MATRICES
WITH
LINEAR AND CIRCULAR SPECTRA

by
LUANG-HUNG CHANG
B.Sc. National Taiwan University, 1958

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Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

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ABSTRACT

Much is known about the eigenvalues of some special types of matrices. For example, the eigenvalues of a hermitian or skew-hermitian matrix lie on a line while those of a unitary matrix lie on a circle; their spectra are "linear" or "circular". This suggests the question: What matrices have this property? Or, more generally, what matrices have their eigenvalues on plane curves of a simple kind? Is it possible to recognize such matrices by inspection?

In this thesis, we make a small start on these problems, exploring some matrices whose eigenvalues lie on one or more lines, or on one or more circles.
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1. Introduction.

Let

\[ A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} = (a_{jk}) \]

be any square matrix of order n with elements in the complex field \( C \). A non-zero vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \) with complex entries \( x_1, x_2, \ldots, x_n \) such that \( AX = \lambda \mathbf{x} \) is said to be an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \); \( \lambda \) is a root of the characteristic equation

\[ \det(\lambda I - A) = 0 \]

of degree n. Counting multiplicities there are n eigenvalues of \( A \). The set of eigenvalues of \( A \) is called the spectrum of \( A \).

There exists a lot of information about the eigenvalues of some special types of matrices. For example, it is easily seen that diagonal \( (a_{jk} = 0 \text{ if } j \neq k) \) and triangular \( (a_{jk} = 0 \text{ if } j > k) \) matrices exhibit their eigenvalues on their main diagonal: \( \lambda_j = a_{jj}, j = 1, 2, \ldots, n \).

It is not quite so trivial that the eigenvalues of a real symmetric matrix \( A (A = A^T) \), where \( A^T \) denotes the transpose of \( A \) lie on the real axis. This was first proved by A. Cauchy [8] in 1829, and many subsequent proofs have been given, by
other eminent mathematicians, including Jacobi and Sylvester. This theorem was generalized by Hermite [12] in 1855 to matrices for which \( A = A^* \) (\( A^* \) denotes the transpose conjugate of \( A \)), and resulted in such matrices being named after him. A well-known simple elegant one-line proof of this result is the following:

For a unit eigenvector \( X \) corresponding to \( \lambda \),

\[
\lambda = \lambda X^* X = X^* AX = (X^* AX)^* = \overline{\lambda}.
\]

In a similar way the eigenvalues of a real skew-symmetric matrix \( (A = -A^T) \) lie on the imaginary axis. This was first proved by A. Clebsch [9], and later by Weierstrass [19]. That the same is true for a skew-hermitian matrix \( (A = -A^*) \) was shown by G. Scorza [18] in 1921. This is immediate if one notes that \( iA \) is hermitian.

The eigenvalues of an orthogonal matrix \( A \) \( (AA^T = I) \), where \( I \) is the identity matrix of order \( n \) have absolute value 1 and occur in reciprocal pairs. This was proved by F. Brioschi [7] in 1854, and again by F. Rahusen [17] in 1894. The eigenvalues of a unitary matrix \( A \) \( (AA^* = A^* A = I) \) also have absolute value 1. The proof was first given by H. Aramata [3] in 1927, and a short proof was given by R. Brauer [5] in 1928. The result is obvious if we observe that for a unit eigenvector \( X \) corresponding to \( \lambda \),

\[
1 = X^* X = X^* A^* AX = (\overline{\lambda} X^*)(\lambda X) = \overline{\lambda} \lambda.
\]
As far as the eigenvalues of a general matrix are concerned, nothing specific can be said about their location; they can obviously lie anywhere in the complex plane. A great many theorems have been proved about the localization of eigenvalues; many have been summarized by M. Marcus and H. Minc [15] and M. Parodi [16].

The three types of matrices mentioned above have something in common. Their eigenvalues each lie on a line or on a circle; or, in other words, their spectra are linear or circular. This suggests the question: What matrices have this property? Or, more generally, what matrices have their eigenvalues on plane curves of a simple sort? Is it possible to recognize such matrices by inspection? These seem to be rather difficult questions to answer. In this thesis, we make a small start, exploring some matrices whose eigenvalues lie on one or more lines, or on one or more circles.

In section 2 we introduce a class of matrices called "HORT". A matrix is said to be HORT if it can be obtained from an hermitian matrix by a suitable rotation and/or translation. Its eigenvalues lie on a line. In a similar way we discuss "UOT" matrices, which can be obtained from unitary matrices by a suitable translation. We obtain necessary and sufficient conditions for a matrix to be HORT and UOT.
With each \( n \times n \) complex matrix \( A \) there is associated in a natural way a \( 2n \times 2n \) real matrix called the Expansion of \( A \). In section 3 we discuss some simple relations between \( A \) and its expansion.

In section 4 we examine \( n \times n \) nonnegative indecomposable matrices. We show that if such a matrix has all its eigenvalues on a line, it has real eigenvalues, and its index of imprimitivity is at most 2 (Theorem 4.2). Its eigenvalues lie on a circle if and only if its index of imprimitivity is \( n \) (Theorem 4.3).

In section 5 we introduce two further classes of matrices called "almost" hermitian (Definition 5.2) and "almost" HORT (Definition 5.3), and obtain necessary and sufficient conditions for a matrix to be "almost" HORT.

In section 6 we discuss the relationship between the eigenvalues of the \( r^{th} \) compound matrix \( C_r(A) \) of \( A \) and those of \( A \).

Finally, in section 7, we looked at the extended polynomial problem on eigenvalues; that is, the determination of the roots \( \lambda \) of

\[
\det( \lambda^r A_0 + \lambda^{r-1} A_1 + \ldots + A_r ) = 0,
\]

where the \( A_i \) are \( n \times n \) complex matrices. In the case where the degree \( r \) is 1 or 2, we give some conditions for the roots to lie on a line or on a circle.
In this thesis we shall use the following notation:

- **$C\#$**: the complex number field.
- **$H$**: Hermitian matrix.
- **$S$**: Skew-Hermitian matrix.
- **$U$**: Unitary matrix.
- **$(p,q)$**: a point in the complex plane.
- **$\text{Exp } A$**: expanded matrix of $A$.
- **$Q_{k,n}$**: totality of strictly increasing sequences of $k$ integers chosen from $1, 2, \ldots, n$.
- **$C_r(A)$**: $r^{th}$ compound matrix of $A$.
- **$A[x|y]$**: submatrix of $A$ using rows numbered $x$ and columns numbered $y$. Here $x$ and $y$ are sequences of integers.
- **$\det(A)$**: determinant of $A$. 
2. Matrices with eigenvalues on lines or circles.

Definition 2.1 A matrix $A$ is $m$-linear ($m$-circular) if all its eigenvalues lie on $m$ lines (circles) in the complex plane.

