REDUCTION OF GAMES
USING DOMINANT STRATEGIES

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B.Sc., University of British Columbia, 1967

A THESIS SUMBITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In the Department

of

MATHEMATICS

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April, 1969

Department of MATHEMATICS

The University of British Columbia
Vancouver 8, Canada

Date APRIL 22, 1969

ABSTRACT

Using the concept of dominant strategies, a method for reducing the strategy spaces of a game is developed. These results are used to reduce some infinite games of the Colonel Blotto type to finite matrix games which are then solved by the Snow-Shapley theorem.

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ACKNOWLEDGEMENTS

The author wishes to express his thanks to Dr. R. A. Restrepo for his invaluable assistance and helpful criticisms in the preparation of this thesis.

The financial support of the University of British Columbia and the National Research Council of Canada is gratefully acknowledged.

CHAPTER 1

INTRODUCTION

1.1 Definition of a Game:

A two-person, zero-sum game is defined by a triplet $G = (X,Y,f)$ where X and Y are measurable spaces and f is a bounded, real valued measurable function defined on $X \times Y$. Associated with the spaces X and Y are two families of sets S and T where S is a σ-field in X and T is a σ-field in in Y. The sets in S (or T) are the measurable subsets of X (or Y) where it is assumed that the individual elements of X (or Y) belong to S (or T). In this paper we will consider only zero-sum games, that is games with two irreconcilable opponents in which one participant wins what the other loses. The space of strategies for Player I is the set

$$U = \{\mu : \mu \text{ is a measure on } S \text{ and } \mu(X) = 1\} .$$

The space of strategies for Player II is the set

$$V = \{\nu : \nu \text{ is a measure on } T \text{ and } \nu(Y) = 1\} .$$

For a particular choice of $\mu \in U$ and $\nu \in V$ the payoff to Player I is

$$F(\mu,\nu) = \iint f(x,y)d\mu d\nu = \iint f(x,y)d\nu d\mu$$

while the payoff to Player II is $-F(\mu,\nu)$. Player I seeks to maximize the payoff while Player II seeks to minimize the payoff.

1.2 Min-max Solutions of Games:

If Player I uses the strategy $\mu_o \in U$ he is certain to receive at least $\inf_{\nu\in V} F(\mu_o,\nu)$ in payoff from Player II . Thus the number $\underline{v} = \sup_{\mu\in U} \inf_{\nu\in V} F(\mu,\nu)$ is the upper limit to the amount Player I can win with certainty independent of Player II's choice of strategy.

If Player II uses the strategy $\nu_o \in V$ he is certain to lose at most $\sup_{\mu\in U} F(\mu,\nu_o)$ in payoff to Player I. Thus the number $\bar{v} = \inf_{\nu\in V} \sup_{\mu\in U} F(\mu,\nu)$ is the lower limit to the amount Player II can restrict his loss with certainty independent of Player I's choice of strategy.

<u>Definition 1.2.1</u> If the relationship

$$\sup_{\mu\in U} \inf_{\nu\in V} F(\mu,\nu) = \inf_{\nu\in V} \sup_{\mu\in U} F(\mu,\nu) = v$$

is valid then v is called the value of the game.

<u>Definition 1.2.2</u> A pair of strategies $\mu^* \in U$ and $\nu^* \in V$ are optimal if and only if $F(\mu,\nu^*) \leq F(\mu^*,\nu^*) \leq F(\mu^*,\nu)$ for all $\mu \in U$ and $\nu \in V$.

Definition 1.2.3 Let v be the value of the game, and let $\epsilon > 0$ be given. A pair of strategies $\mu^* \in U$ and $\nu^* \in V$ are ϵ-optimal if and only if

$$F(\mu^*,\nu) \geq v - \epsilon \quad \text{for all} \quad \nu \in V$$

and

$$F(\mu,\nu^*) \leq v + \epsilon \quad \text{for all} \quad \mu \in U .$$

It can be shown that the validity of

$$\sup_{\mu \in U} \inf_{\nu \in V} F(\mu,\nu) = \inf_{\nu \in V} \sup_{\mu \in U} F(\mu,\nu)$$

guarantees the existence of ϵ-optimal strategies. Furthermore, the relationship

$$\max_{\mu \in U} \min_{\nu \in V} F(\mu,\nu) = \min_{\nu \in V} \max_{\mu \in U} F(\mu,\nu)$$

is valid if and only if there exist optimal strategies $\mu^* \in U$ and $\nu^* \in V$ with $v = F(\mu^*,\nu^*)$.

1.3 Some Classes of Games.

Some types of games frequently considered are as follows:

1. Finite Matrix Games.

$X = \{1,2,\ldots,n\}$, $Y = \{1,\ldots,m\}$, $S = 2^X$, $T = 2^Y$.

$U = \{\mu=(\mu_1,\ldots,\mu_n) \in R^n : \mu_i \geq 0 , \sum_{i=1}^{n} \mu_i = 1\}$

$$V = \{\nu=(\nu_1,\ldots,\nu_n) \in R^m : \nu_j \geq o \ , \sum_{j=1}^{m} \nu_i = 1\}$$

$$f(i,j) = a_{ij} \in R \ , \ i = \{1,\ldots,n\} \quad j \in \{1,\ldots,m\},$$

$$A = [a_{ij}]$$

$$F(\mu,\nu) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_i \ a_{ij} \ \nu_j = \mu A \nu$$

A is called the payoff matrix of the game.

2. Games on the Unit Square.

$X = Y = [0,1]$, $S = T =$ Class of Borel sets of $[0,1]$.

$U = V =$ set of all cumulative distribution functions on $[0,1]$.

For $\mu = \mu(x) \in U$ and $\nu = \nu(y) \in V$,

$$F(\mu,\nu) = \int_o^1\int_o^1 f(x,y)d\mu(x)d\nu(y) \ .$$

3. Games on Convex Subsets of $R^n \times R^m$.

$X \subset R^n$, $Y \subset R^m$, X and Y closed, convex .

S = class of Borel sets generated by the open sets in the relative topology on X .

T = class of Borel sets generated by the open sets in the relative topology on Y .

4. Borel Games.

X and Y are topological spaces.

S and T are the classes of Borel sets (generated by the open sets) of X and Y respectively

The games considered in this thesis will be of types 1 and 3 and techniques will be developed to reduce some games of type 3 to games of type 1.

1.4 Optimal Strategies for Matrix Games.

It can be shown that for any matrix game the relation $\max_{\mu\in U}\min_{\nu\in V} F(\mu,\nu) = \min_{\nu\in V}\max_{\mu\in U} F(\mu,\nu) = v$ is always valid [3, p.26]. That is, matrix games always have optimal strategies. Furthermore, the set of optimal strategies for each player is a closed convex set [3, p.36]. The matrix games that arise in this paper will be solved by means of the following theorem, due to R.N. Snow and L.S. Shapley [3, p.45].

Theorem 1.4.1 If μ^* and ν^* are extreme points of the convex sets U^* and V^* of optimal strategies, and if the value of the game v is not zero, then there exists a non-singular submatrix M of A such that

$$v = \frac{1}{eM^{-1}e^T}$$

$$\mu^* = \frac{eM^{-1}}{eM^{-1}e^T}$$

$$\nu^* = \frac{M^{-1}e^T}{eM^{-1}e^T}$$

where e is a vector of the same dimensions as M all of whose components are 1, and e^T is its transpose.

CHAPTER 2

REDUCTION OF THE GAMES

2.1 Introduction.

It is often possible to simplify a game by reducing the space of strategies of one or both players. That is, if a game G is defined by the triplet (X,Y,f) the reduced game G^1 will be defined by the triplet $(\mathfrak{A},\mathfrak{D},f)$ where $\mathfrak{A} \subset X$ and $\mathfrak{D} \subset Y$ are both measurable sets. If the subsets $\mathfrak{A}$ and $\mathfrak{D}$ are chosen correctly the value of the game and the optimal strategies of the game G^1 will also be the value of the game and optimal strategies of the game G. In particular, this reduction can be very useful if X and Y are infinite sets and $\mathfrak{A}$ and $\mathfrak{D}$ are finite subsets because then the Snow-Shapley theorem can be used to solve the game. It should be noted that this reduction is a generalization of the concept of dominant strategies in finite matrix games, Dresher [2, p.40].

2.2 Definition of Dominance.

For a game G let $\mathfrak{A}$ and $\mathfrak{D}$ be measurable subsets of X and Y respectively, Define $U_{\mathfrak{A}} = \{\mu \in U : \mu(\mathfrak{A}) = 1\}$ and

$V_{\mathcal{D}} = \{ \nu \in V : \nu(\mathcal{D}) = 1\}$.

