MATRICES

WITH

LINEAR AND CIRCULAR SPECTRA

by

LUANG-HUNG CHANG

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Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

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ABSTRACT

Much is known about the eigenvalues of some special types of matrices. For example, the eigenvalues of a hermitian or skew-hermitian matrix lie on a line while those of a unitary matrix lie on a circle; their spectra are "linear" or "circular". This suggests the question : What matrices have this property ? Or, more generally, what matrices have their eigenvalues on plane curves of a simple kind ? Is it possible to recognize such matrices by inspection ?

In this thesis, we make a small start on these problems, exploring some matrices whose eigenvalues lie on one or more lines, or on one or more circles.

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1. Introduction.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{jk})$$

be any square matrix of order n with elements in the complex field C#. A non-zero vector $X = (x_1, x_2, \ldots , x_n)^T$ with complex entries $x_1, x_2, \ldots , x_n$ such that $AX = \lambda X$ is said to be an eigenvector of A corresponding to the eigenvalue λ ; λ is a root of the characteristic equation

$$\det(\lambda I - A) = 0$$

of degree n. Counting multiplicities there are n eigenvalues of A. The set of eigenvalues of A is called the spectrum of A.

There exists a lot of information about the eigenvalues of some special types of matrices. For example, it is easily seen that diagonal($a_{jk}=0$ if $j \neq k$) and triangular($a_{jk}=0$ if $j > k$) matrices exhibit their eigenvalues on their main diagonal : $\lambda_j = a_{jj}$, $j = 1,2,\ldots,n$.

It is not quite so trivial that the eigenvalues of a real symmetric matrix A ($A = A^T$, where A^T denotes the transpose of A) lie on the real axis. This was first proved by A. Cauchy [8] in 1829, and many subsequent proofs have been given, by

other eminent mathematicians, including Jacobi and Sylvester. This theorem was generalized by Hermite [12] in 1855 to matrices for which $A = A^*$ (A^* denotes the transpose conjugate of A), and resulted in such matrices being named after him. A well-known simple elegent one-line proof of this result is the following :

For a unit eigenvector X corresponding to λ,

$$\lambda = \lambda X^* X = X^* \lambda X = X^* A X = (X^* A X)^* = \bar{\lambda}.$$

In a similar way the eigenvalues of a real skew-symmetric matrix ($A = -A^T$) lie on the imaginary axis. This was first proved by A. Clebsch [9] , and later by Weierstrass [19]. That the same is true for a skew-hermitian matrix ($A = -A^*$) was shown by G. Scorza [18] in 1921. This is immediate if one notes that iA is hermitian.

The eigenvalues of an orthogonal matrix A ($AA^T = I$, where I is the identity matrix of order n) have absolute value 1 and occure in reciprocal pairs. This was proved by F. Brioschi [7] in 1854, and again by F. Rahusen [17] in 1894. The eigenvalues of a unitary matrix A ($AA^* = A^*A = I$) also have absolute value 1. The proof was first given by H. Aramata [3] in 1927, and a short proof was given by R. Brauer [5] in 1928. The result is obvious if we observe that for a unit eigenvector X corresponding to λ,

$$1 = X^* X = X^* A^* A X = (\bar{\lambda} X^*)(\lambda X) = \bar{\lambda} \lambda .$$

As far as the eigenvalues of a general matrix are concerned, nothing specific can be said about their location; they can obviously lie anywhere in the complex plane. A great many theorems have been proved about the localization of eigenvalues; many have been summarized by M. Marcus and H. Minc [15] and M. Parodi [16].

The three types of matrices mentioned above have something in common. Their eigenvalues each lie on a line or on a circle; or, in other words, their spectra are linear or circular. This suggests the question : What matrices have this property ? Or, more generally, what matrices have their eigenvalues on plane curves of a simple sort ? Is it possible to recognize such matrices by inspection ? These seem to be rather difficult questions to answer. In this thesis, we make a small start, exploring some matrices whose eigenvalues lie on one or more lines, or on one or more circles.

In section 2 we introduce a class of matrices called "HORT". A matrix is said to be HORT if it can be obtained from an hermitian matrix by a suitable rotation and/or translation. Its eigenvalues lie on a line. In a similar way we discuss "UOT" matrices, which can be obtained from unitary matrices by a suitable translation. We obtain necessary and sufficient conditions for a matrix to be HORT and UOT.

With each $n \times n$ complex matrix A there is associated in a natural way a $2n \times 2n$ real matrix called the Expansion of A. In section 3 we discuss some simple relations between A and its expansion.

In section 4 we examine $n \times n$ nonnegative indecomposable matrices. We show that if such a matrix has all its eigenvalues on a line, it has real eigenvalues, and its index of imprimitivity is al most 2 (Theorem 4.2). Its eigenvalues lie on a circle if and only if its index of imprimitivity is n (Theorem 4.3).

In section 5 we introduce two further classes of matrices called "almost" hermitian (Definition 5.2) and "almost" HORT (Definition 5.3), and obtain necessary and sufficient conditions for a matrix to be "almost" HORT.

In section 6 we discuss the relationship between the eigenvalues of the rth compound matrix $C_r(A)$ of A and those of A.

Finally, in section 7, we looked at the extended polynomial -problem on eigenvalues ; that is, the determination of the roots λ of

$$\det(\lambda^r A_o + \lambda^{r-1} A_1 + \ldots + A_r) = 0,$$

where the A_i are $n \times n$ complex matrices. In the case where the degree r is 1 or 2, we give some conditions for the roots to lie on a line or on a circle.

In this thesis we shall use the following notation :

C# —— the complex number field.

H —— hermitian matrix.

S —— skew-hermitian matrix.

U —— unitary matrix.

(p,q) —— a point in the complex plane.

Exp A —— expanded matrix of A.

$Q_{k,n}$ —— totality of strictly increasing sequences of k integers chosen from 1,2,...,n.

$C_r(A)$ —— r^{th} compound matrix of A.

$A[x|y]$ —— submatrix of A using rows numbered x and columns numbered y. Here x and y are sequences of integers.

det(A) —— determinant of A.

2. Matrices with eigenvalues on lines or circles.

<u>Definition 2.1</u> A matrix A is m-linear (m-circular) if all its eigenvalues lie on m lines (circles) in the complex plane.

This definition implies that if A is m-linear, it is also p-linear for all $p \geq m$. One could argue that A should be m-linear if its eigenvalues lie on m lines but not more than m lines. In that case, however, we should sometimes be faced with some pesky combinatorial considerations, which merit discussion only when we are interested in the least m for which A is m-linear. It appears reasonable to us to define m-linear as in Definition 2.1.

Those matrices with just one distinct eigenvalue (the spectrum is a point) are rether special. We could call them 0-linear and 0-circular, but we shall be content to include them among the 1-linear and 1-circular matrices. These matrices are not the only ones that are both 1-linear and 1-circular. Such matrices are those fro which the spectrum consists of at most two points. Every 2x2 matrix is 1-linear and 1-circular.

