always, and that \( m \cap n = \mathcal{C} \) always. In order to see the last equality, let \( e \) be the identity of \( G \), \( \phi \) the natural isomorphism from \( K \) onto \( K \otimes [\phi_e] \) given by : \( \phi(x) = x \otimes \phi_e \) ([\( \phi_e \)] being the closed linear subspace generated by \( \phi_e \) in \( K_G \)), \( E \) the projection from \( K \otimes K_G \) onto \( K \otimes [\phi_e] \), and finally for each \( A \in \mathcal{a} \), define \( A_e \) to be the operator in \( L(K) \) such that \( A_e x = (\phi^{-1}EA\phi)(x) \) for each \( x \in K \).

Then it is straightforward to compute that \( (U_g \otimes V_g)_e = 1 \) if \( g = e, = 0 \) if \( g \neq e \), and that \( (M \otimes 1)_e = M \) for each \( M \in m \). Thus for any \( A \in \mathcal{m} \), \( A_e \in \mathcal{C} \) and therefore for any \( M \otimes 1 \in m \cap n \):
\[ M = (M \otimes 1)_e \in \mathcal{C} . \]

We conclude that \( m \cap n = \mathcal{C} \).

Before we close this section let us introduce one more definition which will be very useful in the sequel.

**Definition 2.2** Let \( a \supset m \) be von Neumann algebras. Then \((E, F)\) fits in \([a, m]\) if \( E, F \) are von Neumann subalgebras of \( m \) such that for any automorphism \( \phi \) of \( a \) with \( \phi(E) = F \):

\[ \phi(m) = m. \]

An obvious example of such \((E, F)\) is: \( E' \cap a = F' \cap a = m \).

In this thesis we shall consider operators \( A_1, A_2 \) on \( H \) with \( \mathcal{R}(A_1) = \mathcal{R}(A_2) = a \), and such that \((\mathcal{R}(\text{Re } A_1), \mathcal{R}(\text{Re } A_2))\) fits in \([a, m]\), or that \((\mathcal{R}(\text{Im } A_1), \mathcal{R}(\text{Im } A_2))\) fits in \([a, n]\).
3. The calculation of $A(a; m,n)$

We begin with the definition of $A(a; m,n)$.

Definition 3.1 We denote by $A(a; m,n)$ the group of automorphisms of $a$ which keep $m$ pointwise fixed and leave $n$ invariant. Clearly $A(a; m,n)$ is a subgroup of the group $A(a; m)$ of all automorphisms of $a$ which leave $m$ invariant, mentioned in §1.

Lemma 3.1 For each $\sigma \in A(a; m,n)$ there is a character $c : G \to \mathbb{C}$ such that for all $g \in G$:

$$\sigma(U_g \otimes V_g) = c(g)(U_g \otimes V_g).$$

Proof. As $\sigma \in A(a; m,n)$ it keeps $m$ pointwise fixed and leaves $n$ invariant. Now for every $M \in m$:

$$\sigma(U_g \otimes V_g)\sigma(U_g \otimes V_g)^* = \sigma((U_g \otimes V_g)M(U_g \otimes V_g)^*) = (U_g \otimes V_g)M(U_g \otimes V_g)^*.$$

Hence

$$(U_g \otimes V_g)^*\sigma(U_g \otimes V_g) \in m' \cap n$$

$$= m \cap n$$

$$= 0,$$
by the remark preceding Definition 2.2. Therefore there is a complex number $c(g)$ such that

$$\sigma(U_g \otimes V_g) = c(g)(U_g \otimes V_g).$$

It follows readily that the mapping $g \mapsto c(g)$ is a character on $G$. That completes the proof.

**Definition 3.2** Suppose $\mathcal{A}$ is a von Neumann algebra on the Hilbert space $H$, and that $U$ is a unitary operator on $H$. Suppose $U\mathcal{A}U^* = \mathcal{A}$. Then $U$ is said to induce the automorphism of $\mathcal{A}$ given by:

$$A \in \mathcal{A} \mapsto UAU^* \in \mathcal{A}.$$

**Lemma 3.2** For each character $c : G \to \mathbb{C}$ there is a unitary operator $W_c$ on $K_G$ such that the unitary $1 \otimes W_c$ induces an automorphism $\sigma_c \in \mathcal{A}(\mathcal{A}; m, n)$ with

$$\sigma_c(U_g \otimes V_g) = c(g)(U_g \otimes V_g), \quad g \in G.$$

**Proof.** Let $c$ be a character on $G$. Define a unitary operator $W_c$ on $K_G$ by:

$$W_c(\phi_g) = c(g)\phi_g, \quad g \in G.$$

Then obviously $1 \otimes W_c$ commutes with $m$. Furthermore for any $g, h \in G$, we have:
Thus
\[ W_c V_g W_c^* h = W_c V_g c(h) \phi_h \]
\[ = W_c c(h) \phi_{gh} \]
\[ = c(gh) c(h) \phi_{gh} \]
\[ = c(g) V_g \phi_h. \]

Thus
\[ W_c V_g W_c^* = c(g) V_g, \]

and
\[ (1 \otimes W_c) (U_g \otimes V_g) (1 \otimes W_c)^* = c(g) (U_g \otimes V_g). \]

Hence the unitary \( 1 \otimes W_c \) induces an automorphism \( \sigma_c \in A(a; m, n) \) with
\[ \sigma_c(U_g \otimes V_g) = c(g) (U_g \otimes V_g), \quad g \in G. \]

That completes the proof.

**Remark.** Lemma 3.2 holds true with no restriction on the C-system as we do not need any property, like freeness, of it in the above proof.

**Definition 3.3** For a character \( c \) on \( G \) let \( \sigma_c \) denote the automorphism of \( A \) induced by the unitary \( 1 \otimes W_c \) in Lemma 3.2.

**Theorem 3.3** \( A(a; m, n) = \{ \sigma_c : c \text{ a character on } G \} \).
Proof. By Lemma 3.2, each $\sigma_c \in A(a; m,n)$. By Lemma 3.1 for each $\sigma \in A(a; m,n)$ there is a character $c$ on $G$ such that $\sigma|_n = \sigma_c|_n$. But $\sigma|_m = \sigma_c|_m$, both being the identity map on $m$. Since $a = R(m, n)$, so $\sigma = \sigma_c$. That completes the proof.
4. Operators distinguishable by means of $A(a; m,n)$

The following direct consequence of Theorem 3.3 is useful in operator theory.

**Theorem 4.1** Let $A_1$ and $A_2$ be two operators on $H$ such that $R(A_1) = \mathcal{R}(A_2) = a$, $\text{Re} A_1 = \text{Re} A_2$, $\mathcal{R}(\text{Re} A_1) = m$, and $(\mathcal{R}(\text{Im} A_1), \mathcal{R}(\text{Im} A_2))$ fits in $[a, n]$ (cf. Definitions 2.1, 2.2). Then the following statements are equivalent:

(i) $A_1$ and $A_2$ are unitarily equivalent,

(ii) $A_1$ and $A_2$ are (algebraically) equivalent,

(iii) $\text{Im} A_2 = \sigma_c(\text{Im} A_1)$ for some character $c$ on $G$.

[For definition of $\sigma_c$, cf. Definition 3.3]

**Proof.** The theorem follows directly from Theorem 3.3 and the observations that $A_1$ and $A_2$ are equivalent if and only if there is a $\sigma \in A(a; m,n)$ such that $\text{Im} A_2 = \sigma(\text{Im} A_1)$, and that each $\sigma_c$ is implemented by a unitary operator on $H$.

We now illustrate Theorem 4.1 by the following simple example.

Let $m$ be $L_\infty[0, 1]$ acting by multiplication on $K = L_2[0, 1]$ (cf. 14). Let $D$ be the group of dyadic rationals in $[0, 1]$ under addition mod 1. For each $d \in D$, let $\tau_d$ be the automorphism of $m$ given by:
\[ \tau_d M_f = M_d[f] , \]

where \( M_f \in m \) is the multiplication by \( f \in L_{10}[0, 1] \), and \( d[f] \) is the function in \( L_{10}[0, 1] \) given by: \( d[f](y) = f(y - d), \ y \in [0, 1] \). Let \( G = \{ \tau_d : d \in D \} \). Let \( U_d \) be the unitary operator on \( K \) given by:

\[ (U_d h)(x) = h(x - d), \ h \in L_2[0, 1], \ x \in [0, 1]. \]

Then [14] the system \( (m, K, G, \tau_d \mapsto U_d) \) is an abelian, free and ergodic C-system. Let \( g \) be a strictly monotone, continuous and real-valued function defined on \( [0, 1] \). Then \( M_g \otimes 1 \) is self-adjoint, and it generates \( m \) (i.e. \( R(M_g \otimes 1) = m \)). Suppose \( \sum_{d \in D} a_d(U_d \otimes V_d) \) and \( \sum_{d \in D} \beta_d(U_d \otimes V_d) \) are self-adjoint, and suppose each of them generates \( m \). Then by Theorem 4.1 we have:

The operators \( M_g \otimes 1 + i \sum_{d \in D} a_d(U_d \otimes V_d) \) and \( M_g \otimes 1 + i \sum_{d \in D} \beta_d(U_d \otimes V_d) \) are unitarily equivalent if and only if for some character \( \chi \) on \( D \),

\[ \sum_{d \in D} a_d(U_d \otimes V_d) = \sigma_{\chi} \left( \sum_{d \in D} \beta_d(U_d \otimes V_d) \right) , \]

i.e. if and only if, \( a_d = \chi(d) \beta_d \) for all \( d \in D \).
5. The calculation of \( A(m, a; n) \) and \( G(m, a; n) \)

For convenience as well as for later references let us introduce the following definitions.

**Definition 5.1** [cf. 5] Let \( m \) be an abelian von Neumann subalgebra of the von Neumann algebra \( a \).

(i) \( A(m) \equiv \{ \alpha : \alpha \) is an automorphism of \( m \} \); and an \( m \)-group is a subgroup of \( A(m) \).

(ii) \( G(m, a) \equiv \{ \alpha \in A(m) : \) there is a unitary \( U \in a \) such that \( \alpha(M) = UMU^* \) for all \( M \in m \} \).

(iii) \( A(m, a) \equiv \{ \alpha \in A(m) : \) there is an automorphism \( \gamma \) of \( a \) such that \( \gamma|_m = \alpha \} \).

(iv) For a subset \( G \) of \( A(m) \),

\[
G' \equiv \{ \alpha \in A(m) : \alpha \beta = \beta \alpha \text{ for all } \beta \in G \}.
\]

(v) Let \( \alpha, \beta \in A(m) \). We say that \( \alpha \) and \( \beta \) agree on a projection \( F \) of \( m \) if

\[
\alpha(M) = \beta(M) \text{ for all } M \in m \text{ with } FM = M.
\]

We denote by \( E(\alpha, \beta) \) the largest projection of \( m \) on which \( \alpha \) and \( \beta \) agree.

(vi) Let \( S \) be a subset of \( A(m) \). For each \( \alpha \in A(m) \) define

\[
E(\alpha, S) = \sup \{ E(\alpha, \beta) : \beta \in S \},
\]
and let \[ [S] = \{ \alpha \in A(m) : E(\alpha, S) = 1 \} . \]

Call \( S \) full if \[ [S] = S. \]

We shall need the following result.

Lemma 5.1 [cf. 5, 21] Under the assumptions of Definition 2.1 (i)-(iv), we have:

(i) \( G(m, a) = \overline{G} \), where \( \overline{G} = \{ \alpha \in A(m) : \) for some \( g \in G, \alpha(M \otimes 1) = g(M) \otimes 1 \) for all \( M \in m \} . \)

(ii) For each \( \alpha \in G' \), there is a unitary operator \( W_\alpha \) on \( H \) such that \( W_\alpha (M \otimes 1)W_\alpha^* = \alpha(M) \otimes 1, \) for all \( M \in m \), and \( W_\alpha (U_g \otimes V_g)W_\alpha^* = U_g \otimes V_g, \) for all \( g \in G \).

Proof. Part (i) is a well-known result, and it is proved in [5, 20].

Suppose now \( \alpha \in G' \). Since \( m \) is maximal abelian in \( L(K) \), by [9, p.241] there is a unitary operator \( Y \) on \( K \) such that \( YMY^* = \alpha(M) \) for all \( M \in m \).

Define a unitary operator \( W_\alpha \) on \( H \) by:

\[ W_\alpha (x \otimes \phi_g) = (U_g YU_g^* x) \otimes \phi_g, \ x \in K, \ g \in G. \]

Then \( W_\alpha^* (x \otimes \phi_g) = (U_g Y^* U_g^* x) \otimes \phi_g \),

and for all \( M \in m \):
\[ [W_a(M \circ 1)W^*_a](x \circ \phi_g) = [W_a(M \circ 1)](U_gY^*U^*_gx \circ \phi_g) \]
\[ = (U_gY^*U^*_g) \circ \phi_g \]
\[ = [(gag^{-1})M \circ 1](x \circ \phi_g) \]
\[ = [a(M) \circ 1](x \circ \phi_g), \]

also
\[ [W_a(U_g \circ V_g)W^*_a](x \circ \phi_h) = [W_a(U_g \circ V_g)](U_hY^*U^*_hx \circ \phi_h) \]
\[ = W_a[(U_{gh}Y^*U^*_hx) \circ \phi_{gh}] \]
\[ = U_{gh}Y^*U^*_h[U_{gh}Y^*U^*_hx] \circ \phi_{gh} \]
\[ = (U_{gh}Y^*U^*_hx) \circ \phi_{gh} \]
\[ = (U_g \circ V_g)(x \circ \phi_h). \]

Thus
\[ W_a(M \circ 1)W^*_a = a(M) \circ 1, \quad \text{for all } M \in m, \]

and
\[ W_a(U_g \circ V_g)W^*_a = U_g \circ V_g, \quad \text{for all } g \in G. \]

Before we introduce the next theorem, recall that \( A(m, a; n) \) is the group of all automorphisms of \( m \), which extend to automorphisms of \( a \) keeping \( n \) pointwise fixed, and that \( G(m, a; n) \) is the group of all automorphisms of \( m \), which extend to inner automorphisms of \( a \) keeping \( n \) pointwise fixed.
Theorem 5.2  With the assumption of Definition 2.1 (i)-(iv), we have:

\[ A(m,a; n) = \overline{G^r}, \]

and

\[ G(m,a; n) = \overline{G^r} \cap [\overline{G}], \]

where for \( S \subseteq A(m) \), \( \overline{S} = \{ \alpha \in A(m) : \text{for some } s \in S, \alpha(M \otimes 1) = s(M) \otimes 1 \text{ for all } M \in \overline{m} \} \).

Proof. Suppose \( \alpha \in A(m,a; n) \). Then \( \alpha \) extends to an automorphism \( \overline{\alpha} \) of \( a \) with \( \overline{\alpha}|m = \alpha \) and \( \overline{\alpha}(U_g \otimes V_g) = U_g \otimes V_g \) for all \( g \in G \). Let \( \overline{\alpha} \) be the automorphism of \( \overline{m} \) such that for all \( M \in \overline{m} \):

\[ \overline{\alpha}(M) \otimes 1 = \alpha(M \otimes 1). \]

Then for all \( g \in G \), and for all \( M \in \overline{m} \):

\[ ag(M) \otimes 1 = \alpha(g(M) \otimes 1) \]

\[ = \alpha(U_gM U_g^* \otimes 1) \]

\[ = \overline{\alpha}[(U_g \otimes V_g)(M \otimes 1)(U_g \otimes V_g)^*] \]

\[ = (U_g \otimes V_g)\alpha(M \otimes 1)(U_g \otimes V_g)^* \]

\[ = [g\alpha(M)] \otimes 1. \]
Thus \( \alpha \in G' \) and \( \alpha \in \overline{G}' \).

Suppose, on the other hand, that \( \alpha \in \overline{G}' \). Let \( \alpha \) be as above. Then \( \alpha \in G' \). So by Lemma 5.1, there is a unitary operator \( W_\alpha \) on \( H \) such that

\[
W_\alpha (M \otimes 1) W_\alpha^* = \alpha(M) \otimes 1
\]

\[
= \alpha(M \otimes 1),
\]

and

\[
W_\alpha (U_g \otimes V_g) W_\alpha^* = U_g \otimes V_g.
\]

Thus we see that \( \alpha \) extends to an automorphism of \( \alpha \) keeping \( n \) pointwise fixed.

So the first equality of the present theorem is proved. The second equality follows directly from the first and Lemma 5.1 (i). That completes the proof.
6. The calculation of $G' \cap [G]$

In view of Theorem 5.2 we shall compute $G' \cap [G]$ in this section. Before we do that we need a few definitions and an auxiliary result.

