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SEQUENTIAL SPACE METHODS

by

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Abstract

The class of sequential spaces and its successive smaller subclasses, the Fréchet spaces and the first-countable spaces, have topologies which are completely specified by their convergent sequences. Because sequences have many advantages over nets, these topological spaces are of interest. Special attention is paid to those properties of first-countable spaces which can or cannot be generalized to Fréchet or sequential spaces. For example, countable compactness and sequential compactness are equivalent in the larger class of sequential spaces. On the other hand, a Fréchet space with unique sequential limits need not be Hausdorff, and there is a product of two Fréchet spaces which is not sequential. Some of the more difficult problems are connected with products. The topological product of an arbitrary sequential space and a T_3 (regular and T_1) sequential space X is sequential if and only if X is locally countably compact. There are also several results which demonstrate the non-productive nature of Fréchet spaces.

The sequential spaces and the Fréchet spaces are precisely the quotients and continuous pseudo-open images, respectively, of either (ordered) metric spaces or (ordered) first-countable spaces. These characterizations follow from those of the generalized sequential spaces and the generalized Fréchet spaces. The notions

of convergence subbasis and convergence basis play an important role here. Quotient spaces are characterized in terms of convergence subbases, and continuous pseudo-open images in terms of convergence bases. The equivalence of hereditarily quotient maps and continuous pseudo-open maps implies the latter result.

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Introduction

A first-countable space is a topological space whose open sets can be described by its convergent sequences alone. This is so by either of two properties of first-countable spaces ([16], Theorem 2.8):

(1) A set is open if and only if each sequence which converges to a point in the set is, itself, eventually in the set.

(2) A point lies in the closure of a set if and only if there is a sequence in the set converging to the point.

For more general spaces, it is often assumed that sequences are inadequate and that nets or filters must be used. There are, however, many topological spaces which do not satisfy the first axiom of countability and yet sequences suffice to determine open sets. The real line with the integers identified to one point is an example of such a space.

The topological spaces satisfying (1) are called sequential spaces and those satisfying (2), Fréchet spaces. Each first-countable space, and hence each metric space and each discrete space, is both Fréchet and sequential. Moreover, the real line with the integers identified is both a Fréchet space and a sequential space. Consequently, since (2) implies (1) but (1) does not imply (2), the concepts of Fréchet space and sequential space provide successive proper generalizations of first-countable space. In studying sequential spaces, one can restrict oneself to sequential convergence.

Accordingly, since the language of sequences has many advantages over that of nets, it is of interest to know when a topological space is sequential.

A result due to Ponomarev characterizes first-countable T_0-spaces as continuous open images of metric spaces. Analogously, S.P. Franklin [8] establishes that the sequential spaces are precisely the quotients of either metric spaces or first-countable spaces, and Arhangel'skiĭ [2] asserts that "among Hausdorff spaces, Fréchet spaces and only these, are continuous pseudo-open images of metric spaces". (The pseudo-open maps form a class between the open maps and the quotient maps.) In [22], P.R. Meyer extends Arhangel'skiĭ's result by eliminating the Hausdorff hypothesis : the Fréchet spaces are precisely the continuous pseudo-open images of either metric spaces or first-countable spaces. In order to obtain this result, he introduces the notions of convergence subbasis and convergence basis which provide the foundation for studying topological spaces whose open sets are completely specified by any given class of nets. Meyer's generalized sequential space methods are used to derive D.C. Kent's [18] characterizations of "spaces in which well ordered nets suffice."

Recently, many mathematicians have researched sequential spaces and generalized sequential spaces. The purpose of this thesis is to present the more important of their results in a unified theory. The author generalizes a few of these results and proves numerous statements asserted without proof in the original papers.

Chapter 1 is an investigation of sequential spaces, their properties, their characterization as quotients of metric spaces or first-countable spaces, and their relation to other topological properties. Their relation to the first-countable spaces is of particular interest. It is well-known that countable compactness and sequential compactness are equivalent in the class of first-countable spaces. Franklin asserts their equivalence in the larger class of sequential spaces. Franklin proves this result in [8] for Hausdorff spaces and in [10] for spaces with unique sequential limits. In this thesis, the author provides the proof of the same result for arbitrary topological spaces. The author also shows that any countable product of countably compact sequential spaces is countably compact. There are, however, many properties of first-countable spaces which cannot be generalized to sequential spaces. For example, the product of two sequential spaces need not be sequential. A result due to T.K. Boehme [3] shows that this situation cannot occur in the presence of suitable compactness conditions. In addition, E. Michael [23] has proved that for any T_3 sequential space X and sequential space Y, the topological product space $X \times Y$ is sequential if and only if X is locally countably compact.

The second chapter is concerned with Fréchet spaces, their properties, and their relation to sequential spaces. The characterization of sequential spaces is used to prove Arhangel'skiĭ's characterization of Hausdorff Fréchet spaces. In this section, the author proves Arhangel'skiĭ's assertion that the continuous pseudo-open maps and the hereditarily quotient maps are equivalent. The author also provides the proof of a result due to P.W. Harley III [12] concerning the product of two Fréchet spaces.

Chapter 3 is devoted to Meyer's generalized sequential space methods and his study of convergence subbases and m-sequential spaces. (An m-sequential space is a space for which m-nets (i.e., nets whose directed set is of cardinality $\leq$ m) suffice to determine closed sets.) A new characterization of the quotient topology is given in terms of convergence subbasis. This result leads to a characterization of the m-sequential spaces. The author proves an analogous result for continuous pseudo-open images which leads to Meyer's characterization of m-Fréchet spaces.

In the last chapter, the author employs many of the properties of convergence subbases to investigate weakly sequential spaces and m - sequential spaces (i.e., those spaces for which well-ordered nets and well-ordered m-nets, respectively, are sufficient to describe closed sets). These spaces are characterized in terms of ordered topological spaces (i.e., those spaces which have the order topology arising from a total order). Finally, there is a brief coda which demonstrates that the concepts of first-countable space, Fréchet space, and sequential space are equivalent in products of ordered spaces.

Notation

For the most part, the terminology and basic notation used in this thesis follows Kelley [16]. The exceptions are listed below.

(1) $X - A = \{x \in X : x \notin A\}$

(2) For any topological space X and subset A of X, $\mathrm{int}_X(A)$ denotes the interior of A with respect to X and $\mathrm{cl}_X(A)$ is the closure of A with respect to X. When no confusion seems possible these will be abbreviated to int A and cl A.

(3) $\mathcal{R}$ is the set of real numbers, $\mathcal{Z}$ is the set of integers, $\mathcal{Q}$ is the set of rationals, and $\mathcal{N} = \{1,2,3,\ldots\}$ is the set of natural numbers.

(4) ω is the first infinite ordinal and Ω is the first uncountable ordinal.

(5) For any ordinal α, $\alpha+1$ denotes the set of all ordinals which are less than or equal to α; that is, $\alpha+1$ is the successor ordinal of α.

(6) A topological space is said to be first-countable, or a first-countable space, if and only if it satisfies the first axiom of countability. Similarly, a topological space is second-countable, or a second-countable space, if and only if it satisfies the second axiom of countability.

Chapter 1

Sequential Spaces

Sequences have numerous advantages over nets. This is so because many properties of sequences fail to generalize to nets. For example, a converging sequence and its limit is compact, whereas this is not true for nets. Among Hausdorff spaces, each convergent sequence (i.e., the union of the sequence and its limit) satisfies the second axiom of countability and is therefore metrizable. These facts together with other properties of sequences not applicable to nets play a critical role in the investigation of sequential spaces.

This chapter is basically an amplification of Franklin's ([8], [9] and [10]) survey of sequential spaces. There are, however, several important results due to Boehme [3] and Michael [23] related to topological products.

1.1 Definition Let X be a topological space.

(1) A subset U of X is sequentially open if and only if each sequence in X converging to a point in U is eventually in U.

(2) A subset F of X is sequentially closed if and only if no sequence in F converges to a point not in F.

For any topological space, a subset A is closed if and only if no net in A converges to a point not in A ([16], Theorem 2.2). Therefore closed sets are sequentially closed and open sets are sequentially open. The converses need not be true.

<u>1.2 Example</u> There are sequentially open sets which are not open and sequentially closed sets which are not closed.

<u>Proof</u> Consider the ordinal topological space $\Omega + 1$ provided with the order topology. Let S be a sequence in $\Omega + 1$ which is not eventually equal to Ω. Then S is frequently in Ω and hence there is a subsequence S_o of S in Ω. But the supremum of S_o is less than Ω, and therefore S_o cannot converge to Ω. This implies that S cannot converge to Ω. Thus a sequence in $\Omega + 1$ converges to Ω if and only if it is eventually equal to Ω. Additionally, a sequence in Ω can converge only to a member of Ω. It follows that $\{\Omega\}$ is sequentially open and Ω is sequentially closed in $\Omega + 1$. But $\{\Omega\}$ is not open and Ω is not closed in $\Omega + 1$.

<u>1.3 Proposition</u> A subset of a topological space is sequentially open if and only if its complement is sequentially closed.

<u>Proof</u> If U is a sequentially open subset of a topological space X and S is a sequence in X–U converging to x, then $x \in X-U$. This is so because otherwise S is eventually in U. Thus X–U is sequentially closed. Conversely, suppose that F is a sequentially closed subset of X and let S_o be a sequence in X converging to $y \in X-F$. Then S_o is not frequently in F since otherwise there is a subsequence of S_o in F converging to $y \notin F$. Hence S_o is eventually in X–F, and therefore X–F is sequentially open.

<u>1.4 Proposition</u> For any topological space X, the collection of all sequentially open subsets forms a topology for the set X.

<u>Proof</u> Clearly, $\emptyset$ and X are sequentially open. If $\{U_a : a \in A\}$ is any family of sequentially open subsets of X and S is a sequence in X which converges to $x \in \bigcup\{U_a : a \in A\}$, then $x \in U_c$ for some $c \in A$. Consequently S is eventually in U_c and therefore in $\bigcup\{U_a : a \in A\}$. Hence $\bigcup\{U_a : a \in A\}$ is sequentially open. Suppose now that U and V are sequentially open, and let $\{y_n : n \in \omega\}$ be a sequence in X converging to a point in $U \cap V$. Then $\{y_n : n \in \omega\}$ is eventually in both U and V, and there exists $n_o, n_1 \in \omega$ with $\{y_n : n \geq n_o\} \subset U$ and $\{y_n : n \geq n_1\} \subset V$. So $y_n \in U \cap V$ for all $n \geq \sup\{n_o, n_1\}$. Thus $U \cap V$ is sequentially open, and the proof is complete.

1.5 Definition The set of all sequentially open subsets of a topological space is said to be the sequential closure topology.

1.6 Definition A topological space is sequential, or a sequential space, if and only if each sequentially open subset is open. (In view of (1.3) and (1.5), it is clear that a topological space is sequential if and only if each sequentially closed subset is closed, or equivalently, if and only if its topology coincides with the sequential closure topology.)

In first-countable spaces, a set is open if and only if each sequence converging to a point in the set is, itself, eventually in the set ([16], Theorem 2.8). Therefore first-countable spaces, and hence metric spaces and discrete spaces, are sequential. On the other hand, by virtue of (1.2), the ordinal space $\Omega + 1$ provided with the order topology is not sequential.

After a few preliminary results, several equivalent formulations for the notion of sequential space are given.

1.7 Definition Let X and Y be topological spaces, and let T be the topology on X. The space Y divides X if and only if no topology T_α on X which is strictly larger than T leaves every T-continuous function from Y into X T_α-continuous.

<u>1.8 Proposition</u> Let X and Y be topological spaces, let T be the topology on X, and let $T_\alpha = \{B \subset X : f^{-1}(B)$ is open in Y for each T-continuous function $f : Y \longrightarrow X\}$. The space Y divides X if and only if $T = T_\alpha$.

<u>Proof</u> Since inverse set functions preserve set operations, it is clear that T_α is a topology on X. Furthermore, $T \subset T_\alpha$ and every T-continuous function from Y into X is T_α-continuous. If Y divides X, then $T_\alpha \subset T$ and so $T = T_\alpha$. Conversely, suppose that T_β is any topology for X which leaves every T-continuous function from Y into X T_β-continuous. If $g : Y \longrightarrow X$ is a T-continuous function and $B \in T_\beta$, then $g^{-1}(B)$ is open in Y and hence $B \in T_\alpha$. Thus $T = T_\alpha$ implies that Y divides X.

<u>1.9 Lemma</u> A mapping f of the ordinal space $\omega + 1$, provided with its order topology, into a topological space X is continuous if and only if the sequence $\{f(n) : n \in \omega\}$ converges to $f(\omega)$ in X.

<u>Proof</u> If $f : \omega + 1 \longrightarrow X$ is continuous and U is any neighbourhood of $f(\omega)$, $f^{-1}(U)$ is a neighbourhood of ω in $\omega + 1$. Then $f^{-1}(U)$ contains $(m, \omega] = \{n : m < n \in \omega\}$ for some $m \in \omega$. Therefore $\{f(n) : n \in \omega\}$ is eventually in U and hence $\{f(n) : n \in \omega\}$ converges

to $f(\omega)$. Conversely, suppose that V is an open subset of X. If $f(\omega) \notin V$ then $f^{-1}(V) = \{n \in \omega : f(n) \in V\} = \bigcup\{\{n\} : n \in \omega, f(n) \in V\}$, which is the union of open subsets of $\omega + 1$. If $f(\omega) \in V$, $\{f(n) : n \in \omega\}$ is eventually in V and consequently there exists $p \in \omega$ such that $f(n) \in V$ for each $n \geq p$. Therefore $f^{-1}(V) = (p, \omega] \bigcup \{n : p \geq n \in \omega, f(n) \in V\}$ is open in $\omega + 1$. The lemma is proved.

<u>1.10 Definition</u> A convergent sequence is the union of the sequence and all of its limit points. (Let S be a convergent sequence in a topological space X, and let S_o denote the range of S provided with the relative topology. The topology on S is the largest topology in which the natural function $f : S \longrightarrow S_o$ defined by $f(x) = x$ is open.)

<u>1.11 Lemma</u> Every convergent sequence in a Hausdorff space is compact and metrizable.

<u>Proof</u> Let $S = \{x_n : n \in \omega\} \bigcup \{x\}$ be a convergent sequence in a Hausdorff space, and suppose that $\mathcal{U}$ is an open covering for S. Now $x \in U$ for some $U \in \mathcal{U}$. Furthermore, $\{x_n : n \in \omega\}$ is eventually in U and thus $x_n \in U$ whenever $n \geq m$ for some $m \in \omega$. For each $n < m$ choose $U_n \in \mathcal{U}$ such that $x_n \in U_n$. Then $\{U\} \bigcup \{U_n : m > n \in \omega\}$

is a finite subcovering of $\mathcal{U}$ for S, and so S is compact. To see that S is a metric space, let $V_k = S-\{x_n : k > n \in \omega\}$ for each $k \in \omega$. The collection $\{V_n : n \in \omega\}$ is clearly a countable neighbourhood basis at x. Because S is compact Hausdorff and hence a regular T_1-space, there exists open sets U and V satisfying $x \in V \subset \text{cl } V \subset U$ and $x_i \notin U$. Consequently $\{x_i\} = S-(\text{cl } V \cup \{x_n : i \neq n \in \omega, x_n \notin \text{cl } V\})$, which is open in S. The family $\{V_n : n \in \omega\} \cup \{\{x_n\} : n \in \omega\}$ is a countable open basis for the topology on S. Therefore S is a second-countable regular T_1-space. In view of ([16], Theorem 4.17), S is metrizable.

Observe that there is a metric for the convergent sequence S defined by $d(x_m, x_n) = |1/m - 1/n|$ and $d(x_m,x) = 1/m$.

<u>1.12 Theorem</u> For any topological space X, properties (1) and (2) are equivalent. If X is Hausdorff they are also equivalent to (3) and (4).

(1) X is sequential.

(2) $\omega + 1$, provided with its order topology, divides X.

(3) Each subset of X which intersects every convergent sequence in a closed set is closed.

(4) Each subset of X which intersects every compact metric subspace of X in a closed set is closed.

<u>Proof</u> (1) <---> (2) Suppose that U is a subset of X with $f^{-1}(U)$ open in $\omega + 1$ for each continuous function $f : \omega + 1 \longrightarrow X$. Let $\{x_n : n \in \omega\}$ be a sequence in X converging to $x \in U$. Define $g(\omega) = x$ and $g(n) = x_n$ for each $n \in \omega$. Then $\{g(n) : n \in \omega\}$ converges to $g(\omega)$ and it follows from (1.9) that $g : \omega + 1 \longrightarrow X$ is a continuous function. Thus $g^{-1}(U)$ is an open subset of $\omega + 1$ containing ω, which implies that $g^{-1}(U)$ contains $(m,\omega]$ for some $m \in \omega$. So $x_n = g(n) \in U$ for each $n \geq m + 1$, and hence $\{x_n : n \in \omega\}$ is eventually in U. Therefore U is a sequentially open subset of X. If X is sequential, U is open and consequently, by virtue of (1.8), $\omega + 1$ divides X.

Assume now that U is a sequentially open subset of X, and let $f : \omega + 1 \longrightarrow X$ be a continuous function. According to (1.9), $\{f(n) : n \in \omega\}$ converges to $f(\omega)$. If $f(\omega) \in U$ then $f^{-1}(U)$ contains $(k,\omega]$ for some $k \in \omega$. So $f^{-1}(U) = (k,\omega] \cup \{n : k \geq n \in \omega, f(n) \in U\}$ which is open in $\omega + 1$. If $f(\omega) \notin U$, $f^{-1}(U) = \{n \in \omega : f(n) \in U\}$ is open in $\omega + 1$. Therefore $f^{-1}(U)$ is open in $\omega + 1$; then U is open if $\omega + 1$ divides X.

(1) <---> (3) Suppose that F is a subset of X, and let $S = \{x_n : n \in \omega\} \cup \{x\}$ be a convergent sequence in X. Either $F \cap S$ is finite or infinite. In the first case, $F \cap S$ is obviously compact. In the second case, F contains a subsequence of $\{x_n : n \in \omega\}$.

Therefore, if F is sequentially closed, $x \in F$. Hence $F \cap S$ is compact because $(F \cap S)-U$ is finite for any open set U containing x. Now let S_y be a net in $F \cap S$ converging to $y \in X$. Since $F \cap S$ is compact, S_y has a cluster point in $F \cap S$. But, the Hausdorff hypothesis implies that y is the only cluster point of S_y. Evidently, $F \cap S$ is closed. Thus, if a subset of X is sequentially closed, it intersects every convergent sequence in a closed set. Conversely, if $\{x_n : n \in \omega\}$ is contained in F, $\{x_n : n \in \omega\}$ is also a sequence in $F \cap S$. If $F \cap S$ is closed then $x \in F \cap S$ and hence $x \in F$. Consequently, a subset of X is sequentially closed if and only if it intersects every convergent sequence in a closed set. The equivalence of (1) and (3) now follows immediately.

(3) <---> (4) Suppose that F is a subset of X intersecting every compact metric subspace of X in a closed set. According to (1.11), each convergent sequence is a compact metric subspace of X. Therefore F intersects every convergent sequence in a closed set, and hence (3) implies that F is closed. To establish the converse, assume that E is a subset of X intersecting every convergent sequence in a closed set. Let X be a compact metric subspace of X. Since X is Hausdorff, K is also closed. If S is a sequence in $E \cap K$ converging to x, then $x \in K$. Additionally, $x \in E$ because $E \cap (S \cup \{x\})$ is closed. Thus $E \cap K$ is a sequentially closed subset of the closed metric subspace K; consequently $E \cap K$ is closed, and the proof is complete.

The elementary properties of sequential spaces are summarized in the following theorem.

1.13 Theorem (1) A function $f : X \longrightarrow Y$ of a sequential space X into a topological space Y is continuous if and only if $\{f(x_n) : n \in \omega\}$ converges to $f(x)$ whenever $\{x_n : n \in \omega\}$ converges to x.

(2) Every quotient of a sequential space is sequential.

(3) The continuous open or closed image of a sequential space is sequential.

(4) The cartesian product of sequential spaces need not be sequential. However, if the product is sequential, so is each of its coördinate spaces.

(5) The disjoint topological sum of any family of sequential spaces is sequential.

(6) The inductive limit of any family of sequential spaces is sequential.

(7) A subspace of a sequential space need not be sequential. An open or closed subspace, however, is sequential.

(8) Every locally sequential space is sequential.

Proof (1) The necessity of the condition is true for arbitrary topological spaces. If f is continuous and U is an open subset of Y containing $f(x)$, then $f^{-1}(U)$ is an open subset of X containing x. Moreover, $\{x_n : n \in \omega\}$ is eventually in $f^{-1}(U)$ and so $f(x_n) \in U$ for all n sufficiently large. Hence $\{f(x_n) : n \in \omega\}$ converges to $f(x)$. Conversely, suppose that V is an open subset of Y and let $\{y_n : n \in \omega\}$ be a sequence in X converging to $y \in f^{-1}(V)$. By hypothesis, $\{f(y_n) : n \in \omega\}$ converges to $f(y)$ and so eventually, $f(y_n) \in V$. But then $\{y_n : n \in \omega\}$ is eventually in $f^{-1}(V)$. It follows that $f^{-1}(V)$ is a sequentially open subset of X. Then, since X is sequential, $f^{-1}(V)$ is open and hence f is continuous.

(2) Let $f : X \dashrightarrow Y$ be a quotient map of a sequential space X onto a topological space Y. Suppose that U is a sequentially open subset of Y and that $\{x_n : n \in \omega\}$ is a sequence in X converging to $x \in f^{-1}(U)$. Then, since f is continuous, $\{f(x_n) : n \in \omega\}$ converges to $f(x) \in U$. Consequently $\{x_n : n \in \omega\}$ is eventually in $f^{-1}(U)$, which implies that $f^{-1}(U)$ is a sequentially open subset of the sequential space X. Therefore $f^{-1}(U)$ is open and hence, by definition of the quotient topology, U is open.

(3) By ([16], Theorem 3.8), if f is a continuous open or closed map of a topological space X onto a space Y, then Y is the quotient space relative to f and X. It follows from part (2) that if X is a sequential space then the image $f(X) = Y$ is sequential.

