# DIFFERENTIABLE ENGULFING AND COVERINGS OF MANIFOLDS

by

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ABSTRACT

There are now engulfing theorems for topological, piecewise linear, and differentiable manifolds. Differentiable engulfing so far was reduced to piecewise linear engulfing using the J. H. C. Whitehead triangulation of a differentiable manifold and J. R. Munkres' theory of obstructions to the smoothing of piecewise-differentiable homeomorphisms. In the first part of the thesis we observe that the method of proof of M. H. A. Newman's topological engulfing theorem applies, up to a local lemma, simultaneously to all three categories of manifolds. We prove this local lemma in the differentiable case and thus obtain a differentiable engulfing theorem which has a direct proof. Then we solve the problem of the existence of a stretching diffeomorphism between complementary subcomplexes of a simplicial complex in Euclidean space which is crucial for all applications of engulfing. Next we prove a theorem concerning the uniqueness of open differentiable cylinders which is the differentiable analogue of the uniqueness theorem for open cones. A consequence of this theorem is that if $M_1$ and $M_2$ are compact differentiable manifolds with diffeomorphic interiors then $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ are diffeomorphic, where $\mathbb{R}$ denotes the real line. Another consequence is that if a differentiable manifold is the monotone union of open differentiable cells it is diffeomorphic to Euclidean space.

We present several applications of differentiable engulfing which actually hold in all three categories of manifolds.

Our methods are such that they apply also to noncompact manifolds.

Theorem: Let $M$ be a differentiable $n$-dimensional manifold and let $U_1, \ldots, U_m$ be open subsets of $M$ such that $U_i = \bigcup_{j=1}^{\infty} V_{i,j}$, where each $V_{i,j}$ is open in $M$, $\mathrm{Cl}V_{i,j} \subset V_{i,j+1}$, $(M - \mathrm{Cl}V_{i,j}, V_{i,j+1} - \mathrm{Cl}\, V_{i,j})$ is $k_i$-connected, with $k_i \leq n - 3$ if $k_i > 0$, $1 \leq i \leq m$, $j \geq 1$, and $\partial M \subset \bigcup_{i=1}^{m} V_{i,1}$. Then, if $k_1 + \ldots + k_m + m \geq n + 1$, there are diffeomorphisms $h_i$ of $M$ onto itself such that $h_i$ is the identity on $\mathrm{Cl}\, V_{i,1}$, $1 \leq i \leq m$, and $M = \bigcup_{i=1}^{m} h_i(U_i)$.

This theorem has several corollaries. For instance, if $M$ is a $k$-connected differentiable manifold of dimension $n$ without boundary, $k \leq n - 3$ if $k > 0$, and if $m \geq \frac{n+1}{k+1}$, then $M$ may be covered by $m$ open differentiable $n$-cells. Using this result, we give a new and direct proof of the uniqueness of the differentiable structure of Euclidean $n$-space for $n \geq 5$. Finally, we prove a general h-cobordism theorem.

Theorem: Let $M$ be a connected differentiable manifold of dimension $n$, $n \geq 5$, with two connected boundary components $N_1$ and $N_2$ such that the inclusion of $N_i$ into $M$ is a homotopy equivalence, $i = 1,2$. Then there is a diffeomorphism of $N_1 \times [0, \infty)$ onto $M - N_2$.

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# INTRODUCTION

There are now engulfing theorems for topological, piecewise linear, and differentiable manifolds. Differentiable engulfing so far was reduced (in [2]) to piecewise linear engulfing using the J.H.C. Whitehead triangulation of a differentiable manifold ([15]) and J.R. Munkres' theory of obstructions to the smoothing of piecewise differentiable homeomorphisms ([9]). In the first part of this thesis we observe that the method of proof of M.H.A. Newman's topological engulfing theorem ([10]) applies, up to a local lemma, simultaneously to all three categories of manifolds. We prove this local lemma in the differentiable case and thus obtain a differentiable engulfing theorem which has a direct proof.

After proving this differentiable engulfing theorem, we prove a theorem, concerning the existence of a stretching diffeomorphism between complementary subcomplexes of a simplicial complex in Euclidean space, which is crucial for all applications of engulfing. This solves a problem posed in [12], p. 502.