**Definition 6.1** Let $m$ be an abelian von Neumann algebra, and $F$ an $m$-group. Then $F$ is ergodic if $M \in m$, $f(M) = M$ for all $f \in F \Rightarrow M \in C$. $F$ is free if for any $f \in F$, not the identity, there is a family $(E_i)$ of projections of $m$ such that $\sum E_i = 1$ and $f(E_i)E_i = 0$ for each $i$.

**Remark.** It is readily seen that the C-system $[m, K, G, g \mapsto U_g]$ is ergodic (or free) if and only if $G$ is. Also it is well-known and could be easily proved that if $F$ is ergodic, then for any two non-zero projections $P_1, P_2$ of $m$, there is an $f \in F$ such that $f(P_1)P_2 \neq 0$.

**Proposition 6.1** Suppose that $m$ is an abelian von Neumann algebra, and $F$ is an ergodic and abelian $m$-group. Suppose that $G$ is in $F'$. Then if $\alpha_1$ and $\alpha_2$ are in $F$ with $E(\beta, \alpha_1) \neq 0$ and $E(\beta, \alpha_2) \neq 0$, we have:

$$E(\beta, \alpha_1) = E(\beta, \alpha_2).$$

**Proof.** Let $\beta$ agree with $\alpha_1$ on a non-zero projection $P_i$ of $m$ (i=1,2). Since $F$ is ergodic there exists $\alpha \in F$ such that $Q \equiv \alpha(P_1)P_2 \neq 0$. Now if $M \in m$ with $\alpha(M)Q = \alpha(M)$ then $\beta(M) = \alpha_1(M)$. So for $M \in m$ with $MQ = M$ we have first
\[ \beta(M) = a_2(M), \]

and secondly,

\[ \beta(M) = (a\beta)(a^{-1}(M)) \]

\[ = a\alpha_1(a^{-1}(M)) \]

\[ = \alpha_1(M), \]

where we have used both that \( \beta \in F' \) and that \( F \) is abelian. Thus we see that \( \alpha_1 \) and \( \alpha_2 \) agree on \( \alpha(P_1P_2) \). That is, any non-zero projection (of \( M \)) on which \( \beta \) agrees with \( \alpha_2 \) majorizes a non-zero projection (of \( M \)) on which \( \alpha_1 \) agrees with \( \alpha_2 \). Therefore \( E(\beta, \alpha_2)[1 - E(\alpha_1, \alpha_2)] = 0 \), or

\[ E(\beta, \alpha_2) \leq E(\alpha_1, \alpha_2). \]

By the definition of \( E(\alpha_1, \alpha_2) \) we obtain

\[ E(\beta, \alpha_2) \leq E(\beta, \alpha_1). \]

The reverse inequality is obtained by reversing the roles of \( \alpha_1 \) and \( \alpha_2 \), and we conclude that

\[ E(\beta, \alpha_1) = E(\beta, \alpha_2). \]

We shall also need the following result of Bures [5].

Lemma 6.2 [5, Proposition 4.3] Suppose that \( \alpha \) and \( \beta \) are automorphisms of an abelian von Neumann algebra \( M \). Then there exists a family \( (E_i) \)
of projections of $m$ such that

$$\sum E_i = 1 - E(\alpha, \beta)$$

and

$$(\alpha(E_i))(\beta(E_i)) = 0 \text{ for each } i.$$ 

Now we can prove our next theorem. It should be pointed out that part (ii) below is proved in [7] under the superfluous assumption that $F$ is free, and by completely different techniques.

**Theorem 6.3** Suppose that $m$ is an abelian von Neumann algebra, and $F$ is an ergodic and abelian $m$-group. Then:

(i) $F$ is free.

(ii) $F$ is maximal abelian in $[F]$.

(iii) $F' \cap [F] = F$.

(iv) $\beta \in F' \implies E(\beta, \alpha) \neq 0$ for at most one $\alpha \in F$.

**Proof.**

Ad (i). Let $e$ be the identity of $F$, and let $\beta \in F \setminus \{e\}$. Since $E(\beta, e) = 1$ and $\beta \neq e$, we have $E(\beta, e) \neq E(\beta, \beta)$. Now as $F$ is abelian, $\beta \in F'$ and so, by Proposition 6.1, $E(\beta, e) = 0$. By Lemma 6.2, there exists a family $(E_i)$ of projections of $m$ such that

$$\sum E_i = 1 \text{ and } \beta(E_i)E_i = 0 \text{ for each } i.$$
So $F$ is free.

Ad (ii). Let $F_1$ be an abelian subset of $[F]$ containing $F$. Let $\beta \in F_1$. Then as $F_1$ is abelian and $F_1 \supseteq F$, $\beta \in F'$. Now $\beta \in [F]$ also, so

$$\sup \{ E(\beta, a) : a \in F \} = 1,$$

or

$$\sup \{ E(\beta, a) : a \in F \text{ and } E(\beta, a) \neq 0 \} = 1.$$

By Proposition 6.1 this means that for some $a_0 \in F$,

$$E(\beta, a_0) = 1,$$

i.e.

$$\beta = a_0.$$

So $\beta \in F$. We conclude that $F_1 = F$. Thus $F$ is maximal abelian in $[F]$.

Ad (iii). As $F$ is abelian we obviously have $F' \cap [F] \supseteq F$. The above proof of (ii) shows in fact that $F' \cap [F] = F$. Thus we have $F' \cap [F] = F$.

Ad (iv). Suppose that $E(\beta, \alpha_1) \neq 0$ and $E(\beta, \alpha_2) \neq 0$ for $\alpha_1, \alpha_2 \in F$.

Then by Proposition 6.1, $\alpha_1$ and $\alpha_2$ agree on the non-zero projection

$Q = E(\beta, \alpha_1) = E(\beta, \alpha_2)$. Now let $(E_i)$ be any family of orthogonal projections in $m$ such that $\alpha_1^* \alpha_2 (E_i)E_i = 0$ for each $i$. Let $Q_i = Q E_i$.

Then we have $Q_i \leq Q$ and $Q_i \leq E_i$ so that $Q_i = \alpha_1^* \alpha_2 (Q_i)Q_i \leq \alpha_1^* \alpha_2 (E_i)E_i = 0$ for each $i$. As $Q_1 = QE_1$ and $Q \neq 0$, so $\sum E_i \neq 1$. Now by (i), $F$ is free. Thus $\alpha_1^* \alpha_2 = e$, i.e. $\alpha_1 = \alpha_2$. This completes the proof.
7. The calculation of $G'$

In this section we shall treat a number of special cases of the situation considered in §2, and compute the commutant $G'$ of certain group $G$ of automorphisms. [cf. Propositions 7.1, 7.2, 7.3 and 7.4 below.] These results will be used in the next two sections. These cases are concerned with the multiplication algebras on some $L_2$ spaces. The results contained in this section can be obtained by realizing the concrete algebras considered here as tensor products of suitable abelian $W^*$-algebras with respect to suitable normal states, by realizing the group of automorphisms as product groups, and then by appealing to general results for tensor product algebras established in the appendix, but we prefer to prove them here by purely measure theoretical arguments.

7.1 Let $T = [0, 1)$ with addition mod 1 (i.e. the circle group), or $\mathbb{R}$ with usual addition (and in both cases, with usual topology). Let $m$ be $L_\infty(T)$ acting by multiplication on $L_2(T)$ [cf. 15, Chap. III], and let $D$ be a dense subgroup of $T$. For each $x \in T$, denote by $\tau_x$ the automorphism of $m$ given by:

$$(\tau_x f)(y) = f(y - x),$$

for any $f \in L_\infty(T)$, and any $y \in T$.

Proposition 7.1 \(\{ \tau_d : d \in D \}' = \{ \tau_x : x \in T \}\).
Proof. Suppose \( \alpha \in \{ \tau_d : d \in D \} \)'. Since \( D \) is dense in \( T \), \( \{ \tau_d : d \in D \} \) is ergodic on \( m \) [cf. 14, Lemma 13.2.1]. Thus \( \alpha \) preserves the trace induced by the Lebesgue measure \( \lambda \) on \( m \). Now let \( A = [0, \varepsilon] \) where \( 0 < \varepsilon < 1 \), and let \( F \) be a measurable set such that \( \chi_F = \alpha(\chi_A) \). This \( F \) is determined to within a set of measure zero. As \( \lambda(A) = \varepsilon \), so \( \lambda(F) = \varepsilon \). We shall see that for some \( x \in T \), \( F = [x, x + \varepsilon] \) modulo sets of measure zero.

In the case \( T = [0, 1] \), \( F \) is obviously bounded. In the case \( T = \mathbb{R} \), \( F \) is essentially bounded. For if otherwise there will be integers \( m, n \) with \( |m - n| > 2 \) such that

\[
B \equiv [m, m + 1] \cap F \neq 0
\]

and

\[
C \equiv [n, n + 1] \cap F \neq 0 .
\]

But then for some \( d \in D \) with \( |d| > 1 \),

\[
\tau_d(C) \cap B \neq 0 .
\]

[Here, and in what follows, we identify \( \tau_d \) with the translation by \( d \). Also we shall use \( \alpha \) to denote the automorphism \( \alpha \) induces on the ring of measurable subsets of \( T \), and identify sets which differ by a set of measure zero.] Now we have \( \alpha^{-1}(B), \alpha^{-1}(C) \subseteq [0, 1] \) so that for any \( d \in D \) with \( |d| > 1 \),

\[
\tau_d(\alpha^{-1}(C)) \cap \alpha^{-1}(B) = 0 ,
\]
26.

and so \( \tau_d(C) \cap B = 0 \), a contradiction. Thus in any case \( F \) is essentially bounded, and therefore we can choose and fix a bounded \( F \).

We now associate a point \( x_k \in T \) for any bounded, non-zero measurable subset \( K \) of \( T \) as follows. Consider the set \( I \) of points \( y \) of \( \mathbb{R} \) such that \((-\infty, y] \cap K \) has measure zero. \( I \) is clearly non-empty and bounded above, so we may denote by \( x_k \) its supremum. Note that \((-\infty, x_k] \cap K \) has zero measure as \( x_k \) is the limit of a sequence of points in \((-\infty, x_k) \cap I \). Observe also that we always have \( x_k \in T \), and that for any \( \delta > 0 \), \([x_k, x_k + \delta] \cap K \neq 0 \).

To prove the last statement of the first paragraph, let us consider the case \( T = \mathbb{R} \) first. Let \( x \) denote \( x_F \) for the \( F \) mentioned in the first and second paragraph. Let \( F_0 = F \cap (x + \varepsilon + 2, \infty) \), and let \( F_n = F \cap [x + \varepsilon + \frac{1}{n}, x + \varepsilon + 2] \) for each positive integer \( n \). If for some \( n \geq 0 \), \( F_n \) is of positive measure, then there are \( \delta \) with \( 0 < \delta < \min(\varepsilon, \frac{1}{n+1}) \), and \( d \in D \) with \( d \geq \varepsilon \) such that

\[
\tau_d(F \cap [x, x + \delta]) \cap F_n \neq 0 .
\]

But \( \alpha^{-1}(F \cap [x, x + \delta]) \) and \( \alpha^{-1}(F_n) \) are both subsets of \( A \) so that for any \( d \in D \) with \( d \geq \varepsilon \) we have

\[
\tau_d[\alpha^{-1}(F \cap [x, x + \delta])] \cap \alpha^{-1}(F_n) = 0 ,
\]

and so,

\[
\tau_d(F \cap [x, x + \delta]) \cap F_n = 0 ,
\]
a contradiction. Thus each $F_n$ is of measure zero and $F = [x, x + \epsilon]$ modulo sets of zero measure; i.e. $\alpha([0, \epsilon]) = \chi_{[x, x+\epsilon]}$.

Now we turn to the case $T = [0, 1)$. For the sake of convenience, let us take $\epsilon < 1/3$. Let $F$ be as in the first paragraph, and $y$ be $x_F$. If $B = F \setminus [0, y + \epsilon]$ is of zero measure, then we have $\alpha([0, \epsilon]) = \chi_{[y, y+\epsilon]}$. So we may assume $B$ is of positive measure. We shall see $y = 0$ in this case. If $x_B = y + \epsilon$, then by an argument similar to that employed in the preceding paragraph we see that there are measurable subsets $A_1, A_2$ of $A$, and $d \in (\epsilon, 1-\epsilon) \cap D$ such that $\tau_d(A_1) \cap A_2 \neq 0$, which is absurd. So $x_B > y + \epsilon$. Now suppose $y > 0$ and we shall draw some contradiction from this assumption. Let

$$C_1 = \alpha^{-1}([y + \epsilon, x_B]),$$

and

$$C_2 = \alpha^{-1}([0, y]).$$

Then for any positive $\delta$ we have

$$[1 - \delta, 1] \cap C_1 \neq 0 \quad \text{and} \quad [1 - \delta, 1] \cap C_2 \neq 0.$$ 

For if otherwise, we would have, say, $[1 - \delta_0, 1] \cap C_1 = 0$ for some positive $\delta_0$, i.e. $C_1 \subset [\epsilon, 1 - \delta_0]$, and therefore $\tau_d(C_1) \cap [0, \epsilon] = 0$, i.e. $\tau_d([y + \epsilon, x_B]) \cap F = 0$ for all $d \in D \cap (0, \delta_0)$, which is absurd by the definition of $x_B$. A similar argument also applies to the assumption $[1 - \delta_0, 1] \cap C_2 = 0$. Thus we have
for any positive $\delta$. Now we can choose a $\delta_0 > 0$ such that

$$\tau_d([1 - \delta_0, 1]) \cap [1 - \delta_0, 1] = 0,$$

for all $d \in [\varepsilon, x_B] \cap B$. However there must be a $d \in [\varepsilon, x_B] \cap D$ such that

$$\tau_d(a([1 - \delta_0, 1] \cap C_2)) \cap a([1 - \delta_0, 1] \cap C_1) \neq 0,$$

i.e.

$$\tau_d([1 - \delta_0, 1] \cap C_2) \cap ([1 - \delta_0, 1] \cap C_1) \neq 0,$$

a contradiction. Hence $y = 0$.

Now we shall show that $x_B \geq 1 - \varepsilon$. We know already that $x_B > y + \varepsilon = \varepsilon$. If $x_B < 1 - \varepsilon$, then by an argument similar to that employed in the case $T = \mathbb{R}$ we see that there would be a $d \in (\varepsilon, 1 - \varepsilon) \cap D$, and measurable subsets $A', A''$ of $[0, \varepsilon)$ such that

$$\tau_d(A') \cap A'' \neq 0,$$

which is false. So $x_B \geq 1 - \varepsilon$.

Finally we shall show that $F = [0, \varepsilon + x_B - 1] \cup [x_B, 1]$ (mod. measure zero set). If $F \setminus ([0, \varepsilon + x_B - 1] \cup [x_B, 1])$ is of measure zero, then we are done. Otherwise by the definition of $x_B$, one of the sets

$$B_n = F \cap [\varepsilon + x_B - 1 + \frac{1}{n}, \varepsilon], \ n > 1/(1 - x_B),$$

must be of positive measure.
As \( \varepsilon < x_B - x_{B_n} < 1 - \varepsilon \) (recall \( \varepsilon < 1/3 \)), an argument similar to that of above would lead to a contradiction. Thus \( F = [0, \varepsilon + x_B - 1] \cup [x_B, 1] \) to within a set of zero measure. Let \( x = x_B \) then we have

\[
\alpha(\chi_{[0,\varepsilon]}) = \tau_x(\chi_{[0,\varepsilon]}).
\]

So far we have established that in both cases of \( T \), for a given \( \varepsilon \in (0, \frac{1}{3}) \), there is an \( x \in T \) such that

\[
\alpha(\chi_{[0,\varepsilon]}) = \tau_x(\chi_{[0,\varepsilon]}).
\]

As \( \alpha_t \tau_d = \tau_d \alpha \) for any \( d \in D \), we have

\[
\alpha(\chi_{[d,d+\varepsilon]}) = \tau_x(\chi_{[d,d+\varepsilon]})
\]

for any \( d \in D \). As \( \alpha \) and \( \tau_x \) are automorphisms it follows that

\[
\alpha(\chi_{[0,d]}) = \tau_x(\chi_{[0,d]})
\]

for any \( d \in D \cap (0, \varepsilon) \). As the ordering in \( \mathbb{R} \) is archmedian, \( D \) is dense in \( T \) and \( \alpha \) and \( \tau_x \) preserve countable sums of characteristic functions, it follows that

\[
\alpha(\chi_{[0,t]}) = \tau_x(\chi_{[0,t]}),
\]

for any \( t \in T \). Thus \( \alpha \) and \( \tau_x \) agree on every characteristic function in \( L_0(T) \). It follows now from the normality and linearity of \( \alpha \) and \( \tau_x \) that
\[ \alpha(M) = \tau_x(M) \]

for any \( M \in m \), i.e. \( \alpha = \tau_x \).

