ON POSITIVE SOLUTIONS OF A VOLTERRA EQUATION OF THE SECOND KIND

by

DESMOND EDWIN THOMPSON

B.Sc. (Special Hons); University of The West Indies, 1967

A THESIS SUMITTED IN PARTIAL FULFILMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April, 1970

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree tha permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date 22/April/70.

ABSTRACT

Volterra Integral Equations of the second kind occur in many problems in Physics and Engineering. Here we study the conditions for and behaviour of positive solutions of these equations. Examples have been given to point out some of the difficulties that occur in the theory.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I am indebted to my supervisor Dr. G. HUIGE, for his generous and valuable assistance in the research and writing of this paper. My thanks to Dr. G.W. Bluman, who read this thesis and provided excellent suggestions.

I am grateful to the University of B.C. and the National Research Council for their financial support.

Last, but not least, I wish to thank Mrs. Y.S. Choo for typing the thesis.

INTRODUCTION

In this paper we will consider a Volterra equation of the second kind,

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt \qquad (0.1)$$

where $K(x,t)$ is a VOLTERRA TYPE KERNEL that is,

$$K(x,t) = 0 \quad \text{for} \quad t > x \;.$$

In the first part we will be concerned with comparison and approximation theorems. Some compairson theorems have been given in [1] and [7]. Here we apply stronger conditions to $f(x)$ and $K(x,t)$ and obtain a <u>proper inequality</u> between solutions. Their ideas, as well as that of an approximate solution for a differential equation as given in [5], have been combined to give a theorem relating approximate solutions to that of (0.1) .

Some discussion on monotone solutions has been given. Even though the conditions for such solutions are very strong, examples have been given to show that weaker conditions may work.

On the question of upper bounds most of the material has been taken from [1], [3], [5] and [6] . A discussion is given in the first section.

In the last section we deal with conditions for positive solutions of (0.1) when $K(x,t) \leq 0$. In this case the equation then considered is

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt$$

where $K(x,t) \geq 0$. A study of the asymptotic behaviour of positive solutions of this equation is given .

CHAPTER 1

BOUNDS AND MONOTONE SOLUTIONS

For completeness we state here the principal results we need for Volterra equations. Let

$$I = \{ x : 0 \leq x < \infty \} .$$

$$S = \{ t : t \leq x \quad x \varepsilon I \} .$$

and $K(x,t)$ be a VOLTERRA TYPE kernel defined on $I \times I$.

Define $K_1(x,t) = K(x,t)$ and for $n \geq 2$

$$K_n(x,t) = \int_t^x K(x,s)K_{n-1}(s,t)ds .$$

By the resolvent kernel of $K(x,t)$ we mean the function $H(x,t)$ given by the series $H(x,t) = \sum_{k=1}^{\infty} K_n(x,t)$. This series converges uniformly on $I \times I$, provided K is continuous .

We quote here without proof the following theorem.

Let $f(x)$ be continuous on I and $K(x,t)$ be a VOLTERRA TYPE kernel continuous on $I \times S$. Then there is a unique solution to the equation

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt$$

given by

$$u(x) = f(x) + \int_0^x H(x,t)f(t)dt \tag{0.2}$$

For the proof of this theorem see [2] or [4] .

We begin this section with a simple but very important lemma. Unless otherwise stated we will assume that $f \in C(I)$ and

$$K(x,t) \in C(I \times S) .$$

Lemma 1.1 If $K(x,t) \geq 0$ and $f(x) \geq 0$ then $u(x) \geq f(x) \geq 0$.

Proof If $K(x,t) \geq 0$ then the resolvent kernel $H(x,t) \geq 0$ and hence it follows from (0.2) that $u(x) \geq f(x) \geq 0$.

Corollary 1.1 If $K(x,t) > 0$ and $f(x) > 0$ then $u(x) > f(x) > 0$.

Proof Now $u(x) = f(x) + \int_0^x K(x,t)u(t)dt$ and hence from lemma 1.1 $u(x) > 0$ and so $u(x) > f(x) > 0$.

We now give some results on monotone solutions. The first theorem gives conditions on both $f(x)$ and $K(x,t)$. However, it will be seen that these conditions can be relaxed. Even though we have not been able to prove theorems with 'soft' conditions on K , the point is made in examples 1.1 and 1.2 .

<u>THEOREM 1.1</u> If $f(x) \geq 0$ and monotonic increasing and $K(x,t) \geq 0$ and monotonic increasing in x, then $u(x)$ is monotonic increasing .