<u>Definition 2.2.1</u> $\mathfrak{A}$ is dominant with respect to $\mathcal{D}$ iff for each $\nu \in V_{\mathcal{D}}$

$$\sup_{x \in X} \int_{\mathcal{D}} f(x,d)d\nu = \sup_{a \in \mathfrak{A}} \int_{\mathcal{D}} f(a,d)d\nu .$$

$\mathfrak{A}$ is dominant iff it is dominant with respect to Y .

<u>Definition 2.2.2</u> $\mathcal{D}$ is dominant with respect to $\mathfrak{A}$ iff for each $\mu \in U_{\mathfrak{A}}$

$$\inf_{y \in Y} \int_{\mathfrak{A}} f(a,y)d\mu = \inf_{d \in \mathcal{D}} \int_{\mathfrak{A}} f(a,d)d\mu .$$

$\mathcal{D}$ is dominant iff it is dominant with respect to X .

2.3 <u>Reduction of Games using Dominance</u>.

<u>Lemma 2.3.1</u>

1. If for each $x \in X$ there exists $\mu = \mu_x \in U_{\mathfrak{A}}$ such that for all $d \in \mathcal{D}$ $f(x,d) \leq \int_{\mathfrak{A}} f(a,d)d\mu$ (*) then $\mathfrak{A}$ is dominant w.r.t. $\mathcal{D}$.

2. If for each $y \in Y$ there exists $\nu = \nu_y \in V_{\mathcal{D}}$ such that for all $a \in \mathfrak{A}$ $f(a,y) \geq \int_{\mathcal{D}} f(a,d)d\nu$ then $\mathcal{D}$ is dominant w.r.t. $\mathfrak{A}$.

Proof:

1. From (*) we get for each $x \in X$ there exists $\mu \in U_{\mathfrak{A}}$ such that for all $\nu \in V_{\mathfrak{D}}$

$$\int_{\mathfrak{D}} f(x,d)d\nu \leq \int_{\mathfrak{D}} \int_{\mathfrak{A}} f(a,d)d\mu d\nu$$

$$= \int_{\mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu d\mu$$

$$\leq \sup_{a \in \mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu$$

Clearly since $\mathfrak{A} \subset X$

$$\sup_{x \in X} \int_{\mathfrak{D}} f(x,d)d\nu \geq \sup_{a \in \mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu \ .$$

Therefore $\sup_{x \in X} \int_{\mathfrak{D}} f(x,d)d\nu = \sup_{a \in \mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu$ for all $\nu \in V_{\mathfrak{D}}$, which implies $\mathfrak{A}$ is dominant w.r.t. $\mathfrak{D}$.

2. Similarly $\mathfrak{D}$ is dominant w.r.t $\mathfrak{A}$.

Lemma 2.3.2

1. For any $\nu \in V_{\mathfrak{D}}$, $\sup_{\mu \in U_{\mathfrak{A}}} \int_{\mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu d\mu = \sup_{a \in \mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu$.

2. For any $\mu \in U_{\mathfrak{A}}$, $\inf_{\nu \in V_{\mathfrak{D}}} \int_{\mathfrak{D}} \int_{\mathfrak{A}} f(a,d)d\mu d\nu = \inf_{d \in \mathfrak{D}} \int_{\mathfrak{A}} f(a,d)d\mu$.

Proof: That $\sup_{\mu \in U_{\mathfrak{A}}} \int_{\mathfrak{A}} \int_{\mathfrak{D}} f(a,d)d\nu d\mu \leq \sup_{a \in \mathfrak{A}} \int_{\mathfrak{D}} f(a,d)$ is obvious . The inequality in the other direction is true because $\mu(\mathfrak{A}) = 1$ and points were assumed to be measurable sets.

Corollary 2.3.1

1. For any $\nu \in V$, $\sup_{\mu \in U} \int \int f(x,y)d\nu d\mu = \sup_{x \in X} \int f(x,y)d\nu$.

2. For any $\mu \in U$, $\inf_{\nu \in V} \iint f(x,y)d\mu d\nu = \inf_{y \in Y} \int f(x,y)d\mu$.

Lemma 2.3.3

1. $\mathfrak{A}$ is dominant w.r.t $\mathfrak{D}$ iff for all $\nu \in V_{\mathfrak{D}}$

$$\sup_{\mu \in U} \int\int_{\mathfrak{D}} f(x,d)d\nu d\mu = \sup_{\mu \in U_{\mathfrak{A}}} \int_{\mathfrak{A}}\int_{\mathfrak{D}} f(a,d)d\nu d\mu .$$

2. $\mathfrak{D}$ is dominant w.r.t. $\mathfrak{A}$ iff for all $\mu \in U_{\mathfrak{A}}$

$$\inf_{\nu \in V} \int\int_{\mathfrak{A}} f(a,y)d\mu d\nu = \inf_{\nu \in V_{\mathfrak{D}}} \int_{\mathfrak{D}}\int_{\mathfrak{A}} f(a,d)d\mu d\nu$$

Proof: The proof follows immediately from the definition and Lemma 2.3.2.

Theorem 2.3.1 If $\mathfrak{A}$ is dominant w.r.t. $\mathfrak{D}$ and $\mathfrak{D}$ is dominant w.r.t. $\mathfrak{A}$ then the optimal strategies and the value of the game $(\mathfrak{A},\mathfrak{D},f)$ are also optimal strategies and the value of the game (X,Y,f) .

Proof: Let $\mu^* \in U_{\mathfrak{A}}$ and $\nu^* \in V_{\mathfrak{D}}$ be optimal strategies and v be the value of the reduced game $(\mathfrak{A},\mathfrak{D},f)$. Then by the preceeding lemma

$$\inf_{\nu\in V}\sup_{\mu\in U}\iint f(x,y)d\nu d\mu \le \sup_{\mu\in U}\iint_{\mathfrak{D}} f(x,d)d\nu^* d\mu$$

$$= \sup_{\mu\in U_{\mathfrak{A}}}\int_{\mathfrak{A}}\int_{\mathfrak{D}} f(a,d)d\nu^* d\mu$$

$$= \int_{\mathfrak{A}}\int_{\mathfrak{D}} f(a,d)d\nu^* d\mu^* = v$$

$$\sup_{\mu\in U}\inf_{\nu\in V}\iint f(x,y)d\mu d\nu \ge \inf_{\nu\in V}\iint_{\mathfrak{A}} f(a,y)d\mu^* d\nu$$

$$= \inf_{\nu\in V_{\mathfrak{D}}}\int_{\mathfrak{D}}\int_{\mathfrak{A}} f(a,d)d\mu^* d\nu$$

$$= \int_{\mathfrak{D}}\int_{\mathfrak{A}} f(a,d)d\mu^* d\nu^* = v\ .$$

But by Fubini's Theorem

$$\iint f(x,y)d\nu d\mu = \iint f(x,y)d\mu d\nu\ .$$

Thus

$$\inf_{\nu\in V}\sup_{\mu\in U}\iint f(x,y)d\mu d\nu \le v \le \sup_{\mu\in U}\inf_{\nu\in V}\iint f(x,y)d\mu d\nu\ .$$

Theorem 2.3.2

1. Let $X \subset R^n$ be a closed, bounded convex set with a finite set of extreme points $\mathfrak{A}$. If for $\mathfrak{D} \subset Y$ the function $f : X \times Y \to R$ is a convex function of x for each $d \in \mathfrak{D}$, then $\mathfrak{A}$ is dominant w.r.t. $\mathfrak{D}$.

Proof: Let $\nu \in V_{\mathfrak{D}}$, consider the function $g(x) = \int_{\mathfrak{D}} f(x,d)d\nu$. Then g is a convex function and takes its maximum at an extreme point of X.

i.e. $\sup_{x\in X} \int_{\mathcal{D}} f(x,d)d\nu = \max_{a\in \mathfrak{A}} \int_{\mathcal{D}} f(a,d)d\nu$ for all $\nu \in V_{\mathcal{D}}$.

2. Let $Y \subset R^m$ be a closed, bounded set with a finite set of extreme points $\mathcal{D}$. If for $\mathfrak{A} \subset X$ the function $f : X \times Y \to R$ is a concave function of y for each $a \in \mathfrak{A}$, then $\mathcal{D}$ is dominant w.r.t. $\mathfrak{A}$.