Suppose that the matrix A is 1-linear. If the line L of eigenvalues of A passes through $a = r+si$ with angle of inclination $\emptyset$, then parametric equations for L are :

$$\left.\begin{array}{l} x = r + pt \\ y = s + qt \end{array}\right\} , \qquad \text{where } t \text{ is a real parameter.}$$

Let $a=r+si$, $q/p = \tan \emptyset$ if $p \neq 0$, and $\emptyset = \pi/2$ if $p=0$. Now the matrix $B = (A - aI)e^{-i\emptyset}$ has real roots. Conversely, if there exist a complex number a and a real number $\emptyset$ such that B has real roots, then A is 1-linear. Hence

<u>Theorem 2.1</u> A is 1-linear if and only if there exist a complex number a and a real number $\emptyset$, $0 \leq \emptyset < \pi$ such that $(A - aI)e^{-i\emptyset}$ has real roots.

One might be tempted to say that the problem of recognizing 1-linear matrices really amounts to recognizing matrices with real eigenvalues. But it is not this simple. The criterion of Theorem 2.1 may not have too much value in recognizing 1-linear matrices from those with real roots. However, matrices related to hermitian matrices can be recognized.

<u>Definition 2.2</u> Let $A = (a_{jk})$ be an n-square matrix with elements in C#. We say that A is HORT "hermitian on rotation and/or translation" if $(A-aI)^{-i\emptyset}$ is hermitian for some complex number a and real number $\emptyset$.

If A is HORT and $H=(A-aI)e^{-i\emptyset}$ is hermitian, the eigenvalues of A are $\lambda_j(A) = \lambda_j(H)e^{i\emptyset} + a$, $j=1,2,\ldots,n$ and they lie on the line $y = \tan \emptyset \cdot (x-r) + s$ if $\emptyset \neq \pi/2$ and on the line $x = r$ if $\emptyset = \pi/2$.

Note that a skew-hermitian matrix S is HORT since $(S - 0 \cdot I)e^{-i\pi/2} = -iS$ is hermitian.

<u>Theorem 2.2</u> A is HORT if and only if there are complex numbers v and w with $|w| = 1$ such that

$$\bar{a}_{jj} = wa_{jj} + (\bar{v} - vw),$$

and $$\bar{a}_{kj} = wa_{jk} \qquad \text{for } j \neq k, \quad j,k=1,2,\ldots,n.$$

<u>Proof.</u> If A is HORT, there exist, by definition, a complex number a and a real number $\emptyset$ such that $H = (A-aI)e^{-i\emptyset}$ is hermitian. That is

$$(A - aI)e^{-i\emptyset} = (A^* - \bar{a}I)e^{i\emptyset}$$

or
$$A^* = e^{-2i\emptyset}(A - aI) + \bar{a}I,$$

thus
$$\bar{a}_{jj} = e^{-2i\emptyset}a_{jj} + (\bar{a} - ae^{-2i\emptyset})$$

$$\bar{a}_{kj} = e^{-2i\emptyset}a_{jk} \qquad \text{for } j\neq k,\ j,k=1,2,\ldots,n.$$

Put $v = a$ and $w = e^{-2i\emptyset}$.

The converse is immediate by reversing the order of the above argument.

In practice the recognition of an HORT matrix is even simpler than the criterion of Theorem 2.2. If $\emptyset \neq 0$, $a(=v)$ can be assumed real. If $\emptyset = 0$, a can be assumed to be pure imaginary, $a(=v) = bi$ where b is real, $\emptyset = 0$, and $w = 1$. Thus we have

<u>Theorem 2.3</u> A is HORT if and only if either (1) or (2) holds.

(1) $\bar{a}_{jj} = wa_{jj} + v(1-w)$, $\bar{a}_{kj} = wa_{jk}$, $j\neq k$, $j,k=1,2,\ldots,n$ for some real v and complex w such that $|w| = 1$.

(2) $\bar{a}_{jj} = a_{jj} - 2bi$, $\bar{a}_{kj} = a_{jk}$, $j\neq k$, $j,k=1,2,\ldots,n$ for some real number b.

HORT matrices are 1-linear, but of course they are by no means the only 1-linear matrices, just as hermitian matrices are not the only ones with real eigenvalues. In fact the example $A = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$ shows that a matrix with real elements and real eigenvalues can be just about as "unhermitian" as one can conceive.

Suppose that the matrix A is 1-circular. The equation of the circle of eigenvalues must be of the form

$$(x - r)^2 + (y - s)^2 = b^2,$$

where b is a non-negative real number.

Analogous to the case of HORT matrices, we make the following definition.

<u>Definition 2.3</u> Let $A = (a_{jk})$ be an n-square matrix with elements in C#. A is UOT "unitary on translation" if $(A - aI)b^{-1}$ is unitary for some complex number a and real number $b \neq 0$.

If A is UOT and $U = (A - aI)b^{-1}$ is unitary, the eigenvalues of A are

$$\lambda_j(A) = \lambda_j(U)b + a, \qquad j=1,2,\ldots,n$$

and they lie on the circle

$$(x - r)^2 + (y - s)^2 = b^2 \qquad \text{if } a = r+si.$$

<u>Theorem 2.4</u> A is UOT if and only if there exist a complex number v and real number $w \neq 0$ such that

$$\sum_{m=1}^{n} a_{jm}\bar{a}_{jm} - v\bar{a}_{jj} - \bar{v}a_{jj} + |v|^2 - w^2 = 0$$

$$\sum_{m=1}^{n} a_{jm}\bar{a}_{km} - v\bar{a}_{kj} - \bar{v}a_{jk} = 0, \quad j \neq k, \; j,k=1,2,\ldots,n.$$

<u>Proof.</u> If A is UOT there are, by definition, a complex number a and a real number $b \neq 0$ such that $U = (A - aI)b^{-1}$ is unitary. That is

$$UU^* = (A - aI)b^{-1}(A^* - \bar{a}I)b^{-1} = I$$

or

$$AA^* - aA^* - \bar{a}A + |a|^2 I = b^2 I,$$

thus

$$\sum_{m=1}^{n} a_{jm}\bar{a}_{jm} - a\bar{a}_{jj} - \bar{a}a_{jj} + |a|^2 = b^2$$

$$\sum_{m=1}^{n} a_{jm}\bar{a}_{km} - a\bar{a}_{kj} - \bar{a}a_{jk} = 0, \quad j \neq k, \; j,k=1,2,\ldots,n.$$

Put $v = a$ and $w = b$.

The converse is immediate by reversing the order of the above argument.

3. The expanded matrix of a complex matrix

In looking at 1-linear, or more generally m-linear, matrices, there are really two major avenues of investigation : complex matrices and real matrices. We shall not confine ourselves to real matrices, but explore both problems.