Proof: Let $x > s$

$$\text{then } u(x) - u(s) = f(x) - f(s) + \int_0^x K(x,t)u(t)dt - \int_0^s K(s,t)u(t)dt$$

$$\geq \int_0^s (K(x,t) - K(s,t))u(t)dt + \int_s^x K(x,t)u(t)dt \ .$$

but from lemma 1.1 $u(x) \geq 0$ and so $u(x) - u(s) \geq 0$.

<u>Corollary</u> If we replace $f(x) \geq 0$ by $f(x) > 0$ and $K(x,t) \geq 0$ by $K(x,t) > 0$ in the above theorem then $u(x)$ is strictly monotonic increasing.

Proof: From Corollary 1.1 $u(x) > 0$ and so from

$$u(x) - u(s) = f(x) - f(s) + \int_0^s [K(x,t) - K(s,t)]u(t)dt + \int_s^x K(x,t)u(t)dt$$

we have

$$u(x) - u(s) > f(x) - f(s)$$

$$\geq 0 \ .$$

$$\text{i.e } u(x) > u(s) \ .$$

As we can see from Theorem 1.1 the condition that $K(x,t)$ is monotonic increasing in x is quite strong. However, the theorem is not true for arbitrary $K(x,t) \geq 0$ (see example 1.2). On the other hand in some cases (see example 1.1) we can still obtain monotonic increasing solutions when $K(x,t)$ is monotonic decreasing in x. It should be noted that for this to be true it appears that $K(x,t)$ should not decrease too fast.

Before we give the example we quote a lemma due to TRICOMI.

Lemma 1.2 If $K(x,t) = \frac{A(x)}{A(t)}$ then the solution of (0.1) can be written as $u(x) = f(x) + \int_0^x e^{x-t} K(x,t) f(t)dt$.

For the proof see [2] pages 17-18.

EXAMPLE 1.1 Consider the case when $f(x) \in C'(I)$ with $f(x) \geq 0$ and $f'(x) \geq 0$. Take $K(x,t) = \frac{A(x)}{A(t)} = \frac{e^{-x}}{e^{-t}}$ as in lemma 1.2.

then

$$u(x) = f(x) + \int_0^x e^{x-t} e^{-x} e^{t} f(t)dt$$

$$= f(x) + \int_0^x f(t)dt$$

$$u'(x) = f'(x) + f(x).$$

$$\geq 0.$$

EXAMPLE 1.2

Take $A(x) = e^{-x^n}$ and $f(x) = e^x$.

From lemma 1.2

$$u(x) = e^x + \int_0^x e^{-t+x} e^{-x^n} e^{t^n} e^t dt$$

$$u'(x) = e^x + e^{x-x^n} \cdot e^{x^n} + (1 - nx^{n-1}) e^{x-x^n} \int_0^x e^{t^n} dt$$

$$= 2e^x + (1 - nx^{n-1}) e^{x-x^n} \int_0^x e^{t^n} dt .$$

$$u'(1) = 2e + (1 - n) \int_0^1 e^{t^n} dt$$

$$< 0 \quad \text{for} \quad n \geq 7 .$$

THEOREM 1.2 If $u(x)$ is monotonic increasing with $K(x,t) \geq 0$ and $\int_0^x K(x,t)dt < 1$ for all $x \in I$ if $f(x) \geq 0$ then

$$f(x) \leq u(x) \leq \frac{f(x)}{1 - \int_0^x K(x,t)dt} .$$

Proof: $u(x) = f(x) + \int_0^x K(x,t)u(t)dt$

$$\leq f(x) + u(x) \int_0^x K(x,t)dt$$

that is $u(x)(1 - \int_0^x K(x,t)dt) \leq f(x)$

or $$u(x) \leq \frac{f(x)}{1 - \int_0^x K(x,t)dt}$$

that $u(x) \geq f(x)$ follows from lemma 1.1 .

As seen from above, the conditions for a 'nice' upper bound is quite strong. In [7] the case $K(x,t) = u(x)k(x-t)$ was considered and the following theorem stated.

THEOREM 1.3 If the limit $c = \lim_{x\to\infty} f(x)$ exists and $K(x,t) = k(x-t)$ is absolutely integrable i.e $\int_0^\infty |k(t)|dt < \infty$ then the limit of $u(x)$ given by (0.1) is $\lim_{x\to\infty} u(x) = \dfrac{c}{1 - \int_0^\infty k(t)dt}$ if and only if $\int_0^\infty e^{-st}k(t)dt \neq 1$, provided $Re(s) > 0$.