So we have \( \{ \tau_d : d \in D \}' \subseteq \{ \tau_x : x \in T \} \). The reverse inclusion is clear, so the proof is complete.

7.2 Let \( m \) be \( L_2(\mathbb{R}) \) acting by multiplication on \( L_2(\mathbb{R}) \). For each non-zero real number \( r \), define \( s_r \) be the automorphism of \( m \) given by:

\[
(s_r f)(x) = f(r^{-1} x), \quad f \in L_2(\mathbb{R}), \quad x \in \mathbb{R}.
\]

**Proposition 7.2** \( \{ s_r : r \in \mathbb{Q}, r \neq 0 \}' = \{ s_r : r \in \mathbb{R}, r \neq 0 \} \).

**Proof.** Suppose that \( \alpha \in \{ s_r : r \in \mathbb{Q}, r \neq 0 \}' \). We first prove that either \( \alpha(\mathcal{B}_+^r) \subseteq \mathcal{B}_+ \) or \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_- \), where \( \mathcal{B}_+ \) denotes the set of all measurable subsets of the positive reals \( \mathbb{R}_+ \), and \( \mathcal{B}_- \) denotes the set of all measurable subsets of the negative reals \( \mathbb{R}_- \). Suppose \( \alpha(\mathcal{B}_+) \not\subseteq \mathcal{B}_+ \).

Then there is a non-zero \( A_1 \in \mathcal{B}_+ \) such that \( \alpha(A_1) \cap \mathbb{R}_- \neq 0 \). By replacing \( A_1 \) by a subset if necessary we can assume that \( \alpha(A_1) \subseteq \mathbb{R}_- \). Let \( A_2 \) be an arbitrary non-zero member of \( \mathcal{B}_+ \). Since \( \{ s_r : r \in \mathbb{Q}_+, r \neq 0 \} \) is ergodic on \( \mathbb{R}_+ \) (with the usual measure and measurable subsets) by an obvious application of Lemma 13.2.1 of [14], there must be a rational \( r > 0 \) such that

\[
s_r(A_1) \cap A_2 \neq 0.
\]
Thus
\[ s_r[\alpha(A_1)] \cap \alpha(A_2) \neq 0. \]

Therefore \( \alpha(A_2) \cap \mathbb{R}_- \neq 0 \) for any non-zero \( A_2 \in \mathcal{B}_+ \). It follows then \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_- \). Thus we have either \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_+ \) or \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_- \).

We consider the case \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_+ \) first. In this case we have:
\[ \alpha'(\mathcal{B}_+) \subseteq \mathcal{B}_+, \quad \alpha(L_\infty(\mathbb{R}_+)) = L_\infty(\mathbb{R}_+), \quad \text{and} \quad \alpha|_{L_\infty(\mathbb{R}_+)} \text{ is an automorphism of} \ L_\infty(\mathbb{R}_+) \]
(Here an \( f \in L_\infty(\mathbb{R}_+) \) is identified with the \( g \in L_\infty(\mathbb{R}) \) where \( g = f \) on \( \mathbb{R}_+ \) and \( g = 0 \) elsewhere.). Now define \( \phi : L_\infty(\mathbb{R}_+) \to L_\infty(\mathbb{R}) \)
by:
\[ (\phi g)(x) = g(e^{x}), \quad x \in \mathbb{R}, \quad g \in L_\infty(\mathbb{R}_+). \]
This is well-defined because if \( A \) is a zero set in \( \mathbb{R}_+ \) then \( \log(A) \) is a zero set in \( \mathbb{R} \). In fact \( \phi \) is an isomorphism and \( \phi^{-1} : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}_+) \)
is given by:
\[ (\phi^{-1} f)(x) = f(\log x), \quad x \in \mathbb{R}_+, \quad f \in L_\infty(\mathbb{R}). \]
So the map \( \tilde{\alpha} \equiv \phi \alpha \phi^{-1} : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}) \) is an automorphism. It is easily checked that \( \tilde{s}_r \equiv \phi s_r \phi^{-1} = \tau \log r \), and \( \tilde{\alpha} \tilde{s}_r = \tilde{s}_r \tilde{\alpha} \) for any \( r \in \mathcal{Q}_+ \setminus \{0\} \).

As \( \{ \log r : r \in \mathcal{Q}_+, \ r \neq 0 \} \) is dense in \( \mathbb{R} \), it follows from Proposition 7.1 that
\[ \tilde{\alpha} = \tau \log t \]
for some \( t \in \mathbb{R}_+ \). This means \( \alpha|_{L_\infty(\mathbb{R}_+)} = \phi^{-1} \tau \log t \phi = s_t|_{L_\infty(\mathbb{R}_+)} \).
Therefore
\[ \alpha = s_t. \]

Next we consider the remaining case \( \alpha(\mathcal{B}_+) \subseteq \mathcal{B}_- \). In this case
we have \( \alpha(L_\infty(R_+)) = L_\infty(R_-) \), and \( \alpha|L_\infty(R_+) : L_\infty(R_+) \rightarrow L_\infty(R_-) \) is an isomorphism. Now define \( \psi : L_\infty(R_-) \rightarrow L_\infty(R) \) by:

\[
(\psi g)(x) = g(-e^x), \quad x \in R, \quad g \in L_\infty(R_-)
\]

Again \( \psi \) is well-defined, and it is an isomorphism. Put \( \hat{\alpha} = \psi \alpha^{-1} \), where \( \phi^{-1} \) is given in the preceding paragraph. Then \( \hat{\alpha} \) is an automorphism of \( L_\infty(R) \). Also it is easily checked that \( \hat{s}_r \equiv \psi s_{-r} \phi^{-1} = \tau \log r \) for any \( r \in \mathbb{Q}_+ \setminus \{0\} \). Thus for any \( r \in \mathbb{Q}_+ \setminus \{0\} \),

\[
\hat{\alpha} \tau \log r = \hat{\alpha} s_{-r}
\]

\[
= \psi \alpha^{-1} \psi s_{-r} \phi^{-1}
\]

\[
= \psi \alpha s_r \phi^{-1}
\]

\[
= \psi s_r \alpha \phi^{-1}
\]

\[
= \psi s_{-r} \phi^{-1} \psi \phi^{-1}
\]

\[
= \tau \log r \hat{\alpha}
\]

So as in the preceding paragraph we conclude that

\[
\hat{\alpha} = \tau \log t
\]
for some \( t \in \mathbb{R}_+ \). Let \( h \) be an arbitrary element of \( L^Q(L^+_\mathbb{R}) \). Then

\[
\alpha(h)(x) = (\psi^{-1} \tau_{\log t} \phi)(h)(x)
\]

\[
= (\tau_{\log t} \phi)(h)(\log(-x))
\]

\[
= \phi(h)(\log(-x))
\]

\[
= \log(-\frac{x}{t})
\]

\[
= h(-\frac{x}{t})
\]

\[
= s_{-t}(h)(x)
\]

Thus \( \alpha|_{L^0(\mathbb{R}_+)} = s_{-t}|_{L^0(\mathbb{R}_+)} \),

and \( \alpha = s_{-t} \).

We see therefore \( \{ s_r : r \in Q, r \neq 0 \} \subset \{ s_r : r \in \mathbb{R}, r \neq 0 \} \).

The reverse inclusion is clear, and our proof is complete.

7.3 Let \((X_i, S_i, \mu_i)\) be a \( \sigma \)-finite measure space, \( m_i \) be \( L_\omega(X_i, S_i, \mu_i) \) acting by multiplication on \( L_2(X_i, S_i, \mu_i) \), \( i = 1,2 \). Let \( G_i \) be a group of automorphisms of \( m_i \), each element \( g_i \) of which is induced by a point transformation, denoted again by \( g_i \), of \( X_i \). Suppose that each \( G_i \) is countable and ergodic on \( m_i \). Let \( m \) be \( L_\omega(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2) \) acting by multiplication on \( L_2(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2) \). For each pair \( g_1 \in G_1, g_2 \in G_2 \), denote by \( \alpha(g_1, g_2) \) the automorphism of \( m \) given by :
\((a_{(g_1, g_2)} f)(x,y) = f(g_1(x), g_2(y)), (x,y) \in X_1 \times X_2, f \in I_\infty(X_1 \times X_2).\)

Let \(G'_1\) denote the group of automorphisms of \(m_1\) which commute with each element of \(G_1\), and suppose that each \(g'_1 \in G'_1\) is induced by a point transformation, again denoted by \(g'_1\), of \(X_1\). Use the notation \(a_{(g'_1, g'_2)}\) for \((g'_1, g'_2) \in G'_1 \times G'_2\), in exactly the same way as \(a_{(g_1, g_2)}\).

**Proposition 7.3**

\[
\{ a_{(g_1, g_2)} : (g_1, g_2) \in G_1 \times G_2 \}' = \{ a_{(g'_1, g'_2)} : (g'_1, g'_2) \in G'_1 \times G'_2 \}.
\]

**Proof.** Suppose \(a \in \{ a_{(g_1, g_2)} : (g_1, g_2) \in G_1 \times G_2 \}'\). For each \(f \in I_\infty(X_1)\), let \(\phi f \in I_\infty(X_1 \times X_2)\) be defined by:

\[\phi f(x,y) = f(x), (x,y) \in X_1 \times X_2.\]

Then \(\phi : I_\infty(X_1) \rightarrow I_\infty(X_1 \times X_2)\) mapping \(f\) to \(\phi f\) is a monomorphism. Let \(S = \phi(I_\infty(X_1))\). Then it is not hard to see that

\[S = \{ g \in I_\infty(X_1 \times X_2) : \text{for almost all } x \in X_1, g(x,y) \text{ is equivalent to a constant function of } y \in X_2 \}.
\]

Now we show \(a\) maps \(S\) onto \(S\). It suffices to show that \(a(S) = S\).

(For then we have also \(a^{-1}(S) = S\), i.e. \(S \subseteq a(S)\).) Let \(g \in S\), and consider \(a(g)\). As \(g \in S\), \(a_{(0, g_2)} g = g\) for each \(g_2 \in G_2\) (where 0
denotes the identity element of \( G_1 \), and so
\[
\alpha(0, g_2)\alpha(g) = \alpha(g)
\]
for each \( g_2 \in G_2 \). We shall see that this last property implies \( \alpha(g) \in S \).

Now for each \( g_2 \in G_2 \), there is a measure zero set \( E_{g_2} \) of \( X_1 \) such that for each \( x \in X_1 \setminus E_{g_2} \), the set
\[
F_{x, g_2} = \{ y \in X_2 : \alpha(0, g_2)\alpha(g)(x, y) \neq \alpha(g)(x, y) \}
\]
is of zero measure. Let \( E = \bigcup_{g_2 \in G_2} E_{g_2} \). Then, since \( G_2 \) is countable, \( E \) is of zero measure. Furthermore for each \( x \in X_1 \setminus E \), the set
\[
F_x = \bigcup_{g_2 \in G_2} F_{x, g_2}
\]
is of zero measure, and for each \( y \in X_2 \setminus F_x \ (x \in X_1 \setminus E) \) we have:
\[
(*) \quad \alpha(0, g_2)\alpha(g)(x, y) = \alpha(g)(x, y),
\]
for all \( g_2 \in G_2 \). This last property \((*)\) implies that for each fixed \( x \in X_1 \setminus E \), \( \alpha(g)(x, y) \) is equivalent to a constant function of \( y \in X_2 \).

For if otherwise we claim there would be subsets \( A^X_1, A^X_2 \) of \( X_2 \setminus F_x \), each of positive measure, such that
\[
(**) \quad \alpha(g)(x, y_1) \neq \alpha(g)(x, y_2)
\]
for \( y_1 \in A^X_1 \) and \( y_2 \in A^X_2 \). To prove this last assertion \((**)\), let
x \in X_1 \setminus E be such that the real (or imaginary) part \( a(g)_1(x, \cdot) \) of \\
a(g)(x, \cdot) is not equivalent to a constant function in \( X_2 \), and let \( h_x \) \\
be the function on \( X_2 \) defined by:

\[
h_x(y) = \begin{cases} 
  a(g)_1(x, y), & y \in X_2 \cap F_x \\
  0, & y \in F_x 
\end{cases}
\]

For each \( n \in \mathbb{Z} \), let \( B_n = [n, n + 1] \). If for a pair of distinct \( m, n \in \mathbb{Z} \), \\
both \( h_x^{-1}(B_m) \) and \( h_x^{-1}(B_n) \) are of positive measure, then we are done. \\
Otherwise there is a \( n_0 \in \mathbb{Z} \) such that \( h_x^{-1}(B_{n_0}) \) is of positive measure, \\
and that \( h_x^{-1}(B_n) \) is of zero measure for all \( n \in \mathbb{Z} \setminus \{n_0\} \). Now we can \\
apply a similar argument to \( h_x \) and \( B_{n_0} \). Such a procedure must come to \\
an end after finitely many steps, for otherwise, because of the compactness \\
of \( B_{n_0} \), \( h_x \) would be equivalent to a constant function. So (***) is \\
established. Now by the ergodicity of \( G_2 \), there is a \( h \in G_2 \) such that \\

\[
A^x = h_2(A_1^x)A_2^x \neq 0.
\]

Thus for \( y \in A^x \) we have \( y \in A_2^x \), \( h_2(y) \in A_1^x \) and \\

\[
a(0, h_2)a(g)(x, y) = a(g)(x, h_2(y)) \\
\neq a(g)(x, y),
\]

which contradicts (**). Therefore we see that for each fixed \( x \in X_1 \setminus E, \)

\( \alpha(g)(x, y) \) is equivalent to a constant function of \( y \in X_2 \). This completes 
the proof that \( \alpha(S) = S \).
Denote now \( \tilde{\alpha} = \phi^{-1} \alpha : L_\infty(X_1) \to L_\infty(X_1) \). Then \( \tilde{\alpha} \) is an automorphism of \( L_\infty(X_1) \). Now for each \( g_1 \in G_1 \) and each \( g \in S \) we have

\[
\phi g_1 = \alpha(g_1,0)^\phi, \quad \text{(here 0 denotes the identity of } G_2),
\]

and

\[
\phi^{-1} \alpha(g_1,0) g = g_1 \phi^{-1} g.
\]

So we have \( \tilde{\alpha} g_1 = g_1 \tilde{\alpha} \) for each \( g_1 \in G_1 \). Thus

\[
\tilde{\alpha} = g_1
\]

for some \( g_1' \in G_1' \), i.e.

\[
\alpha|S = \alpha(g_1',0)|S.
\]

Dually by setting

\[
T = \{ g \in L_\infty(X_1 \times X_2) : \text{for almost all } y \in X_2, \ g(x,y) \text{ is equivalent to a constant function of } x \in X_1 \},
\]

we have

\[
\alpha|T = \alpha(0,g_2')|T
\]

for some \( g_2' \in G_2 \). Thus for any measurable rectangle \( V \) of \( X_1 \times X_2 \) we have

\[
\alpha(x_{V'}) = \alpha(g_1',g_2')(x_{V'}).
\]

Since \( \alpha, \alpha(g_1',g_2') \) are automorphisms of \( m = L_\infty(X_1 \times X_2) \), by similar
arguments as those employed in §7.1, we conclude
\[ a = a(g_1', g_2') \]
for some \((g_1', g_2') \in G_1' \times G_2'\).

Therefore we see that \( \{ a(g_1', g_2') : (g_1', g_2') \in G_1' \times G_2' \} ' \)
\( \subseteq \{ a(g_1', g_2') : (g_1', g_2') \in G_1' \times G_2' \} \). The reverse inclusion is clear, so
the proof is complete.

7.4 Let \( Z_2 \) be the additive group of two elements 0 and 1, \( S_0 \) the
ring of all subsets of \( Z_2 \), \( \mu_0 \) the measure on \((Z_2, S_0)\) assigning 1 to
1 and \( 1 - q \) to 0 where \( q \in [\frac{1}{2}, 1] \). For each \( n \in \mathbb{Z} \), let \( X_n = Z_2 \),
\( S_n = S_0 \), and \( \mu_n = \mu_0 \). Let \( X = \prod_{n \in \mathbb{Z}} X_n \), \( S' = \prod_{n \in \mathbb{Z}} S_n \), and let \((X, S, \mu_q)\)
be the completion of \( \prod_{n \in \mathbb{Z}} \mu_n \) on \((X, S')\). Let \( \Delta = \prod_{n \in \mathbb{Z}} X_n \). Let \( m \) be
\( L_\mu(X, S, \mu_q) \) acting by multiplication on \( L_2(X, S, \mu_q) \). For each \( \delta \in \Delta \)
the translation in \( X \) by \( \delta \) induces an automorphism \( a_\delta \) of \( m \) (cf. [18]).
For simplicity we write \( a_n \) instead of \( a_\delta_n \), where \( \delta_n \in \Delta(n \in \mathbb{Z}) \) is
such that \( \delta_n(m) = 0 \) if \( m \neq n, = 1 \) if \( m = n \).

