ON THE TOPOLOGIES
OF
THE SAME CLASS OF HOMEOMORPHISMS

BY

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ABSTRACT

Given a topological space $(X,\mathcal{U})$, let $H(X,\mathcal{U})$ be the class of all homeomorphisms of $(X,\mathcal{U})$ onto itself. This paper is devoted to study the following problem posed by Everett and Ulam [1], [11] in 1948. When and how a new topology $\mathcal{V}$ can be constructed on X such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$, i.e., these two topological spaces have exactly the same class of homeomorphisms.

Some of the results obtained are original, and other results agree essentially with the work done previously by Yu-Lee Lee [5], [6], [7], [8], [9].

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INTRODUCTION

Given a topological space $(X,\mathcal{U})$, let $H(X,\mathcal{U})$ be the class of all homeomorphisms of $(X,\mathcal{U})$ onto itself. Everett and Ulam [1], [11] posed the following problem. When and how a topology $\mathcal{V}$ can be constructed on X such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. No results appeared until 1963, when J. V. Whittaker [12] proved the following.

Theorem Suppose X and Y are compact, locally Euclidean manifolds (with or without boundary) and let H(X) and H(Y) be the groups of all homeomorphisms of X and Y onto themselves respectively. If $\mathcal{L}$ is a group isomorphism between H(X) and H(Y), then there exists a homeomorphism β of X onto Y such that

$$\mathcal{L}(h) = \beta h \beta^{-1} \qquad \text{for all } h \in H(X).$$

From this theorem, we have immediately a partial answer to Ulam's problem.

Corollary Suppose $(X,\mathcal{U})$ and $(X,\mathcal{V})$ are compact, locally Euclidean manifolds (with or without boundary) with the class of homeomorphisms $H(X,\mathcal{U})$ and $H(X,\mathcal{V})$ respectively. If $H(X,\mathcal{U}) = H(X,\mathcal{V})$, then $(X,\mathcal{U})$ is homeomorphic to $(X,\mathcal{V})$.

But there do exist many topologies $\mathcal{V}$ such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$ and $(X,\mathcal{U})$ is not homeomorphic with $(X,\mathcal{V})$. The purpose of the first section of this thesis is to construct such topologies and all of them are coarser than the original topology. In addition, we will prove Theorem 1.12 without the condition that the set I of all isolated points is closed though Lee [5] claimed it was necessary.

In the second section of this thesis, we will construct some finer topologies which have the same class of homeomorphisms as the original topology has. A counterexample will be given to show the conditions of Theorem 2 in [7] are not enough, however we can prove this theorem by adding one more condition, and hence show the existence of non-homeomorphic continua with the same class of homeomorphisms in the third section.

In the last section, we will study the problem from a different point of view and show that if $(X, \mathcal{U})$ is the real line with usual topology and $(X, \mathcal{V})$ is any Hausdorff, locally compact (or first countable, or locally connected, or locally arcwise connected) space such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$, then $\mathcal{U} = \mathcal{V}$.

Throughout this thesis, we will use the notation $(X, \mathcal{U})$ as the original topological space and $\mathcal{V}$ as a new topology constructed on X such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$. By X - A, Cl(A) and Int(A), we always mean the complement, closure and the interior of A relative to the original topology $\mathcal{U}$. We will denote the closure of A relative to $\mathcal{V}$ by $Cl_{\mathcal{V}}(A)$ and the neighborhood system of a point p with respect to $\mathcal{U}$ (or $\mathcal{V}$) by $\mathcal{U}_p$ (or $\mathcal{V}_p$).

1. Coarser topology with the same class of homeomorphisms.

Given a topological space $(X,\mathcal{U})$, let $H(X,\mathcal{U})$ be the class of all homeomorphisms of $(X,\mathcal{U})$ onto itself. This section we will devote to the study of when and how a coarser topology $\mathcal{V}$ can be constructed on X such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

First of all, we are going to give two trivial but useful lemmas. As a matter of fact, we will use these two lemmas throughout this section to construct a new topology $\mathcal{V}$ on X such that it has the same class of homeomorphisms as the original topology $\mathcal{U}$ has.

Lemma 1.1 Let $(X,\mathcal{U})$ be a topological space and let P(V) be a topological property possessed by certain subsets V of X. (i.e. $V \in \{U: P(U)\}$ if and only if $f(V) \in \{U: P(U)\}$ for all $f \in H(X,\mathcal{U})$.) Let $\mathcal{S} = \{U: P(U)\}$ and $\mathcal{V}$ be the topology generated by $\mathcal{S}$ as subbase. Then, $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$.

Proof: Let $f \in H(X,\mathcal{U})$ and $V_i \in \mathcal{S}$ for all $i = 1,2,\ldots,N$. Since

$$f(V_1 \cap V_2 \cdots \cap V_N) = f(V_1) \cap f(V_2) \cap \ldots \cap f(V_N) \in \mathcal{V} \quad \text{and}$$

$$f^{-1}(V_1 \cap V_2 \cap \ldots \cap V_N) = f^{-1}(V_1) \cap f^{-1}(V_2) \cap \ldots \cap f^{-1}(V_N) \in \mathcal{V},$$

then $f \in H(X,\mathcal{V})$, and therefore $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$.

Remark 1.2 In particular, if $\mathcal{S} = \{U: P(U)\}$ is a topology for X, then $H(X,\mathcal{U}) \subseteq H(X,\mathcal{S})$.

Lemma 1.3 Let $\mathcal{U}$ be a topology on X and $\mathcal{V}$ be the topology on X generated by some family $\mathcal{S}$ of subsets of X as subbase. Suppose that $U \in \mathcal{U}$ if and only if $U \cup S \in \mathcal{V}$ for all nonempty $S \in \mathcal{S}$. Then, $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V})$.

Proof: Let $U \in \mathcal{U}$ and $\emptyset \neq T = S_1 \cap S_2 \cap \ldots \cap S_N$ where $S_i \in \mathcal{S}$ for all $i = 1,2,\ldots,N$. Then $U \cup T = U \cup (S_1 \cap S_2 \cap \ldots \cap S_N) = (U \cup S_1) \cap (U \cup S_2) \cap \ldots \cap (U \cup S_N) \in \mathcal{V}$. Hence, it is clear that $U \cup V \in \mathcal{V}$ for all nonempty V in $\mathcal{V}$.

Suppose $f \in H(X,\mathcal{V})$, $U \in \mathcal{U}$, and $\emptyset \neq S \in \mathcal{S}$. Then $f^{-1}(S \cup f(U)) = f^{-1}(S) \cup U \in \mathcal{V}$. Thus $S \cup f(U) \in \mathcal{V}$ for all nonempty $S \in \mathcal{S}$ and therefore $f(U) \in \mathcal{U}$. Similarly, $f(S \cup f^{-1}(U)) = f(S) \cup U \in \mathcal{V}$. Thus $f^{-1}(U) \in \mathcal{U}$. Therefore $f \in H(X,\mathcal{U})$ and hence $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V})$.

Remark 1.4 In particular, if $\mathcal{U}$ and $\mathcal{V}$ are two topologies for X such that $U \in \mathcal{U}$ if and only if $U \cup V \in \mathcal{V}$ for all nonempty V in $\mathcal{V}$, then $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V})$.

Theorem 1.5 Let $(X,\mathcal{U})$ be a locally compact space and let

$P_1(V)$ mean that $V \in \mathcal{U}$ and X–V is compact,

$P_2(V)$ mean that $V \in \mathcal{U}$ and X–V is countably compact.

Then, $\mathcal{V}_i = \{U: U = \emptyset \text{ or } P_i(U)\}$ are topologies for X and

$H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ $(i = 1,2)$.

Proof: Let $f \in H(X,\mathcal{U})$ and $\emptyset \neq V \in \mathcal{V}_1$ (or $\emptyset \neq V \in \mathcal{V}_2$), then it is clear that $\emptyset \neq f(V) \in \mathcal{V}_1$ (or $\emptyset \neq f(V) \in \mathcal{V}_2$). Thus, $P_1(V)$ and $P_2(V)$ are topological properties.

Let $\{V_\alpha : \alpha \in \Delta\}$ be any subfamily of $\mathcal{V}_1$ (or $\mathcal{V}_2$). Then $\bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{U}$ and $X - \bigcup_{\alpha \in \Delta} V_\alpha = \bigcap_{\alpha \in \Delta} (X - V_\alpha)$ is compact (or countably compact), hence $\bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{V}_1$ (or $\bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{V}_2$). Clearly, if V_1 and V_2 are any two elements in $\mathcal{V}_1$ (or in $\mathcal{V}_2$), then $V_1 \cap V_2 \in \mathcal{V}_1$ (or $V_1 \cap V_2 \in \mathcal{V}_2$). Thus, $\mathcal{V}_1$ and $\mathcal{V}_2$ are topologies for X and by Remark 1.2 we have $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ ($i = 1,2$).

Now, if $(X,\mathcal{U})$ is compact, then clearly we have $\mathcal{U} = \mathcal{V}_1 = \mathcal{V}_2$ and hence $H(X,\mathcal{U}) = H(X,\mathcal{V}_1) = H(X,\mathcal{V}_2)$.

If $(X,\mathcal{U})$ is not compact, and $U \in \mathcal{U}$ and $\emptyset \neq V \in \mathcal{V}_1$ (or $\emptyset \neq V \in \mathcal{V}_2$), then $U \cup V \in \mathcal{U}$ and $X - (U \cup V) = (X - U) \cap (X - V)$ is compact (or countably compact). Thus $U \cup V \in \mathcal{V}_1$ (or $U \cup V \in \mathcal{V}_2$).

Conversely, suppose $U \notin \mathcal{U}$. Then there exists a point $x \in U - \mathrm{Int}(U)$ and a closed neighborhood W of x such that W is compact in $(X,\mathcal{U})$ and $W \neq X$. Clearly $x \in \mathrm{Cl}(W - U)$ and $W - U \neq \emptyset$, and hence $X - W$ is a nonempty element of $\mathcal{V}_1$ (and hence a nonempty element of $\mathcal{V}_2$). But, $(X - W) \cup U \notin \mathcal{U}$ for $W - U = X - ((X - W) \cup U)$ is not closed in $\mathcal{U}$. This implies that $(X - W) \cup U \notin \mathcal{V}_i \subseteq \mathcal{U}$ for $i = 1,2$. By Remark 1.4 we have $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V}_i)$ $(i = 1,2)$. Therefore, $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ $(i = 1,2)$.

<u>Definition 1.6</u> A Hausdorff space is called metacompact if each open covering has a point-finite open refinement.

<u>Remark 1.7</u> Clearly metacompactness is invariant under homeomorphism and any closed subspace of a metacompact space is metacompact.

By using Lemma 1.1, Lemma 1.3 and the same argument as in Theorem 1.5, we have the following theorem.

<u>Theorem 1.8</u> Suppose $(X,\mathcal{U})$ is a locally compact, Hausdorff space. Let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is paracompact,

$P_2(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is metacompact.

Let $\mathcal{A}_i = \{V: P_i(V)\}$ and $\mathcal{V}_i$ be the topology generated by $\mathcal{A}_i$ as subbase for each i=1,2. Then $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ (i=1,2).

The following two theorems are the generalizations of Theorem 1.5 and Theorem 1.8.

<u>Theorem 1.9</u> Let $(X,\mathcal{U})$ be a locally compact or a regular space and let A be a closed locally compact subset of X such that $f(A) = A$ for all $f \in H(X,\mathcal{U})$. Let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed compact set and C is closed and of first category,

$P_2(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed compact set and C is a closed nowhere dense set,

$P_3(V)$ mean that $V \in \mathcal{U}$ and $A-V = B \cup C$ where B is a closed compact set and C is a closed countably compact set,

$P_4(V)$ mean that $V \in \mathcal{U}$ and $A-V = B \cup C$ where B is a closed countably compact set and C is a closed nowhere dense set,

$P_5(V)$ mean that $V \in \mathcal{U}$ and $A-V= B \cup C$ where B is a closed countably compact set and C is a closed set and of first category.

Then, $\mathcal{V}_i = \{V: V = \emptyset \text{ or } P_i(V)\}$ are topologies for X and $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ (i = 1,2,3,4,5).

<u>Proof</u>: Since the arguments are almost the same, we will prove one of the cases, say, $H(X,\mathcal{U}) = H(X,\mathcal{V}_5)$ only.

First of all we check that $\mathcal{V}_5$ is a topology for X and $P_5(V)$ is a topological property. Let $f \in H(X,\mathcal{U})$ and V be any member in $\mathcal{U}_5$, then $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed countably compact set and C is closed and of first category. Since $f(B)$ is closed countably compact and $f(C)$ is closed and of first category, hence $f(V) \in \mathcal{U}_5$. That is, $P_5(V)$ is a topological property. Let $\{V_\alpha : \alpha \in \Delta\}$ be any subfamily of $\mathcal{U}_5$, then $V_\alpha \in \mathcal{U}$ and $A - V_\alpha = B_\alpha \cup C_\alpha$ where B_α is closed countably compact and C_α is closed and of first category for each $\alpha \in \Delta$. Since $\bigcup_{\alpha\in\Delta} V_\alpha \in \mathcal{U}$ and $A - \bigcup_{\alpha\in\Delta} V_\alpha = \bigcap_{\alpha\in\Delta}(A - V_\alpha) = \bigcap_{\alpha\in\Delta}(B_\alpha \cup C_\alpha) = \bigcap_{\substack{\alpha\in\Delta\\ \alpha\neq\alpha_0}}(B_\alpha \cup C_\alpha) \cap (B_{\alpha_0} \cup C_{\alpha_0}) = D \cup E$ where $D = \bigcap_{\substack{\alpha\in\Delta\\ \alpha\neq\alpha_0}}(B_\alpha \cup C_\alpha) \cap B_{\alpha_0}$, $E = \bigcap_{\substack{\alpha\in\Delta\\ \alpha\neq\alpha_0}}(B_\alpha \cup C_\alpha) \cap C_{\alpha_0}$ and $\alpha_0 \in \Delta$. Clearly, D is closed countably compact and E is closed and of first category. Thus $\bigcup_{\alpha\in\Delta} V_\alpha \in \mathcal{U}_5$. Let $\{V_i : i = 1,2,\dots,N\}$ be any finite subfamily of $\mathcal{U}_5$, then $V_i \in \mathcal{U}$ and $A - V_i = B_i \cup C_i$ where B_i is closed countably compact and C_i is closed and of first category for each $i = 1,2,\dots,N$. Now $\bigcap_{i=1}^{N} V_i \in \mathcal{U}$ and $A - \bigcap_{i=1}^{N} V_i = \bigcup_{i=1}^{N}(A - V_i) = \bigcup_{i=1}^{N}(B_i \cup C_i) = (\bigcup_{i=1}^{N} B_i) \cup (\bigcup_{i=1}^{N} C_i)$. Clearly $\bigcup_{i=1}^{N} B_i$ is closed countably compact and $\bigcup_{i=1}^{N} C_i$ is closed and of first category, thus $\bigcap_{i=1}^{N} V_i \in \mathcal{U}_5$. Therefore, $\mathcal{U}_5$ is indeed a topology for X and hence by Remark 1.2 we have $H(X,\mathcal{U}) \subsetneq H(X,\mathcal{U}_5)$.