(4) An example of a non-sequential product of sequential spaces will be given in (1.19). To prove the second part of (4), let X be the cartesian product of any family $\{X_a : a \in A\}$ of topological spaces. For each $c \in A$ let $P_c : X \longrightarrow X_c$ denote the projection map of X onto its coordinate space X_c. From the definition of the product topology on X, P_c is continuous. Furthermore, according to ([16], Theorem 3.2), the projection of a product space onto each of its coordinate spaces is open. Hence P_c is a continuous open surjection. Thus, if X is sequential, part (3) implies that X_c is sequential.

(5) Let X be the disjoint topological sum of any family $\{X_a : a \in A\}$ of sequential spaces. If U is not open in X, there exists $c \in A$ such that $U \cap X_c$ is not open and hence not sequentially open in X_c. Consequently, there is a point $x \in U \cap X_c$ and a sequence in $X_c - U$ converging to x with respect to X_c and therefore with respect to X. Then U is not sequentially open and the contrapositive of "each sequentially open subset of X being open" is established.

(6) Assuming that $(A, <)$ is a directed set, let $\{X_a, \phi_{ab} : a, b \in A; a < b\}$ denote the family $\{X_a : a \in A\}$ of sequential spaces together with the set of continuous maps

$\phi_{ab} : X_a \dashrightarrow X_b$ satisfying the condition : if $a < b < c$ then $\phi_{ac} = \phi_{bc} \circ \phi_{ab}$. By definition, the inductive limit of $\{X_a : a \in A\}$ is the quotient space $X/_R$ where X is the disjoint topological sum of $\{X_a : a \in A\}$ and R is the equivalence relation : two elements $x_a \in X_a$, $x_b \in X_b$ in X are equivalent if and only if there exists $c \in A$ such that $a < c$, $b < c$ and $\phi_{ac}(x_a) = \phi_{bc}(x_b)$. It follows from parts (5) and (2) that $X/_R$ is sequential.

(7) The non-hereditary nature of sequential spaces will be demonstrated in (1.15) and (1.17). To prove the second part of (7), assume first that Y is an open subspace of a sequential space X and let U be a sequentially open subset of Y. If $S = \{x_n : n \in \omega\}$ is a sequence in X converging to $x \in U \subset Y$ then, since Y is open, S is eventually in Y. There exists $m \in \omega$ such that $x_n \in Y$ for each $n \geq m$. Moreover, $\{x_{m+n} : n \in \omega\}$ is a sequence in Y converging to $x \in U$. Then, since U is sequentially open in Y, $\{x_{m+n} : n \in \omega\}$ is eventually in U. This surely implies that S is eventually in U. Hence U is sequentially open and therefore open in X.

Assume now that Y is a closed subspace of the sequential space X and let F be a sequentially closed subset of Y. Suppose that S is a sequence in F converging to y with respect to X.

Because Y is closed in X, $y \in Y$ and consequently S converges to y in Y. Since F is sequentially closed in Y, $y \in F$. Thus F is sequentially closed in X, and so $F = F \cap Y$ is closed in Y.

(8) Let U be a sequentially open subset of a locally sequential space X. If G is any sequential neighbourhood of $x \in U$, int G is sequential by part (7). Let $V = (\text{int } G) \cap U$. It is clear that V is sequentially open and hence open in int G. But then V is open in X. By hypothesis, there exists a collection $\{G_x : x \in U\}$ of sequential neighbourhoods satisfying $x \in G_x$. For each $x \in U$, $V_x = (\text{int } G_x) \cap U$ is open in X. Therefore $U = \bigcup \{V_x : x \in U\}$ is open, and (8) is established.

As previously stated, first-countable spaces are sequential. The following shows that not all sequential spaces are first-countable.

<u>1.14 Example</u> There is a sequential space which is not first-countable.

<u>Proof</u> Let X be the real line $\mathbb{R}$ with the integers $\mathbb{Z}$ identified to the point 0. From (1.13.2), X is sequential. Suppose that $\{U_i : i \in \omega\}$ is a countable neighbourhood basis at 0 in X. Since each U_i is obviously open in $\mathbb{R}$, there exists a collection $\{V_n : n \in \omega\}$ of open intervals satisfying $n \in V_n \subset U_n$. For each

$n \in \omega$ choose an open interval I_n with $n \in I_n \subsetneq V_n$. Then $\{x \in R : x < 0\} \cup (\bigcup\{I_n : n \in \omega\})$ is an open neighbourhood of 0 in X which does not contain U_n for any $n \in \omega$. Hence X cannot be first-countable.

<u>1.15 Example</u> A subspace of a sequential space need not be sequential.

<u>Proof</u> Let X be the real numbers provided with the topology generated by its usual topology and all sets of the form $\{0\} \cup U$ where U is a usual open neighbourhood of the sequence $\{\frac{1}{n+1} : n \in \omega\}$. The topology of the real line is altered only at 0. For each open subset G of X, $\{0\} \cup G$ is open if and only if $\{\frac{1}{n+1} : n \in \omega\}$ is eventually in G. Accordingly, each sequence in X converging to 0 is either eventually in $\{0\}$ or eventually in every neighbourhood of $\{\frac{1}{n+1} : n \in \omega\}$, and hence is either eventually equal to 0 or a subsequence of $\{\frac{1}{n+1} : n \in \omega\}$.

Define a subspace $Y = \{(x, 0) : 0 \neq x \in R\} \cup \{(\frac{1}{n+1}, 1)\ n \in \omega\} \cup \{(0, 1)\}$ of the plane. The space Y is the disjoint topological sum of the punctured real line $\{(x, 0) : 0 \neq x \in R\}$ and the convergent sequence $\{(\frac{1}{n+1}, 1) : n \in \omega\} \cup \{(0, 1)\}$. Since both $\{(x, 0) : 0 \neq x \in R\}$

and $\{(\frac{1}{n+1}, 1) : n \in \omega\} \cup \{(0, 1)\}$ are first-countable, Y is sequential. The relative topology for Y is generated by sets of the form $\{(\frac{1}{m+1}, 1)\}$, $\{(0, 1)\} \cup \{(\frac{1}{n+1}, 1) : m \leq n \in \omega\}$, and $(U-\{0\}) \times \{0\}$ where $m \in \omega$ and U is a usual open subset of the real line R. Let $P : Y \longrightarrow X$ be the surjection defined by $P(x, y) = x$ for each $(x, y) \in Y$. To establish that X is sequential, it suffices to prove that P is a quotient map. It is first shown that P is continuous. Let U be an open subset of X. If $0 \notin U$, U is open in R and consequently $P^{-1}(U) = (U \times \{0\}) \cup \{(\frac{1}{n+1}, 1) : n \in \omega, \frac{1}{n+1} \in U\}$ is open in Y. If $0 \in U$, $U = \{0\} \cup G$ where G is an open subset of the real line such that $\{\frac{1}{n+1} : n \in \omega\}$ is eventually in G. Assume $\frac{1}{n+1} \in G$ whenever $n \geq m \in \omega$. Then

$P^{-1}(U) = (G-\{0\} \times \{0\}) \cup (\{(0, 1)\} \cup \{(\frac{1}{n+1}, 1) : m \leq n \in \omega\})$

$\cup \{(\frac{1}{n+1}, 1) : m > n \in \omega, \frac{1}{n+1} \in U\}$, which is open in Y. Hence P is continuous. Now let V be a subset of X such that $P^{-1}(V)$ is open in Y. If $0 \notin V$ then $P^{-1}(V) = \{(\frac{1}{n+1}, 1) : n \in \omega, \frac{1}{n+1} \in V\} \cup (V \times \{0\})$, which is open in Y if and only if V is open in R. If $0 \in V$, $(0, 1) \in P^{-1}(V)$ and so there exists $k \in \omega$ such that $(\frac{1}{n+1}, 1) \in P^{-1}(V)$ for each $n \geq k$. Then $P^{-1}(V) = (\{(0,1)\} \cup \{(\frac{1}{n+1}, 1) : k \leq n \in \omega\}) \cup \{(\frac{1}{n+1}, 1) : k > n \in \omega, \frac{1}{n+1} \in V\} \cup ((V-\{0\}) \times \{0\})$. Since $P^{-1}(V)$ is open in Y, $V-\{0\}$ is an open subset of the real line containing $\frac{1}{n+1}$ for each $n \geq k$, and consequently $V = \{0\} \cup (V-\{0\})$ is open in X. Hence P is a quotient map, and therefore X is sequential.

Consider the subspace $X-\{\frac{1}{n+1} : n \in \omega\}$. Because each sequence in $X-\{\frac{1}{n+1} : n \in \omega\}$ converging to 0 must be eventually equal to 0, $\{0\}$ is sequentially open. But then, since $\{0\}$ is not open, $X-\{\frac{1}{n+1} : n \in \omega\}$ is a non-sequential subspace of the sequential space X.

<u>1.16 Example</u> (1) The continuous image of a sequential space need not be sequential.

(2) The open and closed image of a sequential space need not be sequential.

<u>Proof</u> (1) Let (X, T) and (X, T_α) be topological spaces with the discrete topology and a non-sequential topology on X respectively. The identity map $1_X : (X, T) \dashrightarrow (X, T_\alpha)$ is a continuous surjection of the sequential space (X, T) onto the non-sequential space (X, T_α). In particular, the continuous image of the identity map of $\Omega + 1$, provided with its discrete topology, onto itself, provided with its order topology, is not sequential.

(2) Let R be the real line and let X be the topological space of (1.15). The identity mapping of the first-countable space $R-\{\frac{1}{n+1} : n \in \omega\}$ onto the non-sequential space $X-\{\frac{1}{n+1} : n \in \omega\}$ is an open and closed surjection.

The topological space M of the next example is important for later reference.

<u>1.17 Example</u> There is a countable, T_4 (normal and T_1) sequential space with a non-sequential subspace.

<u>Proof</u> Let $M = (N \times N) \cup N \cup \{0\}$ with each $(m, n) \in N \times N$ an isolated point, where N denotes the set of natural numbers. For a basis of neighbourhoods at $n \in N$, take all sets of the form $\{n\} \cup \{(n, m) : m \geq q\}$ where $q \in N$. Define a subset U to be a neighbourhood of 0 if and only if $0 \in U$ and U is a neighbourhood of all but finitely many natural numbers. Clearly, M is countable and Hausdorff. To establish that M is normal, let G be an open subset of M containing the closed subset A. If $0 \notin A$, choose $\{m_n \in N : n \in N \cap A\}$ such that $V_n = \{n\} \cup \{(n, m) : m \geq m_n\}$ is contained in G. Since $0 \notin A$ and A is closed, $N \cap A$ is finite. Therefore, $V = [\, \cup \{V_n : n \in N \cap A\}] \cup [(N \times N) \cap A]$ is open and $A \subset V \subset \mathrm{cl}\, V \subset G$. Suppose now that $0 \in A$. Then, choosing $\{m_n \in N : n \in N \cap G\}$ such that $U_n = \{n\} \cup \{(n, m) : m \geq m_n\}$ is contained in G, $U = [\, \cup \{U_n : n \in N \cap G\}] \cup [(N \times N) \cap A] \cup \{0\}$ is open and $A \subset U \subset \mathrm{cl}\, U \subset G$. Hence M is normal.

To prove that M is sequential, let U be a sequentially open subset of M. For each $x \in (N \times N) \cap U$, $x \in \text{int } U$ since $\{x\}$ is open. For each $x \in N \cap U$, $\{(x, m+1) : m \in \omega\}$ is a sequence in M converging to x. Then, since U is sequentially open, there exists $m_x \in \omega$ such that $V_x = \{x\} \cup \{(x, m) : m \geq m_x\}$ is contained in U. But V_x is a neighbourhood of x, and therefore $x \in \text{int } U$. If $0 \in U$ then $N-U$ is finite because otherwise $N-U$ contains a sequence converging to 0. Consequently $\{0\} \cup (\bigcup\{V_x : x \in N \cap U\})$ is a neighbourhood of 0 contained in U. Hence U is open, and so M is sequential.

Since $0 \in cl_M(N \times N)$, $\{0\}$ is not open in $M-N$. If $\{(n_i, m_i) : i \in \omega\}$ is any sequence in $N \times N$, either there is some $n \in N$ such that $n_i = n$ for infinitely many i or there is no such n. In the first case, $\{(n_i, m_i) : i \in \omega\}$ has a cluster point in the set $\{n\} \cup \{(n, m) : m \in N\}$. Indeed, either there exists $m \in N$ such that $m_i = m$ for infinitely many i or there is no such m. It follows that the subsequence $\{(n, m_i) : i \in \omega\}$ has a cluster point at either (n, m) or n. Then, since M is Hausdorff, $\{(n_i, m_i) : i \in \omega\}$ cannot converge to 0 in the first case. In the second case, $\{(n_i, m_i) : i \in \omega\}$ has a subsequence in which each point has a distinct first coordinate. Without loss of generality it can be assumed that the sequence $\{(n_i, m_i) : i \in \omega\}$ has distinct first coordinates. Choose a sequence $\{k_i \in N : i \in \omega\}$ such that $k_i > m_i$. For each $i \in \omega$ let $V_i = \{n_i\} \cup \{(n_i, m) : m \geq k_i\}$, and for each $n \in N-\{n_i : i \in \omega\}$ let $U_n = \{n\} \cup \{(n, m) : m \in N\}$. Then $(\cup \{V_i : i \in \omega\}) \cup (\cup \{U_n : n \in N-\{n_i : i \in \omega\}\}) \cup \{0\}$ is a neighbourhood of 0 disjoint from $\{(n_i, m_i) : i \in \omega\}$. Accordingly, a sequence in $M-N$ converges to 0 if and only if it is eventually equal to 0. Therefore $\{0\}$ is sequentially open in the subspace $M-N$, and hence $M-N$ is not sequential.

The topological space $M-N$ is a countable Hausdorff space which is not sequential. The following shows that such a space must fail to be locally compact.

<u>1.18 Proposition</u> Every countable, locally compact Hausdorff space is first-countable (and hence sequential).

<u>Proof</u> Let X be a countable, locally compact Hausdorff space. Then X is regular. Let $x \in X$. By hypothesis, there exists a compact neighbourhood K of x. The subspace K is regular and compact. From the regularity condition, there is a collection $\mathcal{U} = \{U_y : x \neq y \in X\}$ of neighbourhoods of x satisfying $y \notin \text{cl } U_y \subset K$. Clearly $\{x\} = \bigcap \mathcal{U}$. The family $\mathcal{B}$ of all finite intersections of

members of $\mathcal{U}$ is a neighbourhood basis at x. This is so because otherwise there exists an open neighbourhood V of x such that no member of $\mathcal{B}$ is contained in V. But then, the intersection of the closed subset K-V with any finite intersection of $\{\mathrm{cl}\ U_y : y \in X-\{x\}\}$ is non-empty and yet $(K-V) \cap \{\mathrm{cl}\ U_y : y \in X-\{x\}\} = \emptyset$; this contradicts K being compact. To complete the proof, it is only necessary to establish that $\mathcal{B}$ is countable. Let $A_i = \{n : i \geq n \in \mathbb{N}\}$. There is a one-to-one correspondence between the set of functions $\mathcal{U}^{A_i} = \{f : A_i \longrightarrow \mathcal{U};\ f \text{ is a function}\}$ and the set of all finite intersections of i elements of $\mathcal{U}$. Consequently the cardinality of $\mathcal{B}$ is less than or equal to the cardinality of $\bigcup \{\mathcal{U}^{A_i} : i \in \mathbb{N}\}$. The cardinality of $\mathcal{U}^{A_i}$ is $\aleph_0^i = \aleph_0$, and hence the cardinality of $\mathcal{B}$ is $\leq \aleph_0 \cdot \aleph_0 = \aleph_0^2 = \aleph_0$ ([16], Theorem 179, page 279).

Since any countable product of first-countable spaces is first-countable, it is natural to ask if there is an analogous result for sequential spaces. This question is answered negatively by the succeeding example. Indeed, the product of two sequential spaces need not be sequential. The construction used in this example is slightly different than that derived by Franklin ([8], Example 1.11). Using this construction, it is also possible to prove that the square of a sequential space need not be sequential.

1.19 Example   There is a product of two sequential spaces which is not sequential.

Proof Let $\mathcal{Q}'$ be the rationals $\mathcal{Q}$ with the integers identified, and let $X = \mathcal{Q} \times \mathcal{Q}'$. The space X is the product of two sequential spaces but contains a sequentially open set W which is not open. To describe W, let $\{x_n : n \in \omega\}$ be a sequence of irrational numbers less than one converging monotonically downward to 0. For each $n \in \omega$, let H_n be the interior of the plane rhombus determined by the points $(-x_n, n)$, $(0, n + \frac{1}{2})$, (x_n, n) and $(0, n - \frac{1}{2})$; let J_n be the interior of the triangle determined by the points (x_n, n), $(1, n + \frac{1}{2})$ and $(1, n - \frac{1}{2})$; and let K_n be the reflection of J_n in the y-axis. Then

$$W_n = H_n \cup J_n \cup K_n \cup \{(x, y) \in R^2 : |x| > x_o\} \cup \{(x, y) \in R^2 : y < 0\}$$

is an open subset of the plane. Thinking of $\cup \{W_n : n \in \omega\}$ as a subset of the plane with the horizontal integer lines identified, let $W = X \cap (\cup \{W_n : n \in \omega\})$. (See Figure 1)

Let $P_1 : X \longrightarrow \mathcal{Q}$ and $P_2 : X \longrightarrow \mathcal{Q}'$ be the canonical projections. For any open neighbourhoods U and U' of 0 in $\mathcal{Q}$ and $\mathcal{Q}'$ respectively, $P_1^{-1}(U) \cap P_2^{-1}(U')$ is not contained in W because there exists $m \in \omega$ such that $P_1^{-1}(U) \cap P_2^{-1}(U' \cap \{x : m - \frac{1}{2} < x < m + \frac{1}{2}\}$ is not contained in W_m. Therefore $(0, 0) \notin \text{int } W$, and hence W is not open in X.

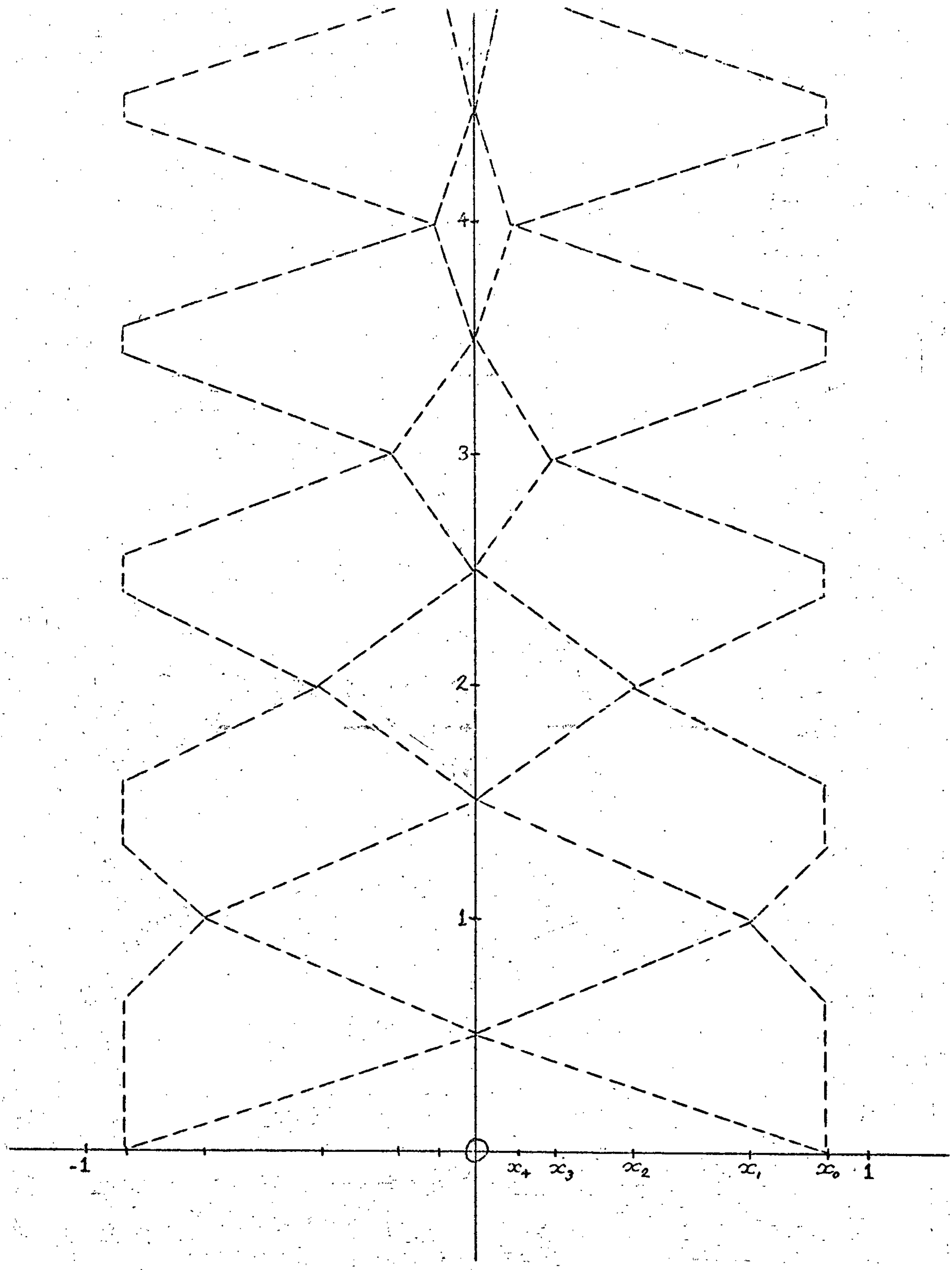

Figure 1: The Graph of $\bigcup\{W_n : n \in \omega\}$ in R^2 (See Example 1.19)

To establish that W is sequentially open, let $\{y_n : n \in \omega\}$ be a sequence in X converging to $y \in W$. If $P_2(y) \neq 0$, convergence in X is simply convergence in $Q \times Q$ and, since $(Q \times Q) \cap (\bigcup\{W_n : n \in \omega\})$ is open in $Q \times Q$, $\{y_n : n \in \omega\}$ is eventually in W for this case. Assuming that $P_2(y) = 0$, if $P_1(y) \neq 0$ then W can be replaced by a scaled down version of itself, in W, with y at the symmetric position. Therefore it can be assumed without loss of generality that $y = (0, 0)$. Now $\{y_n : n \in \omega\} \longrightarrow (0, 0)$ implies that $\{P_2(y_n) : n \in \omega\} \to 0$ in Q'. But then, if P is the quotient map of Q onto Q' and K is the set of integers k such that $\{P^{-1} \circ P_2(y_n) : n \in \omega\}$ is frequently in $U-\{k\}$ for each neighbourhood U of k, $\{P_2(y_n) : n \in \omega\}$ is eventually in $\{0\} \cup V$ where V is any neighbourhood of K. Furthermore, K is finite because otherwise $\{P_2(y_n) : n \in \omega\}$ has a subsequence not converging to 0. To verify that K is finite, let V be a neighbourhood of K and suppose that $K = \{k_n : n \in \omega\}$ where $k_n < k_m$ if and only if $n < m$. There is a sequence $\{I_i : i \in \omega\}$ of open intervals satisfying (1) $I_i \subset V$, (2) $Z \cap I_i = \{k_i\}$, and (3) $I_i \cap I_j \neq \emptyset$ if and only if $i = j$. For each $i \in \omega$ there exists $n_i \in \omega$ such that $y_{n_i} \in I_i$. Next, let $\{U_n : n \in \omega\}$ be a sequence of open sets satisfying $k_i \in U_i \subset I-\{y_{n_i}\}$, and let G be a neighbourhood of Z-K disjoint from $\bigcup \{I_i : i \in \omega\}$. It follows that $G \cup (\bigcup \{U_n : n \in \omega\})$ is a neighbourhood of 0 in Q' disjoint from the sequence $\{P_2(y_{n_i}) : i \in \omega\}$. Therefore, $\{P_2(y_{n_i}) : i \in \omega\}$ is a subsequence of $\{P_2(y_n) : n \in \omega\}$ not converging to 0, and hence K must be finite. Let $q = \sup \{k : k \in K\}$. Since $\{P_1(y_n) : n \in \omega\}$ converges to 0 in Q and $\{P_2(y_n) : n \in \omega\}$ is eventually in $\{0\} \cup V$ for any neighbourhood V of K, $\{y_n : n \in \omega\}$ is eventually in $E = [X \cap (\bigcup \{W_n : n \leq q\})] \cup (Q \times \{0\})$. But E is contained in W. Thus W is sequentially open, and this completes the proof.