2.4 Definition of ϵ-Dominance

Definition 2.4.1 $\mathfrak{A}$ is ϵ-dominant w.r.t. $\mathcal{D}$ iff for all $\nu \in V_{\mathcal{D}}$

$$\sup_{x\in X} \int_{\mathcal{D}} f(x,d)d\nu \leq \sup_{a\in \mathfrak{A}} \int_{\mathcal{D}} f(a,d)d\nu + \epsilon .$$

Definition 2.4.2 $\mathcal{D}$ is ϵ-dominant w.r.t. $\mathfrak{A}$ iff for all $\mu \in U_{\mathfrak{A}}$

$$\inf_{y\in Y} \int_{\mathfrak{A}} f(a,y)d\mu \geq \inf_{d\in \mathcal{D}} \int_{\mathfrak{A}} f(a,d)d\mu - \epsilon .$$

2.5 Reduction of Games using ϵ-Dominance.

Lemma 2.5.1

1. $\mathfrak{A}$ is ϵ-dominant w.r.t. $\mathcal{D}$ iff for all $\nu \in V_{\mathcal{D}}$

$$\sup_{\mu\in U} \iint_{\mathcal{D}} f(x,d)d\nu d\mu \leq \sup_{\mu\in U_{\mathfrak{A}}} \int_{\mathfrak{A}} \int_{\mathcal{D}} f(a,d)d\nu d\mu + \epsilon$$

2. $\mathcal{D}$ is ϵ-dominant w.r.t. $\mathfrak{A}$ iff for all $\mu \in U_{\mathfrak{A}}$

$$\inf_{\nu\in V} \iint_{\mathfrak{A}} f(a,y)d\mu d\nu \geq \inf_{\nu\in V_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathfrak{A}} f(a,d)d\mu d\nu - \epsilon$$

Proof: Follows immediately from Lemma 2.3.2.

Theorem 2.5.1 If $\mathfrak{A}$ is ϵ-dominant w.r.t. $\mathfrak{D}$ and $\mathfrak{D}$ is ϵ-dominant w.r.t. $\mathfrak{A}$ then the optimal strategies for the game $(\mathfrak{A},\mathfrak{D},f)$ are ϵ-optimal for the game (X,Y,f) .

Proof: The proof is the same as the proof in Theorem 2.3.1.

Lemma 2.5.2

1. Let $X \subset R^n$ be a convex bounded set, $f : X \to R$ be a convex, bounded uniformly continuous function and $\bar{X}$ = closure X. Then we can extend f to a function $F : \bar{X} \to R$ such that $F/X = f$ and F is a convex function on $\bar{X}$.

Proof:

$$\text{Let} \quad F(y) = \begin{cases} f(y) & \text{if } \; y \in X \\ \lim\limits_{\substack{x \to y \\ x \in X}} f(x) & \text{if } \; y \in \bar{X} - X \end{cases}$$

First we show that F is well defined. Let $\{x_n\}$ and $\{x_n'\}$ be sequences in X with $\lim\limits_n x_n = \lim\limits_n x_n' = x$, $\lim\limits_n f(x_n) = S$, $\lim\limits_n f(x_n') = t$ with $t \neq S$. If we denote the metric in R^n by ρ then for all $\delta > o$ there exists an integer $N > o$ such that $n > N$ then $\rho(x_n,x_n') < \delta$. Also, for all $\epsilon > o$ there exists $N' > o$ such that if $n > N'$

$$|f(x_n) - S| < \epsilon \quad \text{and} \quad |f(x_n') - t| < \epsilon .$$

If we let $\epsilon = \frac{1}{3}|S-t|$ then for all $\delta > o$ there exists x_n and x_n' with $\rho(x_n,x_n') < \delta$ but

$$|f(x_n) - f(x_n')| \geq \epsilon .$$

This contradicts the fact that f is uniformly continuous. Hence $S = t$ and F is well defined.

Now, let $x, y \in \bar{X}$ and $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\lim_n x_n = x$ and $\lim_n y_n = y$.

Then if $o \leq \alpha \leq 1$

$$\begin{aligned}
\alpha F(x) + (1 - \alpha)F(y) &= \alpha \lim_n f(x_n) + (1 - \alpha) \lim_n f(y_n) \\
&= \lim_n [\alpha f(x_n) + (1 - \alpha) f(y_n)] \\
&\geq \lim_n f(\alpha x_n + (1 - \alpha)y_n) \\
&= F(\alpha x + (1 - \alpha)y) .
\end{aligned}$$

i.e. F is convex on $\bar{X}$ and clearly $F/X = f$ by definition.

2. Let $Y \subset R^m$ be a convex, bounded set, $f : Y \to R$ be a concave, bounded, uniformly continuous function and $\bar{Y}$ = closure Y. Then we can extend f to a function $F : \bar{Y} \to R$ such that $F/Y = f$ and F is a concave function on $\bar{Y}$.

Lemma 2.5.3

1. Let $X \subset R^n$ be a convex, bounded set such that $\bar{X}$ has a finite set of extreme points $\bar{\mathfrak{A}}$, $f : X \to R$ be a convex, bounded, uniformly continuous function. Then

$$\sup_{x \in X} f(x) = \sup_{\bar{a} \in \bar{\mathfrak{A}}} \lim_{\substack{x \to \bar{a} \\ x \in X}} f(x) .$$

Proof: The proof follows immediately from Lemma 2.5.2 and the fact that a convex function on a closed convex set with a finite number of extreme points takes its maximum at an extreme point.

2. Let $Y \subset R^m$ be a convex, bounded set such that $\bar{Y}$ has a finite set of extreme points $\bar{\mathfrak{D}}$, $f : Y \to R$ be a concave, bounded, uniformly continuous function. Then

$$\inf_{y \in Y} f(y) = \inf_{\bar{d} \in \bar{\mathfrak{D}}} \lim_{\substack{y \to \bar{d} \\ y \in Y}} f(y) .$$

Theorem 2.5.2

1. Let $X \subset R^n$ be a convex, bounded set such that $\bar{X}$ has a finite set of extreme points $\bar{\mathfrak{A}}$, and let $\mathfrak{D} \subset Y$ be a finite set. If the function $f : X \times Y \to R$ is a bounded, convex, uniformly continuous function of x for each $d \in \mathfrak{D}$, then for all $\epsilon > o$ there exists a finite set $\mathfrak{A} = \mathfrak{A}(\epsilon) \subset X$ such that $\mathfrak{A}$ is ϵ-dominant w.r.t. $\mathfrak{D}$.

Proof: Let ρ be the metric on R^n. By Lemma 2.5.2 for each $\bar{a} \in \bar{\mathfrak{A}}$, $d \in \mathfrak{D}$ and $\epsilon > o$ there exists $\delta = \delta(\bar{a}, d, \epsilon) > o$ such that

$$\lim_{\substack{x \to \bar{a} \\ x \in X}} f(x,d) \leq f(a,d) + \epsilon \quad \text{when} \quad a \in X \quad \text{and} \quad \rho(a,\bar{a}) < \delta .$$

Since $\mathfrak{D}$ is finite this implies that for $\bar{a} \in \bar{\mathfrak{A}}$ there exists

$a = a(\bar{a},\epsilon) \in X$ such that for all $d \in \mathcal{D}$

$$\varlimsup_{\substack{x\to\bar{a}\\ x\in X}} f(x,d) \leq f(a,d) + \epsilon \ .$$

Then for all $\nu \in V_{\mathcal{D}}$

$$\varlimsup_{\substack{x\to\bar{a}\\ x\in X}} \int_{\mathcal{D}} f(x,d)d\nu \leq \int_{\mathcal{D}} f(a,d)d\nu + \epsilon$$

Let $\mathfrak{A} = \{a = a(\bar{a},\epsilon) : \bar{a} \in \bar{\mathfrak{A}}\}$. But $\int_{\mathcal{D}} f(x,d)d\nu$ is a convex bounded continuous function on X and by Lemma 2.5.3, for all $\nu \in V_{\mathcal{D}}$

$$\sup_{x\in X} \int_{\mathcal{D}} f(x,d)d\nu = \sup_{\bar{a}\in\bar{\mathfrak{A}}} [\varlimsup_{\substack{x\to\bar{a}\\ x\in X}} \int_{\mathcal{D}} f(x,d)d\nu] \leq \sup_{a\in\mathfrak{A}} \int_{\mathcal{D}} f(a,d)d\nu + \epsilon$$

i.e. $\mathfrak{A}$ is dominant w.r.t. $\mathcal{D}$.

Theorem 2.5.3

1. Let $X = \bigcup_{i=1}^{n} X_i$ where for each i , X_i is a bounded, convex set such that closure X_i has a finite number of extreme points, let $\mathcal{D} \subset Y$ be a finite set. If the function $f : X \times Y \to R$ is a bounded, convex, uniformly continuous function of x in each region X_i for each $d \in \mathcal{D}$, then for all $\epsilon > 0$ there exists a finite set $\mathfrak{A} \subset X$ such that $\mathfrak{A}$ is ϵ-dominant w.r.t. .