In general the resolvent kernel cannot be easily calculated and so the need for upper bounds that remain close to the solution is extremely important. If $K(x,t) = g(x)h(t)$ then as shown in [3]

$$H(x,t) = g(x)h(t) \exp\left(\int_t^x g(t)h(t)dt\right) .$$

In some cases we can write $K(x,t) \leq K_1(x)K_2(t)$ and apply GRONWALL'S INEQUALITY and obtain

$$u(x) \leq f(x) + K_1(x)(\exp \int_0^x K_1(s)K_2(s)ds)(\int_0^x K_2(s)f(s)ds) \quad (1.1)$$

For proof see [5] or [1] .

This form however as was pointed out in [6] does not say much about the behaviour of $u(x)$ at infinity. Sometimes (see example given below) it is better to write the upper bound for $K(x,t)$ in the form .

$$K(x,t) \leq \sum_{i=1}^{n} K_i(x)H_i(t) \ .$$

If this is done then we can apply the following theorem due to WILLETT. The proof is given in [6] .

THEOREM 1.4 Suppose that

$$u(x) \leq f(x) + \sum_{i=1}^{n} w_i(x) \int_0^x v_i(s)u(s)ds$$

where $w_i v_j$ $(i,j = 1 \cdots n)$ and $v_i u$ are integrable on I . Then $u \leq E_n f(x)$ where $E_i (i = 0, 1 \cdots n)$ is defined inductively as a composition of $i + 1$ functional operators, that is,

$$E_i = D_i D_{i-1} \cdots D_o \qquad \text{where}$$

$$D_o f(x) = f(x)$$

$$D_i f = f + (E_{i-1}w_i)(\exp \int_o^x v_i E_{i-1} w_i) \cdot (\int_o^x v_i f dt) \ .$$

The following example was considered in [6] .

Let $$u(x) \leq x + \int_0^x (\lambda^2 x e^{-\lambda t} + 1)u(t)dt \; .$$

Here λ is a real parameter and the problem is to determine the asymptotic behaviour of u as $\lambda \longrightarrow \infty$, in particular, to prove that $u = O(1)$ uniformly for x restricted to compact subintervals of $[0, \infty)$.

If we take $w_1(x) = 1$ $v_1(x) = 1$ and $w_2(x) = x$ and $v_2(x) = \lambda^2 e^{-\lambda x}$ then the desired result follows by direct application of theorem 1.4. However if we apply 1.1 directly i.e take $K(x,t) \leq \max(1, \lambda^2 x)e^{-\lambda t}$ or

$$K(x,t) \leq (1 + \lambda^2 x)\max(1, e^{-\lambda t})$$

then this does not produce that u must be bounded as $\lambda \longrightarrow \infty$.

CHAPTER 2

COMPARISON AND APPROXIMATION THEOREMS

In this chapter we will be mainly concerned with the equation

$$u_i(x) = f_i(x) + \int_o^x K_i(x,t)u_i(t)dt \ , \ (i = 1,2) \ . \tag{2.1}$$

Here we assume that $f_i(x) \in L_2(I)$ and

$$K_i(x,t) \in L_2(I\times I) \ , \quad \text{for } i = 1,2$$

THEOREM 2.1. Let $K_i(x,t)$ be a Volterra type kernel with $|K_1(x,t)| \leq K_2(x,t)$ and $|f_1(x)| \leq f_2(x)$; $K_1(x,t)$ and $f_1(x)$ possibly complex. If $u_i(x)$ is the unique solution of the equation

$$u_i(x) = f_i(x) + \int_o^x K_i(x,t)u_i(t)dt \ ,$$

then $|u_1(x)| \leq u_2(x)$. In fact,

$$u_2(x) - |u_1(x)| \geq f_2(x) - |f_1(x)| .$$

Proof: The proof is a consequence of Lemma 1.1. The reader is referred to [1] for details.

We now consider the case when $f_1(x)$ is real-valued.

THEOREM 2.2. Let $K_i(x,t)$ $i = 1,2$ be a Volterra type kernel such that $K_2(x,t) \geq K_1(x,t) \geq 0$. If $f_2(x) \geq f_1(x)$ with $f_2(x) \geq 0$ and $u_i(x)$ is the unique solution of (2.1) then $u_2(x) \geq u_1(x)$. In fact $u_2(x) - u_1(x) \geq f_2(x) - f_1(x)$.

