COMPARISON AND OSCILLATION THEOREMS
FOR ELLIPTIC EQUATIONS

by

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We accept this thesis as conforming
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ABSTRACT

New comparison and Sturm-type theorems are established which enable us to extend known oscillation and non-oscillation criteria to: (1) non-self-adjoint operators, (2) quasi-linear operators, (3) fourth order operators of a type not previously considered.

Since the classical principle of Courant does not hold for some of the operators considered, the comparison theorems involve, in part, new estimates on the location of the smallest eigenvalue of the operators in question. A description of the behaviour of the eigenvalue as the domain is perturbed is also given for such operators by the use of Schauder's "a priori" estimates.

The Sturm-type theorems are proved by topological arguments and extended to quasi-linear as well as to non-self-adjoint operators.

The fourth order operators considered are of a type which does not yield forms identical to those arising in second order problems.

Some examples illustrating the theory are given.
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CHAPTER I

SECOND ORDER NON-SELF-ADJOINT EQUATIONS

1. Introduction.

In this chapter the second order linear elliptic operator $L$ defined by

$$Lu = - \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + \sum_{j=1}^{n} b_j D_ju + cu$$

will be considered in unbounded domains $\Omega$ of the $n$-dimensional Euclidean space $\mathbb{R}^n$. Our main results are a quite general Sturm theorem as well as a comparison theorem for operator (1.1). These will enable us to easily generalize oscillation theorems which are known for self-adjoint second order operators.

Points of $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n)$ and $D_i$ will signify differentiation with respect to $x_i$. The coefficients $a_{ij}$ will always be assumed real and of class $C^1(\Omega)$; furthermore the matrix $(a_{ij}(x))$ will be taken positive definite symmetric in $\Omega$. By $\mathcal{D}(L, \Omega)$ we shall mean the collection of all real functions of class $C^2(\Omega) \cap C(\bar{\Omega})$, where $\Omega$ denotes any subdomain of $\mathbb{R}$. The coefficients $b_j$ and $c$ will also be taken real.

Definition. A (classical) solution $u$ of $Lu = f$ in $\Omega$ is a function $u$ in $\mathcal{D}(L, \Omega)$ for which $Lu(x) = f(x)$ for every $x$. 

Definition A bounded domain $N \subset \mathbb{R}$ is a nodal domain of $L$ iff there exists a non-trivial solution $u$ of $Lu = 0$ in $N$ such that $u = 0$ on $\partial N$.

Notation $R_r = \mathbb{R} \cap \{x : x \in \mathbb{R}^n \text{ and } |x| > r\}$

$\bar{\Omega}$ = closure of $\Omega$ in the $\mathbb{R}^n$ topology.

Definition The operator (1.1) is oscillatory of type 1 (Osc 1) iff for every $r > 0$, $L$ has a nodal domain in $R_r$.

Definition The operator (1.1) is oscillatory of type 2 (Osc 2) iff for every $r > 0$, every solution of $Lu = 0$ in $R$ has a zero in $R_r$.

Definition The operator (1.1) is non-oscillatory iff there exists $r > 0$ such that $L$ has no nodal domains in $R_r$.

2. Relation Between the Two Types of Oscillation.

We shall now show that if $L$ is Osc 1, then it is in fact Osc 2 under mild conditions on the coefficients of $L$ and none at all on the nature of the boundaries of the regions involved. Specifically, we shall show that if $N$ is a nodal domain of $L$, then every solution of $Lu = 0$ in $N$ must vanish somewhere in $\bar{N}$. If the coefficients are continuous in $N$ and $L$ is uniformly elliptic there, i.e., there exist constants $m > 0$ and $k > 0$ such that
\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq m \sum_{i=1}^{n} \xi_i^2, \quad |a_{11}(x)| \leq k, \quad |b_1(x)| \leq k \]

for every \( \xi = (\xi_1, \ldots, \xi_n) \) and for every \( x \) in \( N \), our result follows immediately by methods identical to those of Protter and Weinberger [1].

**Proposition 1.1** Assume that \( L \) obeys the above conditions and that \( N \) is a nodal domain of the operator \( L \). Then every function \( v \) in \( \mathcal{D}(L,N) \) such that \( Lv = 0 \) in \( N \), must vanish somewhere in \( N \).

**Proof:** If this were not the case, we could find a function \( v \) in \( \mathcal{D}(L,N) \) such that \( Lv = 0 \) in \( N \) and, without loss of generality, \( v > 0 \) everywhere in \( N \). Let \( u \) be a non-trivial function such that \( Lu = 0 \) in \( N \) and \( u = 0 \) on \( \partial N \). We define a new function \( w \) by \( w = \frac{u}{v} \). Then clearly \( w \) is in \( \mathcal{D}(L,N) \) and again we may assume \( w > 0 \) somewhere in \( N \). Now

\[ L(vw) = L(u) = 0 \quad \text{in} \quad N, \]

but

\[ L(vw) = L(v)w - \sum_{i,j=1}^{n} D_i (a_{ij} D_j w)v + \sum_{j=1}^{n} (vb_j - 2 \sum_{i=1}^{n} a_{ij} D_i v) D_j w. \]

Hence

\[ - \sum_{i,j=1}^{n} D_i (a_{ij} D_j w) + \sum_{j=1}^{n} (b_j - 2 \sum_{i=1}^{n} a_{ij} \frac{D_i v}{v}) D_j w = 0 \quad \text{in} \quad N. \]

But this violates the classical Hopf maximum principle [2, p. 150] as \( w = 0 \) on \( \partial N \). The contradiction proves that \( v \) must
vanish somewhere in $\bar{N}$.

We shall show more generally:

**Theorem 1.2** Assume that $N$ is any nodal domain of $L$ in $\mathbb{R}$. Furthermore assume that the coefficients $c$ and $b_j$ are bounded in $N$ and that one of the $a_{ii}$ satisfies $a_{ii} > \gamma > 0$ in $N$, for some constant $\gamma$. Then every solution of $Lw = 0$ in $N$ must vanish somewhere in $\bar{N}$.

Note that if $L$ is uniformly elliptic in $N$, then in fact $a_{ii}(x) \geq m > 0$ for every $i = 1, \ldots, n$ and for every $x$ in $N$, as may be seen by choosing a suitable $\xi$ in the definition of uniform ellipticity. This shows that Theorem 1.2 is in fact an extension of Proposition 1.1.

To prove Theorem 1.2, the following propositions will first be shown:

**Proposition 1.3** Assume $w$ is a function in $C^2(N)$ and $L(w) > 0$ in $N$. Then the points of $N$ where $w = 0$ cannot be minima of $w$.

**Proof:** If $w(x_o) = 0$ is a minimum of $w$ in $N$, then
\[
\sum_{i,j=1}^{n} a_{ij}(x_o) D_{ij}w(x_o) > 0
\]

Since the matrix $(a_{ij}(x_o))$ is positive definite, we may assume that at $x_o$ it is diagonal with positive elements. Then
\[ \sum_{i=1}^{n} a_{ii}(x_0) D_{ii} u(x_0) < 0 \]

which is impossible, since \( D_{ii} u(x_0) \geq 0 \) at the minimum \( x_0 \)
for every \( i = 1, \ldots, n \).

The next Lemma, the key part in the proof of Theorem 1.2, will be proved by the use of the classical Schauder "continuity" method [3].

**Lemma 1.4** Assume that \( u, v, w \) are functions in \( S(L,N) \) and that:

(a) \( Lu \geq 0 \), \( u \) somewhere negative in \( N \), \( u = 0 \) on \( \partial N \)

(b) \( w > 0 \) in \( \bar{N} \), \( Lw > 0 \) in \( N \)

(c) \( Lv \geq 0 \) in \( N \), \( v \) somewhere positive in \( N \).

Then \( v \) must vanish somewhere in \( \bar{N} \).

**Proof:** Assume not. Then, since \( w > 0 \) in \( \bar{N} \) and \( u \) is somewhere negative, there exists \( x_0 \) in \( N \) and \( \alpha > 0 \) such that

\[ \alpha w(x_0) + u(x_0) = 0 \]

and \( L(\alpha w + u) > 0 \) in \( N \).

We now define a family of functions as follows:

\[ w_t(x) = \alpha w(x) + tu(x) + (1-t)v(x) \text{ for } x \in \bar{N} \text{ and } t \in [0,1]. \]

Let \( T \) denote the set:

\[ T = \{ t : w_t \text{ vanishes somewhere in } \bar{N} \} \cap [0,1]. \]
We shall now show that $T$ is a non-empty set which is both open and closed in the induced topology on $[0,1]$. By the connectedness of $[0,1]$ this will mean that in fact $T = [0,1]$ and hence $0 \in T$, which is clearly impossible as $\omega + \nu > 0$ in $\bar{N}$.

First, $T$ is non-empty since $1 \in T$. To prove $T$ is closed, let $\{t_i\}_{i=1}^\infty$ be a sequence in $T$ with $\lim t_i = t_0$. Then there exist $x_i \in \bar{N}$ such that $w_{t_i}(x_i) = 0$. As $\bar{N}$ is compact, we may assume, without loss of generality, that there exists a point $x_0 \in \bar{N}$ such that $\lim x_i = x_0$. Now, we also have the estimate:

$$|w_{t_i}(x) - w_{t_0}(x)| < |t_i - t_0| \left( \sup_{x \in \bar{N}} \{|u(x)| + |v(x)|\} \right)$$

and therefore $\{w_{t_i}\}_{i=1}^\infty$ converges uniformly in $\bar{N}$ to $w_{t_0}$.

In the inequality:

$$|w_{t_0}(x_0)| \leq |w_{t_0}(x_0) - w_{t_0}(x_i)| + |w_{t_0}(x_i) - w_{t_i}(x_i)| + |w_{t_i}(x_i)|$$

the first term on the right hand side tends to zero with $i$ by uniform continuity, and the second tends to zero by uniform convergence. Hence $w_{t_0}(x_0) = 0$. Finally, to prove $T$ is open, let $t_0 \in T$ and let $w_{t_0}$ have a zero at $x_0$ in $\bar{N}$.

Clearly $w_{t_0}$ must have a zero in $\bar{N}$, as it is positive on $\partial N$. Since $L(w_{t_0}) > 0$ and $w_{t_0} > 0$ on $\partial N$, by Proposition 1.3 $w_{t_0}$ must have both positive and negative values in $N$. Now,
Hence \( w_t \) also will have both positive and negative values in \( \mathbb{N} \) if \( |t - t_0| \) is taken small enough. By the connectedness of \( \mathbb{N} \), \( w_t \) will have a zero in \( \mathbb{N} \). We have thus arrived at the contradiction \( aw + v = 0 \) somewhere in \( \bar{\mathbb{N}} \). Our original assumption must be false and therefore \( v = 0 \) somewhere in \( \bar{\mathbb{N}} \).

**Corollary 1.5**  Assume that there exists a function \( f \) in \( \mathcal{S}(L,\mathbb{N}) \) such that \( Lf > 0 \) (or \( < 0 \)) in \( \mathbb{N} \). If \( u \) and \( v \) satisfy the conditions of Lemma 1.4, then \( v \) must vanish somewhere in \( \mathbb{N} \).

**Proof:** Again assume not. Then there exists a constant \( \alpha \) such that \( v > \alpha > 0 \) in \( \bar{\mathbb{N}} \). Choose a constant \( \beta \) such that \( \beta > 0 \) and \( \beta \left( \max_{x \in \bar{\mathbb{N}}} |f| \right) < \alpha \). Define a new function \( w \) by \( w(x) = v(x) + \beta f(x) \). Clearly \( Lw > 0 \) and \( w > 0 \) in \( \bar{\mathbb{N}} \).

By Lemma 1.4, \( v \) must be zero somewhere in \( \bar{\mathbb{N}} \), and this contradiction proves the corollary. If \( Lf < 0 \) in \( \mathbb{N} \), look at \(-f\).

We are now in a position to prove Theorem 1.2:

**Proof of Theorem 1.2:** It will now be sufficient to construct a function \( f \) such that \( Lf > 0 \) in \( \mathbb{N} \), \( f \in \mathcal{S}(L,\mathbb{N}) \). For simplicity and without loss of generality we may assume that \( \bar{\mathbb{N}} \) lies in the part of \( \mathbb{R}^n \) where \( x_i > 0 \). Let \( f(x) = -x_i^m \), \( m \) a positive integer to be chosen later. We have, for \( x \in \mathbb{N} \),
\[ Lf(x) = x_i^{m-2} (m^2a_{ii} + m \text{ (bounded terms)}) \]

and clearly \( f \) belongs to \( \mathfrak{D}(L,N) \) for every value of \( m \).
Since \( a_{ii} \) is assumed bounded away from zero in \( \bar{N} \), \( Lf > 0 \) for sufficiently large \( m \).