This definition implies that if $A$ is $m$-linear, it is also $p$-linear for all $p \geq m$. One could argue that $A$ should be $m$-linear if its eigenvalues lie on $m$ lines but not more than $m$ lines. In that case, however, we should sometimes be faced with some pesky combinatorial considerations, which merit discussion only when we are interested in the least $m$ for which $A$ is $m$-linear. It appears reasonable to us to define $m$-linear as in Definition 2.1.

Those matrices with just one distinct eigenvalue (the spectrum is a point) are rather special. We could call them $0$-linear and $0$-circular, but we shall be content to include them among the $1$-linear and $1$-circular matrices. These matrices are not the only ones that are both $1$-linear and $1$-circular. Such matrices are those from which the spectrum consists of at most two points. Every $2\times2$ matrix is $1$-linear and $1$-circular.

Suppose that the matrix $A$ is $1$-linear. If the line $L$ of eigenvalues of $A$ passes through $a = r + si$ with angle of inclination $\phi$, then parametric equations for $L$ are:

$$
\begin{cases}
  x = r + pt \\
  y = s + qt
\end{cases}
$$

where $t$ is a real parameter.

Let $a = r + si$, $q/p = \tan \phi$ if $p \neq 0$, and $\phi = \pi/2$ if $p = 0$.

Now the matrix $B = (A - aI)e^{-i\phi}$ has real roots. Conversely, if there exist a complex number $a$ and a real number $\phi$ such that $B$ has real roots, then $A$ is $1$-linear. Hence
Theorem 2.1  A is 1-linear if and only if there exist a complex number $a$ and a real number $\theta$, $0 \leq \theta < \pi$ such that $(A - aI)e^{-i\theta}$ has real roots.

One might be tempted to say that the problem of recognizing 1-linear matrices really amounts to recognizing matrices with real eigenvalues. But it is not this simple. The criterion of Theorem 2.1 may not have too much value in recognizing 1-linear matrices from those with real roots. However, matrices related to hermitian matrices can be recognized.

Definition 2.2  Let $A = (a_{jk})$ be an $n$-square matrix with elements in $\mathbb{C}$. We say that $A$ is HORT "hermitian on rotation and/or translation" if $(A - aI)e^{i\theta}$ is hermitian for some complex number $a$ and real number $\theta$.

If $A$ is HORT and $H = (A - aI)e^{i\theta}$ is hermitian, the eigenvalues of $A$ are $\lambda_j(A) = \lambda_j(H)e^{i\theta} + a$, $j=1,2,...,n$ and they lie on the line $y = \tan \theta \cdot (x - r) + s$ if $\theta \neq \pi/2$ and on the line $x = r$ if $\theta = \pi/2$.

Note that a skew-hermitian matrix $S$ is HORT since $(S - 0I)e^{-i\pi/2} = -iS$ is hermitian.

Theorem 2.2  A is HORT if and only if there are complex numbers $v$ and $w$ with $|w| = 1$ such that

$$\bar{a}_{jj} = wa_{jj} + (\bar{v} - vw),$$

and

$$\bar{a}_{kj} = wa_{jk} \quad \text{for } j \neq k, j,k=1,2,...,n.$$
8.

Proof. If $A$ is HORT, there exist, by definition, a complex number $a$ and a real number $\phi$ such that $H = (A-aI)e^{-i\phi}$ is hermitian. That is

$$(A-aI)e^{-i\phi} = (A^*-\bar{a}I)e^{i\phi}$$

or

$$A^* = e^{-2i\phi}(A-aI) + \bar{a}I,$$

thus

$$\bar{a}_{jj} = e^{-2i\phi}a_{jj} + (a - ae^{-2i\phi})$$

$$\bar{a}_{kj} = e^{-2i\phi}a_{jk} \quad \text{for } j \neq k, j,k=1,2,...,n.$$

Put $v = a$ and $w = e^{-2i\phi}$.

The converse is immediate by reversing the order of the above argument.

In practice the recognition of an HORT matrix is even simpler than the criterion of Theorem 2.2. If $\phi \neq 0$, $a(=v)$ can be assumed real. If $\phi = 0$, $a$ can be assumed to be pure imaginary, $a(=v) = bi$ where $b$ is real, $\phi = 0$, and $w = 1$.

Thus we have

Theorem 2.3  A is HORT if and only if either (1) or (2) holds.

(1) $\bar{a}_{jj} = wa_{jj} + v(1-w), \quad \bar{a}_{kj} = wa_{jk}, \quad j \neq k, j,k=1,2,...,n$

for some real $v$ and complex $w$ such that $|w| = 1$.

(2) $\bar{a}_{jj} = a_{jj} - 2bi, \quad \bar{a}_{kj} = a_{jk}, \quad j \neq k, j,k=1,2,...,n$

for some real number $b$. 
HORT matrices are 1-linear, but of course they are by no means the only 1-linear matrices, just as hermitian matrices are not the only ones with real eigenvalues. In fact the example $A = \begin{pmatrix} 1 & a \\ 0 & 2 \end{pmatrix}$ shows that a matrix with real elements and real eigenvalues can be just about as "unhermitian" as one can conceive.

Suppose that the matrix $A$ is 1-circular. The equation of the circle of eigenvalues must be of the form

$$(x - r)^2 + (y - s)^2 = b^2,$$

where $b$ is a non-negative real number.

Analogous to the case of HORT matrices, we make the following definition.

**Definition 2.3** Let $A = \left( a_{jk} \right)$ be an $n$-square matrix with elements in $\mathbb{C}^\#$. $A$ is UOT "unitary on translation" if $(A - aI)b^{-1}$ is unitary for some complex number $a$ and real number $b \neq 0$.

If $A$ is UOT and $U = (A - aI)b^{-1}$ is unitary, the eigenvalues of $A$ are

$$\lambda_j(A) = \lambda_j(U)b + a, \quad j = 1, 2, \ldots, n$$

and they lie on the circle

$$(x - r)^2 + (y - s)^2 = b^2 \quad \text{if} \quad a = r + si.$$
Theorem 2.4 A is UOT if and only if there exist a complex number \( v \) and real number \( w \neq 0 \) such that
\[
\sum_{m=1}^{n} a_{jm} \bar{a}_{jm} - va_{jj} - \bar{v}a_{jj} + |v|^2 - w^2 = 0
\]
\[
\sum_{m=1}^{n} a_{jm} \bar{a}_{km} - va_{kj} - \bar{v}a_{jk} = 0, \quad j \neq k, \ j, k = 1, 2, \ldots, n.
\]

Proof. If \( A \) is UOT there are, by definition, a complex number \( a \) and a real number \( b \neq 0 \) such that \( U = (A - aI)b^{-1} \) is unitary. That is
\[
UU^* = (A - aI)b^{-1}(A^* - \bar{a}I)b^{-1} = I
\]
or
\[
AA^* - aA^* - \bar{a}A + |a|^2 I = b^2 I,
\]
thus
\[
\sum_{m=1}^{n} a_{jm} \bar{a}_{jm} - a\bar{a}_{jj} - \bar{a}a_{jj} + |a|^2 = b^2
\]
\[
\sum_{m=1}^{n} a_{jm} \bar{a}_{km} - a\bar{a}_{kj} - \bar{a}a_{jk} = 0, \quad j \neq k, \ j, k = 1, 2, \ldots, n.
\]
Put \( v = a \) and \( w = b \).