We note, however, that corresponding to each complex matrix A there is a related real matrix called the "expanded matrix of A", denoted by $\operatorname{Exp} A$, and defined for

$$A = (a_{jk}) = (b_{jk} + c_{jk} i), \qquad b_{jk},\ c_{jk} \text{ real,}$$

by

$$\operatorname{Exp} A = \begin{pmatrix} b_{11} & c_{11} & b_{12} & c_{12} & \cdots & b_{1n} & c_{1n} \\ -c_{11} & b_{11} & -c_{12} & b_{12} & \cdots & -c_{1n} & b_{1n} \\ b_{21} & c_{21} & b_{22} & c_{22} & \cdots & b_{2n} & c_{2n} \\ -c_{21} & b_{21} & -c_{22} & b_{22} & \cdots & -c_{2n} & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & c_{n1} & b_{n2} & c_{n2} & \cdots & b_{nn} & c_{nn} \\ -c_{n1} & b_{n1} & -c_{n2} & b_{n2} & \cdots & -c_{nn} & b_{nn} \end{pmatrix} .$$

In 1960, E. Gott [10] proved that

$$\det(\operatorname{Exp} A) = |\det(A)|^2 .$$

A one-line proof of this theorem was given by J. L. Brenner [6] in 1961 based on an interesting theorem of S. N. Afriat [1] :

$$\det(\operatorname{Exp} A) = \det \begin{pmatrix} \operatorname{Re} \det(A) & \operatorname{Im} \det(A) \\ -\operatorname{Im} \det(A) & \operatorname{Re} \det(A) \end{pmatrix} = |\det(A)|^2 .$$

Brenner also showed that the collection of the eigenvalues of Exp A consists of the eigenvalues of A and their conjugates.

This is particularly interesting for matrices A with real roots. The roots of Exp A are just those of A counted twice, —— certainly real and 1-linear.

We give some simple results relating A and Exp A in

<u>Theorem 3.1</u> Let $A = (a_{jk})$ be an n-square matrix with complex elements. Exp A is normal if and only if A is normal. Exp A is symmetric if A is hermitian, Exp A is skew-symmetric if A is skew-hermitian and Exp A is unitary if A is unitary.

<u>Proof.</u> First we show that Exp A is normal if and only if A is normal. Suppose $a_{jk} = b_{jk} + c_{jk}i$, and let

$$A_{jk} = \begin{pmatrix} b_{jk} & c_{jk} \\ -c_{jk} & b_{jk} \end{pmatrix} .$$

Since

$$\sum_{m=1}^{n} A_{jm}A_{km}^{*} = \sum_{m=1}^{n} A_{jm}^{*}A_{km} \qquad j,k=1,2,\ldots,n$$

if and only if

$$\sum_{m=1}^{n} a_{jm}\bar{a}_{km} = \sum_{m=1}^{n} \bar{a}_{jm}a_{km} \qquad j,k=1,2,\ldots,n.$$

$$(\text{Exp } A)(\text{Exp } A)^{*} = \left(\left(\sum_{m=1}^{n} A_{jm}A_{km}^{*}\right)_{jk}\right)$$

$$= \left(\left(\sum_{m=1}^{n} A_{jm}^{*}A_{km}\right)_{jk}\right) = (\text{Exp } A)^{*}(\text{Exp } A)$$

if and only if

$$AA^* = ((\sum_{m=1}^{n} a_{jm}\bar{a}_{km})_{jk}) = ((\sum_{m=1}^{n} \bar{a}_{jm}a_{km})_{jk}) = A^*A.$$

Thus A is normal if and only if Exp A is normal.

If A is hermitian, its eigenvalues are real, and, since the eigenvalues of Exp A are those of A and their conjugates, Exp A has real roots. However A being hermitian also implies Exp A normal. Now a normal matrix with real eigenvalues must be hermitian. Hence Exp A is hermitian, and indeed, since its elements are real, Exp A is symmetric.

Similarly, one can prove the last two statements, if one notes that the conjugate of a pure imaginary is again pure imaginary and $|a+bi| = 1 = |a-bi|$.

<u>Theorem 3.2</u> Let $A = (a_{jk})$ be an n-square matrix with complex elements. Assume that A is HORT; i.e., $(A - aI)e^{-i\emptyset}$ is hermitian for some complex number $a=r+si$ and real number $\emptyset$ with $0 \leqq \emptyset < \pi$. Exp A is 1-linear if (1) $\emptyset=\pi/2$, or (2) $s=0$ and $\emptyset=0$, or (3) spectrum of A is a point. Exp A is 2-linear otherwise.

<u>Proof.</u> If $\emptyset=\pi/2$, the eigenvalues of A lie on $x = r$ and so do those of Exp A. If $s=0$ and $\emptyset=0$, the eigenvalues of A and Exp A are all real and lie on $y=0$. If the spectrum of A is a point, Exp A is obviously 1-linear. Otherwise, the eigenvalues of A lie on the line with equation

$$y = \tan\emptyset\cdot(x - r) + s.$$

Those of Exp A lie on

$$y = \tan\emptyset\cdot(x - r) + s$$

and $$y = -\tan\emptyset\cdot(x - r) - s.$$

This proves our theorem.

<u>Theorem 3.3</u> Let $A = (a_{jk})$ be an n-square matrix with complex entries. Assume that A is UOT; i.e., there are complex number $a=r+si$ and real number $b\neq 0$ such that $(A - aI)b^{-1}$ is unitary. Exp A is 1-circular if $s=0$, or if the spectrum of A is a point. Exp A is 2-circular otherwise.

<u>Proof.</u> By the remarks shown in Section 2, the eigenvalues of A lie on the circle

$$(x - r)^2 + (y - s)^2 = b^2 .$$

Now, if $s=0$, the above equation may be written

$$(x - r)^2 + y^2 = b^2$$

which is symmetric with respect to the real axis. Then the eigenvalues of Exp A all lie on this circle for they are conjugate in pairs. If $s\neq 0$, the eigenvalues of Exp A will lie on two circles :

$$(x - r)^2 + (y - s)^2 = b^2$$

and $$(x - r)^2 + (y + s)^2 = b^2.$$

More generally, one can see that if A is 1-linear then $\operatorname{Exp} A$ is either 1-linear or 2-linear. Conversely, if $\operatorname{Exp} A$ is 1-linear, then A is 1-linear; if $\operatorname{Exp} A$ is 2-linear, then A can be either 1-linear or 2-linear. It is obvious, however, that if $\operatorname{Exp} A$ is α-linear, $\alpha = 1,2,3,\ldots$, then $\operatorname{Exp}(\operatorname{Exp} A) = \operatorname{Exp}^2(A)$ is also α-linear. No new lines of eigenvalues are added. This is not surprising since $\operatorname{Exp}^2(A)$ is A "blown up" with each root duplicated. A similar discussion applies to α-circular matrices.

A more interesting question is to ask for the number of lines on which the eigenvalues of B_m lie, where

(1) $B_o = B$ is 1-linear,

(2) $B_k = \operatorname{Exp}(a_k B_{k-1} + b_k I)$; a_k, b_k complex, $k=1,2,\ldots,m$.

In general, each time we make an Expansion, we double the number of lines of eigenvalues, hence the maximum linearity is 2^m. But of course B_m can be 2^q-linear for each q, $0 \leq q \leq m$, for suitable choices of B, a_k and b_k. Conceivably, B_m could also be α-linear, where α is not a power of 2. Just when B_m is α-linear for any integer $\alpha \leq 2^m$ seems to be a complex combinatorial problem. To illustrate the possibilities we give the linearity of B_1 and B_2 when B_o is hermitian, skew hermitian and unitary (Tables 1 and 2).