Remarks  The above results also hold if the matrix \( (a_{ij}(x)) \) is merely assumed non-negative in \( \bar{N} \), as long as one of the \( a_{ii} \) is positive in \( \bar{N} \). Also it will be sufficient to assume \( L \) elliptic in any domain \( R \) such that \( \bar{N} \subset R \), as then \( a_{ii} > 0 \) in \( \bar{N} \) for every \( i = 1, \ldots, n \) as a trivial consequence of the definition of ellipticity. Some of these results could possibly be obtained by using a more general form of the maximum principle.

Corollary 1.6  Assume that \( u,v \) are functions in \( \mathfrak{D}(L,N) \) and that in \( N \) we have \( Lu \geq f, L v \geq g \) with \( f \) and \( g \) bounded functions. Furthermore if:

(i) \( u = 0 \) on \( \partial N \), \( u \) somewhere negative in \( N \)

(ii) \( v \) somewhere positive in \( N \)

(iii) \( A(u,v) = \begin{vmatrix} f & g \\ u & v \end{vmatrix} \geq 0 \) in \( \bar{N} \)

(iv) The coefficients of \( L \) satisfy the conditions of Theorem 1.2.

Then \( v \) must vanish somewhere in \( \bar{N} \).

Proof:  If \( v \) were always positive in \( \bar{N} \), we could define a new elliptic operator \( L_1 \), by
\( L_1w = (L - \frac{\xi}{v})w \), \( w \in \mathcal{D}(L,N) \)

Then

\[
L_1u = (L - \frac{\xi}{v})u \geq f - g \frac{u}{v} = \frac{1}{v} \left| \frac{f}{u} \frac{g}{v} \right| \geq 0
\]

and \( L_1v \geq 0 \) in \( N \). But as a consequence of Lemma 1.4 and Theorem 1.2, \( v \) must vanish somewhere in \( \bar{N} \). Contradiction.

**Corollary 1.7** Assume that \( u,v \) are functions in \( \mathcal{D}(L,N) \) and \( Lu = 0 \) in \( N \), \( u = 0 \) on \( \partial N \), \( u \) non-trivial. Let 
\( L_2v = Lv + cv = 0 \) in \( N \) and \( c \leq 0 \) bounded in \( \bar{N} \). Then \( v \) must vanish somewhere in \( \bar{N} \) if the coefficients of \( L \) satisfy the conditions of Theorem 1.2.

**Proof:** We may assume that \( u \) is always negative in \( N \), for if it were not, we could consider a subdomain of \( N \) where it was. If \( v \) is never zero we can assume \( v > 0 \) in \( \bar{N} \). Then

\[
Lu = 0 \ , \ Lv = -cv \ \text{in} \ \bar{N}
\]

and

\[
\Delta(u,v) = \begin{vmatrix} 0 & -cv \\ u & v \end{vmatrix} = cvu \geq 0 \ \text{in} \ \bar{N}
\]

Hence \( v \) must vanish somewhere in \( \bar{N} \) by Corollary 1.6. Contradiction.

In view of the above results, it seems natural to concentrate on determining some conditions for the Osc 1 behaviour of \( L \).
3. A Comparison Theorem for $L$.

It is well known that oscillation results may be obtained for an operator $L$ if we can relate $L$ to another operator $L_1$ with suitable oscillatory behaviour. We shall do this under the computationally simplifying assumption that the coefficients of $L$ and the boundaries of the domains involved are sufficiently smooth. Specifically we shall assume that every bounded domain in question has a $C^\infty$ boundary [4, p. 128] and that the coefficients of $L$ are of class $C^\omega$ in a convex set containing the domain. If $G$ denotes any such domain, then we know [4, p. 131] that generalized solutions of $Lu = f$ in $G$, $u = 0$ on $\partial G$, are classical (in fact $C^\infty(\overline{G})$) for $f$ of class $C^\omega(\overline{G})$ and that the generalized operator $(L + \lambda)^{-1}$ is completely continuous in the $L^2$ norm as a map from $L^2(G)$ to $H^1_0(G)$, for $\lambda$ sufficiently large [2, p. 199]. Furthermore, we shall assume $L$ to be uniformly elliptic.

**Definition** An operator $A$ will be called positive iff $f \in \mathcal{D}(A)$ and $f \geq 0$ (a.e.) implies that $Af \geq 0$ (a.e).

**Proposition 1.8** For $\lambda$ sufficiently large, the generalized operator $(L + \lambda)^{-1}$ is positive.

**Proof:** First assume that $f \in C^\omega(\overline{G})$, $f \geq 0$. Then $u = (L + \lambda)^{-1}f$ iff $(L + \lambda)u = f$, $u = 0$ on $\partial G$ in the classical sense. It follows that $u \geq 0$ for $\lambda$ sufficiently large [1]. If $f \geq 0$ (a.e) and $f$ is an arbitrary member of
we choose a sequence \( \{f_n\}_{n=1}^{\infty} \) of \( \mathcal{C}^\infty(\bar{G}) \) functions such that \( \lim f_n = f \) in the \( L^2(G) \) norm and \( f_n \geq 0 \). Hence by continuity, \( \lim (L + \lambda)^{-1} f_n = (L + \lambda)^{-1} f \) in the \( L^2(G) \) norm and therefore \( (L + \lambda)^{-1} f \geq 0 \) (a.e).

From now on we shall always assume that when the operator \( (L + \lambda)^{-1} \) is considered, \( \lambda \) has been chosen sufficiently large so that the above results hold. By the smoothness assumption on the coefficients of \( L \) we are assured of the existence of a \( \mathcal{C}^\infty \) fundamental solution for \( (L + \lambda) \), [2, p. 214]. Since, furthermore, the problem \( (L + \lambda)u = f \), \( u = 0 \) on \( \partial G \) has classical solutions for every \( f \) of class \( \mathcal{C}^\infty(\bar{G}) \), a Green's function \( K(x,y) \) may be constructed for \( (L + \lambda) \). Clearly \( K(x,y) \) must be non-negative, otherwise \( (L + \lambda)^{-1} \) could not be a positive operator. This is a fact which also follows from the maximum principle.

Consider now the operator \( \mathcal{A} : L^p(G) \to C(\bar{G}) \) defined by

\[
\mathcal{A}(f) = \int_{\bar{G}} K(x,y) f(y) dy
\]

where \( p > n \). Then [5, p. 259] \( \mathcal{A} \) is a completely continuous positive operator which has a positive eigenvalue which is simple, bigger than the absolute value of any other eigenvalue, and whose corresponding eigenvector is non-negative in \( G \). Furthermore, \( \mathcal{A} \) has no other linearly independent non-negative eigenvector.

From this we may conclude that the operator \( L \) has a
simple real eigenvalue $\lambda_0$, with a non-negative eigenvector, such that all other real eigenvalues are bigger than $\lambda_0$. Furthermore $L$ has no other linearly independent non-negative eigenvector.

**Definition** The eigenvalue $\lambda_0$ will be called the **smallest eigenvalue** of $L$.

**Proposition 1.9** The eigenvector $u_0$ associated with $\lambda_0$ is positive in $G$.

**Proof:** We have $(L - \lambda_0)u_0 = 0$ in $G$. If $u_0$ were identically zero on a non-empty subset $S$ of $G$, then not all of its partial derivatives could be zero at all points of $S$, as follows by a trivial modification of a result of Kreith [6].

Our next concern will be to determine bounds for $\lambda_0$ and its behaviour as the domain $G$ is perturbed.

**Proposition 1.10.[1]** Let $w$ be any smooth function, $w > 0$ in $G$ and $w = 0$ on $\partial G$. Then

$$\lambda_0 \leq \sup_{x \in G} \left[ \frac{Lw(x)}{w(x)} \right]$$

**Proof:** If not, we could find a function $w$ such that

$$Lw \not< \lambda_0 w \text{ in } G \quad w = 0 \text{ on } \partial G \quad \text{or}$$
(L - \lambda_0)w < 0 \text{ in } G, w = 0 \text{ on } \partial G

Since the extended operator \((L + \lambda)^{-1}\) is positive and completely continuous, we can conclude by a Theorem of Krein and Rutman [7, p. 65] that there exists a function \(v > 0\) in \(G\) such that

\[(L^* - \lambda_0)v = 0 \text{ in } G, \quad v = 0 \text{ on } \partial G\]

where \(L^*\) is the formal adjoint of \(L\). But this implies

\[0 > ((L - \lambda_0)w,v) = (w,(L^* - \lambda_0)v) = 0\]

which is impossible.

**Proposition 1.11** \(\lambda_0 \geq \mu_0\), where \(\mu_0\) denotes the smallest eigenvalue of the formally self-adjoint operator \(\frac{L + L^*}{2}\) with zero boundary conditions.

**Proof:** Under the regularity assumptions at the beginning of this section, we are assured of the validity of Courant's Principle [8, Vol. 1, p. 398]. Therefore

\[\mu_0 = \inf_{u \in D} \frac{B(u,u)}{(u,u)}\]

where

\[B(u,u) = \sum_{i,j=1}^{n} a_{ij} D_{i,j} u + (c - \sum_{i=1}^{n} D_{1}(b_i)/2)u^2\]

\[D = \{u : u \in C(G), \text{ piecewise } C^1(G) \text{ and } u = 0 \text{ on } \partial G\}\]

Let \(v\) be the eigenvector corresponding to \(\lambda_0\). We may assume \(\|v\|^2 = (v,v) = 1\), and by the known regularity properties of
solutions of elliptic equations with smooth coefficients, [4,p.131]

\[ B(v,v) = \left( \frac{L + L^*}{2} v, v \right) = (Lv, v) = \lambda_0 \]

Hence \( \lambda_0 \geq \mu_0 \).

Proposition 1.11 may be sharpened as follows:

**Proposition 1.12** If \( \Sigma D_i(b_i) \) is never zero in \( G \), then \( \lambda_0 > \mu_0 \).

**Proof:** Let

\[
\frac{L + L^*}{2} u = \mu_0 u \quad \text{in } G \\
u = v = 0 \text{ on } \partial G, \quad u > 0, \quad v > 0 \text{ in } G.
\]

As stated above, \( \frac{L + L^*}{2} \) and \( L \) have only one positive linearly independent eigenvector each. Now \( u + v > 0 \) in \( G \), hence if \( u + v \neq cu \) for some constant \( c \), \( u + v \) cannot be an eigenvector of \( \frac{L + L^*}{2} \). Therefore if \( u + v \neq cu \) we have

\[
\mu_0 \int_G (u+v)^2 < \int_G (u+v) L(u+v) = \mu_0 \int_G u^2 + \int_G vLu + \lambda_0 \int_G uv + \lambda_0 \int_G v^2
\]

and

\[
\mu_0 \int_G (u-v)^2 < \int_G (u-v) L(u-v) = \mu_0 \int_G u^2 - \int_G vLu - \lambda_0 \int_G uv + \lambda_0 \int_G v^2
\]

adding we obtain
15.

\[ \mu_o \left( \int_G u^2 + v^2 \right) < \mu_o \int_G u^2 + \lambda_o \int_G v^2 \]

or \( \mu_o < \lambda_o \).

Hence it will be sufficient to show that if \( \sum D_1(b_i) \) is never zero in \( G \) then \( v \neq cu \) in \( G \). Now, if \( \sum D_1(b_i) \neq 0 \) in \( G \) and \( v = cu \) then

\[ Lu = \lambda_o u, \quad \frac{L + L^*}{2} u = \mu_o u \]

and

\[ (u, Lu) = (u, \frac{L + L^*}{2} u) \quad \text{or} \quad \lambda_o = \mu_o. \]

Therefore

\[ \sum_{j=1}^{n} b_j D_j u + \frac{1}{2} \sum_{j=1}^{n} D_j(b_j)u = (\lambda_o - \mu_o)u = 0 \]

or

\[ 2 \sum_{j=1}^{n} b_j D_j u = - \sum_{j=1}^{n} D_j(b_j)u \]

Since \( u \) takes on its maximum at a point inside \( G \), this is impossible.

Proposition 1.11 and 1.12 enable us to conclude:

**Corollary 1.13** Let \( \{G_t\}_{t=1}^{\infty} \) be a family of domains whose diameters tend to zero with \( t \). Then the smallest eigenvalue \( \lambda_o(t) \) corresponding to \( G_t \) tends to \( +\infty \) with \( t \).
Proof: We know [9, p.7] that this is the case for the eigenvalues $\mu_0(t)$ of the operator $\frac{L + L^*}{2}$. Corollary 1.13 then follows from the fact that $\lambda_0(t) \geq \mu_0(t)$ for every $t$.

Our next objective is to establish a suitable upper bound for $\lambda_0$.

**Definition**

$$h(x) = -\sum_{i=1}^{n} \frac{b_i(x)b_i^*(x)}{2 \det(a_{ij}(x))}$$

where $b_i^*$ is the cofactor of $\frac{b_i}{2}$ in the matrix

$$\begin{pmatrix}
(a_{ij}(x)) & b_1 \\
. & \ddots \\
. & . & b_n \\
\frac{b_1}{2}, \ldots, \frac{b_n}{2} & h
\end{pmatrix}$$

**Definition**

$$F[u] = \sum_{i,j=1}^{n} a_{ij}D_iuD_ju + u \sum_{i=1}^{n} b_iD_iu + (h + c)u^2$$

$$M[u] = \int_{G} F[u]$$

for every function $u \in D$.