Proposition 7.4

(i) When \( q > \frac{1}{2} \), \( \{ a_n : n \in \mathbb{Z} \} ' = \{ a_\delta : \delta \in \Delta \} \).

(ii) When \( q = \frac{1}{2} \), the translation in \( X \) by any \( x \in X \) induces an
automorphism \( a_x \) of \( m \), and \( \{ a_n : n \in \mathbb{Z} \} ' = \{ a_x : x \in X \} \).
Proof. Suppose $\alpha \in \{ \alpha_n : n \in \mathbb{Z} \}$. Let $P_n$ denote the projection of $X$ onto $X_n$, and let $A_n = P_n^{-1}(1)$, $B_n = P_n^{-1}(0)$, $C_n = \alpha(A_n)$ and $D_n = \alpha(B_n)$. [Here $C_n$, $D_n$ are chosen, and fixed, among a class of sets, the difference of any two members of which is of zero measure. Hereafter equalities involving sets of $X$ means equalities modulo sets of zero measure.] Then as $A_n \cap B_n = 0$, $A_n \cup B_n = X$, $\alpha_n(A_n) = B_n$, $\alpha_m(B_n) = B_n$, $\alpha_m(A_n) = A_n$ for each integer $m \neq n$, and $\alpha_k \alpha = \alpha \alpha_k$, we have $C_n \cap D_n = 0$, $C_n \cup D_n = X$, $\alpha_n(C_n) = D_n$, and for each integer $m \neq n$, $\alpha_m(C_n) = C_n$ and $\alpha_m(D_n) = D_n$. We shall see these properties imply that either $C_n = A_n$ or $C_n = B_n$ (and therefore $D_n = B_n$ or $D_n = A_n$ resp.). For this purpose assume that there is a point $x \in C_n$ with $P_n(x) = 1$, and we shall show that under this additional condition, $C_n = A_n$. It suffices to show $A_n \subseteq C_n$ (modulo sets of zero measure), as this implies that $B_n \subseteq D_n$ and finally, together with other properties of $A_n$, $B_n$, $C_n$, $D_n$, that $A_n = C_n$. Suppose $A_n \not\subseteq C_n$ (modulo sets of zero measure). Then there is a set $E_n \subseteq A_n \setminus C_n$ of positive measure. If $C_n \subseteq A_n$ then $D_n \subseteq B_n$ so that $E_n \subseteq A_n \setminus D_n$, which is impossible as $C_n \cup D_n = X$. So there is a point $y \in C_n$ with $P_n(y) = 0$. Now as in addition we have $x \in C_n$ with $P_n(x) = 1$, and as $\alpha_m(C_n) = C_n$ for all integer $m \neq n$, it follows that

$$\{ z \in X : \text{for some } z' \in C_n, P_k(z) = P_k(z') \text{ for all integer } k > n \} \subseteq C_n.$$  

Since $D_n = \alpha_n(C_n)$ it follows that $D_n \subseteq C_n$. Thus again we would have $E_n \subseteq A_n \setminus (C_n \cup D_n)$, which is absurd. So we have proved that if there is a point $x \in C_n$ with $P_n(x) = 1$, then $A_n \subseteq C_n$ and so
Now similarly we can prove that if there is a point \( x \in C_n \) with \( P_n(x) = 0 \), then

\[ C_n = B_n. \]

Thus we have:

\[
\begin{cases}
\alpha(A_n) = A_n & \text{or} \ (2) \ \\
\alpha(B_n) = B_n & \text{or} \ (2) \ \\
\alpha(B_n) = A_n & \text{or} \ (2)
\end{cases}
\]

Now define an \( x \in X \) by

\[ x(n) = \begin{cases} 0 & \text{for such } n \text{ that } (1) \text{ is the case,} \\ 1 & \text{for such } n \text{ that } (2) \text{ is the case.} \end{cases} \]

Then we have

\[ \alpha(A_n) = A_n + x \]

for each \( n \in \mathbb{Z} \). Thus we have

\[ \alpha(S) = S + x \]

for any \( S \in S \). [In particular the translation by \( x \) maps a measure zero set to a set of zero measure.]

Consider now the case where \( q = \frac{1}{2} \). In this case, the Kakutani theorem [11] implies that the measure by \( \mu_q(z \in X) \) defined by:
is equivalent to \( \mu_q \). Thus the translation in \( X \) by an arbitrary \( z \in X \) induces an automorphism \( \alpha_z \) of \( m \). Combining the result of the preceding paragraph we have

\[
\alpha = \alpha_x .
\]

Now we turn to the case where \( q > \frac{1}{2} \). In this case, the Kakutani theorem [11] implies that the measure \( \mu_z \), defined above, is equivalent to \( \mu_q \) only if \( z \in \Delta \); in other words, the translation in \( X \) by \( z \in X \) maps a measure zero set to a measure zero set only if \( z \in \Delta \). Thus, in view of the result in a previous paragraph, we have \( x \in \Delta \) and \( \alpha = \alpha_x \).

We see therefore \( \{ \alpha_n : n \in \mathbb{Z} \} \subset \{ \alpha_\delta : \delta \in \Delta \} \) in the first case (i.e. when \( q > \frac{1}{2} \)), and \( \{ \alpha_n : n \in \mathbb{Z} \} \subset \{ \alpha_x : x \in \Delta \} \) in the second case (i.e. when \( q = \frac{1}{2} \)). As we have seen that in the second case, the translation in \( X \) by an arbitrary \( x \in X \) induces an automorphism \( \alpha_x \) of \( m \), the reverse inclusions are clear. Our proof is thus complete.

We now define a measure \( \lambda_q \) on \([0, 1]\), and show that there is an isometric isomorphism from \( L^2[0, 1] \) onto \( L^2(X) \), which maps \( L^\infty([0,1], \lambda_q) \) onto \( L^\infty(X, \mu_q) \). To this end define \( \psi : X \to [0, 1] \) by:

\[
\psi(x) = \sum_{n \in \mathbb{Z}} \frac{x(n)}{2^n}, \quad x \in X .
\]

It is easy to check that \( \psi \) is measurable. Define \( \lambda_q \) on \([0, 1]\) by:
\[
\lambda_q(B) = \mu_q(\psi^{-1}(B)), \text{ B Lebesque measurable subset of } [0, 1].
\]

Notice that \(\lambda_{\frac{1}{2}} = \lambda\), the Lebesque measure on \([0, 1]\) because they agree on sets of the form \([0, \sum_{i=1}^{k} \frac{1}{2^n}]\). In general, an isometric isomorphism \(\phi\) is defined on \(L_2([0, 1], \lambda_q)\) to \(L_2(X, \mu_q)\) by:

\[
(\phi f)(x) = f(\psi(x)), \quad f \in L_2[0, 1], x \in X.
\]

Since the set of all dyadic rationals in \([0, 1]\) is a \(\lambda_q\)-zero set, \(\phi\) is indeed surjective. Clearly \(\phi\) maps \(L_\infty([0, 1], \lambda_q)\) onto \(L_\infty(X, \mu_q)\).

Now we define, for each \(n \in \mathbb{Z}\), an automorphism \(\beta_n\) of \(L_\infty([0, 1], \lambda_q)\) by:

\[
\beta_n = \phi^{-1} a_n \phi,
\]

where \(a_n\) is defined previously. These \(\beta_n\)'s are induced by the point transformation \(\psi a_n \psi^{-1}\) on \([0, 1]\). The ambiguity of \(\psi a_n \psi^{-1}\) at dyadic rationals will be settled by setting:

\[
(\psi a_n \psi^{-1})(\frac{k}{2^n}) = \frac{k}{2^n} \quad \text{for } k = 0, 1, \ldots, 2^n.
\]

With this agreement we have

\[
(\psi a_n \psi^{-1})(y) = \begin{cases} 
y + \frac{1}{2^n}, & \text{if } y \in \left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) 
y - \frac{1}{2^n}, & \text{if } y \in \left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right), i=0,1,\ldots,2^n-1 
y, & \text{otherwise}
\end{cases}
\]
We shall also refer to these point transformations by $\beta_n$.

The automorphisms $\phi^{-1} \alpha_x \phi$ ($x \in X$) are also induced by point transformations, even though their expressions are not as neat as the $\beta_n$'s. However for convenience we shall refer to them by the following method. For each dyadic irrational number $s$ let $\alpha_s = \psi \alpha_x(s)^{-1}$ where $x(s)$ is the unique element in $X$ such that $\psi(x(s)) = s$, and where the ambiguity of $\alpha_s$ at dyadic rationals is settled as before. For a dyadic rational $t$, let $x(t)$ be the unique element in $X \setminus \Delta$ such that $\psi(x(t)) = t$, and let $\alpha_t = \psi \alpha_x(t)^{-1}$, with the ambiguity of $\alpha_t$ settled as before. Then the automorphism $\phi^{-1} \alpha_x \phi$ is either a finite product of the $\beta_n$'s, or is induced by the point transformation $\alpha_r$ for some $r \in [0, 1]$. With these notations we have

**Corollary 7.5.** Let $m$ be $L_\infty([0, 1], \lambda_q)$ acting by multiplication on $L_2([0, 1], \lambda_q)$. Then an automorphism $\alpha$ of $m$ commutes with all $\beta_n$'s ($n \in \mathbb{Z}$) if and only if:

(i) in the case $q = \frac{1}{2}$, i.e. $\lambda_q = \lambda$, $\alpha$ is either a finite product of the $\beta_n$'s, or is induced by an $\alpha_r$ for some $r \in [0, 1]$;

(ii) in the case $q > \frac{1}{2}$, $\alpha$ is a finite product of the $\beta_n$'s.
8. Operators distinguishable by $G' \cap [G]$ and $G'$

We summarize here the results obtained in §§5, 6, 7, in a form most suitable for applications to operator theory.

**Theorem 8.1** Suppose $A_1$, $A_2$ are two operators on a Hilbert space $H$ such that $R(A_1) = R(A_2) = \mathcal{A}[m, K, G, g \mapsto U_g]$ for some ergodic and abelian C-system $[m, K, G, g \mapsto U_g]$, that $(R(\text{Re } A_1), R(\text{Re } A_2))$ fits in $[a, m \otimes 1]$, that $\text{Im } A_1 = \text{Im } A_2 = T$, and that $R(T) = R(U_g \otimes V_g : g \in G)$ [cf. Definition 2.1, 2.2 for notations]. Then $A_1$ and $A_2$ are unitarily equivalent if and only if they are (algebraically) equivalent, and that is the case if and only if there is an $\alpha \in \overline{G'}$ such that

$$\alpha(\text{Re } A_1) = \text{Re } A_2,$$

where $\overline{G'} = \{ \alpha \in \mathcal{A}(m \otimes 1) : \text{for some } s \in G', \alpha(M \otimes 1) = s(M) \otimes 1 \text{ for all } M \in m \}$. Moreover $A_1$ and $A_2$ are inner equivalent if and only if for some $g \in \overline{G}$:

$$g(\text{Re } A_1) = \text{Re } A_2.$$

**Proof.** The present theorem follows directly from Lemma 5.1 (ii), Theorem 5.2 and Definition 2.2.
Theorem 8.2. Suppose that

(i) \( m \) is \( L^\omega[0,1] \) acting on \( K = L_2[0,1] \), \( D \) a dense subgroup of \( [0,1] \) under the addition mod 1, \( G \) the group of all automorphisms on \( m \) induced by the translation in \( [0,1] \) by \( d \in D \), and \( g \in G \mapsto U_g \) the usual (cf. [15]) unitary representation of \( G \) on \( K \);

or (ii) \( m \) is \( L^\omega(\mathbb{R}) \) acting on \( K = L_2(\mathbb{R}) \), \( D \) a dense subgroup of \( \mathbb{R} \), \( G \) the group of all automorphisms on \( m \) induced by the translation in \( \mathbb{R} \) by \( d \in D \), and \( g \in G \mapsto U_g \) the usual unitary representation of \( G \) on \( K \);

or (iii) \( m \) is \( L^\omega(\mathbb{R}) \) acting on \( K = L_2(\mathbb{R}) \), \( G \) the group of all automorphisms \( s_\tau \) on \( m \) given by:

\[
(s_\tau f)(x) = f(x - \tau), \quad f \in L^\omega(\mathbb{R}), \quad x \in \mathbb{R},
\]

for \( \tau \in \mathbb{Q} \) with \( \tau \neq 0 \), and \( g \in G \mapsto U_g \) the usual unitary representation of \( G \) on \( K \);

or (iv) \( m \) is \( L^\omega(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2) \) acting on \( K = L_2(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2) \), \( G \) the group of all automorphisms \( \alpha(g_1, g_2) \), \( (g_1, g_2) \in G_1 \times G_2 \), of \( m \) given by:

\[
(\alpha(g_1, g_2) f)(x, y) = f(g_1(x), g_2(y)), \quad (x, y) \in X_1 \times X_2, \quad f \in L^\omega(X_1 \times X_2),
\]

where each \( G_i \) \( (i=1,2) \) is a countable, abelian and ergodic group of automorphisms on \( m_i \) \( [= L^\omega(X_i, S_i, \mu_i)] \) acting on \( L_2(X_i, S_i, \mu_i) \) such
that each element \( e_1 \in G_1 \) and each \( e'_1 \in G'_1 \) are induced by point transformations, denoted again by \( g_1 ; g'_1 \) respectively, of \( X_1 \). Suppose that each \( G_i \) has a unitary representation \( g_1 \mapsto U_{g_1} \) on \( L^2(X_i, S_i, \mu_i) \).

Let \( U(g_1, g_2) \), \((g_1, g_2) \in G_1 \times G_2 \), be the unitary operator on \( L^2(X_1 \times X_2, S_1 \times S_2, \mu_1 \times \mu_2) \) given by:

\[
U(g_1, g_2)f_{1,2} = (U_{g_1}f_1)(U_{g_2}f_2),
\]

where \( f_1 \in L^2(X_1, S_1, \mu_1) \), \( f_2 \in L^2(X_2, S_2, \mu_2) \) and \( f_{1,2}(x,y) = f_1(x)f_2(y) \), for all \((x,y) \in X_1 \times X_2\). Suppose \( G \) is with the unitary representation \( \alpha(g_1, g_2) \mapsto U(g_1, g_2) \).

Or (v) \( m \) is \( L_\infty[0,1] \) acting by multiplication on \( K = L^2[0,1] \), \( G = \{ \beta_\delta : \delta \in \Delta \} \) with the usual unitary representation \( \beta_\delta \mapsto U_\delta \) on \( K \) (cf. the paragraph preceding Corollary 7.5 for the definition of \( \beta_\delta \));

Or (vi) \( m \) is \( L_\infty([0,1], \lambda_q) \) \((q > \frac{1}{2})\) acting by multiplication on \( K = L^2([0,1], \lambda_q) \), \( G = \{ \beta_\delta : \delta \in \Delta \} \) with the usual unitary representation; and that the Hilbert space \( H = K \otimes K_G \) and the unitaries \( V_g \) \((g \in G)\) are as defined in §2 [Definition 2.1]. Suppose \( A_1, A_2 \) are operators on \( H \) such that \( R(A_1) = R(A_2) = \alpha[m, K, G, g \mapsto U_g] \),

\[
\text{Im} A_1 = \text{Im} A_2, \quad R(\text{Im} A_1) = R(U_g \Theta V_g : g \in G),
\]

and that \((R(\text{Re} A_1), R(\text{Re} A_2)) \) fits in \([\alpha, m \Theta I] \).
Denote $\text{Re} \ A_i = M_{f_i} \theta 1$, $M_{f_i}$ being the multiplication by the essentially bounded measurable function $f_i$. Then $A_1$ and $A_2$ are unitarily equivalent if and only if they are (algebraically) equivalent, and they are unitarily equivalent if and only if, in the case:

(i), $f_2 = \tau_r(f_1)$ for some $r \in [0, 1]$, where $\tau_r$ is the translation mod 1 by $r$;

(ii), $f_2 = \tau_r(f_1)$ for some $r \in \mathbb{R}$, where $\tau_r$ is the translation by $r$;

(iii), $f_2 = s_r(f_1)$ for some non-zero real number $r$, where $s_r$ is defined by: $(s_r f)(x) = f(r^{-1} x), x \in \mathbb{R}$;

(iv), $f_2 = \alpha(g'_1, g'_2)(f_1)$ for some $(g'_1, g'_2) \in \mathcal{G}_1 \times \mathcal{G}_2$, where

$$(\alpha(g'_1, g'_2) f)(x, y) = f(g'_1(x), g'_2(y));$$

(v), $f_2 = \alpha_r(f_1)$ for some $r \in \mathbb{R}$, or $f_2 = \beta_{n_1} \beta_{n_2} \cdots \beta_{n_m}(f_1)$ for some finite sequence of integers $n_1, n_2, \ldots, n_m$, where $\alpha_r$'s are defined in the paragraph preceding Corollary 7.5;

(vi), $f_2 = \beta_{n_1} \beta_{n_2} \cdots \beta_{n_m}(f_1)$ for some finite sequence of integers $n_1, n_2, \ldots, n_m$.