Proof: Let $H_i(x,t)$ be the resolvent kernel for (2.1). Then it is easily seen that $0 \leq H_1(x,t) \leq H_2(x,t)$.

$$\begin{aligned} u_2(x) - u_1(x) &= f_2(x) - f_1(x) + \int_o^x H_2(x,t)f_2(t)dt \\ &\qquad - \int_o^x H_1(x,t)f_1(t)dt . \\ &= f_2(x) - f_1(x) + \int_o^x [H_2(x,t) - H_1(x,t)]f_2(t)dt \\ &\qquad + \int_o^x H_1(x,t)[f_2(t) - f_1(t)]dt \end{aligned}$$

and so $u_2(x) - u_1(x) \geq f_2(x) - f_1(x)$.

This completes the proof.

It should be noted that Theorem 2.2 is applicable when Theorem 2.1 is not.

THEOREM 2.3. Let $f(x) \varepsilon L_2(I)$, $K(x,t) \geq 0$ and $K(x,t) \varepsilon L_2(I \times I)$. If $u(x)$ is the unique solution to (0.1) and $v(x) \varepsilon L_2(I)$ with

$$v(x) \leq f(x) + \int_0^x K(x,t)v(t)dt$$

then $u(x) \geq v(x)$.

Proof: Let $v(x) = g(x) + \int_0^x K(x,t)v(t)dt$, then $f(x) \geq g(x)$.

$$u(x) - v(x) = f(x) - g(x) + \int_0^x K(x,t)[u(t) - v(t)]dt$$

and so from Lemma 1.1 $u(x) - v(x) \geq 0$, i.e. $u(x) \geq v(x)$.

THEOREM 2.4. Let $K(x,t) = \sum_{i=1}^{n} K_i(x,t)$, $F(x) = \sum_{i=1}^{n} f_i(x)$ with $K_i(x,t) \geq 0$, $f_i(x) \geq 0$ and $K_i(x,t) \in L_2(I\times I)$ $f_i(x) \in L_2(I)$ for $i = 1,\ldots,n$. If $u_i(x)$ is the unique solution of

$$u_i(x) = f_i(x) + \int_0^x K_i(x,t)u_i(t)dt$$

and $u(x)$ is the unique solution of

$$u(x) = F(x) + \int_0^x K(x,t)u(t)dt$$

then $\sum_{i=1}^{n} u_i(x) \leq u(x)$.

Proof: The proof is by induction on n .

Theorem true for $n = 1$.

Assume true for $n = k-1$. i.e. $\sum_{i=1}^{n-1} u_i(x) \leq v(x)$ where

$$v(x) = \sum_{i=1}^{k-1} f_i(x) + \int_o^x \sum_{i=1}^{k-1} K_i(x,t)v(t)dt$$

$$u_n(x) + v(x) = f_n(x) + \sum_{i=1}^{k-1} f_i(x) + \int_o^x \sum_{i=1}^{k-1} K_i(x,t)v(t)dt$$

$$+ \int_o^x K_n(x,t)u_n(t)dt$$

since from Lemma 1.1 $u_n(x) \geq 0$. From Theorem 2.3 we have

$$u_n(x) + v(x) \leq u(x) .$$

i.e. $$u_n(x) + \sum_{i=1}^{k-1} u_i(x) \leq u(x)$$

or $$\sum_{i=1}^{n} u_i(x) \leq u(x) .$$

This completes the proof.

We should note here that even if we deal with distinct functions in L_2 (i.e. $f_2(x) > f_1(x)$ say) we cannot in general obtain a direct inequality for solutions. If however, the solutions are continuous then this can be done.

Let us consider the case when $f(x)$ and $K(x,t)$ are continuous.

THEOREM 2.5. Let $u(x) = f(x) + \int_o^x K(x,t)u(t)dt$, $v(x) < f(x) + \int_o^x K(x,t)v(t)dt$ with $f(x) \geq 0$ and $K(x,t) \geq 0$. If $v(x)$ is continuous then $u(x) > v(x)$ for $x \varepsilon I$.