Next we prove a theorem concerning the uniqueness of open differentiable cylinders which is the differentiable analogue of the uniqueness theorem for open (topological) cones ([5]). A consequence of this theorem is that if $M_1$ and $M_2$ are compact differentiable manifolds with diffeomorphic interiors, then $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ are diffeomorphic, where

$\mathbb{R}$ denotes the real line. Another consequence is that if a differentiable manifold is the monotone union of open differentiable cells it is diffeomorphic to Euclidean space.

We present several applications of differentiable engulfing which actually hold in all three categories of manifolds. Our methods are such that they apply also to noncompact manifolds.

<u>Theorem 5.1.</u> Let $M$ be a differentiable $n$-dimensional manifold, and let $U_1, \ldots, U_m$ be open subsets of $M$ such that $U_i = \bigcup_{j=1}^{\infty} V_{i,j}$, where each $V_{i,j}$ is open in $M$, $\mathrm{Cl}\ V_{i,j} \subset V_{i,j+1}$, $(M - \mathrm{Cl}\ V_{i,j}, V_{i,j+1} - \mathrm{Cl}\ V_{i,j})$ is $k_i$-connected, with $k_i \leq n-3$ if $k_i > 0$, $1 \leq i \leq m$, $j \geq 1$, and $\partial M \subset \bigcup_{i=1}^{m} V_{i,1}$. Then if $k_1 + \ldots + k_m + m \geq n+1$, there are diffeomorphisms $h_i$ of $M$ onto itself such that $h_i$ is the identity on $\mathrm{Cl}\ V_{i,1}$, $1 \leq i \leq m$, and $M = \bigcup_{i=1}^{m} h_i(U_i)$.

This theorem has several corollaries. For instance, if $M$ is a $k$-connected differentiable manifold of dimension $n$ without boundary, $k \leq n-3$ if $k > 0$, and if $m \geq \frac{n+1}{k+1}$, then $M$ may be covered by $m$ open differentiable $n$-cells. Using this result, we give a new and direct proof of the uniqueness of the differentiable structure of Euclidean $n$-space for $n \geq 5$. Finally, we prove a general h-cobordism theorem.

<u>Theorem 5.3.</u> Let $M$ be a connected differentiable manifold of dimension $n$, $n \geq 5$, with two connected boundary components $N_1$ and $N_2$ such that the inclusion of $N_i$ into $M$ is a homotopy equivalence, $i = 1,2$. Then there is a diffeomorphism of $N_1 \times [0,\infty)$ onto $M-N_2$.

# CHAPTER 0

## Notation and Fundamental Definitions

In this paper, $\mathbb{R}$ will denote the set of real numbers, $I$ will denote the unit interval $[0,1]$, $\mathbb{R}^n$ will denote Euclidean $n$-space, $H^n$ will denote the half-space $\{(x_1,\ldots,x_n) \in \mathbb{R}^n: x_n \geq 0\}$, $S^{n-1}$ will denote the unit $(n-1)$-sphere in $\mathbb{R}^n$, and $D^n$ will denote the closed unit $n$-ball in $\mathbb{R}^n$. By the word <u>map</u> we shall always mean a continuous map. If $X$ is a topological space, $id_X$ will denote the identity map of $X$.

<u>Definition 0.1.</u> If $X$ is a topological space and $A \subset X$ is a subset, we say that the pair $(X,A)$ is k-connected if $\pi_n(X,A) = 0$ for all $n \leq k$. If $A$ is $(k-1)$-connected and $X$ is k-connected, then $(X,A)$ is k-connected.

<u>Definition 0.2.</u> Let $Y$ be a metric space with metric $d$. If $A$ and $B$ are subsets of $Y$, the <u>distance</u>, dist $(A,B)$, between $A$ and $B$ is defined to be $\inf\{d(x,y): x \in A, y \in B\}$. If $X$ is a topological space, and $f$ and $g$ are maps of $X$ into $Y$, the <u>distance</u>, $d(f,g)$, between $f$ and $g$ is defined to be $\sup\{d(f(x),g(x)): x \in X\}$.