If $(X,\mathcal{U})$ is compact, then A and $A - V$ are closed compact sets for any open set V. Hence $A - V = (A - V) \cup \emptyset$ and $\mathcal{U} = \mathcal{U}_5$. Therefore we may assume that $(X,\mathcal{U})$ is locally compact but not compact. If $U \in \mathcal{U}$ and $V \in \mathcal{U}_5$, $V \neq \emptyset$ then it is clear that $U \cup V \in \mathcal{U}$ and

$A - (U \cup V) = (A - U) \cap (A - V) = (A - U) \cap (B \cup C) = ((A-U) \cap B) \cup ((A-U) \cap C)$ where $(A-U) \cap B$ is closed countably compact and $(A-U) \cap C$ is closed and of first category. Therefore, $U \cup V \in \mathcal{V}_\varsigma$ for any nonempty V in $\mathcal{V}_\varsigma$. If $U \notin \mathcal{U}$, then there exists a point $x \in U - \text{Int}(U)$ and a closed compact neighborhood W of x. We have $W \neq X$ and $x \in \text{Cl}(W - U)$. Since $X - W \in \mathcal{U}$ and $A - (X - W) = A \cap W$ is closed compact, thus $X - W$ is a nonempty element of $\mathcal{V}_\varsigma$. Clearly, $(X - W) \cup U \notin \mathcal{U}$ and hence $(X - W) \cup U \notin \mathcal{V}_\varsigma$ for $\mathcal{V}_\varsigma \subseteq \mathcal{U}$. Thus by Remark 1.4 we have $H(X, \mathcal{U}) \supseteq H(X, \mathcal{V}_\varsigma)$, and therefore $H(X, \mathcal{U}) = H(X, \mathcal{V}_\varsigma)$.

If $(X, \mathcal{U})$ is regular, then $U \in \mathcal{U}$ implies $U \cup V \in \mathcal{V}_\varsigma$ for all nonempty V in $\mathcal{V}_\varsigma$ is still true. If $U \notin \mathcal{U}$, then there exists a point $x \in U - \text{Int}(U)$. If $x \in A$, then since A is a closed locally compact subset of $(X, \mathcal{U})$ thus there exists a closed compact neighborhood W of x in the relative topology of A. But since A is closed in $(X, \mathcal{U})$, thus W is a closed compact neighborhood of x in $(X, \mathcal{U})$. Therefore $X - W$ is a nonempty element in $\mathcal{V}_\varsigma$ but $(X - W) \cup U \notin \mathcal{V}_\varsigma$. If $x \notin A$, then by regularity, there exists a closed neighborhood W_x of x such that $W_x \cap A = \emptyset$. Since $X - W_x \in \mathcal{U}$ and $A - (X - W_x) = A \cap W_x = \emptyset$, thus $X - W_x$ is a nonempty element in $\mathcal{V}_\varsigma$. But $(X - W_x) \cup U$ is not in $\mathcal{U}$ and hence it is not in $\mathcal{V}_\varsigma$. Thus by Remark 1.4 we have $H(X, \mathcal{U}) \supseteq H(X, \mathcal{V}_\varsigma)$. That is, $H(X, \mathcal{U}) = H(X, \mathcal{V}_\varsigma)$.

<u>Theorem 1.10</u> Let $(X, \mathcal{U})$ be a locally compact , Hausdorff space and let A be a closed locally compact subset of X such that $f(A) = A$ for all f in $H(X, \mathcal{U})$. Let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed paracompact set and C is a closed nowhere dense set,

$P_2(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed paracompact set and C is a closed set and of first category,

$P_3(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed metacompact set and C is a closed nowhere dense set,

$P_4(V)$ mean that $V \in \mathcal{U}$ and $A - V = B \cup C$ where B is a closed metacompact set and C is closed and of first category.

Let $\mathcal{E}_i = \{V: P_i(V)\}$ and $\mathcal{V}_i$ be the topology generated by $\mathcal{E}_i$ as subbase for each $i=1,2,3,4$. Then $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ $(i=1,2,3,4)$.

<u>Proof</u>: By using Lemma 1.1, Lemma 1.3 and the same argument as in Theorem 1.9.

<u>Theorem 1.11</u> Let $(X,\mathcal{U})$ be a first countable, Hausdorff space and let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is compact,

$P_2(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is countably compact,

$P_3(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is paracompact,

$P_4(V)$ mean that $V \in \mathcal{U}$ and $X - V$ is metacompact,

$P_5(V)$ mean that $V \in \mathcal{U}$ and $\mathrm{Card}(X - V) \leq \alpha$ where α is any fixed cardinal number greater than or equal to $\aleph_0$,

$P_6(V)$ mean that $V \in \mathcal{U}$, $\mathrm{Card}(X - V) \leq \alpha$ and $X - V$ is compact.

Let $\mathcal{E}_i = \{V: P_i(V)\}$ and $\mathcal{V}_i$ be the topology generated by $\mathcal{E}_i$ as subbase $(i = 3,4)$ and let $\mathcal{V}_j = \{V: V = \emptyset \text{ or } P_j(V)\}$ $(j=1,2,5,6)$.

Then $\mathcal{V}_j$ $(j=1,2,5,6)$ are topologies for X and

$H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ for all $i = 1,2,3,4,5,6$.

<u>Proof</u>: Clearly, $P_i(V)$ (i=1,2,3,4,5,6) are topological properties and $\mathcal{V}_j$ (j=1,2,5,6) are topologies for X, thus by Lemma 1.1 and Remark 1.2 we have $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V}_i)$ (i = 1,2,3,4,5,6).

Let $U \in \mathcal{U}$ and $V_i \in \mathcal{Q}_i$, $V_i \neq \emptyset$ (i = 3,4), then it is clear that $U \cup V_i \in \mathcal{U}$ and $X - (U \cup V_i) = (X - U) \cap (X - V_i) \in \mathcal{Q}_i \subseteq \mathcal{V}_i$ (i = 3,4). Thus, $U \cup V_i \in \mathcal{V}_i$ (i = 3,4). If $U \notin \mathcal{U}$, then there exists a point $x_o \in U - \mathrm{Int}(U)$ and a sequence $x_n \in X - U$ (n = 1,2,3,...) such that $\{x_n\}$ converges to x_o. Clearly, $B = X - \{x_n: n = 0,1,2,3,...\} \in \mathcal{U}$, $X - B = \{x_n: n = 0,1,2,3,...\}$ is compact and Card$(X - B) \leq \alpha$. If $X = \{x_n: n = 0,1,2,3,...\}$ we may choose $B = X - \{x_n: n = 0,2,3,...\} \neq \emptyset$. Thus B is a nonempty element in $\mathcal{Q}_i$ (i = 3,4). But, since $X - (U \cup B) = \{x_n: n = 1,2,3,...\}$ is not closed in $\mathcal{U}$, hence $U \cup B \notin \mathcal{V}_i \subseteq \mathcal{U}$ (i = 3,4). Therefore, by Lemma 1.3 we have $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V}_i)$ (i = 3,4). By Remark 1.4 and the same argument, we will also have $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V}_j)$ (j = 1,2,5,6). Hence the proof is completed.

Yu-Lee Lee [5] proved the following theorem under one additional condition that the set I of all isolated points of $(X,\mathcal{U})$ is closed. Furthermore, he claimed that the condition was necessary. However, it seems to us that we can prove the following theorem without this additional condition.

<u>Theorem 1.12</u> Let A be a closed subset of a first countable, Hausdorff space $(X,\mathcal{U})$ such that $f(A) = A$ for each f in $H(X,\mathcal{U})$ and A contains no isolated point relative to the relative topology. Let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $Cl(V \cap A) = A$,

$P_2(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $A - V$ is compact,

$P_3(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $A - V$ is countably compact,

$P_4(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $A - V$ is paracompact,

$P_5(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $A - V$ is metacompact,

$P_6(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $Card(A - V) \leqslant \alpha$ where α is any fixed cardinal number greater than or equal to $\aleph_0$,

$P_7(V)$ mean that $V \in \mathcal{U}$, $Cl(V \cap A) = A$, $Card(A - V) \leqslant \alpha$ and $A - V$ is compact.

Let $\mathcal{A}_j = \{V: P_j(V)\}$ and $\mathcal{V}_j$ be the topology generated by $\mathcal{A}_j$ as subbase for each $j = 4,5$; and let $\mathcal{V}_i = \{V: V = \emptyset \text{ or } P_i(V)\}$ for each $i = 1,2,3,6,7$. Then $\mathcal{V}_i$ $(i = 1,2,3,6,7)$ are topologies for X and $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ for each $i = 1,2,3,4,5,6,7$.

Proof: Since the argument is almost the same for each case, thus we will prove one case, say, $H(X,\mathcal{U}) = H(X,\mathcal{V}_3)$ only.

First of all we show that $P_3(V)$ is a topological property and $\mathcal{V}_3$ is a topology for X, then by Remark 1.2 we have $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V}_3)$. Let $f \in H(X,\mathcal{U})$ and $V \in \mathcal{V}_3$, $V \neq \emptyset$. Then $V \in \mathcal{U}$, $Cl(V \cap A) = A$ and $A - V$ is countably compact. Clearly, $f(V) \in \mathcal{U}$, $A - f(V) = f(A - V)$ is countably compact and $Cl(A \cap f(V)) = Cl(f(A \cap V)) = f(Cl(A \cap V)) = f(A) = A$. Thus $f(V) \in \mathcal{V}_3$ and hence $P_3(V)$ is a topological property. Let $\{V_\alpha : \alpha \in \Delta\}$ be any subfamily of $\mathcal{V}_3$, then $\bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{U}$, $A \supseteq Cl(A \cap \bigcup_{\alpha \in \Delta} V_\alpha) \supseteq Cl(A \cap V_{\alpha_0}) = A$ where $\alpha_0 \in \Delta$ and $A - \bigcup_{\alpha \in \Delta} V_\alpha = \bigcap_{\alpha \in \Delta}(A - V_\alpha)$ is countably compact. Therefore $\bigcup_{\alpha \in \Delta} V_\alpha \in \mathcal{V}_3$. Let V_1 and V_2 be any two elements in $\mathcal{V}_3$, then it is clear that $V_1 \cap V_2 \in \mathcal{U}$, $A - (V_1 \cap V_2) = (A - V_1) \cup (A - V_2)$ is countably compact. Since $Cl(V_1 \cap V_2 \cap A) \subseteq A$ and

$Cl(V_1 \cap A) = Cl(V_2 \cap A) = A$, thus for any $x \in A$ and any open neighborhood N of x there exists a point $y \in N \cap V_1 \cap A$, but since $y \in A$ and $N \cap V_1$ is an open neighborhood of y, hence $N \cap V_1 \cap V_2 \cap A \neq \emptyset$. therefore $x \in Cl(V_1 \cap V_2 \cap A)$. Thus $V_1 \cap V_2 \in \mathcal{V}_3$ and hence $\mathcal{V}_3$ is indeed a topology for X.

Now, if $U \in \mathcal{U}$ and $V \in \mathcal{V}_3$, $V \neq \emptyset$, then $U \cup V \in \mathcal{U}$, $Cl(A \cap (U \cup V)) = A$ abd $A - (U \cup V) = (A - U) \cap (A - V)$ is countably compact. Hence $U \cup V \in \mathcal{V}_3$ for any $V \neq \emptyset$ in $\mathcal{V}_3$.

Conversely, if $U \notin \mathcal{U}$, then there exists a point $p_o \in U\text{-Int}(U)$, thus p_o is not an isolated point in $(X, \mathcal{U})$, i.e. $\{p_o\} \notin \mathcal{U}$. By first countability, there exists a sequence $\{p_i\}$ of distinct points in $X - U$ such that $\{p_i\}$ converges to p_o. Clearly $\{p_i: i=0,1,2,\dots\}$ is closed and compact in $(X, \mathcal{U})$ and hence $V = X - \{p_i: i=0,1,2,\dots\} \in \mathcal{U}$. We may assume that $V \neq \emptyset$, for if $X - \{p_i: i=0,1,2,\dots\} = \emptyset$ then we let $V = X - \{p_i: i=0,2,3,\dots\}$. Since $A - V = A \cap \{p_i: i=0,1,\dots\}$ is compact and since $Cl(A \cap V) = Cl(A - \{p_i: i=0,1,2,\dots\})$, thus

(1) if $A \cap \{p_i: i=0,1,2,\dots\} = \emptyset$, then clearly $Cl(A \cap V) = Cl(A) = A$;

(2) if $A \cap \{p_i: i=0,1,2,\dots\} \neq \emptyset$ and $p_o \notin A$, then clearly there are at most finite points, say, $p_{i_1}, p_{i_2}, \dots, p_{i_N}$ in A where $p_{i_j} \in \{p_i: i=1,2,\dots\}$ for each $j=1,2,\dots,N$. If there is a point, say, $p_{i_{n_o}} \notin Cl(A \cap V) = Cl(A - \{p_i: i=0,1,2,\dots\})$, $n_o \in \{1,2,\dots,N\}$, then there exists an open neighborhood N_1 of $p_{i_{n_o}}$ such that

$N_1 \cap A - \{p_i: i=o,1,2\ldots\} = \emptyset$. By the Hausdorff property there exists an open neighborhood N_2 of $p_{i_{n_o}}$ such that $p_{i_j} \notin N_2$ for all $j=1,2,\ldots N$, $j \neq n_o$. Let $N = N_1 \cap N_2$, then N is an open neighborhood of $p_{i_{n_o}}$ and $N \cap A = \{p_{i_{n_o}}\}$. Thus $p_{i_{n_o}}$ is an isolated point relative to the relative topology of A, this is a contradiction. Hence, we have $Cl(A \cap V) = Cl(A - \{p_i: i=o,1,2,\ldots\}) = A$.

(3) if $p_o \in A'$ and if there are at most finite points of $\{p_i: i=o,1,2,\ldots\}$ in A. Then since every neighborhood of a point in A contains infinite points of A, hence it is clear that
$Cl(A \cap V) = Cl(A - \{p_i: i=o,1,2,\ldots\}) = A$.