The converse is immediate by reversing the order of the above argument.
3. The expanded matrix of a complex matrix

In looking at 1-linear, or more generally m-linear, matrices, there are really two major avenues of investigation: complex matrices and real matrices. We shall not confine ourselves to real matrices, but explore both problems.

We note, however, that corresponding to each complex matrix \( A \) there is a related real matrix called the "expanded matrix of \( A \)" denoted by \( \text{Exp} A \), and defined for \( A = (a_{jk}) = (b_{jk} + c_{jk}i) \), \( b_{jk}, c_{jk} \) real, by

\[
\text{Exp} A = \\
\begin{pmatrix}
 b_{11} & c_{11} & b_{12} & c_{12} & \cdots & b_{1n} & c_{1n} \\
-c_{11} & b_{11} & -c_{12} & b_{12} & \cdots & -c_{1n} & b_{1n} \\
b_{21} & c_{21} & b_{22} & c_{22} & \cdots & b_{2n} & c_{2n} \\
-c_{21} & b_{21} & -c_{22} & b_{22} & \cdots & -c_{2n} & b_{2n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{n1} & c_{n1} & b_{n2} & c_{n2} & \cdots & b_{nn} & c_{nn} \\
-c_{n1} & b_{n1} & -c_{n2} & b_{n2} & \cdots & -c_{nn} & b_{nn}
\end{pmatrix}
\]

In 1960, E. Gott [10] proved that

\[ \det(\text{Exp} A) = |\det(A)|^2. \]

A one-line proof of this theorem was given by J. L. Brenner [6] in 1961 based on an interesting theorem of S. N. Afriat [1]:

\[ \det(\text{Exp} A) = \det \begin{pmatrix} \text{Re} \det(A) & \text{Im} \det(A) \\ -\text{Im} \det(A) & \text{Re} \det(A) \end{pmatrix} = |\det(A)|^2. \]
Brenner also showed that the collection of the eigenvalues of Exp A consists of the eigenvalues of A and their conjugates.

This is particularly interesting for matrices A with real roots. The roots of Exp A are just those of A counted twice, certainly real and 1-linear.

We give some simple results relating A and Exp A in Theorem 3.1

Let A = (a_{jk}) be an n-square matrix with complex elements. Exp A is normal if and only if A is normal. Exp A is symmetric if A is hermitian, Exp A is skew-symmetric if A is skew-hermitian and Exp A is unitary if A is unitary.

Proof. First we show that Exp A is normal if and only if A is normal. Suppose a_{jk} = b_{jk} + c_{jk}i, and let

\[ A_{jk} = \begin{pmatrix} b_{jk} & c_{jk} \\ -c_{jk} & b_{jk} \end{pmatrix}. \]

Since

\[ \sum_{m=1}^{n} A_{jm} A_{km}^* = \sum_{m=1}^{n} A_{jm} A_{km} \]

if and only if

\[ \sum_{m=1}^{n} a_{jm} a_{km} = \sum_{m=1}^{n} a_{jm} a_{km} \]

j, k = 1, 2, ..., n.

\[ (\text{Exp } A)(\text{Exp } A)^* = \left( \sum_{m=1}^{n} A_{jm} A_{km}^* \right)_{jk} \]

\[ = \left( \sum_{m=1}^{n} A_{jm} A_{km}^* \right)_{jk} = (\text{Exp } A)^* (\text{Exp } A) \]
if and only if

\[ \text{AA}^* = (\sum_{m=1}^{n} a_{jm} a_{km})_{jk} = (\sum_{m=1}^{n} -a_{jm} a_{km})_{jk} = A^* A. \]

Thus A is normal if and only if \( \text{Exp} A \) is normal.

If A is hermitian, its eigenvalues are real, and, since the eigenvalues of \( \text{Exp} A \) are those of A and their conjugates, \( \text{Exp} A \) has real roots. However A, being hermitian also implies \( \text{Exp} A \) normal. Now a normal matrix with real eigenvalues must be hermitian. Hence \( \text{Exp} A \) is hermitian, and indeed, since its elements are real, \( \text{Exp} A \) is symmetric.

Similarly, one can prove the last two statements, if one notes that the conjugate of a pure imaginary is again pure imaginary and \( |a+bi| = 1 = |a-bi| \).

**Theorem 3.2** Let \( A = (a_{jk}) \) be an \( n \)-square matrix with complex elements. Assume that A is HORT, i.e., \( (A - aI)e^{-i\phi} \) is hermitian for some complex number \( a=r+si \) and real number \( \phi \) with \( 0 \leq \phi < \pi \). \( \text{Exp} A \) is 1-linear if (1) \( \phi = \pi/2 \), or (2) \( s=0 \) and \( \phi=0 \), or (3) spectrum of A is a point. \( \text{Exp} A \) is 2-linear otherwise.

**Proof.** If \( \phi = \pi/2 \), the eigenvalues of A lie on \( x = r \) and so do those of \( \text{Exp} A \). If \( s=0 \) and \( \phi=0 \), the eigenvalues of A and \( \text{Exp} A \) are all real and lie on \( y=0 \). If the spectrum of A is a point, \( \text{Exp} A \) is obviously 1-linear. Otherwise, the eigenvalues of A lie on the line with equation

\[ y = \tan\phi \cdot (x - r) + s. \]
Those of Exp A lie on
\[ y = \tan \phi \cdot (x - r) + s \]
and \[ y = -\tan \phi \cdot (x - r) - s. \]
This proves our theorem.

**Theorem 3.3** Let \( A = (a_{jk}) \) be an \( n \)-square matrix with complex entries. Assume that \( A \) is UOT; i.e., there are complex number \( a = r + si \) and real number \( b \neq 0 \) such that \((A - aI)b^{-1}\) is unitary. Exp A is 1-circular if \( s = 0 \), or if the spectrum of \( A \) is a point. Exp A is 2-circular otherwise.