The following Lemma is known [10, p. 192].
Lemma 1.14  If there exists a sufficiently smooth function \( u \), \( u \neq 0 \), \( u = 0 \) on \( \partial G \), such that \( M[u] < 0 \), then every solution \( v \) of \( Lv \geq 0 \), \( v \) somewhere positive, must vanish at a point in \( \bar{G} \).

From this we can prove:

Theorem 1.15  Assume that in \( G \) the smallest eigenvalue of \( \frac{L + L^*}{2} + h \) is non-positive. Then if \( D_1 \) is any smooth domain such that \( \bar{G} \subset D_1 \), the smallest eigenvalue for \( L \) in \( D_1 \) must be negative.

Proof:  Choose a smooth domain \( D_2 \) such that
\[
\bar{G} \subset D_2 \subset \bar{D}_2 \subset D_1.
\]

Then in \( D_2 \) the smallest eigenvalue of the formally self-adjoint operator \( \frac{L + L^*}{2} + h \) must be negative by the monotonicity property of eigenvalues (see Lemma 1.18). Therefore there exists a function \( w \) in \( \mathcal{D}(L,D_2) \) such that \( w > 0 \) in \( D_2 \), \( w = 0 \) on \( \partial D_2 \) and \( (\frac{L + L^*}{2} + h)w < 0 \) in \( D_2 \) or
\[
M[w] = \int_{D_2} (wLw + hw^2) < 0
\]

Now let \( \lambda_0 \) be the smallest eigenvalue of \( L \) in \( D_1 \) and \( v \) its associated positive eigenfunction. If \( \lambda_0 \geq 0 \), then \( Lv \geq 0 \) in \( D_2 \) and \( v > 0 \) in \( \bar{D}_2 \), contradicting Lemma 1.14.
Corollary 1.16. If \( \mu_0 \) denotes the smallest eigenvalue of \( \frac{L + L^*}{2} + h \) in a domain \( G \), then \( L \) has an eigenvalue \( < \mu_0 \) in any smooth domain \( D_1 \) such that \( \overline{G} \subset D_1 \).

A determination of the continuity of the smallest eigenvalue \( \lambda_0 \) of \( L \), as the domain varies in a reasonable fashion, will enable us to say more than what is stated in Corollary 1.16. We first have:

Proposition 1.17. Let \( w \) be any function \( \mathcal{D}(L,G) \) such that \( w > 0 \) in \( G \) and \( w = 0 \) on \( \partial G \). Then

\[
\lambda_0 = \sup_{x \in G} \left[ \frac{Lw(x)}{w(x)} \right]
\]

iff \( w = cu \) for some constant \( c \), where \( u \) is the eigenvector corresponding to \( \lambda_0 \).

Proof: By Proposition 1.10 we know that

\[
\lambda_0 \leq \sup_{x \in G} \left[ \frac{Lw(x)}{w(x)} \right].
\]

Clearly if \( w = cu \) then the equality will hold. Conversely if the equality holds, then \( (L - \lambda_0)w \leq 0 \). Again letting \( v \) be the positive eigenvector corresponding to \( L^* \) and \( \lambda_0 \), we have:

\[
((L - \lambda_0)w, v) = 0 \quad \text{i.e.} \quad (L - \lambda_0)w = 0 \quad (a.e.)
\]

Therefore, \( (L - \lambda_0)w = 0 \) in \( G \). Hence \( w \) is an eigenfunction of \( L \), but since \( \lambda_0 \) is simple, there exists a constant \( c \) such
Lemma 1.18  If \( \bar{G} \subset G \), \( G, G \) smooth domains, then the smallest eigenvalue of \( L \) for \( G \) strictly exceeds that for \( G \).

Proof: Let \( Lu = \lambda u \) in \( G \), \( u = 0 \) on \( \partial G \), \( u > 0 \) in \( G \)

\( Lv = \mu v \) in \( G \), \( v = 0 \) on \( \partial G \), \( v > 0 \) in \( G \).

Then in \( G \) we have, by a trivial calculation,

\[
L(\frac{u^2}{2v}) = \frac{uLu}{v} - \frac{u^2Lv}{2v^2} - v \sum_{i,j=1}^{n} a_{ij}D_i(u)D_j(u)
\]

or

\[
L(\frac{u^2}{2v}) \leq (\frac{u^2}{2v}) (2\lambda - \mu)
\]

By Proposition 1.17,

\[
\lambda < \sup_{x \in G} \left[ \frac{L(\frac{u^2}{2v})}{\frac{u^2}{2v}} \right] \leq 2\lambda - \mu
\]

Therefore \( \lambda > \mu \).

We will now consider the problem of the continuity of \( \lambda \) as the domain varies. It is clearly sufficient to consider the continuity of the eigenvalues of the operator \( L + \lambda \), where \( \lambda \) is a sufficiently large constant so that \( c(x) + \lambda \geq 0 \) for every \( x \). It will also be useful to introduce the following spaces and norms \([8, \text{Vol. II, p. 332}]\).

Definition \( C_{m+\alpha}(G) \) will denote the class of all functions \( u \)}
which have partial derivatives up to order $m$ which are continuous in $\bar{G}$ and all the $m$th partial derivatives satisfy a Hölder condition with exponent $\alpha$.

**Definition** For every $u \in C_{m+\alpha}(G)$ we define

$$||u||_m = \sum_{i=0}^{n} \left( \sup_{|\ell| = 1} \sup_{x \in \bar{G}} |D^{\ell}u(x)| \right)$$

where $\ell$ is an $n$-tuple $(\ell_1, \ldots, \ell_n)$ of non-negative integers, $|\ell| = \sum_{i=1}^{n} \ell_i$ and $D^{\ell}u$ signifies $D_{1}^{\ell_1}D_{2}^{\ell_2} \ldots D_{n}^{\ell_n}u$.

Denoting by $H_{\alpha}[D^{m}u]$ the smallest constant $K$ with the property that all the $m$th order derivatives of $u$ satisfy a Hölder condition in $\bar{G}$ with exponent $\alpha$ $(0 < \alpha < 1)$ and coefficient $K$, we define:

**Definition** $||u||_{m+\alpha} = ||u||_m + H_{\alpha}[D^{m}u]$.

Under the above regularity conditions we may assume that the Schauder "estimate to the boundary" [8, p. 335]

$$(1.2) \quad ||u||_{2+\alpha} \leq K_1(||u||_0 + ||f||_\alpha)$$

will hold for every $C_{2+\alpha}(G)$ solution of $Lu = f$ in $G$ with zero boundary values. Since the "c" coefficient of $L$ may be taken positive, we can invoke the maximum principle, to reduce (1.2) to

$$(1.3) \quad ||u||_{2+\alpha} \leq K_1(||f||_\alpha)$$
The constant \( K_1 \) depends only on \( G, \alpha \), the ellipticity constant of \( L \), and the bounds on the coefficients of \( L \), but not on the coefficients themselves.

We also find convenient to introduce the concept of a "strong barrier function" [8, Vol. II, p. 340].

**Definition** A strong barrier function \( w_Q \) corresponding to a point \( Q \) on \( \partial G \) is a function which is of class \( C^2(G) \cap C(\overline{G}) \), non-negative in \( G \), zero only at \( Q \) and satisfies \( L[w_Q] \geq 1 \) in \( G \).

**Theorem 1.19** Let \( G \) be any bounded domain such that
\[
G = \bigcup_{m=1}^{\infty} G_m, \quad \overline{G}_m \subset G_{m+1}, \quad \text{each } G_m \text{ convex with smooth boundary and } G_1 \text{ non-empty.}
\]
Assume that at each point \( Q \) on \( \partial G \) there exists a strong barrier function \( w_Q \). Let \( \{u_m, u_m^*\} \) be the smallest eigenvalue and associated normalized corresponding positive eigenvector for \( G_m \). Furthermore assume that a constant \( K_1 \) in Schauder's estimate (1.3) can be chosen independent of \( G_m \). Then the operator \( L \) has an eigenvalue \( \mu \) in \( G \) which is the limit of the \( u_m \). The limit of a subsequence of the \( u_m \), uniformly in the compacta of \( G \), will be an eigenvector corresponding to \( \mu \).

**Proof:** The convexity of \( G_m \) implies that all \( u_m \) are of class \( C^{2+\alpha}(G_m) \) for every \( \alpha \). Since \( G_1 \) is open and non-void it must contain a sphere. Also \( G \) is bounded so it may be placed
inside a sphere. In any case, by Lemma 1.18, \( \{u_m\}^\infty_1 \) is a monotone sequence bounded below. Hence we can find a number \( \mu \) such that \( \lim u_m = \mu \).

Now without loss of generality, assume \( \| u_m \|_\alpha (G_m) = 1 \) where \( \| \cdot \|_\alpha (G_m) \) indicates that the norm is to be taken over \( G_m \). Then,

\[
\| u_m \|_{2+\alpha}(G_m) \leq K^1 (\| Lu \|_\alpha (G_m))
\]

which obviously reduces to

\[
(1.4) \quad \| u_m \|_{2+\alpha}(G_m) \leq K^1 |u_m| \cdot \| u_m \|_\alpha (G_m) \leq K
\]

with \( K \) independent of \( m \). By equicontinuity we can find a subsequence of \( \{u_m\}^\infty_1 \), also called \( \{u_m\}^\infty_1 \), and a function \( u \) such that \( \lim u_m = u \) uniformly on compacta \( G \), together with their respective first and second partial derivatives. Hence if \( x \in G \), then \( \lim (Lu_m(x)) = Lu(x) \) and \( \lim (\mu u_m(x)) = \mu u(x) \). Therefore \( Lu = \mu u \) in \( G \). Note that \( u \geq 0 \) in \( G \) as each \( u_m \) is positive in \( G_m \). Once again by [6], we can state that either \( u > 0 \) in \( G \) or \( u \equiv 0 \) in \( G \). We claim that in fact \( u \equiv 0 \) is impossible. To see this, note that \( (1.4) \) implies that the first and second partials of each \( u_m \) are bounded above by \( K \) in absolute value. Hence for \( x, y \in G_m \) we have

\[
|u_m(x) - u_m(y)| = \left| \sum_{i=1}^n \frac{\partial u_m}{\partial x_i} (\theta) (x_i - y_i) \right| \leq Kn |x - y|
\]

where \( \theta \) denotes a point on the line between \( x \) and \( y \). Hence
Identically,

\[(1.6) \quad |D_i u_m(x)| \leq Kn \, |x - y| + |D_i u_m(y)| \quad \text{for every } i.\]

Also, \( \|u_m\|_{\alpha(G_m)} = 1 \) for every \( m \). Hence we can find a constant \( c_1 \) independent of \( m \) such that

\[\|u_m\|_{\alpha(G_m)} \leq c_1 \|u_m\|_1(G_m)\]

or,

\[\frac{1}{c_1} \leq \|u_m\|_1(G_m)\]

Now let \( G_N \) denote a subset of \( G \) such that given any point \( x \) of \( G \) we can find a point of \( G_N \) inside the sphere with centre \( x \) and radius \( \frac{1}{4c_1 Kn} \). Then for \( m \) sufficiently large, we have from \((1.5)\) that

\[\sup_{x \in G_m} |u_m(x)| \leq Kn \left( \frac{1}{4c_1 Kn} \right) + \sup_{y \in G_N} |u_m(y)|\]

and, from \((1.6)\)

\[\sup_{x \in G_m} |D_i u_m(x)| \leq Kn \left( \frac{1}{4c_1 Kn} \right) + \sup_{y \in G_N} |D_i u_m(y)|\]

Therefore

\[\|u_m\|_1(G_m) \leq \frac{1}{2c_1} + \|u_m\|_1(G_N)\]

And as a consequence
\[ \frac{1}{2c_1} \leq \| u_m \|_1(G_N) . \]

Since \( \lim (\| u_m - u \|_1(G_N)) = 0 \) we must have
\[ \frac{1}{2c_1} \leq \| u \|_1(G_N) . \]

Finally, set \( u = 0 \) on \( \partial G \). We claim that with this definition \( u \) is continuous in \( \bar{G} \). For, let \( Q \) be an arbitrary point of \( \partial G \) and let \( \varepsilon > 0 \) be chosen arbitrarily. If \( w_Q \) is the barrier function associated with \( Q \), define a new function \( w \) by \( w = \varepsilon + \beta w_Q \) with \( \beta \) a positive constant to be later determined. Then
\[ L[w] = \xi \varepsilon + \beta L[w_Q] \geq \beta \]

Now on \( \partial G_m \), \( w \pm u_m \geq 0 \) and in \( G_m \)
\[ L(w \pm u_m) \geq \beta \pm u_m u_m \]

Since \( \| u_m \|_{\alpha}(G_m) = 1 \) and \( u_m \) is a bounded sequence, \( \beta \) can be chosen so that \( L(w \pm u_m) \geq 0 \) in \( G_m \) for every \( m \). By the Hopf maximum principle [2, p. 150], we conclude that \( w \pm u_m \geq 0 \) in \( G_m \), or
\[ |u(x)| \leq w(x) = \varepsilon + \beta w_Q(x), \text{ for every } x \in G. \]

But, for \( x \) in \( G \) and near \( Q \), by the continuity of \( w_Q \) we must have \( |u(x)| < 2\varepsilon \).