Defining Q' and W_n as above and thinking of $\bigcup \{W_n : n \in \omega\}$ as a subset of the plane with the horizontal integer lines identified and the vertical integer lines identified, it is not difficult to see from (1.19) that $(Q' \times Q') \cap (\bigcup \{W_n : n \in \omega\})$ is a sequentially open subset of $Q' \times Q'$ which is not open. Hence $Q' \times Q'$ is not sequential, and therefore the square of a sequential space need not be sequential.

After a few preliminary results it will be shown that the situation described in (1.19) cannot occur in the presence of suitable compactness and separation conditions. First, it is convenient to prove that countable compactness and sequential compactness are equivalent in sequential spaces. As is well-known

([28], Proposition 9.8), these concepts are equivalent in the class of first-countable spaces. Since sequentially compact spaces are always countably compact, the following establishes their equivalence in the larger class of sequential spaces. The proof is provided by the author.

<u>1.20 Theorem</u> Every countably compact sequential space is sequentially compact.

<u>Proof</u> Let X be sequential and countably compact, and suppose that $S = \{x_n : n \in \omega\}$ is a sequence in X with no convergent subsequence. Let $A = \bigcup \{\mathrm{cl}\, x_n : n \in \omega\}$. If $S_o = \{y_n : n \in \omega\}$ is a sequence in A converging to y, either S_o is frequently in $\mathrm{cl}\, x_m$ for some $m \in \omega$, or no such m exists. In the first case, $y \in \mathrm{cl}\, x_m \subset A$. In the second case, there exists a subsequence $\{y_{n_k} : k \in \omega\}$ of S_o with $y_{n_k} \in \mathrm{cl}\, x_{n_k}$. But then $\{x_{n_k} : k \in \omega\}$ is a subsequence of S converging to y. The second case, therefore, cannot occur and so $y \in A$. From this it follows that A is sequentially closed and hence closed. Since X is countably compact, A is countably compact and consequently S has a cluster point $x \in A$. Now $x \in \mathrm{cl}\, x_n$ for only finitely many $n \in \omega$ because otherwise S would have a convergent sequence. Let $k \in \omega$ be such that $x \notin \mathrm{cl}\, x_n$ whenever $n \geq k$. But then, applying the same argument at above,

$\bigcup\{\mathrm{cl}\ x_n : n \geq k\}$ is sequentially closed and yet does not contain its accumulation point x.

A result due to Novák [27] demonstrates that the product of two countably compact spaces need not be countably compact. The following shows that one of the spaces being sequential is enough.

<u>1.21 Corollary</u> The product of two countably compact spaces, one of which is sequential, is countably compact.

<u>Proof</u> Let $\{(x_n, y_n) : n \in \omega\}$ be a sequence in the topological product space $X \times Y$ of a countably compact space X and a countably compact sequential space Y. By virtue of (1.20), Y is sequentially compact. Accordingly, the sequence $\{y_n : n \in \omega\}$ has a subsequence $\{y_{n_k} : k \in \omega\}$ which converges to some point $y \in Y$. Since X is countably compact, $\{x_{n_k} : k \in \omega\}$ has a cluster point $x \in X$. Then (x, y) is a cluster point of the sequence $\{(x_n, y_n) : n \in \omega\}$.

<u>1.22 Corollary</u> Let X be the topological product of any countable family $\{X_n : n \in \omega\}$ of sequential spaces. Then X is countably compact if and only if each X_n is countably compact.

Proof The necessity of the condition is obvious since the continuous image of a countably compact space is countably compact ([7], Theorem 11.3.6). To establish that X is sequentially compact and hence countably compact, let $\{x_n : n \in \omega\}$ be a sequence in X. For each $i \in \omega$ let P_i be the projection map of X onto X_i. If each X_n is countably compact then, by (1.20), each X_n is sequentially compact. Hence there exists a sequence $\{k_i : i \in \omega\}$ of functions mapping ω into ω such that $\{x_{k_o(n)} : n \in \omega\}$ is a subsequence of $\{x_n : n \in \omega\}$ and $\{P_o(x_{k_o(n)}) : n \in \omega\}$ is a convergent subsequence of $\{P_o(x_n) : n \in \omega\}$, and for $1 \leq i \in \omega$, $\{x_{k_i(n)} : n \in \omega\}$ is a subsequence of $\{x_{k_{i-1}(n)} : n \in \omega\}$ and $\{P_i(x_{k_i(n)}) : n \in \omega\}$ is a convergent subsequence of $\{P_i(x_{k_{i-1}(n)}) : n \in \omega\}$. The sequence $\{x_{k_n(n)} : n \in \omega\}$ is the desired convergent subsequence of $\{x_n : n \in \omega\}$.

1.23 Corollary Let X be an uncountable set, and let 2 denote the set $\{0, 1\}$ provided with the discrete topology. Then the product space 2^X is not sequential.

Proof Suppose that 2^X is sequential. Since any product of compact topological spaces is compact, 2^X is (countably) compact and hence, by (1.20), sequentially compact. Let $f : X \longrightarrow 2^{\omega}$ be a surjection.

Define a sequence $\{x_n : n \in \omega\}$ in 2^X by $x_n(\alpha) = [f(\alpha)](n)$ for each $\alpha \in X$. Let $\{x_{n_k} : k \in \omega\}$ be a subsequence of $\{x_n : n \in \omega\}$. Now there exists $y \in 2^\omega$ such that $y(n_{2k}) = 0$ and $y(n_{2k+1}) = 1$ for each $k \in \omega$. Since f is surjective, $f(\beta) = y$ for some $\beta \in X$. Therefore, if P_β is the canonical projection map of 2^X onto the β-th coordinate space, then $\{P_\beta(x_{n_k}) : k \in \omega\}$ cannot converge since 2 is discrete; clearly $P_\beta(x_{n_k}) = x_{n_k}(\beta) = [f(\beta)]\ (n_k) = y(n_k)$. Thus $\{x_{n_k} : k \in \omega\}$ does not converge, and hence 2^X cannot be sequentially compact. The contradiction shows that 2^X must not be sequential.

1.24 Theorem Let X and Y be sequential spaces, and assume that each point of X has a neighbourhood basis consisting of sequentially compact sets. Then the topological product space $X \times Y$ is sequential.

Proof Let G be a sequentially open subset of $X \times Y$. To prove that G is open, suppose that $(u, v) \in G$ and let $G_v = \{x : (x, v) \in G\}$. Clearly $u \in G_v$. If $\{s_n : n \in \omega\}$ is a sequence in X converging to $s \in G_v$ then $\{(s_n, v) : n \in \omega\}$ is a sequence in $X \times Y$ converging to $(s, v) \in G$. Since G is sequentially open, $\{(s_n, v) : n \in \omega\}$ is eventually in G, and consequently $\{s_n : n \in \omega\}$ is eventually in G_v. Hence G_v is sequentially open and therefore open in X. By hypothesis, there exists a sequentially compact neighbourhood U of u with $U \times \{v\}$ contained in G. Let V be the largest subset of Y such that $U \times V \subset G$; that is, $V = \{z : U \times \{z\} \subset G\}$. If V is not open, there is a sequence $\{y_n : n \in \omega\}$ in Y–V converging to $y \in V$. But then, for each $n \in \omega$ there exists $x_n \in U$ with $(x_n, y_n) \notin G$. Since U is sequentially compact, $\{x_n : n \in \omega\}$ has a subsequence $\{x_{n_k} : k \in \omega\}$ which converges to some point $x \in U$. It follows that $\{(x_{n_k}, y_{n_k}) : k \in \omega\}$ converges to $(x, y) \in G$ and hence that $(x_{n_k}, y_{n_k}) \in G$ for all k sufficiently large. The contradiction shows that V must be open. Then, since $(u, v) \in U \times V \subset G$, $(u, v) \in \text{int } G$ and so G is open.

1.25 Corollary (1) The product of two sequential spaces, one of which is regular and either locally countably compact or locally sequentially compact, is sequential.

(2) The product of two sequential spaces, one of which is locally compact and either Hausdorff or regular, is sequential.

<u>Proof</u> (1) Let X and Y be sequential spaces, and assume that X is regular and locally countably compact. Each point $x \in X$ has a countably compact neighbourhood K. Let $\{U_a : a \in A\}$ be a neighbourhood basis at x such that each U_a is a subset of K. Since X is regular, for each $a \in A$ there exists an open set V_a with $x \in V_a \subset \mathrm{cl}\, V_a \subset U_a$. Then each cl V_a is countably compact and $\{\mathrm{cl}\, V_a : a \in A\}$ is a basis of countably compact closed neighbourhoods of x. By virtue of (1.13.7) and (1.20), cl V_a is sequential and hence sequentially compact. Accordingly, each $x \in X$ has a neighbourhood basis $\{\mathrm{cl}\, V_a : a \in A\}$ consisting of sequentially compact subsets. The preceding theorem implies that the product space $X \times Y$ is sequential. The second part of (1) is now clear since sequentially compact spaces are countably compact.

(2) This follows from (1) because every locally compact Hausdorff space is regular and every compact space is locally countably compact.

As seen in Example 1.19, the product space $Q \times Q'$ is not sequential. Although both coordinate spaces Q and Q' are regular, neither topological space is locally countably compact. It is clear that Q' fails to be locally countably compact at 0. The space Q is not locally countably compact because regular locally countably compact spaces are Baire spaces and Q is not a Baire space ([7], pp. 249-250).

<u>1.26 Corollary</u> If X and Y are sequential Hausdorff spaces then the product spaces $X \times Y$ and $(X \times Y)_s$, provided with the usual product topology and the sequential closure topology respectively, have the same compact sets.

<u>Proof</u> Since the sequential closure topology is larger than the product topology, it is only necessary to show that each compact subset of $X \times Y$ is compact in $(X \times Y)_s$. Let K be compact in $X \times Y$, and let $K_x = \{x : (x, y) \in K\}$ and $K_y = \{y : (x, y) \in K\}$ be the projections of K into X and Y respectively. The subspaces K_x and K_y are compact Hausdorff and hence closed. Thus K_x and K_y are also sequential spaces. It follows from (1.25) that the product space $K_x \times K_y$ is sequential. Consequently, the topology induced on K by $K_x \times K_y$ is the same as that induced on K by either the usual product topology or the sequential closure topology. Therefore $U \cap K$ is open in K whenever U is open in $(X \times Y)_s$, and hence K is a compact subset of $(X \times Y)_s$.

The foregoing corollary is of interest in studying k-spaces. (A topological space X is a k-space if and only if a subset A of X is closed whenever $A \cap K$ is closed in K for every compact subset K of X.)

1.27 Proposition Every sequential space is a k-space.

Proof Suppose that A is a subset of a sequential space X with $A \cap K$ closed in K for every compact subset K of X. Let S be a sequence in A converging to x. Since $S \cup \{x\}$ is compact, $A \cap (S \cup \{x\})$ is closed in $S \cup \{x\}$. This implies that $x \in A$ and hence that A is sequentially closed.

There are, however, k-spaces which are not sequential. For example, the ordinal space $\Omega + 1$ provided with the order topology is a k-space which is not sequential. The space $\Omega + 1$ is a k-space because it is compact ([7], pp. 222, 162) and the locally compact spaces are k-spaces ([7], 11.9.3).

In view of (1.26) and (1.27), it is not difficult to see that the product of two k-spaces need not be a k-space. The non-sequential space $Q \times Q'$ is, in fact, a product of two Hausdorff sequential spaces which is not a k-space. This is so because there exists a non-closed sequentially closed subset A of $Q \times Q'$ such that $A \cap K$ is closed in K for every compact subset K of $(Q \times Q')_s$.

The next two theorems are important results concerning the product of quotient maps and the product of sequential spaces respectively. For each cardinal number m, let D_m denote the discrete space of cardinality m, let Y_m be the quotient space obtained from $D_m \times [0, 1]$ by identifying all points in $D_m \times \{0\}$, let $g_m : D_m \times [0, 1] \dashrightarrow Y_m$ be the quotient map, and let y_o denote the point $D_m \times \{0\}$ in Y_m. For any topological space X, let 1_X be the identity function on X. (For any two functions $f : X \dashrightarrow Z$ and $g : Y \dashrightarrow Z_o$, define $(f \times g)(x, y) = (f(x), g(y))$ for each $(x, y) \in X \times Y$.)

<u>1.28 Theorem</u> The following properties of a regular space X are equivalent.

(1) X is locally countably compact.

(2) $1_X \times g$ is a quotient map for every quotient map g with sequential domain.

(3) $h = 1_X \times g_m$ is a quotient map, where m is the smallest cardinal such that each $x \in X$ has a neighbourhood basis of cardinality $\leq m$.

<u>Proof</u> (1) ---> (2) Let $g : Y \dashrightarrow Z$ be a quotient map with sequential domain Y, and let f denote the product map $1_X \times g$. Clearly f is continuous; if $A \times B$ is a basic open subset of $X \times Z$ then $f^{-1}(A \times B) = A \times g^{-1}(B)$, which is open since g is continuous. Let G be a subset of $X \times Z$ with $f^{-1}(G)$ open in $X \times Y$. Suppose that $(u, v) \in G$ and let $r \in g^{-1}(v)$. There is a basic open set $U \times V$ in $X \times Y$ such that $(u, r) \in U \times V \subset f^{-1}(G)$. Since X is locally countably compact, there exists a countably compact neighbourhood K_1 of u.

Then, since X is regular, there is an open subset U_1 of X satisfying $u \in U_1 \subset \text{cl } U_1 \subset U \cap K_1$. The set $K = \text{cl } U_1$ is a countably compact neighbourhood of u contained in U. Let $E = \{z \in Z : K \times \{z\} \subset G\}$. Since $(u, r) \in K \times V \subset f^{-1}(G)$, $K \times \{r\} \subset f^{-1}(G)$ and $G \supset f(K \times \{r\}) = K \times \{g(r)\} = K \times \{v\}$, which implies that $v \in E$. It remains to prove that E is open. But since g is a quotient map, it suffices to show that $g^{-1}(E)$ is open. If $a \in Y$ and $K \times \{a\} \subset f^{-1}(G)$ then $f(K \times \{a\}) \subset G$, which implies that $K \times g(a) \subset G$ and hence that $g(a) \in E$. Thus $g^{-1}(E) = \{y \in Y : K \times \{y\} \subset f^{-1}(G)\}$. Suppose that $g^{-1}(E)$ is not open. Then, since Y is sequential, there is a sequence $\{y_n : n \in \omega\}$ in $Y - g^{-1}(E)$ converging to some $y \in g^{-1}(E)$. So $K \times \{y_n\} \not\subset f^{-1}(G)$ for each $n \in \omega$. Hence there exists a sequence $\{x_n : n \in \omega\}$ in K with each $(x_n, y_n) \notin f^{-1}(G)$. Because K is countably compact, $\{x_n : n \in \omega\}$ has a cluster point $x \in K$. Then the sequence $\{(x_n, y_n) : n \in \omega\}$ has a cluster point $(x, y) \in f^{-1}(G)$. Since $f^{-1}(G)$ is open, $\{(x_n, y_n) : n \in \omega\}$ is frequently in $f^{-1}(G)$ contradicting $(x_n, y_n) \notin f^{-1}(G)$ for all $n \in \omega$. Thus $g^{-1}(E)$ must be open, and (1) implies (2).

(2) ---> (3) Both D_m and $[0, 1]$ are sequential spaces. Since D_m is discrete, D_m is locally compact Hausdorff and it follows from (1.25) that $D_m \times [0, 1]$ is sequential. Thus g_m is a quotient map with sequential domain $D_m \times [0, 1]$ and so (2) implies (3).

(3) ---> (1) Suppose that X is not locally countably compact at some point x_o. Let $\{U_a : a \in D_m\}$ be a neighbourhood basis at x_o. For each $a \in D_m$, cl U_a is not countably compact and thus has a countable family $\{F_n^a : n \in N\}$ of distinct non-empty closed subsets satisfying the finite intersection property whose intersection is empty. Let $E_n^a = \bigcap \{F_k^a : n \geq k \in N\}$ for each $n \in N$. It is clear that $\bigcap \{E_n^a : n \in N\} = \emptyset$, $E_n^a \supset E_{n+1}^a$, and each E_n^a is closed and non-empty. Thus, for each $a \in D_m$ there exists a countable well-ordered family $\{E_n^a : n \in N\}$ of distinct non-empty closed subsets of cl U_a satisfying the finite intersection property whose intersection is empty.

To establish that h is not a quotient map, for each $a \in D_m$ define $S_a \subset X \times (D_m \times [0, 1])$ by $S_a = \bigcup \{E_n^a \times \{(a, \frac{1}{n})\} : n \in N\}$, and define $S \subset X \times Y_m$ by $S = \bigcup \{h(S_a) : a \in D_m\}$. It suffices to show that $h^{-1}(S)$ is closed in $X \times (D_m \times [0, 1])$ and that S is not closed in $X \times Y_m$. Note that $(x_o, y_o) \notin S$ since $(x, (a, 0)) \notin S_a$ for any $x \in X$. If $U \times V$ is a neighbourhood of (x_o, y_o) in $X \times Y_m$ then, since X is regular, cl $U_c \subset U$ for some $c \in D_m$. Thus choosing $(c, \frac{1}{p}) \in g_m^{-1}(V) \cap \{(c, \frac{1}{n}) : n \in N\}$, $\emptyset \neq h(E_p^c \times \{(c, \frac{1}{p})\}) \subset (U \times V) \cap S$; the point $(c, \frac{1}{p})$ exists because V is a neighbourhood of y_o and

$\{\frac{1}{n+1} : n \in \omega\}$ converges to 0 in $[0, 1]$. It follows that $(x_o, y_o) \in cl_{X\times Y_m}(S)$ and hence that S is not closed in $X \times Y_m$. Since D_m is discrete, it remains to prove that $h^{-1}(S) \cap (X \times (\{a\} \times [0, 1]))$ is closed in $X \times (\{a\} \times [0, 1])$ for each $a \in D_m$. But since $(X \times \{(a, 0)\}) \cap S_a = \emptyset$, $h^{-1}(S) \cap (X \times (\{a\} \times [0,1]))$ $= S_a$. Assume that the point $(x, (a, \alpha))$ is contained in the closure of S_a with respect to $X \times (\{a\} \times [0, 1])$. Since each E_n^a is closed, it is clear that $x \in E_1^a$. Then, since $\{E_n^a : n \in N\}$ is well-ordered and $\bigcap \{E_n^a : n \in N\} = \emptyset$, there is a smallest set E_p^a containing x. It follows that $\alpha \geq \frac{1}{p}$ because otherwise $G \times (\{a\} \times (\alpha - \varepsilon, \alpha + \varepsilon) \cap [0, 1])$, where $0 < \varepsilon < \frac{1}{p(p+1)}$ and G is a neighbourhood of x disjoint from E_{p+1}^a, is a neighbourhood of $(x, (a, \alpha))$ disjoint from S_a. Moreover, since $[0, 1]$ is Hausdorff, $\alpha \in \{\frac{1}{n} : n \leq p\}$. Therefore $(x,(a, \alpha)) \in E_\alpha^a \times \{(a, \alpha)\} \subset S_a$. Thus $h^{-1}(S)$ is closed in $X \times (D_m \times [0, 1])$, and the proof is complete.

<u>1.29 Theorem</u> The following properties of a T_3 sequential space X are equivalent.

(1) X is locally countably compact.

(2) $X \times Y$ is sequential for each sequential space Y.

(3) $X \times Y_m$ is a k-space, where m is the smallest cardinal such that each point of X has a neighbourhood basis of cardinality $\leq m$.