<u>Definition 0.3.</u> If $K$ is a simplicial complex, the i-th barycentric subdivision of $K$ will be denoted by $\beta^i(K)$, and the n-skeleton of $K$ will be denoted by $K^{(n)}$. If $S$ is a subset of $|K|$, the <u>neighborhood of $S$ in $K$</u> is defined to

be the subcomplex

$$N(S,K) = \{\Delta \in K\colon \Delta \text{ is a face of } \tilde{\Delta} \text{ in } K \text{ and } \tilde{\Delta} \cap S \neq \emptyset\}.$$

Definition 0.4. If $A \subset \mathbb{R}^n$, and $f\colon A \to \mathbb{R}^m$ is a map, we say that $f$ is a $C^\infty$-map if it can be extended to a $C^\infty$-map of a neighborhood of $A$ into $\mathbb{R}^m$.

Definition 0.5. A $C^\infty$-n manifold M is a locally Euclidean Hausdorff space with a countable basis and a $C^\infty$-structure $\mathcal{S}$. $\mathcal{S}$ is a collection of pairs $(U,h)$ satisfying four conditions:

(1) Each $(U,h) \in \mathcal{S}$ consists of an open set $U \subset M$ together with a homeomorphism $h$ which maps $U$ onto an open subset of $H^n$.

(2) The coordinate neighborhoods in $\mathcal{S}$ cover $M$.

(3) If $(U_1,h_1), (U_2,h_2) \in \mathcal{S}$, then $h_1 \circ h_2^{-1}\colon h_2(U_1 \cap U_2) \to H^n$ is a $C^\infty$-map with nonzero Jacobian.

(4) $\mathcal{S}$ is maximal with respect to (3).

The boundary, $\partial M$, of $M$ is defined to be the set of points of $M$ which do not have a neighborhood which is $C^\infty$-diffeomorphic to $\mathbb{R}^n$.

Definition 0.6. If $M$ is a $C^\infty$-manifold without boundary, a family of maps $\{h_t\colon t \in I\}$ (usually written $h_t$) is said to be $C^\infty$-isotopy of $M$ if each $h_t$ is a $C^\infty$-diffeomorphism of $M$ onto itself, and the map $H\colon M \times I \to M$ defined by $H(m,t) = h_t(m)$ is a $C^\infty$-map.

Definition 0.7. A topological space $X$ is said to be 1-connected at $\infty$ if for each compact set $C \subset X$ there is a compact set $D \supset C$ such that $X-D$ is simply connected.

Definition 0.8. If $A, B \subset R^n$ are joinable subsets then $A*B$ denotes the join of $A$ and $B$.

# CHAPTER ONE

## Local $C^\infty$-Engulfing

Lemma 1.1. Let $T = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 < \alpha < \beta < 1\}$. There is a $C^\infty$-map $\theta: \mathbb{R} \times T \to \mathbb{R}$ such that if $\theta_\alpha^\beta(x) = \theta(x,\alpha,\beta)$, then

$\theta_\alpha^\beta(x) = x$ if $x \notin (0,1)$, $\theta_\alpha^\beta([0,\alpha]) = [0,\beta]$ and $\dfrac{d\theta_\alpha^\beta}{dx}(x) > 0$ for all $x \in \mathbb{R}$.

Proof: We let $\theta(x,\alpha,\beta)$ be of the form $\theta(x,\alpha,\beta) = x + g(x,\alpha,\beta)$. We construct a $C^\infty$-map $g: \mathbb{R} \times T \to \mathbb{R}$ such that if $g_\alpha^\beta(x) = g(x,\alpha,\beta)$, then $g_\alpha^\beta(x) = 0$ if $x \notin (0,1)$,

$g_\alpha^\beta(\alpha) = \beta - \alpha$, and $\dfrac{dg_\alpha^\beta}{dx}(x) > -1$. See Figure 1.