Remark. Theorem 1.19 shows the existence of eigenvalues for any domain \( G \) of the above type.
Theorem 1.20  Assume now that $\mathcal{G} = \bigcap_{l=1}^{\infty} G_m$, $G$ non-empty, $G_1$ bounded, $\mathcal{G}_{m+1} \subset G_m$ with $G_m$, $u_m$, $m$, $K$ and the conditions upon them as for Theorem 1.19. Then again we can find $u, v$ such that $Lu = \mu u$ in $G$, $u = 0$ on $\partial G$, $u \geq 0$ in $G$ and $u = \lim u_m$, if we also assume $\partial G = \partial \mathcal{G}$.

**Proof:** Again without loss of generality, assume that $\|u_m\|_{\alpha(G_m)} = 1$. Clearly once again $\{u_m\}$ is a monotone bounded sequence. Proceeding exactly as in Theorem 1.19, we find that there exists a $u$, the limit of a subsequence of $\{u_m\}$ such that $Lu = \mu u$ in $G$. Following the steps of Theorem 1.19, we can state that $u \neq 0$ in $G$, as exactly the same procedure as before may be used to show the existence of a compact subset $G^*$ of $G$ such that $\|u_m\|_{\alpha(G^*)} \geq \frac{1}{2c_1}$ for all $m$ sufficiently large.

There remains to show that $u = 0$ on $\partial G$. To see this, note that $\|u_m\|_{2+\alpha(G_m)} < K$ and hence the first derivatives of $u_m$ are bounded in $G_m$ independently of $m$. Then if $Q \in \partial G$, $|u(Q)| \leq |u(Q) - u_m(Q)| + |u_m(Q)|$.

Now, by taking $m$ sufficiently large, we can find a point of $\partial G_m$ arbitrarily close to $Q$. Hence by equicontinuity and uniform convergence, we can conclude that $u(Q) = 0$.

**Remark**  Clearly $\mu$ is the smallest eigenvalue of $L$ in $G$ as its eigenvector is positive in $G$. 
Proposition 1.21 Let \( \{G_i\}_{i=1}^{\infty} \) denote a family of concentric spheres with \( r_i \) denoting the radius of \( G_i \). If there exists constants \( \delta, \beta \) such that
\[
0 < \delta < \inf_i r_i < \sup_i r_i < \beta
\]
then this family satisfies the condition on the constant in Schauder's estimate imposed on the family \( \{G_m\} \) of Theorems 1.19 and 1.20.

Proof: Let \( G_i \) denote an arbitrary member of the family and \( \theta_i \) the translating and contracting bijective map from \( E^n \) to \( E^n \) which maps the unit sphere \( U \) onto \( G_i \). Let \( y = (y_1, \ldots, y_n) \) and \( x = (x_1, \ldots, x_n) \) denote a generic point of the domain and range space of \( \theta_i \) respectively. Then \( \theta_i \) may be explicitly given as:
\[
x_j = y_j r_i + \gamma_j \quad j = 1, \ldots, n
\]
where \( \gamma = (\gamma_1, \ldots, \gamma_n) \) denotes the centre of the spheres. For simplicity assume that the operator \( L \) is written in non-divergent form,
\[
L u(x) = \sum_{\ell,m=1}^{n} A_{\ell m}(x) D_{\ell m} u(x) + \sum_{\ell=1}^{n} B_{\ell}(x) D_{\ell} u(x) + C(x) u(x)
\]
Now let \( u \) be a function in \( \mathcal{S}(L,G_i) \) such that \( u = 0 \) on \( \partial G_i \) and \( L u(x) = f(x) \) in \( G_i \). Define new functions \( g, h \) by \( g(y) = u(\theta_i(y)) \) and \( h(y) = f(\theta_i(y)) \). Then clearly we have \( g(y) = 0 \) on \( \partial U \) and
\[ \sum_{l,m} \varphi_i A_{lm}(\theta_i(y)) \frac{\partial g(y)}{\partial y_l \partial y_m} + \sum_{l} \varphi_i B_{l}(\theta_i(y)) \frac{\partial g}{\partial y_l}(y) + \]
\[ + C(\theta_i(y))g(y) = h(y) \text{ for } y \in U. \]

These changes in co-ordinates translate our "many domains, one operator" problem into a "many operators, one domain" problem which can be dealt with. Now by the assumptions on the radii of the \( G_i \) it is clear that Hölder constants and exponents can be specified for the coefficients \( A_{lm}(\theta_i(\cdot)) \), \( B_{l}(\theta_i(\cdot)) \), and \( c(\theta_i(\cdot)) \) which are independent of \( i \). For example, for \( y_1, y_2 \in Y : \]
\[ \frac{|A_{lm}(\theta_i(y_1)) - A_{lm}(\theta_i(y_2))|}{|y_1 - y_2|^\alpha} \leq \beta^\alpha \cdot \omega \]
where \( \omega \) denotes \( \sup_{x_1, x_2 \in G^*} \left( \frac{|A_{lm}(x_1) - A_{lm}(x_2)|}{|x_1 - x_2|^\alpha} \right) \) and \( G^* \) is a sphere with centre \( y \) and radius \( \beta \). Since an ellipticity constant for each operator may also be taken independently of \( i \), we may conclude that
\[ \| g \|_{2+\alpha} (u) \leq K \| h \|_{\alpha}(G) \]
by the ordinary "estimate to the boundary", with \( K \) not a function of \( i \) or of \( g \). However,
\[
\sup_{y \in U} |g(y)| = \sup_{x \in G_1} |u(x)|
\]
\[
\sup_{y \in U} \left| \frac{\partial g}{\partial y} (y) \right| = r_1 \sup_{x \in G_1} \left| \frac{\partial u}{\partial x} (x) \right|
\]

Therefore,
\[
\delta \sup_{x \in G_1} \left| \frac{\partial u}{\partial x} (x) \right| \leq \sup_{y \in U} \left| \frac{\partial g}{\partial y} (y) \right| \leq \beta \sup_{x \in G_1} \left| \frac{\partial u}{\partial x} (x) \right|
\]

Hence, from these and similar results we can show the existence of two positive constants \( k_1, k_2 \) such that
\[
\| g \|_{2+\alpha(U)} \geq k_1 \| u \|_{2+\alpha(G_1)}
\]
and
\[
\| h \|_{\alpha(U)} \leq k_2 \| f \|_{\alpha(G_1)}
\]
Combining our results, we obtain
\[
\| u \|_{2+\alpha(G_1)} \leq K_3 \| f \|_{\alpha(G_1)}
\]
with \( K_3 \) a combination of all the previous constants, and independent of \( i \) and \( u \).

4. Oscillation Criteria for \( L \).

We are now in a position to apply some of the above results so as to obtain oscillation criteria for the non-self-adjoint operator \( L \). As has been done before [9], our aim will be accomplished by finding conditions on the coefficients of \( L \)
which will assume the "majorization" of $L$ by an operator which is known to be $\text{Osc} \ L$, or conversely, conditions which will ensure that $L$ "majorizes" a known non-oscillatory operator. We shall assume for simplicity that the operator $L$ has coefficients defined on a set $R$ containing the half-space \( \{x : x_n \geq 0\} \) in $\mathbb{E}^n$. We recall that

$$R_r = R \cap \{x : |x| > r\}$$

and we further define as in [11, p. 3] the cone $C_\alpha$ by

$$C_\alpha = \{x : x_n \geq |x| \cos \alpha\}$$

and

$$S_r = R \cap \{x : |x| = r\}$$

Let $\Lambda(x)$ denote the largest eigenvalue of $(a_{ij}(x))$. A majorant [11] of $(a_{ij}(x))$ is a positive valued function $f$ of class $C^1(0, \infty)$ such that

$$f(r) \geq \max_{x \in S_r} \Lambda(x)$$

Furthermore, any smooth function $g(r)$ such that:

$$g(r) \geq \max_{x \in S_r} [c(x) + h(x) - \frac{\Sigma b_i}{2}]$$

will be called a majorant of $c + h - \frac{\Sigma b_i}{2}$ where $h$ is the function previously defined. We also introduce the spherical co-ordinates [12, p. 58]
\[ x_1 = r \cos \theta_1, \quad x_n = r \cos \theta_1 \]
\[ x_i = r \cos \theta_{n-1+i} \sin \theta_j, \quad i = 2, \ldots, n-1 \]

Theorem 1.22: \( L \) is Osc 1 if there exists a cone \( C_\alpha \),
\( (0 < \alpha < \frac{\pi}{2}) \) and \( (a_{ij}(x)) \), \( c + h - \sum_{i} \frac{b_i}{2} \) have majorants \( f, g \)
respectively such that

\[ \int_{1}^{\infty} \frac{dr}{r^{n-1} f(r)} = +\infty, \quad \int_{1}^{\infty} r^{n-1} [g(r) + \lambda \alpha r^{-2} f(r)] \, dr = -\infty \]

where \( \lambda_\alpha \) is the smallest number for which the problem

\[ \frac{d}{d\theta_1}[\sin^{-2}\theta_1 \frac{d\omega}{d\theta_1}] + \lambda \alpha \phi \sin^{-2}\theta_1 = 0, \quad 0 \leq \theta_1 \leq \alpha \]
\[ \phi(\alpha) = 0 \]

has a non-trivial solution (such a \( \lambda_\alpha \) is known to exist [13]).

Proof: Clearly the operator

\[ L_1 u = - \sum_{i=1}^{n} D_i (f(r)D_i u) + g(r)u \]

majorizes \( \frac{L + L^*}{2} + h \). Let \( S = \{ x : |x| < q \} \) be a given sphere. We will show that \( L \) has a nodal domain outside \( S \),
i.e. in \( R_q \). Writing \( L_1 \) in hyperspherical co-ordinates, we
note [11] that it is Osc 1, and in particular it has nodal domains in the form of truncated cones. Now, by choosing a
nodal domain $N$ of $L_1$ sufficiently far from the origin, it is possible to find a sphere $S_1$ such that $S \cap S_1 = \emptyset$ and $N \subset S_1$. It follows immediately that $L_1$ will have a negative smallest eigenvalue for $S_1$, and hence so will $\frac{L_1 + L^*_1}{2} + h$ and therefore $L$. Let $p = (p_1, \ldots, p_n)$ denote the centre of $S_1$ and $t$ its radius. Consider the family $\{S_t\}_{t \in [1, \infty)}$ of concentric spheres given by:

$$S_t = \{x : |x - p|^2 < \frac{t^2}{t^2 - 1}\} \quad t \in [1, \infty)$$

and let $\mu(t)$ denote the smallest eigenvalue of $L$ in $S_t$. We define

$$t_o = \sup \{t : \mu(t) < 0\}$$

Clearly $t_o$ exists. Let $t_o = \lim_{n} t_n$ monotonically from below. Then $\bigcap_{n} S_{t_o} = S_{t_n}$ and from the above results,

$$\mu(t_o) = \lim \mu(t_n) < 0$$

Conversely, let $t_o = \lim_{m} t_m$ monotonically from above. Then $S_{t_o} = \bigcup_{m} S_{t_m}$ and therefore

$$\mu(t_o) = \lim \mu(t_m) \geq 0.$$ 

Combining these results, we see that $S_{t_o}$ is a nodal domain for $L$.

Several other theorems may now be proved which are analogous to those known for self-adjoint operators [9, 11]. They differ only in the conditions which are imposed to ensure the oscillation of the majorizing operator. These reduce to different conditions postulated to ensure the oscillation of an ordinary differential equation. In particular, we can obtain
non-self-adjoint equivalents of the following theorems which are known [11] to hold for self-adjoint operators:

**Theorem 1.23** \( L \) is Osc 1 if

\[
\int_{1}^{\infty} \frac{dr}{r^{n-1}f(r)} < \infty, \quad \int_{1}^{\infty} r^{n-1} h_n^m(r) [g(r) + \lambda_2 r^{-2}f(r)]dr = -\infty
\]

for some \( m > 1 \), where \( h_n(r) = \int_{r}^{\infty} \frac{dt}{t^{n-1}f(t)} \) and all the other functions are as previously defined.

**Proof:** The conditions of the Theorem ensure the oscillation of the radial equation in the comparison operator [14]. The rest of the proof is identical to that of Theorem 1.22.