<u>Proof</u> The proof that (1) implies (2) is given in (1.25), and it is obvious that (2) implies (3) since Y_m is sequential and every sequential space is a k-space. By virtue of (1.28), to establish that (3) implies (1) it suffices to prove that $h = 1_X \times g_m$ is a quotient map. First, it is convenient to prove a preliminary lemma. Let K be a compact subset of $X \times Y_m$ and let P be the first coordinate projection map of $X \times Y_m$ onto X. If $(X \times \{y_o\}) \cap K = \emptyset$, $h^{-1}(K) = K$ and K is also compact in $X \times (D_m \times [0, 1])$. If $(X \times \{y_o\}) \cap K \neq \emptyset$, for any $a \in D_m$ the set $E = (K-(X \times \{y_o\})) \cup (P(K) \times \{(a, 0)\})$ is a compact subset of $X \times (D_m \times [0, 1])$ because any open neighbourhood of $(a, 0)$ in $D_m \times [0, 1]$ is contained in some open neighbourhood of y_o in Y_m. In addition, $h(E) = K$. It follows that every compact subset of $X \times Y_m$ is the image under h of a compact subset of $X \times (D_m \times [0, 1])$. Suppose now that B is a subset of $X \times Y_m$ with $h^{-1}(B)$ closed in $X \times (D_m \times [0, 1])$. Since $X \times Y_m$ is a Hausdorff k-space, to prove that B is closed it is sufficient to show that $B \cap K$ is compact in K for every compact subspace K of $X \times Y_m$.

But $K = h(C)$ for some compact subset C of $X \times (D_m \times [0, 1])$. Thus $E = h^{-1}(B) \cap C$ is a compact subset of C. Clearly $h(E)$ is contained in $B \cap K$. If $b \in B \cap K$ then $b \in K$ and so there exists some $a \in C$ such that $h(a) = b$. Furthermore, $b \in B$ implies that $a \in h^{-1}(B)$ and hence that $a \in h^{-1}(B) \cap C = E$. Therefore $b = h(a) \in h(E)$ and consequently $h(E) = B \cap K$. Then, since E is compact in C and h is continuous, $B \cap K$ is a compact subset of $h(C) = K$. Thus B is closed and h is a quotient map.

A characterization of the sequential spaces follows from the next theorem.

<u>1.30 Theorem</u> Every sequential space is a quotient of a disjoint topological sum of convergent sequences.

<u>Proof</u> Let X be a sequential space. For each $x \in X$ and for each sequence $\{x_n : n \in \omega\}$ in X converging to x, let $S(x_n, x) = \{x_n : n \in \omega\} \cup \{x\}$ be a Hausdorff topological space in which each x_n is isolated and the sequence $\{x_n : n \in \omega\}$ converges to x. Although the elements of $S(x_n, x)$ need not be distinct in X, they are taken to be distinct in $S(x_n, x)$. Thus $S(x_n, x)$ is homeomorphic to $\omega + 1$ provided with the order topology. Clearly $S(x_n, x)$ is a convergent sequence in $S(x_n, x)$. Let W be the disjoint topological sum of all possible $S(x_n, x)$.

Since for each $x \in X$ the convergent sequence $\{x_n : x_n = x, n \in \omega\} \cup \{x\}$ is a summand of W, the natural function $f : W \dashrightarrow X$ defined by $f(x) = x$ is a surjection. In addition, f is continuous because it is continuous on each summand. To complete the proof, it remains to establish that f is a quotient map. Let U be a subset of X with $f^{-1}(U)$ open in W. If $\{y_n : n \in \omega\}$ is a sequence in X converging to $y \in U$, $y \in f^{-1}(U) \cap S(y_n, y)$ which is open in $S(y_n, y)$. Then $\{y_n : n \in \omega\}$ as a subset of $S(y_n, y)$ is eventually in $f^{-1}(U)$, and hence $\{y_n : n \in \omega\}$ as a subset of X is eventually in U. Consequently U is sequentially open and therefore open in X.

<u>1.31 Corollary</u> A Hausdorff space is sequential if and only if it is a quotient of the disjoint topological sum of its convergent sequences.

<u>Proof</u> The necessity of the condition is clear from (1.30). It is only necessary to remark that if X is Hausdorff then W is precisely the disjoint topological sum of all the convergent sequences in X. Conversely, (1.11) implies that each summand of W is a metric space and hence a sequential space. Then X is sequential by (1.13.5) and (1.13.2).

<u>1.32 Corollary</u> Every sequential space is the quotient of a zero-dimensional, locally compact, complete metric space.

<u>Proof</u> It suffices to show that W is a zero-dimensional, locally compact, complete metric space. Suppose that $\mathcal{U}$ is a neighbourhood basis at $y \in W$. Each $U \in \mathcal{U}$ is both open and closed in W because $U \cap S(x_n, x)$ is both open and closed in $S(x_n, x)$. Hence W is zero-dimensional. According to (1.11), each $S(x_n, x)$ is a compact metric space. Obviously W is locally compact. If $d_{S(x_n,x)}$ is a metric on $S(x_n, x)$, then

$$d(u, v) = \begin{cases} d_{S(x_n,x)}(u, v) & \text{if } u, v \in S(x_n, x) \\ 1 & \text{otherwise} \end{cases}$$

is a metric on W. Lastly, W is complete by virtue of ([7], Corollary 14.2.4).

<u>1.33 Corollary</u> The following are equivalent.

(1) X is sequential.

(2) X is the quotient of a metric space.

(3) X is the quotient of a first-countable space.

<u>Proof</u> In view of the preceding corollary, (1) implies (2). Clearly (2) implies (3) because metric spaces are first-countable. Since first-countable spaces are sequential, (3) implies (1) by (1.13.2).

<u>1.34 Example</u> There is a separable sequential space which is not the quotient of a separable metric space.

<u>Proof</u> Let H be the real numbers provided with the half-open interval topology. Sets of the form $\{x : a \leq x < b\} = [a, b)$ are a basis for this topology. Since for each $x \in H$ the collection $\{[x, \frac{1}{n+1}) : n \in \omega\}$ is a countable neighbourhood basis at x, H is first-countable. Then $H \times H$ is first-countable and hence sequential. The space $H \times H$ is also separable because $\{(x, y) : x, y \text{ rational}\}$ is a countable dense subset. Suppose that $H \times H$ is the quotient of a separable metric space X, and let $P : X \dashrightarrow H \times H$ be the quotient map. According to ([7], 9.5.6), every separable metric space is Lindelöf. Thus X is Lindelöf and consequently so is $H \times H$. Indeed, if $\{U_a : a \in A\}$ is an open covering of $H \times H$, $\{P^{-1}(U_a) : a \in A\}$ is an open covering of X and hence it has a countable subcovering $\{P^{-1}(U_{a_n}) : n \in \omega\}$; then $\{U_a : a \in A\}$ has a countable subcovering $\{U_{a_n} : n \in \omega\}$ of $H \times H$.

Consider the subspace $K = \{(x, -x) : x \text{ irrational}\}$ of $H \times H$. For any $\varepsilon > 0$ each $(x, y) \in H \times H$ such that $x + y \geq 0$, $([x, x+\varepsilon) \times [y, y+\varepsilon)) \cap K \neq \emptyset$ if and only if $(x, y) \in K$. In addition, for each $(x, y) \in H \times H$ such that $x + y < 0$, $([x, x+\delta) \times [y, -x-\delta)) \cap K = \emptyset$ whenever $0 < \delta < -x -y$. Thus K is a closed subspace of the Lindelöf space

$H \times H$. Since K is discrete and uncountable, K is not Lindelöf. However, if $\{V'_a : a \in A\}$ is an open covering of K, $\{V_a : a \in A\} \cup \{(H \times H)-K\}$ is an open covering of $H \times H$. Then, since $H \times H$ is Lindelöf, there is a countable subcovering $\{V_{a_n} : n \in \omega\} \cup \{(H \times H)-K\}$ of $H \times H$. It follows that K is covered by $\{V_{a_n} : n \in \omega\}$ and hence that K is Lindelöf. The contradiction shows that $H \times H$ must not be the quotient of a separable metric space.

It has been shown that the notion of sequential space is neither hereditary nor productive. The following is a characterization of those subspaces and those products of sequential spaces which are themselves sequential.

<u>1.35 Proposition</u> For X sequential, let ϕ_X denote the quotient map of X^* onto X, where X^* is the disjoint topological sum of convergent sequences in X as derived in (1.30).

(1) A subspace Y of a sequential space X is sequential if and only if $\phi_X|\phi_X^{-1}(Y)$ is a quotient map.

(2) The product of two sequential spaces X and Y is sequential if and only if $\phi_X \times \phi_Y$ is a quotient map.

Proof (1) Let $Y_1 = \phi_X^{-1}(Y)$ and $\phi = \phi_X|_{Y_1}$. Let $g : Y^* \longrightarrow Y$ be the function defined by $g(x) = x$ for each $x \in Y^*$. Then g is a quotient map if and only if Y is sequential. It suffices to show that ϕ is a quotient map if and only if g is a quotient map. Let U be a subset of Y. If $\phi^{-1}(U)$ is open in Y_1 then $\phi^{-1}(U) \cap S$ is open in S for each summand S of Y_1. From the definition of the relative topology on Y it is clear that each convergent sequence in Y is a convergent sequence in X. Therefore each summand of Y^* is a summand of Y_1, and consequently $g^{-1}(U) = \phi^{-1}(U) \cap Y^*$ is open in Y^*. Conversely, suppose that $g^{-1}(U)$ is open in Y^*. Then $\phi^{-1}(U) \cap S = g^{-1}(U) \cap S$ is open in S for each summand S of Y_1 which is also a summand of Y^*. Let $S(x_n, x)$ be a summand of X^*; then $S_1 = S(x_n, x) \cap Y$ is a summand of Y_1. The topological space S_1 is either finite or infinite. In the first case, each point of S_1 is isolated and thus $S_1 \cap \phi^{-1}(U)$ must be open in S_1. Assume now that the second case occurs. If $x \notin U$ then $S_1 \cap \phi^{-1}(U) = \{x_n : n \in \omega\} \cap \phi^{-1}(U)$ which is certainly open in S_1. If $x \in U$, each sequence in S_1 converging to $x \in S_1 \cap \phi^{-1}(U)$ is a subsequence $\{x_{n_k} : k \in \omega\}$ of $\{x_n : n \in \omega\}$. But since $\phi^{-1}(U) \cap S(x_{n_k}, x)$ is open in $S(x_{n_k}, x)$, $\{x_{n_k} : k \in \omega\}$ is eventually in $S_1 \cap \phi^{-1}(U)$. It follows that $S_1 \cap \phi^{-1}(U)$ is a sequentially open subset of the first-countable space S_1. Therefore $\phi^{-1}(U)$ is open in Y_1, and the proof of (1) is complete.

(2) Since X^* and Y^* are first-countable, the topological product space $X^* \times Y^*$ is first-countable and hence sequential. Then if $h = \phi_X \times \phi_Y$ is a quotient map, $X \times Y$ is sequential. To establish the converse, assume that $X \times Y$ is sequential and let G be a subset of $X \times Y$ with $h^{-1}(G)$ open in $X^* \times Y^*$. Suppose $\{(x_n, y_n) : n \in \omega\}$ is a sequence in $X \times Y$ converging to $(x, y) \in G$. Then $(x, y) \in h^{-1}(G)$ and there exists a basic open subset $U \times V$ of $X^* \times Y^*$ such that $(x, y) \in U \times V \subset h^{-1}(G)$. Accordingly, $U \cap S(x_n, x)$ is open in $S(x_n, x)$ and $V \cap S(y_n, y)$ is open in $S(y_n, y)$. This implies that the sequences $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are eventually in U and V respectively, and hence that $(x_n, y_n) \in h^{-1}(G)$ for all n sufficiently large. Therefore $(x_n, y_n) = (\phi_X(x_n), \phi_Y(y_n)) = h(x_n, y_n) \in G$ for all n sufficiently large, and so G is a sequentially open subset of the sequential space $X \times Y$. Thus G is open and h is a quotient map.

A first-countable space with unique sequential limits is Hausdorff since otherwise it is possible to find a sequence converging to two distinct points. That is, if x and y are distinct points which cannot be separated by disjoint open sets and $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ are countable neighbourhood bases of x and y

respectively, then the sequence $\{x_n : n \in \omega\}$ satisfying $x_n \in U_n \cap V_n$ converges to both x and y. The succeeding examples show that this result cannot be generalized to sequential spaces.

The construction used in the first example is based on Sorgenfrey's [29] well-known result concerning the product of normal spaces. It will be shown that the square of the normal space H is not normal.

1.36 Example There is a sequential space with unique sequential limits which is not Hausdorff.

Proof Let H be the real numbers provided with the half-open interval topology. The topological product space $H \times H$ is sequential. If $A = \{(x, y) : x + y = 1\}$ is the antidiagonal of $H \times H$, let A_q and A_i be those points of A with rational and irrational coordinates respectively. For any $\varepsilon > 0$ and each $(x, y) \in H \times H$ such that $x + y \geq 1$, $([x, x+\varepsilon) \times [y, y+\varepsilon)) \cap A_q \neq \emptyset$ if and only if $(x, y) \in A_q$; similarly, $([x, x+\varepsilon) \times [y, y+\varepsilon)) \cap A_i \neq \emptyset$ if and only if $(x, y) \in A_i$. And for each $(x, y) \in H \times H$ such that $x + y < 1$, $([x, x+\delta) \times [y, 1-x-\delta)) \cap A = \emptyset$ whenever $0 < \delta < 1 - x - y$. Therefore A_q and A_i are disjoint closed subsets of $H \times H$.

To prove that A_q and A_i cannot be separated by disjoint open sets, let U be an open neighbourhood of A_i. For each irrational x, let $f(x) = \sup\{\varepsilon > 0 : [x, x+\varepsilon) \times [1-x, 1-x+\varepsilon) \subset U\}$. Then f is a function on the set of irrational numbers and f is never zero. The set of irrationals is the countable union of sets of the form $\{x : f(x) \geq \frac{1}{n}\}$ where $n \in N$. In the real line R, the irrationals are of the second category ([7], pp. 249-251) and consequently there exists $m \in N$ such that $\{x : f(x) \geq \frac{1}{m}\}$ is not nowhere dense in R. Hence there is a rational number r which is an accumulation point of $\{x : f(x) \geq \frac{1}{m}\}$. For any neighbourhood V of the point (r, 1-r), there exists $p \in R$ such that $p < \frac{1}{m}$ and $[r, r+p) \times [1-r, 1-r+p) \subset V$. But there is an irrational number $s \in (r-p, r+p)$ such that $[s, s+\frac{1}{m}) \times [1-s, 1-s+\frac{1}{m}) \subset U$. Then $([r, r+p) \times [1-r, 1-r+p)) \cap ([s, s+\frac{1}{m}) \times [1-s, 1-s+\frac{1}{m})) \neq \emptyset$, and hence $U \cap V \neq \emptyset$ for every neighbourhood V of (r, 1-r). Therefore $(r, 1-r) \in \text{cl } U$, and so $A_q \cap \text{cl } U \neq \emptyset$ for every neighbourhood U of A_i.

The set $E = (A_q \times A_q) \cup (A_i \times A_i) \cup \{(x, x) : x \in (H \times H) - (A_i \cup A_q)\}$ is an equivalence relation in the first-countable space $H \times H$. The quotient space $X = (H \times H)/_E$ is sequential and T_1 but not Hausdorff. Let $\phi : H \times H \longrightarrow X$ be the quotient map, and let

$q = \phi(A_q)$ and $i = \phi(A_i)$. Then q and i are the only pair of distinct points of X which cannot be separated by open sets, and consequently if some sequence in X converges to two distinct points, they must be q and i. Suppose that $S = \{x_n : n \in \omega\}$ converges to q. Since X is T_1, it can be assumed that $x_n \neq i$ for all $n \in \omega$. Again since X is T_1, if frequently $x_n = q$ then S cannot converge to i. However, if x_n is eventually different from q, there must be some $q_1 \in A_q$ and a subsequence S_o of S converging to q_1 in H × H. But then there is a neighbourhood of A_i disjoint from S_o, and thus S cannot converge to i. Hence X has unique sequential limits.

1.37 Example There is a countable, compact, sequential space with unique sequential limits which is not Hausdorff.

Proof Let M be the Hausdorff sequential space of (1.17). Let p be some point not in M and let $M_1 = M \cup \{p\}$ with M open in M_1 and where the basic neighbourhoods of p are of the form $\{p\} \cup ((N \times N)-F)$ with F the union of the ranges of a finite number of convergent sequences in M. Since M is Hausdorff and a convergent sequence in M cannot also converge to p, M_1 has unique sequential limits. However, M_1 is not Hausdorff because

0 and p have no disjoint neighbourhoods. It is also clear that M_1 is countable and compact. To verify that M_1 is compact, let U and V be any open neighbourhoods of 0 and p respectively. Then $N-(U \cup V)$ is finite and $M_1-(U \cup V)$ is the union of the ranges of a finite number of convergent sequences in $N \times N$. But each sequence in $M_1-(U \cup V)$ converges to some member of $N-(U \cup V)$. Hence for any finite collection $\{U_n : n \in N-(U \cup V)\}$ of open sets satisfying $n \in U_n$, $M_1-(U \cup V \cup [\bigcup \{U_n : n \in N-(U \cup V)\}])$ is finite.

To see that M_1 is sequential, let V be a sequentially open subset of M_1. If $p \notin V$, V is sequentially open in the sequential space M. Then, since M is open in M_1, V is also open in M_1. Assume now that $p \in V$. Clearly, V is a neighbourhood of each point in $V-\{p\}$ because $V-\{p\}$ is sequentially open in M. Since any subset $\{(m, n) : m \in A \subset N, n \in B \subset N\}$ contains a sequence converging to p whenever A is infinite, M_1-V must contain points of $N \times N$ having only finitely many first-coordinates. Thus V contains a basic neighbourhood of p and so $p \in \text{int } V$. Therefore V is an open subset of M_1 and consequently M_1 is sequential.

<u>1.38 Proposition</u> (1) A sequential space with unique sequential limits is T_1.

(2) If X is a topological space with unique sequential limits and $X \times X$ is sequential, then the diagonal $\Delta = \{(x,x) : x \in X\}$ of $X \times X$ is closed (and hence X is Hausdorff).

<u>Proof</u> (1) For each member y of a sequential space Y, the singleton $\{y\}$ is sequentially closed and hence closed in Y.

(2) Every sequence in Δ is of the form $\{(x_n, x_n) : n \in \omega\}$. Since X has unique sequential limits, $\{(x_n, x_n) : n \in \omega\}$ converges to (x, x) if and only if $\{x_n : n \in \omega\}$ converges to x. Therefore Δ is sequentially closed and hence closed in $X \times X$.

It follows from part (2) that the product spaces $X \times X$ and $M_1 \times M_1$, where $X = (H \times H)_{/E}$ and M_1 are the non-Hausdorff sequential spaces of (1.36) and (1.37) respectively, are not sequential. If these products were sequential, X and M_1 would be Hausdorff.

After a preliminary result it will be shown that a sequential space with unique sequential limits in which each point has a neighbourhood basis consisting of countably compact sets is Hausdorff. The topological spaces $X = (H \times H)_{/E}$ and M_1 do not satisfy this compactness condition. For any neighbourhood U of q in X, let $\{x_n : n \in \omega\}$ be a sequence in U such that $x_n \in (n, n+\varepsilon_n) \times (1-n, 1-n+\varepsilon_n) \subset U$ where $0 < \varepsilon_n < \frac{1}{2}$. The sequence $\{x_n : n \in \omega\}$ has no cluster point in the neighbourhood U. Therefore U is not countably compact, and consequently X is not even locally countably compact. Suppose now that V is any neighbourhood of p in M_1. Then, since $\{n+1 : n \in \omega\}$ is a sequence in M converging to 0, $V-(N \cup \{0\})$ is also a neighbourhood of p. It is clear that there exists $m \in N$ such that $\{(m, y_n) : n \in \omega\}$ is a sequence in $V-(N \cup \{0\})$ with $y_n = y_k$ if and only if $n = k$. But $\{(m, y_n) : n \in \omega\}$ has no cluster point in $V-(N \cup \{0\})$. So $V-(N \cup \{0\})$ is not countably compact, and M_1 does not have a countably compact neighbourhood basis at p.

<u>1.39 Proposition</u> A sequential space has unique sequential limits if and only if each countably compact subset is closed (and hence sequential).

<u>Proof</u> Let X be a sequential space. Suppose X has unique sequential limits and K is a countably compact subset of X. Let $S = \{x_n : n \in \omega\}$ be a sequence in K converging to x. Since X has unique sequential limits, $\{x\} \cup \text{range } S$ is sequentially closed and hence closed in X. Then if y is a cluster point of S, either $y = x$ or frequently $x_n = y$.

Again since X has unique sequential limits, $y = x$. Thus x is the only cluster point of S, and consequently $x \in K$. Therefore K is a sequentially closed subset of the sequential space X.

Assume now that S is a sequence in X converging to two distinct points x and y. Since $\{x\} \cup$ range S is compact, $\{x\} \cup$ range S is closed and hence contains y. This implies that S is frequently equal to y. But then $\{y\}$ is a non-closed compact subset of X.

<u>1.40 Corollary</u> A sequential space has unique sequential limits if and only if each sequentially compact subset is closed.

<u>Proof</u> Since every sequentially compact set is countably compact, the necessity of the condition follows from (1.39). Conversely, suppose that $S = \{x_n : n \in \omega\}$ is a sequence in a sequential space X converging to points x and y. Then $\{x\} \cup$ range S is sequentially compact because each sequence in the set has a subsequence which is either a subsequence of S or eventually equal to x_m for some $m \in \omega$. Therefore $\{x\} \cup$ range S is closed, and consequently either $y = x$ or frequently $x_n = y$. The latter case cannot occur since otherwise $\{y\}$ is a non-closed sequentially compact subset of X.

<u>1.41 Proposition</u> Let X be a sequential space with unique sequential limits. If each point has a neighbourhood basis consisting of countably compact sets, then X is Hausdorff.

Proof Each countably compact subset of X is closed (by 1.39), sequential (by 1.13.7), and hence sequentially compact (by 1.20). Thus each point of X has a neighbourhood basis consisting of sequentially compact sets, and so (1.24) implies that $X \times X$ is sequential. Then, according to (1.38.2), X is Hausdorff.

1.42 Corollary If X is a sequential space with unique sequential limits and each point has a neighbourhood basis consisting of compact or sequentially compact sets, then X is Hausdorff.