Let $a_1=p_1+q_1 i\neq 0$, $b_1=r_1+s_1 i\neq 0$, $a_2=p_2+q_2 i\neq 0$ and $b_2=r_2+s_2 i\neq 0$.

Table 1. $B_o = H(S)$

Conditions on p_1,q_1,r_1,s_1	Exp B_1	Conditions on p_2,q_2,r_2,s_2	Exp B_2
$q_1=0$ & $s_1=0$ ($p_1=0$ & $s_1=0$)	1-linear($y=0$)	$q_2=0$ & $s_2=0$	1-linear($y=0$)
		$p_2=0$ & $s_2\neq 0$	1-linear(perp. to $y=0$)
		otherwise	2-linear
$p_1=0$ & $r_1=0$ ($q_1=0$ & $r_1=0$)	1-linear($x=0$)	$p_2=0$ & $s_2=0$	1-linear($y=0$)
		$q_2=0$ & $s_2\neq 0$	1-linear(perp. to $y=0$)
		otherwise	2-linear
$p_1=0$ & $s_1=0$ ($q_1=0$ & $s_1=0$)	1-linear (perp. to $y=0$)	$q_2=0$	1-linear(perp. to $y=0$)
		$p_2=0$	2-linear(paral.to $y=0$)
$q_1=0$ & $r_1=0$ ($p_1=0$ & $r_1=0$)	2-linear (paral.to $y=0$)	$q_2=0$ & $s_2=0$	2-linear(paral.to $y=0$)
		$q_2=0$ & $s_2\neq 0$	4-linear(paral.to $y=0$)
		$p_2=0$	2-linear(perp. to $y=0$)
otherwise	2-linear	$q_2=0$ & $s_2=0$	2-linear
		$q_2=0$ & $s_2\neq 0$	4-linear
		$p_2=0$ & $s_2=0$	2-linear
		$p_2=0$ & $s_2\neq 0$	4-linear

Table 2. $B_o = U$

Conditions on p_1,q_1,r_1,s_1	Exp B_1	Conditions on p_2,q_2,r_2,s_2	Exp B_2
$s_1=0$	1-circular	$s_2=0$	1-circular
		otherwise	2-circular
otherwise	2-circular	$s_2=0$	2-circular
		otherwise	4-circular

4. Nonnegative matrices.

<u>Definition 4.1</u> A real n-square matrix $A = (a_{jk})$ is called nonnegative, if $a_{jk} \geqq 0$ for $j,k = 1,2,\ldots,n$. We write $A \geqq 0$.

<u>Definition 4.2</u> A nonnegative n-square matrix $A = (a_{jk})$ $(n > 1)$ is said to be decomposable if there exists permutation matrix P such that $PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$ where B and C are square matrices. Otherwise A is indecomposable.

The fundamental theorem on indecomposable nonnegative matrices is the Perron-Frobenius theorem [15], which we state as follows :

<u>Theorem 4.1</u> Let A be an n-square nonnegative indecomposable matrix. Then :

(1) A has a real positive eigenvalue r (the maximal eigenvalue of A) which is a simple root of the characteristic equation of A. If $\lambda_j(A)$ is any eigenvalue of A, then $|\lambda_j(A)| \leqq r$.

(2) If A has h eigenvalues of modulus r : $\lambda_1 = r, \lambda_2, \ldots, \lambda_h$, then they are the h distinct roots of $\lambda^h - r^h = 0$; h is called the index of imprimitivity of A.

(3) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the eigenvalues of A and $\theta = e^{i2\pi/h}$, then $\lambda_1\theta, \lambda_2\theta, \ldots, \lambda_n\theta$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$

in some order.

(4) If $h > 1$, then there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & A_{h-1,h} \\ A_{h,1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where the zero blocks down the main diagonal are square.

Suppose that the nonnegative indecomposable matrix A is 1-linear. Let r be the maximal (positive real) eigenvalue of A and suppose that there were a complex eigenvalue, say $\lambda \neq r$. Since the coefficients of $\det(\lambda I - A) = 0$ are real, $\bar{\lambda}$ is also a root of $\det(\lambda I - A) = 0$ and $|\lambda| \leqq r$. Then λ and $\bar{\lambda}$ lie on or in the circle $x^2 + y^2 = r^2$ and are symmetrically placed with respect to the x-axis. Thus r, λ and $\bar{\lambda}$ can only be on a line if $\lambda = \bar{\lambda}$. Hence we have :

<u>Theorem 4.2</u> If an n-square nonnegative indecomposable matrix A is 1-linear, then all its eigenvalues are real. Furthermore, $h \leqq 2$.

It is trivial that if A is nonnegative indecomposable and 1-linear, then Exp A is also 1-linear; in fact, its roots are all real.

Part(3) of Theorem 4.1 indicates that A is al most n/h-circular.

If A is 1-circular, then all the eigenvalues of A satisfy $|\lambda_j(A)| = r$, $j = 1,2,\ldots,n$, where r is the positive real maximal eigenvalue of A. Thus all $\lambda_j(A)$ are roots of $x^n - r^n = 0$, and $h = n$. Conversely, if $h = n$, the roots of $\lambda^n - r^n = 0$ coincide with the eigenvalues of A, and A has 1-circular eigenvalues. Hence

<u>Theorem 4.3</u> An n-square nonnegative indecomposable matrix A is 1-circular if and only if $n = h$.

An example of the matrices appearing in Theorem 4.3 is the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix} .$$

The eigenvalues of P are the nth roots of unity :

$$\lambda_j(P) = e^{i(2\pi/n)j}, \qquad j = 1,2,\ldots,n,$$

for P is the companion matrix of the polynomial $\lambda^n - 1$.

P is a special case of the more general matrix

$$B = \begin{pmatrix} a & b_1 & 0 & \ldots & 0 \\ 0 & a & b_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & a & b_{n-1} \\ b_n & \ldots & 0 & 0 & a \end{pmatrix} .$$

Here, $\det(\lambda I - B) = (\lambda - a)^n + (-1)^n \prod_{k=1}^{n} b_k$. The eigenvalues of B are :

$$\lambda_j(B) = re^{i(2\pi/n)j} + a, \qquad j = 1,2,\ldots,n, \text{ n even}$$

or

$$\lambda_j(B) = re^{i(\pi/n + 2\pi j/n)} + a, \quad j=1,2,\ldots,n, \text{ n odd},$$

where $r = \sqrt[n]{\prod_{k=1}^{n} b_k}$.

The eigenvalues of Exp B are just those of B counted twice; hence Exp B is again 1-circular.

B is interesting in that it is 1-circular even if the non-zero entries are non-real. Its eigenvalues are

$$\lambda_j(B) = \sqrt[n]{|c|}\; e^{i((\phi+2\pi j)/n)}, \qquad j = 1,2,\ldots,n,$$

where $c = \prod_{k=1}^{n} b_k = |c|e^{i\phi}$. Let $a = p+qi$, p,q are real, then Exp B is 1-circular if $q = 0$ and Exp B is 2-circular otherwise.