Proof: $u(0) = f(0) > v(0)$ that is $u(0) > v(0)$. Suppose the theorem is false, By continuity of $u(x)$ and $v(x)$ there exists $x_1 > 0$ such that

$$v(x_1) = u(x_1)$$

where $$v(x) < u(x) \quad \text{for} \quad 0 \leq x < x_1 .$$

Now
$$\begin{aligned} u(x_1) &= f(x_1) + \int_o^{x_1} K(x_1,t)u(t)dt \\ &\geq f(x_1) + \int_o^x K(x_1 t)v(t)dt \\ &> v(x_1) \end{aligned}$$

This contradicts $u(x_1) = v(x_1)$. Hence the theorem is true.

COROLLARY 2.1. If $w(x) > f(x) + \int_o^x K(x,t)w(t)dt$

$$u(x) = f(x) + \int_o^x K(x,t)u(t)dt$$

$$v(x) < f(x) + \int_o^x K(x,t)v(t)dt$$

where $w(x)$ is continuous, then

$$w(x) > u(x) > v(x) .$$

COROLLARY 2.2. Let $u_i(x)$ be the unique solutions of

$$u_i(x) = f_i(x) + \int_0^x K_i(n,t)u_i(t)dt$$

for $i = 1,2$. If $f_i(x) \varepsilon C(I)$ and $K_i(x,t) \varepsilon C(I\times I)$ with $f_1(x) > f_2(x) \geq 0$, $K_1(x,t) \geq K_2(x,t) \geq 0$ then $u_1(x) > u_2(x)$ for $x \varepsilon I$.

Proof: $u_1(x) = f_1(x) + \int_0^x K_1(x,t)u_1(t)dt > f_2(x) + \int_0^z K_2(x,t)u_1(t)dt$

hence from Theorem 2.4 we have $u_1(x) > u_2(x)$ for $x \varepsilon I$.

DEFINITION 2.1. We say $y(x,\delta)$ is a δ-approximate solution of (0.1) if $\left|y(x,\delta) - f(x) - \int_0^x K(x,t)y(t)dt\right| \leq \delta(x)$.

THEOREM 2.6. Let $G(x,t)$, $K(x,t) \varepsilon L_2(I\times I)$ with $G(x,t) \geq K(x,t) \geq 0$. If $v(x)$ is the solution of $v(x) = \delta(x) + \int_0^x G(x,t)v(t)dt$ and $u(x)$ is the unique solution of (0,1) then $|y(x,\delta) - u(x)| \leq v(x)$.

Proof: Consider the case when $y(x) \leq \delta(x) + f(x) + \int_0^x K(x,t)y(t)dt$.

Since $u(x)$ is a solution of (0.1) we have

$$y(x) - u(x) \leq \delta(x) + \int_0^x K(x,t)[y(t) - u(t)]dt$$

Let $$w(x) = \delta(x) + \int_0^x K(x,t)w(t)dt$$

then from Theorem 2.3 we have

$$y(x) - u(x) \leq w(x) \leq v(x) ,$$

that is $$y(x) - u(x) \leq v(x) .$$

If we take $y(x) \geq -\delta(x) + f(x) + \int_0^x K(x,t)y(t)dt$

then $$y(x) - u(x) \geq -\delta(x) + \int_0^x K(x,t)y(t) - u(t)dt$$

and so $$u(x) - y(x) \leq \delta(x) + \int_0^x K(x,t)[u(t) - y(t)]dt$$

Hence from Theorem 2.3

$$u(x) - y(x) \leq v(x)$$

therefore we have

$$|u(x) - y(x,\delta)| \leq v(x) .$$

DEPENDENCE ON INITIAL VALUE

In the case of differential equation of the form $y' = f(x,y)$ where f is continuous and $y(x_o) = y_o$ we know that solutions depend continuously on the initial value x_o. With this is mind we can write the solution of (0.1) as a function of two variables that is, the solution of $u(x) = f(x) + \int_s^x (K(x,t)u(t)dt$ can be written as $u(x,s)$, for $x \geq s$. The next theorem shows that $u(x,s)$ is monotone decreasing in the second variable.

THEOREM 2.3 Let $u(x,s)$ be the solution of

$$u(x,s) = f(x) + \int_s^x K(x,t)u(t,s)dt$$

where $f(x) \geq 0$ and $K(x,t) \geq 0$; then for $s_1 > s_2$, $u(x,s_1) \leq u(x,s_2)$.

Proof: $$u(x,s_1) = f(x) + \int_{s_1}^x K(x,t)u(t,s_1)dt$$

$$\leq f(x) + \int_{s_2}^x K(x,t)u(t,s_1)dt$$

hence from theorem 2.1 we have $u(x,s_1) \leq u(x,s_2)$.