Proof. The present theorem follows readily from Theorem 8.1, Propositions 7.1, 7.2, 7.3 and Corollary 7.5.
We now illustrate Theorem 8.2 by the following simple example. Let $\mathcal{M}$ be $L_\infty(\mathbb{R})$ acting by multiplication on $L_2(\mathbb{R})$, $D$ the dyadic rationals in $\mathbb{R}$, and $G$ the group of all automorphisms of $\mathcal{M}$ induced by translations (in $\mathbb{R}$) by $d \in D$. Suppose that $\sum_{d \in D} \alpha_d(U_d \otimes V_d)$, $\alpha_d \in \mathbb{C}$, is self-adjoint, and it generates $\mathcal{R}(U_d \otimes V_d : d \in D)$. Let $f_1, f_2$ be two strictly monotone, continuous, bounded and real-valued functions defined on $\mathbb{R}$. Then by Theorem 8.2 we have:

$$M_{f_1} \otimes 1 + i \sum_{d \in D} \alpha_d(U_d \otimes V_d) \quad \text{and} \quad M_{f_2} \otimes 1 + i \sum_{d \in D} \alpha_d(U_d \otimes V_d)$$

are unitarily equivalent if and only if for some $r \in \mathbb{R}$:

$$f_2(x) = f_1(x - r) \quad \text{a.e.} \quad x \in \mathbb{R}.$$
9. Examples of non-equivalent operators of various type
with $\mathcal{R}(\text{Re } A)$ thick and of various type in $\mathcal{R}(A)$

In this section we shall construct numerous non-equivalent operators
on the separable Hilbert space, by applying the results stated in §8 and the
theory of abelian subalgebras developed by Bures [5]. More precisely, we
shall prove the following theorem. [For the terminology see §9.1 below.]

Theorem 9.1 [cf. 5, §15.]

I. Given a positive integer $n \geq 2$, there exists a family $(A_i)_{i \in I}$
(Card I the continuum) of pairwise unitarily non-equivalent operators on
the separable Hilbert space such that:

1. $\mathcal{R}(A_i)$ is the hyperfinite factor $\mathcal{A}$ on the separable Hilbert
space,
2. $\mathcal{R}(<\text{Re } A_i>) = E$ for all $i \in I$,
3. $E$ is thick in $\mathcal{A}$, of deficiency type $I_n$ and uniform
multiplicity $n$, and $E^c$ is regular maximal abelian in $\mathcal{A}$.

II. Given a positive integer $n \geq 2$ and a real number $x > n$, there exists
a family $(A_i)_{i \in I}$ (Card I the continuum) of pairwise unitarily non-equivalent
operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i)$ is the hyperfinite factor $\mathcal{A}$ on the separable Hilbert
space,
2. $\mathcal{R}(\text{Re } A_i) = E$ for all $i \in I$.

3. $E$ is thick in $\mathcal{A}$, of deficiency type $I_n$ and uniform multiplicity $x$, and $E^c$ is regular maximal abelian in $\mathcal{A}$.

III. There exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i)$ is the hyperfinite factor $\mathcal{A}$ on the separable Hilbert space,

2. $\mathcal{R}(\text{Re } A_i) = E$ for all $i \in I$,

3. $E$ is thick in $\mathcal{A}$, of deficiency type $I_\infty$ and uniform multiplicity $\omega$, and $E^c$ is regular maximal abelian in $\mathcal{A}$.

IV. Given $\Lambda : [1, \omega) \to [0, 1]$ non-decreasing, right continuous, and with $\Lambda(2) = 0$ and $\lim_{x \to \omega} \Lambda(x) = 1$, there exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i)$ is the hyperfinite factor $\mathcal{A}$ on the separable Hilbert space,

2. $\mathcal{R}(\text{Re } A_i) = E$ for all $i \in I$,

3. $E$ is thick in $\mathcal{A}$, of deficiency type $I_2$ and with multiplicity function $\Lambda$, and $E^c$ is regular maximal abelian in $\mathcal{A}$. 
V. Given a positive integer $n \geq 2$, there exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i) = \mathcal{A}$ for all $i \in I$,
2. $\mathcal{A}$ is a factor of type $\text{II}_\infty$,
3. $\mathcal{R}(\text{Re } A_i) = \mathcal{E}$ for all $i \in I$,
4. $\mathcal{E}$ is thick in $\mathcal{A}$ and is of deficiency type $\text{I}_n$, and $\mathcal{E}^c$ is regular maximal abelian in $\mathcal{A}$.

VI. There exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i) = \mathcal{A}$ for all $i \in I$,
2. $\mathcal{A}$ is a factor of type $\text{II}_\infty$,
3. $\mathcal{R}(\text{Re } A_i) = \mathcal{E}$ for all $i \in I$,
4. $\mathcal{E}$ is thick in $\mathcal{A}$ and is of deficiency type $\text{I}_\infty$, and $\mathcal{E}^c$ is regular maximal abelian in $\mathcal{A}$.

VII. There exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $\mathcal{R}(A_i) = \mathcal{A}$ for all $i \in I$,
2. $\mathcal{A}$ is a factor of type $\text{II}_\infty$,
3. $\mathcal{R}(\text{Re } A_i) = \mathcal{E}$ for all $i \in I$,.
4. $E$ is thick in $a$ and is of deficiency type III, and $E^c$ is regular maximal abelian in $a$.

VIII. Given a positive integer $n > 2$, there exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $R(A_i) = a$ for all $i \in I$,
2. $a$ is a factor of type III,
3. $R(\text{Re } A_i) = E$ for all $i \in I$,
4. $E$ is thick in $a$ and is of deficiency type $I_n$, and $E^c$ is regular maximal abelian in $a$.

IX. There exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $R(A_i) = a$ for all $i \in I$,
2. $a$ is a factor of type III,
3. $R(\text{Re } A_i) = E$ for all $i \in I$,
4. $E$ is thick in $a$ and is of deficiency type $I_\infty$, and $E^c$ is regular maximal abelian in $a$.

X. There exists a family $(A_i)_{i \in I}$ (Card $I$ the continuum) of pairwise unitarily non-equivalent operators on the separable Hilbert space such that:

1. $R(A_i) = a$ for all $i \in I$,
2. $a$ is a factor of type III,
3. \( \mathcal{R}(\text{Re } A_i) = \mathcal{E} \) for all \( i \in I \).

4. \( \mathcal{E} \) is thick in \( \mathcal{A} \) and is of deficiency type III, and \( \mathcal{E}^c \) is regular maximal abelian in \( \mathcal{A} \).

The detailed proof of the above theorem is given in §9.2 below. However the construction can be outlined as follows. We begin with a suitable C-system \([m, K, G, g \mapsto U_g]\) and construct von Neumann algebras \( \mathcal{A}, m \) and \( n \) as outlined in §2. Then we construct a suitable thick subalgebra \( \mathcal{E} \) in \( m \) with \( \mathcal{R}(\mathcal{E} \otimes 1, n) = \mathcal{A} \). We then find for each of \( \mathcal{E} \) and \( n \) a self-adjoint single generator (i.e. a self-adjoint element \( S \) of \( \mathcal{E} \) with \( \mathcal{R}(S) = \mathcal{E}_i \)); they will respectively be the real part and the imaginary part of our operator. By varying the real part of this operator according to the results of §8 and by varying the \( \mathcal{E}, m \) and \( G \), we obtain numerous non-equivalent operators of type II\(_1\), II\(_\infty\) and III on the separable Hilbert space.

This section is divided into two subsections. In the first subsection we give a summary of what we need from the theory of thick subalgebras developed in [5]. In the second subsection we construct those promised operators.

9.1 We introduce here a few (algebraic) invariants and results from the the theory of abelian subalgebras developed by Bures in [5], which will be used in the next subsection.
Definition 9.1.1 Suppose that \( \alpha \) is a factor of type \( \text{II}_1 \) with the normalized dimension function \( d \), and \( E \) a fixed abelian von Neumann subalgebra of \( \alpha \).

(i) If \( P \) is a projection of \( E^c \equiv E' \cap \alpha \), define:

\[
C(P) = \inf \{ E : E \text{ a projection of } E \text{ with } E \geq P \}.
\]

(ii) If \( P \) is a non-zero projection of \( E^c \), define:

\[
s(P) = \frac{d(C(P))}{d(P)}.
\]

(iii) If \( E \) is a non-zero projection of \( E \), define:

\[
r(E) = \sup \{ s(P) : P \text{ a projection of } E^c \text{ with } 0 < P \leq E \}.
\]

(iv) If \( E \) is a non-zero projection of \( E \), define:

\[
m(E) = \inf \{ r(F) : F \text{ a projection of } E \text{ with } 0 < F \leq E \}.
\]

(v) A projection \( E \) of \( E \) is said to have uniform multiplicity (with respect to \( [E, \alpha] \)) \( x \) if \( r(E) = m(E) = x \). \( E \) is said to have uniform multiplicity \( x \) in \( \alpha \) if \( r(1) = m(1) = x \).

(vi) It has been proved in [5, Proposition 1.14] that for each real number \( x \geq 1 \) there exists a unique projection \( E_x \) of \( E \) such that:

(a) \( r(E_x) \leq x \)

and (b) if \( F \) is a projection of \( E \) with \( 0 < F \leq 1 - E_x \), then \( r(F) > x \).

We define the multiplicity function of \( E \) as the function \( \Lambda \) from \( [1, \infty) \) to \( [0, 1] \) given by:
\[ A(x) = d(E_x) \quad \text{for all } x > 1. \]

**Definition 9.1.2.** Let \( m \) be a von Neumann algebra and \( G \) a full \( m \)-group (cf. Definition 5.1(v)).

(i) A \( G \)-trace is a trace \( \omega \) on \( m^+ \) which satisfies:
\[
\omega(\alpha(M)) = \omega(M) \quad \text{for all } \alpha \in G \text{ and all } M \in m^+. 
\]

(ii) \( G \) will be called finite (resp. semi-finite) if, for every \( T \in m \) with \( T > 0 \), there exists a finite (resp. semi-finite) normal \( G \)-trace \( \omega \) satisfying \( \omega(T) > 0 \).

(iii) \( G \) will be called properly-infinite (resp. of type III) if there exists no non-zero finite (resp. semi-finite) normal \( G \)-trace.

(iv) A projection \( E \) of \( m \) will be called \( G \)-abelian if
\[
\alpha(F) \leq F + (1 - E) 
\]
for all \( \alpha \in G \) and for all projections \( F \) of \( m \) with \( F \leq E \).

(v) \( G \) will be called continuous if every non-zero projection of \( m \) fails to be \( G \)-abelian.

(vi) If \( E \) and \( F \) are projections of \( m \), write
\[ E \sim F \quad (G) \]
to mean that there exists a family \( (\alpha_i, E_i)_{i \in I} \) with each \( \alpha_i \in G \) and each \( E_i \) a projection of \( m \), such that
\[
\sum_{i \in I} E_i = E \quad \text{and} \quad \sum_{i \in I} \alpha_i(E_i) = F.
\]
(vii) $G$ will be said to be of type $I$ if, for every non-zero projection $E$ of $m$, there exists a $G$-abelian projection $E_1$ of $m$ satisfying $0 < E_1 < E$.

(viii) $G$ will be said to be of type $II_1$ (resp. $II_{\infty}$) if $G$ is continuous and finite (resp. continuous, semifinite and properly infinite).

(ix) Let $n$ be a cardinal number. $G$ will be said to be of type $I_n$ if there exists a family $(E_i)_{i \in I}$ of $G$-abelian projections of $m$ such that $\sum_{i \in I} E_i = 1$, the cardinality of $I$ is $n$, and $E_i \sim E_j (G)$ for all $i, j \in I$.

**Definition 9.1.3** Let $E$ be an abelian subalgebra of a von Neumann algebra $\mathfrak{a}$ (not necessarily a factor). $E$ is said to be of deficiency type $I$ (resp. type $I_n$, type $II_1$, type $II_{\infty}$ and type $III$) if so is the $E^{cc}$-group $D$, which consists of all automorphisms of $E^{cc}$ keeping $E$ pointwise fixed. (Note that $E^{cc} = (E' \cap a)' \cap a$, that $E^{cc}$ is abelian and $D$ is full.)

**Definition 9.1.4** An abelian von Neumann subalgebra $E$ of $\mathfrak{a}$ is called thick if $E^{cc}$ is maximal abelian in $\mathfrak{a}$.

Note that Definitions 9.1.1, 9.1.3 and 9.1.4 are (algebraic) invariants of $E$ with respect to $\mathfrak{a}$ in the sense that if $E_1$ and $E_2$ are abelian subalgebras of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively, and if there is an isomorphism $\phi$ of $\mathfrak{a}_1$ onto $\mathfrak{a}_2$ with $\phi(E_1) = E_2'$, then $E_1$ has one of those properties if and only if so has $E_2$. We now turn to methods of producing thick subalgebras having various invariants. Lemmas 9.1.2 to 9.1.8
are taken from [5], and on these lemmas our construction of thick subalgebras in §9.2 will be based. We need two more notions before we can introduce these lemmas.

**Definition 9.1.5** Suppose that $m$ is an abelian von Neumann algebra, $H$ is an $m$-group, $\alpha \in A(m)$ (cf. Definition 5.1 (i)), and $P$ is a projection of $m$. We will say that $\alpha$ is strongly orthogonal to $H$ on $P$ (in symbols, $\alpha \perp H$ on $P$) if there exists a family $(P_i)_{i \in I}$ of projections of $m$ such that $\sup_{i \in I} P_i \geq P$ and:

$$\left( (\alpha \xi)(P_i) \right) \left( \xi'(P_i) \right)^* P = 0 \text{ for all } \xi, \xi' \in H \text{ and all } i \in I.$$ 

We define $E'(\alpha, H)$ to be $1 - P(\alpha, H)$ where $P(\alpha, H)$ is

$$\sup\{ \text{Projection P of m : } \alpha \perp H \text{ on P} \}.$$ 

**Definition 9.1.6** Suppose $\mathfrak{a}$ is a von Neumann algebra, $S \subset A(\mathfrak{a})$ and $S \subset \mathfrak{a}$. We define:

$$S^o = \{ A \in \mathfrak{a} : \alpha(A) = A \text{ for all } \alpha \in S \},$$

and

$$S^o = \{ \alpha \in A(\mathfrak{a}) : \alpha(A) = A \text{ for all } A \in S \}.$$ 

**Lemma 9.1.1** Suppose that $m$ is an abelian von Neumann algebra, $H$ is an $m$-group, $\alpha \in A(m)$ and $P$ is a projection of $m$. Then $\alpha \perp H$ on $P$ if and only if there exists a family $(E_i)_{i \in I}$ of projections of $H^o$ such
that \( \sup_{i \in I} P_i \geq P \) and \( \alpha(E_i)E_iP = 0 \) for each \( i \in I \).

**Lemma 9.1.2** Suppose that \([m, K, G, g \mapsto U_g]\) is a C-system. Let \( a = a[m, K, G, g \mapsto U_g] \). Suppose now that \( H \) is an \( m \)-group satisfying:

\[ E'(g, H) = E(g, 1) \quad \text{for all} \quad g \in G. \]

Then \( E = H^* \otimes 1 \) is thick in \( a \) with \( E^c = m \otimes 1 \).

**Corollary 9.1.3** Suppose \( \xi \in A(m) \) be such that either one of the following holds:

(i) \( \xi^n = 1 \) for some \( n \in \mathbb{Z} \);

(ii) there exists a family \( (Q_k)_{k \in \mathbb{Z}} \) of projections of \( m \) such that

\[ \sum_{k \in \mathbb{Z}} Q_k = 1 \quad \text{and} \quad \xi(Q_k) = Q_{k+1} \quad \text{for all} \quad k \in \mathbb{Z}. \]

Suppose also that for any integer \( m \) and any \( g \in G \):

\[ E(\xi^m, g) \leq E(1, g), \]

where \( 1 \) is the identity of \( G \). Then \( E = \xi^* \otimes 1 \) is thick in \( a \) with \( E^c = m \otimes 1 \).