Proof: From Theorem 2.5.2, in each region X_i there exists a finite set $\mathfrak{A}_i$ such that $\mathfrak{A}_i$ is ϵ-dominant w.r.t $\mathcal{D}$ in X_i .

i.e. for all $\nu \in V_{\mathcal{D}}$

$$\sup_{x \in X_i} \int_{\mathcal{D}} f(x,d)\,d\nu \leq \sup_{a \in \mathfrak{A}_i} \int_{\mathcal{D}} f(a,d)\,d\nu + \epsilon \ .$$

Letting $\mathfrak{A} = \bigcup_{i=1}^{n} \mathfrak{A}_i$ we get

$$\sup_{x \in X} \int_{\mathcal{D}} f(x,d)\,d\nu \leq \sup_{a \in \mathfrak{A}} \int_{\mathcal{D}} f(a,d)\,d\nu + \epsilon \ .$$

CHAPTER 3

ATTACK AND DEFENSE PROBLEMS

3.1 Introduction

In this Chapter, the results of Chapter 2 are used to reduce some infinite games to finite matrix games. The solutions of these matrix games are then found with the use of the Snow-Shapley theorem. The games considered are all of the Colonel Blotto type and hence the terminology of attack and defense problems is used throughout this Chapter.

In each model, n cities are to be defended by D units and attacked by A units. The outcome of a battle at one city does not affect the outcome of a battle at another city. The attacking player seeks to distribute his forces among the n cities so as to maximize the expected payoff, while the defending player seeks to distribute his forces among the n cities so as to minimize the expected loss. Each game is defined by its sets X and Y of pure strategies and its payoff function f where

$$X = \{x = (x_1, x_2, \ldots, x_n) : x_i \geq o \; , \; \sum_{i=1}^{n} x_i = A\}$$

$$Y = \{y = (y_1, y_2, \ldots, y_n) : y_i \geq o \; , \; \sum_{i=1}^{n} y_n = D\}$$

$$f(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i) \quad \text{and}$$

$f_i(x_i,y_i)$ is the payoff at city i if the attacking force consists of x_i units.

The methods will be applied to two types of models: first, models with a continuous payoff whose solutions have been obtained before by other methods; secondly, to some games with discontinuous payoffs, previously unsolved.

3.2 A Model of 2 Cities with a Continuous, Convex Payoff Function

The first game considered is defined by

$$X = \{(x,1-x) \mid 0 \leq x \leq 1\}$$

$$Y = \{(y,1-y) \mid 0 \leq y \leq 1\}$$

$$f(x,y) = f_1(x,y) + f_2(1-x,1-y) \quad \text{where}$$

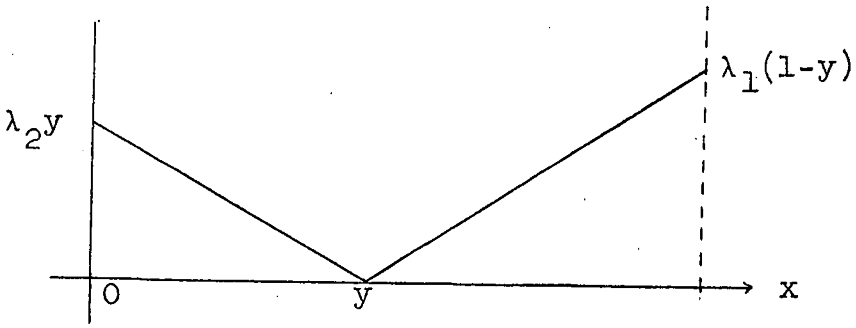
$$f_1(x,y) = \begin{cases} \lambda_1(x-y) & \text{if } x \geq y \\ 0 & \text{if } x \leq y \end{cases}$$

$$f_2(1-x,1-y) = \begin{cases} \lambda_2(1-x-(1-y)) & \text{if } 1-x \geq 1-y \\ 0 & \text{if } 1-x \leq 1-y \end{cases}$$

Adding f_1 and f_2 we get

$$f(x,y) = \begin{cases} \lambda_1(x-y) & \text{if } x \geq y \\ \lambda_2(y-x) & \text{if } x \leq y \end{cases}$$

Graph of $f(x,y)$ for fixed y :



From the graph it is clear that for each y in $[o,1]$, $f(x,y)$ is a convex function of x. Therefore, by Theorem 2.3.2 the set of extreme points of X, $\mathfrak{A} = \{(o,1),(1,o)\}$, is dominant. But $f(o,y) = \lambda_2 y$ and $f(1,y) = \lambda_1(1-y)$ are both linear functions of y. By T heorem 2.3.2 the set of extreme points of Y, $\mathfrak{D} = \{(o,1),(1,o)\}$, is dominant with respect to $\mathfrak{A}$. Using Theorem 2.3.1, we have reduced our infinite game to the game with matrix

$$B = \begin{bmatrix} f(o,o) & f(o,1) \\ f(1,0) & f(1,1) \end{bmatrix} = \begin{bmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{bmatrix}$$

The Snow-Shapley theorem gives us the value of the game as

$$v = \frac{\lambda_1 \lambda_2}{\lambda_1+\lambda_2} .$$

The optimal strategy for the attacking player is

$\alpha^* = (\frac{\lambda_1}{\lambda_1+\lambda_2}, \frac{\lambda_2}{\lambda_1+\lambda_2})$ where he attacks only city 2 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and attacks only city 1 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$. The optimal strategy for the defending player is

$\beta^* = (\frac{\lambda_2}{\lambda_1+\lambda_2}, \frac{\lambda_1}{\lambda_1+\lambda_2})$ where he defends only city 2 with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and defends only city 1 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

3.3 A Model of n-Cities with a Continuous, Convex Payoff.

This game is defined by

$$X = \{x = (x_1,\dots,x_n) | x_i \geq o \ , \ \sum_{i=1}^{n} x_i = A\}$$

$$Y = \{y = (y_1,\dots,y_n) | y_i \geq o \ , \ \sum_{i=1}^{n} y_i = D\}$$

$$f(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i) \quad \text{with} \quad f_i(x_i,y_i) = \lambda_i \max(o, x_i - y_i)$$

and $A \geq D$.

It is apparent that $f(x,y)$ is a convex function in x for each $y \in Y$. Therefore by Theorem 2.3.2 the set of extreme points of X, $\mathfrak{A} = \{a_i = (a_{i1}, a_{i2}, \dots, a_{in}) | a_{ij} = o$ if $i \neq j$ $a_{ij} = A$ if $i = j$, $i = 1,\dots,n\}$ is dominant. But $f(a_i,y) = \lambda_i(A-y_i)$ is a linear function of y for each $a_i \in$. Therefore by Theorem 2.3.2, the set of extreme points of Y,

$$\mathfrak{D} = \{d_i = (d_{i1},\dots,d_{in}) | d_{ij} = o \ \text{if} \ i \neq j ,$$

$$d_{ij} = D \ \text{if} \ i = j \ , \ i = 1,\dots,n\} ,$$

is dominant with respect to $\mathfrak{A}$.

We have reduced our infinite game to the game with matrix

$$B = [f(a_i,d_j)] = \begin{bmatrix} \lambda_1(A-D) & \lambda_1 A & \lambda_1 A \cdots\cdots & \lambda_1 A \\ \lambda_2 A & \lambda_2(A-D) & \lambda_2 A \cdots\cdots & \lambda_2 A \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \cdot \\ & & & \lambda_{n-1}A \\ \lambda_n A \cdots & & \cdots \lambda_n A & \lambda_n(A-D) \end{bmatrix}$$

This matrix game is equivalent to the game solved in Dresher [2, p. 54]. If we assume that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > o$ and let $A_k = \sum_{i=1}^{k} \frac{1}{\lambda_i A}$, $1-p = \frac{A-D}{A}$ then

(1) The optimal strategy for the attacking player is

$$\alpha^* = (\alpha_1,\ldots,\alpha_n) \quad \text{where}$$

$$\alpha_i = \frac{1}{\lambda_i A A_t} \qquad \text{for } i \leq t ,$$

$$= o \qquad \text{for } i > t .$$

(2) The optimal strategy for the defending player is

$$\beta^* = (\beta_1,\ldots,\beta_n) \quad \text{where}$$

$$\beta_i = \frac{1}{p}\left(1 - \frac{1}{\lambda_i A} \cdot \frac{t-p}{A_t}\right) \qquad \text{for } i \leq t ,$$

$$= o \qquad \text{for } i > t .$$

(3) The value of the game is

$$v = \frac{t-p}{A_t} = \max_{k \leq n} \frac{k-p}{A_k} .$$

3.4 A Model of 2-Cities with a Payoff with Large Discontinuities.

The remaining models considered in this chapter are in some ways similar to the game solved by Cooper and Restrepo [1]. However, because of the large discontinuities in the payoff, the optimal strategies and the value of the game considered here are very sensitive to changes in the relative sizes of the attacking force A and the defending force D . The 2-city model was solved for the cases where $\frac{3}{2}A \leq D < 2A$ and where $\frac{4}{3}A \leq D < \frac{3}{2}A$ in which the infinite games could be reduced to finite matrix games. The n-city model was only solved for the case where $(\frac{2n-1}{2})A \leq D < nA$. Because of the large discontinuities in the payoff, we were unable to find finite dominant subsets for the other models. However, it is felt that similar techniques can be used to extend the results obtained.