(4) if $p_o \in A$ and if there are infinite points, say, $\{p_{i_1}, p_{i_2}, \ldots, p_{i_n}, \ldots\}$ of $\{p_i: i=0,1,2,\ldots\}$ in A. First of all we claim that $p_o \in Cl(A \cap V) = Cl(A - \{p_i: i=0,1,2,\ldots\})$. If not, then there exists a open neighborhood N_1 of p_o such that $N_1 \cap (A-\{p_i: i=0,1,\ldots\}) = \emptyset$ and there exists $n_o > 0$ such that $p_{i_n} \in N_1$ for all $n \geq n_o$. By the Hausdorff property, there exist open neighborhoods N_2 of $p_{i_{n_o}}$ and N_3 of p_o such that $N_2 \cap N_3 = \emptyset$. Then $N_2 \cap N_1$ is an open neighborhood of $p_{i_{n_o}}$ such that $N_1 \cap N_2$ contain only finite points of $\{p_{i_n}: n=1,2,\ldots\}$. By the Hausdorff property again, there exists open neighborhood N_4 of $p_{i_{n_o}}$ such that $N_4 \cap N_2 \cap N_1$ contains no

point except $p_{i_{n_o}}$ in $\{p_{i_n}: n=1,2,\ldots\}$. Thus $N_4 \cap N_2 \cap N_1 \cap A=\{p_{i_{n_o}}\}$ contradicts the assumption that A has no isolated point relative to the relative topology. Now, we claim also that $p_{i_n} \in Cl(A \cap V)=$ $Cl(A - \{p_i: i=0,1,2,\ldots\})$ for all n=1,2,.... If there were a point, say $p_{i_{m_o}} \notin Cl(A - \{p_i: i=0,1,2,\ldots\})$, then there exists an open neighborhood M_1 of $p_{i_{m_o}}$ such that $M_1 \cap A - \{p_i: i=0,1,2,\ldots\} = \emptyset$. By the Hausdorff property there are open neighborhoods M_2 of $p_{i_{m_o}}$ and M_3 of p_o such that $M_2 \cap M_3 = \emptyset$ and hence there are at most finite points of $\{p_{i_n}: n=1,2,\ldots\}$ in M_2. Thus there is an open neighborhood M_4 of $p_{i_{m_o}}$ such that $M_4 \cap M_2$ contains no point except $p_{i_{m_o}}$ in $\{p_{i_n}: n=1,2,\ldots\}$. Then, $M_4 \cap M_2 \cap M_1$ is an open neighborhood of $p_{i_{m_o}}$ and $M_4 \cap M_2 \cap M_1 \cap A = \{p_{i_{m_o}}\}$. This is a contradiction. Hence, in any case we have $Cl(A \cap V) = A$. That is, $V = X - \{p_i: i=0,1,2,\ldots\}$ is a nonempty element in $\mathcal{V}_3$. But, it is clear that $U \cup X - \{p_i: i=0,1,2,\ldots\} \notin \mathcal{U}$ and hence $U \cup V \notin \mathcal{V}_3$. Therefore, by Remark 1.4 we have $H(X,\mathcal{U}) \supsetneqq H(X,\mathcal{V}_3)$. Hence the proof is completed.

<u>Theorem 1.13</u> Suppose $(X,\mathcal{U})$ is a first countable, Hausdorff space and the set I of all isolated points of $(X,\mathcal{U})$ is closed. Let $P_1(V)$ mean that $V \in \mathcal{U}$ and $Cl(V) = X$,

$P_2(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is compact,
$P_3(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is paracompact,
$P_4(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is metacompact,
$P_5(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is countably compact,
$P_6(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $Card(X - V) \leq \alpha$ where α is any fixed cardinal number greater than or equal to $\aleph_0$,
$P_7(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$, $Card(X - V) \leq \alpha$ and $X - V$ is compact.

Let $\mathcal{A}_j = \{V: P_j(V)\}$ and $\mathcal{V}_j$ be the topology generated by $\mathcal{A}_j$ as subbase for each $j=3,4$; and let $\mathcal{V}_i = \{V: V = \emptyset \text{ or } P_i(V)\}$ $(i=1,2,5,6,7)$. Then $\mathcal{V}_i$ $(i=1,2,5,6,7)$ are topologies for X and $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ for all $i=1,2,3,4,5,6,7$.

<u>Proof</u>: If $I = \emptyset$, then it is the special case $A = X$ in Theorem 1.12.

If $I \neq \emptyset$, then the argument in Theorem 1.12 still hold except one part which we will argue as follows:

If $U \notin \mathcal{U}$. Then there exists a point $p_0 \in U - Int(U)$ and a sequence $\{p_i\}$ in $X - U$ such that $\{p_i\}$ converges to p_0 and $p_i \notin I$ for each i. Clearly, $X - \{p_n: n=0,1,2,\dots\}$ is a nonempty element in $\mathcal{V}_i$ for each $i=1,2,5,6,7$; and is also a nonempty element in $\mathcal{A}_j$, $j=3,4$. But $U \cup (X - \{p_n: n=0,1,2,\dots\}) \notin \mathcal{U}$ and hence $U \cup (X - \{p_n: n=0,1,2,\dots\}) \notin \mathcal{V}_i$ for all $i=1,2,3,4,5,6,7$.

<u>Remark 1.14</u> The condition that the set I of all isolated points in $(X,\mathcal{U})$ be closed is necessary in Theorem 1.13.
For let $X = [-1, 0] \cup \{\frac{1}{n}: n=1,2,\dots\}$
with the relative topology $\mathcal{U}$ inherited from the real line.
Let

$$f(x) = \begin{cases} -x-1, & \text{if } x \in [-1, 0] \\ x, & \text{if } x \in \{\frac{1}{n}: n=1,2,\dots\}. \end{cases}$$

Now $[-1, 0) \in \mathcal{U}$ and $f([-1, 0)) = (-1, 0] \notin \mathcal{U}$. Hence $f \notin H(X,\mathcal{U})$.

Let V be any open set in $(X,\mathcal{U})$ such that $Cl(V) = X$, then clearly $\{\frac{1}{n}: n=1,2,\ldots\} \subseteq V$ and hence $f(V) \in \mathcal{U}$ and $Cl(f(V)) = X$. Thus, we have $f \in H(X,\mathcal{V}_1)$. It is also clear that $f \in H(X,\mathcal{V}_i)$ for all i=2, 3,...,7.

We can set another condition in order to permit that I has exactly one limit point.

Theorem 1.15 Let $(X,\mathcal{U})$ be a first countable, Hausdorff space and the set I of all isolated points of $(X,\mathcal{U})$ has exactly one limit point e and Cl(I) be compact. Suppose in addition that $f(e) = e$ for any f in $H(X-I, \mathcal{U}|X-I)$. Let

$P_1(V)$ mean that $V \in \mathcal{U}$ and $Cl(V) = X$,

$P_2(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is compact,

$P_3(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $X - V$ is countably compact,

$P_4(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$ and $Card(X - V) \leq \alpha$ where α is any fixed cardinal number greater than or equal to $\aleph_0$,

$P_5(V)$ mean that $V \in \mathcal{U}$, $Cl(V) = X$, $Card(X - V) \leq \alpha$ and $X - V$ is compact.

Then, $\mathcal{V}_i = \{V: V = \emptyset \text{ or } P_i(V)\}$ are topologies for X and $H(X,\mathcal{U}) = H(X,\mathcal{V}_i)$ (i=1,2,3,4,5).

Proof: Since the arguments are almost the same we will prove one of the cases, say $H(X,\mathcal{U}) = H(X,\mathcal{V}_5)$ only.

By the same argument as in Theorem 1.12 we know that $\mathcal{V}_i$ (i=1, 2,3,4,5) are topologies for X and $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V}_i)$ (i=1,2,3,4,5).

Now, we are going to show that $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V}_5)$.

(Claim 1): $p \in V$ for all nonvoid V in $\mathcal{V}_5$ if and only if $p \in I$.

If $p \in I$, then $\{p\} \in \mathcal{U}$ and hence $p \in V$ for all nonvoid $V \in \mathcal{V}_\zeta$ for otherwise $Cl(V) \neq X$. If $p \notin I$, then $\{p\}$ is closed in $\mathcal{U}$ and hence $V = X - \{p\} \in \mathcal{U}$, $Cl(V) = Cl(X - \{p\}) = X$ and $Card(X - V) = Card(\{p\}) = 1 \leq \alpha$, $X - V = \{p\}$ is compact. Thus $V \in \mathcal{V}_\zeta$ but $p \notin V$.

(Claim 2): If $f \in H(X, \mathcal{V}_\zeta)$, then $f(I) = I$ and hence $f \mid X - I \in H(X-I, \mathcal{V}_\zeta \mid X-I)$.

If there exists a point $x \in I$ and $f(x) \notin I$, then by (Claim 1) there is a nonvoid $V \in \mathcal{V}_\zeta$ such that $f(x) \notin V$. Therefore $x \notin f^{-1}(V) \in \mathcal{V}_\zeta$, contradicts the assumption $x \in I$. Hence $f(I) = I$. By a similar argument, f is onto and hence $f(I) = I$.

Case 1: Suppose that e is not an isolated point of $X - I$. Then it is clear that $X - I$ has no isolated point with respect to $\mathcal{U} \mid X-I$. Let $\mathcal{W}_1 = \{V \in \mathcal{U} \mid X-I : V = \emptyset$ or $Cl_{\mathcal{U}|X-I}(V) = X - I$, $Card(X-I - V) \leq \alpha$ and $(X - I) - V$ is compact in $\mathcal{U} \mid X-I\}$. Then by Theorem 1.12 we know that $\mathcal{W}_1$ is a topology for $X - I$ and $H(X-I, \mathcal{U} \mid X-I) = H(X-I, \mathcal{W}_1)$.

(Claim 3): $\mathcal{W}_1 = \mathcal{V}_\zeta \mid X-I$ and therefore $H(X-I, \mathcal{U} \mid X-I) = H(X-I, \mathcal{V}_\zeta \mid X-I)$.

First of all we note that $X-I$ is closed in $\mathcal{U}$ and hence for any $A \subseteq X - I$ we have $Cl_{\mathcal{U}|X-I}(A) = Cl(A)$. Now, let $\emptyset \neq V_1 \in \mathcal{W}_1$. Then $V_1 = V_2 \cap X-I$ where $V_2 \in \mathcal{U}$, $Cl(V_1) = X-I$, $Card(X-I - V) \leq \alpha$ and $(X - I) - V_1$ is compact in $\mathcal{U} \mid X-I$ (and hence $(X - I) - V_1$ is compact in $\mathcal{U}$). Let $V_3 = V_2 \cup I$. Then $V_3 \in \mathcal{U}$, $V_1 = V_3 \cap X-I$, $Cl(V_3) \supseteq Cl(V_2) \supseteq X-I$ (i.e. $Cl(V_3) = X$), $Card(X-V_3) = Card(X-I - V_1) \leq \alpha$ and $X - V_3 = (X - I) - V_1$ is compact in $\mathcal{U}$, therefore $V_3 \in \mathcal{V}_\zeta$. Hence $V_1 \in \mathcal{V}_\zeta \mid X-I$. That is, $\mathcal{W}_1 \subseteq \mathcal{V}_\zeta \mid X-I$. Conversely, let $\emptyset \neq U_1 \in \mathcal{V}_5 | X-I$. Then $U_1 = U_2 \cap X-I$ where $U_2 \in \mathcal{U}$, $Cl(U_2) = X$, $Card(X - U_2) \leq \alpha$ and $X - U_2$ is compact in $\mathcal{U}$. Thus, it is clear that $U_1 \in \mathcal{U} | X-I$, $Cl(U_1) = Cl(U_2 \cap X-I) = X - I$, $Card(X-I - U_1) = Card(X - U_2) \leq \alpha$ and $X - U_2 = X-I - U_1$ is compact in $\mathcal{U} | X-I$. That is $U_1 \in \mathcal{W}_1$. Therefore $\mathcal{W}_1 = \mathcal{V}_5 | X-I$.

Thus $f | X-I \in H(X-I, \mathcal{V}_5 | X-I) = H(X-I, \mathcal{U} | X-I)$ and hence $f(e)= e$ by hypothesis. If $f \notin H(X, \mathcal{U})$, then either there exists a sequence $\{p_i\}$ and p_o in X such that $\{p_i\}$ converges to p_o but $\{f(p_i)\}$ does not converge to $f(p_o)$, or there exists a sequence $\{p_i\}$ and p_o in X such that $\{p_i\}$ does not converge to p_o but $\{f(p_i)\}$ converges to $f(p_o)$. In the first case, if $p_o \neq e$, then there are at most finite many points of $\{p_i\}$ in I, for otherwise p_o will be a limit point of I which contradicts the uniqueness of limit point of I. Thus, if $p_o \neq e$ we may choose $p_i \notin I$ for each i. Then $\{p_i\}$ converges to p_o in $(X-I, \mathcal{U}|X-I)$ and $\{f(p_i)\}$ does not converge to $f(p_o)$ in $(X-I, \mathcal{U}|X-I)$ and therefore $f | X-I \notin H(X-I, \mathcal{U}|X-I)$. This is a contradiction. If $p_o = e$ and all subsequences of $\{p_i\}$ are in I (i.e. at most finite points of $\{p_i\}$ are not in I). Since $f(I)=I$, hence there are at most finitely many points of $\{f(p_i)\}$ that are not in I. Since Cl(I) is compact, then every neighborhood of $f(p_o)=e$ contains all but a finite number of points of I, hence clearly $\{f(p_i)\}$ converges to $f(p_o)$. If $p_o = e$ and there is a subsequence $\{p_{n_i}\} \subseteq X - I$ such that $\{p_{n_i}\}$ converges to e but $\{f(p_{n_i})\}$ does not converge to e, then $f \notin H(X-I, \mathcal{U}|X-I)$. This is also a contradiction. Therefore f is a continuous function of $(X, \mathcal{U})$ onto itself. By a similar argument f^{-1} is continuous. Hence, $f \in H(X, \mathcal{U})$ and we have $H(X, \mathcal{U}) = H(X, \mathcal{V}_\varsigma)$.

Case 2: Now, suppose e is an isolated point of X - I.

(Claim 4): e is the only point in X - I such that $V \in \mathcal{V}_\varsigma$ if and only if $V \cup \{e\} \in \mathcal{V}_\varsigma$ for any nonempty V. Therefore $f(e) = e$.

Suppose $\emptyset \neq V \in \mathcal{V}_\varsigma$. Let U be the open neighborhood of e such that $U \cap X-I = \{e\}$. Then clearly $I \subseteq V$ (since $Cl(V)=X$) and $V \cup \{e\} = V \cup U \in \mathcal{V}_\varsigma$. On the other hand, if $V \neq \emptyset$ and $V \cup \{e\} \in \mathcal{V}_\varsigma$ i.e. $V \cup \{e\} \in \mathcal{U}$, $Cl(V \cup \{e\}) = X$, $Card(X - (V \cup \{e\})) \leq \alpha$ and $X - (V \cup \{e\})$ is compact in $\mathcal{U}$. If $e \in V$, then clearly $V = V \cup \{e\} \in \mathcal{V}_\varsigma$. If $e \notin V$, then $V = V \cup \{e\} - \{e\} \in \mathcal{U}$, $Card(X - V) \leq \alpha$ and $X - V = (X - (V \cup \{e\})) \cup \{e\}$ is compact in $\mathcal{U}$ but since $Cl(V \cup \{e\}) = X$, thus $I \subseteq V$ and hence $Cl(V) = X$. That is, $V \in \mathcal{V}_\varsigma$.