**Proof.** By the remarks shown in Section 2, the eigenvalues of A lie on the circle
\[ (x - r)^2 + (y - s)^2 = b^2. \]
Now, if \( s = 0 \), the above equation may be written
\[ (x - r)^2 + y^2 = b^2, \]
which is symmetric with respect to the real axis. Then the eigenvalues of Exp A all lie on this circle for they are conjugate in pairs. If \( s \neq 0 \), the eigenvalues of Exp A will lie on two circles:
\[ (x - r)^2 + (y - s)^2 = b^2 \]
and \[ (x - r)^2 + (y + s)^2 = b^2. \]
More generally, one can see that if $A$ is 1-linear then \( \exp A \) is either 1-linear or 2-linear. Conversely, if \( \exp A \) is 1-linear, then $A$ is 1-linear; if \( \exp A \) is 2-linear, then $A$ can be either 1-linear or 2-linear. It is obvious, however, that if \( \exp A \) is $\alpha$-linear, $\alpha=1,2,3,\ldots$, then \( \exp(\exp A) = \exp^2(A) \) is also $\alpha$-linear. No new lines of eigenvalues are added. This is not surprising since \( \exp^2(A) \) is $A$ "blown up" with each root duplicated. A similar discussion applies to \( \alpha \)-circular matrices.

A more interesting question is to ask for the number of lines on which the eigenvalues of $B_m$ lie, where

1. $B_0 = B$ is 1-linear,
2. $B_k = \exp(a_k B_{k-1} + b_k I)$; $a_k, b_k$ complex, $k=1,2,\ldots,m$.

In general, each time we make an expansion, we double the number of lines of eigenvalues, hence the maximum linearity is $2^m$. But of course $B_m$ can be $2^q$-linear for each $q$, $0 \leq q \leq m$, for suitable choices of $B, a_k$ and $b_k$. Conceivably, $B_m$ could also be $\alpha$-linear, where $\alpha$ is not a power of 2. Just when $B_m$ is $\alpha$-linear for any integer $\alpha \leq 2^m$ seems to be a complex combinatorial problem. To illustrate the possibilities we give the linearity of $B_1$ and $B_2$ when $B_0$ is hermitian, skew hermitian and unitary (Tables 1 and 2).
Let \( a_1 = p_1 + q_1 i \neq 0, \ b_1 = r_1 + s_1 i \neq 0, \ a_2 = p_2 + q_2 i \neq 0 \) and \( b_2 = r_2 + s_2 i \neq 0 \).

### Table 1. \( B_0 = H(S) \)

<table>
<thead>
<tr>
<th>Conditions on ( p_1, q_1, r_1, s_1 )</th>
<th>( \text{Exp B}_1 )</th>
<th>Conditions on ( p_2, q_2, r_2, s_2 )</th>
<th>( \text{Exp B}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 = 0 ) &amp; ( s_1 = 0 )</td>
<td>1-linear ( y = 0 )</td>
<td>( q_2 = 0 ) &amp; ( s_2 = 0 )</td>
<td>1-linear ( y = 0 )</td>
</tr>
<tr>
<td>( p_1 = 0 ) &amp; ( s_1 = 0 )</td>
<td>( p_2 = 0 ) &amp; ( s_2 \neq 0 )</td>
<td>otherwise</td>
<td>1-linear (perp. to ( y = 0 ))</td>
</tr>
<tr>
<td>( p_1 = 0 ) &amp; ( r_1 = 0 )</td>
<td>1-linear ( x = 0 )</td>
<td>( q_2 = 0 ) &amp; ( s_2 = 0 )</td>
<td>1-linear ( y = 0 )</td>
</tr>
<tr>
<td>( q_1 = 0 ) &amp; ( r_1 = 0 )</td>
<td>( q_2 = 0 ) &amp; ( s_2 \neq 0 )</td>
<td>otherwise</td>
<td>1-linear (perp. to ( y = 0 ))</td>
</tr>
<tr>
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<td>1-linear (perp. to ( y = 0 ))</td>
<td>( q_2 = 0 )</td>
<td>2-linear</td>
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<tr>
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<td>( p_2 = 0 ) &amp; ( s_2 \neq 0 )</td>
<td>otherwise</td>
<td>2-linear (perp. to ( y = 0 ))</td>
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<tr>
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<td>2-linear (paral. to ( y = 0 ))</td>
<td>( q_2 = 0 ) &amp; ( s_2 = 0 )</td>
<td>2-linear (paral. to ( y = 0 ))</td>
</tr>
<tr>
<td>( p_1 = 0 ) &amp; ( r_1 = 0 )</td>
<td>( q_2 = 0 ) &amp; ( s_2 \neq 0 )</td>
<td>( p_2 = 0 )</td>
<td>4-linear (paral. to ( y = 0 ))</td>
</tr>
<tr>
<td>( p_1 = 0 ) &amp; ( s_1 = 0 )</td>
<td>( p_2 = 0 ) &amp; ( s_2 \neq 0 )</td>
<td>otherwise</td>
<td>2-linear (perp. to ( y = 0 ))</td>
</tr>
<tr>
<td>( q_1 = 0 ) &amp; ( r_1 = 0 )</td>
<td>otherwise</td>
<td>2-linear</td>
<td>4-linear</td>
</tr>
</tbody>
</table>

### Table 2. \( B_0 = U \)

<table>
<thead>
<tr>
<th>Conditions on ( p_1, q_1, r_1, s_1 )</th>
<th>( \text{Exp B}_1 )</th>
<th>Conditions on ( p_2, q_2, r_2, s_2 )</th>
<th>( \text{Exp B}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 = 0 )</td>
<td>1-circular</td>
<td>( s_2 = 0 )</td>
<td>1-circular</td>
</tr>
<tr>
<td>otherwise</td>
<td>otherwise</td>
<td>otherwise</td>
<td>2-circular</td>
</tr>
<tr>
<td>otherwise</td>
<td>2-circular</td>
<td>( s_2 = 0 )</td>
<td>2-circular</td>
</tr>
<tr>
<td>otherwise</td>
<td>otherwise</td>
<td>otherwise</td>
<td>4-circular</td>
</tr>
</tbody>
</table>

**Definition 4.1** A real n-square matrix $A = (a_{jk})$ is called nonnegative, if $a_{jk} \geq 0$ for $j,k = 1,2,\ldots,n$.

We write $A \geq 0$.

**Definition 4.2** A nonnegative n-square matrix $A = (a_{jk})$ $(n > 1)$ is said to be decomposable if there exists permutation matrix $P$ such that $PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$ where $B$ and $C$ are square matrices. Otherwise $A$ is indecomposable.

The fundamental theorem on indecomposable nonnegative matrices is the Perron-Frobenius theorem \[15\], which we state as follows:

**Theorem 4.1** Let $A$ be an n-square nonnegative indecomposable matrix. Then:

1. $A$ has a real positive eigenvalue $r$ (the maximal eigenvalue of $A$) which is a simple root of the characteristic equation of $A$. If $\lambda_j(A)$ is any eigenvalue of $A$, then $|\lambda_j(A)| \leq r$.
2. If $A$ has $h$ eigenvalues of modulus $r : \lambda_1 = r, \lambda_2, \ldots, \lambda_h$, then they are the $h$ distinct roots of $\lambda^h - r^h = 0$; $h$ is called the index of imprimitivity of $A$.
3. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the eigenvalues of $A$ and $\theta = e^{i2\pi/h}$, then $\lambda_1 \theta, \lambda_2 \theta, \ldots, \lambda_n \theta$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$.
in some order.