CHAPTER 3

ASYMPTOTIC BEHAVIOUR

From the above discussion we can see that $f(x)$ acts as 'boundary line' depending on whether $K(x,t) \geq 0$ or $K(x,t) \leq 0$.

In the case when $f(x)$ and $K(x,t)$ are continuous it is impossible to obtain solutions $u(x) \leq f(x)$ when $K(x,t)$ is positive.

The natural question therefore arises: are there solutions $u(x)$ such that $0 \leq u(x) \leq f(x)$? In the next theorem we give sufficient conditions for such solutions .

Instead of considering equation (0.1) with $K(x,t) \leq 0$ we will consider

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt \tag{3.1}$$

with $K(x,t) \geq 0$.

THEOREM 3.1 Let $f(x) > 0$ and $K(x,t) > 0$ if $\frac{f(x)}{f(y)} < \frac{K(x,t)}{K(y,t)}$ for $x < y$ $0 \leq t \leq x$ then the solution to (3.1) is such that $f(x) \geq u(x) \geq 0$.

Proof: $u(x) = f(x) - \int_0^x K(x,t)u(t)dt$ that is $f(x) \geq u(x)$.

It remains to prove that $u(x)$ cannot be negative anywhere on I .

$$u(0) = f(0) > 0 .$$

Suppose the theorem is false. Then by continuity there exists $x_1 > 0$ and $\delta > 0$ such that

$$u(x) \geq 0 \qquad 0 < x \leq x_1$$

$$u(x_1) = 0$$

$$u(x) < 0 \qquad x_1 < x \leq x_1 + \delta$$

i.e for $\qquad x_1 < x \leq x_1 + \delta$

we have

$$0 > u(x) > f(x) - \int_0^{x_1} K(x,t)u(t)dt$$

$$= \frac{f(x)}{f(x_1)}\{ f(x_1) - \frac{f(x_1)}{f(x)} \int_0^{x_1} K(x,t)u(t)dt \}$$

$$\geq \frac{f(x)}{f(x_1)}\{ f(x_1) - \int_0^{x_1} K(x_1,t)u(t)dt \}$$

$$= \frac{f(x)}{f(x_1)}u(x_1)$$

$$= 0 .$$

i.e. $\qquad u(x) \geq 0 \quad$ for $\quad x_1 < x \leq x_1 + \delta$

contradiction to the fact that $u(x) < 0$ in that interval .

Therefore the theorem is true .

$$\text{i.e.}\quad f(x) \geq u(x) \geq 0 \quad .$$

Corollary 3.1   If $K(x,t)$ is monotonic decreasing in x and $f(x)$ montonic increasing, the conditions for the above theorem is satisfied.

Corollary 3.2   If we take the case $K(x,t) = k(x - t)$ all we need is that $k(t)$ be monotonic decreasing .

We now study the asymptotic behaviour of $u(x)$ given certain conditions on $K(x,t)$ and $f(x)$.

THEOREM 3.2   If $u(x) \geq 0$ is a solution of (3.1) and if $\infty > K(x,t) > 0$, $K(x,t)$ monotone increasing in x and $f(x)$ monotone increasing and $f(x) \leq B < \infty$ then $\lim_{x\to\infty} u(x) = 0$.

Proof: Suppose that $u(x) \not\to 0$ then there exist a sequence $\{x_n\}$ such that $x_n < x_{n+1} \longrightarrow \infty$ and

$$u(x_n) \geq \alpha > 0 \quad \text{for some} \quad \alpha > 0 \ .$$

We claim there exist $\beta > 0$ such that

$$u(x) \geq \tfrac{1}{2}\alpha \quad \text{if} \quad x_n < x < x_n + \beta.$$

Now $u(x) - u(x_n)$

$$= f(x) - f(x_n) - \int_o^x K(x,t)u(t)dt + \int_o^{x_n} K(x,t)u(t)dt$$

$$= f(x) - f(x_n) + \int_o^{x_n} K(x_n,t)u(t)dt - \int_o^{x_n} K(x,t)u(t)dt - \int_{x_n}^x K(x,t)u(t)dt$$

$$\geq \int_o^{x_n} K(x_n,t)u(t)dt - \int_o^{x_n} K(x,t)u(t)dt - \int_{x_n}^x K(x,t)u(t)dt$$

$$\geq - \int_{x_n}^x K(x,t)u(t)dt$$

$$\geq - B\int_{x_n}^x K(x,t)dt \qquad M = \sup_{x_n \leq t \leq x} u(t) \leq B < \infty .$$

i.e. $u(x) > u(x_n) - B\int_{x_n}^x K(x,t)dt > \frac{\alpha}{2}$

provided $$\int_{x_n}^{x_n+\beta} K(x,t)dt \leq \frac{\alpha/2}{B} .$$

Since we may take $|x_n - x_{n+1}| \geq 1$ for all $n \geq 0$ and $\beta < 1$.