Now, for any $x_o \in X-I$, $x_o \neq e$, then x_o is not a limit point of I and hence there exists a sequence $\{x_i\}$ converges to x_o and $x_i \notin I$ for each i. Let $V = X - \{x_i\colon i=0,1,2,\ldots\}$, then clearly $V \in \mathcal{V}_\varsigma$ but $V \cup \{x_o\} \notin \mathcal{U}$ and hence $V \cup \{x_o\} \notin \mathcal{V}_\varsigma$. Therefore, we complete the assertion that e is the only point in X-I such that $V \cup \{e\} \in \mathcal{V}_\varsigma$ if and only if V is a nonvoid element of $\mathcal{V}_\varsigma$. Hence $f(e) = e$.

Now, since X - Cl(I) contains no isolated points relative to $\mathcal{U}|X - Cl(I)$, thus if we let

$\mathcal{W}_2 = \{V \in \mathcal{U}|X-Cl(I)\colon V=\emptyset \text{ or } Cl_{\mathcal{U}|X-Cl(I)}(V)=X-Cl(I),\ Card(X-Cl(I) - V) \leq \alpha \text{ and } X-Cl(I) - V \text{ is compact in } \mathcal{U}|X - Cl(I)\}$

then $\mathcal{W}_2$ is a topology for $X - Cl(I)$ and $H(X-Cl(I), \mathcal{U}|X-Cl(I)) = H(X-Cl(I), \mathcal{W}_2)$ by Theorem 1.12. Since e is an isolated point of $X - I$, there is an open neighborhood N of e such that $X-I \cap N = \{e\}$, then $X - Cl(I) = X - (I \cup \{e\}) = X - (I \cup N)$ is closed in $\mathcal{U}$. Hence it is easy to check that $\mathcal{W}_2 = \mathcal{V}_s|X - Cl(I)$ and therefore $f|X-Cl(I) \in H(X-Cl(I), \mathcal{V}_s|X-Cl(I)) = H(X-Cl(I), \mathcal{U}|X-Cl(I))$. By the same argument as in Case 1, we will have $H(X,\mathcal{U}) = H(X,\mathcal{V}_s)$. Therefore, in any case we have $H(X,\mathcal{U}) = H(X,\mathcal{V}_s)$.

<u>Remark 1.16</u> If e is an isolated point of $X - I$, then $\mathcal{W}_1 \neq \mathcal{V}_s|X-I$. For let $X = [-2, -1] \cup \{0\} \cup \{\frac{1}{n}: n=1,2,\ldots\}$ with relative topology inherited from the real line. Then clearly $I = \{\frac{1}{n}: n = 1,2,\ldots\}$ and $e = 0$. Let $V_1 = [-2, -1] = V_2 \cap X-I$ where $V_2 = [-2, -1] \cup I$, then $V_1 \in \mathcal{V}_s|X - I$. But, clearly $V_1 \notin \mathcal{W}_1$ for $Cl(V_1) = [-2, -1] \neq X - I$.

<u>Remark 1.17</u> The condition $f(e) = e$ for any $f \in H(X-I,\mathcal{U}|X-I)$ is necessary as shown in Remark 1.14.

<u>Remark 1.18</u> The condition that Cl(I) be compact is also necessary. For let $X = [-2, -1] \cup \{0\} \cup I$ where $I = \{\frac{1}{n}: n=3,4,5,\ldots\} \cup \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\}$ with the relative topology $\mathcal{U}$ inherited from the real line. Then $(X,\mathcal{U})$ is a first countable, Hausdorff space and I is the set of all isolated points of $(X,\mathcal{U})$ which has only one limit point $e = 0$ and $f(0) = 0$ for all f $H(X-I,\mathcal{U}|X-I)$. But Cl(I) is not compact. Let

$$f(x) = \begin{cases} x & , \text{ if } x \in X-I \\ x - \frac{1}{2} & , \text{ if } x \in \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\} \\ x + \frac{1}{2} & , \text{ if } x \in \{\frac{1}{n}: n=3,4,5,\ldots\} \end{cases}$$

Since $\{\frac{1}{n}: n=3,4,5,\ldots\}$ is a sequence converging to 0 but $\{f(\frac{1}{n}): n=3,4,5,\ldots\} = \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\}$ does not converge to $f(0) = 0$, hence $f \notin H(X,\mathcal{U})$. But it is clear that $f \in H(X,\mathcal{U}_i)$ for each $i=1,2,3,4,5$.

Remark 1.19 The condition that I has only one limit point is necessary. For let $X = [-2, -1] \cup \{0, \frac{1}{2}\} \cup I$ where $I = \{\frac{1}{n}: n=3,4,5,\ldots\} \cup \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\}$ with the relative topology $\mathcal{U}$ inherited from the real line.

Let

$$f(x) = \begin{cases} x & , \text{ if } x \in X - I \\ x - \frac{1}{2}, & \text{ if } x \in \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\} \\ x + \frac{1}{2}, & \text{ if } x \in \{\frac{1}{n}: n=3,4,5,\ldots\}. \end{cases}$$

Since $\{\frac{1}{n}: n=3,4,5,\ldots\}$ is a sequence converging to 0 but $\{f(\frac{1}{n}): n=3,4,5,\ldots\} = \{\frac{1}{2}+\frac{1}{n}: n=3,4,5,\ldots\}$ does not converge to $f(0) = 0$, thus $f \notin H(X,\mathcal{U})$. But it is clear that $f \in H(X,\mathcal{U}_i)$ for each $i=1,2,3,4,5$.

Theorem 1.20 Let $(X,\mathcal{U})$ be a compact Hausdorff space and let $p_0 \in X$ such that $f(p_0) = p_0$ for any $f \in H(X,\mathcal{U})$. Let $P(V)$ mean that $V \in \mathcal{U}$, $p_0 \notin V$ and $(X - \{p_0\}) - V$ is compact.

If $$\mathcal{V} = \{V: V = \emptyset \text{ or } V = X \text{ or } P(V)\}$$
then $(X,\mathcal{V})$ is a topological space and $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

Proof: It is easy to check that $P(V)$ is a topological property and $\mathcal{V}$ is indeed a topology for X. Thus, we have $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$ by Remark 1.2.

Now, we are going to show that $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V})$. First of all we claim that for any $f \in H(X,\mathcal{V})$ we have $f(p_0) = p_0$.

Since $X - \{p_0\} \in \mathcal{U}$, $p_0 \notin X - \{p_0\}$ and $X - \{p_0\} - (X - \{p_0\}) = \emptyset$, thus $X - \{p_0\} \in \mathcal{V}$ and hence $f(X - \{p_0\}) = X - \{f(p_0)\} \in \mathcal{V}$ for any $f \in H(X, \mathcal{V})$. Since by our construction the only neighborhood of p_0 in $H(X, \mathcal{V})$ is the whole space X, thus it is clear that $p_0 \notin X - \{f(p_0)\} \in \mathcal{V}$. Therefore, $f(p_0) = p_0$ and hence $f | X - \{p_0\} \in H(X - \{p_0\}, \mathcal{V} | X - \{p_0\})$ for all $f \in H(X, \mathcal{V})$. Clearly for any $V \subseteq X - \{p_0\}$, $V \in \mathcal{U} | X - \{p_0\}$ if and only if $V \in \mathcal{U}$. Thus it is clear that for any $A \subseteq X - \{p_0\}$, A is closed and compact in $(X, \mathcal{U})$ if and only if A is closed and compact in $(X - \{p_0\}, \mathcal{U} | X - \{p_0\})$. Now since $(X, \mathcal{U})$ is a compact Hausdorff space, hence $(X - \{p_0\}, \mathcal{U} | X - \{p_0\})$ is a locally compact Hausdorff space. Thus by Theorem 1.5

$$\mathcal{W} = \{ V \subseteq X - \{p_0\} : V = \emptyset \text{ or } V \in \mathcal{U} | X - \{p_0\} \text{ and } (X - \{p_0\}) - V \text{ is compact in } \mathcal{U} | X - p_0 \}$$

is a topology for $X - \{p_0\}$ and $H(X - \{p_0\}, \mathcal{U} | X - \{p_0\}) = H(X - \{p_0\}, \mathcal{W})$. But, by the above discussion it is clear that

$$\mathcal{W} = \{ V \in X - \{p_0\} : V = \emptyset \text{ or } V \in \mathcal{U} \text{ and } (X - \{p_0\}) - V \text{ is compact in } \mathcal{U} \}.$$

Thus, $\mathcal{W} = \mathcal{V} | X - \{p_0\}$ and therefore $f | X - p_0 \in H(X - \{p_0\}, \mathcal{U} | X - \{p_0\}) = H(X - \{p_0\}, \mathcal{V} | X - \{p_0\})$ for any $f \in H(X, \mathcal{V})$. Now, if $U \in \mathcal{U}$ and $p_0 \notin U$, then $U \subseteq X - \{p_0\}$ and hence $U \in \mathcal{U} | X - \{p_0\}$, $f(U) \in \mathcal{U} | X - \{p_0\} \subseteq \mathcal{U}$. If $U \in \mathcal{U}$ and $p_0 \in U$, then X-U is closed and compact in $\mathcal{U}$ and hence is closed and compact in $(X - \{p_0\}, \mathcal{U} | X - \{p_0\})$. Thus, $f(X - U) = X - f(U)$ is closed and compact in $(X - \{p_0\}, \mathcal{U} | X - \{p_0\})$

and hence it is closed and compact in $(X,\mathcal{U})$. Therefore, $f(U)\in\mathcal{U}$. By a similar argument we have $f^{-1}(U)\in\mathcal{U}$ for any $U\in\mathcal{U}$. Hence $f\in H(X,\mathcal{U})$. That is, $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

<u>Theorem 1.21</u> Let $(X,\mathcal{U})$ be a compact Hausdorff space without isolated points and let $p_0\in X$ such that $f(p_0) = p_0$ for any f in $H(X,\mathcal{U})$. Let $P(V)$ mean that $V\in\mathcal{U}$ and $p_0\in V$ or $p_0\notin V\in\mathcal{U}$ and $(X - \{p_0\}) - V$ is compact. Then $\mathcal{V}=\{V: V = \emptyset \text{ or } P(V)\}$ is a topology for X and $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

<u>Proof</u>: Clearly, $P(V)$ is a topological property. Let $\{V_\alpha : \alpha\in\Delta\}$ be any subfamily of $\mathcal{V}$. If $\bigcup_{\alpha\in\Delta} V_\alpha = X$ or $p_0\in V_\beta$ for a $\beta\in\Delta$, then it is clear that $\bigcup_{\alpha\in\Delta} V_\alpha\in\mathcal{V}$. If $\bigcup_{\alpha\in\Delta} V_\alpha \neq X$ and $p_0\notin V_\alpha$ for all $\alpha\in\Delta$, then $p_0\notin\bigcup_{\alpha\in\Delta} V_\alpha$ and $(X - \{p_0\}) - \bigcup_{\alpha\in\Delta} V_\alpha = \bigcap_{\alpha\in\Delta}(X - \{p_0\}) - V_\alpha$ is compact. Therefore, $\bigcup_{\alpha\in\Delta} V_\alpha\in\mathcal{V}$. Let V_1 and V_2 be any two elements in $\mathcal{V}$. If $V_1\cap V_2 = \emptyset$ or $p_0\in V_1$ and $p_0\in V_2$, then it is clear that $V_1\cap V_2\in\mathcal{V}$. If $V_1\cap V_2\neq\emptyset$ and $p_0\notin V_1$, $p_0\notin V_2$, then clearly $p_0\notin V_1\cap V_2$ and $(X - \{p_0\}) - V_1\cap V_2 = ((X - \{p_0\}) - V_1)\cup((X - \{p_0\}) - V_2)$ is compact, therefore $V_1\cap V_2\in\mathcal{V}$. If $V_1\cap V_2\neq\emptyset$ and $p_0\in V_1$, $p_0\notin V_2$, then $p_0\notin V_1\cap V_2$ and $(X - \{p_0\}) - V_1\cap V_2 = (X - V_1)\cup((X - \{p_0\}) - V_2)$ is compact. Thus in any case $V_1\cap V_2\in\mathcal{V}$. Therefore, $\mathcal{V}$ is a topology for X and hence $H(X,\mathcal{U})\subseteq H(X,\mathcal{V})$ by Remark 1.2.

We claim that p_0 is the only point in X such that $\{p_0\}\cup V\in\mathcal{V}$ for any nonvoid V in $\mathcal{V}$. If $V\in\mathcal{V}$ and $p_0\in V$, then clearly

$\{p_o\} \cup V = V \in \mathcal{V}$. If $\emptyset \neq V \in \mathcal{V}$ and $p_o \notin V$, then $(X - \{p_o\}) - V = X - (\{p_o\} \cup V)$ is compact in $\mathcal{U}$. Thus $\{p_o\} \cup V \in \mathcal{U}$ and hence $\{p_o\} \cup V \in \mathcal{V}$. That is, $\{p_o\} \cup V \in \mathcal{V}$ for any $\emptyset \neq V \in \mathcal{V}$. Now, for any $x \neq p_o$, there exist open neighborhoods V_1 of x and V_2 of p_o such that $V_1 \cap V_2 = \emptyset$, thus $V_2 \neq \emptyset$ and $V_2 \in \mathcal{V}$. But $\{x\} \cup V_2 \notin \mathcal{U}$ ($\because \{x\} \notin \mathcal{U}$) and hence $\{x\} \cup V_2 \notin \mathcal{V}$. Therefore, p_o is the only point in X such that $\{p_o\} \cup V \in \mathcal{V}$ for any nonvoid V in $\mathcal{V}$. Hence, it is clear that $f(p_o) = p_o$ for any $f \in H(X, \mathcal{V})$ and therefore $f \,|\, X - \{p_o\} \in H(X - \{p_o\}, \mathcal{V} \,|\, X - \{p_o\})$.

Proceeding by the argument we used in Theorem 1.20, we can easily obtain that $H(X,\mathcal{U}) = H(X, \mathcal{V})$.

<u>Theorem 1.22</u> Let $(X,\mathcal{U})$ be a first countable, compact, Hausdorff space and let $p_o \in X$ such that $f(p_o) = p_o$ for any f in $H(X,\mathcal{U})$. Let $P(U)$ mean that $U \in \mathcal{U}$, $p_o \notin U$ and $(X - \{p_o\}) - U$ is countably compact. Then, $\mathcal{V} = \{U: U = \emptyset \text{ or } U = X \text{ or } P(U)\}$ is a topology for X and $H(X,\mathcal{U}) = H(X, \mathcal{V})$.

<u>Proof</u>: By using the property that any countably compact subset of a first countable space is closed, it is clear that $P(U)$ is a topological property and $\mathcal{V}$ is a topology for X. Thus, we have $H(X,\mathcal{U}) \subseteq H(X, \mathcal{V})$ by Remark 1.2.

By the same argument as in Theorem 1.20, we have $f(p_o) = p_o$ and hence $f \,|\, X - \{p_o\} \in H(X - \{p_o\}, \mathcal{V} \,|\, X - \{p_o\})$ for all $f \in H(X,\mathcal{V})$.