5. Tridiagonal matrices.

<u>Definition 5.1</u> A matrix $A = (a_{jk})$ is called a tridiagonal or Jacobi matrix if $a_{jk} = 0$ whenever $|j - k| \geqq 2$.

Thus

$$(5.1) \qquad L_n = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \cdots\cdots\cdots & 0 \\ d_2 & b_2 & c_2 & 0 & \cdots\cdots\cdots & 0 \\ 0 & d_3 & b_3 & c_3 & 0 \;\; \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots\cdots & 0 & d_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots\cdots\cdots & 0 & 0 & d_n & b_n \end{pmatrix}$$

is a general n-square complex Jacobi matrix. Since any matrix A is similar to a Jacobi matrix L_n [14] , we are interested in conditions under which L_n is m-linear; particularly when L_n is 1-linear.

The following theorem was proved by F. M. Arscott in 1961 [4] :

<u>Theorem 5.1</u> If all the entries of the Jacobi matrix L_n are real and $c_j d_{j+1} > 0$, $j = 1,2,\ldots,n-1$, then all its eigenvalues are real and simple.

We shall generalize Theorem 5.1 as follows :

<u>Theorem 5.2</u> Let M_n be the matrix L_n with real b_k, $1 \leqq k \leqq n$, and complex c_j, d_{j+1} such that $c_j d_{j+1} > 0$, $1 \leqq j \leqq n-1$. Then the eigenvalues of M_n are real and simple.

<u>Proof.</u> The coefficient of λ^k in $\det(\lambda I - M_n)$ is the sum of the $(n-k)$-square principal minors. Wherever c_j occurs as a factor in such a minor so does d_{j+1}, and vice versa. Thus these elements always occur together as a product $c_j d_{j+1}$. It follows that the characteristic equation $\det(\lambda I - M_n) = 0$ has the same coefficients and roots as when all c_j and d_{j+1} are real and $c_j d_{j+1} > 0$. By Theorem 5.1 the eigenvalues of M_n are real and simple.

By Theorem 5.1 and Theorem 5.2, given a Jacobi matrix L_n, if $b_1, b_2, \ldots, b_n$ are real, $c_1, c_2, \ldots, c_{n-1}$ and $d_2, d_3, \ldots, d_n$ are complex numbers such that $c_j d_{j+1} > 0$, $j=1,2,\ldots,n-1$, then its eigenvalues are all real. Analogous to the case of 1-linear matrices discussed in Section 2, we make the following definitions :

<u>Definition 5.2</u> Let L_n be a Jacobi matrix with complex entries. We call L_n "almost" hermitian if $b_1, b_2, \ldots, b_n$ are real, $c_1, c_2, \ldots, c_{n-1}$ and $d_2, d_3, \ldots, d_n$ are complex numbers such that $c_j d_{j+1} > 0$, $j=1,2,\ldots,n-1$.

Note that if L_n is "almost" hermitian and $d_{j+1} = \bar{c}_j$, $j=1,2,\ldots,n-1$, then L_n is hermitian.

<u>Definition 5.3</u> Let L_n be a Jacobi matrix. We call L_n "almost" HORT if $(L_n - aI)e^{-i\phi}$ is "almost" hermitian, for some complex number a and real number ϕ such that $0 \leqq \phi < \pi$.

<u>Theorem 5.3</u> L_n is "almost" HORT if and only if there are complex numbers v and w with $|w| = 1$ such that $(b_j - v)w$ is real, $j=1,2,\ldots,n$ and $c_j d_{j+1} w^2 > 0$, $j=1,2,\ldots,n-1$.

<u>Proof.</u> If L_n is "almost" HORT, by definition, there exist a complex number a and a real number ϕ with $0 \leqq \phi < \pi$ such that $B = (L_n - aI)e^{-i\phi}$ is "almost" hermitian; i.e., $(b_j - a)e^{-i\phi}$ is real and $c_j d_{j+1} e^{-2i\phi} > 0$, $j=1,2,\ldots,n-1$. Put $v = a$ and $w = e^{-i\phi}$.

The converse is immediate by reversing the order of the above argument.

<u>Theorem 5.4</u> Let L_n be a Jacobi matrix that is "almost" HORT; i.e., $(L_n - aI)e^{-i\phi}$ is "almost" hermitian for some complex number $a=r+si$ and real number ϕ with $0 \leqq \phi < \pi$. Then Exp A is 1-linear if (1) $\phi = \pi/2$ or (2) $s = 0$ and $\phi = 0$, and Exp A is 2-linear otherwise.

<u>Proof.</u> We omit the proof here since it is similar to that of Theorem 3.2.

6. Compound matrices and some other theorems.

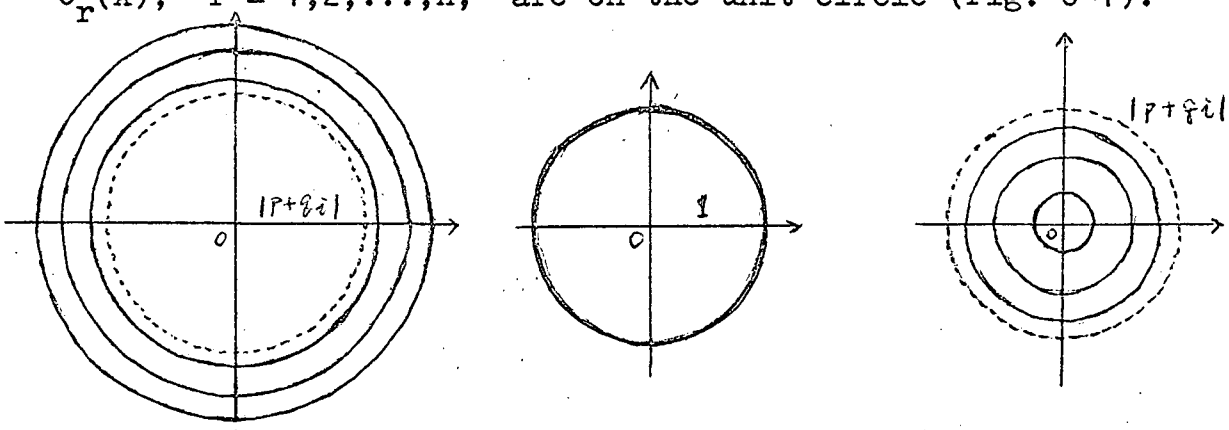
<u>Definition 6.1</u> Let $A = (a_{jk})$ be an n-square matrix and $1 \leqq r \leqq n$. The rth compound matrix of A, $C_r(A)$, is the $\binom{n}{r} \times \binom{n}{r}$ matrix whose entries are $\det(A[x|y])$, $x \in Q_{r,n}$ and $y \in Q_{r,n}$ arranged lexicographically in x and y.