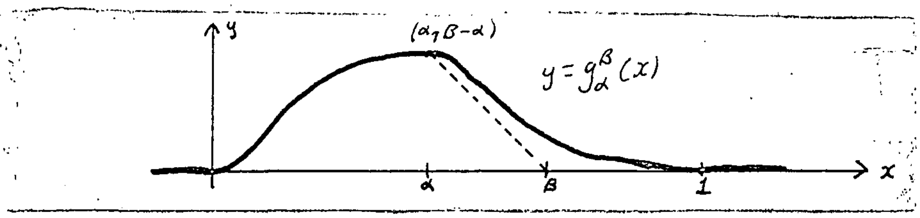


FIGURE 1

To construct such a map we use the following $C^\infty$-map as a building block: let $\epsilon: \mathbb{R} \times T \to \mathbb{R}$ be defined by

$$\epsilon(x,\alpha,\beta) = \frac{1}{C}\int_\alpha^x e^{\frac{1}{\alpha-t} + \frac{1}{t-\beta}}\,dt, \text{ if } x \in (\alpha,\beta),\ \epsilon(x,\alpha,\beta) = 0 \text{ if}$$

$x \le \alpha$, and $\epsilon(x,\alpha,\beta) = 1$ if $x \ge \beta$, where

$C = \int_{\alpha}^{\beta} e^{\frac{1}{\alpha-t} + \frac{1}{t-\beta}} dt.$ See Figure 2.

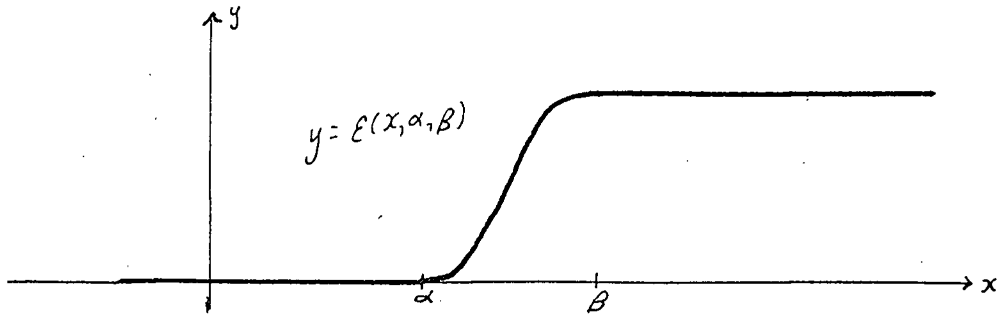


FIGURE 2

To define $g_{\alpha}^{\beta}(x)$ for $x \geq \alpha$, we need a modified version of the map $\varepsilon$. We define

$$\delta\colon \mathbb{R}\times T \to \mathbb{R} \text{ by } \delta(x,\alpha,\beta) = \begin{cases} \varepsilon(x,\alpha,\alpha + \frac{1-\beta}{2}), & \text{if } x \leq \alpha + \frac{1-\beta}{2}, \\ 1, & \text{if } \alpha + \frac{1-\beta}{2} \leq x \leq 1 - \frac{1-\beta}{2}, \\ 1-\varepsilon(x,1-\frac{1-\beta}{2},1), & \text{if } x \geq 1 - \frac{1-\beta}{2}. \end{cases}$$

See Figure 3.

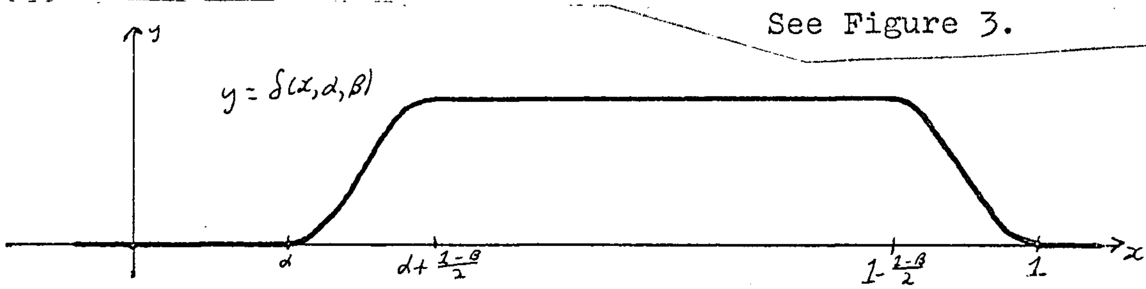


FIGURE 3

We define $\tilde{\varepsilon}\colon \mathbb{R} \times T \to \mathbb{R}$ by $\tilde{\varepsilon}(x,\alpha,\beta) = \frac{1}{\tilde{C}} \int_0^x \delta(t,\alpha,\beta)dt$, where $\tilde{C} = \int_0^1 \delta(t,\alpha,\beta)dt > \beta-\alpha$. Now define