Clearly, for any $A \subseteq X - \{p_o\}$, A is closed and countably compact in $(X,\mathcal{U})$ if and only if A is closed and countably compact in $(X - \{p_o\}, \mathcal{U}|X - \{p_o\})$. Now, since $(X - \{p_o\}, \mathcal{U}|X - \{p_o\})$ is a first countable Hausdorff space, thus by Theorem 1.11

$$\mathcal{W} = \{V \subseteq X - \{p_o\} : V = \emptyset \text{ or } V \in \mathcal{U}|X - \{p_o\} \text{ and } (X - \{p_o\}) - V \text{ is countably compact in } \mathcal{U}|X - \{p_o\}\}$$

is a topology for $X - \{p_o\}$ and $H(X - \{p_o\}, \mathcal{U}|X - \{p_o\}) = H(X - \{p_o\}, \mathcal{W})$.

But, it is clear that

$$\mathcal{W} = \{V \in X - \{p_o\} : V = \emptyset \text{ or } V \in \mathcal{U} \text{ and } (X - \{p_o\}) - V \text{ is countably compact in } \mathcal{U}\}.$$

Thus, $\mathcal{W} = \mathcal{V}|X - \{p_o\}$ and therefore

$$f|X - \{p_o\} \in H(X - \{p_o\}, \mathcal{U}|X - \{p_o\}) = H(X - \{p_o\}, \mathcal{V}|X - \{p_o\})$$

for all $f \in H(X, \mathcal{V})$. Now, if $U \in \mathcal{U}$ and $p_o \notin U$, then $U \subseteq X - \{p_o\}$ and hence $U \in \mathcal{U}|X - \{p_o\}$, $f(U) \in \mathcal{U}|X - \{p_o\} \subseteq \mathcal{U}$. If $U \in \mathcal{U}$ and $p_o \in U$, then $X - U$ is closed and countably compact in $\mathcal{U}$ and hence is closed and countably compact in $(X - \{p_o\}, \mathcal{U}|X - \{p_o\})$. Thus $f(X - U) = X - f(U)$ is closed and countably compact in $(X - \{p_o\}, \mathcal{U}|X - \{p_o\})$ and hence it is closed and countably compact in $(X, \mathcal{U})$. Therefore, $f(U) \in \mathcal{U}$. By a similar argument we have $f^{-1}(U) \in \mathcal{U}$ for any $U \in \mathcal{U}$. Hence $f \in H(X, \mathcal{U})$ and therefore $H(X, \mathcal{U}) = H(X, \mathcal{V})$.

<u>Theorem 1.23</u> Let $(X, \mathcal{U})$ be a first countable, compact, Hausdorff space without isolated points and let $p_o \in X$ such that $f(p_o) = p_o$ for any f in $H(X, \mathcal{U})$. Let $P(V)$ mean that $V \in \mathcal{U}$ and $p_o \in V$ or $p_o \notin V \in \mathcal{U}$ and $(X - \{p_o\}) - V$ is countably compact.

Then, $\mathcal{V} = \{V: \ V = \emptyset \text{ or } P(V)\}$ is a topology for X and $H(X, \mathcal{U}) = H(X, \mathcal{V})$.

<u>Proof</u>: We can use the same arguments as those of Theorem 1.21 and Theorem 1.22.

2. Finer topologies with the same class of homeomrophisms.

Given a topological space $(X,\mathcal{U})$, let $H(X,\mathcal{U})$ be the class of all homeomorphisms of $(X,\mathcal{U})$ onto itself. In section one we have constructed many topologies $\mathcal{V}$ on X such that $H(X,\mathcal{U})=H(X,\mathcal{V})$. However, all topologies constructed in section one are coarser than the original topology. Therefore it is natural to ask the question, given a topological space $(X,\mathcal{U})$, can we construct topologies $\mathcal{V}\supseteq\mathcal{U}$ such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. This section is devoted to investigating this problem.

<u>Definition 2.1</u> Let $(X,\mathcal{U})$ be a topological space and A not an open subset of X. Let $\mathcal{V} = \{O_1 \cup (O_2 \cap A) : O_1 \in \mathcal{U},\ O_2 \in \mathcal{U}\}$. Then $\mathcal{V}$ is a topology for X and is call the simple extension of $\mathcal{U}$ with respect to A.

It seems that N. Levine introduced this concept in [10] and proved some results such as when $\mathcal{V}$ has the same topological property as $\mathcal{U}$ has. Naturally, we will ask the question that under what conditions we have $H(X,\mathcal{U}) = H(X,\mathcal{V})$. It is hard to find result in general; however we have the following two results:

<u>Theorem 2.2</u> Let $(X,\mathcal{U})$ be a topological space and A not an open subset of X such that $A\cap V = \emptyset$ for all $V\in\mathcal{U}$, $V \neq X$, and Card $A \neq$ Card V for every $V\in\mathcal{U}$ which contains no proper nonvoid subset $U\in\mathcal{U}$. Let $\mathcal{V} = \{O_1 \cup (O_2 \cap A) : O_1 \in \mathcal{U},\ O_2 \in \mathcal{U}\}$ be the simple extension of $\mathcal{U}$ with respect to A. Then, $H(X,\mathcal{U}) = H(X,\mathcal{V})$ if and only if $f(A) = A$ for all $f\in H(X,\mathcal{U})$.

Proof: Clearly we have $\mathcal{V} = \mathcal{U} \cup \{A \cup V : V \in \mathcal{U}\}$. We claim that $f(A) = A$ for every $f \in H(X, \mathcal{V})$ and $H(X, \mathcal{V}) \subseteq H(X, \mathcal{U})$.

Let $f \in H(X, \mathcal{V})$. Suppose that there were a nonvoid $V \in \mathcal{U}$ such that $f(A \cup V) = A$. Since $\emptyset \neq f(V) \subseteq A$ and $f(V) \in \mathcal{V}$, thus we have either $f(V) \in \mathcal{U} \subseteq \mathcal{V}$ or $f(V) = A$. If $f(V) \in \mathcal{U}$ then $f(V) \cap A = f(V) \neq \emptyset$ which contradicts our hypothesis. If $f(V) = A$, then f is not a one to one function since $f(A \cup V) = A$ and $A \cap V = \emptyset$. Thus, there does not exist a nonvoid $V \in \mathcal{U}$ such that $f(A \cup V) = A$. Now suppose that there were a nonvoid $V \in \mathcal{U}$ such that $f(V) = A$. It is clear that there does not exist a nonvoid $U \in \mathcal{U}$ such that $U \subsetneqq V$, for otherwise we must have $\emptyset \neq f(U) \in \mathcal{U} \subseteq \mathcal{V}$ and $f(U) \subsetneq A$ which is impossible. Therefore, $f(V) \neq A$ since $\text{Card } A \neq \text{Card } V$ by hypothesis. Thus, the only possibility is $f(A) = A$ for any $f \in H(X, \mathcal{V})$.

Now, suppose $f \in H(X, \mathcal{V})$ and $f(V) = A \cup U$ for some nonvoid V, U in $\mathcal{U}$. Since $f(A) = A$ by the above discussion, thus we have $V \cap A \neq \emptyset$ which is a contradiction. Hence the only possibility is $f(V) \in \mathcal{U} \subseteq \mathcal{V}$ and therefore $H(X, \mathcal{V}) \subseteq H(X, \mathcal{U})$.

Now, if $H(X, \mathcal{U}) = H(X, \mathcal{V})$, then we must have $f(A) = A$ for all $f \in H(X, \mathcal{U})$ by the above discussion. On the other hand, if $f(A) = A$ for all $f \in H(X, \mathcal{U})$, then it is clear that $H(X, \mathcal{U}) \subseteq H(X, \mathcal{V})$ and hence $H(X, \mathcal{U}) = H(X, \mathcal{V})$. Thus, the proof is completed.

Example 2.3 Let $X = \{1,2,3,4,5,\ldots\}$, $A = \{3,4,5\}$ and $\mathcal{U} = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,2,6,7,8,9\}, \{1,2,6,7,8,9,10\}, \{1,2,6,7,8,9,10,11\}, \ldots\}$.

Let $\mathcal{V}$ be the simple extension of $\mathcal{U}$ with respect to A. Then it is clear that $f(A) = A$ for all $f \in H(X, \mathcal{U})$ and $H(X, \mathcal{U}) = H(X, \mathcal{V})$.

<u>Remark 2.4</u> The hypothesis that Card A $\neq$ Card V for every $V \in \mathcal{U}$ which contains no proper nonvoid $U \in \mathcal{U}$ is necessary.
For let $X = \{1,2,3,\ldots\}$, $A = \{1,3,5,\ldots\}$ and $\mathcal{U} = \{\emptyset, X, \{2,4,6,\ldots\}\}$
Then, clearly $\mathcal{V} = \{\emptyset, X, \{1,3,5,\ldots\}, \{2,4,6,\ldots\}\}$ and $f(A) = A$ for all $f \in H(X,\mathcal{U})$ and hence $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$.
Let

$$f(x) = \begin{cases} x + 1 &, \text{ if } x \in \{1,3,5,\ldots\} \\ x - 1 &, \text{ if } x \in \{2,4,6,\ldots\} \end{cases}$$

Then, it is clear that $f \in H(X,\mathcal{V})$ but $f \notin H(X,\mathcal{U})$.

<u>Theorem 2.5</u> Let $(X,\mathcal{U})$ be a topological space such that $\mathcal{U}$ has at least four elements and for any V_1, V_2 in $\mathcal{U}$ we have either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. Suppose A is not an open subset of X and $A \cap V = \emptyset$ for any $V \in \mathcal{U}$, $V \neq X$. If $\mathcal{V}$ is the simple extension of $\mathcal{U}$ with respect to A, then
$H(X,\mathcal{U}) = H(X,\mathcal{V})$ if and only if $f(A) = A$ for all $f \in H(X,\mathcal{U})$.
<u>Proof:</u> The proof is almost the same as that of Theorem 2.2. It is sufficient to prove that $f(A) = A$ for every $f \in H(X,\mathcal{V})$.

Let $f \in H(X,\mathcal{V})$. By the same argument of Theorem 2.2, we know that there does not exist a nonvoid $V \in \mathcal{U}$ such that $f(A \cup V) = A$. Now, if there were a nonvoid $V \in \mathcal{U}$ such that $f(V) = A$, then it is clear that there does not exist nonvoid $U \in \mathcal{U}$ such that $U \subsetneq V$, for otherwise $f(U) \in \mathcal{U}$ and $f(U) \cap A = f(U) \neq \emptyset$ which contradicts the hypothesis. Since $\mathcal{U}$ has at least four elements, thus there exists $V_1 \in \mathcal{U}$ such that $V \subsetneq V_1$ and $V_1 \neq X$. Now, if $f(V_1) \in \mathcal{U} \subseteq \mathcal{V}$, then $f(V_1) \cap A \supseteq f(V) \cap A = A \neq \emptyset$ which is impossible. If $f(V_1) = A \in \mathcal{V}$, then clearly f is not a one to one function since $f(V) = A$ and $V_1 \supsetneq V$. Thus, the only possibility is $f(V_1) = A \cup V_2$ for some $\emptyset \neq V_2 \in \mathcal{U}$. Then, $f(V_1) = f(V) \cup f(V_1 - V) = A \cup V_2$ and we must have $f(V_1 - V) = V_2$ for $f(V) = A$ and $A \cap V_2 = \emptyset$. Therefore, $V_1 - V \in \mathcal{U}$. But it is clear that neither $V_1 - V \subseteq V$ nor $V \subseteq V_1 - V$. Therefore $f(V) \neq A$ for any nonvoid $V \in \mathcal{U}$ and hence the only possibility is $f(A) = A$.

Example 2.6 Let $X = \{1,2,3,\ldots\}$, $A = \{3,4\}$, and $\mathcal{U} = \{\emptyset, X, \{1,2\}, \{1,2,5\}, \{1,2,5,6\}, \ldots\}$.
Let $\mathcal{V}$ be the simple extension of $\mathcal{U}$ with respect to A.
Then clearly $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

Remark 2.7 The hypothesis that $\mathcal{U}$ has at least four elements is necessary as shown by the example in Remark 2.4.

In the remainder of this section we will prove that there does exist new topology $\mathcal{V}$ on X such that $\mathcal{U} \subseteq \mathcal{V}$ and $H(X,\mathcal{U}) = H(X,\mathcal{V})$ in case $(X,\mathcal{U})$ is a regular T_1 Baire space.

Definition 2.8 A family of subsets of $\mathcal{N}$ in $(X,\mathcal{U})$ is called an I-family if the following three conditions are satisfied:

(1) The empty set $\emptyset$ is in $\mathcal{N}$.

(2) If $N \in \mathcal{N}$ then $\operatorname{Int} N = \emptyset$

(3) If $\{N_1, N_2, \ldots, N_k\} \subseteq \mathcal{N}$ and $\{f_1, f_2, \ldots, f_k\} \subseteq H(X,\mathcal{U})$ then $\bigcup\{f_i(N_i): i=1,2,\ldots,k\} \in \mathcal{N}$ for every k.

Lemma 2.9 If $\mathcal{N}$ is an I-family of $(X,\mathcal{U})$ then $\{U-N: U \in \mathcal{U}, N \in \mathcal{N}\}$

forms a base of a topology $\mathcal{V}$ on X. In this case, we say that $\mathcal{V}$ is generated by $\mathcal{A}$.

<u>Proof:</u> Let $U_1, U_2 \in \mathcal{U}$ and $N_1, N_2 \in \mathcal{A}$. Then it is clear that $(U_1 - N_1) \cap (U_2 - N_2) = (U_1 \cap U_2) - (N_1 \cup N_2)$. Since the identity mapping of X onto X is a homeomorphism, thus $N_1 \cup N_2 \in \mathcal{A}$ by condition (3) of Definition 2.8. Therefore, $\{U - N : U \in \mathcal{U}, N \in \mathcal{A}\}$ forms a base of a topology $\mathcal{V}$ on X.

<u>Remark 2.10</u> Let $\mathcal{A}$ be an I-family of $(X, \mathcal{U})$ and $\mathcal{V}$ be the topology generated by $\mathcal{A}$. If every member of $\mathcal{A}$ is closed in $\mathcal{U}$, then we have $\mathcal{U} = \mathcal{V}$.

<u>Theorem 2.11</u> Let $\mathcal{A}$ be an I-family of subsets of a regular T_1 space $(X, \mathcal{U})$. Let $\mathcal{V}$ be the topology generated by $\mathcal{A}$. Then, $\mathcal{U} \subseteq \mathcal{V}$ and $H(X, \mathcal{U}) = H(X, \mathcal{V})$.

<u>Proof</u>: Let $f \in H(X, \mathcal{U})$ and $U \in \mathcal{U}, N \in \mathcal{A}$. Since $f(U - N) = f(U) - f(N)$ and $f(U) \in \mathcal{U}$, $f(N) \in \mathcal{A}$, thus f is continuous in $(X, \mathcal{V})$. Similarly f^{-1} is continuous in $(X, \mathcal{V})$ and therefore $H(X, \mathcal{U}) \subseteq H(X, \mathcal{V})$.