**Lemma 9.1.4** Suppose that \( m \) is an abelian von Neumann algebra, that \( n \) is a positive integer or \( \omega \), and that \( \xi \in A(m) \). If \( n \) is finite assume that \( \xi^n = 1 \). Assume that there exists a family \( (Q_i)_{i \in \mathbb{Z}_n} \) of projections
of $m$ such that $\sum_{i \in \mathbb{Z}_n} Q_i = 1$ and $\xi(Q_i) = Q_{i+1}$ for all $i \in \mathbb{Z}_n$. (We take $\mathbb{Z}_0 = \mathbb{Z}$.) Then $(\xi^o)^o$ is of type $I_n$.

Lemma 9.1.5 Suppose that the conditions of Lemma 9.1.4 hold. Suppose also that $E = \xi^o$ is thick in a factor $a$ of type $II_1$ with $E^c = m$, and denote the normalized dimension function by $d$. Suppose the following condition holds:

$$d[\xi(PQ)] = \frac{d(Q_{i+1})}{d(Q_i)} \frac{d(PQ)}{d(Q_i)}$$

for all projections $P$ of $m$, and all $i \in \mathbb{Z}_n$. Then $E$ has uniform multiplicity $x$, where $x = \sup \{ 1/d(Q_i) : i \in \mathbb{Z}_n \}$. In particular, if $n = \infty$, then $E$ has uniform multiplicity $\infty$.

Lemma 9.1.6 Suppose that $m$ is an abelian von Neumann algebra, that $\xi \in A(m)$ satisfies $\xi^2 = 1$, and that $E = \xi^o$ is thick in a factor of type $II_1$ with $E^c = m$. Denote by $d$ the normalized dimension function on $a$. Suppose also that there exists a projection $Q$ of $m$ such that:

$$\xi(Q) = 1 - Q,$$

and $d[\xi(PQ)] \geq d(PQ)$ for all projections $P$ of $m$.

Finally, suppose that for each $x \geq 1$ there exists a projection $E_x$ of $E$ such that, for all projections $P$ of $m$:

$$d[\xi(PQE_x)] \leq (x - 1)[d(PQE_x)],$$

and $d[\xi(PQ(1 - E_x))] \geq (x - 1)[d(PQ(1 - E_x))]$ whenever $PQ(1 - E_x) \neq 0$.

Then $A$, defined by:

$$A(x) = d(E_x) \quad \text{for all } x \geq 1,$$

is the multiplicity function of $E$. 
Lemma 9.1.7 Suppose that $m_2$ is algebraically isomorphic to $L_\infty[0, 1]$ acting by multiplication on $L_2[0, 1]$, that $m_1$ is of countable decomposibility type [i.e. there is a countably infinite family $(E_i)_{i \in \mathbb{Z}}$ of projections of $m_1$ with $\sum_{i \in \mathbb{Z}} E_i = 1$, and any family $(F_i)_{i \in I}$ of orthogonal projections of $m_1$ is countable], that $m = m_1 \otimes m_2$ and $E = m_1 \otimes \mathbb{C}$. Then $E^\circ$ is of type III.

Lemma 9.1.8 Let $m, n$ be abelian $W^*$-algebras, and $H$ an ergodic $\mathfrak{n}$-group. Then

$$(1 \otimes H)^\circ = m \otimes \mathbb{C},$$

where $1 \otimes H = \{ \alpha \in A(m \otimes n) : \alpha = 1 \otimes h \text{ for some } h \in H \}$ (cf. [9], p.56).

We shall also use the following notion and lemma from [5].

Definition 9.1.7 Suppose that $m$ is a von Neumann algebra, that $\alpha \in A(m)$, and that $\omega$ is a trace on $m^+$. We will say that $\alpha$ is locally non-$\omega$-preserving if there is no non-zero projection $E$ of $m$ such that

$$\omega(\alpha(EM)) = \omega(EM) \quad \text{for all } M \in m^+.\)$$

We say that $\alpha$ preserves $\omega$ if

$$\omega(\alpha(M)) = \omega(M) \quad \text{for all } M \in m^+.\)$$
Lemma 9.1.9 Suppose that $\omega$ is a trace on the von Neumann algebra $m$ and that $\alpha, \beta \in A(m)$. If $\alpha$ is locally non-$\omega$-preserving and $\beta$ preserves $\omega$, then

$$E(\alpha, \beta) = 0$$

Definition 9.1.8 Let $m$ be an abelian von Neumann subalgebra of the von Neumann algebra $a$. Then $m$ is regular in $a$ if

$$R(U : U \text{ unitary in } a \text{ with } U^*mU \subseteq m) = a.$$ 

9.2 In this subsection we shall follow the outline in the beginning of §9 to construct the promised operators, and thus prove Theorem 9.1. We shall adhere to the use of notations $m, G, a$, etc. therein. Note that (cf. [5]) with the conditions we already have on $G$, $a$ will be a factor of type $II_1$, $II_\infty$ or $III$ if so is the type of $G$. Furthermore (cf. [5]) $a$ will be the hyperfinite factor of type $II_1$ on the separable Hilbert space if $G$ is type $II_1$ and abelian. By a theorem of von Neumann [13], the abelian von Neumann algebra $R(U_g \theta V_g : g \in G)$ on the separable Hilbert space $H$ has a self-adjoint single generator. So our first task is to construct a thick abelian subalgebra $E$ of $a$ so that $R(E, U_g \theta V_g : g \in G) = a$.

We now establish a lemma which will be useful for this purpose.

Lemma 9.2.1 Let $\xi \in A(m)$. Suppose that there is a non-zero projection $Q$ of $m$ such that for every non-zero projection $P$ of $m$ with $P \subseteq Q$, there exists a non-zero projection $R$ of $R(\xi, U_g : g \in G) \cap m$ with
$R < P$. Then all projections of $m$ are in $R(\xi^0, U_g: g \in G)$ and,

$$R(\xi^0 \otimes 1, U_g \otimes V_g: g \in G) = a.$$  

**Proof.** Since $a = R(m \otimes 1, U_g \otimes V_g: g \in G)$ it suffices to prove that all projections of $m$ are in $R(\xi^0, U_g: g \in G)$. Let $P$ be a non-zero projection of $m$. Let $R$ be the largest projection in $R(\xi^0, U_g: g \in G) \cap \cap m$ with respect to the property that $R < P$. Assume that $P - R \neq 0$. As $G$ is ergodic on $m$, there is a $g \in G$ such that $Q[U_g(P - R)U_g^*] = 0$. By assumption there is a non-zero projection $Q_0$ of $R(\xi^0, U_g: g \in G) \cap \cap m$ such that $Q_0 \leq Q[U_g(P - R)U_g^*]$. But then $0 \neq U_g^*(Q_0)U_g \leq P - R$, and $U_g^*(Q_0)U_g$ is a projection of $R(\xi^0, U_g: g \in G) \cap m$. This contradicts the choice of $R$. Therefore $P - R = 0$, and $P \in R(\xi^0, U_g: g \in G)$. That completes the proof.

We can now prove Theorem 9.1.

**Proof of Theorem 9.1** We shall prove the theorem by constructing the required operators according to the outline given in the beginning of §9. The construction is an imitation of that of the corresponding thick subalgebras in [5], with some modification to suit the present purpose. Paragraph indexed by I etc. will serve to establish statement I etc. of Theorem 9.1. Since in each case the $m$ will be $L_\infty(X)$ acting by multiplication on $L_2(X)$ for some $\sigma$-finite measure space $X$, we specify only the group of "invariant" transformations, with the unitary representation
$g \mapsto U_g$ of $G$ on $L^2(X)$ understood as the usual one in our case (cf. [15]).

These $G$'s will be abelian, ergodic and free on $\mathfrak{m}$. We shall write $\mathcal{R}(E, G)$ instead of $\mathcal{R}(E, U_g \otimes V_g : g \in G)$, and write $m = m \otimes 1_{X_0}$. For $f \in L^\infty_{\omega}(X)$, $M_f$ will be the operator in $\mathfrak{m}$ corresponding to multiplication by $f$.

I. Take $\mathfrak{m}$ to be $L^\infty_{\omega}[0, 1]$ acting on $L^2[0, 1]$, $G = \{ \tau_x : x$ p-adic rational in $[0, 1] \}$, where $p$ is a fixed prime $> 2$ such that $p$ does not divide $n$, and $\tau_x$ is the automorphism of $\mathfrak{m}$ given by

$$(\tau_x f)(y) = f(y - x)$$

for any $f \in L^\infty_{\omega}[0, 1]$ and $y \in [0, 1]$ (addition in $[0, 1]$ is mod 1).

It is well-known that $G$ is of type $\text{II}_1$.

Let $\xi$ be the automorphism of $\mathfrak{m}$ given by:

$$(\xi f)(y) = \begin{cases} f(\frac{2}{n} - y), & y \in [\frac{1}{n}, \frac{2}{n}] \\ f(y - \frac{1}{n}), & y \in (\frac{2}{n}, 1] \\ f(1 - y), & y \in [0, \frac{1}{n}] \end{cases}$$

where $f \in L^\infty_{\omega}[0, 1]$. Then $\xi^n = 1$, and

$$\xi(Q_i) = Q_{i+1} \quad \text{for } i \in \mathbb{Z}_n,$$

where $Q_i$ is the projection of $\mathfrak{m}$ corresponding to the characteristic
function of \([\frac{i}{n}, \frac{i+1}{n}]\). Observing that \(\frac{m}{n}\) is never a p-adic rational, it
is easy to verify that
\[
E(\xi^n, \tau_x) = 0,
\]
for all integers \(m\) and all p-adic rationals \(x \in (0, 1)\). By Corollary
9.1.3, Lemmas 9.1.4, 9.1.5, the subalgebra \(E = \xi \odot \mathbb{1}\) of \(m\) is thick in
\(\alpha\), of deficiency type \(n\), with uniform multiplicity \(n\) and \(E^c = m\).

To show that \(R(E, \mathbb{G}) = \alpha\), let \(x\) be a p-adic rational in
\((0, \frac{1}{2n})\), and let \(P_1\) be the projection of \(m\) corresponding to \([\frac{1}{n} - x, \frac{1}{n}]\).
Let
\[
\overline{P}_1 = P_1 + \xi(P_1) + \cdots + \xi^{n-1}(P_1).
\]
Then \(\overline{P}_1 \in \xi\) and \(0 < \overline{P}_1\tau_{1-x}(\overline{P}_1) < Q_0\), and \(\overline{P}_1\tau_{1-x}(\overline{P}_1)\) is in \(R(\xi^o, U_g : g \in \mathbb{G}) \cap m\). So by Lemma 9.2.1 \(R(E, \mathbb{G}) = \alpha\).

In order to have a self-adjoint single generator \(M_x \odot \mathbb{1}\) of \(E\)
\((M_x \in m : the\ multiplication\ by\ f \in L_\infty[0, 1])\), we may take any strictly
monotone, continuous and real-valued function \(g\) defined on \([0, \frac{1}{n}]\), and
let
\[
f = g + \xi(g) + \cdots + \xi^{n-1}(g),
\]
where \(g\) is extended to \([0, 1]\) with 0 as its value on \([0, 1] \setminus [0, \frac{1}{n}]\).
Thus in particular if for each strictly positive real number \(r\) we denote
by \(g_r\) the function \(x \mapsto x^r\) on \([0, \frac{1}{n}]\), and let
then each $M_r \ast 1$ is a self-adjoint single generator of $E$. Let $T$ be a self-adjoint single generator of the abelian $R(U_g \ast V_g : g \in G)$, and let $A_r$ be the operator on $H$ such that $\text{Re} \ A_r = M_r \ast 1$, and $\text{Im} \ A_r = T$. Then by Theorem 8.2 $(A_r)$ is a family with the desired properties.

II. Take $m$ to be $L^0[0, 1]$ acting on $L^2[0, 1]$, $G = \{ \tau_g : g \text{ p-adic rational in } [0, 1] \}$, where $p$ is a fixed integer $\geq 2$. $G$ is of type $\text{II}_1$. Select $(\delta_k)_{k \in \mathbb{Z}}$ such that

$$1 = \delta_0 < \delta_1 < \cdots < \delta_{n-2} < \delta_{n-1},$$

and

$$\sum_{k \in \mathbb{Z}} \delta_k = x.$$ 

This is possible as $x > n$. Let $(x_k)_{k \in \mathbb{Z}}$ be a partition of $[0, 1)$:

$$0 = x_0 < x_1 < \cdots < x_{n-1} < 1 = x_n,$$

such that $x_{k+1} - x_k = \delta_k/x$ for $k \in \mathbb{Z}$. Then we define an automorphism $\xi$ of $m$ by

$$(\xi f)(x_k + y) = f(x_{k-1} + \frac{\delta_{k-1}}{\delta_k} y) \text{ for all } y \in [0, \delta_k/x],$$

where $f \in L^0[0, 1]$. (We take $x_{-1} = x_{n-1}$ and $\delta_{-1} = \delta_{n-1}$.) It is easy to see that for $m = 1, 2, \cdots, n-1, \xi^m$ is locally non-$\lambda$-preserving ($\lambda$ Lebesque measure). By Lemmas 9.1.3, 9.1.4, 9.1.5, the abelian algebra
$E = \xi^n \otimes I$ is thick in $\alpha$ with $E^c = \mu$, and is of deficiency type $I_n$ and uniform multiplicity $x$.

To prove that $R(E, G) = \alpha$, let $Q_k, P_1, P_2$ be respectively the projections of $m$ corresponding to the characteristic functions of $[x_k, x_{k+1}]$, $[0, 1/4x]$ and $[3/4x, 1/x]$, and let

$$
\overline{P}_1 = P_1 + \xi(P_1) + \xi^2(P_1) + \cdots + \xi^{n-1}(P_1),
$$

and

$$
\overline{P}_2 = P_2 + \xi(P_2) + \xi^2(P_2) + \cdots + \xi^{n-1}(P_2).
$$

Select a $p$-adic rational $g$ so that $g \in (1/2x, \delta_1/2x)$. Then

$$
0 \neq \tau_g(\overline{P}_1)\overline{P}_2 \leq Q_\alpha.
$$

By Lemma 9.2.1 we see that $R(E, G) = \alpha$.

In order to have a self-adjoint single generator $M_f \otimes I$ of $E$, we may take any strictly monotone, continuous and real-valued function $g$ defined on $[0, x_1]$, and let

$$
f = g + \xi(g) + \cdots + \xi^{n-1}(g),
$$

where $g$ is extended to $[0, 1]$ with 0 as its values on $[0, 1]\setminus[0, x_1]$. Thus in particular if for each strictly positive real number $r$ we denote by $g_r$ the function $x \mapsto x^r$ on $[0, x_1]$ and let
\[ f_r = s_r + \xi(g_r) + \xi^2(g_r) + \ldots + \xi^{n-1}(g_r), \]

then \( M_r \) is a self-adjoint single generator of \( E \). Let \( T \) be a self-adjoint single generator of \( R(U_g \otimes V_g : g \in G) \) and let \( A_r \) be the operator on \( H \) with \( \text{Re}(A_r) = M_r \otimes 1 \) and \( \text{Im}(A_r) = T \). Then by Theorem 8.2 the family \( \{A_r\} \) has the desired properties.

III. Take \( m \) to be \( L_\infty[0, 1] \) acting on \( L_2[0, 1] \), \( G = \{\tau_g : g \text{ dyadic rational in } [0, 1]\} \). Let \( (x_n)_{n \in \mathbb{Z}} \) be defined by:

\[ x_0 = 0, \quad x_n = x_{n-1} + \frac{1}{2^n} \quad \text{for } n \in \mathbb{Z}. \]

Define an automorphism \( \xi \) of \( m \) by:

\[
(\xi f)(x) = \begin{cases} 
  f(x), & x \in [x_0, x_1) \cup \{1\} \\
  f(x_{n-1} + 2(x - x_n)), & x \in [x_n, x_{n+1}), n \in \mathbb{Z}.
\end{cases}
\]

Then clearly \( \xi^m \) is locally non-\( \lambda \)-preserving (\( \lambda \) : Lebesgue measure). Thus by Lemmas 9.1.9, 9.1.3, 9.1.4, 9.1.5, the abelian subalgebra \( E = \xi^o \otimes 1 \) is thick in \( a \) with \( E^c = m \), and is of deficiency type \( I_\infty \) and uniform multiplicity \( \infty \).

To prove that \( R(E, G) = a \), let \( P_1, P_2 \) be respectively the projections of \( m \) corresponding to the characteristic functions of \([0, 1/8] \)
and $[3/8, 1/2]$. Let

$$\overline{P}_i = P_i + \sum_{k \in \mathbb{Z}} n_k(P_i), \ i = 1, 2,$$

where for each $k \in \mathbb{Z}$, $n_k$ is the automorphism of $m$ given by

$$\eta_k(x) = \begin{cases} f(x_{k-1} + 2(x - x_k)), & x \in [x_k, x_{k+1}) \\ 0, & x \in [0, 1] \setminus [x_k, x_{k+1}) \end{cases}.$$

Then $\overline{P}_1$ and $\overline{P}_2$ are in $\xi^\circ$. Now

$$\tau_{3/16}(\overline{P}_1)\overline{P}_2 Q_1 = 0, \ \tau_{3/16}(\overline{P}_1)\overline{P}_2 \neq 0,$$

where $Q_1$ is the projection of $m$ corresponding to the characteristic function of $[x_0, x_1]$. Thus

$$0 < \tau_2[\tau_{3/16}(\overline{P}_1)\overline{P}_2] \leq Q_1.$$

So by Lemma 9.2.1, $R(E, G) = \alpha$.