(4) If $h > 1$, then there exists a permutation matrix $P$ such that

$$
PAP^T = \begin{pmatrix}
0 & A_{12} & 0 & \ldots & 0 & 0 \\
0 & 0 & A_{23} & \ldots & 0 & 0 \\
& \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & A_{n-1,h} \\
A_{n,1} & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}
$$

where the zero blocks down the main diagonal are square.

Suppose that the nonnegative indecomposable matrix $A$ is 1-linear. Let $r$ be the maximal (positive real) eigenvalue of $A$ and suppose that there were a complex eigenvalue, say $\lambda \neq r$. Since the coefficients of $\det(\lambda I - A) = 0$ are real, $\overline{\lambda}$ is also a root of $\det(\lambda I - A) = 0$ and $|\lambda| \leq r$. Then $\lambda$ and $\overline{\lambda}$ lie on or in the circle $x^2 + y^2 = r^2$ and are symmetrically placed with respect to the $x$-axis. Thus $r$, $\lambda$ and $\overline{\lambda}$ can only be on a line if $\lambda = \overline{\lambda}$. Hence we have:

**Theorem 4.2** If an $n$-square nonnegative indecomposable matrix $A$ is 1-linear, then all its eigenvalues are real. Furthermore, $h \leq 2$.

It is trivial that if $A$ is nonnegative indecomposable and 1-linear, then $\text{Exp} A$ is also 1-linear; in fact, its roots are all real.
Part(3) of Theorem 4.1 indicates that $A$ is almost $n/h$-circular.

If $A$ is 1-circular, then all the eigenvalues of $A$ satisfy $|\lambda_j(A)| = r, \ j = 1,2,\ldots,n$, where $r$ is the positive real maximal eigenvalue of $A$. Thus all $\lambda_j(A)$ are roots of $x^n - r^n = 0$, and $h = n$. Conversely, if $h = n$, the roots of $\lambda^n - r^n = 0$ coincide with the eigenvalues of $A$, and $A$ has 1-circular eigenvalues. Hence

Theorem 4.3 An $n$-square nonnegative indecomposable matrix $A$ is 1-circular if and only if $n = h$.

An example of the matrices appearing in Theorem 4.3 is the permutation matrix

$$
P = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}.
$$

The eigenvalues of $P$ are the $n^{th}$ roots of unity:

$$
\lambda_j(P) = e^{i(2\pi/n)j}, \quad j = 1,2,\ldots,n,
$$

for $P$ is the companion matrix of the polynomial $\lambda^n - 1$.

$P$ is a special case of the more general matrix

$$
B = \begin{pmatrix}
a & b_1 & 0 & \ldots & 0 \\
0 & a & b_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & a & b_{n-1} \\
b_n & \ldots & 0 & 0 & a
\end{pmatrix}.
$$
Here, \( \det(\lambda I - B) = (\lambda - a)^n + \frac{n}{1} \sum_{k=1}^{n} b_k \). The eigenvalues of \( B \) are:

\[
\lambda_j(B) = re^{i(2\pi/n)}j + a, \quad j = 1, 2, \ldots, n, \quad n \text{ even}
\]
or

\[
\lambda_j(B) = re^{i(\pi/n + 2\pi j/n)} + a, \quad j = 1, 2, \ldots, n, \quad n \text{ odd},
\]

where \( r = \sqrt{\frac{1}{1} \sum_{k=1}^{n} b_k} \).

The eigenvalues of \( \exp B \) are just those of \( B \) counted twice; hence \( \exp B \) is again 1-circular.

\( B \) is interesting in that it is 1-circular even if the non-zero entries are non-real. Its eigenvalues are

\[
\lambda_j(B) = n \sqrt{|c|} e^{i((\psi + 2\pi j)/n)}, \quad j = 1, 2, \ldots, n,
\]

where \( c = \frac{1}{1} \sum_{k=1}^{n} b_k = |c| e^{i\psi} \). Let \( a = p+qi, \quad p, q \) are real, then

\( \exp B \) is 1-circular if \( q = 0 \) and \( \exp B \) is 2-circular otherwise.
5. Tridiagonal matrices.

**Definition 5.1** A matrix $A = (a_{jk})$ is called a tridiagonal or Jacobi matrix if $a_{jk} = 0$ whenever $|j - k| \geq 2$.

Thus

$$L_n = \begin{pmatrix}
    b_1 & c_1 & 0 & 0 & \cdots & 0 \\
    d_2 & b_2 & c_2 & 0 & \cdots & 0 \\
    0 & d_3 & b_3 & c_3 & 0 & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & 0 & d_{n-1} & b_{n-1} & c_{n-1} \\
    0 & \cdots & 0 & 0 & d_n & b_n
\end{pmatrix}$$

is a general $n$-square complex Jacobi matrix. Since any matrix $A$ is similar to a Jacobi matrix $L_n$ [14], we are interested in conditions under which $L_n$ is $m$-linear; particularly when $L_n$ is $1$-linear.

The following theorem was proved by F. M. Arscott in 1961 [4]:

**Theorem 5.1** If all the entries of the Jacobi matrix $L_n$ are real and $c_j d_{j+1} > 0$, $j = 1, 2, \ldots, n - 1$, then all its eigenvalues are real and simple.

We shall generalize Theorem 5.1 as follows:
Theorem 5.2  Let $M_n$ be the matrix $L_n$ with real $b_k$, $1 \leq k \leq n$, and complex $c_j$, $d_{j+1}$ such that $c_j d_{j+1} > 0$, $1 \leq j \leq n-1$. Then the eigenvalues of $M_n$ are real and simple.

Proof. The coefficient of $\lambda^k$ in $\det(\lambda I - M_n)$ is the sum of the $(n-k)$-square principal minors. Wherever $c_j$ occurs as a factor in such a minor so does $d_{j+1}$, and vice versa. Thus these elements always occur together as a product $c_j d_{j+1}$. It follows that the characteristic equation $\det(\lambda I - M_n) = 0$ has the same coefficients and roots as when all $c_j$ and $d_{j+1}$ are real and $c_j d_{j+1} > 0$. By Theorem 5.1 the eigenvalues of $M_n$ are real and simple.