**Theorem 1.24** Assume \( \Lambda(x) \) is bounded in \( \mathbb{R} \). Then \( L \) is Osc 1 for \( n = 2 \) if

\[
\int_{1}^{\infty} r[g(r) + \lambda_2 r^{-2}f(r)]dr = -\infty
\]

and for \( n \geq 3 \) if there exists \( \delta > 0 \) such that:

\[
\int_{1}^{\infty} r^{1-\delta}[g(r) + \lambda_2 r^{-2}f(r)]dr = -\infty
\]

**Proof:** Identical to what was done in [11] for the self-adjoint case.

**Theorem 1.25** Let \( \Lambda(x) \leq \Lambda_1 \) in \( \mathbb{R} \). Then \( L \) is Osc 1 if
\[
\liminf_{r \to \infty} \left[-r^2 g(r)\right] > \Lambda_1 \left[ \lambda_\alpha + \frac{(n-2)^2}{4} \right]
\]

Proof: Again as was done in [11].

Remark Theorems 1.24 and 1.25 represent further extensions of classical results such as Glazman's criterion [15, p. 158] for the Schrödinger operator.

To obtain non-oscillation results, we use the fact that \( L + L^* \) majorizes \( \frac{L + L^*}{2} \). Clearly then, if conditions are so chosen that \( \frac{L + L^*}{2} \) is not Osc 1, neither will be \( L \). This fact has already been implicitly used by Swanson [16], and some results are given in his paper.

Remarks All the results in this Chapter have dealt with oscillations "at \( \infty \)". Osc 1 behaviour may also of course arise due to the misbehaviour of the operator at a finite point on the boundary of \( \mathbb{R} \). It should therefore be possible to extend all our results to some such situations by a suitable transformation of co-ordinates.

The results of Section 2 were obtained not only for elliptic equations but also for a class of parabolic equations. This leads us to believe that it may be possible to extend some of the later results to such operators and therefore obtain Osc 1 criteria for them.
CHAPTER II

QUASILINEAR ELLIPTIC EQUATIONS

Introduction

Under consideration in this Chapter will be operators of type

\[ Lu = L_1 u + c(x, u) \]

where \( L_1 \) denotes the linear elliptic operator defined by:

\[ L_1 u = - \sum_{i,j=1}^{n} D_i (a_{ij} D_j u) + \sum_{j=1}^{n} b_j D_j u \]

The domain of \( L_1 \) and the terminology describing the oscillatory behaviour of \( L \) will be identical to those introduced in Chapter I. Our main results will be: (1) Comparison theorems of Sturm's type; (2) Oscillation theorems, i.e., sufficient conditions on the coefficients of \( L \) for \( L \) to be Osc 2; and (3) Theorems giving bounds for the eigenvalues of \( L \).

Whenever a domain \( G \subset \mathbb{R}^n \) is under consideration, we shall assume that \( c(x, \xi) \) is a continuous real valued function with domain \( \bar{G} \times I \), for some real interval \( I \) containing the origin, and that the partial derivative \( c_2(x, \xi) \) [i.e. \( \frac{\partial c}{\partial \xi} \)] exists as a continuous function in \( \bar{G} \times I \). We shall denote by \( \mathcal{D}(G, L) \) the collection of functions of class \( C^2(\bar{G}) \cap C^1(\bar{G}) \)
which map \( G \) into \( I \). Finally, the conditions on the coefficients of \( L_1 \) will be assumed to be those imposed at the beginning of Chapter I.

2. A Sturm Theorem and a Comparison Theorem for \( L \).

We begin by giving an extension of a Sturm theorem which is known to hold for linear operators. Then, under the assumption that \( L_1 \) is formally self-adjoint (i.e. \( b_j = 0 \)), we shall prove a comparison theorem relating \( L \) to a linear operator.

Theorem 2.1 Let \( G \) be a bounded domain of \( \mathbb{R}^n \) in which one of the \( a_{jj} \) is positive in \( G \) and \( b_j \) is bounded for \( j = 1, \ldots , n \). Furthermore assume that there exists a non-trivial function \( u \in \mathcal{D}(G,L) \) such that \( Lu \geq 0 \) in \( G \), \( u \leq 0 \) in \( G \), \( u = 0 \) on \( \partial G \). Finally, let \( c_2(x,\xi) \) be monotonically non-increasing as a function of \( \xi \) for each \( x \) in \( G \), and let \( c(x,0) \leq 0 \). If \( v \) is any function in \( \mathcal{D}(G,L) \) such that \( Lv \geq 0 \) in \( G \), \( v > 0 \) somewhere in \( G \), then \( v \) must vanish somewhere in \( G \).

Proof: If not, then by the above assumptions \( v \) must always be positive in \( G \). For each \( x \in G \) we define the function \( f \) by \( f(t) = L(tu) \) for \( 0 \leq t \leq 1 \). This is possible as under the assumptions on \( c(x,\xi) \), \( L(tu) \) is well defined for \( 0 \leq t \leq 1 \). Then

\[
f(1) - f(0) = \int_0^1 \frac{df}{dt} \, dt
\]
or:

\[ L_1 u + \int_0^1 c_2(x, tu) dt u = Lu - c(x, 0) \geq -c(x, 0) \]

Identically:

\[ L_1 v + \int_0^1 c_2(x, tv) dt v \geq -c(x, 0) . \]

Setting \( c(u) = \int_0^1 c_2(x, tu) dt \), \( c(v) = \int_0^1 c_2(x, tv) dt \) we have:

\[ L_1 u \geq - c(u) u - c(x, 0) \]
\[ L_1 v \geq - c(v) v - c(x, 0) \]

and \( u = 0 \) on \( \partial G \), \( u \leq 0 \) in \( G \), \( v > 0 \) in \( G \). Hence,

\[ \Delta(u, v) = \begin{vmatrix} - c(u) u - c(x, 0) & - c(v) v - c(x, 0) \\ u & v \end{vmatrix} \]
\[ = uv[\int_0^1 (c_2(x, tv) - c_2(x, tu)) dt] + c(x, 0)(u - v) \]

Since \( tv(x) \geq tu(x) \) for every \( t \in [0,1] \), for every \( x \in G \), the first term on the right hand side is non-negative by the fact that \( c_2 \) is monotonically non-increasing. Since \( c(x, 0) \leq 0 \), the second term also is non-negative. Therefore \( \Delta(u, v) \geq 0 \) in \( G \). This is a contradiction of the result known from Chapter 1 [Corollary 1.6], for the class of linear operators of which \( L_1 \) is a member.
Consequence  Under all the above conditions, if \( L \) has a non-positive eigenvalue with a non-positive eigenvector in \( G \), then every \( v \), as above, must vanish somewhere in \( \bar{G} \).

Remark  Similar results may be proved if the conditions in Theorem 2.1 are reversed in sign.

A comparison theorem will now be proved in the case that the operator \( L \) is formally self-adjoint.

Theorem 2.2  Assume that in a bounded domain \( G \) there exist functions \( u \) and \( v \) such that:

\[
lu = l_1 u + c^*(x, u) \geq 0 \text{ in } G
\]

\[
Lv = L_1 v + c(x, v) \geq 0 \text{ in } G
\]

where \( L, l \) are operators of type (2.1) and \( L_1 \) is formally self-adjoint. Furthermore, let the following conditions hold:

(a) \( u \leq 0 \text{ in } G \), \( u = 0 \text{ on } \partial G \), \( v \) somewhere positive in \( G \).

(b) \( c_2^*(x, \xi) \) monotonically non-increasing as a function of \( \xi \) for fixed \( x \) in \( G \).

(c) \( c_2^*(x, \xi) \geq c_2(x, \xi) \) for every \( (x, \xi) \) in \( \bar{G} \times I \)

(d) \( c(x, 0) \leq 0 \), \( c^*(x, 0) \leq 0 \)

(e) \( \int_G u(L_1 u - \ell_1 u) \leq 0 \)
(f) The boundary of $G$ is such that Green's formula may be applied.

(g) The domain of $c^*$ contains the domain of $c$.

Then $v$ must vanish somewhere in $G$.

**Proof:** Assume that $v$ is never zero. Then the function $\frac{u^2}{2v}$ is well defined and smooth in $\bar{G}$. Furthermore, it vanishes together with its first partials on $\partial G$. We have:

$$L_1 \left( \frac{u^2}{2v} \right) = \frac{u L_1 u}{v} - \frac{u^2}{2v} L_1 v - v \sum_{i,j} a_{ij} D_i \left( \frac{u}{v} \right) D_j \left( \frac{u}{v} \right)$$

where $(a_{ij})$ denotes the matrix associated with $L_1$. Then, using condition (f) and the assumed symmetry of $L_1$, we obtain:

$$\int_G \frac{u^2}{v} L_1 v = \int_G u L_1 u - \int_G v^2 \sum_{i,j} a_{ij} D_i \left( \frac{u}{v} \right) D_j \left( \frac{u}{v} \right)$$

or, as $u$ and $v$ obviously cannot be linearly dependent in $\bar{G}$,

$$\int_G \frac{u^2}{v} L_1 v < \int_G u L_1 u$$

Now, once again we write:

$$L_1(u) + \left[ \int_0^1 c_2^*(x, tu) dt \right] u \geq - c^*(x, 0)$$

and

$$L_1(v) + \left[ \int_0^1 c_2(x, tv) dt \right] v \geq - c(x, 0)$$

Set $c(u) = \int_0^1 c_2^*(x, tu) dt$, $c(v) = \int_0^1 c_2(x, tv) dt$
Then,
\[ \int \frac{u^2}{v} [-c(x,0) - c(v)v] < \int u(L_1 u - \tau_1 u) + \int (-c(x,0) - c(u)u)u \]

or
\[ (2.2) \int \frac{u^2}{v} [-c(x,0)] + \int c(x,0)u + \int u^2(c(u) - c(v)) < \int u(L_1 u - \tau_1 u) \]

Now the first two terms on the left hand side are clearly non-negative, and the third term may be expressed as:
\[ \int u^2(c(u) - c(v)) = \int u^2 [\int_0^1 \{c_u^*(x,tu) - c_2(x,tv)\} dt] dG \]
\[ = \int u^2 [\int_0^1 \{c_u^*(x,tu) - c_u^*(x,tv)\} dt + \int_0^1 \{c_2(x,tv) - c_2(x,tv)\} dt] dG \]

By the use of conditions (b) and (c), we see that it is also non-negative. Hence the left side in (2.2) is non-negative while the right side is non-positive. The contradiction establishes Theorem 2.2.

Remark The same result follows if we assume that:

(a) \( u \geq 0 \) in \( G \), \( u = 0 \) on \( \partial G \), \( v < 0 \) somewhere in \( G \), and \( \tau u \leq 0 \), \( \tau v \leq 0 \) in \( G \).

(b) \( c_u^*(x,\xi) \) monotonically non-decreasing as a function of \( \xi \).

(c) \( c_2(x,\xi) \geq c_2(x,\xi) \) for every \( (x,\xi) \in \bar{G} \times I \).
(d) \( c(x,0) \geq 0, \quad c^*(x,0) \geq 0 \).

(e) \( \int_G u(L_1 u - t_1 u) \leq 0 \)

(f) The boundary of \( G \) such that Green's formula may be applied.

(g) The domain of \( c^* \) contains that of \( c \).

It is a consequence of these results that if \( t \) admitted a non-positive eigenvalue with a non-positive eigenvector in a domain \( G \), then \( L \) could not have in a domain \( G_1 \) such that \( \bar{G} \subset G_1 \), a non-negative eigenvalue to which there corresponded an eigenvector positive in \( G_1 \). The same conclusion holds if the conditions are reversed in sign.

Consequence Of particular interest is the case \( c^*(x,u) = \gamma(x)u \), that is: the operator \( t \) is linear. Then clearly \( c^*(x,0) = 0 \) and \( c^*_2 \) satisfies all the above monotonicity conditions. Furthermore, under reasonable smoothness conditions on the coefficients of \( t \) and on the boundary of \( G \), the eigenvector corresponding to the smallest eigenvalue has constant sign in \( G \). It is clear therefore that if the operator \( t \) admits in a domain \( G \) a non-positive smallest eigenvalue then every function \( v \), of class \( \mathcal{E}(G,L) \) such that \( Lv \) is of fixed sign or zero in \( G \) and \( v \) and \( Lv \) do not have opposite sign at every point of \( G \), must vanish somewhere in \( G \) if \( c(x,0) = 0 \), \( c^*_2(x,\xi) \leq \gamma(x) \) for every \( x \in G \), for every \( \xi \in I \) and
\[ \int_G v(L_1 v - t_1 v) \leq 0. \]

As an example, let \( c_2(x,\xi) = \gamma(x) - \sum_{i=1}^{m} g_i(x) \xi^{2i} \) where the \( g_i \) are continuous non-negative functions.

**Theorem 2.3** Assume that the operator \( t \) defined in an unbounded domain \( R \) by:

\[ t(u) = - \sum_{i,j} D_i (A_{ij} D_j u) + \gamma(x) u \]

has a smooth nodal domain outside of any given sphere and in \( R \). If: \( c(x,0) = 0 \), \( c_2(x,\xi) \leq \gamma(x) \) for every \( \xi \in I \), for every \( x \in R \), and \( (A_{ij}(x) - a_{ij}(x)) \) is non-negative definite in \( R \), then every function \( v \) in \( \mathcal{D}(R,\mathcal{L}) \) which is such that \( L v = 0 \) in \( R \), must vanish somewhere in the complement of any given sphere.