For example, if $n = 3$ and $r = 2$ then

$$C_2(A) = \begin{pmatrix} \det(A[1,2|1,2]) & \det(A[1,2|1,3]) & \det(A[1,2|2,3]) \\ \det(A[1,3|1,2]) & \det(A[1,3|1,3]) & \det(A[1,3|2,3]) \\ \det(A[2,3|1,2]) & \det(A[2,3|1,3]) & \det(A[2,3|2,3]) \end{pmatrix} .$$

In particular, $C_1(A) = A$ and $C_n(A) = \det(A)$.

<u>Theorem 6.1</u> Let $A = (p+qi)U$, $p+qi \neq 0$. The eigenvalues of $C_r(A)$, $1 \leqq r \leqq n$, lie on a circle with center at the origin and radius $|p+qi|^r$. When $r = n$, the eigenvalues coalesce at the point $(p+qi)^n$. If $|p+qi| = 1$, all the eigenvalues of $C_r(A)$, $r = 1,2,\ldots,n$, are on the unit circle (Fig. 6-1).



$|p+qi|>1$ $\qquad$ $|p+qi|=1$ $\qquad$ $|p+qi|<1$

Fig. 6-1

<u>Proof.</u> Since the eigenvalues of $C_r(A)$ are the $\binom{n}{r}$ products $\lambda_{j_1}(A)\lambda_{j_2}(A)\ldots\lambda_{j_r}(A)$, where $1 \leqq j_1 < j_2 < \ldots < j_r \leqq n$, and since the $\lambda_j(A)$ are on the circle with center at the origin and radius $|p+qi|$, the eigenvalues of $C_r(A)$ lie on the circle with center at the origin and radius $|p+qi|^r$.

<u>Theorem 6.2</u> The eigenvalues of $C_r(S)$, $1 \leqq r \leqq n$, lie on the real axis if r is even and on the imaginary axis if r is odd, while those of $C_r(H)$ lie on the real axis.

<u>Proof.</u> Since the eigenvalues of S are pure imaginaries, the eigenvalues of $C_r(S)$ are real if r is even and pure imaginary if r is odd.

The last statement follows immediate from the fact that the set of real numbers is a field.

One notes that the set $\{C_r(A) : r=1,2,\ldots,n\}$ is 2-linear if $A = S$ and 1-linear if $A = H$.

The following two theorems tell us that we can obtain 1-circular matrix from 1-linear matrix by certain transformation, and vice versa.

<u>Theorem 6.3</u> $A = (I - S)(I + S)^{-1}$ is a unitary matrix; i.e., we can obtain a matrix whose eigenvalues lie on a unit circle from a skew-hermitian matrix by this transformation (Fig. 6-2).

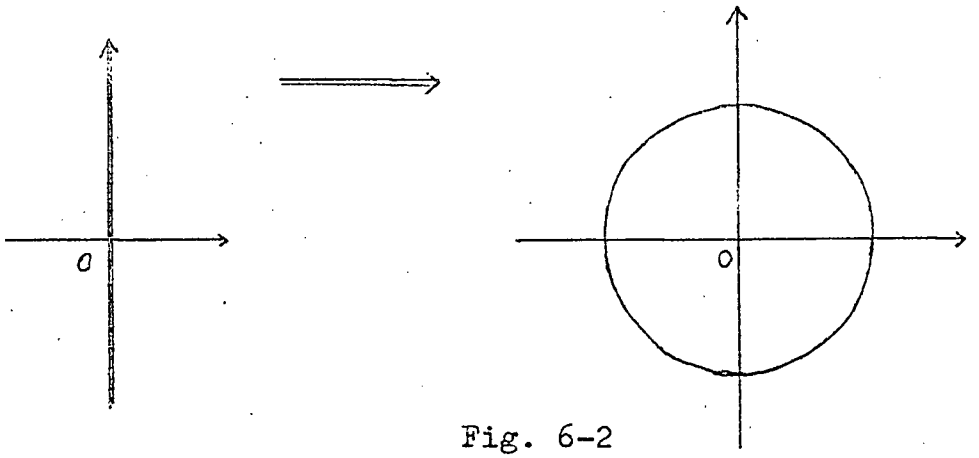


Fig. 6-2

<u>Proof.</u> First note that $I + S$ is nonsingular since the roots of S are pure imaginaries. Also, $(I+S)(I-S) = (I-S)(I+S)$.

$$\begin{aligned} AA^* &= (I-S)(I+S)^{-1}((I+S)^{-1})^*(I-S)^* \\ &= (I-S)(I+S)^{-1}(I+S^*)^{-1}(I-S^*) \\ &= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S) \qquad \text{(since } S^* = -S\text{)} \\ &= (I-S)((I-S)^{-1}(I+S))^{-1}(I+S) \\ &= I. \end{aligned}$$

<u>Theorem 6.4</u> If $\det(I+U) \neq 0$, then

$$A = (\ I - U\ )(\ I + U\ )^{-1}$$

is skew-hermitian; i.e., we can obtain a matrix whose eigenvalues lie on the imaginary axis from a unitary matrix by this transformation (Fig. 6-3).

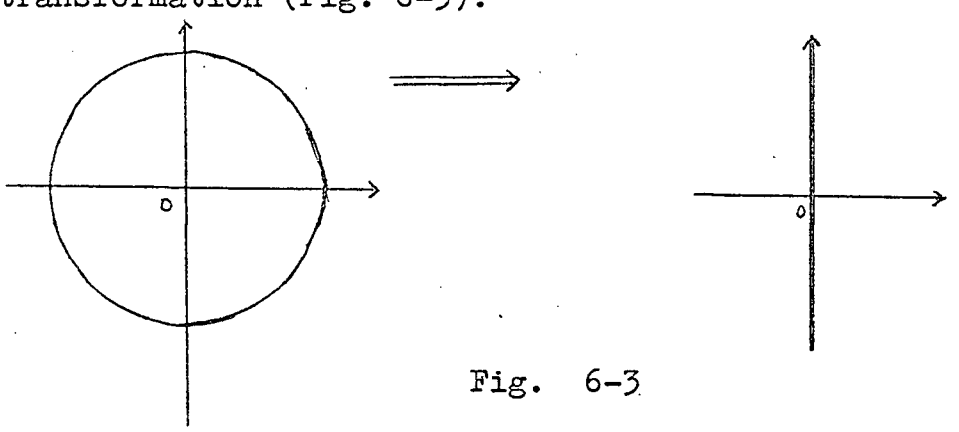


Fig. 6-3

<u>Proof.</u> We need only to show that $A = -A^*$:

$$
\begin{aligned}
A^* &= ((I+U)^{-1})^*(I-U)^* \\
&= (I+U^*)^{-1}(I-U^*) \\
&= (I+U^{-1})^{-1}(I-U^{-1}) && \text{(since } U^* = U^{-1}\text{)} \\
&= (U^{-1}(U+I))^{-1}(U^{-1}(U-I)) \\
&= (U+I)^{-1}UU^{-1}(U-I) \\
&= (U+I)^{-1}(U-I) \\
&= (U+I)^{-1}(U-I)(U+I)(U+I)^{-1} \\
&= (U+I)^{-1}(U+I)(U-I)(U+I)^{-1} \\
&= -(I-U)(I+U)^{-1} \\
&= -A^*.
\end{aligned}
$$

7. Extended polynomial-problem on eigenvalues.

Let $A_o, A_1, \ldots, A_r$ be $r+1$ n-square complex matrices. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a nonzero vector with complex entries. Consider

$$(7.1) \qquad (\lambda^r A_o + \lambda^{r-1} A_1 + \ldots + A_r)X = 0.$$

The determination of λ such that equation (7.1) holds is called the extended polynomial-problem on eigenvalues. X is said to be an eigenvector belonging to the eigenvalue λ. One can see that the eigenvalues of the polynomial-problems are exactly the roots of

$$(7.2) \qquad \det(\lambda^r A_o + \lambda^{r-1} A_1 + \ldots + A_r) = 0.$$

The determinant is a polynomial in λ of degree not exceeding $r \cdot n$. If A_o is nonsingular, the degree of (7.2) is exactly $r \cdot n$.