The 2-city model is defined by

$$X = \{(x,A-x) : x \in [o,A]\}$$

$$Y = \{(y,D-y) : y \in [o,D]\}$$

$$f(x,y) = f_1(x,y) + f_2(A-x,D-y) \quad \text{where}$$

$$f_1(x,y) = \begin{cases} \lambda_1 & \text{if} \quad x > y \\ -hx & \text{if} \quad x \leq y \end{cases}$$

$$f_2(A-x,D-y) = \begin{cases} \lambda_2 & \text{if} \quad A-x > D-y \\ -h(A-x) & \text{if} \quad A-x \leq D-y \end{cases}$$

with $\lambda_1 \geq \lambda_2$.

If $o < A < D < 2A$ we get

$$f(x,y) = \begin{cases} \lambda_1 - h(A-x) & \text{if } x > y \\ \lambda_2 - hx & \text{if } x < A-D+y \\ -hA & \text{if } A-D+y \leq x \leq y \end{cases}$$

<u>Case 1</u>: <u>$3/2\ A \leq D < 2A$</u>

It will be shown that in this case the infinite game can be reduced to the finite game with payoff matrix:

$$\begin{bmatrix} -hA & \lambda_2 \\ \lambda_1 & -hA \end{bmatrix}$$

corresponding to the dominant sets

$$\mathfrak{A} = \{(o,A),\ (A,o)\}$$

$$\mathfrak{D} = \{(D-A,A),\ (A,D-A)\}\ .$$

To establish this result, consider

$$(1) \qquad \begin{aligned} f(x,D-A) &= \begin{cases} -hA & \text{if } x \leq D-A \\ \lambda_1 - h(A-x) & \text{if } x > D-A \end{cases} \\ f(x,A) &= \begin{cases} -hA & \text{if } x \geq A-(D-A) \\ \lambda_2 - hx & \text{if } x < A-(D-A) \end{cases} \ . \end{aligned}$$

From (1) it is clear that since $A-(D-A) \leq D-A$:

(i) for $x \in [o,D-A]$ $\quad f(x,D-A) = f(o,D-A)$

$f(x,A) \leq f(o,A)$

(ii) for $x \in (D-A,A]$ $\quad f(x,D-A) \leq f(A,D-A)$

$f(x,A) = f(A,A)$.

By Lemma 2.3.1, $\mathfrak{a}$ is dominant w.r.t. $\mathfrak{D}$.

Now consider:

$$f(o,y) = \begin{cases} -hA & \text{if } y \leq D-A \\ \lambda_2 & \text{if } y > D-A \end{cases}$$

(2)

$$f(A,y) = \begin{cases} -hA & \text{if } y \geq A \\ \lambda_1 & \text{if } yy < A . \end{cases}$$

From (2) it is clear that

(i) for $y \in [o,A]$ $\quad f(o,y) \geq f(o,D-A)$

$f(A,y) \geq f(A,D-A)$

(ii) for $y \in [A,D]$ $\quad f(o,y) \geq f(o,A)$

$f(A,y) \geq f(A,A)$

By Lemma 2.3.1, $\mathfrak{a}$ is dominant w.r.t. $\mathfrak{D}$.

Therefore, we need only consider the game with matrix

$$B = \begin{bmatrix} f(o,D-A) & f(o,A) \\ f(A,D-A) & f(A,A) \end{bmatrix} = \begin{bmatrix} -hA & \lambda_2 \\ \lambda_1 & -hA \end{bmatrix} ,$$

for which

$$B^{-1} = \frac{1}{\lambda_1\lambda_2 - (hA)^2} \begin{bmatrix} hA & \lambda_2 \\ \lambda_1 & hA \end{bmatrix}$$

Then, by the Snow-Shapley theorem,

(1) The value of the game is

$$v = \frac{\lambda_1\lambda_2 - (hA)^2}{\lambda_1 + \lambda_2 + 2hA} .$$

(2) The optimal strategy for the attacking player is

$$\alpha^* = (\lambda_1 + hA, \lambda_2 + hA) \cdot \frac{1}{\lambda_1+\lambda_2+2hA} .$$

(3) The optimal strategy for the defending player is

$$\beta^* = (\lambda_2 + hA, \lambda_1 + hA) \cdot \frac{1}{\lambda_1+\lambda_2+2hA} .$$

Case 2: $4/3A \leq D < 3/2A$

It will be shown that in this case the infinite game can be reduced to the finite game with payoff matrix

$$\begin{bmatrix} -hA & \lambda_2 & \lambda_2 \\ \lambda_1 - h[A-(D-A)^+] & -hA & \lambda_2 - h(D-A)^+ \\ \lambda_1 - h(D-A)^+ & -hA & \lambda_2 - h[A-(D-A)^+] \\ \lambda_1 & \lambda_1 & -hA \end{bmatrix}$$

corresponding to the ϵ-dominant sets.

$$\mathfrak{A} = \{(o,A),((D-A)^+,A-(D-A)^+),\ (A-(D-A)^+,\ (D-A)^+),\ (A,o)\}$$

$$\mathfrak{D} = \{(D-A,A),(A-(D-A),2(D-A)),(A,D-A)\}\ ,$$

where $(D-A)^+ = (D-A) + \delta$ for small $\delta > o$.

To establish dominance we consider now the payoffs for strategies in the sets $\mathfrak{A}$ and $\mathfrak{D}$.

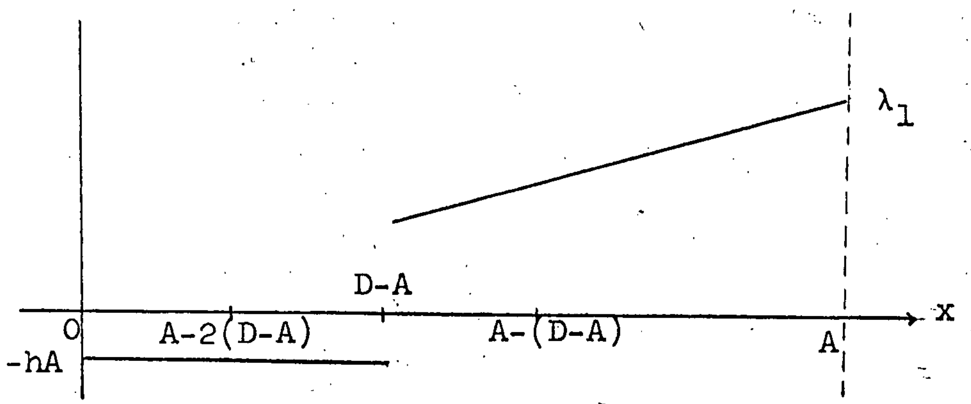
Consider first

$$f(x,D-A) = \begin{cases} -hA & \text{if } x \leq D-A \\ \lambda_1 - h(A-x) & \text{if } x > D-A \end{cases}$$

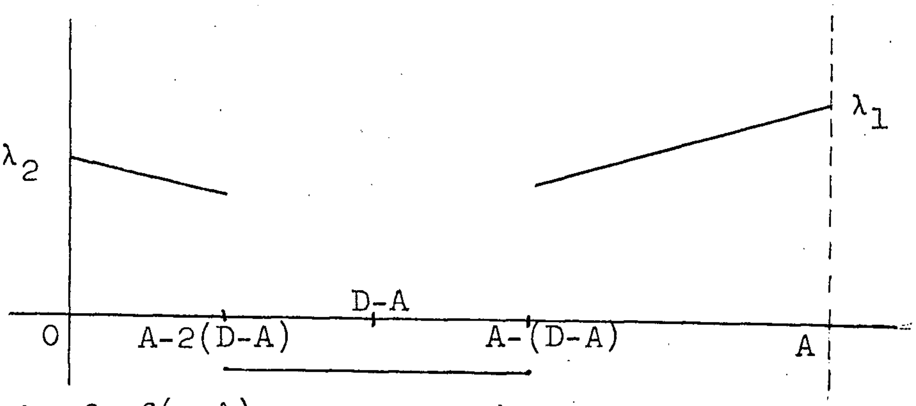
$$(1)\quad f(x,A-(D-A) = \begin{cases} \lambda_2 - hx & \text{if } x < A-2(D-A) \\ -hA & \text{if } A-2(D-A) \leq x \leq A-(D-A) \\ \lambda_1 - h(A-x) & x > A-(D-A) \end{cases}$$

$$f(x,A) = \begin{cases} \lambda_2 - hx & \text{if } x < A-(D-A) \\ -hA & \text{if } x \geq A-(D-A) \end{cases}$$