**Proof:** Since \( t \) has a nodal domain outside of every sphere, it is clear that we may construct a domain \( N \) outside of every sphere for which the smallest eigenvalue is non-positive. Then since obviously \( L v = 0 \) in \( N \) and therefore \( v \) and \( L v \) cannot have opposite sign at any chosen point of \( N \), regardless of \( v \), the theorem follows.

A small variation in the calculation of Theorem 2.2 leads to non-oscillation criteria which apply to the more general non-self-adjoint operator (2.1). We state one such result.
Theorem 2.4  Let the formally self-adjoint linear operator \( \tau \) be such that the equation

\[
\tau v = \tau_1 v + \gamma(x)v = 0
\]

is satisfied in \( \mathbb{R} \) by a non-trivial function \( v \) all of whose zeros are inside a fixed sphere \( S \). Then if:

(a) \( c_2(x,\xi) \geq \gamma(x) \) for every \( x \in \mathbb{R} , \) for every \( \xi \in I \)

(b) \( c(x,0) = 0 \)

(c) \( \int_G u(t_1 u - \tau_1 u) \leq 0 \) for every bounded subdomain \( G \) of \( \mathbb{R} \)

and every \( u \in \mathfrak{S}(G,L) \),

the operator \( L \) cannot have smooth nodal domains outside the sphere \( S \).

Proof:  The idea of the proof is to show that in the closure of a smooth nodal domain of \( L \) every solution of \( \tau v = 0 \) must vanish at least once. To do this, we repeat exactly the proof of Theorem 2.2 except for the use of the following estimate:

\[
\int_0^1 [c_2^*(x,tu) - c_2(x,cv)]dt = \int_0^1 [c_2^*(x,tu) - c_2(x,tu)]dt + \\
+ \int_0^1 [c_2(x,tu) - c_2(x,tv)]dt
\]

to replace the one in the proof of Theorem 2.3.
3. Oscillation and Non-Oscillation Theorems.

We shall begin by giving two specific examples and then we shall state other results of a more general nature.

Example 1  Duffing's Equation (Hard Spring Case) [17, p.16]. The ordinary differential equation under consideration is:

\[- \frac{d^2y}{dx^2} - y - k^2y^3 = 0 \quad (k^2 > 0)\]

If we choose the comparison equation:

\[- \frac{d^2y}{dx^2} - y = 0\]

which has solutions \( y = \pm \sin(x-a) \) for every constant \( a \), we see that every solution of Duffing's Equation must vanish in every closed interval of length \( \pi \).

Example 2  Mathieu's Equation [18, p. 401]. We consider the ordinary equation:

\[- \frac{d^2y}{dx^2} - b \frac{dy}{dx} - (1+a \cos 2x)y - cy^3 = 0\]

where \( a,b,c \) are constants. Multiplication by the integrating factor \( e^{bx} \) reduces the equation to:

\[- \frac{d}{dx} [e^{bx} \frac{dy}{dx}] - e^{bx} [(1+a \cos 2x)y + cy^3] = 0\]

so that in this case:
\[ c(x,y) = -e^{bx}[(1+a \cos 2x)y + cy^3] \]

and

\[ c_2(x,y) = -e^{bx}[1+a \cos 2x + 3cy^2] \]

If \( c > 0 \), then

\[ c_2(x,y) \leq -e^{bx}[1+a \cos 2x] \leq -e^{bx}[1 - |a|] \]

We are therefore led to the comparison equation:

\[ -\frac{d}{dx}[e^{bx} \frac{dy}{dx}] - e^{bx}[1 - |a|]y = 0 \]

or

\[ \frac{d^2y}{dx^2} + b \frac{dy}{dx} + [1 - |a|]y = 0 \]

which will have oscillatory solutions if \( b^2 < 4(1 - |a|) \). We have thus proved the following theorem:

**Theorem 2.5** If \( c \geq 0 \) and \( b^2 < 4(1 - |a|) \), then every solution of Mathieu's equation must vanish outside of every bounded interval.

Conversely, if \( c \leq 0 \), then

\[ c_2(x,y) \geq -e^{bx}[1+a \cos 2t] \geq -e^{bx}[1 + |a|] \]

and hence the comparison equation may be chosen to be:
\[
\frac{d^2y}{dx^2} + b \frac{dy}{dx} + [1 + |a|]y = 0
\]

which will have never vanishing solutions if \( b^2 \geq 4(1 + |a|) \).

Theorem 2.6 If \( c \leq 0 \) and \( b^2 \geq 4(1 + |a|) \), no solution of Mathieu's equation can vanish more than once.

To obtain more general results, we make the following definitions similar to those of Chapter 1.

**Definition** A function \( f \) of class \( C^1(0,\infty) \) is a majorant of \((a_{ij})\) iff
\[
f(r) \geq \max_{x \in S_r} (\Lambda(x))
\]

where \( \Lambda(x) \) denotes the largest eigenvalue of \((a_{ij}(x))\).

**Definition** A smooth function \( g \) such that
\[
g(r) \geq \sup_{x \in S_r} \sup_{\xi \in I} [c_2(x, \xi)]
\]
is a majorant of \( c_2 \).

Theorem 2.7 If \( R \) contains a cone \( C_\alpha (\alpha > 0) \), \( c(x,0) = 0 \) and \((a_{ij}), c_2\) admit majorants \( f, g \), respectively, such that
\[
\int_1^{\infty} \frac{dr}{r^{n-1}f(r)} = +\infty \quad \int_1^{\infty} r^{n-1}[g(r) + \lambda_\alpha r^{-2}f(r)] \, dr = -\infty
\]
then every solution of

\[- \sum_{i,j} D_i(a_{ij}D_ju) + c(x,u) = 0\]

in \(\mathbb{R}\) must vanish outside every sphere. Here \(\lambda_\alpha\) denotes the same number which was defined in Chapter 1 and in [11].

**Proof:** Under the above conditions, the operator

\[Lu = - \sum_{i=1}^{n} D_i(f(r)D_iu) + g(r)u\]

has a nodal domain outside every sphere. Our result then follows immediately from Theorem 2.3.

**Theorem 2.8** Assume that \(\mathbb{R}\) contains the cone \(C_\alpha (\alpha > 0)\) and that \((a_{ij}), c_2\) admit majorants \(f, g\), respectively, such that:

\[\int_{1}^{\infty} \frac{dr}{r^{n-1}f(r)} < \infty, \quad \int_{1}^{\infty} r^{n-1}h_n^m(r)[g(r) + \lambda_\alpha r^{-2}f(r)]dr = -\infty\]

for some number \(m > 1\), where \(h_n(r) = \int_{r}^{\infty} \frac{dt}{t^{n-1}f(t)}\). Then the same conclusion as in Theorem 2.7 holds.

**Proof:** Again the conditions on \(f\) and \(g\) assure that the equation

\[- \sum_{i=1}^{n} D_i(f(r)D_iu) + g(r)u = 0\]

is Osc 1. The rest follows as in Theorem 2.7.
It is clear that the other known oscillation results for linear operators may similarly be extended, and we therefore omit their proof. Of course, we can also extend some of the non-oscillation results. Once again we draw attention to a paper by Swanson [19] where some such theorems are mentioned for a class of formally "self-adjoint" operators. It is clear that similar theorems exist for operators whose linear part is not self-adjoint. We state one as an example:

**Theorem 2.9** The operator (2.1) is non-oscillatory in $\mathbb{R}$ if $c(x,0) = 0$, if the differential equation:

$$\sum_{i=1}^{n} D_i (A(r)D_i v) + C(r)v = 0$$

admits a solution all of whose zeros are inside a fixed sphere and $\sum D_i \left( -\frac{b_i}{2} \right) \leq 0$, where

$$A(r) \leq \inf_{x \in S_r} \{ \Lambda_1(x) \}$$

$$C(r) \leq \inf_{x \in S_r} \inf_{\xi \in I} \{ c_2(x,\xi) \}$$

and $\Lambda_1(x)$ denotes the smallest eigenvalue of $(a_{ij}(x))$.

**Proof:** Immediate from Theorem 2.4.

**Remark** Theorem 2.9 leads to obvious conditions on the coefficients of (2.1) which will ensure non-oscillation and which we shall omit.
Remark The problem of finding sufficient conditions for $L$ to be Osc 1 still remains open.


We shall conclude this chapter by extending to the operator (2.1) some known estimates for the first real eigenvalue of uniformly elliptic linear operators.

Theorem 2.10 Let $\lambda$ be a real eigenvalue for $L$ with real eigenvectors $u,v$ such that $u \leq 0$ and $v \geq 0$ in $G$. The two eigenvectors may be linearly dependent. Furthermore, assume that all functions involved in the definition of $L$ are $C^\infty$ and also that the boundary of $G$ is $C^\infty$. Finally, let $c(x,0) = 0$ and $c_2(x,\xi)$ be monotonically non-increasing as a function of $\xi$. Then if $w$ is any function in $\mathcal{D}(G,L)$ such that $-w$ is also in $\mathcal{D}(G,L)$ and such that $w \geq 0$ in $G$, $w = 0$ on $\partial G$, we have:

\begin{equation}
\sup_{x \in G} \left[ \frac{L(-w)}{-w} \right] \geq \lambda \geq \inf_{x \in G} \left[ \frac{L(w)}{w} \right]
\end{equation}

Proof: Again we have:

$$L(u) = L_1(u) + \left[ \int_0^1 c_2(x,tu) dt \right] u = \lambda u$$

$$L(w) = L_1(w) + \left[ \int_0^1 c_2(x,tw) dt \right] w$$

$$= L_1(w) + \left[ \int_0^1 c_2(x,tu) dt \right] w + \left[ \int_0^1 (c_2(x,tw) - c_2(x,ru)) dt \right] w$$
Since $tw \geq tu$,

$$L_1[w] + \left[ \int_0^1 c_2(x, tu) dt \right] w \geq L[w]$$

Now, from the theory of linear operators [1], we have:

$$\lambda \geq \inf_{x \in G} \left[ \frac{L_1[w] + \left[ \int_0^1 c_2(x, tu) dt \right] w}{w} \right] \geq \inf_{x \in G} \left[ \frac{Lw}{w} \right]$$

To prove the other inequality in (2.3), we proceed identically. We have:

$$L_1(v) + \left[ \int_0^1 c_2(x, tv) dt \right] v = \lambda v$$

and again we obtain:

$$L_1(-w) + \left[ \int_0^1 c_2(x, -tw) dt \right] (-w) \geq L(-w)$$

The fact that $v \geq 0$ implies that $\lambda$ must be the smallest eigenvalue of the linear operator $\mathcal{L}$ defined by:

$$\mathcal{L}(f)(x) = L_1(f)(x) + \left[ \int_0^1 c_2(x, tv(x)) dt \right] f(x)$$

Once again by [1],

$$\lambda \leq \sup_x \left[ \frac{L_1(w) + \left[ \int_0^1 c_2(x, tv) dt \right] w}{w} \right]$$

$$= \sup_x \left[ \frac{L_1(-w) + \left[ \int_0^1 c_2(x, tv) dt \right] (-w)}{(-w)} \right] \leq \sup_x \left[ \frac{L(-w)}{(-w)} \right]$$
where in the last inequality we use the fact that \(-w \leq 0\).

Remark. As the inequalities in (2.3) were proved independently of each other, if only the existence of \(u\) (or \(v\)) is known, the corresponding inequality in (2.3) will still hold. We also remark that the lower bound for \(\lambda\) actually only requires \(w \geq 0\) in \(G\), as it is a direct consequence of the similar bound in [1].

Under the assumption of formal self-adjointness of the linear part of the operator (2.1), other bounds may be found.

**Theorem 2.11** ("Self-adjoint" case). Assume now that \(b_j = 0\) for all \(j\), that is:

\[
Lu = L_1u + c(x,u) = -\sum_{i,j} D_i(a_j)D_ju + c(x,u).
\]

Let \(c_2(x,\xi)\) be monotonically non-increasing as a function of \(\xi\), \(c(x,0) \leq 0\). If \(Lv = \lambda v\) in \(G\) with \(\lambda\) real, \(v > 0\) in \(G\) and \(v = 0\) on \(\partial G\), then \(\lambda \leq \mu\) where \(\mu\) denotes the smallest eigenvalue of \(L_1 + c_2(x,0)\).