If $r = 1$ and $A_o = I$, then the roots of $\det(\lambda I + A_1) = 0$ are the eigenvalues of $-A_1$.

If $r = 1$ and A_o and A_1 are arbitrary matrices the problem is referred to as a generalized eigenvalue problem [13].

<u>Theorem 7.1</u> The roots of (7.2) are the generalized eigenvalues of $\lambda B_o + B_1$ such that

$$(7.3) \qquad (\lambda B_o + B_1)X = 0,$$

where X is a nonzero vector,

$$B_o = \begin{pmatrix} -I & 0 & 0 & \dots & 0 & 0 \\ 0 & -I & 0 & \dots & 0 & 0 \\ 0 & 0 & -I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -I & 0 \\ 0 & 0 & 0 & \dots & 0 & -A_o \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I \\ -A_r & -A_{r-1} & -A_{r-2} & \dots & -A_2 & -A_1 \end{pmatrix}.$$

If $A_o = I$, then the roots of (7.2) are the eigenvalues of B_1. If A_o is nonsingular, then the roots of (7.2) are the eigenvalues of B_2, where

$$B_2 = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I \\ -A_rA_o^{-1} & -A_{r-1}A_o^{-1} & -A_{r-2}A_o^{-1} & \dots & -A_2A_o^{-1} & -A_1A_o^{-1} \end{pmatrix}.$$

Proof.

$$\lambda B_o + B_1 = \begin{pmatrix} -\lambda I & I & 0 & \dots & 0 & 0 \\ 0 & -\lambda I & I & \dots & 0 & 0 \\ 0 & 0 & -\lambda I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\lambda I & I \\ -A_r & -A_{r-1} & -A_{r-2} & \dots & -A_r & -A_1-\lambda A_o \end{pmatrix}$$

To the first column add λ times the second column, λ^2 times the third column,..., λ^{r-1} times the last column, we obtain

$$\lambda B_o + B_1 = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & -\lambda I & I & \dots & 0 & 0 \\ 0 & 0 & -\lambda I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\lambda I & I \\ -P(\lambda) & -A_{r-1} & -A_{r-2} & \dots & -A_2 & -A_1-\lambda A_o \end{pmatrix}$$

where $P(\lambda) = \lambda^r A_o + \lambda^{r-1} A_1 + \dots + \lambda A_{r-1} + A_r$.

Since $\det(\lambda B_o + B_1)$

$$= (-1)^{r-1} \det(-P(\lambda)) \det \begin{pmatrix} I & 0 & 0 & \dots & 0 & 0 \\ -\lambda I & I & 0 & \dots & 0 & 0 \\ 0 & -\lambda I & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & -\lambda I & I \end{pmatrix}$$

$$= (-1)^{r-1} \det(-P(\lambda)) = (-1)^{n+r-1} \det(\lambda^r A_o + \lambda^{r-1} A_1 + \dots + A_r),$$

the roots of (7.2) are the generalized eigenvalues of $\lambda B_o + B_1$.

Now if $A_o = I$, $-B_o$ is an identity matrix of order $r \cdot n$,

and the roots of (7.2) are the eigenvalues of B_1.

If A_o is nonsingular, then A_o^{-1} exists, and (7.2) may be written

$$\det((\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1}) A_o) = 0,$$

or

$$\det(\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1}) \det(A_o) = 0.$$

Since $\det(A_o) \neq 0$, the roots of (7.2) are the roots of

$$\det(\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1}) = 0. \tag{7.4}$$

From what we have proved above, the roots of (7.4) are the eigenvalues of B_2, where

$$B_2 = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & I \\ -A_r A_o^{-1} & -A_{r-1} A_o^{-1} & -A_{r-2} A_o^{-1} & \ldots & -A_2 A_o^{-1} & -A_1 A_o^{-1} \end{pmatrix}.$$

Q.E.D.

Let $P(\lambda) = \lambda^r A_o + \lambda^{r-1} A_1 + \ldots + \lambda A_{r-1} + A_r$.

<u>Definition 7.1</u> The polynomial $P(\lambda)$ is said to be m-linear (m-circular) if the roots of $\det P(\lambda) = 0$ lie on m, but not fewer than m lines (circles).

We shall give a few results on 1-linear and 1-circular polynomials when $r = 1$ and 2.

For the case $r = 1$, we are primarily interested in the case when the roots of $\det P(\lambda) = 0$ are real. One might guess that if A_o and A_1 are both hermitian, then $\lambda A_o + A_1$ has real eigenvalues. It is not quite this simple. Some restrictions are needed on A_o and A_1.

<u>Theorem 7.2</u> Let H_o and H_1 be hermitian matrices of order n and H_o be positive or negative definite, then the roots of $\det(\lambda H_o - H_1) = 0$ are real [2].

<u>Proof.</u> Let X be a nonzero eigenvector corresponding to the generalized eigenvalue of $\lambda H_o - H_1$; i.e., $(\lambda H_o - H_1)X = 0$, or $\lambda H_o X = H_1 X$. Premultiplying by X^*, we have $\lambda(X^* H_o X) = X^* \lambda H_o X = X^* H_1 X$. Since $X^* H_1 X$ is real, and $X^* H_o X$ is real and not zero, λ is real. Therefore the roots of $\det(\lambda H_o - H_1) = 0$ are real.

<u>Theorem 7.3</u> If H_o and H_1 are hermitian matrices such that $H_o H_1 = H_1 H_o$ and $\det(H_o) \neq 0$, then the roots of $\det(\lambda H_o - H_1) = 0$ are real.

<u>Proof.</u> We shall first prove that $H_o H_1$ is hermitian if and only if H_o and H_1 commute. On the one hand, if $(H_o H_1)^* = H_o H_1$, then $H_1^* H_o^* = H_1 H_o = H_o H_1$. On the other hand, if $H_o H_1 = H_1 H_o$,

then $(H_oH_1)^* = H_1^*H_o^* = H_1H_o = H_oH_1$.