$$g(x,\alpha,\beta) = \begin{cases} (\beta-\alpha)\cdot\varepsilon(x,0,\alpha), & x \leq \alpha \\ (\beta-\alpha)\cdot[1-\tilde{\varepsilon}(x,\alpha,\beta)], & x \geq \alpha. \end{cases}$$

$g$ is a $C^\infty$ map, and, if $x \geq \alpha$,

$$\frac{dg_\alpha^\beta}{dx}(x) = (\beta-\alpha)\,[-\frac{1}{\tilde{C}}\delta(x,\alpha,\beta)] > -1. \qquad \text{Q.E.D.}$$

The following lemma is Corollary 4.3. of [11], p. 129.

<u>Lemma 1.2.</u> Let $f_1,\ldots,f_n$ be $n$ real-valued differentiable functions of $n$ real variables. Necessary and sufficient conditions that the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = (f_1(x),\ldots,f_n(x))$ be a diffeomorphism of $\mathbb{R}^n$ onto itself are:

(1) $\det(\frac{\partial f_i}{\partial x_j})$ never vanishes

(2) $\lim\limits_{\|x\|\to\infty} \|f(x)\| = \infty$.

<u>Theorem 1.1.</u> Let $\Delta^m \subset \mathbb{R}^m \subset \mathbb{R}^n$ be an $m$-simplex, $\Delta^m = v*\Delta^{m-1}$, where $\Delta^{m-1} \subset \mathbb{R}^{m-1} \subset \mathbb{R}^m$, $b_{\Delta^{m-1}} = 0$, and $v = (0,\ldots,0,1) \in \mathbb{R}^m$ lies on the $x_m$-axis. Let $p\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ be the orthogonal projection. Let $A \subset \Delta^m$ be a closed subset such that $A = p^{-1}(p(A)) \cap \Delta^m$. Let $U$ be an open set in $\mathbb{R}^n$ such that $A \cup v * \partial\Delta^{m-1} \subset U$, and let $F$ be a closed subset of $\mathbb{R}^n$ such that $F \cap \Delta^m \subset A \cup v * \partial\Delta^{m-1}$. Then there is a compact set

$C \subset \mathbb{R}^n - F$ and a $C^\infty$-isotopy $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h_o = id_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and $\Delta^m \subset h_1(U)$.

Proof:

(1) Let $c = \frac{1}{3}\text{dist}(\mathbb{R}^n - U, A \cup v * \partial\Delta^{m-1}) > 0$. Let

$N_1 = \{x \in \Delta^{m-1} \colon \text{dist}(x, A \cup v * \partial\Delta^{m-1}) \geq c\}$, and

$N_2 = \{x \in \Delta^{m-1} \colon \text{dist}(x, A \cup v * \partial\Delta^{m-1}) \geq 2c\}$. Then

$N_1$ and $N_2$ are compact, and $N_2 \subset N_1$. Further,

$p^{-1}(\Delta^{m-1} - N_2) \cap \Delta^m \subset U$, since $\Delta^{m-1} - N_2 \subset U$, and if

$x \in p^{-1}(\Delta^{m-1} - N_2) \cap \Delta^m$, then $\text{dist}(x, A \cup v * \partial\Delta^{m-1}) \leq$

$\text{dist}(p(x), A \cup v * \partial\Delta^{m-1}) < c$. If $N_2 = \emptyset$, then $\Delta^m \subset U$,

so we may let $C = \emptyset$ and $h_t = id_{\mathbb{R}^n}$. From now on, we assume that $N_2 \neq \emptyset$. Let $d = \frac{1}{2}\text{dist}(N_1, F) > 0$.