By Theorem 5.1 and Theorem 5.2, given a Jacobi matrix $L_n$, if $b_1, b_2, \ldots, b_n$ are real, $c_1, c_2, \ldots, c_{n-1}$ and $d_2, d_3, \ldots, d_n$ are complex numbers such that $c_j d_{j+1} > 0$, $j=1,2,\ldots,n-1$, then its eigenvalues are all real. Analogous to the case of 1-linear matrices discussed in Section 2, we make the following definitions:

Definition 5.2  Let $L_n$ be a Jacobi matrix with complex entries. We call $L_n$ "almost" hermitian if $b_1, b_2, \ldots, b_n$ are real, $c_1, c_2, \ldots, c_{n-1}$ and $d_2, d_3, \ldots, d_n$ are complex numbers such that $c_j d_{j+1} > 0$, $j=1,2,\ldots,n-1$. 
Note that if $L_n$ is "almost" hermitian and $d_{j+1} = \bar{c}_j$, $j=1,2,...,n-1$, then $L_n$ is hermitian.

**Definition 5.3** Let $L_n$ be a Jacobi matrix. We call $L_n$ "almost" HORT if $(L_n - aI)e^{-i\phi}$ is "almost" hermitian, for some complex number $a$ and real number $\phi$ such that $0 \leq \phi < \pi$.

**Theorem 5.3** $L_n$ is "almost" HORT if and only if there are complex numbers $v$ and $w$ with $|w| = 1$ such that $(b_j - v)w$ is real, $j=1,2,...,n$ and $c_jd_{j+1}w^2 > 0$, $j=1,2,...,n-1$.

**Proof.** If $L_n$ is "almost" HORT, by definition, there exist a complex number $a$ and a real number $\phi$ with $0 \leq \phi < \pi$ such that $B = (L_n - aI)e^{-i\phi}$ is "almost" hermitian; i.e., $(b_j - a)e^{-i\phi}$ is real and $c_jd_{j+1}e^{-2i\phi} > 0$, $j=1,2,...,n-1$.

Put $v = a$ and $w = e^{-i\phi}$.

The converse is immediate by reversing the order of the above argument.

**Theorem 5.4** Let $L_n$ be a Jacobi matrix that is "almost" HORT; i.e., $(L_n - aI)e^{-i\phi}$ is "almost" hermitian for some complex number $a=r+si$ and real number $\phi$ with $0 \leq \phi < \pi$. Then $\text{Exp } A$ is 1-linear if (1) $\phi = \pi/2$ or (2) $s = 0$ and $\phi = 0$, and $\text{Exp } A$ is 2-linear otherwise.
Proof. We omit the proof here since it is similar to that of Theorem 3.2.
6. Compound matrices and some other theorems.

**Definition 6.1** Let $A = (a_{jk})$ be an $n$-square matrix and $1 \leq r \leq n$. The $r^{th}$ compound matrix of $A$, $C_r(A)$, is the $\binom{n}{r} \times \binom{n}{r}$ matrix whose entries are $\det(A[x|y])$, $x \in Q_{r,n}$ and $y \in Q_{r,n}$ arranged lexicographically in $x$ and $y$.

For example, if $n = 3$ and $r = 2$ then

$$C_2(A) = \begin{pmatrix}
\det(A[1,2|1,2]) & \det(A[1,2|1,3]) & \det(A[1,2|2,3]) \\
\det(A[1,3|1,2]) & \det(A[1,3|1,3]) & \det(A[1,3|2,3]) \\
\det(A[2,3|1,2]) & \det(A[2,3|1,3]) & \det(A[2,3|2,3])
\end{pmatrix}$$

In particular, $C_1(A) = A$ and $C_n(A) = \det(A)$.

**Theorem 6.1** Let $A = (p+qi)U$, $p+qi \neq 0$. The eigenvalues of $C_r(A)$, $1 \leq r \leq n$, lie on a circle with center at the origin and radius $|p+qi|^r$. When $r = n$, the eigenvalues coalesce at the point $(p+qi)^n$. If $|p+qi| = 1$, all the eigenvalues of $C_r(A)$, $r = 1, 2, \ldots, n$, are on the unit circle (Fig. 6-1).

Fig. 6-1
Proof. Since the eigenvalues of $C_r(A)$ are the $\binom{n}{r}$ products
\[\lambda_{j_1}(A)\lambda_{j_2}(A)\cdots\lambda_{j_r}(A),\]
where $1 \leq j_1 < j_2 < \cdots < j_r \leq n$, and since the $\lambda_j(A)$ are on the circle with center at the origin and radius $|p+qi|$, the eigenvalues of $C_r(A)$ lie on the circle with center at the origin and radius $|p+qi|^r$.

Theorem 6.2 The eigenvalues of $C_r(S)$, $1 \leq r \leq n$, lie on the real axis if $r$ is even and on the imaginary axis if $r$ is odd, while those of $C_r(H)$ lie on the real axis.

Proof. Since the eigenvalues of $S$ are pure imaginaries, the eigenvalues of $C_r(S)$ are real if $r$ is even and pure imaginary if $r$ is odd.

The last statement follows immediate from the fact that the set of real numbers is a field.

One notes that the set $\{C_r(A) : r=1,2,\ldots,n\}$ is 2-linear if $A = S$ and 1-linear if $A = H$.

The following two theorems tell us that we can obtain 1-circular matrix from 1-linear matrix by certain transformation, and vice versa.

Theorem 6.3
\[A = (I - S)(I + S)^{-1}\]
is a unitary matrix; i.e., we can obtain a matrix whose eigenvalues lie on a unit circle from a skew-hermitian matrix by this transformation (Fig. 6-2).
Proof. First note that $I + S$ is nonsingular since the roots of $S$ are pure imaginaries. Also, $(I+S)(I-S) = (I-S)(I+S)$.

$$AA^* = (I-S)(I+S)^{-1}((I+S)^{-1})^*(I-S)^*$$

$$= (I-S)(I+S)^{-1}(I+S^*)^{-1}(I-S^*)$$

$$= (I-S)(I+S)^{-1}(I-S)^{-1}(I+S) \quad \text{(since } S^* = -S)$$

$$= (I-S)((I-S)^{-1}(I+S))^{-1}(I+S)$$

$$= I.$$

Theorem 6.4 If $\det(I+U) \neq 0$, then

$$A = (I - U)(I + U)^{-1}$$

is skew-hermitian; i.e., we can obtain a matrix whose eigenvalues lie on the imaginary axis from a unitary matrix by this transformation (Fig. 6-3).
Proof. We need only to show that $A = -A^*$:

$$A^* = ((I+U)^{-1})^*(I-U)^*$$

$$= (I+U^*)^{-1}(I-U^*)$$

$$= (I+U^{-1})^{-1}(I-U^{-1})$$  \text{(since $U^* = U^{-1}$)}

$$= (U^{-1}(U+I))^{-1}(U^{-1}(U-I))$$

$$= (U+I)^{-1}UU^{-1}(U-I)$$

$$= (U+I)^{-1}(U-I)$$

$$= (U+I)^{-1}(U-I)(U+I)(U+I)^{-1}$$

$$= (U+I)^{-1}(U+I)(U-I)(U+I)^{-1}$$

$$= -(I-U)(I+U)^{-1}$$

$$= -A^*.$$
7. Extended polynomial-problem on eigenvalues.