Suppose there exists $g \in H(X, \mathcal{V}) - H(X, \mathcal{U})$. Then either there exists a $V \notin \mathcal{U}_{p_0}$ (neighborhood system of p_0 in $(X, \mathcal{U})$) with $g^{-1}(V) \in \mathcal{U}_{g^{-1}(p_0)}$ or $V \notin \mathcal{U}_{p_0}$ but $g(V) \in \mathcal{U}_{g(p_0)}$ for some $p_0 \in X$. We need only show that the first alternative leads to a contradiction.

It is clear that $\{p_0\} \notin \mathcal{U}$ because $V \notin \mathcal{U}_{p_0}$. Since $g^{-1} \in H(X, \mathcal{V})$ and $g^{-1}(V) \in \mathcal{U}_{g^{-1}(p_0)} \subseteq \mathcal{V}_{g^{-1}(p_0)}$, we have $V \in \mathcal{V}_{p_0}$.

Hence there exists $U \in \mathcal{U}$ and $N \in \mathcal{N}$ such that $p_o \in U - N \subseteq V$ and therefore we must have $p_o \in Cl(N) - N$ for if $p_o \notin Cl(U)$ then $p_o \in U - Cl(U) \subseteq V$ and hence $V \in \mathcal{U}_{p_o}$ which is a contradiction. Since $g^{-1}(p_o) \in g^{-1}(V) \in \mathcal{U}_{g^{-1}(p_o)}$ and $(X, \mathcal{U})$ is regular, there exists $g^{-1}(V_1) \in \mathcal{U}_{g^{-1}(p_o)}$ such that $Cl(g^{-1}(V_1)) \subseteq g^{-1}(V)$. Since $g^{-1}(V_1) \in \mathcal{U}_{g^{-1}(p_o)} \subseteq \mathcal{V}_{g^{-1}(p_o)}$, it follows that $V_1 \in \mathcal{V}_{p_o}$ and $V_1 \supseteq U_1 - N_1$ for some $U_1 \in \mathcal{U}_{p_o} \cap \mathcal{U}$, $N_1 \in \mathcal{N}$ and $p_o \in Cl(N_1) - N_1$. It is clear that $((N \cap U) - V) \cap ((N_1 \cap U_1) - V) \neq \emptyset$ for otherwise $((N \cap U) \cap (N_1 \cap U_1)) - V = \emptyset$, thus $(N \cap N_1) \cap (U \cap U_1) \subseteq V$; but $(U \cap U_1) - (N \cap N_1) \subseteq V$ and hence we could have $V \supseteq U \cap U_1 \in \mathcal{U}_{p_o}$, a contradiction. Let $q \in ((N \cap U) - V) \cap ((N_1 \cap U_1) - V)$. Then $g^{-1}(q) \in g^{-1}((N \cap U) - V) \subseteq X - g^{-1}(V) \subseteq X - Cl(g^{-1}(V_1)) \subseteq X - Cl_{\mathcal{V}}(g^{-1}(V_1))$ and thus $g^{-1}(q) \notin Cl_{\mathcal{V}}(g^{-1}(V_1))$.

But $q \in (N_1 \cap U_1) - V$; hence if $q \notin Cl_{\mathcal{V}}(U_1 - N_1)$, then there exists $U_2 - N_2 \in \mathcal{V}_q$ such that $U_2 \in \mathcal{U}_q \cap \mathcal{U}$, $N_2 \in \mathcal{N}$ and $(U_2 - N_2) \cap (U_1 - N_1) = (U_2 \cap U_1) - (N_2 \cup N_1) = \emptyset$. Since $\emptyset \neq U_2 \cap U_1 \in \mathcal{U} \cap \mathcal{U}_q$ and $U_2 \cap U_1 \subseteq N_2 \cup N_1$ which would imply that $N_2 \cup N_1 \notin \mathcal{N}$, a contradiction. Therefore, $q \in Cl_{\mathcal{V}}(U_1 - N_1) \subseteq Cl_{\mathcal{V}}(V_1)$. Hence $g^{-1}(q) \in Cl_{\mathcal{V}}(g^{-1}(V_1))$. This contradiction shows that

$g^{-1}(V) \in \mathcal{U}_{g^{-1}(p_o)}$ implies $V \in \mathcal{U}_{p_o}$ for each p_o in X. Thus g^{-1} is continuous. Similarly g is continuous and we have $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

The following corollary serves as an example to illustrate the preceding theorem.

<u>Corollary 2.12</u> If $(X,\mathcal{U})$ is a regular T_1 Baire space and $\mathcal{N}$ is the family of subsets of the first category in X. Then $\mathcal{N}$ is an I-family of $(X,\mathcal{U})$. Let $\mathcal{V}$ be the topology generated by $\mathcal{N}$, then $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

<u>Proof</u>: It suffices to show that $\mathcal{N}$ is an I-family of $(X,\mathcal{U})$. Clearly condition (1) and (3) of Definition 2.8 are satisfied by $\mathcal{N}$. Let $N \in \mathcal{N}$, $N = \bigcup_{i=1}^{\infty} N_i$ where N_i are nowhere dense subsets of $(X,\mathcal{U})$ and let U be any open subset of $(X,\mathcal{U})$ such that $U \subseteq N = \bigcup_{i=1}^{\infty} N_i$. We are to prove $U = \emptyset$. Since $U \subseteq \bigcup_{i=1}^{\infty} N_i \subseteq \bigcup_{i=1}^{\infty} \mathrm{Cl}(N_i)$, then we have $\bigcap_{i=1}^{\infty} (X - \mathrm{Cl}(N_i)) \subseteq X - U$. But, since $X - \mathrm{Cl}(N_i)$ are open dense sets and $(X,\mathcal{U})$ is a Baire space, thus the closed set $X - U$ is dense in X, so that $X = X - U$ and therefore $U = \emptyset$. Hence $\mathcal{N}$ is an I-family indeed. Therefore, $H(X,\mathcal{U}) = H(X,\mathcal{V})$ by Theorem 2.11.

<u>Remark 2.13</u> In Corollary 2.12, if $(X,\mathcal{U})$ is a first countable, regular T_1 Baire space and $\mathcal{N}$ contains at least one element which is not closed in $(X,\mathcal{U})$, then $(X,\mathcal{U})$ and $(X,\mathcal{V})$ are not homeomorphic.

<u>Proof</u>: Let $N \in \mathcal{N}$ be a nonclosed subset of $(X, \mathcal{U})$. Then there exists a point $p_o \in X - N$ and a sequence of points $\{p_i: i=1,2,\ldots\} \subseteq N$ which converges to p_o. Then clearly $\{p_i: i=1,2,3,\ldots\} \in \mathcal{N}$, then $X - \{p_i: i=1,2,\ldots\} \in \mathcal{V}$ and hence $\{p_i: i=1,2,3,\ldots\}$ is closed in $(X, \mathcal{V})$. Suppose $(X, \mathcal{V})$ is regular, then there exist open sets A_1, A_2 in $(X, \mathcal{V})$ such that $p_o \in A_1$, $\{p_i: i=1,2,\ldots\} \subseteq A_2$ and $A_1 \cap A_2 = \emptyset$. Therefore, there exist $\{U_i: i=0,1,2,\ldots\}$ in $\mathcal{U}$ and $\{N_i: i=0,1,2,\ldots\}$ in $\mathcal{N}$ such that $p_i \in U_i - N_i$ for $i=0,1,\ldots$ and $\emptyset = (U_o - N_o) \cap \bigcup_{i=1}^{\infty}(U_i - N_i) = \bigcup_{i=1}^{\infty}((U_o \cap U_i) - (N_o \cup N_i))$. Hence $U_o \cap U_i \subseteq N_o \cup N_i$ for each $i=1,2,3,\ldots$. But, since $U_o \in \text{Nbhd } p_o$, thus U_o contains all but finitely many p_i. Therefore there exists $i \neq 0$ such that $U_o \cap U_i \neq \emptyset$. Hence $N_o \cup N_i$ contains nonvoid interior and therefore $N_o \cup N_i \notin \mathcal{N}$ which is a contradiction. Hence $(X, \mathcal{V})$ is not a regular space and thus $(X, \mathcal{U})$ and $(X, \mathcal{V})$ are not homeomorphic.

3. Continua with the same class of homeomorphisms.

Let $H(X,\mathcal{U})$ be the class of all homeomorphisms of a topological space $(X,\mathcal{U})$ onto itself. In the previous two sections many different topologies $\mathcal{V}$ for X have been constructed such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$ and $(X,\mathcal{U})$ and $(X,\mathcal{V})$ are not homeomorphic. However all topologies constructed for X ever since are either non-Hausdorff or non-compact. In this section, we will show the existence of non-homeomorphic continua with the same class of homeomorphisms by repeatedly applying the following two theorems.

<u>Lemma 3.1</u> Let p be a point in a Hausdorff space $(X,\mathcal{U})$ and $\mathcal{V} = \{V \in \mathcal{U}: p \notin V \text{ or } X - V \text{ is compact}\}$. Then $(X,\mathcal{V})$ is a topological space and $\mathcal{V} \subseteq \mathcal{U}$, and moreover, $(X,\mathcal{V})$ is a Hausdorff space if and only if $(X,\mathcal{U})$ is locally compact at all $q \neq p$.

<u>Proof</u>: It is clear that $(X,\mathcal{V})$ is a topological space and $\mathcal{V} \subseteq \mathcal{U}$. Now suppose that $(X,\mathcal{V})$ is a Hausdorff space; then for any $q \neq p$ there exist V_1, V_2 in $\mathcal{V}$ such that $q \in V_1$, $p \in V_2$ and $V_1 \cap V_2 = \emptyset$. By the construction of $\mathcal{V}$ it is clear that $q \in X - V_2$ and $X - V_2$ is compact. Therefore $(X,\mathcal{U})$ is locally compact at all $q \neq p$. Now suppose that $(X,\mathcal{U})$ is locally compact at all $q \neq p$. Since $(X,\mathcal{U})$ is a Hausdorff space, thus for any $q_1 \in X$, $q_2 \in X$ and $q_1 \neq q_2 \neq p$, there exist $V_1 \in \mathcal{U}$, $V_2 \in \mathcal{U}$ such that $q_1 \in V_1$, $q_2 \in V_2$, $p \notin V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$; therefore $V_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}$ and hence q_1 and q_2 can be separated by two open sets in $\mathcal{V}$. Now, consider the point p and any other point $q \neq p$. Then there exist $V_1 \in \mathcal{U}$

and $V_2 \in \mathcal{U}$ such that $p \in V_1$, $q \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since $(X, \mathcal{U})$ is locally compact at $q \neq p$, thus there exists $V_3 \in \mathcal{U}$ such that $Cl(V_3)$ is compact and $q \in V_3 \subseteq Cl(V_3) \subseteq V_2$. It is clear that $V_3 \in \mathcal{V}$, $X - Cl(V_3) \in \mathcal{V}$ and hence p and q are separated by open sets in $\mathcal{V}$. Therefore $(X, \mathcal{V})$ is indeed a Hausdorff space.

Theorem 3.2. Let X, $\mathcal{U}$, $\mathcal{V}$, and p be as in Lemma 3.1. Suppose the following two conditions are satisfied:

(a) $f(p) = p$ for all $f \in H(X,\mathcal{U}) \cup H(X,\mathcal{V})$,

(b) If $p \in Cl(A)$, then $p \in Cl(g(A))$ for each $A \subseteq X - \{p\}$ and $g \in H(X - \{p\}, \mathcal{U}|X - \{p\})$.

Then $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

Proof: Since $f(p) = p$ for all f in $H(X,\mathcal{U})$, thus it is clear that $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$.

Now we are going to show that $H(X,\mathcal{U}) \supseteq H(X,\mathcal{V})$. Since $X - \{p\} \in \mathcal{U}$, thus by the construction of $\mathcal{V}$ it is clear that $\mathcal{V}|X - \{p\} = \mathcal{U}|X - \{p\}$. Now if $f \in H(X,\mathcal{V})$, then by (a), $f(p) = p$, we have $f \in H(X - \{p\}, \mathcal{V}|X - \{p\})$ and hence $f \in H(X - \{p\}, \mathcal{U}|X - \{p\}) = H(X - \{p\}, \mathcal{V}|X - \{p\})$. That is, $f|X - \{p\}$ is bicontinuous at every q in $X - \{p\}$ relative to $\mathcal{U}|X - \{p\}$. Since $(X,\mathcal{U})$ is Hausdorff, thus f is bicontinuous at each q in $X - \{p\}$ relative to $\mathcal{U}$. By (b), f and f^{-1} are also continuous at p relative to $\mathcal{U}$ and hence f is also in $H(X,\mathcal{U})$. Therefore $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

The next theorem is to reverse the order of constructing the topology, but the proof is essentially the same as Theorem 3.2.

<u>Theorem 3.3</u> Let p be a point in a Hausdorff space $(X,\mathcal{U})$ and $V \in \mathcal{U}$ such that $p \notin V$ and $f(V) = V$ for all f in $H(X,\mathcal{U})$. By $\mathcal{U}_q$ we denote the neighborhood system (not necessarily open) at q in $(X,\mathcal{U})$. Let $\mathcal{V}_q = \mathcal{U}_q$ if $q \neq p$ and $\mathcal{V}_p = \{U - V: \ U \in \mathcal{U}_p\}$ and let $\mathcal{V}$ be the topology generated by taking $\mathcal{V}_q$ as a base of the neighborhood system at q. Suppose the following two conditions are satisfied:

(a) $f(p) = p$ for all f in $H(X,\mathcal{U}) \cup H(X,\mathcal{V})$,

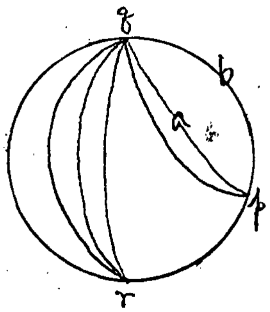
(b) If $p \in Cl(A)$, then $p \in Cl(g(A))$ for each $A \subseteq X - \{p\}$ and $g \in H(X - \{p\}, \mathcal{U}|X - \{p\})$.

Then $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

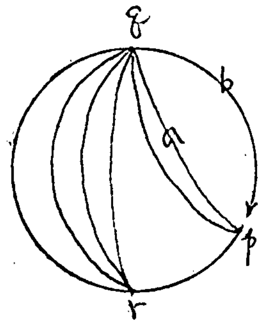
<u>Proof</u>: First of all we show that $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$. If V_1 is a neighborhood of $q \in X$, $q \neq p$ in $\mathcal{V}$, then there exists $U \in \mathcal{U}_q$ such that $q \in U \subseteq V_1$. Therefore, for any $f \in H(X,\mathcal{U})$ it is clear that $f(V_1)$ is a neighborhood of $f(q)$ in $\mathcal{V}$. If V_1 is a neighborhood of p in $\mathcal{V}$, then there exists $U \in \mathcal{U}_p$ such that $p \in U - V \subseteq V_1$. Since $f(p) = p$ and $f(V) = V$ for all $f \in H(X,\mathcal{U})$, thus it is clear that $f(V_1)$ is a neighborhood of $f(p)$ in $\mathcal{V}$. Therefore, f^{-1} is continuous in $\mathcal{V}$ for all $f \in H(X,\mathcal{U})$. Similarly, f is continuous in $\mathcal{V}$ for all $f \in H(X,\mathcal{U})$. Hence $H(X,\mathcal{U}) \subseteq H(X,\mathcal{V})$.