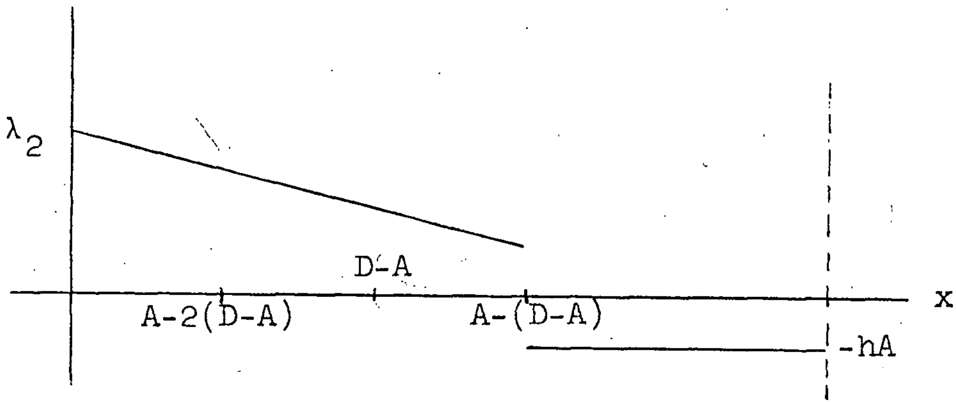
Graph of $f(x,D-A)$



Graph of $f(x,A-(D-A))$



Graph of $f(x,A)$



Let $X_1 = \{(x,A-x) \mid o \leq x \leq D-A\}$

$X_2 = \{(x,A-x) \mid D-A < x < A-(D-A)\}$

$X_3 = \{(x,A-x) \mid A-(D-A) \leq x \leq A\}$

Then $X = X_1 \cup X_2 \cup X_3$ and:

(i) $x \in [o, D-A]$

$$f(x,D-A) \leq f(o,D-A)$$

$$f(x,A-(D-A)) \leq f(o,A-(D-A))$$

$$f(x,A) \leq f(o,A)\ .$$

That is, in the region X_1 , $\mathfrak{A}_1 = \{(o,A)\}$ is dominant w.r.t. $\mathfrak{D}$.

(ii) For $x \in (D-A, A-(D-A))$, $f(x,D-A)$, $f(x,A-(D-A))$ and $f(x.A)$ are all linear functions. By Theorem 2.5.2, for $\epsilon > o$ there exists $\delta > o$ such that if $(D-A)^+ = (D-A) + \delta$ then the set $\mathfrak{A}_2 = \{((D-A)^+, A-(D-A)^+), ((A-(D-A)^+, (D-A)^+)\}$ is ϵ-dominant w.r.t. $\mathfrak{D}$ in the region X_2.

(iii) For $x \in [A-(D-A), A]$

$$f(x,D-A) \leq f(A,D-A)$$

$$f(x,A-(D-A)) \leq f(A,A-(D-A)) .$$

$$f(x,A) = F(A,A)$$

By Lemma 2.3.1, $\mathfrak{A}_3 = \{(A,o)\}$ is dominant w.r.t. $\mathfrak{D}$ in the region X_3.

Using Theorem 2.5.3, we see that $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3$ is ϵ-dominant w.r.t. $\mathfrak{D}$.

Now consider:

$$f(o,y) = \begin{cases} -hA & \text{if } y \leq D-A \\ \lambda_2 & \text{if } y > D-A \end{cases}$$

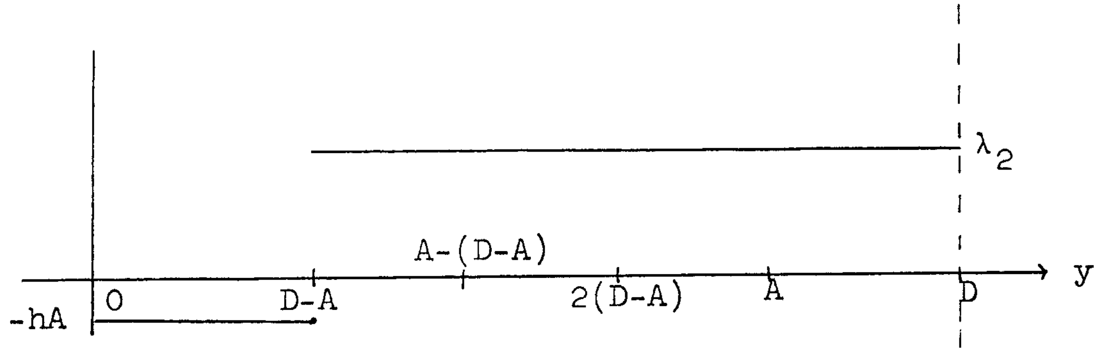
$$f((D-A)^+,y) = \begin{cases} \lambda_1 - h(A-(D-A)^+) & \text{if } y < (D-A)^+ \\ -hA & \text{if } (D-A)^+ \leq y \leq 2(D-A) \\ \lambda_2 - h(D-A)^+ & \text{if } y > 2(D-A) \end{cases}$$

(2)

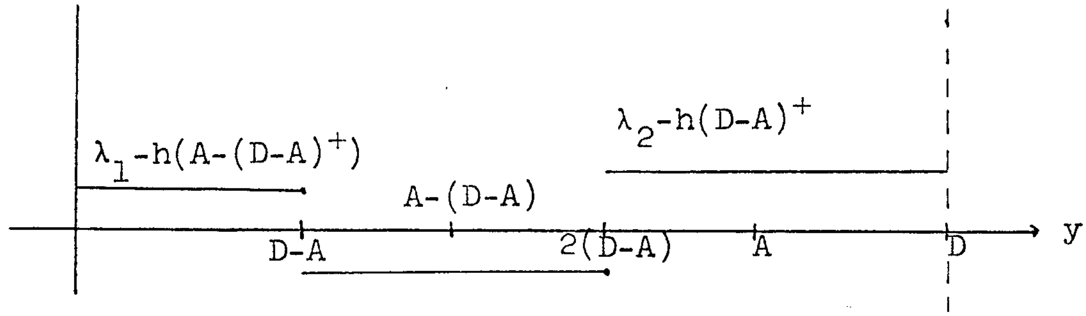
$$f(A-(D-A)^+,y) = \begin{cases} \lambda_1 - h(D-A)^+ & \text{if } y < A-(D-A)^+ \\ -hA & \text{if } A-(D-A)^+ \leq y \leq A-\delta \\ \lambda_2 - h(A-(D-A)^+) & \text{if } y > A-\delta \end{cases}$$

$$f(A,y) = \begin{cases} \lambda_1 & \text{if } y < A \\ -hA & \text{if } y \geq A \end{cases}$$