we have

$$\int_o^x K(x,t)u(t)dt \geq \frac{\alpha}{2} A \sum_{x_n < x-\beta} 1 . \qquad A = \inf_{(x,t)\varepsilon I\times I} K(x,t)$$

$$\longrightarrow \infty \quad \text{as} \quad x \to \infty .$$

i.e. $$B \geq f(x) = u(x) + \int_o^x K(x,t)u(t)dt$$

and so $f(x) \longrightarrow \infty$ as $x \to \infty$.

This contradicts the hypothesis that $f(x) \leq B < \infty$.

Therefore $u(x) \longrightarrow 0$ as $x \to \infty$.

THEOREM 3.3 Let $u(x)$ be a solution of (3.1). If $\lim_{x\to\infty} \int_0^A K(x,t)dt = 0$ with $A < \infty$, $\lim_{x\to\infty} \int_0^x K(x,t)dt = C < \infty$ and $f(x) \leq B < \infty$ with

$$\limsup_{x\to\infty} f(x) > 0 \quad \text{then}$$

$$u(x) \not\longrightarrow 0 .$$

Proof: Suppose $u(x) \longrightarrow 0$ then for $\varepsilon > 0$ there exist $N > 0$ such that $u(x) < \varepsilon$ and $\int_0^A K(x,t)dt < \varepsilon \quad x \geq N$.
and hence

$$f(x) = u(x) + \int_0^x K(x,t)u(t)dt$$

$$= u(x) + \int_0^A K(x,t)u(t)dt + \int_A^x K(x,t)u(t)dt$$

$$\leq u(x) + B\int_0^A K(x,t)dt + \varepsilon\int_A^x K(x,t)dt \qquad \text{for } x > N .$$

$$< \varepsilon + B\varepsilon + \varepsilon C \qquad \text{for } x > N .$$

that is $\qquad f(x) \longrightarrow 0$

this contradicts $\limsup_{x\to\infty} f(x) > 0$.

Therefore $\qquad u(x) \not\to 0$.

Corollary If we consider the case $K(x,t) = k(x - t)$ and $\int_0^\infty k(t)dt < \infty$ then the above theorem is still true .

THEOREM 3.3 If $u(x) \geq 0$ is a solution of (3.1) and $u(x)$ is monotonic decreasing with $\lim_{x\to\infty} \int_0^x K(x,t)dt = \infty$ and $f(x) \leq B < \infty$ then $u(x) \longrightarrow 0$.

Proof: $\quad u(x) = f(x) - \int_0^x K(x,t)u(t)dt$

$$u(x) + u(x)\int_0^x K(x,t)dt \leq f(x)$$

$$u(x) \leq \frac{f(x)}{1 + \int_0^x K(x,t)dt}$$

$$\longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty \ .$$

BIBLIOGRAPHY

[1] P.R. BEESACK : Comparison theorems and inequalities for Volterra Integral Equations. Bull. Amer. Math. Soc., 20, (1969), 61-66 .

[2] F.G. TRICOMI Integral Equations, Interscience New York, 1957 .

[3] S.C. CHU and F.T. METCALF. On Gronwall's Inequality, Proc. Amer. Math. Soc., 18, (1967) 439-440 .

[4] W. POGORZELSKI, Integral Equations and their Applications Vol 1, Pergamon Prees, 1966 .

[5] E.A. CODDINGTON and N. LEVINSON, Theory of Ordinary Differential Equation McGRAW-HILL New York, 1955 .

[6] D. WILLETT A linear generalization of Gronwall's Inequalility. Proc, Amer. Math. Soc., 16, (1965) 774-778 .

[7] N. DISTEFANO A Volterra Integral Equation in the stability of some linear hereditary phenomena. J. Math. Anal and Appl, 23, (1968) 365-383 .