Let $A_0, A_1, \ldots, A_r$ be $r+1$ $n$-square complex matrices. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a nonzero vector with complex entries. Consider

\begin{equation}
(\lambda^r A_0 + \lambda^{r-1} A_1 + \ldots + A_r)X = 0.
\end{equation}

The determination of $\lambda$ such that equation (7.1) holds is called the extended polynomial-problem on eigenvalues. $X$ is said to be an eigenvector belonging to the eigenvalue $\lambda$. One can see that the eigenvalues of the polynomial-problems are exactly the roots of

\begin{equation}
\det(\lambda^r A_0 + \lambda^{r-1} A_1 + \ldots + A_r) = 0.
\end{equation}

The determinant is a polynomial in $\lambda$ of degree not exceeding $r \cdot n$. If $A_0$ is nonsingular, the degree of (7.2) is exactly $r \cdot n$.

If $r = 1$ and $A_0 = I$, then the roots of $\det(\lambda I + A_1) = 0$ are the eigenvalues of $-A_1$.

If $r = 1$ and $A_0$ and $A_1$ are arbitrary matrices the problem is referred to as a generalized eigenvalue problem [13].
Theorem 7.1  The roots of (7.2) are the generalized eigenvalues of $\lambda B_0 + B_1$ such that

$$(7.3) \quad (\lambda B_0 + B_1)X = 0,$$

where $X$ is a nonzero vector,

$$B_0 = \begin{pmatrix} -I & 0 & 0 & \ldots & 0 & 0 \\ 0 & -I & 0 & \ldots & 0 & 0 \\ 0 & 0 & -I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -I & 0 \\ 0 & 0 & 0 & \ldots & 0 & -A_0 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ -A_r & -A_{r-1} & -A_{r-2} & \ldots & -A_2 & -A_1 \end{pmatrix}.$$

If $A_0 = I$, then the roots of (7.2) are the eigenvalues of $B_1$.

If $A_0$ is nonsingular, then the roots of (7.2) are the eigenvalues of $B_2$, where

$$B_2 = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ -A_rA_0^{-1} & -A_{r-1}A_0^{-1} & -A_{r-2}A_0^{-1} & \ldots & -A_2A_0^{-1} & -A_1A_0^{-1} \end{pmatrix}.$$
Proof.

\[ \lambda \mathbf{B}_0 + \mathbf{B}_1 = \begin{pmatrix} -\lambda \mathbf{I} & \mathbf{I} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda \mathbf{I} & \mathbf{I} & \ldots & 0 & 0 \\ 0 & 0 & -\lambda \mathbf{I} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda \mathbf{I} & \mathbf{I} \\ -\mathbf{A}_r & -\mathbf{A}_{r-1} & -\mathbf{A}_{r-2} & \ldots & -\mathbf{A}_2 & -\mathbf{A}_1 - \lambda \mathbf{A}_0 \end{pmatrix} \]

To the first column add \( \lambda \) times the second column, \( \lambda^2 \) times the third column, \ldots, \( \lambda^{r-1} \) times the last column, we obtain

\[ \lambda \mathbf{B}_0 + \mathbf{B}_1 = \begin{pmatrix} 0 & \mathbf{I} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda \mathbf{I} & \mathbf{I} & \ldots & 0 & 0 \\ 0 & 0 & -\lambda \mathbf{I} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -\lambda \mathbf{I} & \mathbf{I} \\ -\mathbf{P}(\lambda) & -\mathbf{A}_{r-1} & -\mathbf{A}_{r-2} & \ldots & -\mathbf{A}_2 & -\mathbf{A}_1 - \lambda \mathbf{A}_0 \end{pmatrix} \]

where \( \mathbf{P}(\lambda) = \lambda \mathbf{A}_0 + \lambda^{r-1} \mathbf{A}_1 + \ldots + \lambda \mathbf{A}_{r-1} + \mathbf{A}_r \).

Since

\[ \det(\lambda \mathbf{B}_0 + \mathbf{B}_1) = \det \begin{pmatrix} \mathbf{I} & 0 & 0 & \ldots & 0 & 0 \\ -\lambda \mathbf{I} & \mathbf{I} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda \mathbf{I} & \mathbf{I} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{I} & 0 \\ 0 & 0 & 0 & \ldots & -\lambda \mathbf{I} & \mathbf{I} \end{pmatrix} \]

\[ = (-1)^{r-1} \det(-\mathbf{P}(\lambda)) \det \begin{pmatrix} \mathbf{I} & 0 & 0 & \ldots & 0 & 0 \\ -\lambda \mathbf{I} & \mathbf{I} & 0 & \ldots & 0 & 0 \\ 0 & -\lambda \mathbf{I} & \mathbf{I} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{I} & 0 \\ 0 & 0 & 0 & \ldots & -\lambda \mathbf{I} & \mathbf{I} \end{pmatrix} \]

\[ = (-1)^{r-1} \det(-\mathbf{P}(\lambda)) = (-1)^{n+r-1} \det(\lambda \mathbf{A}_0 + \lambda^{r-1} \mathbf{A}_1 + \ldots + \mathbf{A}_r), \]

the roots of (7.2) are the generalized eigenvalues of \( \lambda \mathbf{B}_0 + \mathbf{B}_1 \).

Now if \( \mathbf{A}_0 = \mathbf{I} \), \(-\mathbf{B}_1\) is an identity matrix of order \( r \cdot n \),
and the roots of (7.2) are the eigenvalues of $B_1$.

If $A_o$ is nonsingular, then $A_o^{-1}$ exists, and (7.2) may be written

$$\det((\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1})A_o) = 0,$$

or

$$\det(\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1})\det(A_o) = 0.$$

Since $\det(A_o) \neq 0$, the roots of (7.2) are the roots of

$$(7.4) \quad \det(\lambda^r I + \lambda^{r-1} A_1 A_o^{-1} + \ldots + A_r A_o^{-1}) = 0.$$

From what we have proved above, the roots of (7.4) are the eigenvalues of $B_2$, where

$$B_2 = \begin{pmatrix}
0 & I & 0 & \ldots & 0 & 0 \\
0 & 0 & I & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 0 & 0 & \ldots & 0 & 0 \\
-A_r A_o^{-1} & -A_{r-1} A_o^{-1} & -A_{r-2} A_o^{-1} & \ldots & -A_2 A_o^{-1} & -A_1 A_o^{-1}
\end{pmatrix}.$$

Q.E.D.

Let $P(\lambda) = \lambda^r A_o + \lambda^{r-1} A_1 + \ldots + \lambda A_{r-1} + A_r$.

**Definition 7.1** The polynomial $P(\lambda)$ is said to be $m$-linear ($m$-circular) if the roots of $\det P(\lambda) = 0$ lie on $m$, but not fewer than $m$ lines (circles).