Chapter 2

Fréchet Spaces

The Fréchet spaces form an important subclass of the sequential spaces which contains the first-countable spaces. The study of Fréchet spaces is closely related to that of both first-countable spaces and sequential spaces. For example, every subspace of a Fréchet space is Fréchet and the quotient of a Fréchet space need not be Fréchet. On the other hand, there is a Fréchet space with unique sequential limits which is not Hausdorff and the product of two Fréchet spaces need not be Fréchet. This chapter emulates Franklin's ([8], [9] and [10]) investigation of Fréchet spaces. There are, however, several results concerned with Arhangel'skiĭ's [2] study of pseudo-open maps and a result due to Harley [12] connected with the product of Fréchet spaces.

2.1 Definition A topological space X is Fréchet, or a Fréchet space, if and only if the closure of any subset A of X is the set of limits of sequences in A.

In first-countable spaces, a point x is an accumulation point of a set A if and only if there is a sequence in $A-\{x\}$ which converges to x ([16], Theorem 2.8). Therefore first-countable spaces, and hence metric spaces and discrete spaces, are Fréchet.

2.2 Proposition Every Fréchet space is sequential. However, there are sequential spaces which are not Fréchet.

Proof By definition of sequentially closed and Fréchet space, it is obvious that every sequentially closed subset of a Fréchet space is closed. Thus every Fréchet space is sequential. On the other hand, the topological spaces M and M_1 of (1.17) and (1.37) respectively are examples of sequential spaces which are not Fréchet. In both spaces, $0 \in \mathrm{cl}(N \times N)$ but no sequence in $N \times N$ converges to 0.

2.3 Theorem (1) Every subspace of a Fréchet space is Fréchet.

(2) The disjoint topological sum of any family of Fréchet spaces is Fréchet.

(3) Every locally Fréchet space is Fréchet.

(4) If A is any subset of a Fréchet space X then $Y = X_{/A}$, the topological space X with the points in A identified, is Fréchet.

<u>Proof</u> (1) Let Y be a subspace of a Fréchet space X and let A be a subset of Y. Then $cl_X(A)$ is the set of limits in X of sequences in A. Hence $cl_Y(A) = cl_X(A) \cap Y$ is the set of limits in Y of sequences in A.

(2) Let X be the disjoint topological sum of the family $\{X_a : a \in A\}$ of Fréchet spaces. Let B be a subset of X and let B' be the set of limits of sequences in B. For each $c \in A$, $B' \cap X_c$ is the set of limits in X_c of sequences in $B \cap X_c$; because X_c is Fréchet, $B' \cap X_c$ is closed in X_c. Then B' is closed and (2) is proved.

(3) Let B' be the set of limit points of sequences in a subset B of a locally Fréchet space X. For each $x \in X-B'$ there is a neighbourhood G of x which is Fréchet. By part (1), int G is Fréchet. Let $V = (\text{int } G) \cap (X-B')$. The intersection $(\text{int } G) \cap B'$ is the set of limits of sequences in $(\text{int } G) \cap B$. Then $(\text{int } G) \cap B'$ is closed in the subspace int G, and consequently V is an open subset of int G. It follows that V is open in X. By hypothesis, there exists a collection $\{G_x : x \in X-B'\}$ of Fréchet subspaces of X such that each G_x is a neighbourhood of x. Each $V_x = (\text{int } G_x) \cap (X-B')$ is open in X. Therefore $X-B' = \bigcup \{V_x : x \in X-B'\}$ is open in X and so B' is a closed subset of X. Hence X is Fréchet.

(4) If $g : X \dashrightarrow Y$ is the quotient map, let $i = g(A)$. Let B' be the set of limit points of sequences in a subset B of Y. If $i \notin cl_Y(B)$ then, since X is Fréchet, $cl_Y(B) = cl_X(B) = B'$. If $i \in cl_Y(B)$, either $k \in cl_X(B)$ for some $k \in A$ or no such k exists. In the second case, there is a collection $\{U_x : x \in A\}$ of open subsets of X satisfying $x \in U_x$ and $U_x \cap B = \emptyset$. But then $\bigcup\{U_x : x \in A\}$ is a neighbourhood of i disjoint from B. The contradiction shows that the first case must occur. Thus $k \in cl_X(B)$ and consequently there exists a sequence in B converging to k. Hence $i \in B'$ and the proof is complete.

2.4 Example (1) There is a Fréchet space which is not first-countable.

(2) The product of two Fréchet spaces need not be Fréchet.

Proof (1) The real line with the integers identified is Fréchet (by 2.3.4) but does not satisfy the first axiom of countability (by 1.14).

(2) It follows from (2.3.4) that the topological space $\mathcal{Q}'$, the rationals with the integers identified, is Fréchet. Then $\mathcal{Q} \times \mathcal{Q}'$ is the product of two Fréchet spaces which, by (1.19), is not sequential and hence not Fréchet. Similarly, the square $\mathcal{Q}' \times \mathcal{Q}'$ is not a Fréchet space.

2.5 Example (1) The open and closed image of a Fréchet space need not be Fréchet.

(2) The quotient of a Fréchet space need not be Fréchet.

Proof (1) Since every Fréchet space is sequential and every first-countable space is Fréchet, the proof of (1) is the same as that of (1.16.2).

(2) Let X be the topological space of (1.15) and let $A = X - (\{\frac{1}{n+1} : n \in \omega\} \cup \{0\})$. For each $n \in \omega$ there is a sequence $\{x_i^n : i \in \omega\}$ in A converging to $\frac{1}{n+1}$. By the theorem on iterated limits ([16], Theorem 2.4), $0 \in \mathrm{cl}\, A$. However, every sequence in X converging to 0 is eventually equal to 0 or a subsequence of $\{\frac{1}{n+1} : n \in \omega\}$. Hence X is not Fréchet. But X is a quotient of the first-countable space Y.

2.6 Definiton A surjective function $f : X \dashrightarrow Y$ of the topological space X onto the topological space Y is pseudo-open if and only if for any $y \in Y$ and any open neighbourhood U of $f^{-1}(y)$, $y \in \mathrm{int}\, f(U)$.

2.7 Definition A map $f : X \dashrightarrow Y$ satisfying some property $\mathcal{C}$ is said to be hereditarily $\mathcal{C}$ if and only if for each subspace Y_1 of $f(X)$ and $X_1 = f^{-1}(Y_1)$ the induced map $f|_{X_1} : X_1 \dashrightarrow Y_1$ satisfies property $\mathcal{C}$.

<u>2.8 Proposition</u> (1) Every open or closed surjection is pseudo-open.

(2) Every continuous pseudo-open map is a quotient map.

(3) Every pseudo-open map is hereditarily pseudo-open.

<u>Proof</u> Let $f : X \dashrightarrow Y$ be a surjective mapping of the topological space X onto the topological space Y.

(1) Let $y \in Y$ and let U be an open neighbourhood of $f^{-1}(y)$. If f is an open map, $y \in f(f^{-1}(y)) \subset f(U) = \text{int } f(U)$. Thus open surjections are pseudo-open. Suppose now that f is a closed map. Then, since X-U is closed and $f^{-1}(y) \not\subset X-U$, y is not contained in the closed set $f(X-U)$. Therefore $Y-f(X-U)$ is open and consequently, since $f(X-U) \supset f(X)-f(U)$, $y \in Y-f(X-U) \subset Y-(f(X)-f(U)) = f(U)$ implies that $y \in \text{int } f(U)$.

(2) Let V be a subset of Y such that $f^{-1}(V)$ is open in X. For each $y \in V$, $f^{-1}(V)$ is an open neighbourhood of $f^{-1}(y)$. Then, since f is pseudo-open, $y \in \text{int } f(f^{-1}(V)) = \text{int } V$ and hence V is open.

(3) The function f induces a map $h : X_1 \dashrightarrow Y_1$ where Y_1 is a subspace of Y, $X_1 = f^{-1}(Y_1)$, and $h = f|_{X_1}$. Suppose that f is pseudo-open. Let $y \in Y_1$ and let U_1 be an open neighbourhood of $h^{-1}(y)$ in X_1. Then $U_1 = U \cap X_1$ where U is open in X and $f^{-1}(y) = h^{-1}(y) \subset U_1 \subset U$. Accordingly, $y \in \text{int}_Y f(U)$. But $y \in (\text{int}_Y f(U)) \cap Y_1 = (\text{int}_{Y_1} f(U)) \cap Y_1 = \text{int}_{Y_1} (f(U) \cap Y_1) = \text{int}_{Y_1} h(U_1)$. Hence h is pseudo-open.

2.9 Example (1) There is a pseudo-open map which is not open.

(2) There is a quotient map which is not hereditarily quotient.

Proof (1) The quotient map from the real line onto the real line with the integers identified is a pseudo-open map which is not open.

(2) Let X and Y be as in (1.15), and let P be the quotient map of Y onto X. If $X_1 = X - \{\frac{1}{n+1} : n \in \omega\}$, then

$Y_1 = P^{-1}(X_1) = \{(0, 1)\} \cup \{(x, 0) : 0 \neq x \in R\} - \{(\frac{1}{n+1}, 0) : n \in \omega\}$

and P induces the map $P_1 = P|_{Y_1}$. But P_1 is not a quotient map since $P_1^{-1}(\{0\}) = \{(0, 1)\}$ is open in Y_1 and yet $\{0\}$ is not open in X_1.

The next result was asserted without proof by Arhangel'skiĭ [2]. The proof is provided by the author.

2.10 Proposition A function is continuous pseudo-open if and only if it is hereditarily quotient.

Proof Let $f : X \dashrightarrow Y$ be a continuous surjection, and let $f_1 = f|_{X_1}$ where Y_1 is a subspace of Y and $X_1 = f^{-1}(Y_1)$. If U_1 is open in Y_1, $U_1 = U \cap Y_1$ for some open subset U of Y. Then, since f is continuous, $f_1^{-1}(U_1) = f_1^{-1}(U \cap Y_1) = f^{-1}(U \cap Y_1) = f^{-1}(U) \cap X_1$ is open in X_1. Thus continuous maps are hereditarily continuous. It follows from (2.8) that if f is continuous pseudo-open then f_1 is a continuous pseudo-open map and therefore a quotient map. Conversely, assume that f is hereditarily quotient. Suppose $y \in Y$ and V is an open neighbourhood of $f^{-1}(y)$. Let $Y_2 = (Y-f(V)) \cup \{y\}$, $X_2 = f^{-1}(Y_2) = [X-f^{-1}(f(V))] \cup f^{-1}(y)$, and $f_2 = f|_{X_2}$. Then $f_2^{-1}(y) = f^{-1}(y) = V \cap X_2$, which is open in X_2. Since f_2 is a quotient map, $\{y\}$ is open in Y_2 and thus $\{y\} = G \cap Y_2$ for some open subset G of Y. This implies that G is contained in f(V) and hence that $y \in \text{int } f(V)$.

<u>2.11 Proposition</u> Every continuous pseudo-open image of a Fréchet space is Fréchet.

<u>Proof</u> Let $f : X \longrightarrow Y$ be a continuous pseudo-open function of a Fréchet space X onto a topological space Y. Let B be a subset of Y and suppose that $y \in \text{cl } B$. If $f^{-1}(y) \cap \text{cl } f^{-1}(B) = \emptyset$, $U = X - \text{cl } f^{-1}(B)$ is an open neighbourhood of $f^{-1}(y)$. Then, since f is pseudo-open, $y \in \text{int } f(U) \subset f(U) = f(X-\text{cl } f^{-1}(B)) \subset f(X-f^{-1}(B)) \subset Y-B$ contradicting $y \in \text{cl } B$. Hence there is some $x \in f^{-1}(y) \cap \text{cl } f^{-1}(B)$ and, since X is Fréchet, there exists a sequence $\{x_n : n \in \omega\}$ in $f^{-1}(B)$ converging to x. Thus $\{f(x_n) : n \in \omega\}$ is contained in B and, since f is continuous, $\{f(x_n) : n \in \omega\}$ converges to $f(x) = y$. Therefore Y is a Fréchet space.

<u>2.12 Corollary</u> (1) The continuous open or closed image of a Fréchet space is Fréchet.

(2) If a product space is Fréchet, so is each of its coordinate spaces.

<u>Proof</u> (1) follows from (2.8.1) and (2.11), and (1) implies (2).

The following is a slight generalization of the necessity condition of Franklin's Proposition 2.3 ([8]). The Hausdorff hypothesis is replaced by "unique sequential limits".

<u>2.13 Proposition</u> If $f : X \dashrightarrow Y$ is a quotient map of a topological space X onto a Fréchet space Y having unique sequential limits, then f is pseudo-open.

<u>Proof</u> Let $y \in Y$ and suppose that U is an open neighbourhood of $f^{-1}(y)$. Assume that $y \notin \text{int } f(U)$. Then $y \in Y\text{-int } f(U) = \text{cl } (Y-f(U))$, and consequently there exists a sequence S in $Y-f(U)$ converging to y. Because Y has unique sequential limits, $\text{cl (range } S) = \{y\} \cup \text{range } S$. If $F = f^{-1}(S)$ then, since f is continuous, $\text{cl } F = \text{cl}(f^{-1}(S)) \subset f^{-1}(\text{cl } S) = f^{-1}(S \cup \{y\}) = f^{-1}(S) \cup f^{-1}(y) = F \cup f^{-1}(y)$. But $f^{-1}(y) \subset U$ and $U \cap F = \emptyset$. This implies that $f^{-1}(y) \cap \text{cl } F = \emptyset$ and therefore that F is closed. Hence $f^{-1}(Y-S) = f^{-1}(Y)-f^{-1}(S) = X-F$ is open. Then, since f is a quotient map, $Y-S$ is an open neighbourhood of y, contradicting the supposition that S converges to y.

2.14 Theorem A Hausdorff space is Fréchet if and only if it is a continuous pseudo-open image of the disjoint topological sum of its convergent sequences.

Proof Each Fréchet Hausdorff space is sequential Hausdorff and hence, by (1.31), a quotient of the disjoint topological sum of its convergent sequences. Then, by (2.13), the quotient map must be pseudo-open. Conversely, for any Hausdorff space X each convergent sequence in X is a metric space and hence a Fréchet space. It follows from (2.3.2) and (2.11) that X is Fréchet.

2.15 Corollary Among Hausdorff spaces, the following statements are equivalent.

(1) X is a Fréchet space.

(2) X is the continuous pseudo-open image of a metric space.

(3) X is the continuous pseudo-open image of a first-countable space.

Proof By virtue of (1.32) and (2.14), X is the continuous pseudo-open image of a zero-dimensional, locally compact, complete metric space. Since metric spaces are first-countable and first-countable spaces are Fréchet, (2) implies (3) and (3) implies (1) by (2.11).

As previously stated, first-countable T_o-spaces are precisely the continuous open images of metric spaces. In view of (1.30) and (2.14), Franklin posed and answered negatively the question of whether every first-countable (Hausdorff) space is the continuous open image of a disjoint topological sum of convergent sequences. Any such sum is a Baire space as are continuous open images of Baire spaces ([5], p.767). J.de Groot's Corollary is applicable here because every convergent sequence in a Hausdorff space is metrizable. However, many spaces are first-countable Hausdorff but not Baire spaces. The rationals Q is an example of such a topological space.

An unanswered question of Alexandroff asks whether or not there is a first-countable compact Hausdorff space with cardinality > c. The corresponding question for Fréchet spaces is trivially answered by the following.

<u>2.16 Proposition</u> The one point compactification of any discrete space is a Fréchet space.

<u>Proof</u> Let $X^* = X \cup \{\infty\}$ be the one point compactification of the discrete space X. For any subset A of X^*, $\infty \in (\text{cl } A) - A$ if and only if A contains infinitely many points. Moreover, any sequence $\{x_n : n \in \omega\}$ in A satisfying $x_n = x_m$ if and only if $n = m$ converges to ∞. Therefore, if $\infty \in \text{cl } A$, A contains a sequence converging to ∞.

If $x \in (\mathrm{cl}\ A) \cap X$ then, since X is discrete, $x \in A$. Hence X^* is Fréchet.

The topological spaces $X = (H \times H)_{/E}$ and M_1, of (1.36) and (1.37) respectively, are sequential spaces with unique sequential limits which are not Hausdorff. Although M_1 is not Fréchet, the next result shows that X is.

<u>2.17 Example</u> There are Fréchet spaces with unique sequential limits which are not Hausdorff.

<u>Proof</u> Let H and $X = (H \times H)_{/E}$ be as in (1.36). Recall that $\phi : H \times H \longrightarrow X$ is a quotient map of the first-countable space $H \times H$ onto the non-Hausdorff sequential space X which has unique sequential limits. To establish that X is Fréchet, it suffices to prove that ϕ is a pseudo-open map. Let $x \in X$ and suppose that U is an open neighbourhood of $\phi^{-1}(x)$. If $x \in X - \{q, i\}$ then $\phi^{-1}(x) = x$ and for any neighbourhood V of x such that $V \cap A = \emptyset$, $x \in U \cap V = \phi(U \cap V) \subset \phi(U)$. If $x = q$ and G is an open neighbourhood of A_q disjoint from A_i, $q \in \phi(U \cap G) = \mathrm{int}\ \phi(U \cap G) \subset \mathrm{int}\ \phi(U)$. Similarly, if $x = i$ then $i \in \phi(U)$. Thus ϕ is continuous pseudo-open and X is Fréchet.

Let Y be a Fréchet space with unique sequential limits. It follows from (1.41) that if each point of Y has a neighbourhood basis consisting of countably compact sets, then Y is Hausdorff. The succeeding example shows that simple compactness is not enough to ensure that Y is Hausdorff.

2.18 Example There is a countable, compact, Fréchet space with unique sequential limits which is not Hausdorff.

Proof Let $Y = (N \times N) \cup \{p, q\}$ with $p \neq q$ and $\{p, q\} \cap (N \times N) \neq 0$. Let each $(i, j) \in N \times N$ be an isolated point. For a basis of neighbourhoods of p take all sets of the form $\{p\} \cup (\bigcup \{(i,j) : i,j \in N; i \geq k\})$ where $k \in N$, and for q take all sets of the form $\{q\} \cup (\bigcup \{i, j) : i, j \in N; j \geq j_i\})$ where each $j_i \in N$. The topological space Y is compact because if U and V are open neighbourhoods of p and q respectively, then $Y-(U \cup V)$ is finite. It is also clear that Y is not Hausdorff since p and q cannot be separated by disjoint open sets. Then, since $Y-\{p, q\}$ is discrete, if some sequence converges to two distinct points, they must be p and q. However, a sequence $\{(i_n, j_n) : n \in \omega\}$ in $N \times N$ can converge to p only when $\{i_n : n \in \omega\}$ is unbounded, and to q only when $\{i_n : n \in \omega\}$ is bounded. Therefore Y has unique sequential limits. It remains to prove that Y is Fréchet. Let A be a subset of Y. Each point in $Y-\{q\}$ has a countable neighbourhood basis. Thus for each $y \in Y-\{q\}$, $y \in \mathrm{cl}\, A$ if and only if there exists a sequence in A converging to y. If for each $i \in N$, $A \cap (\{i\} \times N)$ is finite, $q \notin \mathrm{cl}\, A$. If for some i it is infinite, there is a sequence in A converging to q. Hence Y is a Fréchet space.

The topological product spaces $X \times X$ and $Y \times Y$, where X and Y are the non-Hausdorff Fréchet spaces of (2.17) and (2.18) respectively, are not Fréchet. In fact, these products are not even sequential. If these product spaces were sequential then, by (1.38.2), X and Y would be Hausdorff. Similarly, the spaces $Q \times Q'$ and $Q' \times Q'$ of (2.4.2) are products of Fréchet spaces which are not sequential. Example 2.19 shows that this need not always be the case. This example also demonstrates that the term "sequential" cannot be replaced by "Fréchet" in (1.24) and (1.25).

2.19 Example There is a product of two Hausdorff Fréchet spaces which is sequential but not Fréchet. In addition, one of the spaces is normal, compact, and first-countable.

Proof Let X be the real line with the integers identified, and let $I = [0, 1]$ be the closed unit interval. Both X and I are Hausdorff. Furthermore, X is Fréchet and I is a normal, compact first-countable space. It follows from (1.25.2) that $X \times I$ is sequential. To see that $X \times I$ is not Fréchet, define $A \subset X \times I$ by $A = \bigcup \{A_n : n \in N\}$ where $A_n = \{(n - \frac{1}{k}, \frac{1}{n}) : k \in N\}$. Then $(0, 0) \in \text{cl } A$ since $(0, 0) \in \text{cl } \{(n-1, \frac{1}{n}) : n \in N\}$ and $\{(n-1, \frac{1}{n}) : n \in N\} \subset A$. But no sequence in A converges to (0, 0) because no sequence contained in A converges in $R \times I$ to (k, 0) for any $k \in Z$.

<u>2.20 Example</u> The product of two continuous pseudo-open maps may be a quotient map without being continuous pseudo-open.

<u>Proof</u> Let X and I be the Hausdorff Fréchet spaces of (2.19). By (2.13), the quotient maps $\phi_X : X^* \text{---}> X$ and $\phi_I : I^* \text{---}> I$ are continuous pseudo-open. Since $X \times I$ is sequential, (1.35.2) implies that $\phi_X \times \phi_I : X^* \times I^* \text{---}> X \times I$ is a quotient map. However, $\phi_X \times \phi_I$ cannot be continuous pseudo-open because $X^* \times I^*$ is first-countable but $X \times I$ is not Fréchet.

The next two results, which further illustrate the non-productive nature of Fréchet spaces, generalize Harley's Theorem ([12]). The author provides the proof.

<u>2.21 Lemma</u> The product of two Fréchet spaces, one of which is discrete, is Fréchet.

Proof Let X and Y be Fréchet spaces, and assume that Y is discrete. Let B be a subset of the topological product space $X \times Y$, and suppose that $(u, v) \in \mathrm{cl}\, B$. Since Y is discrete, $U \times \{v\}$ is a neighbourhood of (u, v) for any neighbourhood U of u. Therefore $v \in \{y \in Y : (x,y) \in B\}$. Then $u \in \mathrm{cl}\, \{x \in X : (x,v) \in B\}$ and there is a sequence $\{u_n : n \in \omega\}$ in $\{x : (x,v) \in B\}$ converging to u. The sequence $\{(u_n, v) : n \in \omega\}$ is contained in B and converges to (u, v).