(2) Let $\hat{g} \colon \Delta^{m-1} \to \mathbb{R}$ be the continuous function defined by $\hat{g}(x) = \|s(x) - x\|$, where $s(x)$ is the intersection of the line through $x$ parallel to the $x_m$-axis with $v * \partial\Delta^{m-1}$. Let $g \colon \Delta^{m-1} \to \mathbb{R}$ be a $C^\infty$-$\frac{c}{4}$-approximation to $\hat{g}$. If $x \in N_1$, then $\hat{g}(x) \geq c$ and hence

$g(x) - \frac{c}{2} > \hat{g}(x) - \frac{c}{4} - \frac{c}{2} \geq \frac{c}{4}$. For each $x \in N_1$ we define a "vertical stretching interval". Let

$$v_1(x) = x + (g(x) - \tfrac{c}{4})\cdot v,$$

$$v_2(x) = x + (g(x) - \tfrac{c}{2})\cdot v$$

$$v_3(x) = x, \quad \text{and}$$

$$v_4(x) = x - d\cdot v.$$

Then $[v_1(x),v_2(x)] \subset U$. The stretching interval will be $[v_1(x),v_4(x)]$ and by "stretching" we will map $[v_1(x),v_2(x)]$ onto $[v_1(x),v_3(x)]$. The interval $[v_1(x),v_4(x)]$ has length $\gamma(x) = g(x) - \frac{c}{4} + d > 0$. To apply Lemma 1.1, we map the interval $[v_1(x),v_4(x)]$ linearly onto $[0,1]$ such that $v_1(x)$ is mapped onto $0$ and $v_4(x)$ is mapped onto 1. Then $v_2(x)$ is mapped onto $\alpha(x) = \frac{c}{4\cdot\gamma(x)}$ and $0$ onto $\beta(x) = \frac{\gamma(x)-d}{\gamma(x)}$. Note that $\alpha(x) < \beta(x)$. See Figure 4.

(3) Before constructing $h_t$ we must construct a $C^\infty$-function $\mu : R^n \to I$ with proper compact support.

Let $\nu_1 : R^{m-1} \to I$ be a $C^\infty$-function such that

$\nu_1(x) = 1$ if $x \in N_2$ and

$Cl(\nu_1^{-1}((0,1])) \subset \text{Int } N_1 = \{x \in \Delta^{m-1}: \text{dist}(x, A \cup v * \partial\Delta^{m-1}) > c\}$

which is open in $R^{m-1}$.

Consider next the compact set

$C_o = \{x = (x_1,\ldots,x_m) \in R^m: p(x) \in N_1 \text{ and } -d \leq x_m \leq g(x) - \frac{c}{4}\}$.