Secondly, we note that $H_1H_o^{-1}$ is hermitian, for H_o^{-1} is hermitian and $H_oH_1 = H_1H_o$ implies $H_1H_o^{-1} = H_o^{-1}H_1$.

Now, the equation $\det(\lambda H_o - H_1) = 0$ may be written $\det(\lambda I - H_1H_o^{-1})\det(H_o) = 0$. The roots of $\det(\lambda I - H_1H_o^{-1}) = 0$ are real and so are those of $\det(\lambda H_o - H_1) = 0$.

<u>Theorem 7.4</u> If the generalized eigenvalues of $\lambda B_o + B_1$ are real, $(\lambda + a)B_o - bB_1$ for a, b complex numbers is 1-linear provided $b \neq 0$ and B_o nonsingular.

<u>Proof.</u> The result is immediate if one notes that the roots of $\det((\lambda + a)I - bB_1B_o^{-1}) = 0$ are of the form $a + \lambda_j b$, where λ_j, $j=1,2,\ldots,n$, are the generalized eigenvalues of $\lambda B_o + B_1$.

We now look at the case $r = 2$.

<u>Theorem 7.5</u> If $A_o = H_o$, $A_1 = H_1$, $A_2 = -I$ and H_o is positive definite, then the roots of

$$\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$$

are real.

<u>Proof.</u> One notes that

$$-B_o = \begin{pmatrix} I & 0 \\ 0 & H_o \end{pmatrix} \text{ is positive definite and } B_1 = \begin{pmatrix} 0 & I \\ I & -H_1 \end{pmatrix}$$

is hermitian. By Theorem 7.2, the roots of $\det(-B_o\lambda + B_1) = 0$ or $\det(B_o\lambda - B_1) = 0$ are real. Thus the roots of $\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$ are real by Theorem 7.1.

<u>Remark</u> If A_2 is nonsingular and

$$\det(\lambda^2 A_o(-A_2)^{-1} + \lambda A_1(-A_2)^{-1} - I) = 0$$

satisfies the conditions in Theorem 7.5, then all its roots are real.

<u>Theorem 7.6</u> If $A_o = H_o$, $A_1 = H_1$ and $A_2 = -H_o$ such that $\det(H_o) \neq 0$ and $H_oH_1 = H_1H_o$, then the eigenvalues of $\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$ are real.

<u>Proof.</u> The equation $\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$ may be written $\det(\lambda^2 I + \lambda H_1 H_o^{-1} - I)\det(H_o) = 0$, since $\det(H_o) \neq 0$. By Theorem 7.1, the roots of the last equation are the generalized eigenvalues of $\lambda B_o + B_1$, where

$$B_o = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} 0 & I \\ I & H_1H_o^{-1} \end{pmatrix}.$$

Since B_o, B_1 are hermitian, $\det(B_o) \neq 0$ and $B_oB_1 = B_1B_o$, by Theorem 7.3, the roots are real.

Turning to 1-circular polynomials we have the following results :

<u>Theorem 7.7</u> If U_o and U_1 are unitary matrices, then the roots of $\det(\lambda U_o + U_1) = 0$ lie on the unit circle in the complex plane.

<u>Proof.</u> The equation $\det(\lambda U_o + U_1) = 0$ may be written

$$\det(\lambda I + U_1 U_o^{-1})\det(U_o) = 0.$$

Since $\det(U_o) \neq 0$ and $U_1 U_o^{-1}$ is unitary, the roots of the above equation are on the unit circle and so are those of $\det(\lambda U_o + U_1) = 0$.

<u>Theorem 7.8</u> If $A_o = U_o$, $A_1 = 0$ and $A_2 = U_1$, then the eigenvalues of $\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$ lie on the unit circle in the complex plane.

<u>Proof.</u> Since $B_o = \begin{pmatrix} -I & 0 \\ 0 & -U_o \end{pmatrix}$ and $B_1 = \begin{pmatrix} 0 & I \\ -U_1 & 0 \end{pmatrix}$ are unitary, by Theorems 7.1 and 7.7 the eigenvalues of $\det(\lambda^2 A_o + \lambda A_1 + A_2) = 0$ are on the unit circle.

<u>Theorem 7.9</u> If the generalized eigenvalues of $\lambda B_o + B_1$ are on the unit circle, $(\lambda + a)B_o - B_1$ is 1-circular, where $a = r+si$ is a complex number and b is a nonzero real number.

<u>Proof.</u> Since the roots of $\det((\lambda + a)I - bB_1B_o^{-1}) = 0$ are of the form $a + b\lambda_j$, where λ_j, $j=1,2,\ldots,n$, are the generalized eigenvalues of $\lambda B_o + B_1$, they are on a circle with center at (r,s) and radius b; i.e., $(\lambda+a)B_o - bB_1$ is 1-circular.

BIBLIOGRAPHY

[1] Afriat, S. N., Composite matrices, Quart. J. Math., Oxford, Ser.(2), 5, 1954, pp.81-98.

[2] Aitken, A. C., Determinants and matrices. 4th ed.,1946.

[3] Aramata, H., Über einen Satz für unitäre Matrizen, Tohoku Math. J. Vol.28, 1927, pp.281.

[4] Arscoot, F. M., Latent roots of tri-diagonal matrices, Proc. Edinburgh Math. Soc. Vol.12, 1961, Edinburgh Math. Notes, No. 44, pp.5-7.

[5] Brauer, R., Über einen Satz für unitäre Matrizen, Tohoku Math. J. Vol.30, 1928, pp.72.

[6] Brenner, J. L., Expanded matrices from matrices with complex elements, SIAM Review Vol.3, No.2, April, 1961, pp.165-166.

[7] Brioschi, F., J. Math. pures appl. Vol.19, 1854, pp.253-256.

[8] Cauchy, A., Anciens Exercises. 1829-1830-Coll. Works II, Vol.9, pp.174-195.

[9] Clebsch, A., J. reine. angew. Math., Vol.62, 1863, pp.232-245.

[10] Goot, E., A theorem on determinants, SIAM Review, Vol.2, No.4, October, 1960, pp.288-291.

[11] Hadeler, K. P., Eigenwerete von Operatorpolynormen, Arch. Rat. Mech. Anal. 20, 1965, pp.72-80.

[12] Hermite, C. R. Acad. Sci., Paris, Vol.41, 1855, pp.181-183.

[13] Hojdar, J., Eigenwertabschätzungen für ein Polynomial-problem, Aplikace Math. Svazek 14, 1969, pp.120-133.

[14] Householder, A. S., The theory of matrices in numerical analysis. Blaisdell, New York, 1964.

[15] Marcus, M. & Minc, H., A survey of matrix theory and matrix inequalities. Allyn and Bacon, Inc., Boston, 1964.

[16] Parodi, M., La localisation des valeurs caracteristiques des matrices. Gauthier-villars, Editeur-imprimeur-libraire, Paris, 1959.

[17] Rahusen, A. E., Wiskundige Opgaven, Vol.5, 1893, pp.392-394.

[18] Scorza, G., Corpi numerici e Algebre, Messina, 1921, pp.133-179.

[19] Weierstrass, S. B. preuss. Akad. Wiss., 1879.