**Proof:** Let \(u \in C^\infty_0(G)\). Then:

\[
\int_G vL_1 \left(\frac{u^2}{2v}\right) \leq \int_G uL_1u - \frac{\lambda}{2} \int_G u^2 + \int_G \frac{u^2}{v} c(x,v)
\]

or

\[
\lambda \int_G u^2 \leq \int_G uL_1u + \int_G \left[\left(\int_0^1 c_2(x,tv)dt\right)v + c(x,0)\right]
\]
Therefore,
\[ \lambda \int_G u^2 \leq \int_G u L_1 u + \int_G u^2 c_2(x,0) \]
and hence,
\[ \lambda \leq \inf_{u \in C^\infty_0(G), u \neq 0} \left[ \frac{\int_G u (L_1 u + c_2(x,0)) u}{\int_G u^2} \right] \]

Now the eigenvector corresponding to \( u \) is a member of the Hilbert space \( H^1_0(G) \), a space where by definition the \( C^\infty_0(G) \) functions are dense in the topology induced by the norm:
\[ \| u \|_1^2 = \int_G (\sum |D_i u|^2 + |u|^2) \]

Furthermore, for smooth functions \( u, v \) vanishing on \( \partial G \), we have:
\[ |(L_1 u, u) - (L_1 v, v)| \leq \int_G \sum |a_{i,j}| \left( |D_i u| \cdot |D_j u - D_j v| + |D_j v| \cdot |D_i u - D_i v| \right) \mathrm{d}G + C \| u - v \|_0 \| u + v \|_0 \]
\[ \leq M(\| u \|_1 \| u - v \|_1 + \| v \|_1 \| u - v \|_1) + C \| u - v \|_0 \| u + v \|_0 \]
where \( \| u \|_0^2 = \int_G |u|^2 \), and \( M \) and \( C \) are constants. It is evident therefore, that the infimum taken over all \( C^\infty_0(G) \) functions does in fact give the smallest eigenvalue of \( L_1 + c_2(x,0) \).

Remark This result sharpens Theorem 2.2 for the special choice of operators \( l, L \) given by:
\[ Lu = L_1 u + c_2(x,0)u ; \quad Lu = L_1 u + c(x,u) \]

**Theorem 2.12** Assume that \( Lu = \lambda u \), \( u \leq 0 \) in \( G \), \( u = 0 \) on \( \partial G \). Furthermore assume \( c_2(x,\xi) \) is monotonically non-increasing as a function of \( \xi \) and that \( c(x,0) \leq 0 \). Then \( \lambda \geq \mu \), where \( \mu \) denotes the smallest eigenvalue of \( L_1 + c_2(x,0) \).

**Proof:**
\[
\int_G uL_1udx + \int_G u^2 \left[ \int_0^1 c_2(x,tu)dt \right] dx + \int_G uc(x,0)dx = \lambda \int_G u^2 dx
\]

But \( \int_G uc(x,0) \geq 0 \) and as \( t \geq 0 \), \( u \leq 0 \) we have \( c_2(x,tu) \geq c_2(x,0) \)

Therefore,
\[
\lambda \int_G u^2 \geq \int_G (L_1 u + c_2(x,0)u).
\]

Combining the results of Theorems 2.11 and 2.12 we are led to the following conclusion: If for any operator (2.1) with self-adjoint linear part, such that \( c_2(x,\xi) \) is monotone non-increasing as a function of \( \xi \) and \( c(x,0) \) is non-positive for all \( x \), there exists a real eigenvalue \( \lambda \) to which there corresponds both a positive and negative eigenvector, then this eigenvalue must equal the smallest eigenvalue of the operator defined by:

\[
L_\delta u = - \sum_1^\delta D_1 (a_{ij} D_j u) + c_2(x,0)u.
\]

**Consequence** No operator of type \( Lu = - \sum_1^\delta D_1 (a_{ij} D_j u) + c(x,u) \)
can have a real eigenvalue in a domain \( G \) to which there corresponds both a positive and a negative eigenvector if \( c_2(x,\xi) \) is strictly decreasing as a function of \( \xi \) and \( c(x,0) \leq 0 \).

**Proof:** If not, let \( \lambda \) be such an eigenvalue of \( L \) and \( u \) its positive eigenvector. By the above considerations, we can conclude that there exists a function \( w \) which is positive in \( G \), vanishes on \( \partial G \) and such that:

\[
- \sum_{i,j} D_i (a_{ij} D_j w) + c_2(x,0)w = \lambda w
\]

or

\[
- \sum_{i,j} D_i (a_{ij} D_j w) + \left[ \int_0^1 c_2(x,0)dt \right] w = \lambda w
\]

and

\[
- \sum_{i,j} D_i (a_{ij} D_j u) + \left[ \int_0^1 c_2(x,tu)dt \right] u = \lambda u - c(x,0)
\]

Multiplying the above equations by \( u, w \) respectively, subtracting and integrating, we obtain:

\[
\int_G \left[ \int_0^1 [c_2(x,0) - c_2(x,tu)]dt \right] uwdG = \int_G c(x,0)wdG
\]

Now under the assumptions, the left-hand side is positive, while the right is non-positive. Contradiction.

As an example of such operators we may take

\[
L_u = - \sum_{i,j} (a_{ij} D_j u) + u - u^2
\]
Remark Identical results hold if the conditions in the above considerations are reversed in sign.

Remark The methods developed in this chapter together with those developed in Chapter I may, perhaps, also be used to describe the oscillatory behaviour of operators defined by:

\[ Lu = L_\perp u + c(x,u, D_\perp u, \ldots, D_n u) \]

where \( L_\perp \) is a linear second order operator.
1. Introduction

It has already been observed [10, p. 115], that solutions of the equation

$$Lu = \sum_{i,j,k,l} D_{ijkl} u - \sum_j D_{ij} D_{j} u + \sum_j c_j D_j u - du = 0$$

need not vanish in the closure of a bounded domain $G$ for which there exists a function $v$ such that $Lv = 0$ in $G$, $v = 0$ on $\partial G$. If we define a nodal domain for the operator $L$ to be a bounded domain $N$ for which we can find a non-trivial function $w$ such that:

$$Lw = 0 \text{ in } N, \quad w = w_1 = 0 \text{ on } \partial N, \quad i = 1, \ldots, n$$

we can show that the same behaviour may occur even when $G$ is a nodal domain for $L$. For example, the function $v = \frac{1}{2}(1 - \cos 2x)$ satisfies:

$$L_\beta v = \frac{d^4 v}{dx^4} + \frac{4d^2 v}{dx^2} = 0, \quad v(0) = v(\pi) = \frac{dv}{dx}(0) = \frac{dv}{dx}(\pi) = 0$$

but any function $u = \text{constant}$ is also a solution of $L_\beta = 0$.

For this reason, the oscillatory behaviour of fourth order elliptic operators will be considered from a nodal domain
viewpoint only.

Definition L is **oscillatory** iff L has a nodal domain outside of every sphere centered at the origin.

Definition L is **non-oscillatory** iff L is not oscillatory.

Oscillation results for a special class of fourth order operators have already been proved [9, p. 40]. The operators considered there were of a type which gives rise to forms that are identical to those of second order operators. We shall consider operators for which this is not the case. Specifically, we shall examine the operators $L$, $L_\perp$ defined by:

\begin{equation}
(3.1) \quad L u = \sum_{i,j,k,l} D_{ij}(a_{ij}k_l D_{kl}u) - \sum_{i,j} D_{ij}(b_{ij} D_{ij}u) + \sum_{j} c_{j}D_{j}u - du
\end{equation}

and

\begin{equation}
(3.2) \quad L_\perp u = \sum_{i,j} D_{ij}(m_{ij} D_{ij}u) - \sum_{i,j} D_{ij}(l_{ij} D_{ij}u) + \sum_{j} p_{j}D_{j}u - qu
\end{equation}

whose coefficients are defined in an unbounded domain $\mathbb{R}$ of $\mathbb{E}^n$. We shall assume that the coefficients are so smooth that all the differentiations involved in the operations defining $L$ and $L_\perp$, may be performed, and that the resulting partial derivatives are continuous in $\mathbb{R}$.

Given any subdomain $F$ of $\mathbb{R}$ we shall denote by $\mathcal{S}(F,L)$ (or $\mathcal{S}(F,L_\perp)$) the collection of functions of class $C^2(F) \cap C^2(\overline{F})$. Finally, we shall assume that $(a_{ij}(x))$ is a positive definite
symmetric matrix in \( \mathbb{R} \). The matrix \((m_{ij}(x))\) will be assumed to be either positive definite or a symmetric matrix all of whose terms are positive in \( \mathbb{R} \).

By the above assumptions, it is clear that at each point of \( \mathbb{R} \) the matrices \((a_{ij}(x)), (m_{ij}(x))\) will each have a positive eigenvalue. If we further assume that:

\[
\sum_{i,j} a_{ij}(x) \xi_i \xi_j > c \sum_{i} \xi_i^2, \quad \sum_{i,j} m_{ij} \xi_i \xi_j > c_1 \sum_{i} \xi_i^2
\]

for every \( x \in \mathbb{R} \) and for every \( \xi = (\xi_1, \ldots, \xi_n) \), when \((a_{ij}(x)), (m_{ij}(x))\) are positive definite, then the operators defined by (3.1) and (3.2) will be uniformly elliptic. This will also be the case when the matrix \((m_{ij}(x))\) has positive entries, if we assume that \( m_{ij}(x) \) is bounded away from zero for every \( i,j = 1, \ldots, n \).

Under the above conditions, and the further assumption that \( p_j = c_j = 0, j = 1, \ldots, n \), the theory of Courant may be used to predict for operators (3.1) and (3.2) the value of the smallest eigenvalue whose eigenvector satisfies Dirichlet boundary conditions.

2. **Comparison Theorems for Formally Self-Adjoint Operators.**

We shall begin by relating formally self-adjoint operators of type (3.1) to those of type (3.2). Since the matrix \((a_{ij}(x))\) is symmetric, it is possible to reduce it, by a suitable
transformation, to diagonal form with the eigenvalues $\lambda_i(x)$ as entries. Without loss of generality, we may assume $\lambda_i(x) \leq \lambda_{i+1}(x)$ for $i = 1, \ldots, n-1$.

**Definition** The matrix $\Lambda(x) = (\lambda_i(x)\lambda_j(x))$ is the characteristic matrix of $(a_{ij}(x))$.

Clearly the characteristic matrix is non-negative symmetric and therefore diagonalizable.

**Theorem 3.1** Let $G$ be a bounded domain and $u$ a function of class $\mathcal{S}(G,L)$ such that $u = u_i = 0$ on $\partial G$ for $i = 1, \ldots, n$. Let $L_1^u$, $L_2^u$ be given by:

$$L_1^u = \sum_{i,j,k,l} D_{ij} (a_{ij} a_{kl} D_{kl} u)$$

and

$$L_2^u = \sum_{i,j} D_{ij} (m D_{ij} u)$$

If $m(x) \geq \lambda(x)$ where $\lambda(x)$ denotes the largest eigenvalue of $\Lambda(x)$, then:

$$\int_G (uL_2^u) dx \geq \int_G (uL_1^u) dx$$

**Proof:**

$$\int_G uL_2^u - \int_G uL_1^u = \int_G [m \Sigma_{i,j} (D_{ij} u)^2 - (\Sigma_{i,j} a_{ij} D_{ij} u)^2]$$

Let $U(x)$ denote the matrix $(D_{ij} u(x))$. Then for any $x$ in $G$, ...
(3.3) \[ m(x) \sum_{i,j} (D_{ij}u)^2 - (\sum a_{ij} D_{ij}u)^2 = m(x) \operatorname{Trace} [U^2] - \{\operatorname{Trace} [(a_{ij}(x) \cdot U)]\}^2 \]

Let \( S \) denote the real orthogonal matrix which diagonalizes \( (a_{ij}(x)) \). Since the trace is invariant under such transformations [20, p.18], the left-hand side of (3.3) equals:

\[ m(x) \operatorname{Trace} [(S^TUS)^2] - \{\operatorname{Trace} [S^T(a_{ij}(x))S \cdot S^TUS]\}^2 \]

where \( S^T \) denotes the transpose of \( S \). Let \( S^TUS = (\eta_{ij}(x)) \). Since \( U \) is symmetric, so is \( S^TUS \), and therefore:

\[ m(x) \sum_{i,j} (D_{ij}u)^2 - (\sum a_{ij} D_{ij}u)^2 = m(x) \sum_{i,j} \eta_{ij}^2 - (\sum \lambda_i \eta_{ii})^2 \]

\[ \geq m(x) \sum_{i} \eta_{ii}^2 - \sum_{i,j} \lambda_i \lambda_j \eta_{ii} \eta_{jj} \]

The right-hand side of the inequality is non-negative, as may be seen by performing one more change of co-ordinates and appealing to the invariance under such transformations of matrices which are multiples of the identity matrix.

**Corollary 3.2** Let the operator \( L \) be as defined in (3.1) with \( c_j = 0 \), and let

\[ L_1u = \sum_{i,j} D_{ij}(m(x) D_{ij}u) - \sum_{i,j} D_{ij}(B_{ij} D_{ij}u) - Eu. \]

If: (i) \( m(x) \geq \lambda(x) \)
(ii) \((B_{ij}(x) - b_{ij}(x))\) is non-negative

(iii) \(E(x) \leq d(x)\)

for every \(x\) in a domain \(G\), then the smallest eigenvalue of \(L\) in \(G\) cannot exceed that of \(L_1\).