Then $C_o \cap F = \emptyset$. Let $\eta = \text{dist}(C_o, F) > 0$. Let

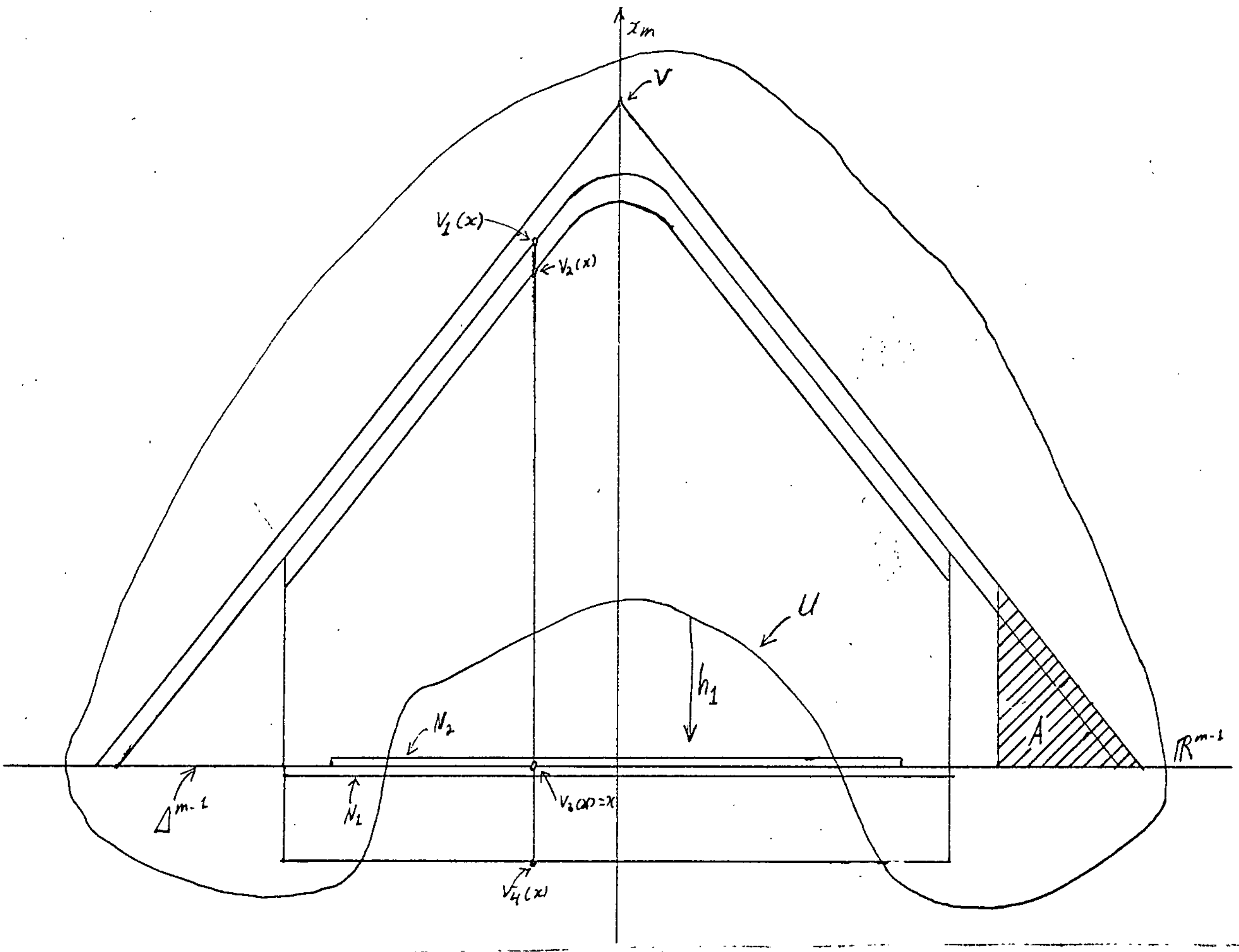


FIGURE 4

$\nu_2 \colon \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $0 < \nu_2(t) \leq 1$, if $t < \eta$, $\nu_2(0) = 1$ and $\nu_2(t) = 0$ if $t \geq \eta$.

Let $\pi \colon \mathbb{R}^n \to \mathbb{R}^m$ be the orthogonal projection and let $r = p \circ \pi$. We define

$$\mu(x) = \nu_1(r(x)) \cdot \nu_2(2\|x - \pi(x)\|) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Notice that $\frac{\partial \mu}{\partial x_m}(x) = 0$.

Let $C = \{x \in \mathbb{R}^n \colon \pi(x) \in C_0 \text{ and } \|x - \pi(x)\| \leq \frac{\eta}{2}\}$. Then $C$ is compact and $C \cap F = \emptyset$. We note that

$$\mathrm{Cl}(\mu^{-1}((0,1])) \cap \pi^{-1}(C_0) \subset C.$$

(4) We define $h_t^m \colon \mathbb{R}^n \to \mathbb{R}$ as follows. Let $x \in \mathbb{R}^n$. If $r(x) \in N_1$, let $h_t^m(x) = x_m + t \cdot \mu(x) \cdot [\{$stretching $C^\infty$-diffeomorphism with respect to $[v_1(x), v_4(x)]$ applied to $x_m\} - x_m] = (1 - t \cdot \mu(x)) \cdot x_m + t \cdot \mu(x) \times$

$$[-\gamma(r(x)) \cdot \theta_{\alpha(r(x))}^{\beta(r(x))}\left(-\frac{1}{\gamma(r(x))}[x_m - (g(r(x)) - \tfrac{c}{4})]\right) + g(x) - \tfrac{c}{4}].$$

If $r(x) \notin N_1$, let $h_t^m(x) = x_m$. We note that $h_t^m$ is a $C^\infty$-map.