For each strictly positive real number $r$, denote by $g_r$ the function $x \mapsto x^r$ on $[0, 1/2]$ and $x \mapsto 0$ on $[1/2, 1]$, and let

$$f_r = g_r + \sum_{k \in \mathbb{Z}} n_k(g_r).$$
Let $T$ be a self-adjoint single generator of $\mathcal{R}(U_g \otimes V_g : g \in G)$ and let $A_r$ be the operator on $H$ with $\text{Re} \, A_r = M_r \otimes 1$ and $\text{Im} \, A_r = T$. Then by Theorem 8.2, $(A_r)$ is a family with the desired properties.

IV. Take $m = L^\infty([0, 1]^2)$ on $L^2([0, 1]^2)$, and take $G = \{ \tau_{(x, y)} : x, y \text{ dyadic rationals in } [0, 1] \}$, where $\tau_{(x, y)}$ is the translation in $[0, 1]^2$ by $(x, y) \pmod{1}$. Clearly $G$ is of type $\text{II}_1$.

Let $\phi : [0, 1] \rightarrow [2, \infty)$ be defined by:

$$\phi(y) = \inf\{ t \geq 2 : A(t) \geq y \}.$$ 

Define the subset $S$ of $[0, 1]^2$ by:

$$S = \{ (x, y) \in [0, 1]^2 : x < [\phi(y)]^{-1} \}.$$ 

Define the automorphism $\xi$ of $m$ by:

$$\xi f(x, y) = \begin{cases} f([\phi(y)]^{-1} + [\phi(y) - 1]x, y) & \text{for all } (x, y) \in S, \\ f([\phi(y) - 1]^{-1}[x - (\phi(y))^{-1}], y) & \text{for all } (x, y) \not\in S, \end{cases}$$

where $f \in L_\infty([0, 1]^2)$. Let $Q$ be the projection of $m$ corresponding to $S$. Then clearly $\xi$ is locally non-$\lambda$-preserving ($\lambda : \text{Lebesgue measure}$), $\xi^2 = 1$ and $\xi(Q) = 1 - Q$. Thus by Lemmas 9.1.9, 9.1.3, 9.1.4, the abelian von Neumann algebra $E = \xi^0 \otimes 1$ is thick in $\mathfrak{a}$ with $E^c = m$, and $E$ is of deficiency type $\text{I}_2$. 
For each $x \geq 1$ let $E_x$ be the projection of $m$ corresponding to $[0, 1] \times [0, A(x)]$. Then it is easy to see that the conditions of Lemma 9.1.6 are satisfied, and we conclude that $A$ is the multiplicity function of $E$.

In order to prove that $R(E, G) = a$, let $x_0 = 1/\phi(\frac{3}{n})$. Since $A(2) = 0$ and since $A$ is right continuous, $x_0 < 1/2 - \delta$ for some $\delta \in (0, 1)$. Let $P_1$ be the projection of $m$ corresponding to $\{ (x, y) : 0 < x < \delta/\phi(y), \ y \in [3/4, 1] \}$. Let $P_1 = P_1 + \xi(P_1)$.

Then $P_1 \in \xi^*$ and $0 < \tau(\frac{1}{2}, 0)(P_1) \leq 1 - Q$. Thus by Lemma 9.2.1 $R(E, G) = a$.

Let $M_f$ (resp. $T$) be a self-adjoint single generator of $\xi^*$ (resp. $R(U_g \otimes V_g : g \in G)$), where $f \in L_\infty([0, 1]^2)$ and $f \geq 0$. For each strictly positive real number $r$, let $g_r = (f)^r$ and let $A_r$ be the operator on $H$ with $\text{Re } A_r = M_{g_r} \otimes 1$ and $\text{Im } A_r = T$. Then by Theorem 8.2, the family $(A_r)$ has the desired properties.

V. Take $m = L_\infty([0, 1] \times \mathbb{R})$ acting by multiplication on $L_2([0, 1] \times \mathbb{R})$, and take $G = \{ \tau(x, y) : x \text{ p-adic rational in } [0, 1], \ y \text{ rational in } \mathbb{R} \}$, where $p$ is a fixed prime $> 2$ such that $p$ does not divide $n$. Clearly $G$ is of type $\text{II}_\infty$. 
In order to define our thick subalgebra let us recall the automorphism $\xi$ of $I$. Let $\xi'$ be the unique automorphism of the present $m$ such that for any $f_1 \in L_\infty[0,1]$ and for any $f_2 \in L_\infty(\mathbb{R})$:

$$\xi'(f_1,f_2)(x,y) = [(\xi f_1)(x)]f_2(y),$$

where $f_1,f_2(x,y) = f_1(x)f_2(y)$, for all $(x,y) \in [0,1] \times \mathbb{R}$. Now it is easy to see that for exactly the same reasons as in $I$, the abelian algebra $E = (\xi')^\circ \otimes 1$ is thick in $\mathcal{A}$ with $E^c = m$, and is of deficiency type $I_n$.

In order to prove that $R(E, G) = a$, we first notice that all projections of $m$ corresponding to $[0,1] \times S$ ($S \subset \mathbb{R}$ measurable) are in $(\xi')^\circ$. It then follows that all projections of $m$ are in $R((\xi')^\circ, U_g : g \in G)$ because a similar property holds for $\xi$. Thus $R(m \otimes 1, U_g \otimes V_g : g \in G) \subseteq R((\xi')^\circ \otimes 1, U_g \otimes V_g : g \in G)$, i.e. $a = R(E, G)$.

Let $M_f$ (resp. $T$) be a self-adjoint single generator of $(\xi')^\circ$ (resp. $R(U_g \otimes V_g : g \in G)$), where $f \in L_\infty([0,1] \times \mathbb{R})$ and $f \geq 0$. For each strictly positive real number $r$, let $g_r = (f)^r$ and let $A_r$ be the operator on $H$ with $\text{Re} A_r = M_{g_r} \otimes 1$ and $\text{Im} A_r = T$. Then the family $(A_r)$ has the required properties.

VI. Take $m$ to be $L_\infty([0,1] \times \mathbb{R})$ acting on $L_2([0,1] \times \mathbb{R})$, and $G = \{ \tau(x,y) : x \text{ dyadic rational in } [0,1], y \text{ rational in } \mathbb{R} \}$. Clearly $G$ is of type $\text{II}_\infty$. 
To define our thick subalgebra let us recall the automorphism $\xi$ in III. Let $\xi'$ be the unique automorphism of the present $m$ such that for any $f_1 \in L_0([0,1])$ and any $f_2 \in L_0(\mathbb{R})$:

$$\xi'(f_{1,2})(x, y) = [(\xi f_1)(x)] f_2(y),$$

where $f_{1,2}(x, y) = f_1(x)f_2(y)$, for all $(x, y) \in [0,1] \times \mathbb{R}$. Now it is easy to see that $\xi'$ is locally non-$\lambda$-preserving as so is $\xi$ ($\lambda$: Lebesgue measure). Thus for exactly the same reasons as in III, the abelian algebra $E = (\xi')^0 \otimes l$ is thick in $a$ with $E^c = m$, and is of deficiency type $I_\infty$.

In exactly the same way as in V above, we can prove that $R(E, G) = a$.

Let $M_\xi$ (resp. $T$) be a self-adjoint single generator of $(\xi')^0$ (resp. $R(U_\Xi \otimes V_\Xi : \Xi \in G)$), where $f \in L_0([0,1] \times \mathbb{R})$ and $f \geq 0$. For each strictly positive integer $r$ let $g_r = (f)^r$, and let $A_r$ be the operator on $H$ with $\text{Re } A_r = M_{g_r} \otimes l$, and $\text{Im } A_r = T$. Then the family $(A_r)$ has the required properties.

VII. Take $m$ to be $L_0(\mathbb{R}^2)$ acting on $L^2(\mathbb{R}^2)$, and $G_1 = \{ \alpha_{(r_1, r_2)} : r_1 \text{ non-zero rational, } r_2 \text{ rational} \}$, where $\alpha_{(r_1, r_2)}$ is the automorphism of $m$ defined by:

$$(\alpha_{(r_1, r_2)} f)(x, y) = f(r_1^{-1} x, y - r_2), \text{ for all } x, y \in \mathbb{R},$$
where \( f \in L_\infty(\mathbb{R}^2) \). Now the functional \( \mu \) defined on \( m^+ \) (to \([0, \infty]\)) by:

\[
\mu(f) = \int_{\mathbb{R}^2} \frac{1}{|x|} f(x, y) \, dx \, dy
\]

is a semi-finite, normal \( G \)-trace on \( m \). So \( G \) is of type \( II_\infty \). Also \( G \) is ergodic (and abelian). Thus \( \alpha \) is a factor of type \( II_\infty \), acting on the separable Hilbert space.

Let \( H_1 = \{ \tau_{(r, mr)} : r \text{ rational} \} \), where \( m \) is a fixed irrational positive number, and \( \tau_{(r, mr)} \) is the automorphism of \( m \) defined by:

\[
(\tau_{(r, mr)} f)(x, y) = f(x - r, y - mr)
\]

for all \( x, y \in \mathbb{R} \) and \( f \in L_\infty(\mathbb{R}^2) \). Let

\[
E = \mathbb{H}_1 \otimes 1.
\]

To see that \( E \) is thick in \( \alpha \), we let, for real numbers \( a, b \) with \( a < b \):

\[
E_{a, b} = \{ (x, y) \in \mathbb{R}^2 : mx + a \leq y \leq mx + b \},
\]

and let \( P_{a, b} \) be the projection of \( m \) corresponding to \( E_{a, b} \). Then for any \( \alpha(r_1, r_2) \in G \), not the identity:

\[
E_{a, b} \alpha(r_1, r_2)(E_{a, b}) \to 0 \text{ strongly as } a \to b \text{ from the left. As } E_{a, b} \in \mathbb{H}_1 \text{ we conclude by Lemma 9.1.1 that } E'(\alpha(r_1, r_2), H_1) = 0 \text{. So by }
Lemma 9.1.2, $E$ is thick in $a$ with $E^c = m$.

We use Bure's argument to determine the deficiency type of $E$. Let $m_1 = m_2$ be $L^2(\mathbb{R})$ acting on $L^2(\mathbb{R})$. Then by rotation it is seen that there is an isomorphism of $m$ with $m_1 \otimes m_2$ which takes $H_1$ onto $1 \otimes H_2$ for some ergodic $m_2$-group $H_2$. It then follows from Lemmas 9.1.8, 9.1.7 that $H_1^0$ is of deficiency type III in $m$. So $E$ is of deficiency type III in $m$.

In order to see that $R(E, G) = a$, let

$$Q_{a,b} = E_{a,b} \alpha(2^{-n}, 1)(E_{a,b}).$$

Then it is easy to see that each $Q_{a,b}$ ($a < b$) is of positive Lebesgue measure and that within each open circle $C$ inside $Q_{1,2}$ there is a $Q_{a,b}$ within $C$ for some suitable $a, b$ satisfying $a < b$. Thus we see that the projection in $m$ corresponding to any open measurable subset of $Q_{1,2}$ is in $R(H^0, U_g : g \in G)$, and therefore (Lemma 9.2.1), $R(E, G) = a$.

Let $S$ (resp. $T$) be a self-adjoint single generator of $H^0_1$ (resp. $R(U_g \otimes V_g : g \in G)$). For each $r \in \mathbb{R}$ let $\alpha_r = \tau(r, r)$ on $m$ and let $A_r$ be the operator on $H$ such that $\text{Re} A_r = \alpha_r(S) \otimes 1$ and $\text{Im} A_r = T$. Then $R(\text{Re} A_r) = H^0_1 \otimes 1$ as $\alpha_r$ commutes with each element of $H_1$. By Theorem 8.2 the family $(A_r)$ has the desired properties.
VIII. Take $m$ to be $L_\infty([0, 1]^2, \lambda \times \mu)$ acting on $L_2([0, 1]^2, \lambda \times \mu)$, $G = \{ \alpha(x, \delta) : x \text{ p-adic rational in } [0, 1], \text{ and } \delta \text{ a finite sequence of positive integers } \}$, where $\mu$ can be any one of the measure $\mu_q(q \in (\frac{1}{2}, 1))$ on $[0, 1]$ defined in the paragraph preceding Corollary 7.5 ($\lambda$ is the Lebesque measure), and $\alpha(x, \delta)$ is the automorphism of $m$ given by:

$$\alpha(x, \delta)f(y, z) = f(y - x, \beta_\delta(z)), \ y, z \in [0, 1],$$

where $\beta_\delta$ is as defined in the paragraph preceding Corollary 7.5. Then $G$ is abelian and ergodic (and free) on $m$. Also $G$ is of type III [14 or 7] so $\alpha$ is a factor of type III on the separable Hilbert space.

To define our thick subalgebra let us recall the automorphism $\xi$ in $I$. Let $\xi'$ be the unique automorphism of the present $m$ such that for any $f_1 \in L_\infty([0, 1], \lambda)$ and for any $f_2 \in L_\infty([0, 1], \mu)$:

$$[\xi'(f_1 f_2)](y, z) = [(\xi f_1)(y)]f_2(z),$$

where $f_{1,2}(y, z) = f_1(y)f_2(z)$, for all $y, z \in [0, 1]$. Now it is easy to see that for exactly the same reasons as in V, the abelian algebra $E = (\xi')^\circ \otimes I$ is thick in $\alpha$ with $E^c = m$, and is of deficiency type $I_n$.

For almost verbally the same reason as in $V$, we have $R(E, G) = \alpha$.

Let $M_f$ (resp. $T$) be a self-adjoint single generator of $(\xi')^\circ$ (resp. $R(U_g \otimes V_g : g \in G)$), where $f \in L_\infty([0, 1]^2, \lambda \times \mu)$ and $f \geq 0$. 
For each strictly positive real number \( r \), let \( g_r = (f)^r \), and let \( A_r \) be the operator on \( H \) with \( \text{Re} \, A_r = M_{g_r} \otimes 1 \) and \( \text{Im} \, A_r = T \). Then the family \( (A_r) \) has the required properties.

IX. Take \( m \) to be \( L_\infty([0, 1]^2, \lambda \times \mu) \) acting on \( L_2([0, 1]^2, \lambda \times \mu) \), and \( G = \{ \alpha_{(g, \delta)} : g \) dyadic rational in \([0, 1]\), and \( \delta \) a finite sequence of positive integers\} , where \( \lambda, \mu \) are the same measures used in VIII above, and \( \alpha_{(g, \delta)} \) is the automorphism of \( m \) given by:

\[
\alpha_{(g, \delta)} f(x, y) = f(x - g, \beta_\delta(y)),
\]

for all \( x, y \in [0, 1] \) and \( f \in L_\infty([0, 1]^2, \lambda \times \mu) \) (\( g, \delta \) being as in VIII above). For verbally the same reasons as in VIII, \( \alpha \) is a factor of type III on the separable Hilbert space.

Recall the automorphism \( \xi \) in III, and define the automorphism \( \xi' \) on the present \( m \) by (for \( f_1 \in L_\infty([0, 1], \lambda), f_2 \in L_\infty([0, 1], \mu) \)):

\[
[(\xi'(f_{1,2}))(x, y) = [(\xi f_1)(x)f_2(y), x, y \in [0, 1]],
\]

where \( f_{1,2}(x, y) = f_1(x)f_2(y), x, y \in [0, 1] \). Then by exactly the same reasons as in III, the abelian algebra \( E = (\xi')^\circ \otimes 1 \) is thick in \( \alpha \) with \( E^c = m \), and is of deficiency type \( I_\infty \). Furthermore the same argument used in V applies here and we conclude that \( R(E, G) = a \).
Exactly as in VIII above we can easily obtain a family of operators on $H$ with the desired properties.

X. Take $m$ to be $L_0 QR^2 \times [0,1]$, $\lambda \times \mu$) acting on $L_2 QR^2 \times [0,1]$, $\lambda \times \mu$), and $G = \{ \alpha (r_1, r_2, \delta) : r_1$ strictly positive rational, $r_2$ rational, $\delta$ a finite sequence of positive integers $\}$, where $\alpha (r_1, r_2, \delta)$ is the automorphism of $m$ defined by:

$$(\alpha (r_1, r_2, \delta)f)(x, y, z) = f(r_1^{-1}x, y - r_2, \beta_\delta(z)),$$

for all $f \in L_0 QR^2 \times [0,1]$, $\lambda \times \mu$), and all $(x, y, z) \in IR^2 \times [0,1]$. (The $\mu$ and $\beta_\delta$ are as those in VIII above, and $\lambda$ is the Lebesque measure on $IR^2$.) Then as in VIII we see that $a$ is a factor of type III on the separable Hilbert space $H$.