2.22 Theorem Let X be a Fréchet space. Let A be a subset of X satisfying the property : there is a sequence $\{U_n : n \in \omega\}$ of open subsets of X such that (1) $U_n \subset U_{n+1}$, (2) A is contained in $U = \bigcup \{U_n : n \in \omega\}$, and (3) $A \cup U_n$ is not open. Let $X_{/A}$ denote the quotient space obtained from X by identifying the points in A. Then if X is T_1 and Y is Hausdorff, $X_{/A} \times Y$ is Fréchet if and only if Y is discrete. If A is closed, the T_1 hypothesis may be replaced by regularity.

Proof If Y is discrete then, since $X_{/A}$ is Fréchet (by 2.3.4), the topological product space $X_{/A} \times Y$ is Fréchet (by 2.21). To establish the converse, let $i = g(A)$ where $g : X \dashrightarrow X_{/A}$ is the quotient map, and suppose that y is not an isolated point of Y. Then $y \in cl(Y-\{y\})$ and consequently, since Y is Fréchet (by 2.3.2), there is a sequence $\{y_n : n \in \omega\}$ in $Y-\{y\}$ converging to y. Since Y is Hausdorff, it can be assumed without loss of generality that the y_n are distinct. Let $W = \bigcup \{U-(U_n \cup A) \times \{y_n\} : n \in \omega\}$. Then $(i, y) \in cl\ W$. Indeed, this is so because $A \cup U_n$ being not open implies that $(i, y_n) \in cl\ (U-(U_n \cup A) \times \{y_n\})$. Since $X_{/A} \times Y$ is Fréchet, there exists a sequence $\{(r_k, s_k) : k \in \omega\}$ in W converging to (i, y). The Hausdorff hypothesis on Y implies that $\{s_k : k \in \omega\}$ is a subsequence of $\{y_n : n \in \omega\}$. Let $y_{n_k} = s_k$ for each $k \in \omega$. Thus $\{(r_k, y_{n_k}) : k \in \omega\}$ is a sequence in W converging to (i, y) with each $r_k \in U-(U_{n_k} \cup A)$. Since X is T_1 (or X is regular and A is closed), for each $k \in \omega$ there is an open neighbourhood G_k of A such that $r_k \notin G_k$. Let $U_n^k = U_n \cap (\bigcap \{G_j : j \leq k\})$ for each n, $k \in \omega$; and let m_k be the largest member of ω satisfying $r_k \notin U_{(m_k-1)}$. It follows that $\bigcup \{U_{m_k}^k : k \in \omega\}$ is an open neighbourhood of A disjoint from the sequence $\{r_k : k \in \omega\}$. But then $\{r_k : k \in \omega\}$ cannot converge to i which contradicts $\{(r_k, y_{n_k}) : k \in \omega\}$ converging to (i, y). Therefore $\{y\}$ must be open, and the proof is complete.

Several examples of sequential spaces which are not Fréchet have already been given. After another such example, a characterization of those sequential spaces which are also Fréchet follows. The non-Fréchet sequential space M of (1.17) is used to give a characterization in the Hausdorff case.

2.33 Example There is a compact sequential Hausdorff space which is not Fréchet.

Proof Let $\mathcal{F}$ be an infinite maximal pairwise almost disjoint family of infinite subsets of the natural numbers $\mathcal{N}$ (Two sets U and V are almost disjoint if and only if $U \cap V$ is finite). To establish the existence of $\mathcal{F}$, let $\mathcal{G}$ be the collection of all infinite pairwise almost disjoint families of infinite subsets of $\mathcal{N}$. The set $\mathcal{G}$ is partially ordered by set inclusion. Note that $\mathcal{G} \neq \emptyset$. Indeed, for each real number r there is a sequence $\{x_n^r : n \in \omega\}$ of rational numbers converging to r. Then if $f : \mathcal{Q} \longrightarrow \mathcal{N}$ is a bijection between the rationals and the natural numbers, $\{\{f(x_n^r) : n \in \omega\} : r \in \mathcal{R}\} \in \mathcal{G}$. Now, let $\{\mathcal{E}_a : a \in A\}$ be a chain in $\mathcal{G}$ and let $\mathcal{E} = \bigcup \{\mathcal{E}_a : a \in A\}$. For any pair E, $F \in \mathcal{E}$ there is some $c \in A$ such that $E, F \in \mathcal{E}_c$ which implies that $E \cap F$ is finite. Therefore $\mathcal{E} \in \mathcal{G}$ and consequently $\mathcal{E}$ is an upper bound of the chain $\{\mathcal{E}_a : a \in A\}$. Then, by Zorn's lemma, there is a maximal element $\mathcal{F}$ of the set $\mathcal{G}$.

Let $\psi = \mathcal{F} \cup N$ with points of N isolated and neighbourhoods of $F \varepsilon \mathcal{F}$ those subsets of ψ containing F and all but finitely many points of F. Clearly ψ is Hausdorff. Furthermore, ψ is locally compact because $F \cup \{F\}$ is a compact neighbourhood of F in ψ. It follows from ([16], Theorem 5.21) that $\psi^* = \psi \cup \{\infty\}$, the one point compactification of ψ, is a compact Hausdorff space. By definition of the topology on ψ^*, it is clear that $\infty \varepsilon \text{ cl } N$. However, if $\{x_n : n \varepsilon \omega\}$ is a sequence of distinct points in N then, since $\mathcal{F}$ is maximal, $\{x_n : n \varepsilon \omega\} \cap F$ is infinite for some $F \varepsilon \mathcal{F}$ and hence $\{x_n : n \varepsilon \omega\}$ converges to F. Therefore no sequence in N converges to ∞, and so ψ^* is not Fréchet.

It remains to show that ψ^* is a sequential space. Suppose that V is a sequentially open subset of ψ^*, and let $x \varepsilon V$. If $x \varepsilon N$, $x \varepsilon \text{ int } V$ because $\{x\}$ is open. If $x \varepsilon \mathcal{F}$ then, since any sequence of distinct points in x converges to x, x-V is finite and hence $\{x\} \cup \{n : n \varepsilon x \cap V\}$ is a neighbourhood of x contained in V. Assume now that $x = \infty$. Since $F \cup \{F\}$ is a neighbourhood of $F \varepsilon \mathcal{F}$ and ψ^*-U is compact for each open neighbourhood U of ∞, any sequence of distinct points of $\mathcal{F}$ converges to ∞. Accordingly, V contains all but finitely many members of $\mathcal{F}$. Let $\mathcal{F}-V = \{F_i : i \leq m\}$ where $m \varepsilon N$. If $\{U_i : i \leq m\}$ is any finite collection of open sets satisfying $F_i \subset U_i$ then, since $\mathcal{F}$ is maximal, $N-[V \cup (\bigcup \{U_i : i \leq m\})]$ is finite and consequently ψ^*-V is compact. Hence ψ^* being Hausdorff implies that ψ^*-V is a closed compact subset of ψ. Therefore V is open whenever $\infty \in V$. Thus every sequentially open subset of ψ^* is a neighbourhood of each of its points, and so ψ^* is sequential.

Since no sequence in N converges to ∞, the singleton $\{\infty\}$ is sequentially open but not open in the subspace ψ^*-F. Hence ψ^*-F is a non-sequential subspace of ψ^*. The next result proves that such a subspace must always exist in sequential spaces which are not Fréchet.

<u>2.24 Proposition</u> A sequential space is Fréchet if and only if it is hereditarily sequential.

<u>Proof</u> If a sequential space is Fréchet, every subspace is Fréchet and hence sequential. Conversely, if a topological space X is hereditarily sequential, (1.35.1) implies that ϕ_X is a hereditarily quotient map with Fréchet domain. Then ϕ_X is continuous pseudo-open and therefore X is Fréchet.

<u>2.25 Theorem</u> A Hausdorff sequential space is Fréchet if and only if it contains no subspace which, with the sequential closure topology, is homeomorphic to the topological space M of (1.17).

<u>Proof</u> Let Y be a subspace of a sequential space X, and let $h : M \dashrightarrow Y_s$ be a homeomorphism of M onto the subspace Y provided with the sequential closure topology. If $Y \neq Y_s$, Y is a non-sequential subspace of X and hence, by (2.24), X is not Fréchet. If $Y = Y_s$ then, since $M - N$ is a non-sequential subspace of M, $Y - h(N) = h(M) - h(N) = h(M - N)$ is a non-sequential subspace of X. Again by (2.24), X is not Fréchet.

Conversely, suppose that X is not Fréchet. Then there exists a subset B of X such that cl $B \neq B'$ where B' is the set of limits of sequences in B. Since X is sequential, there is a sequence $S = \{x_i : i \in \omega\}$ in B' converging to some point $x \in (\text{cl } B) - B'$. The sequence S is not frequently in B because otherwise it has a subsequence in B converging to $x \notin B'$. Consequently, it can be assumed that the x_i are distinct and $S \subset B' - B$. Then, since $x_i \in B' - B$, there exists a sequence $\{x_{ij} : j \in \omega\}$ in B converging to x_i. The x_{ij} (for $i, j \in \omega$) may be taken all distinct. This is so because X is Hausdorff and S converges to x; that is, there exists a family $\{U_i : i \in \omega\}$ of pairwise disjoint open sets satisfying

$x_i \in U_i$. Since $x \notin B'$, no sequence consisting of the x_{ij} converges to x. Thus the topological space Y_s where the subspace $Y = \{x\} \cup \{x_i : i \in N\} \cup \{x_{ij} : i, j \in N\}$ is homeomorphic to M.

It is easy to verify that the non-Fréchet Hausdorff sequential space ψ^* of (2.23) satisfies (2.25). If $\{E_i : i \in \omega\}$ is any collection of pairwise disjoint subsets of N then, since F is maximal, for each $i \in \omega$ there exists $F_i \in F$ such that $F_i \cap E_i$ is infinite. Thus $\{F_i \cap E_i : i \in \omega\}$ is a pairwise disjoint sequence in F converging to ∞. Let $\{n_{ij} : j \in \omega\}$ be any sequence in $F_i \cap E_i$ such that the $n_{ij}(j \in \omega)$ are distinct. Clearly, each $\{n_{ij} : j \in \omega\}$ converges to $F_i \cap E_i$ and the $n_{ij}(i, j \in \omega)$ are all distinct. Then, since no sequence in N converges to ∞, the subspace $\{\infty\} \cup \{F_i \cap E_i : i \in N\} \cup \{n_{ij} : i, j \in N\}$ provided with the sequential closure topology is homeomorphic to M.

Chapter 3

Generalized Sequential Space Methods

When a topology is specified by its open sets, the importance of basis and subbasis is well-known. In the same way, the concepts of convergence basis and convergence subbasis are prominent in the study of topological spaces whose topologies are determined by their convergence classes. For example, one can study convergence subbases and convergence bases consisting of convergent sequences in lieu of studying sequential spaces and Fréchet spaces respectively. The notion of convergence subbasis is also useful in the investigation of generalized sequential spaces. In this chapter, the topological spaces in which m-nets are adequate to describe open sets are examined; these spaces are called m-sequential (m-Fréchet) spaces. It will be shown that any topological space can, for sufficiently large m, be so described.

<u>3.1 Definition</u> Let X be a set and let $\mathcal{C}$ be a class of pairs (S, x) where S is a net and x is a point in X. The class $\mathcal{C}$ is a p-convergence class on X if and only if it satisfies : If $(\{x_n : n \in D\}, x) \in \mathcal{C}$ and E is a cofinal subset of D, then $(\{x_n : n \in E\}, x) \in \mathcal{C}$. A p-convergence class in which all of the nets are sequences is called a sequential p-convergence class.

Observe that each convergence class ([16], p.74) is a p-convergence class but the converse need not be true. The convergence associated with a p-convergence class C on a set X can be studied topologically by means of the largest topology on X in which the C-nets (i.e., all of the nets in C) are convergent.

3.2 Theorem Let C be a p-convergence class on a set C. For any subset A of X, let t-cl A be the smallest set containing A and closed with respect to the formation of limits of C-nets. Then t-cl is a closure operator and hence defines a topology T(t) for X ([16], Theorem 1.8). This is the largest topology on X in which the C-nets converge.

Proof It is first shown that t-cl is a closure operator. Since a net is a function on a directed set, and the set is non-empty by definition, t-cl ϕ is empty. By definition of t-cl, $A \subset$ t-cl A for each subset A of X. Then t-cl $A \subset$ t-cl(t-cl A). Again by definition of t-cl, t-cl(t-cl A) is the smallest set containing t-cl A and closed with respect to the formation of limits of C-nets. But t-cl A is closed with respect to the formation of limits of C-nets and so t-cl(t-cl A) $\subset$ t-cl A. Hence t-cl A = t-cl(t-cl A). It remains to prove that t-cl(A $\cup$ B) = (t-cl A) $\cup$ (t-cl B). Clearly A $\cup$ B $\subset$ (t-cl A) $\cup$ (t-cl B) $\subset$ t-cl(A $\cup$ B). To establish the opposite inclusion, let S = $\{x_n : n \in D\}$ be a net in (t-cl A) $\cup$ (t-cl B) with (S, x) $\in C$. Let $D_A = \{n \in D : x_n \in$ t-cl A$\}$ and $D_B = \{n \in D : x_n \in$ t-cl B$\}$.

Since $D_A \cup D_B = D$, either D_A or D_B is cofinal in D. It can be assumed without loss of generality that D_A is cofinal in D. Then $S_1 = \{x_n : n \in D_A\}$ is a subnet of S in t-cl A and $(S_1, x) \in C$. Hence $x \in$ t-cl A $\subset$ (t-cl A) $\cup$ (t-cl B) and consequently (t-cl A) $\cup$ (t-cl B) is closed with respect to the formation of limits of C-nets.

Let $(S, x) \in C$. If S does not converge to x in (X, T(t)), there is an open neighbourhood U of x such that S is not eventually in U. Then S is frequently in X-U and there is a subnet S_1 in X-U with $(S_1, x) \in C$. But since U is open, X-U = t-cl(X-U) and hence $x \in$ X-U. The contradiction shows that S must converge to x with respect to T(t), and hence that each C-net converges in (X, T(t)). Suppose now that T_α is a topology on X in which the C-nets converge. If $V \in T_\alpha$ then for each net pair $(S, x) \in C$ such that $S \subset X-V$, $x \in X-V$. Thus X-V = t-cl(X-V) which implies that $V \in T(t)$.

<u>3.3 Corollary</u> Let X be the topological space provided with the topology T(t) derived from a p-convergence C. Then X is T_1 if and only if C satisfies : If $S = \{x_n : n \in D\}$ is a net in X such that $x_n = x$ for each $n \in D$ and $y \neq x$, then $(S, y) \notin C$.

Consequently, if C is a sequential p-convergence class and X has unique sequential limits, X is T_1.

<u>Proof</u> If X is T_1 and $y \neq x$, S cannot converge to y and so (3.2) implies that $(S, y) \notin C$. Conversely, if S_o is a net in $\{x\}$ and $z \neq x$ then, since $(S_o, z) \notin C$, $z \notin$ t-cl$\{x\}$. Hence $\{x\}$ is closed and therefore X is T_1.

<u>3.4 Proposition</u> Let X be the topological space provided with the topology T(t) derived from a sequential p-convergence class C. Let $C(T(t))$ denote the class of convergent sequences in X. Then $C = C(T(t))$ if C satisfies:

(1) If $S = \{x_n : n \in \omega\}$ is a sequence such that $x_n = x$ for each $n \in \omega$, then $(S, x) \in C$.

(2) If S is a sequence and $(S, x) \notin C$ then there is a subsequence of S, no subsequence of which together with x is a member of C.

(3) If $(S, x) \in C$ and $(S, y) \in C$ then $x = y$.

<u>Proof</u> It is clear that $C \subset C(T(t))$. To prove the opposite inclusion, suppose that $(S, x) \notin C$. By (2), there is a subsequence $S_o = \{y_n : n \in \omega\}$ of S, no subsequence of which together with x is a member of C. It can be assumed without loss of generality that $y_n \neq x$ for each $n \in \omega$.

Either there exists $(S_1, z) \varepsilon C$ such that S_1 is a subsequence of S_o or no such sequence pair exists. In the first case, any C-net in $S_1 \cup \{z\}$ is either a subsequence of S_1 or frequently equal to some point in S_1. Then conditions (2) and (3) imply that $S_1 \cup \{z\}$ is a closed subset of X disjoint from x. In the second case, S_o is a closed subset of X disjoint from x. In both cases, S has a subsequence not converging to x. Hence $(S, x) \not\varepsilon C(T(t))$.

Another closure operator is defined in the following proposition.

<u>3.5 Proposition</u> Let C be a p-convergence class on a set X, and for each subset A of X let c-cl A be the union of A and the limits of those C-nets contained in A. Then if C is a convergence class, c-cl is a closure operator, and $(S, x) \varepsilon C$ if and only if S converges to x with respect to the topology associated with c-cl.

<u>Proof</u> This is given in ([16], Theorem 2.9). In the presence of a convergence class, c-cl is precisely the closure operator described in that theorem.

<u>3.6 Corollary</u> If C_α and C_β are convergence classes on a set X and T_α and T_β are the associated topologies, then $C_\alpha \subset C_\beta$ if and only if $T_\alpha \supset T_\beta$.

<u>Proof</u> If $(S, x) \in C_\alpha$, S is eventually in each neighbourhood of x in (X, T_α). Thus $T_\beta \subset T_\alpha$ implies that S is eventually in each neighbourhood of x in (X, T_β) and hence that $(S, x) \in C_\beta$. Conversely, suppose that $C_\alpha \subset C_\beta$ and let $U \in T_\beta$. If $(S, x) \in C_\alpha$ and $S \subset X-U$, then $(S, x) \in C_\beta$ and $U \in T_\beta$ implies that $x \in X-U$. Therefore X-U is closed in (X, T_α) and so $U \in T_\alpha$.

<u>3.7 Proposition</u> Let C be a p-convergence class on a set X, and let T(t) be the topology for X associated with the t-closure operator. Then T(t) is the topology with the smallest convergence class containing C.

<u>Proof</u> Let $C(T(t))$ denote the convergence class for (X, T(t)). According to (3.2), T(t) is the largest topology on X in which the C-nets converge. Then, if T is any topology on X whose convergence class $C(T)$ contains C, $T \subset T(t)$. The preceding result implies that $C(T(t)) \subset C(T)$.

If C is a p-convergence class on a set X, then c-cl need not be idempotent and hence not a closure operator. Let T(c) denote the topology associated with c-cl whenever c-cl(c-cl A) = c-cl A for each subset A of X. Clearly, c-cl A is a subset of t-cl A, and it can be a proper subset. Observe that a topological space is Fréchet if and only if for each subset A of X, cl A = c-cl A with respect to the convergent sequences in X; similarly, X is sequential if and only if cl A = t-cl A. Therefore, since not all sequential spaces are Fréchet, c-cl and t-cl need not coincide. In particular, consider the topological space M of (1.17). As previously observed, no sequence in $N \times N$ converges to 0. Hence c-cl$(N \times N)$ = M-{0} whereas t-cl$(N \times N)$ = M.

The t-closure operator, however, can be constructed inductively by iteration of c-cl. Define $A^{\circ} = A$ and for each successor ordinal α let A^{α} = c-cl A^{β} where $\alpha = \beta + 1$ for some ordinal β. ($\beta + 1$ denotes the ordinal successor of β). If α is a limit ordinal define $A^{\alpha} = \bigcup \{A^{\beta} : \beta < \alpha\}$. It is clear that $A^{\alpha} \subset$ t-cl A for each ordinal α, and if A^{α} = t-cl A then A^{β} = t-cl A whenever $\beta \geq \alpha$. For any subset A of X the cardinality of the number of iterations of c-cl to obtain t-cl A is $\leq 2^{m}$ where m is the cardinality of the set X. Then, since the ordinals are well-ordered, for each $x \in$ t-cl A there is a smallest ordinal η such that x belongs to the η-th iterate of c-cl on A; that is, $x \in A^{\alpha}$ whenever $\alpha \geq \eta$ and $x \notin A^{\alpha}$ whenever $\alpha < \eta$.

<u>3.8 Definition</u> Let C be a p-convergence class on a set X and let A be a subset of X.

(1) A point $x \in$ t-cl A is said to be of Baire order η (write ord $x = \eta$) with respect to C and A if and only if η is the smallest ordinal such that x is a member of the η-th iterate of c-cl on A.

(2) The Baire order of a set A (ord A) is defined as $\sup\{\text{ord } x : x \in \text{t-cl } A\}$.

<u>3.9 Proposition</u> Let C be a p-convergence class for a set X. Then $T(t) = T(c)$ if and only if every subset of X has Baire order ≤ 1.

<u>Proof</u> The topology $T(t)$ coincides with $T(c)$ if and only if t-cl A = c-cl A for each subset A of X, which occurs if and only if every subset of X has Baire order ≤ 1.

<u>3.10 Definition</u> Let C be a p-convergence class on a topological space X.

(1) C is a convergence subbasis for X, or for the topology on X, if and only if the topology on X is the topology with the smallest convergence class containing C.

(2) C is a convergence basis for X, or for the topology on X, if and only if C is a convergence subbasis for X and every subset of X has Baire order ≤ 1.

<u>3.11 Proposition</u> Let C be a p-convergence class on a topological space X.

(1) C is a convergence subbasis for X if and only if X has topology $T(t)$.

(2) C is a convergence basis for X if and only if X has topology $T(c)$.

Proof By virtue of (3.7) and (3.10), (1) is clear. Then (1) together with (3.9) implies (2).

3.12 Definition Let m be an infinite cardinal number. An m-net is a net whose directed set is of cardinality $\leq$ m.

3.13 Definition Let X be a topological space, and let m be an infinite cardinal number.

(1) X is m-sequential, or an m-sequential space, if and only if it has a convergence subbasis in which all of the nets are m-nets.

(2) X is m-Fréchet, or an m-Fréchet space, if and only if it has a convergence basis consisting of m-nets.

3.14 Proposition (1) Every m-Fréchet space is m-sequential.

(2) If a topological space is m-sequential then it is m_1-sequential whenever $m_1 \geq m$. Similarly, an m-Fréchet space is m_1-Fréchet if $m_1 \geq m$.

Proof The proof of (1) is obvious because, by definition, every convergence basis is a convergence subbasis. Since every m-net is an m_1-net for $m_1 \geq m$, (2) is also clear.

The next two results give several equivalent formulations of the definitions of m-sequential space and m-Fréchet space respectively.

3.15 Proposition The following statements about an arbitrary topological space X are equivalent.

(1) X is m-sequential.