Now, if $f \in H(X,\mathcal{V})$, then by (a), $f(p) = p$ for all $f \in H(X,\mathcal{V})$ and by the construction of $\mathcal{V}$, we have $\mathcal{V}|X - \{p\} = \mathcal{U}|X - \{p\}$ and hence $f \in H(X - \{p\}, \mathcal{V}|X - \{p\}) = H(X - \{p\}, \mathcal{U}|X - \{p\})$. Since $(X,\mathcal{U})$ is Hausdorff, hence it is clear that f is bicontinuous at each q in $X - \{p\}$ relative to $\mathcal{U}$. By (b), f and f^{-1} are also continuous at p relative to $\mathcal{U}$. Hence $f \in H(X,\mathcal{U})$ and therefore $H(X,\mathcal{U}) = H(X,\mathcal{V})$.

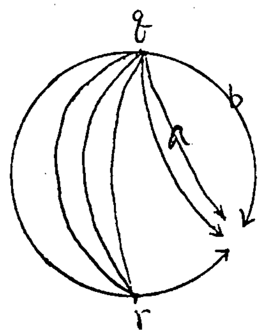
<u>Remark 3.4</u> Yu-Lee Lee [7] gave the above theorem without the condition that $f(V) = V$ for all $f \in H(X,\mathcal{U})$. However, the condition is necessary as shown by the following example:



(Figure 1)



(Figure 2)



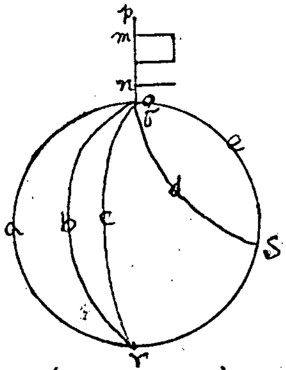
(Figure 3)

Let $(X,\mathcal{U})$ be the topological space as shown in (Figure 1) with the usual topology. Let V be the open arc $\widehat{pbq}$. Clearly, the topology $\mathcal{V}$ constructed by the method as in Theorem 3.3 with respect to the point p and open set V can be described by (Figure 2) with the usual topology. Thus, it is clear that $f(p) = p$ for all $f \in H(X,\mathcal{U}) \cup H(X,\mathcal{V})$ and hence condition (a) of Theorem 3.3 is satisfied.

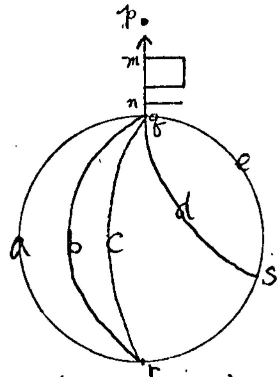
Clearly $(X - \{p\}, \mathcal{U}|X - \{p\})$ can be described by (Figure 3) with the usual topology. By a simple argument it is clear that condition (b) of Theorem 3.3 is also satisfied.

Now, we let f be the function that maps the arc $\overset{\frown}{qbp}$ onto the arc $\overset{\frown}{qap}$ (i.e. rotating the arcs $\overset{\frown}{qbp}$ and $\overset{\frown}{qap}$ with respect to points p and q.) and $f(x) = x$ elsewhere. Then, it is clear that $f \in H(X,\mathcal{U})$ but $f \notin H(X,\mathcal{V})$. Hence $H(X,\mathcal{U}) \neq H(X,\mathcal{V})$.

Now, we are going to apply Theorem 3.2 and Theorem 3.3 to construct non-homeomorphic continua topologies for a set but with the same class of homeomorphisms.



(Figure 4)



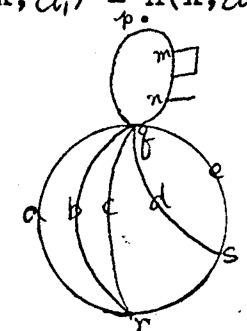
(Figure 5)

Let $(X,\mathcal{U}_1)$ be a plane continuum as shown in (Figure 4). Let $V = X - \{p\}$ and $\mathcal{U}_2$ be the topology constructed as in Theorem 3.3 with respect to the point p and open set V. Then, clearly $(X,\mathcal{U}_2)$ can be described by (Figure 5) with the usual topology.

(Claim 1): $H(X,\mathcal{U}_1) = H(X,\mathcal{U}_2)$.

If U is a neighborhood of p in $\mathcal{U}_1$, it is clear that $U - \{p\}$ is connected. But, for a suitable neighborhood U of $t \in X$, $t \neq p$ in $\mathcal{U}_1$ we know that $U - \{t\}$ contains at least two components in $\mathcal{U}_1$, thus it is clear that $f(p) = p$ for all $f \in H(X,\mathcal{U}_1)$ and hence $f(V) = V$ for all $f \in H(X,\mathcal{U}_1)$. Since $p \in X$ is the only point in $(X,\mathcal{U}_2)$ such that $\{p\} \in \mathcal{U}_2$, hence $f(p) = p$ for all $f \in H(X,\mathcal{U}_2)$. Thus, the condition (a) of Theorem 3.3 is satisfied.

If $p \in Cl_{\mathcal{U}_1}(A)$ for some $A \subseteq X - \{p\}$, then it is clear that there exists a sequence $\{x_n\}$ in $A \cap \overrightarrow{mp}$ such that $\{x_n\}$ converges to p . Since for any $g \in H(X - \{p\}, \mathcal{U}_1 | X - \{p\})$, $g | \overrightarrow{mp}$ is the identity mapping, thus it is clear that $p \in Cl_{\mathcal{U}_1}(g(A))$. Therefore, the condition (b) of Theorem 3.3 is satisfied and hence $H(X,\mathcal{U}_1) = H(X,\mathcal{U}_2)$.



(Figure 6)

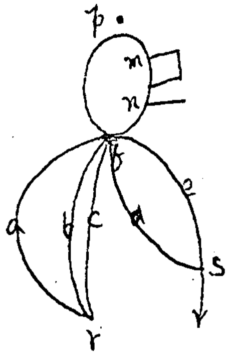
Let $(X,\mathcal{U}_3)$ be the topology constructed by the method as in Lemma 3.1 with respect to the point q in $(X,\mathcal{U}_2)$. Clearly $(X,\mathcal{U}_3)$ can be described by (Figure 6) with the usual topology.

(Claim 2): $H(X,\mathcal{U}_2) = H(X,\mathcal{U}_3)$.

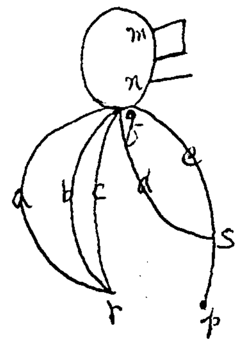
If V_1 is a neighborhood of q in $(X,\mathcal{U}_2)$, then it is clear that $V_1 - \{q\}$ contains six compoments in $\mathcal{U}_2$. But, for a suitable neighborhood U of $t \in X$, $t \neq q$ it is clear that $U - \{t\}$ contains at most four compoments of $\mathcal{U}_2$. Thus, $f(q) = q$ for all $f \in H(X,\mathcal{U}_2)$. By the same argument, it is clear that $f(q) = q$ for all $f \in H(X,\mathcal{U}_3)$. Therefore, the condition (a) of Theorem 3.2 is satisfied.

Now, if $q \in Cl_{\mathcal{U}_2}(A)$ for some $A = X - \{q\}$. Then, there exists a sequence $\{x_k\}$ in the intersection of A and segment $\overrightarrow{nq}$ or one of arcs $\overset{\frown}{seq}$, $\overset{\frown}{sdq}$, $\overset{\frown}{raq}$, $\overset{\frown}{rbq}$ and $\overset{\frown}{rcq}$ such that $\{x_k\}$ converges to q. For convenience, we assume that $\{x_k\}$ is in the intersection of A and arc $\overset{\frown}{seq}$. But, since for any $g \in H(X - \{q\}, \mathcal{U}_2 | X - \{q\})$, we have either $g | (\overset{\frown}{seq})$ be the identity mapping or g rotating arc $\overset{\frown}{seq}$ and arc $\overset{\frown}{sdq}$ with respect to the point s , thus it is clear that $\{g(x_k)\}$ converges to q and hence $q \in Cl_{\mathcal{U}_2}(g(A))$.

Therefore, the condition (b) of Theorem 3.2 is satisfied and hence $H(X,\mathcal{U}_2) = H(X,\mathcal{U}_3)$.



(Figure 7)



(Figure 8)

Let $(X,\mathcal{U}_4)$ be the topology constructed by the method as in Theorem 3.3 with respect to the point r and V = the open arc $\overset{\frown}{rs}$ in $(X,\mathcal{U}_3)$. Then it is clear that $(X,\mathcal{U}_4)$ can be described by (Figure 7) with the usual topology.

By the same argument as in (Claim 1) and (Claim 2), it is clear that the conditions (a) and (b) in Theorem 3.3 are satisfied. Hence we have $H(X,\mathcal{U}_3) = H(X,\mathcal{U}_4)$.

Let $(X,\mathcal{U}_5)$ be the topology constructed by the method as in Lemma 3.1 with respect to point p in $(X,\mathcal{U}_4)$. Clearly $(X,\mathcal{U}_5)$ can be described by (Figure 8) with the usual topology. By the same argument as in (Claim 1) and (Claim 2) we have $H(X,\mathcal{U}_4) = H(X,\mathcal{U}_5)$. Therefore $(X,\mathcal{U}_5)$ is a continuum and $H(X,\mathcal{U}_1) = H(X,\mathcal{U}_5)$.

(Claim 3): $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_5)$ are not homeomorphic.

Suppose that f is a homeomorphism between $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_5)$. If U is a suitable neighborhood of r in $\mathcal{U}_1$, then it is clear that $U - \{r\}$ contains four components. But, for any neighborhood

U of $f(r)$ in $\mathcal{U}_5$ it is clear that $U - \{f(r)\}$ does not contain four components. Therefore, f can not be a homeomorphism between $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_5)$. That is, $(X,\mathcal{U}_1)$ and $(X,\mathcal{U}_5)$ are not homeomorphic.

4. Characterizing the topology by the class of homeomorphisms:

In this section, we will consider the following problem:

Suppose X is the real line and $\mathcal{U}$ is the usual topology on X. Let $(X,\mathcal{V})$ be a topological space such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$ and which satisfies some additional conditions. Then what can we say about the topology $\mathcal{V}$?

<u>Lemma 4.1</u> Let $(X,\mathcal{U})$ be the real line with usual topology. Let $U = (a, b)$ be any open interval in X and x, z be any two points in U. Then there exists a homeomorphism $f \in H(X,\mathcal{U})$ such that $f(x) = z$ and $f(y) = y$ for all $y \in X - U$.

<u>Proof</u>: Since translations and scalar multiplication are homeomorphisms of $(X,\mathcal{U})$ onto itself, thus it suffices to show that if $U = (-1, 1)$ and $z \in U$, then there exists $f \in H(X,\mathcal{U})$ such that $f(0) = z$ and $f(y) = y$ for all $y \in X - U$.

Let

$$f(y) = \begin{cases} y + z(1 - |y|) , & \text{if } |y| \leq 1 \\ y , & \text{if } |y| \geq 1 . \end{cases}$$

It is clear that the following inequalities hold.

$$(1 + |z|)\ |y - x| \geq |f(y) - f(x)| \geq (1 - |z|)\ |y - x| .$$

These immediately imply that f is the homeomorphism required.

<u>Remark 4.2</u> Let $(X,\mathcal{U})$ be the real line with usual topology and $a < b < c < d$ be any four points in X. By Lemma 4.1 , it is clear that there exists $f \in H(X,\mathcal{U})$ such that

$f((c, d)) = (b, d)$, $f((a, c)) = (a, b)$ and $f(x) = x$ for all $x \in X - (a,d)$.

<u>Lemma 4.3</u> Let $(X,\mathcal{U})$ be the real line with usual topology and let $(X,\mathcal{V})$ be any Hausdorff space such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. Then $\mathcal{U} \subseteq \mathcal{V}$.

<u>Proof</u>: Let $U = (a, b)$ be any open interval in X and x be an arbitrarily fixed point in U. Let $c \in U$ and $c \neq x$. Since $(X,\mathcal{V})$ is a Hausdorff space, hence there exist V_1 and V_2 in $\mathcal{V}$ such that $c \in V_1$, $x \in V_2$ and $V_1 \cap V_2 = \emptyset$. By Lemma 1, there exists $f \in H(X,\mathcal{U})$ such that $f(c) = x$ and $f(y) = y$ for all $y \in X - U$. Since $H(X,\mathcal{U}) = H(X,\mathcal{V})$ by hypothesis, thus $f \in H(X,\mathcal{V})$ and hence $x \in f(V_1) \cap V_2 \in \mathcal{V}$ and $f(V_1) \cap V_2 \subseteq U$. Therefore, $U \in \mathcal{V}$ and hence $\mathcal{U} \subseteq \mathcal{V}$.

<u>Lemma 4.4</u> Let $(X,\mathcal{U})$ be the real line with usual topology and $(X,\mathcal{V})$ be a Hausdorff space. If $H(X,\mathcal{U}) = H(X,\mathcal{V})$, then each member V of $\mathcal{V}$ can be written in the form $U - N$ where $U \in \mathcal{U}$, $N \subseteq U$ and $\text{Int}(N) = \emptyset$.

<u>Proof</u>: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. If $V \in \mathcal{V} - \mathcal{U}$, then we claim that $(V \cap \text{Cl}(X - V)) \cap \text{Cl}(\text{Int}(X - V)) = \emptyset$. For otherwise, if p is a point in this intersection, then $p \in V \cap \text{Cl}(\text{Int}(X-V))$ and thus every open $\mathcal{U}$-neighborhood U of p contains points in $\text{Int}(X - V)$ and therefore $U \supseteq I$ for some interval $I = (a, b) \subseteq X-V$. Since $(X,\mathcal{U})$ is a first countable space, hence there exists a sequence of intervals $\{(a_i, b_i): i=1,2,\ldots\}$ in $X - V$ such that $\{a_i\}$ and $\{b_i\}$ are converging to p . Without loss of generality, we may assume that $p < a_{i+1} < b_{i+1} < a_i$ for each i. Let $\{c_i\}$ and $\{d_i\}$ be two sequences of points in X such that

$$a_{i+1} < c_i < d_i < b_{i+1} \quad \text{for each } i=1,2,\ldots$$

For convenience, let $c_o = b_1$. Then by Remark 4.2, there exists a sequence of homeomorphisms $\{f_i\}$ in $H(X,\mathcal{U})$ such that $f_i((a_i, c_{i-1})) = (d_i, c_{i-1})$, $f_i((c_i, a_i)) = (c_i, d_i)$ and $f_i(x) = x$ for all $x \in X - (c_i, c_{i-1})$, $i=1,2,\ldots$.