Fix a positive irrational number $m$ and let $K_1 = \{ \alpha_r : r$ rational $\}$, where $\alpha_r$ is the automorphism of $m$ defined by:

$$(\alpha_r f)(x, y, z) = f(x - r, y - mr, z)$$

for all $x, y \in IR$, $z \in [0,1]$ and $f \in L_0 QR^2 \times [0,1], \lambda \times \mu$). Then it follows for reasons similar to those in VII that $E'(\alpha (r_1, r_1, \delta), K_1) = 0$ for every $(r_1, r_2, \delta)$ distinct from the identity. Therefore $E = K_1 \Theta 1$ is thick in $a$ with $E^c = m$. As in VIII we see that $R(E, G) = a$. Now we use Bure's argument to determine the deficiency type of $E$. By rotation
of axes there is an isomorphism of \( m \) with \( m_1 \otimes m_2 \) taking \( K_1 \) onto 
\( 1 \otimes H_2 \), where \( m_1 \) is \( L_\infty(\mathbb{R} \times [0, 1], \lambda \times \mu) \) acting on \( L_2(\mathbb{R} \times [0, 1], \lambda \times \mu) \), \( m_2 \) is \( L_\infty(\mathbb{R}, \lambda) \) acting on \( L_2(\mathbb{R}, \lambda) \) (\( \lambda \) being the Lebesgue measure on \( \mathbb{R} \)), and \( H_2 \) is an ergodic \( m_2 \)-group. It then follows from Lemmas 9.1.8, 9.1.7 that \( E \) is of deficiency type III.

For each real number \( s \) let \( \alpha_s \) be the automorphism of \( m \) defined by:

\[
[\alpha_s f](x, y, z) = f(x - s, y - s, z), \quad x, y \in \mathbb{R}, z \in [0, 1],
\]

where \( f \in L_\infty(\mathbb{R}^2 \times [0, 1], \lambda \times \mu) \). Let \( S \) (resp. \( T \)) be a self-adjoint single generator of the abelian \( K_1^0 \) (resp. \( R(U_g \otimes V_g : g \in G) \)). Then each \( \alpha_s(S) \) is a self-adjoint single generator of \( K_1^0 \) since \( \alpha_s \) commutes with each \( \alpha_r \in K_1 \). Let \( A_s \) be the operator on \( H \) such that \( \text{Re} \ A_s = \alpha_s(S) \otimes 1 \) and \( \text{Im} \ A_s = T \). Then the family \( (A_s)_{s \in \mathbb{R}} \) has the desired properties.

So we complete the proof of Theorem 9.1.
Appendix

We have seen in §5 that $G'$ plays an important role in the study of $A(m, a; n)$. In this appendix we shall compute $G'$ when $m = \otimes_{i \in I} (m_i, \omega_i)$ and $G = \bigotimes_{i \in I} (G_i, \omega_i)$ (where each $m_i$ is an abelian $W^*$-algebra, $\omega_i$ is a normal state on $m_i$ with $\omega_i(1) = 1$, and each $G_i$ is an $m_i$-group). These notations will be explained later (cf. Definition A.1 below). The main result below (Theorem A.4) generalizes those results (Propositions 7.1, 7.2, 7.3, 7.4) we obtained in §7.

In order to have a set of standard notations let us briefly recall von Neumann's (infinite) tensor product of a family of von Neumann algebras. Let $I$ be an arbitrary index set, and for each $i \in I$, let $a_i$ be a von Neumann algebra acting on the Hilbert space $H_i$, and let $x_i \in H_i$ with $\|x_i\| = 1$. Then by von Neumann's construction (see [12]) we have the tensor product $H = \bigotimes_{i \in I} (H_i, x_i)$ (in von Neumann's terminology, incomplete direct product) of $(H_i)_{i \in I}$ with respect to $(x_i)_{i \in I}$. There exist canonical isomorphisms $A_i \mapsto \overline{A_i}$ from $L(H_i)$ into $L(H)$. The tensor product $\otimes_{i \in I} (a_i, x_i)$ of $(a_i)_{i \in I}$ with respect to $(x_i)_{i \in I}$ is defined to be the von Neumann algebra on $H$ generated by the set $\{ \overline{A_i} : i \in I, A_i \in a_i \}$.

We now introduce the following definitions.

Definition A.1 [6] Let $a$ be a $W^*$-algebra.

(i) A representation $\phi$ of $a$ on a Hilbert space $K$ is an
isomorphism of $a$ onto a von Neumann algebra acting on $K$.

(ii) For a representation $\phi$ of $a$ (on $K$), and a normal state $\mu$ on $a$:

$$S(\phi, \mu) = \{ x \in K : (\phi(A)x|x) = \mu(A) \text{ for all } A \in a \} .$$

(iii) For normal states $\mu, \nu$ on $a$, define:

$$d_{\phi}(\mu, \nu) = \begin{cases} [\mu(1) + \nu(1)]^{1/2} & \text{if either } S(\phi, \mu) \text{ or } S(\phi, \nu) \text{ is empty,} \\ \inf \{ \| x - y \| : x \in S(\phi, \mu), y \in S(\phi, \nu) \} & \text{otherwise,} \end{cases}$$

for any representation $\phi$ of $a$, and define:

$$d(\mu, \nu) = \inf \{ d_{\phi}(\mu, \nu) : \phi \text{ a representation of } a \} .$$

(iv) Suppose that $I$ is an arbitrary index family, and suppose that for each $i \in I$, $a_i$ is a $\mathcal{W}$-algebra, $\omega_i$ is a normal state on $a_i$ with $\omega_i(1) = 1$. We say that a $\mathcal{W}$-algebra $a$, together with $(a_i)_{i \in I}$, where $a_i$ is an isomorphism from $a_i$ into $a$, is a tensor product of $a_i$ with respect to $\omega_i_{i \in I}$ if the following condition holds:

For every family $(\phi_i, x_i)_{i \in I}$ where $\phi_i$ is a representation of $a_i$ and $x_i \in S(\phi_i, \omega_i)$, there is an isomorphism $\phi$ of $a$ onto $\prod_{i \in I}(\phi_i(a_i), x_i)$ with

$$\phi(a_i(A_i)) = \phi_i(A_i)$$

for all $A_i \in a_i$ and for all $i \in I$.

It has been proved [6] that the tensor product $a$ of $(a_i)_{i \in I}$ with respect to $(\omega_i)_{i \in I}$ exists and is unique up to isomorphism preserving
the injections \((a_i)_{i \in I}\). We write \(a = \prod_{i \in I} (a_i, \omega_i)\).

(v) Let \(a_i\), with canonical injections \((a_i)_{i \in I}\), be the tensor product of \((a_i)_{i \in I}\) with respect to \((\omega_i)_{i \in I}\). Let \(g_i\) be an automorphism of \(a_i\). If there is an automorphism \(g\) of \(a\) such that

\[ ga_i = a_i g_i \]

for all \(i \in I\), we write \(g = \prod_{i \in I} g_i\), and say that \(\prod_{i \in I} g_i\) exists on \(a\).

Note that \(\prod_{i \in I} g_i\), if it exists, is uniquely determined by the \((g_i)_{i \in I}\).

(vi) For each \(i \in I\) let \(a_i\) be a \(W^*\)-algebra, \(\omega_i\) a normal state of \(a_i\) with \(\omega_i(1) = 1\), and \(G_i\) a group of automorphisms of \(a_i\). Define:

\[ \prod_{i \in I} (a_i, \omega_i) = \{ \alpha \in A(\prod_{i \in I} (a_i, \omega_i)) : \alpha = \prod_{i \in I} g_i \text{ for some } g_i \in \prod_{i \in I} G_i \} \]

and

\[ \bigcup_{i \in I} (a_i, \omega_i) = \{ \alpha \in A(\prod_{i \in I} (a_i, \omega_i)) : \alpha = \prod_{i \in I} g_i \text{ for some } (g_i)_{i \in I} \in \bigcup_{i \in I} G_i \} \]

where \(\prod_{i \in I} G_i\) and \(\bigcup_{i \in I} G_i\) are, respectively, the direct product and the weak direct product of the family \((G_i)_{i \in I}\) of groups.

It follows readily from the definition that \(\prod_{i \in I} (a_i, \omega_i)\) is a subgroup of \(A(\prod_{i \in I} (a_i, \omega_i))\), and that \(\bigcup_{i \in I} (G_i, \omega_i)\) is a subgroup of \(\prod_{i \in I} (G_i, \omega_i)\).

We shall need a special case of the following result from [6 and 16], e.g. when each \(a_i\) is abelian.
Lemma A.1 [6, 16] For each $i \in I$ let $a_i$ be a $W^\ast$-algebra, $\omega_i$ a normal state of $a_i$ with $\omega_i(1) = 1$, and $g_i$ an automorphism of $a_i$. Then $\varnothing g_i$ exists on $\varnothing(a_i, \omega_i)$ if and only if

$$\sum_{i \in I} [d(\omega_i, \omega_i g_i)]^2 < \infty.$$ 

Corollary A.2 With the notation of Lemma A.1 we have

$$\varnothing_\mathcal{I}(G_i, \omega_i) = \left\{ \varnothing g_i : (g_i)_{i \in I} \in \bigsqcup_i G_i \text{ with } \sum_{i \in I} [d(\omega_i, \omega_i g_i)]^2 < \infty \right\},$$

and

$$\bigsqcup_i (G_i, \omega_i) = \left\{ \varnothing g_i : (g_i)_{i \in I} \in \bigsqcup_i G_i \right\}.$$ 

Furthermore if $I = \{1, 2\}$ then $\varnothing_\mathcal{I}(a_i, \omega_i)$ can be identified with $a_1 \varnothing a_2$, and $\varnothing_\mathcal{I}(G_i, \omega_i) = \bigsqcup_i (G_i, \omega_i) = G_1 \varnothing G_2$ under the identification.

We shall also need the following result from [5].

Lemma A.2 [5]. Let $m, n$ be abelian $W^\ast$-algebras, and $H$ an ergodic $n$-group. Then

$$\left\{ A \in m \varnothing n : (1 \varnothing h)(A) = A \text{ for all } h \in H \right\} = m \varnothing n.$$ 

Proposition A.3 Suppose that $I$ is an arbitrary index set, and that for each $i \in I$, $m_i$ is an abelian $W^\ast$-algebra, $\omega_i$ a normal state of $m_i$ with $\omega_i(1) = 1$, and $G_i$ an ergodic $m_i$-group. Then $\bigsqcup_i (G_i, \omega_i)$ is ergodic on $\varnothing_\mathcal{I}(m_i, \omega_i)$. 
Proof. Let $\phi_j(j \in I)$ be a representation of $m_j$ on $H_j$ with $x_j \in S(\phi_j, \omega_j)$. Let $\alpha_j$ be the canonical injection of $L(H_j)$ into $\Theta_i(L(H_i), x_i)$. Let $G = \bigotimes_{i \in I} (G_i, \omega_i)$ act on $\Theta_i(\phi_i(m_i), x_i)$. For each fixed $j \in I$, let

$$n_j = \bigotimes_{i \in I \setminus \{j\}} (\phi_i(m_i), x_i).$$

Let $G^\circ = \{ A \in \bigotimes_{i \in I} (\phi_i(m_i), x_i) : g(A) = A \text{ for all } g \in G \}$. Then by Lemma A.2, $G^\circ \subseteq \Theta \Theta n_j$ so that $(G^\circ)' = L(H_j)$. It follows that $(G^\circ)' \supseteq \bigotimes_{i \in I} (L(H_i), x_i)$. By a well-known theorem (see [4] or [12]),

$$\bigotimes_{i \in I} (L(H_i), x_i) = L(\bigotimes_{i \in I} (H_i, x_i)),$$

so we have $(G^\circ)' = L(\bigotimes_{i \in I} (H_i, x_i))$, i.e. $G^\circ = \mathcal{C}$, and $G$ is ergodic.

**Theorem A.4** Suppose that for each $i \in I$, $m_i$ is an abelian $W^*$-algebra, $\omega_i$ a normal state on $m_i$ with $\omega_i(1) = 1$, and $G_i$ an ergodic $m_i$-group. Then

$$\bigotimes_{i \in I} (G_i, \omega_i)' = \Theta_i(G_i, \omega_i)' = \Theta_i(G_i', \omega_i).$$

Proof. Let $m$ be tensor product of $(m_i)_{i \in I}$ with respect to $(\omega_i)_{i \in I}$, and for each $i \in I$, let $\alpha_i$ be the canonical injection from $m_i$ into $m$. Suppose $h \in (\bigotimes_{i \in I} (G_i, \omega_i))'$, i.e. $h$ is an automorphism of $m$ with $h \circ (\bigotimes_{i \in I} g_i) = (\bigotimes_{i \in I} g_i) \circ h$ for each $(g_i)_{i \in I} \in \bigotimes_{i \in I} G_i$. Let $M_j \in m_j$ be arbitrary. Then for any $(g_j^i)_{i \in I} \in \bigotimes_{i \in I} G_i$ with $g_j^i = e_j$ the identity of $G_j$:
(\Theta e^j)_{\left( h \circ \alpha_j \right)(M_j)} = h[(\Theta e^j)_{\alpha_j(M_j)}]

= h[(\alpha_j \circ e^j)(M_j)]

= (h \circ \alpha_j)(M_j),

i.e. \((h \circ \alpha_j)(M_j)\) is a fixed point of \(\Theta e^j \in \bigsqcup_{\mathcal{I}} G_i\) with \(e^j = e_j\).

Now it is known (cf. [12], Theorem VI) that there is an isomorphism \(\Lambda\) from \(\Theta e^j(m_i, \omega_i)\) onto \(m_j \Theta_{\left( \mathcal{I} \setminus \{j\} \right)}(m_i, \omega_i)\) such that

\[(\Lambda \circ \alpha_j)(M_j) = \overline{M_j} \quad \text{for any } M_j \in m_j,\]

where \(M_j \mapsto \overline{M_j}\) is the canonical injection from \(m_j\) into \(m_j \Theta_{\left( \mathcal{I} \setminus \{j\} \right)}(m_i, \omega_i)\).

But then we have for any \(e^j \in \bigsqcup_{\mathcal{I}} G_i\) with \(e_j = e_j:\)

\[\left[ 1 \Theta_{\left( \mathcal{I} \setminus \{j\} \right)} e^j \right] \circ \Lambda = \Lambda \circ \Theta e^j ,\]

so that \(\Lambda[(h \circ \alpha_j)(M_j)]\) is fixed by any such \(1 \Theta_{\left( \mathcal{I} \setminus \{j\} \right)} e^j\). Now by

Lemma A.3 \(\bigsqcup_{\mathcal{I} \setminus \{j\}} (G_i, \omega_i)\) is ergodic on \(m_j \Theta_{\left( \mathcal{I} \setminus \{j\} \right)}(m_i, \omega_i)\), so by Lemma A.2,

\[\Lambda[(h \circ \alpha_j)(M_j)] \in m_j \Theta C .\]

Thus \((h \circ \alpha_j)(M_j) \in \Lambda^{-1}(\overline{M_j}) = \alpha_j(m_j)\). This shows that
\[ h[\alpha_j(m_j)] = \alpha_j(m_j) ; \]

since \( h^{-1} \in \prod_{i \in I} (G_i, \omega_i) \)' also we have

\[ h[\alpha_j(m_j)] = \alpha_j(m_j) . \]

Thus we may put \( h_j = \alpha_j^{-1} \circ h \circ \alpha_j \). Then \( h_j \) is an automorphism of \( m_j \) such that

\[ h \circ \alpha_j = \alpha_j \circ h_j , \]

and that for any \( g_j \in G_j \),

\[ h_j \circ g_j = g_j \circ h_j \]

by a direct simple calculation. Therefore we see that

\[ \prod_{i \in I} (G_i, \omega_i) \)' \subseteq \prod_{i \in I} (G_i, \omega_i) \).

But obviously we have

\[ (\prod_{i \in I} (G_i, \omega_i))' \subseteq (\prod_{i \in I} (G_i, \omega_i))' , \]

and

\[ (\prod_{i \in I} (G_i, \omega_i))' \subseteq (\prod_{i \in I} (G_i, \omega_i))' . \]

Thus we have

\[ \prod_{i \in I} (G_i, \omega_i) \)' = (\prod_{i \in I} (G_i, \omega_i))' = \prod_{i \in I} (G_i, \omega_i) . \]
References