(2) X has topology T(t) with respect to some p-convergence class consisting of m-nets in X.

(3) A subset F of X is closed if and only if no m-net in F converges to a point not in F.

(4) A subset U of X is open if and only if each m-net in X converging to a point in U is eventually in U.

(5) The class C of all pairs (S, x) where S is an m-net in X converging to the point x is a convergence subbasis for X.

Proof If X is m-sequential then, by definition, X has a convergence subbasis C_1 in which all of the nets are m-nets. According to (3.11), X has topology T(t) with respect to C_1 and therefore (1) implies (2). If F is a subset of X with no m-net in F converging to a point not in F, no C_1-net in F converges to a point not in F; consequently F = t-cl F = cl F and (2) implies (3). Suppose that U is a subset of X such that each m-net in X converging to a point in U is eventually in U. Let S be an m-net in X-U converging to a point x. Then $x \in X-U$

because otherwise S is eventually in U. Thus (3) implies that X-U is closed and hence that U is open. To establish that (4) implies (5), let T be any topology for X in which the C-nets converge. Then if V ε T, every m-net in X converging to a point in V is eventually in V, and so V is open by (4). Accordingly X has the largest topology in which the C-nets converge. It follows from (3.6) that (4) implies (5). By definition, (5) obviously implies (1).

3.16 Proposition For any arbitrary topological space X, the following are equivalent.

(1) X is m-Fréchet.

(2) The class C of all pairs (S, x) where S is an m-net in X converging to x is a convergence basis for X.

(3) The closure of any subset A of X is the set of limits of m-nets in A.

(4) X has topology T(c) with respect to some p-convergence class consisting of m-nets in X.

Proof If X is m-Fréchet, X is m-sequential and hence the class C is a convergence subbasis for X. Moreover, every subset of X has Baire order ≤ 1. Therefore C is a convergence basis and (1) implies (2). It follows from (2) that X has the topology T(c) associated with C. Consequently (2) implies (4). In addition, (2) is equivalent to (3) because for each subset A of X, x ε cl A if and only if there exists an m-net in A converging to x. The proof that (4) implies (1) is clear form (3.11).

The following corollary together with (3.15) and (3.16) shows:

(1) A topological space is sequential if and only if it has a convergence subbasis in which all of the nets are sequences.

(2) A topological space is Fréchet if and only if it has a convergence basis in which all of the nets are sequences. Furthermore, this result implies that every Fréchet space is m-Fréchet and that every sequential space is m-sequential.

<u>3.17 Corollary</u> (1) A topological space is sequential if and only if it is $\aleph_0$-sequential.

(2) A topological space is Fréchet if and only if it is $\aleph_0$-Fréchet.

<u>Proof</u> Since every sequence is an $\aleph_0$-net, the sequential spaces and the Fréchet spaces obviously satisfy (3.15.3) and (3.16.3) respectively. To prove the converses, it is first shown that every non-trivial $\aleph_0$-net has a cofinal sequence. Let $\{x_n : n \in D\}$ be an $\aleph_0$-net and let $g : \omega \longrightarrow D$ be a bijection. Because D is a directed set, for each $k \in \omega$ there exists $n_k \in \omega$ such that $g(n_k) \geq g(i)$ for every $i \leq k$. Then $\{x_{g(n_k)} : k \in \omega\}$ is a subnet of $\{x_n : n \in D\}$. Suppose that F is a sequentially closed subset of a topological space X. If S is an $\aleph_0$-net in F converging to some point x, $x \in F$ since otherwise S has a cofinal sequence which is eventually in X-F. Then

if X is $\aleph_0$-sequential, F is closed by (3.15). Thus (1) is proved. To complete the proof of (2), suppose that A is a subset of an $\aleph_0$-Fréchet space X and let $x \in \mathrm{cl}\, A$. By virtue of (3.16), there is an $\aleph_0$-net in A converging to x. Then, since every non-trivial $\aleph_0$-net has a cofinal sequence, there exists a sequence in A converging to x.

In view of (3.15) and (3.16), it is easy to see that many of the properties of sequential spaces and Fréchet spaces can be generalized to m-sequential spaces and m-Fréchet spaces respectively, by simply replacing "sequences" with "m-nets". This is so whenever those properties of sequences used, can be generalized to m-nets. Nevertheless, for greater generality it is convenient to state results in terms of a convergence subbasis or a convergence basis.

Let $\mathcal{C}$ be the class of convergent m-net pairs in a topological space X. According to (3.15), $\mathcal{C}$ is a convergence subbasis for X if and only if X is m-sequential. However, it is possible to have a convergence subbasis which is a proper subset of $\mathcal{C}$. Although using a smaller convergence subbasis may increase Baire order, there is an upper bound.

3.18 Proposition If X is an m-sequential space with any convergence subbasis, then no element of X has Baire order equal to the least ordinal of cardinality m^+. (m^+ denotes the cardinal successor of m)

Proof Let ω_α denote the least ordinal of cardinality m^+. Then α is of the form $\beta + 1$ for some ordinal β, where ω_β is the least ordinal of cardinality m. Thus ω_α is regular and hence not the supremum of any set B of strictly smaller ordinals if the cardinality of B is $\leq$ m. Assume that A is a subset of X and let $x \in$ cl A. Since X is m-sequential, there exists an m-net $\{x_n : n \in D\}$ such that $x_n \in$ cl A and ord $x_n <$ ord x. Then ord x = sup {ord $x_n : n \in D\}$, and consequently ord $x < \omega_\alpha$.

3.19 Theorem Let $\mathcal{C}$ be a convergence subbasis for a topological space X, let Y be a subset of X, and let $\mathcal{D}$ be the trace of $\mathcal{C}$ on Y. (i.e., $\mathcal{D} = \{(\{x_n : n \in D\}, x) \in \mathcal{C} : x_n \in Y$ for each $n \in D$, $x \in Y\}$.) Then $\mathcal{D}$ is a convergence subbasis for a topology on Y which is larger than the relative topology. This induced topology coincides with the relative topology on Y if Y is closed or open in X. The two topologies coincide for all subsets of X if and only if $\mathcal{C}$ is a convergence basis.

Proof The space X has the topology T(t) associated with $\mathcal{C}$, and the trace $\mathcal{D}$ is clearly a p-convergence class on Y. For each subset A of Y define u-cl A to be the smallest set containing A and closed with respect to the formation of limits of $\mathcal{D}$-nets. By (3.2), u-cl is a closure operator on Y and hence defines a topology T(u) for Y. It follows from (3.11.1) that $\mathcal{D}$ is a convergence subbasis for T(u). Furthermore, T(u) is larger than the relative topology on Y because the $\mathcal{D}$-nets converge in the relative topology and T(u) is the largest topology on Y in which the $\mathcal{D}$-nets converge.

To establish that these topologies on Y coincide when Y is closed or open, or when $\mathcal{C}$ is a convergence basis, it suffices to show that the two topologies have the same closed sets. Let F be a subset of Y. By definition of the u-closure operator, u-cl $F \subset$ (t-cl F) $\cap$ Y. To prove the opposite inclusion, assume that $x \in$ (t-cl F) $\cap$ Y with ord $x = \lambda$ with respect to $\mathcal{C}$ and F, and proceed by transfinite induction on λ. If $\lambda = 0$, (t-cl F) $\cap Y = F^\circ \cap Y = F \cap Y = F \subset$ u-cl F. If $\lambda = 1$ then (t-cl F) $\cap Y = F^1 \cap Y =$ (c-cl F) $\cap Y \subset$ u-cl F. Thus the proof is complete for the case in which $\mathcal{C}$ is a convergence basis. For $\lambda > 1$ consider separately the cases Y is closed and Y is open. First, suppose that Y is closed. Then t-cl $F \subset$ t-cl $Y = Y$. By the induction hypothesis, there exists a net pair $(\{x_n : n \in D\}, x) \in \mathcal{C}$ with $x_n \in$ (t-cl F) $\cap$ Y and ord $x_n < \lambda$ for each $n \in D$. Consequently each $x_n \in$ u-cl F and $(\{x_n : n \in D\}, x) \in \mathcal{D}$, which implies that $x \in$ u-cl F. Assume now that Y is open. By the induction hypothesis, there is a net pair $(\{y_n : n \in D\}, x) \in \mathcal{C}$ with $y_n \in$ t-cl F and ord $y_n < \lambda$ for each $n \in D$. Since Y is open and $x \in Y$, the net $\{y_n : n \in D\}$ is eventually in Y. Thus $E = \{n \in D : x_n \in Y\}$ is a cofinal subset of D and so $(\{y_n : n \in E\}, x) \in \mathcal{D}$. Therefore $y_n \in$ u-cl F for each $n \in E$ implies that $x \in$ u-cl F.

It remains to show that if the two topologies are the same then $\mathcal{C}$ is a convergence basis. If $\mathcal{C}$ is not a convergence basis for X, there is at least one Baire order 2 situation. That is, there exists a subset B of X with $x \in$ t-cl B and ord $x = 2$. Let $Y = B \cup \{x\}$. Then $x \in$ (t-cl B) $\cap$ Y but, since ord $x = 2$, (c-cl B) $\cap$ Y = B and no net in B converges to $x \in Y$. Thus $x \notin$ u-cl B and the topologies are different.

<u>3.20 Corollary</u> Every open or closed subspace of an m-sequential space is m-sequential. A topological space is m-Fréchet if and only if it is hereditarily m-Fréchet if and only if it is hereditarily m-sequential.

<u>Proof</u> Let Y be an open or closed subspace of an m-sequential space X with a convergence subbasis $\mathcal{C}$ consisting of m-nets. According to (3.19), the trace of $\mathcal{C}$ on Y is a convergence subbasis for the relative topology on Y. Hence Y is m-sequential.

It is obvious that every hereditarily m-Fréchet space is m-Fréchet. Conversely, if X is an m-Fréchet space with a convergence basis $\mathcal{C}$ consisting of m-nets then, by (3.19), for every subspace Y of X the trace of $\mathcal{C}$ on Y is a convergence subbasis for the relative topology on Y. Thus Y is m-sequential and consequently every m-Fréchet space is hereditarily m-sequential. In addition, any subspace of an m-Fréchet space is hereditarily m-sequential. It remains to show that every hereditarily m-sequential space is m-Fréchet. Assume that X is a hereditarily m-sequential space with a convergence subbasis $\mathcal{C}$ in which all of the nets are m-nets. If Y is a subspace of X, then Y is m-sequential and therefore has a convergence subbasis $\mathcal{D}$ consisting of m-nets. The trace of $\mathcal{C}$ on Y surely coincides with $\mathcal{D}$. The preceding theorem implies that $\mathcal{C}$ is a convergence basis for X and hence that X is m-Fréchet.

<u>3.21 Proposition</u> If X is the disjoint topological sum of any family $\{X_a : a \in A\}$ of topological spaces where each X_a has a convergence subbasis $\mathcal{C}_a$, then $\mathcal{C} = \cup\{\mathcal{C}_a : a \in A\}$ is a convergence subbasis for X. If each $\mathcal{C}_a$ is a convergence basis, so is $\mathcal{C}$.

<u>Proof</u> Let $(S, x) \in \mathcal{C}$ and suppose that U is an open neighbourhood of x in X. Then $(S, x) \in \mathcal{C}_a$ for some $a \in A$ and therefore, since $U \cap X_a$ is open in X_a, $x_n \in U \cap X_a$ for all n sufficiently large. Thus the convergence class on X contains $\mathcal{C}$. Now let T denote the usual topology on X and let T_α be any topology for X whose convergence class $\mathcal{C}(T_\alpha)$ contains $\mathcal{C}$. If $V \notin T$, $V \cap X_c$ is not open in X_c for some $c \in A$. From this, it follows that there exists a

C-net S_1 in $X_c - V$ converging to a point $y \in V \cap X_c$. Then, since $(S_1, y) \in C$ and $C \subset C(T_\alpha)$, $V \notin T_\alpha$. Consequently, T is the largest topology on X in which the C-nets converge, or equivalently by (3.6), T is the topology with the smallest convergence class containing C. Hence C is a convergence subbasis for X.

Assume now that each C_a is a convergence basis. To complete the proof, it suffices to show that every subset of X has Baire order ≤ 1 with respect to the convergence subbasis C. Let F be a subset of X. Then $(c\text{-cl } F) \cap X_a = c\text{-cl}_{X_a}(F) = \text{cl}_{X_a}(F) = (\text{cl } F) \cap X_a$. Therefore $(c\text{-cl } F) \cap X_a$ is closed in X_a for each $a \in A$, and hence c-cl F is closed in X.

<u>3.22 Corollary</u> The disjoint topological sum of any family of m-sequential spaces is m-sequential. The disjoint topological sum of any family of m-Fréchet spaces is m-Fréchet.

<u>3.23 Definition</u> Let C be a p-convergence class on a set X. For any function with domain X let fC denote the set of all net pairs $(\{f(x_n) : n \in D\}, f(x))$ for $(\{x_n : n \in D\}, x) \in C$.

<u>3.24 Theorem</u> Let $f : X \longrightarrow Y$ be a function of a topological space X into a topological space Y, and let $\mathcal{C}$ be a convergence subbasis for X.

(1) The function f is continuous if and only if $f\mathcal{C}$ is contained in the convergence class of Y.

(2) If f is surjective, $f\mathcal{C}$ is a convergence subbasis for Y if and only if Y is a quotient space.

<u>Proof</u> (1) Let $(\{f(x_n) : n \in D\}, f(x)) \in f\,\mathcal{C}$. Since $\mathcal{C}$ is a convergence subbasis for X, each $\mathcal{C}$-net belongs to the convergence class of X. Accordingly, $\{x_n : n \in D\}$ converges to x in X. Then, since f is continuous, $\{f(x_n) : n \in D\}$ converges to $f(x)$ in Y. Conversely, let A be a closed subset of Y and suppose that $(\{x_n : n \in D\}, x) \in \mathcal{C}$ with $x_n \in f^{-1}(A)$ for every $n \in D$. Clearly each $f(x_n) \in A$. Because A is closed and by hypothesis $\{f(x_n) : n \in D\}$ converges to $f(x)$, $f(x) \in A$. Thus $x \in f^{-1}(A)$ and $f^{-1}(A) = \text{t-cl } f^{-1}(A)$. Then, since $\mathcal{C}$ is a convergence subbasis for X, $f^{-1}(A)$ is closed.

(2) By definition, $f\mathcal{C}$ is a convergence subbasis for Y if and only if the topology on Y is the topology with the smallest convergence class containing $f\mathcal{C}$. According to (3.6) and part (1), $f\mathcal{C}$ is a convergence subbasis for Y if and only if the topology on Y is the largest topology such that f is continuous.

3.25 Corollary Every quotient of an m-sequential space is m-sequential.

Proof Let $f : X \dashrightarrow Y$ be a quotient map of an m-sequential space X onto a topological space Y. The space X has a convergence subbasis $\mathcal{C}$ in which all of the nets are m-nets. It is obvious that each net pair in $f\mathcal{C}$ is an m-net pair. Then, since (3.24) implies that $f\mathcal{C}$ is a convergence subbasis for Y, Y is m-sequential.

Example 2.5 shows that the quotient of a Fréchet space need not be Fréchet. Consequently, if $\mathcal{C}$ is a convergence basis for a topological space X and f is a quotient map with domain X, it is only possible to conclude that $f\mathcal{C}$ is a convergence subbasis for the quotient space. However, $f\mathcal{C}$ is a convergence basis whenever f is continuous pseudo-open.

3.26 Theorem Let $f : X \dashrightarrow Y$ be a surjection of the topological space X onto the topological space Y, and let $\mathcal{C}$ be a convergence basis for X. Then f is continuous pseudo-open if and only if $f\mathcal{C}$ is a convergence basis for Y.

Proof Let Y_1 be a subspace of Y and let $\mathcal{D}$ be the trace of $\mathcal{C}$ on $f^{-1}(Y_1)$. By virtue of (3.19), $\mathcal{D}$ is a convergence subbasis for the relative topology on $f^{-1}(Y_1)$. If f is continuous pseudo-open, then f is hereditarily quotient and so $f\mathcal{D}$ is a convergence subbasis for the relative topology on Y_1. Moreover, $f\mathcal{D}$ coincides with the trace of fC on Y_1 and hence (3.19) implies that fC is a convergence basis for Y.

Assume now that fC is a convergence basis for Y. By (3.24.1), f is continuous. Let $x \in Y$ and let U be an open neighbourhood of $f^{-1}(y)$ in X. If f is not pseudo-open, $y \in cl(Y-f(U))$ and hence there is a net pair $(\{f(x_n) : n \in D\}, f(x)) \in f\,C$ such that $f(x) = y$ and each $f(x_n) \in Y-f(U)$. Consequently $x_n \notin U$ for every $n \in D$. Then, since $x \in U$, the net $\{x_n : n \in D\}$ does not converge to x in X. Therefore $(\{x_n : n \in D\}, x) \notin C$ and the theorem is proved by contradiction.

3.27 Corollary Every continuous pseudo-open image of an m-Fréchet space is m-Fréchet.

Proof This follows from (3.26) in the same way that (3.25) followed from (3.24).

3.28 Definition Let m be an infinite cardinal number. A topological space is m-first-countable, or an m-first-countable space, if and only if each point has a neighbourhood basis of cardinality $\leq$ m. (Note that $\aleph_0$-first-countable and first-countable are equivalent concepts.)

<u>3.29 Proposition</u> If X is a topological space and m is an infinite cardinal, then each of the following implies the next.

(1) X is m-first-countable.

(2) X is m-Fréchet.

(3) X is m-sequential.

(4) For any subset A of X, each point in cl A is in cl B for some subset B of A with the cardinality of B $\leq$ m.

(5) X is 2^m-Fréchet.

<u>Proof</u> (1) ---> (2) Let F be a subset of X and suppose that $x \in$ cl F. By hypothesis, x has a neighbourhood basis $\{U_a : a \in A\}$ with the cardinality of A $\leq$ m. Defining a < c if and only if $U_c \subset U_a$, A is a directed set with order <. Because $x \in$ cl F, there exists $x_a \in U_a \cap F$ for each $a \in A$. Then $\{x_a : a \in A\}$ is an m-net in F converging to x, and hence X is m-Fréchet.

(2) ---> (3) This is clear since every convergence basis is a convergence subbasis.

(3) ---> (4) If X is m-sequential, X has a convergence subbasis $\mathcal{C}$ in which all of the nets are m-nets. For $x \in$ cl A = t-cl A, the existence of a subset B satisfying (4) is established by transfinite induction on the Baire order λ of x with respect to A and $\mathcal{C}$. If $\lambda = 0$ then $x \in A$ and $x \in$ cl $\{x\}$. If $\lambda = 1$, $x \in$ c-cl A and there is an m-net pair $(S, x) \in \mathcal{C}$ such that S is an m-net in A converging to x. Clearly $x \in$ cl S and the cardinality of S is $\leq$ m. By the

induction hypothesis, there exists an m-net pair $(\{x_n : n \in D\}, x) \in C$ with $x_n \in \text{cl } A$ and $\text{ord } x_n < \lambda$ for each $n \in D$. Consequently each $x_n \in \text{cl } B_n$ for some subset B_n of A with the cardinality of $B_n \leq m$. But then $x \in \text{cl } (\bigcup \{B_n : n \in D\})$, $\bigcup\{B_n : n \in D\} \subset A$, and the cardinality of $\bigcup\{B_n : n \in D\}$ is $\leq m^2 = m$.

(4) ---> (5) Let A be a subset of X and suppose that $x \in \text{cl } A$. By hypothesis, $x \in \text{cl } B$ for some subset B of A with the cardinality of $B \leq m$. If $\{U_i : i \in I\}$ is a neighbourhood basis at x, $U_i \cap B \neq \emptyset$ for every $i \in I$. Since B has cardinality $\leq m$, there are at most 2^m distinct sets $U_i \cap B$. Define an equivalence relation on I by identifying a and c whenever $U_a \cap B = U_c \cap B$. Let D be the index set I under this equivalence relation. The cardinality of D is $\leq 2^m$. Order D by specifying $a < c$ if and only if $U_a \supset U_c$, and for each $n \in D$ choose $x_n \in U_n \cap B$. Then $\{x_n : n \in D\}$ is a 2^m-net in B converging to x.

For the case $m = \aleph_0$, there are examples which show that all of the conditions in the foregoing proposition are distinct. As previously observed, the real line with the integers identified is a Fréchet space which is not first-countable and the space M of (1.17) is a sequential space which is not Fréchet. The countable space $Q \times Q'$ of (1.19) clearly satisfies (4) but it is not sequential.

Finally, the ordinal space $\Omega + 1$ with the order topology is a $2^{\aleph_0}$-Frechet space which does not satisfy (4).

The next result is a characterization of m-sequential spaces which generalizes the characterization of sequential spaces. An interesting corollary to this theorem is a characterization of m-Fréchet spaces which leads to an extension of the characterization of Hausdorff Fréchet spaces given in (2.15). The Hausdorff hypothesis is eliminated.

<u>3.30 Theorem</u> A topological space is m-sequential if and only if it is a quotient of an m-first-countable space.

<u>Proof</u> By virtue of (3.29) and (3.25), a quotient of an m-first-countable space is m-sequential. Conversely, let X be an m-sequential space with a convergence subbasis in which all of the nets are m-nets. For each $(\{x_n : n \in D\}, x) \in \mathcal{C}$, let $S(x_n, x) = \{x_n : n \in D\} \cup \{x\}$ be a topological space in which the x_n are taken to be distinct and $x_n \neq x$ for every $n \in D$, and which has the convergence basis generated by the $\mathcal{C}$-net pair $(\{x_n : n \in D\}, x)$. Each x_n is isolated, and x has a neighbourhood basis indexed by the directed set D whose cardinality is $\leq m$. Thus each $S(x_n, x)$ is m-first-countable. The disjoint topological sum W of all such $S(x_n, x)$ is therefore m-first-countable and has, by (3.21), a convergence basis $\mathcal{E}$ formed by taking the union

of the convergence bases for the $S(x_n, x)$. The theorem now follows from (3.24). The surjection $f : W \dashrightarrow X$ defined by $f(x) = x$ is a quotient map since $f\mathcal{E} = \mathcal{C}$.

<u>3.31 Corollary</u> A topological space is m-Fréchet if and only if it is a continuous pseudo-open image of an m-first-countable space.