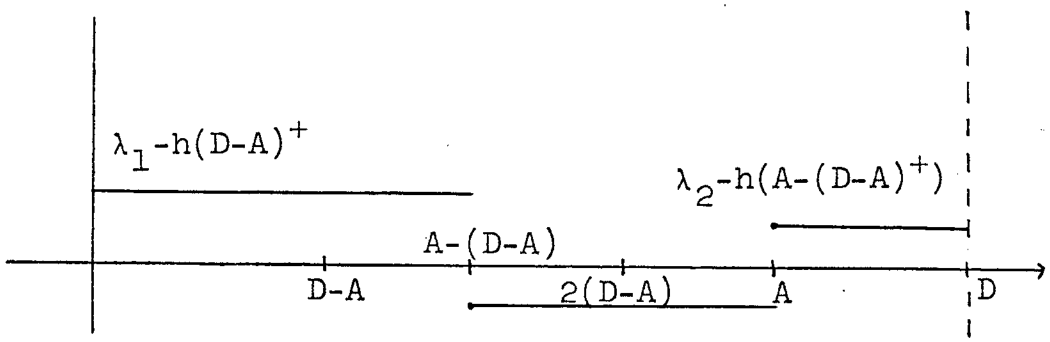
Graph of $f(o,y)$



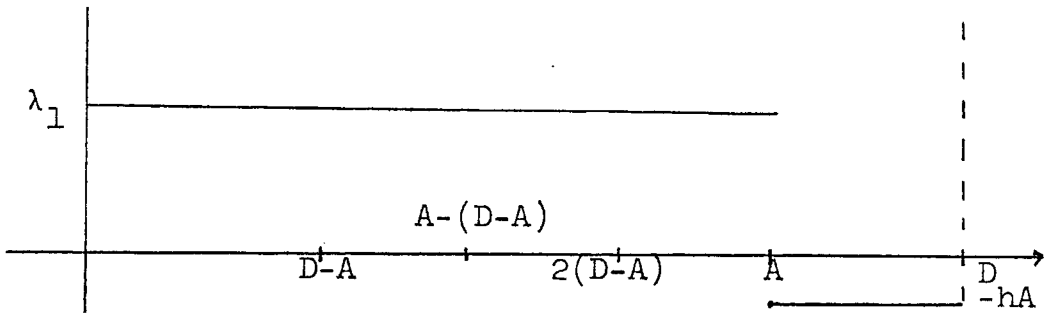
Graph of $f((D-A)^+,y)$



Graph of $f(A-(D-A)^+,y)$



Graph of $f(A,y)$



From (2) and the graphs we see that

(i) For $y \in [o, D-A]$

$$f(o,y) = f(o,D-A)$$

$$f((D-A)^+,y) = f((D-A)^+,D-A)$$

$$f(A-(D-A)^+,y) = f(A-(D-A)^+,D-A)$$

$$f(A,y) = f(A,D-A)$$

(ii) For $y \in (D-A, A)$

$$f(o,y) = f(o,A-(D-A))$$

$$f((D-A)^+,y) \geq f((D-A)^+,A-(D-A))$$

$$f(A-(D-A)^+,y) \geq f(A-(D-A)^+,A-(D-A))$$

$$f(A,y) = f(A,A-(D-A))$$

(iii) For $y \in [A, D]$

$$f(a,y) = f(a,A) \quad \text{for all } a \in \mathfrak{A}$$

Therefore by Lemma 2.3.1, $\mathfrak{D}$ is dominant with respect to $\mathfrak{A}$. The infinite game has been reduced to the game with matrix

$$B = \begin{bmatrix} f(o,D-A) & f(o,A-(D-A)) & f(o,A) \\ f((D-A)^+,D-A) & f((D-A)^+,A-(D-A)) & f((D-A)^+,A) \\ f(A-(D-A)^+,D-A) & f(A-(D-A)^+,A-(D-A)) & f(A-(D-A)^+,A) \\ f(A,D-A) & f(A,A-(D-A)) & f(A,A) \end{bmatrix}$$

$$B = \begin{bmatrix} -hA & \lambda_2 & \lambda_2 \\ \lambda_1 - h[A-(D-A)^+] & -hA & \lambda_2 - h(D-A)^+ \\ \lambda_1 - h(D-A)^+ & -hA & \lambda_2 - h[A-(D-A)^+] \\ \lambda_1 & \lambda_1 & -hA \end{bmatrix}$$

Let $\mu = \lambda_2 + hA$, $\nu = \lambda_1 + hA$, $a = h[A-(D-A)^+]$, $b = h(D-A)^+$. Then letting $C = [b_{ij} + hA]$ which has the same optimal strategies as B ,

$$C = \begin{bmatrix} o & \mu & \mu \\ \nu - a & o & \mu - b \\ \nu - b & o & \mu - a \\ \nu & \nu & o \end{bmatrix}$$

According to the Snow-Shapley theorem it is sufficient to consider for the determination of the optimal strategies only square matrices of dimension less than or equal to 3 . Considering the matrix

$$M = \begin{bmatrix} o & \mu & \mu \\ \nu - a & o & \mu - b \\ \nu & \nu & o \end{bmatrix}$$

Then

$$M^{-1} = \begin{bmatrix} -\nu(\mu-b) & \mu\nu & \mu(\mu-b) \\ \nu(\mu-b) & -\mu\nu & \mu(\nu-a) \\ \nu(\nu-a) & \mu\nu & -\mu(\nu-a) \end{bmatrix} \cdot \frac{1}{\mu\nu(\mu-b)+\mu\nu(\nu-a)}$$

From the Snow-Shapley theorem

$$v = \frac{\mu\nu(\mu-b) + \mu\nu(\nu-a)}{\rho}$$

where $\rho = \mu(\mu-b) + \mu\nu + \nu(\nu-a)$.

$$\alpha^* = (\nu(\nu-a),\ \mu\nu,\ o,\ \mu(\mu-b)) \cdot \frac{1}{\rho}$$

$$\beta^* = \begin{bmatrix} \mu(\mu-b) + \mu\nu - \nu(\mu-b) \\ \mu(\nu-a) + \nu(\mu-b) - \mu\nu \\ \nu(\nu-a) + \mu\nu - \mu(\nu-a) \end{bmatrix} \cdot \frac{1}{\rho}$$

Then

$$\alpha^* C = (v,v,v)$$

$$C\beta^* = \begin{bmatrix} v \\ v \\ v + \dfrac{(\mu-\nu)[(\mu-b)(\nu-b) - (\mu-a)(\nu-a)]}{\rho} \\ v \end{bmatrix}$$

But $\mu-\nu = \lambda_2-\lambda_1 \leq o$ since $\lambda_1 \geq \lambda_2$,

$$(\mu-b)(\nu-b) - (\mu-a)(\nu-a) = (\lambda_2-h(D-A)^+ + hA)(\lambda_1-h(D-A)^+ + hA)$$
$$- (\lambda_2+h(D-A)^+)(\lambda_1+h(D-A)^+) \geq o$$

since $-h(D-A)^+ + hA \geq h(D-A)^+$ if $D < \frac{3}{2}A$.

That is,

(1) The value of the game with matrix B is

$$v = \frac{\mu\nu(\mu+\nu - (a+b))}{\rho} - hA$$

(2) The ϵ-optimal strategy for the attacking player is α^*.

(3) The ϵ-optimal strategy for the defending player is β^*.

Case 3: $5/4A \leq D < 4/3A$

In this case we were unable to find finite dominant subsets. Based on Case 1 and 2, the intuitive most likely sets are

$$\mathfrak{A} = \{(o,A),\ ((D-A)^+,\ A-(D-A)^+),\ (A-2(D-A)^+,\ 2(D-A)^+),$$
$$(2(D-A)^+,\ A-2(D-A)^+),\ (A-(D-A)^+,\ (D-A)^+)\}$$

$$\mathfrak{D} = \{(o,D),\ (D-A,A),\ (2-(D-A),A-2(D-A)),\ (A-2(D-A),2(D-A)),$$
$$(A,D-A),\ (D,o)\}\ .$$

We note that the first components of points in $\mathfrak{A}$ and $\mathfrak{D}$ partition the intervals $[0,A]$ and $[o,D]$ into subintervals. However, either for some $d \in \mathfrak{D}$, $f(x,d)$ is not convex on a subinterval of $[o,A]$ or for some $a \in \mathfrak{A}$, $f(a,y)$ is not concave on a subinterval of $[0,D]$. Therefore the results of Chapter 2 do not apply for the choice of $\mathfrak{A}$ and $\mathfrak{D}$.

3.5 A Model of n-Cities with a Payoff with Large Discontinuities.

This game is a generalization of the game considered in Section 3.4. The game is defined by

$$X = \{x = (x_1,x_2,\ldots,x_n) \mid x_i \geq o\ ,\ \sum_{i=1}^{n} x_i = A\}$$

$$Y = \{y = (y_1,y_2,\ldots,y_n) \mid y_i \geq o\ ,\ \sum_{i=1}^{n} y_i = D\}$$

$$f(x,y) = \sum_{i=1}^{n} f_i(x_i,y_i) \quad \text{where}$$

$$f_i(x_i,y_i) = \begin{cases} \lambda_i & \text{if} \quad x_i > y_i \\ -hx_i & \text{if} \quad x_i \le y_i \end{cases} \qquad \text{and}$$

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ .$$

The only model considered here is the case where $(\frac{2n-1}{2})A \le D < nA$. It will be shown that optimal strategies for this game can be obtained by finding optimal strategies for the reduced game $(\mathfrak{A},\mathfrak{D},f)$ where

$$\mathfrak{A} = \{(A,o,\ldots,o),(o,A,o,\ldots,o),\ldots,(o,\ldots,o,A)\} = \{a_o,\ldots,a_n\}$$

$$\mathfrak{D} = \{(d-(n-1)A,A,\ldots,A),\ldots,(A,\ldots,A,D-(n-1)A)\} = \{d_1,\ldots,d_n\} \ .$$

To prove this let

$$X_o = \{x \in X \mid x_i \le D-(n-1)A \ , \text{ for all } \ i\}$$

$$X_i = \{x \in X \mid x_i > D-(n-1)A \ge x_j \ \text{ for all } \ j \ne i\} \ .$$

Then $X = X_o \bigcup_{i=1}^{n} X_i$ since if $x_i > D-(n-1)A$ then $x_i > \frac{1}{2}A$.

But 1. For $x \in X_o$, $f(x,d) = -hA$ for all $d \in \mathfrak{D}$.