We shall give a few results on $1$-linear and $1$-circular polynomials when $r = 1$ and 2.
For the case $r = 1$, we are primarily interested in the case when the roots of $\det P(\lambda) = 0$ are real. One might guess that if $A_0$ and $A_1$ are both hermitian, then $\lambda A_0 + A_1$ has real eigenvalues. It is not quite this simple. Some restrictions are needed on $A_0$ and $A_1$.

Theorem 7.2 Let $H_0$ and $H_1$ be hermitian matrices of order $n$ and $H_0$ be positive or negative definite, then the roots of $\det(\lambda H_0 - H_1) = 0$ are real [2].

Proof. Let $X$ be a nonzero eigenvector corresponding to the generalized eigenvalue of $\lambda H_0 - H_1$; i.e., $(\lambda H_0 - H_1)X = 0$, or $\lambda H_0 X = H_1 X$. Premultiplying by $X^*$, we have

\[ \lambda( X^* H_0 X ) = X^* \lambda H_0 X = X^* H_1 X. \]

Since $X^* H_1 X$ is real, and $X^* H_0 X$ is real and not zero, $\lambda$ is real. Therefore the roots of $\det(\lambda H_0 - H_1) = 0$ are real.

Theorem 7.3 If $H_0$ and $H_1$ are hermitian matrices such that $H_0 H_1 = H_1 H_0$ and $\det(H_0) \neq 0$, then the roots of $\det(\lambda H_0 - H_1) = 0$ are real.

Proof. We shall first prove that $H_0 H_1$ is hermitian if and only if $H_0$ and $H_1$ commute. On the one hand, if $(H_0 H_1)^* = H_0 H_1$, then $H_1^* H_0^* = H_1 H_0 = H_0 H_1$. On the other hand, if $H_0 H_1 = H_1 H_0$,
then \((H_0^{-1}H_1)^* = H_1^*H_0 = H_1H_0 = H_0^{-1}H_1\).

Secondly, we note that \(H_1H_0^{-1}\) is hermitian, since \(H_0^{-1}\) is hermitian and \(H_1H_0 = H_1H_0\) implies \(H_1H_0^{-1} = H_0^{-1}H_1\).

Now, the equation \(\det(\lambda H_0 - H_1) = 0\) may be written as \(\det(\lambda I - H_1H_0^{-1})\det(H_0) = 0\). The roots of \(\det(\lambda I - H_1H_0^{-1}) = 0\) are real and so are those of \(\det(\lambda H_0 - H_1) = 0\).

**Theorem 7.4** If the generalized eigenvalues of \(\lambda B_0 + B_1\) are real, \((\lambda + a)B_0 - bB_1\) for \(a, b\) complex numbers is 1-linear provided \(b\neq 0\) and \(B_0\) nonsingular.

**Proof.** The result is immediate if one notes that the roots of \(\det((\lambda + a)I - bB_1B_0^{-1}) = 0\) are of the form \(a + \lambda_jb\), where \(\lambda_j, j=1,2,\ldots,n\), are the generalized eigenvalues of \(\lambda B_0 + B_1\).

We now look at the case \(r = 2\).

**Theorem 7.5** If \(A_0 = H_0, A_1 = H_1, A_2 = -I\) and \(H_0\) is positive definite, then the roots of \(\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0\) are real.
Proof. One notes that

\[-B_0 = \begin{pmatrix} I & 0 \\ 0 & H_0 \end{pmatrix}\]
is positive definite and \(B_1 = \begin{pmatrix} 0 & I \\ I & -H_1 \end{pmatrix}\)
is hermitian. By Theorem 7.2, the roots of \(\det(-B_0 \lambda + B_1) = 0\) or \(\det(B_0 \lambda - B_1) = 0\) are real. Thus the roots of \(\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0\) are real by Theorem 7.1.

Remark If \(A_2\) is nonsingular and

\[\det(\lambda^2 A_0(-A_2)^{-1} + \lambda A_1(-A_2)^{-1} - I) = 0\]
satisfies the conditions in Theorem 7.5, then all its roots are real.

Theorem 7.6 If \(A_0 = H_0\), \(A_1 = H_1\) and \(A_2 = -H_0\) such that \(\det(H_0) \neq 0\) and \(H_0 H_1 = H_1 H_0\), then the eigenvalues of \(\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0\) are real.

Proof. The equation \(\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0\) may be written

\[\det(\lambda^2 I + \lambda H_1 H_0^{-1} - I)\det(H_0) = 0, \text{ since } \det(H_0) \neq 0.\]

By Theorem 7.1, the roots of the last equation are the generalized eigenvalues of \(\lambda B_0 + B_1\), where

\[B_0 = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}\] and \(B_1 = \begin{pmatrix} 0 & I \\ I & H_1 H_0^{-1} \end{pmatrix}\). Since \(B_0\) and \(B_1\) are hermitian, \(\det(B_0) \neq 0\) and \(B_0 B_1 = B_1 B_0\), by Theorem 7.3, the roots are real.
Turning to 1-circular polynomials we have the following results:

**Theorem 7.7** If $U_0$ and $U_1$ are unitary matrices, then the roots of $\det(\lambda U_0 + U_1) = 0$ lie on the unit circle in the complex plane.

**Proof.** The equation $\det(\lambda U_0 + U_1) = 0$ may be written

$$\det(\lambda I + U_1 U_0^{-1}) \det(U_0) = 0.$$ 

Since $\det(U_0) \neq 0$ and $U_1 U_0^{-1}$ is unitary, the roots of the above equation are on the unit circle and so are those of $\det(\lambda U_0 + U_1) = 0$.

**Theorem 7.8** If $A_0 = U_0$, $A_1 = 0$ and $A_2 = U_1$, then the eigenvalues of $\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0$ lie on the unit circle in the complex plane.

**Proof.** Since $B_0 = \begin{pmatrix} -I & 0 \\ 0 & -U_0 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 0 & I \\ -U_1 & 0 \end{pmatrix}$ are unitary, by Theorems 7.1 and 7.7 the eigenvalues of $\det(\lambda^2 A_0 + \lambda A_1 + A_2) = 0$ are on the unit circle.
Theorem 7.9  If the generalized eigenvalues of $\lambda B_0 + B_1$ are on the unit circle, $(\lambda + a)B_0 - B_1$ is 1-circular, where $a = r + si$ is a complex number and $b$ is a nonzero real number.

Proof. Since the roots of $\det((\lambda + a)I - bB_1B_0^{-1}) = 0$ are of the form $a + b\lambda_j$, where $\lambda_j$, $j = 1, 2, \ldots, n$, are the generalized eigenvalues of $\lambda B_0 + B_1$, they are on a circle with center at $(r, s)$ and radius $b$; i.e., $(\lambda + a)B_0 - bB_1$ is 1-circular.
BIBLIOGRAPHY