<u>Proof</u> The continuous pseudo-open image of an m-first-countable space is m-Fréchet by (3.29) and (3.27). The converse coincides with that of (3.30) with the exception that $\mathcal{C}$ is a convergence basis and the fact that $f\mathcal{E} = \mathcal{C}$ together with (3.26) implies f is continuous pseudo-open.

<u>3.32 Corollary</u> For any topological space X, the following statements are equivalent.

(1) X is Fréchet.

(2) X is a continuous pseudo-open image of a first-countable space.

(3) X is a continuous pseudo-open image of a metric space.

<u>Proof</u> According to (3.31) and (3.17.2), (1) is equivalent to (2). Clearly (3) implies (1). To establish the opposite implication, let X be a Fréchet space with a convergence basis $\mathcal{C}$ in which all of the nets are sequences. Then W has a convergence basis $\mathcal{E}$ consisting

of sequences, and $f\mathcal{E} = \mathcal{C}$ implies that f is a continuous pseudo-open map of W onto X. Each summand $S(x_n, x)$ of W is a convergent sequence in the Hausdorff space $S(x_n, x)$. Hence, by (1.32), W is metrizable.

In general, the product of m-sequential or m-Fréchet spaces need not be m-sequential. Several examples for the case $m = \aleph_0$ have already been given. However, the product of two m-sequential spaces, one of which is such that each point has a neighbourhood basis consisting of m-sequentially compact sets, is m-sequential. (A topological space is m-sequentially compact if and only if every m-net has a convergent m-subnet.) The proof of this result is analogous to that of (1.24). The following is a generalization of (1.23).

<u>3.33 Proposition</u> Let X be the product of any family $\{X_a : a \in A\}$ of non-trivial topological spaces (each space has at least one non-empty proper open set). If the cardinality of A is > m, then X is not m-sequential. In particular, no uncountable product of non-trivial spaces is sequential.

<u>Proof</u> By hypothesis, each coordinate space X_a contains two points, denoted by 0 and 1, and a neighbourhood of 1 not containing 0. Let e be the function in X whose a-th value is 1 for each $a \in A$, and

let E be the subset of X consisting of all characteristic functions of finite subsets of A. Clearly $e \in \text{cl } E$. It suffices to prove that no iteration of m-nets in E can converge to e. For convenience, define the cozero set of a function f, denoted by coz f, to be the set $\{a \in A : f(a) \neq 0\}$. The functions in E have finite cozero sets. Suppose that $\{f_n : n \in D\}$ is an m-net of functions converging to f with the cardinality of each coz $f_n \leq m$. Since coz $f \subset \bigcup\{\text{coz } f_n : n \in D\}$, the cardinality of coz f is $\leq m^2 = m$. Thus by forming iterated limits of m-nets in E it is only possible to obtain functions whose cozero sets have cardinality $\leq m$. Consequently, if the cardinality of A is $> m$, then the cardinality of coz e is $> m$ and hence no m-net in $(\text{cl } E)-\{e\}$ converges to e.

<u>3.34 Proposition</u> If X is the product of any family $\{X_a : a \in A\}$ of non-trivial topological spaces, then each point of X has a neighbourhood basis of cardinality less than or equal to the maximum r of the cardinality p of A and $q = \sup \{\chi(X_a) : a \in A\}$. (Write $\chi(Y) = m$ if and only if Y is m-first-countable.)

<u>Proof</u> Let $x \in X$ and assume that $\{U_i^a : i \in I_a\}$ is a neighbourhood basis for the a-th coordinate of x with the cardinality of $I_a \leq \chi(X_a)$. For each $a \in A$ let P_a denote the canonical projection map of X onto X_a.

Then the set $\mathcal{B}$ of all finite intersections of elements in $B = \bigcup\{\{P_a^{-1}(U_1^a) : i \in I_a\} : a \in A\}$ is a neighbourhood basis for x. Since the cardinality of B is $\leq pq \leq r^2 = r$, $\mathcal{B}$ has cardinality $\leq \aleph_0 \cdot r = r$. (See 1.18.)

Chapter 4

Generalized Sequential Spaces and their Properties in Ordered Topological Spaces

The properties of convergence subbasis and convergence bases are applied, in this chapter, to the investigation of topological spaces whose open sets are specified by well-ordered nets.

__4.1 Definition__ A well-ordered net is a net whose directed set is well-ordered. (A well-ordered m-net is a net whose directed set is well-ordered and of cardinality $\leq$ m.)

__4.2 Definition__ (1) A topological space is weakly sequential, or a weakly sequential space, if and only if it has a convergence subbasis in which all of the nets are well-ordered.

(2) A topological space is weakly Fréchet, or a weakly Fréchet space, if and only if it has a convergence basis consisting of well-ordered nets.

__4.3 Definition__ Let X be a topological space, and let m be an infinite cardinal.

(1) X is __m__-sequential, or an __m__-sequential space, if and only if it has a convergence subbasis in which all of the nets

are well-ordered m-nets.

(2) X is m-Fréchet, or an m-Fréchet space, if and only if it has a convergence basis consisting of well-ordered m-nets.

Since sequences are well-ordered $\aleph_0$-nets, these generalized sequential spaces and generalized Fréchet spaces clearly contain the sequential and Fréchet spaces respectively. In particular, a topological space is (sequential, Fréchet) if and only if it is ($\underline{\aleph_0}$-sequential, $\underline{\aleph_0}$-Fréchet). Observe that a space is m-sequential if and only if it is both weakly sequential and m-sequential. Similarly, a topological space is m-Fréchet if and only if it is both weakly Fréchet and m-Fréchet.

The aim of the first part of this chapter is to characterize the generalized sequential spaces of (4.2) and (4.3). Their characterizations lead to new characterizations of the Fréchet spaces and the sequential spaces in terms of orderable spaces. To avoid tedious repetition, the elementary properties of these generalized sequential spaces will not be formally stated. The preceding chapter's survey of convergence subbases greatly facilitates their study. It is easy to see that the investigation of these spaces is analogous to that of the m-sequential spaces.

<u>4.4 Definition</u> (1) A topological space is weakly first-countable, or a weakly first-countable space, if and only if each of its points has a well-ordered neighbourhood basis. (A collection $\{F_a : a \in A\}$ of sets is called well-ordered whenever A is well-ordered and $F_a \subset F_c$ if and only if $a > c$ in A.)

(2) A topological space is $\underline{m}$-first-countable, or an $\underline{m}$-first-countable space, if and only if each of its points has a well-ordered neighbourhood basis of cardinality $\leq m$. (Note that first-countable, $\underline{\aleph}_0$ -first-countable, and $\aleph_0$-first-countable are equivalent concepts.)

<u>4.5 Proposition</u> (1) Every weakly first-countable space is weakly Fréchet and hence weakly sequential.

(2) Every $\underline{m}$-first-countable space is $\underline{m}$-Fréchet and hence $\underline{m}$-sequential.

<u>Proof</u> Let F be a subset of a weakly first-countable space X and suppose that $x \in \text{cl } F$. By definition, x has a well-ordered neighbourhood basis $\{U_a : a \in A\}$. Then, since $x \in \text{cl } F$, there exists a well-ordered net $\{x_a : a \in A\}$ which satisfies $x_a \in U_a \cap F$ and therefore converges to x. If X is $\underline{m}$-first-countable, the cardinality of A is $\leq m$.

4.6 Example For any uncountable cardinal m, there is an m-first-countable space which is not weakly sequential and hence not m-sequential.

Proof Let D be the family of all finite subsets of a set whose cardinality is m, and order D by $\supset$. Then D is a directed set of cardinality m. Let $S = \{x_n : n \in D\} \cup \{x\}$ be a topological space in which the x_n are distinct and $x_n \neq x$ for every $n \in D$, and which has a convergence basis generated by the net pair $(\{x_n : n \in D\}, x)$. Each x_n is isolated and x has a neighbourhood basis indexed by D. Consequently S is m-first-countable. However, S is not weakly sequential because $x \in cl(S-\{x\})$ and yet no well-ordered net in $S-\{x\}$ converges to x. To verify that this is so, suppose that $\{x_{N(k)} : k \in K\}$ is a well-ordered subnet of $\{x_n : n \in D\}$. Choose a countable collection $\{n_i : i \in \omega\}$ of distinct elements in D. For each $i \in \omega$ there exists $k_i \in K$ such that, if $k \geq k_i$ then $N(k) \supset n_i$. From the description of D, it is obvious that there is no supremum of $\{k_i : i \in \omega\}$ in K. But then $\{k_i : i \in \omega\}$ is a cofinal subset of K and hence $\{N(k_i) : i \in \omega\}$ is a cofinal subset of D. This is impossible since $\bigcup\{N(k_i) : i \in \omega\}$ is only a countable subset of the given set of cardinality m.

4.7 Definition An ordered topological space is a space which has the order topology arising from a total order on the set. A topological space is orderable if and only if some total order can be imposed on the set relative to which the given topology coincides with the order topology.

4.8 Proposition Let A be a subset of an ordered topological space X. If an m-net in A converges to a point $x \in X-A$, then there is a strictly monotone well-ordered m-net in A converging to x.

Proof It is first shown that every totally ordered set has a cofinal well-ordered subset. Let $\mathcal{F} = \{F_i : i \in I\}$ be the family of all well-ordered subsets of a totally ordered set Y. Partially order $\mathcal{F}$ by defining $F_i < F_j$ whenever $F_i = F_j$ or F_i is an initial segment of F_j. Note that $F_i < F_j$ implies that $F_i \subset F_j$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$ and suppose that B is a subset of $\bigcup \mathcal{C}$. There exists $C \in \mathcal{C}$ such that $C \cap B \neq \emptyset$, and $C \cap B$ has a least element b since C is well-ordered. The total order < on $\mathcal{C}$ implies that b is the least element of B and hence that $\bigcup \mathcal{C} \in \mathcal{F}$. By Zorn's lemma, $\mathcal{F}$ has a maximal well-ordered element Y_1. Then Y_1 is also a cofinal subset of Y because otherwise there exists $y \in Y-Y_1$ with no element of Y_1 greater than y; from this, $Y_1 \cup \{y\} \in \mathcal{F}$

contradicting the maximality of Y_1.

Let A_o be the intersection of A with the range of the m-net in the hypothesis. The cardinality of A_o is clearly $\leq m$. Let $A_1 = \{y \in A_o : y < x\}$ and $A_2 = \{y \in A_o : y > x\}$. Since X has the order topology, $x \in \mathrm{cl}\, A_i$ for $i = 1$ or $i = 2$; assume the former. The set A_1 is directed by the total order inherited from X, and thus the identity map on A_1 is a strictly monotone m-net converging to x. Moreover, A_1 has a cofinal well-ordered subset A_3 and the identity map on A_3 is the desired net.

4.9 Theorem The following statements about an arbitrary topological space X are equivalent.

(1) X is weakly sequential.

(2) X is the quotient of a weakly first-countable orderable space.

(3) X is the quotient of a weakly first-countable space.

(4) X is the quotient of an orderable space.

Proof Clearly (2) implies both (3) and (4). In addition, (3) implies (1) by (4.5.1) and (3.24). It remains to show that (1) ---> (2) and (4) ---> (1).

To establish the latter implication, let $f : y \longrightarrow X$ be the quotient map of an ordered space Y onto a topological space X. Suppose U is a subset of X such that any well-ordered net converging to a point in U is eventually in U. It suffices to prove that $f^{-1}(U)$ is open in Y. Let $y \in f^{-1}(U)$. If $y \notin \text{int } f^{-1}(U)$, there is a net $\{y_n : n \in D\}$ which is disjoint from $f^{-1}(U)$ and converges to y. According to (4.8), it can be assumed that D is well-ordered. But then $\{f(y_n) : n \in D\}$ is a well-ordered net in X-U converging to $f(y) \in U$. Hence (4) implies (1) by contradiction.

Assume now that X is a weakly sequential space with a convergence subbasis C in which all of the nets are well-ordered nets. For each $(\{x_n : n \in D\}, x) \in C$, let $S(x_n, x) = \{x_n : n \in D'\} \cup \{x\}$ be a topological space in which the x_n are taken to be distinct and $x_n \neq x$ for every $n \in D'$, and which has the order topology arising from the total order defined as follows. Let a_o be the least element of D and let $D' = (\omega \times \{a_o\}) \cup (Z \times (D-\{a_o\}))$ with ω and Z ordered in the usual way. Totally order D' by specifying $(i, n) < (j, m)$ whenever $n < m$ or $i < j$ and $n = m$. Each element of D' has an immediate successor, and each element other than $(0, a_o)$ has an immediate predecessor. (Such an order is called a discrete order.) Now let $x_{(i,n)} = x_n$ for each $(i, n) \in D'$, and totally order $S(x_n, x)$ by defining $x_n < x_m$ if $n < m$ in D' and $x_n < x$ for each $n \in D'$.

In the ordered space $S(x_n, x)$, each x_n is isolated and x has a well-ordered neighbourhood basis indexed by a set order isomorphic to D. Then $S(x_n, x)$ is weakly first-countable and has a convergence basis generated by the net pair $(\{x_n : n \in D'\}, x)$. The disjoint topological sum W of all such $S(x_n, x)$ is also weakly first-countable. The natural mapping of W onto X defined by $x \dashrightarrow x$ is a quotient map because the net pairs $(\{x_n : n \in D'\}, x)$ form a convergence subbasis for X.

To demonstrate that W is orderable, let $\{S_a : a \in A\}$ denote the set of all $S(x_n, x)$ and define a discrete order on $Z \times A$ in the same way as D'. (In this case, A is assigned an arbitrary total order with least element a_o.) Using the existence of a one-to-one correspondence between A and $Z \times A$, this discrete order can be imposed on A. Let W be totally ordered by specifying $x < y$ whenever $x < y$ in S_a where $x, y \in S_a$, or $a < b$ in A where $x \in S_a$ and $y \in S_b$. Because of the discrete orderings and the fact that each S_a has a greatest element and a least element, the order topology on W coincides with its usual disjoint topological sum topology. Thus W is orderable, and the theorem is proved.

4.10 Corollary For any topological space X and any infinite cardinal m, the first three statements are equivalent. If m = $\aleph_0$ they are also equivalent to (4).

(1) X is m-sequential.

(2) X is the quotient of an orderable m-first-countable space.

(3) X is the quotient of an m-first-countable space.

(4) X is the quotient of an orderable metric space.

Proof This is analogous to (4.9). It is only necessary to remark that each $S(x_n, x)$ is m-first-countable and hence so is W.

4.11 Proposition The following are equivalent.

(1) X is weakly Fréchet.

(2) X is the continuous pseudo-open image of an orderable weakly first-countable space.

(3) X is the continuous pseudo-open image of a weakly first-countable space.

(4) X is the continuous pseudo-open image of an orderable space.

Proof Clearly (2) implies both (3) and (4), and (3) implies (1) by (4.5.1) and (3.26). The fact that (1) ---> (2) follows from (4.9) in the same way that (3.31) followed from (3.30).

To prove that (4) ---> (1), let $f : Y \longrightarrow X$ be a continuous pseudo-open function of an ordered space Y onto a topological space X. Let A be a subset of X and suppose that $x \in \mathrm{cl}\ A$. Then some $y \in f^{-1}(x) \cap \mathrm{cl}\ f^{-1}(A)$. This is so because otherwise there is an open neighbourhood U of $f^{-1}(x)$ disjoint from $f^{-1}(A)$; from this, $A \cap f(U) = \emptyset$ contradicting $x \in (\mathrm{cl}\ A) \cap \mathrm{int}\ f(U)$. Since Y is an ordered topological space, there exists a well-ordered net $\{y_n : n \in D\}$ in $f^{-1}(A)$ converging to y. Then $\{f(y_n) : n \in D\}$ is a well-ordered net in A converging to $f(y) = x$.

<u>4.12 Corollary</u> For any topological space X and any infinite cardinal m, the first three statements are equivalent. If $m = \aleph_0$ they are also equivalent to (4).

(1) X is <u>m</u>-Fréchet.

(2) X is the continuous pseudo-open image of an orderable <u>m</u>-first-countable space.

(3) X is the continuous pseudo-open image of an <u>m</u>-first-countable space.

(4) X is the continuous pseudo-open image of an orderable metric space.

The final results are concerned with the sequential properties of ordered topological spaces and the relation between the notions of first-countable space, Fréchet space, and sequential space in products of these spaces. It is now known that Fréchet spaces and sequential spaces are successive proper generalizations of first-countable spaces, and that the product of two Fréchet spaces need not be sequential. For topological spaces which are products of ordered spaces, the situation is quite different.

<u>4.13 Theorem</u> If X is an ordered topological space, the following are equivalent. The first three statements are equivalent whenever X is a product of ordered spaces.

(1) X is m-first-countable.

(2) X is m-Fréchet.

(3) X is m-sequential.

(4) X is $\underline{m}$-Fréchet.

(5) X is $\underline{m}$-sequential.

<u>Proof</u> From (3.29), (1) ---> (2) ---> (3). Obviously (4) ---> (5) ---> (3) and (4) ---> (2). Furthermore, (2) ---> (4) by (4.8). It remains to show that (2) ---> (1) and (3) ---> (2). Assume first that X is an ordered topological space.

(2) ---> (1). Let $x \in X$. If x is an isolated point then x has a neighbourhood basis consisting of the singleton $\{x\}$. Suppose that x is not isolated but has either an immediate predecessor or an immediate successor. In the former case, $x \in \mathrm{cl}\,\{y \in X : y > x\}$. By hypothesis, there exists an m-net $\{x_n : n \in D\}$ in $\{y : y > x\}$ converging to x. Let $A_n = \{y : x \leq y < x_n\}$. The collection

$\{A_n : n \in D\}$ is a neighbourhood basis at x and the cardinality of D is $\leq m$. Similarly, x has a neighbourhood basis of cardinality $\leq m$ in the latter case. Suppose now that x is not isolated and has neither an immediate successor nor an immediate predecessor. By hypothesis, there is an m-net $\{x_n : n \in D\}$ converging to x with $x_n < x$ for each n. There is also an m-net $\{y_n : n \in E\}$ converging to x with each $y_n > x$. The open sets $\{z : x_n < z < y_m\}$ where $(n, m) \in D \times E$ form a neighbourhood basis for x and the cardinality of $D \times E$ is $\leq m^2 = m$.

(3) ---> (2). Suppose that the nets $S_n = \{x_i^n : i \in E_n\}$ converge to x_n and $S = \{x_n : n \in D\}$ converges to x, with the cardinalities of E_n and $D \leq m$. It suffices to construct an m-net in the union of the ranges of the nets S_n converging to x. According to (4.8), it can be assumed that all of the given nets are strictly monotone and directed by ordinal numbers. Either S is increasing or decreasing, and the nets S_n are either frequently increasing or frequently decreasing with respect to the directed set D; that is, the nets S_n are frequently (increasing, decreasing) if and only if for each $p \in D$ there exists $q \geq p$ such that S_q is (increasing, decreasing). There are four cases to consider. First, assume that the nets S_n are increasing and the net S is decreasing. Since $x_n > x$

for each n, there is $i(n) \in E_n$ with $x_n > x^n_{i(n)} \geq x$. Then $\{x^n_{i(n)} : n \in D\}$ is the desired m-net; it converges to x because it is bounded above by the net S which converges downward to x. For the second case, suppose that all of the nets are strictly increasing. Let D' denote the set of all isolated ordinals in D. Clearly D' is a cofinal subset of D; for each $n \in D$ the successor ordinal n + 1 is isolated. Then $x_{n-1} < x_n$ for each $n \in D'$, and there exists $i(n) \in E_n$ satisfying $x_{n-1} < x^n_{i(n)}$. It follows that $\{x^n_{i(n)} : n \in D'\}$ is an m-net converging to x. The remaining two cases are similar to the first and second cases. This completes the proof for the case in which X consists of one ordered space.

Assume now that X is the product of any family $\{X_a : a \in A\}$ of ordered topological spaces. It is only necessary to prove that (3) implies (1). By virtue of (3.33), the cardinality of A is $\leq$ m. Then, since each X_a is m-sequential and hence m-first-countable, (3.34) implies that X is m-first-countable.

<u>4.14 Corollary</u> If X is the product of any family $\{X_a : a \in A\}$ of non-trivial ordered m-sequential (or equivalently m-Fréchet) spaces, then X is m-sequential (or equivalently m-Fréchet) if and only if the cardinality of A is $\leq$ m.

Proof If X is m-sequential then, by (3.33), the cardinality of A is $\not>$ m and hence $\leq$ m. To establish the converse, observe that each X_a is m-first-countable by the preceding theorem. Then if the cardinality of A is $\leq$ m, in view of (3.34), X is m-first-countable.

4.15 Corollary An ordered topological space is weakly Fréchet if and only if it is weakly sequential.

Proof This is the same as (3) <---> (2) of (4.13). In this case, the cardinalities of D and E_n are not important.

4.16 Example There is an ordered topological space which is not weakly first-countable. Moreover, for any uncountable cardinal m, an m-first-countable ordered space need not be $\underline{m}$-first-countable.

Proof Let X = $(\alpha + 1) + \omega^*$ where α is the initial ordinal of cardinality m and ω^* has the reverse order to that of ω. By definition, X has the order : $x < y$ if

	a)	$x, y \in \alpha + 1$	and $x < y$ in $\alpha + 1$,
or	b)	$x, y \in \omega^*$	and $x < y$ in ω^*,
or	c)	$x \in \alpha + 1$	and $y \in \omega^*$.

Let X have the order topology arising from this total order. It is clear that X is m-first-countable. To establish that X is not weakly first-countable and hence not m-first-countable, assume the opposite and let $\{U_a : a \varepsilon A\}$ be a well-ordered neighbourhood basis at the point α. Obviously α has neither an immediate predecessor nor an immediate successor. For each $a \varepsilon A$, let x_a be the least element of $(\alpha + 1) \cap U_a$, and let y_a be the greatest element of $\omega^* \cap U_a$. The range of $\{y_a : a \varepsilon A\}$ is surely countable and, since $\{U_a : a \varepsilon A\}$ is well-ordered, there are less than m elements in the range of $\{x_a : a \varepsilon A\}$ associated with each element in the range of $\{y_a : a \varepsilon A\}$. The supremum of $\{x_a : a \varepsilon A\}$ is therefore less than α, and so $\{x_a : a \varepsilon A\}$ cannot converge to α.

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