2. For $x \in X_i$, $f(x,d_j) = -hA$ if $j \ne i$

$$f(x,d_i) = \lambda_i - h \sum_{\substack{j=1 \\ j\ne i}}^{n} x_j$$

$$f(a_i,d_i) = \lambda_i \ .$$

i.e. for all $x \in X_i$, $f(x,d) \le f(a_i,d)$ for all $d \in \mathfrak{D}$.

By Lemma 2.3.1 $\mathfrak{A}$ is dominant w.r.t. $\mathfrak{D}$.

For the proof that $\mathfrak{D}$ is dominant w.r.t. $\mathfrak{A}$ see Cooper and Restrepo [1, Lemma 2.1].

Now let $w_i = hA + \lambda_i$ and

$$P_{ij} = \begin{cases} w_i & \text{if } i = j \\ o & \text{if } i \neq j \end{cases} .$$

Then $f(a_i, d_j) = -hA + P_{ij}$ and the finite game $(\mathfrak{A}, \mathfrak{D}, f)$ has the same optimal strategies as the matrix game with payoff P_{ij} . See Karlin [3, p.28].

1. The value of the game is

$$v = -hA + [\sum_{i=1}^{n} \frac{1}{w_i}]^{-1} .$$

2. The optimal strategy for the attacking player is

$$\alpha^* = (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \alpha_i = \frac{1}{w_i} \{\sum_{i=1}^{n} \frac{1}{w_i}\}^{-1}$$

3. The optimal strategy for the defending player is

$$\beta^* = (\beta_1, \ldots, \beta_n) \quad \text{with} \quad \beta_i = \frac{1}{w_i} \{\sum_{i=1}^{n} \frac{1}{w_i}\}^{-1} .$$

CHAPTER 4

SOME MATRIX GAMES

4.1 Introduction

As stated previously the 2-city game model considered in Section 3.4 could only be solved for two particular cases. If we consider their dominant sets, we note that for a point in a dominant set, one component is always an integral multiple of D-A . For this reason we are led to consider the same game model but formulated as a discrete game with

$$X = \{(x, A-x) : x \text{ an integer}, \ o \leq x \leq A\}$$

$$Y = \{(y, D-y) : y \text{ an integer}, \ o \leq y \leq D\}, \ D = A+1.$$

By definitions of X and Y, all these games are finite matrix games. As may be expected the size of the payoff matrix for this model increased as A increased and therefore the game was only solved for $A \leq 8$, $D \leq 9$. The computation of the payoff matrix is straight forward. In each case, by choice of suitable notation and by means of standard dominance arguments the resulting games can be reduced to one of the games considered below.

4.2 Matrix Games from the Discrete 2-City Model

In the following games, v will be the value of the game, α^* will be the optimal mixed strategy for Player I, and β^* will be the optimal mixed strategy for Player II. All the games were solved by using the Snow-Shapley theorem.

Game No. 1:

$$M = \begin{bmatrix} 0 & \mu \\ \nu & 0 \end{bmatrix} \qquad \mu,\nu > 0 .$$

This game has already been solved in Section 3.2.

$$v = \frac{\mu\nu}{\mu+\nu}$$

$$\alpha^* = (\nu,\mu)\,\frac{1}{\mu+\nu}$$

$$\beta^* = (\mu,\nu)\,\frac{1}{\mu+\nu}$$

Game No. 2:

$$M = \begin{bmatrix} 0 & \mu & \mu \\ \nu-b & 0 & \mu-a \\ \nu & \nu & 0 \end{bmatrix} \qquad \begin{array}{l} 0 \leq a < \mu \\ 0 \leq b < \nu \end{array}$$

This game has been solved in Section 3.4.

Let $\quad d = \mu\nu(\mu-a) + \mu\nu(\nu-b)$

$\rho = \mu(\mu-a) + \mu\nu + \nu(\nu-b)$, then

$$v = d/\rho$$

$$\alpha^* = (\nu(\nu-b),\mu\nu,\mu(\mu-a))1/\rho$$

$$\beta^{*} = (-\nu(\mu-a) + \mu\nu + \mu(\mu-a), \nu(\mu-a) - \mu\nu + \mu(\nu-b),$$
$$\nu(\nu-b) + \mu\nu - \mu(\nu-b))1/\rho$$

Game No. 3:

$$M = \begin{bmatrix} o & \mu & \mu & \mu \\ \nu-d & o & \mu-a & \mu-a \\ \nu-b & \nu-b & o & \mu-c \\ \nu & \nu & \nu & o \end{bmatrix} \qquad \begin{matrix} o \le a,c < \mu \\ o \le b,d < \nu \end{matrix}$$

$$dM^{-1} = \begin{bmatrix} -\nu(\mu-a)[\mu+\nu-(b+c)] & \mu\nu(\nu-b) & \mu\nu(\mu-a) & \mu(\mu-a)(\mu-c) \\ \nu(\mu-a)(\mu+\nu-b-c) - \nu(\mu-c)(\nu-d) & -\mu\nu(\mu+\nu-b-c) & \mu\nu(\nu-d) & \mu(\mu-c)(\nu-d) \\ \nu(\mu-c)(\nu-d) & \mu\nu(\mu-c) & -\mu\nu(\mu+\nu-a-d) & \mu(\nu-b)(\mu+\nu-a-d) - \mu(\mu-c)(\nu-d) \\ \nu(\nu-b)(\nu-d) & \mu\nu(\nu-b) & \mu\nu(\mu-a) & -\mu(\nu-b)(\mu+\nu-a-d) \end{bmatrix}$$

where $d = \mu\nu[(\mu-a)(\mu-c) + (\mu-a)(\nu-b) + (\nu-b)(\nu-d)]$.

Let $\rho = \mu(\mu-a)(\mu-c) + \mu\nu(\mu-a) + \mu\nu(\nu-b) + \nu(\nu-b)(\nu-d)$,

$e = (1,1,1,1)^{T}$

Then $v = d/\rho$

$$\alpha^{*} = (\nu(\nu-b)(\nu-d), \mu\nu(\nu-b), \mu\nu(\mu-a), \mu(\mu-a)(\mu-c))1/\rho$$

$$\beta^{*} = dM^{-1}e/\rho$$

Game No. 4:

$$M = \begin{bmatrix} o & \mu & \mu & \mu & \mu \\ \nu-f & o & \mu-a & \mu-a & \mu-a \\ \nu-d & \nu-d & o & \mu-c & \mu-c \\ \nu-b & \nu-b & \nu-b & o & \mu-e \\ \nu & \nu & \nu & \nu & o \end{bmatrix} \qquad \begin{array}{l} o \le a,c,e < \mu \\ o \le b,d,f < \nu \end{array}$$

The inverse matrix of M was calculated, but both M^{-1} and β^* were too complicated to include here. However, if we let

$$d = \mu\nu[(\mu-a)(\mu-c)(\mu-e)+(\mu-a)(\mu-c)(\nu-b)+(\mu-a)(\nu-b)(\nu-d) + (\nu-b)(\nu-d)(\nu-f)]$$

$$\rho = \mu(\mu-a)(\mu-c)(\mu-e)+\mu\nu[(\mu-a)(\mu-c)+(\mu-a)(\nu-b)+(\nu-b)(\nu-d)] + \nu(\nu-b)(\nu-d)(\nu-f)$$

then $v = d/\rho$

$$\alpha^* = (\nu(\nu-b)(\nu-d)(\nu-f),\mu\nu(\nu-b)(\nu-d),\mu\nu(\mu-a)(\nu-b), \mu\nu(\mu-a)(\mu-c),\mu(\mu-a)(\mu-c)(\mu-e)) .$$

4.3 A Method of Inverting Matrices.

When using the Snow-Shapley theorem to solve a game with payoff matrix M, the inverse of square submatrices of M must be computed. The strategies given by the theorem for a particular submatrix must then be checked to see if they are optimal. If they are not optimal, the submatrix is discarded and the process is repeated. The following method was found to be a convenient way for finding the inverse of a matrix.

Let M be a square matrix, I be the unit matrix of the same dimension as M. Consider the tabloid $I|M|I$. First we apply row operations to the tabloid $I|M$ yielding the tabloid, say $B|C|I$. Then we apply column operations to $C|I$ resulting in the tabloid $B|D|E$. We apply row and column operations until the matrix D is in such a form that its inverse can easily be found by the standard techniques. But then $D = BME$ which implies $M^{-1} = ED^{-1}B$.

REFERENCES

[1] J. N. Cooper and R. A. Restrepo, "Some Problems of Attack and Defense", SIAM Review, Vol. 9, No. 4, (October 1967), pp. 680-691.

[2] M. Dresher, Games of Strategy: Theory and Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.

[3] S. Karlin, Mathematical Models in the Theory of Games, Programming and Economics, Addison-Wesley, Reading, Massachusetts, 1959.