Proof: Under the given assumptions and by the use of Theorem 3.1, we have that

\[
\int_{G} wL_1w \geq \int_{G} wLw
\]

for any function \(w\) of class \(\mathcal{D}(G,L)\) such that \(w = w_1 = 0\) on \(\partial G\) for \(i = 1, \ldots, n\). The Corollary is then an immediate consequence of Courant's Principle.

We have thus been led to the problem of determining the behaviour of operators defined by operation (3.2). This will be done by first comparing the operator \(L_0\) given by

\[
L_0u = \sum_{i,j} D_{ij}(m_{ij}D_{ij}u)
\]

with the operator \(L_{oo}\) given by

\[
L_{oo}u = \sum_{i,j} D_{ii}(m_{ij}D_{jj}u)
\]

whose behaviour can be more readily determined. We introduce the following notation:
Notation \[ m_{ij}^k(x) = D_k^l(m_{ij})(x) \]

\[ \Lambda_j(x) = \sum_i m_{ij}^i(x) \]

A simple calculation then gives:

\[ L_{oo}u - L_o u = \sum_{i,j} (m_{ij}D_j^iu - m_{ij}D_i^ju) \]

and

\[ \int_G u(L_{oo}u - L_o u) = \sum_{i,j} \int_G m_{ij}D_i^juD_j^iu - \int_G \sum_j \Lambda_j(D_ju)^2 \]

for any function \( u \in \mathcal{D}(G,L) \) such that \( u = u_i = 0 \) on \( \partial G \) for \( i = 1, \ldots, n \).

**Theorem 3.3** If the matrix

\[
(3.4) \quad (m_{ij}(x)) - (\Lambda_1(x), \ldots, \Lambda_n(x))
\]

is non-negative in a bounded domain \( G \), then in that domain the smallest eigenvalue of \( L_{oo} \) is not smaller than that of \( L_o \).

**Proof:** The conclusion of the Theorem is an immediate consequence of Courant's Principle and the previous considerations.

Because of our results on operators of type (3.1), the case \( m_{ij}(x) = m(x) \) is of special interest. In this case, Theorem 3.3 becomes:
Corollary 3.4  If the matrix $(m_{ij}(x)) - (\sum m_{ii}(x))I$ is non-negative in $G$, then the smallest eigenvalue of

$$L_0 u = \sum_{i,j} D_{ij}(m D_{ij} u)$$

is not smaller than that of

$$L_0 u = \sum_{i,j} D_{ij}(m D_{ij} u).$$

Proposition 3.5  The conditions of Corollary 3.4 are fulfilled if the matrix $(m_{ij}(x))$ is non-positive in $G$.

Proof: Let $x_0$ be an arbitrary point of $G$. Since $(m_{ij}(x_0))$ is symmetric, we may reduce it to diagonal form at $x_0$ by a suitable change of coordinates. Since this is accomplished by an orthogonal transformation, the trace of $(m_{ij}(x_0))$ will be left unchanged. Let the matrix $(m_{ij}(x_0))$ be given by:

$$\begin{pmatrix}
\mu_1 & \ldots & \mu_n
\end{pmatrix}$$

Then if $\xi = (\xi_1, \ldots, \xi_n)$ denotes any $n$-tuple, the matrix (3.4) in this special case gives rise to the form:

$$\sum_{i} \mu_i \xi_i^2 - (\sum \mu_i)(\sum \xi_i^2)$$

Since $\mu_i \leq 0$ for every $i$, the Proposition follows.

Finally, of interest is the case $m(x) = m(r)$. Under this additional hypothesis, the sufficiency condition that
Let \((m_{ij}(x))\) be non-positive may be further investigated and simplified. We have:

\[
(m_{ij}(x)) = \left[ \frac{d^2m}{dr^2} \frac{1}{r^2} - \frac{dm}{dr} \frac{1}{r^3} \right] (x_i x_j) + \frac{dm}{dr} \frac{1}{r} I
\]

where \(x_i\) denotes the \(i\)th coordinate of \(x\) and \(I\) denotes the identity matrix. Since the matrix \((x_i x_j)\) is non-negative, to ensure the non-positivity of \((m_{ij}(x))\) in this case, it is sufficient to assume:

\[
r \frac{d^2m}{dr^2} \leq \frac{dm}{dr} \leq 0
\]

To summarize some of these results, we state:

**Theorem 3.6** Assume that there exists a smooth function \(m\) of \(r\) such that:

(i) \(m(r(x)) \geq \lambda(x)\), where \(\lambda(x)\) denotes the biggest eigenvalue of \(\Lambda (x)\).

(ii) \(r \frac{d^2m}{dr^2} \leq \frac{dm}{dr} \leq 0\)

for every \(x \in G\). Furthermore assume that the matrix \((B_{ij}(x) - b_{ij}(x))\) is non-negative and \(E \leq d\) for every \(x \in G\). Then the smallest eigenvalue of the formally self-adjoint operator \(L\) does not exceed that of \(L_\alpha\), where \(L_\alpha\) is defined by:

\[
L_\alpha(u) = \sum_{i,j} D_{ii}(m D_{jj} u) - \sum_{i} D_{i} (B_{ij} D_{j} u) - Eu.
\]
Remark The conditions on the derivatives of \( m \) are trivially satisfied when \( m \) is a constant function.


In this section we shall use the results of Section 2 to obtain several oscillation theorems. Before stating any such theorems, we shall make some brief remarks on the behaviour of the smallest eigenvalue of the operators under consideration.

As a consequence of the structure of the Hilbert space \( H_0^2 \) in which eigenvectors lie and of Gårding's Inequality [2, p. 198], and in view of the estimates on the Laplacian [12, p. 7], it is clear that the smallest eigenvalue of formally self-adjoint operators of type (3.1) and (3.2), must tend to infinity as the domain is so perturbed that its diameter tends to zero. Furthermore, by a long and tedious calculation following exactly what was done by Courant and Hilbert [8, Vol. I, p. 421], we can assume that the eigenvalue varies continuously when the domain \( G \) is deformed "continuously" in a sense similar to that specified by Courant and Hilbert.

We shall begin by considering the operator defined by (3.1) with \( b_{ij} = c_j = 0 \); \( i,j = 1,\ldots,n \). We shall also for simplicity assume that \( \mathbb{R} \) contains the half-space \( \{x : x_n \geq 0\} \).
Theorem 3.7  The operator $L$ given by

\begin{equation}
L u = \sum_{i,j,k,l} D_{ij} (a_{ij} a_{kl} D_{kl} u) - du
\end{equation}

is oscillatory in $\mathbb{R}$ if the biggest eigenvalue of $\Lambda (x)$ is bounded above in $\mathbb{R}$ by some number $\lambda_1$ such that

$$
\int_0^\infty \frac{g(t)}{n\lambda_1} - (n-1)u \, dt = \infty
$$

where $u$ denotes the smallest eigenvalue of the problem:

\begin{align*}
\frac{d^2 \psi}{dt^2}(t) &= \mu \psi(t) \\
\psi(t) = \frac{d \psi}{dt}(t) &= 0 \\
\text{for some bounded interval } I,
\end{align*}

and

$$
g(t) = \min \{ d(x) : x \in (\pi I) \times \{ t \} \}
$$

Proof: We compare (3.5) with the operator $L_c$ defined by:

$$
L_c u = n\lambda_1 \sum_i \frac{d}{dt} u - g(x_n) u .
$$

By a trivial separation of variables and an application of a theorem of Glazman [15, p. 104], we find that $L_c$ is oscillatory, with nodal domains in the shape of rectangles. Given any sphere $S$, we choose a nodal domain $N$ of $L_c$ and a sphere $S_1$ such that $N \subset S_1$ and $S \cap S_1 = \emptyset$. Clearly the smallest eigenvalue for $S_1$ of the operator defined by (3.5) is non-positive. If we define a family of concentric spheres as was done in Chapter I,
we find that one of them is a nodal domain for $L$.

**Theorem 3.8** The operator given by (3.5) is oscillatory if for $r$ sufficiently large,

$$\frac{g(r)}{\lambda_1 n} - (n-1) \mu > 0$$

and

$$\limsup_{r \to \infty} r^{2p-1} \int_r^\infty \left[\frac{g(r)}{\lambda_1 n} - (n-1) \mu\right] > A_p^2$$

where

$$A_p^{-1} = \sqrt[2p-1]{(p-1)!} \sum_{k=1}^{p} \frac{(-1)^{k-1}(p-1)_k}{2p-k}$$

and $\mu, g, \lambda_1$ are as defined in Theorem 3.7.

**Proof:** The proof is identical to that of Theorem 3.7, except that a different theorem of Glazman [15 p. 100] is used to ensure the oscillation of the ordinary differential equation arising in the comparison.

It is clear that several other theorems may be proved by using different oscillation criteria for fourth order ordinary differential equations. As these theorems are analogous in statement and method of proof to Theorems 3.7 and 3.8, they will be omitted. Several of these may be found in [9].

Similarly, oscillation criteria may be found for formally self-adjoint operators of type (3.2), by using the assumptions of Theorem 3.3 on $m_{ij}$. 


For the elliptic operator defined by the operation (3.1), the standard methods used to prove non-oscillation criteria for second order operators no longer work, as the operators do not give rise to forms all of which are positive definite. We shall, nevertheless, prove some non-oscillation results for such operators under the assumption that the matrix \((b_{ij}(x))\) is positive definite.

Theorem 3.9 If the matrix \((b_{ij} - B_{ij})\) is non-negative in \(G\), and the function \(-d - \sum_{j} D_j(c_{ij}) + \gamma\) is also non-negative in \(G\), then the smallest eigenvalue \(\mu\) of the operator \(\mathcal{L}\) in \(G\) defined by

\[
(3.6) \quad \mathcal{L}v = -\sum_{i,j} D_i (b_{ij} D_j v) - \gamma v
\]

cannot exceed that of the operator \(L\) defined by (3.1).

Proof: Let \(\lambda\) be the smallest eigenvalue of \(L\) and \(u\) its associated eigenvector. We then have:

\[
\int_{G} \lambda |u|^2 = \int_{G} \left[ \left( \sum_{i,j} a_{ij} D_i u D_j u \right)^2 + \sum_{i,j} b_{ij} D_i u D_j u - \sum_{j} \frac{D_j(c_{ij})}{2} u^2 - \gamma u^2 \right]
\]

\[
\geq \int_{G} \sum_{i,j} \beta_{ij} D_i u D_j u - \gamma u^2
\]

and therefore \(\lambda \geq \mu\), by Courant's Principle.
It is therefore sufficient to find conditions such that the second order operator $\mathcal{L}$ is non-oscillatory to ensure that $L$ is also non-oscillatory. Such criteria are as given in [9], and will not be repeated here.

**Corollary 3.10** If the matrix $(m_{ij})$ has positive coefficients in $G$ and the matrix $(t_{ij} - \beta_{ij})$ is non-negative as is the function $-q - \sum \frac{D_j(p_j)}{2} + \gamma$, then the smallest eigenvalue of $\mathcal{L}$ cannot exceed that of the operator $L_1$ defined by (3.2).

We shall now consider the operator $L_1$ under the assumption that the matrix $(m_{ij})$ is positive definite. We begin by stating the opposite of Theorem 3.5.

**Proposition 3.11** If the matrix (3.4) is non-positive in a bounded domain $G$, then the smallest eigenvalue of $L_0$ is not smaller than that of $L_{oo}$.

**Proof:** Identical to that of Theorem 3.5.

Proposition 3.11 enables us to prove:

**Theorem 3.12** Let $(m_{ij})$ be a positive definite matrix in $G$ and assume that matrix (3.4) is non-positive in $G$. Furthermore assume that the matrix $(t_{ij} - \beta_{ij})$ is non-negative and that the function $-q - \sum \frac{D_j(p_j)}{2} + \gamma$ is non-negative. Then the smallest eigenvalue of the operator $\mathcal{L}$ cannot exceed that of $L_1$. 

Proof: Under the assumptions of the theorem,

\[(3.7) \quad \int_G u(\sum_{i,j} D_{ij}(m_{ij}D_{ij}u)) \geq \int_G u(\sum_{i,j} D_{ii}(m_{ij}D_{ij}u))\]

for every smooth function \( u \) such that \( u = u_i = 0 \) on \( \partial G \) for \( i = 1, \ldots, n \). However, the right-hand side of \( (3.7) \) is clearly non-negative. We may therefore employ the procedure used to prove Theorem 3.9.

We have again reduced the problem of finding non-oscillation criteria for \( L_1 \) to the problem of finding such criteria for the operator \( \mathcal{L} \). For a collection of such criteria, we refer the reader to [9].

Remarks The methods developed in this chapter are clearly applicable to higher order operators defined by:

\[ Lu = \sum_{i,j,k,l} D_{i,j}^m D_{k,l}^m (a_{ij} a_{kl} D_{i,j}^m u) , \]

where \( m \) denotes a positive integer.

The problem of majorizing arbitrary fourth order operators and the problem of obtaining \( \text{Osc } 2 \) criteria for fourth order operators remain open, as does the problem of obtaining oscillation criteria for higher order non-linear operators.
BIBLIOGRAPHY