Let f be a function defined by

$$f(x) = \begin{cases} x \quad , & \text{if } x \in X - (p, b_1) \\ f_i(x), & \text{if } x \in (c_i, c_{i-1}). \end{cases}$$

Then, f is a one-to-one function from $(X,\mathcal{U})$ onto itself such that $f(x) = x$ for all $x \in X - (p, b_1)$, $f((a_i, b_i)) \supseteq [b_{i+1}, a_i]$ for each $i=1,2,\ldots$ and

$$f^{-1}(x) = \begin{cases} x, & \text{if } x \in X - (p, b_1) \\ f_i^{-1}(x), & \text{if } x \in (c_i, c_{i-1}). \end{cases}$$

Clearly f and f^{-1} are continuous at points $x \neq p$. Now, let (c, d) be any open interval containing the point p. Then, there is an integer N such that $c < p < c_{N+1} < d \leqslant c_N$. Thus,

$$\begin{aligned} f^{-1}((c, d)) &= f^{-1}((c, p]) \cup \bigcup_{i=N+2}^{\infty} f_i^{-1}((c_i, c_{i-1})) \cup f_{N+1}^{-1}((c_{N+1}, d)) \cup \\ &\quad \{f^{-1}(c_i): i=N+1, N+2, \ldots\} \\ &= (c, p] \cup \bigcup_{i=N+2}^{\infty} (c_i, c_{i-1}) \cup \{c_i: i=N+1, N+2, \ldots\} \cup f_{N+1}^{-1}((c_{N+1}, d)) \\ &= (c, c_{N+1}) \cup f_{N+1}^{-1}((c_{N+1}, d)). \end{aligned}$$

Hence $f^{-1}((c, d))$ is open in $(X,\mathcal{U})$ and therefore f is also continuous at p. Similarly, f^{-1} is continuous at p. Thus, f is a homeomorphism from $(X,\mathcal{U})$ onto itself. Let g be the reflection

of X about p, i.e., $g(p + x) = p - x$ for each $x \in X$ and let (a, b) be an open interval about p such that $g(a_1) < a < b < a_1$.
Since $\{f, g\} \subseteq H(X,\mathcal{U}) = H(X,\mathcal{V})$ and $(a_i, b_i) \subseteq X - V$ for each i, thus it is clear that
$(a, b) \cap \{X - [(X-V) \cup f(X-V) \cup g(X-V) \cup g(f(X-V))]\} = \{p\} \in \mathcal{V}$.
Clearly this implies that $\mathcal{V}$ is the discrete topology and hence $H(X,\mathcal{U})$ ($= H(X,\mathcal{V})$) contains all one-to-one functions of X onto itself, which is a contradiction. Therefore, it is true that $(V \cap Cl(X - V)) \cap Cl(Int(X - V)) = \emptyset$.

Let $U = X - Cl(Int(X - V))$. If $x \in V$ and $x \in Cl(X - V)$, then $x \in V \cap Cl(X - V)$ and hence $x \in X - Cl(Int(X - V)) = U$. If $x \in V$ and $x \notin Cl(X - V)$, then $x \notin Cl(Int(X - V))$ and hence $x \in X - Cl(Int(X - V)) = U$. Therefore $V \subseteq U$. Set $N = U - V$, and we have $V = U - N$ and $Int(N) = Int(U \cap (X - V)) = U \cap Int(X - V) = \emptyset$. Therefore, the proof is completed.

<u>Theorem 4.5</u> Let $(X,\mathcal{U})$ be the real line with usual topology and let $(X,\mathcal{V})$ be a locally compact Hausdorff space such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. Then $\mathcal{U} = \mathcal{V}$.

<u>Proof</u>: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{U} \neq \mathcal{V}$; then there exist a point p in X and a set $V \in \mathcal{V}_p - \mathcal{U}_p$ such that $p \in V$ and $Cl_{\mathcal{V}}(V)$ is compact in $(X,\mathcal{V})$. By Lemma 4.4, we know that $p \in V = U - N$ for some $U \in \mathcal{U}$, $N \subseteq U$ and $Int(N) = \emptyset$. Clearly $p \in Cl(N) - N$, for otherwise $p \in V = U - N \supseteq U - Cl(N)$ implies that $V \in \mathcal{U}_p$. Choose $\{p_i\}$ in N such that $\{p_i\}$ converges to p in $\mathcal{U}$. Now $Cl_{\mathcal{V}}(V)$ is $\mathcal{V}$-compact, hence $\mathcal{U}$-compact, hence $\mathcal{U}$-closed, so that $\{p_i\} \subseteq Cl_{\mathcal{V}}(V)$. But, since $p \in V$ and

$\{p_i\} \subseteq X - V$, hence p is not a $\mathcal{V}$-accumulation point of $\{p_i\}$. Furthermore, if x is a $\mathcal{V}$-accumulation point of $\{p_i\}$, then it is clear that x is a $\mathcal{U}$-accumulation point of $\{p_i\}$ and hence $x = p$, a contradiction. Therefore $\{p_i\}$ has no $\mathcal{V}$-accumulation point, which contradicts the compactness of $Cl_{\mathcal{V}}(V)$. Hence we must have $\mathcal{U} = \mathcal{V}$.

<u>Theorem 4.6</u> Let $(X, \mathcal{U})$ be the real line with usual topology and let $(X, \mathcal{V})$ be a first countable Hausforff space such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$. Then $\mathcal{U} = \mathcal{V}$.

<u>Proof</u>: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{U} \neq \mathcal{V}$, then there exist a point $p \in X$ and a set $V_o \in \mathcal{V}_p - \mathcal{U}_p$ such that $p \in V_o = U_o \dot{-} N_o$ for some $U_o \in \mathcal{U}_p$, $N_o \subseteq U_o$, $Int(N_o) = \emptyset$ and $p \in Cl(N_o) - N_o$. Clearly, for any $V \in \mathcal{V}$ such that $p \in V \subseteq V_o$ we have $V \in \mathcal{V}_p - \mathcal{U}_p$, for otherwise $p \in V_o \in \mathcal{U}_p$, a contradiction. Since $(X, \mathcal{V})$ is first countable, hence there exists a decreasing sequence $\{V_i = U_i - N_i : i = 1, 2, \ldots\}$ in $\mathcal{V} - \mathcal{U}$ such that $\{V_i\}$ forms a local base at p. By the Hausdorff property, it is clear that $\cap \{V_i : i = 1, 2, \ldots\} = \{p\}$. We choose p_i in V_i for each i. Then $\{p_i\}$ is a sequence converging to p in $(X, \mathcal{V})$ and therefore a sequence converging to p in $(X, \mathcal{U})$. Clearly $p \in Cl(N_1) - N_1$, and hence there exists a sequence $\{a_i\} \subseteq N_1$ converging to p in $(X, \mathcal{U})$. Without loss of generality, we may choose both sequences $\{a_i\}$ and $\{p_i\}$ with $a_i > p_i > a_{i+1} > p$ for each i. Let $\{c_i\}$ be a sequence in X such that $c_1 > a_1$ and

$a_i > p_i > c_{i+1} > a_{i+1}$ for each $i=2,3,\ldots$. By Lemma 4.1, there exists a sequence of homeomorphisms $\{f_i\} \subseteq H(X,\mathcal{U})$ such that $f_i(a_i) = p_i$ and $f_i(x) = x$ for all $x \in X - (c_{i+1}, c_i)$, $i=1,2,\ldots$.

Let

$$f(x) = \begin{cases} x \quad , & \text{if } x \in X - (p, c_1) \\ f_i(x), & \text{if } x \in (c_{i+1}, c_i). \end{cases}$$

Then, by the same argument as in Lemma 4.4, it is clear that f is a homeomorphism from $(X,\mathcal{U})$ onto itself. That is, we can construct a homeomorphism $f \in H(X,\mathcal{U})$ such that $f(N_1) \supseteq \{p_i\}$ and $f(p) = p$. Therefore $p \in f(V_1) \in \mathcal{V}_p$ and $f(V_1) \cap \{p_i\colon i=1,2,\ldots\} = \emptyset$. Hence there is no V_i such that $V_i \subseteq f(V_1)$. That means $f(V_1) \notin \mathcal{V}_p$, a contradiction. Hence we must have $\mathcal{U} = \mathcal{V}$.

<u>Theorem 4.7</u> Let $(X,\mathcal{U})$ be the real line with usual topology and let $(X,\mathcal{V})$ be a locally arcwise connected, Hausdorff space such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. Then $\mathcal{U} = \mathcal{V}$.

<u>Proof</u>: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{U} \neq \mathcal{V}$, then there exist a point $p \in X$ and a set $V \in \mathcal{V}_p - \mathcal{U}_p$ such that V is arcwise connected in $(X,\mathcal{V})$ and $V = U - N$ for some $U \in \mathcal{U}$, $N \subseteq U$, $\mathrm{Int}(N) = \emptyset$. If q is a point in V, then there exists a closed arc $\widehat{pq}$ in V. Let $\{p_i\}$ in the closed arc $\widehat{pq}$ be such that $\{p_i\}$ converges to p in the arc and hence in $(X,\mathcal{V})$ and $(X,\mathcal{U})$. Then there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that the distances $\{d(p_{n_i},p)\colon i=1,2,\ldots\}$ are strictly decreasing. Without loss of generality, we may assume $p_{n_i} > p$ for each i.

By the same argument as in Theorem 4.6 we know that there exists a homeomorphism f of $(X,\mathcal{U})$ onto itself such that $f(p) = p$ and $f(N) \supseteq \{p_{n_i} : i=1,2,\ldots\}$. Hence $f(V) \cap \{p_{n_i} : i=1,2,\ldots\} = \emptyset$, and $\{p_{n_i}\}$ does not converge to p in $(X,\mathcal{V})$. This is a contradiction. Hence $\mathcal{U} = \mathcal{V}$.

<u>Theorem 4.8</u> Let $(X,\mathcal{U})$ be the real line with usual topology and let $(X,\mathcal{V})$ be a locally connected Hausdorff space such that $H(X,\mathcal{U}) = H(X,\mathcal{V})$. Then $\mathcal{U} = \mathcal{V}$.

<u>Proof</u>: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. If $\mathcal{B}$ is a base for $\mathcal{V}$ consisting of $\mathcal{V}$-connected sets, it is sufficient to show that $\mathcal{B} \subseteq \mathcal{U}$. Suppose not, then there exists a $V \in \mathcal{B}$ such that $V \notin \mathcal{U}$. Therefore $V \cap Cl(X - V) \neq \emptyset$.

Let $p \in V \cap Cl(X - V)$. Then there exists a sequence $\{p_i\}$ in $X - V$ such that $\{p_i\}$ converges to p in $(X,\mathcal{U})$. Without loss of generality, we may assume $\{p_i\}$ is strictly monotone, say monotone increasing, i.e., $p_1 < p_2 < \ldots < p_n < \ldots < p$. Since V is $\mathcal{V}$-connected, it is also $\mathcal{U}$-connected; and since each $p_i \in X - V$, thus it is clear that $V \subseteq [p, \infty)$. Now, let f be the reflection of X about p, i.e., $f(p + x) = p - x$ for each $x \in X$. It is clear that $f \in H(X,\mathcal{U}) = H(X,\mathcal{V})$ and $V \cap f(V) = \{p\} \in \mathcal{V}$. Moreover, for any $x \in X$ let $f_x(X) = X$ be the homeomorphism defined by $f_x(y) = x - y + p$. Then, $f_x(\{p\}) = \{x\}$ and therefore $\mathcal{V}$ is the discrete topology. But this implies that $H(X,\mathcal{U})$ contains all one-to-one functions of

X onto itself, which is a contradiction. Hence $\mathcal{U} = \mathcal{V}$.

Definition 4.9 A space $(X, \mathcal{V})$ is called semi-locally connected if it has a basis $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $X - U$ has only a finite number of components.

Theorem 4.10 Let $(X, \mathcal{U})$ be the real line with usual topology and let $(X, \mathcal{V})$ be a semi-locally connected, Hausdorff space such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$. Then $\mathcal{U} = \mathcal{V}$.

Proof: By Lemma 4.3, we have $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{U} \neq \mathcal{V}$, then there exist a point $p \in X$ and a set $V \in \mathcal{V}_p - \mathcal{U}_p$ such that $X - V$ has only a finite number of components in $(X, \mathcal{V})$. By Lemma 4.4, we have $p \in V = U - N$ for some $U \in \mathcal{U}_p$, $N \subseteq U$, $Int(N) = \emptyset$ and $p \in Cl(N) - N$. Clearly N contains infinitely many points, for otherwise N is closed in $\mathcal{U}$ and hance $V \in \mathcal{U}_p$, a contradiction. Since $Int(N) = \emptyset$, it is also clear that N is totally disconnected in $(X, \mathcal{U})$.

Now, since $p \in Cl(N) - N$, thus there exists a sequence $\{p_i\}$ in N such that $\{p_i\}$ converges to p in $(X, \mathcal{U})$. We can choose a sequence of points $\{q_i\}$ in V such that $\{q_i\}$ converges to p in $(X, \mathcal{U})$ and q_i lies between p_i and p_{i+1} for each i. Hence $X - V$ has infinitely many components with respect to $\mathcal{U}$ and therefore also respect to $\mathcal{V}$. This is a contradiction. Hence $\mathcal{U} = \mathcal{V}$.

BIBLIOGRAPHY

[1] C. J. Everett and S. M. Ulam, "On the problems of determination of mathematical structures by their endomorphisms" Abstract 285T, Bull. Amer. Math. Soc. 54 (1948), 646.

[2] L. R. Ford, Jr., "Homeomorphism groups and coset spaces" Trans. Amer. Math. Soc. 77 (1954), 490-497.

[3] J. De Groot and R. J. Wille, "Rigid continua and topological group-pictures" Archiv. Der Math. 9 (1958), 441-446.

[4] J. L. Kelley, General Topology, D. Van Nostrand Co. Inc., Princeton, N. J., 1955.

[5] Y. L. Lee, "Topologies with the same class of homeomorphisms" Pacific Journal of Mathematics, 20 (1967), 77-83.

[6] Y. L. Lee, "On a class of finer topologies with the same class of homeomorphisms", Proc. Amer. Math. Soc. 21 (1969), 129-133.

[7] Y. L. Lee, "Continua with the same class of homeomorphisms", Kyungpook Mathematical Journal, Vol. 7, No. 1, March 1967, 1-4.

[8] Y. L. Lee, "Characterizing the topology by the class of homeomorphisms", Duke Math. Journal, 35 (1968) 625-630.

[9] Y. L. Lee, "Homeomorphisms on Manifolds", Kyungpook Math. Journal, 7 (1967), 31-36.

[10] N. Levine, "Simple extensions of topologies", Amer. Math. Monthly, 71 (1964), 22-25.

[11] S. M. Ulam, "A collection of mathematical problems", Interscience, N. Y., 1960.

[12] J. V. Whittaker, "On isomorphic groups and homeomorphic spaces", Ann. Math., 78 (1963), 74-91.