Finally, let $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$h_t(x) = (x_1, \ldots, x_{m-1}, h_t^m(x), x_{m+1}, \ldots, x_n) \quad \text{for} \quad x \in \mathbb{R}^n.$$

We compute

$$\frac{\partial h_t^m}{\partial x_m}(x) = (1-t\cdot\mu(x)) + t\cdot\mu(x)\cdot\theta'\frac{\beta(r(x))}{\alpha(r(x))}\left(-\frac{1}{\gamma(r(x))}[x_m-(g(r(x))-\tfrac{c}{4})]\right) > 0.$$

Therefore the rank of the Jacobian of $h_t$ is $n$.

Obviously, $\lim_{\|x\|\to\infty} \|h_t(x)\| = \infty$. By Lemma 1.2, $h_t$ is a $C^\infty$-diffeomorphism. By construction, $h_o = id_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and $\Delta^m \subset h_1(U)$.

<u>Corollary 1.1.</u> Let $\Delta^m = v*\Delta^{m-1} \subset \mathbb{R}^n$ be an arbitrary m-simplex, let $T^{m-1}$ be the hyperplane in $\mathbb{R}^n$ spanned by $\Delta^{m-1}$, and $T^m$ the hyperplane spanned by $\Delta^m$, let $p\colon T^m \to T^{m-1}$ be the projection such that $p(v) = b_{\Delta^{m-1}}$, let $A \subset \Delta^m$ be a closed set such that $A = p^{-1}(p(A)) \cap \Delta^m$, let $U$ be an open subset of $\mathbb{R}^n$ such that $A \cup v*\partial\Delta^{m-1} \subset U$, and let $F$ be a closed set in $\mathbb{R}^n$ such that $F \cap \Delta^m \subset A \cup v*\partial\Delta^{m-1}$. Then there is a compact set $C \subset \mathbb{R}^n - F$ and a $C^\infty$-isotopy $h_t\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h_o = id_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and $\Delta^m \subset h_1(U)$.

# CHAPTER TWO

## The $C^\infty$-Engulfing Theorem

If $\alpha > 0$, let $C_\alpha^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : |x_i| \leq \alpha\}$, and let $\operatorname{Int} C_\alpha^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : |x_i| < \alpha\}$.

Definition 2.1. If $M$ is a $C^\infty$-n-manifold, a set $X \subset M$ is said to be k-dominated if there is a system $\{\varphi_i\}$ of $C^\infty$-diffeomorphisms $\varphi_i : C_1^n \to M$ such that

(1) $X \subset \bigcup \varphi_i(\operatorname{Int} C_1^n)$

(2) For each $i$, $\varphi_i^{-1}(\varphi_i(C_1^n) \cap X) \subset P_i$, where $P_i$ is a k-dimensional subpolyhedron of $C_1^n$.

The set $\{\varphi_i\}$ is called a k-dominating system for $X$, and each $\varphi_i$ is called k-dominating coordinate map for $X$.

Definition 2.2. If $M$ is a $C^\infty$-n-manifold and $K$ is a finite simplicial complex, a map $f: |K| \to M$ is said to be locally linearizable if there is a system $\{\psi_i\}$ of $C^\infty$-diffeomorphisms $\psi_i : C_1^n \to M$ such that

(1) $f(|K|) \subset \bigcup \psi_i(\operatorname{Int} C_1^n)$

(2) For each $i$, there is a subdivision $\sigma_i(K)$ of $K$ such that $f^{-1}(\psi_i(C_1^n)) = |H_i|$, where $H_i$ is a subcomplex of $\sigma_i(K)$, and $\psi_i^{-1} \circ f|_{|H_i|} : H_i \to C_1^n \subset \mathbb{R}^n$ is linear.

The set $\{\psi_i\}$ is called a <u>linearizing system</u> for $f$. Note that if $f\colon |K| \to M$ is linearizable, then $f(|K|)$ is k-dominated.

<u>Definition 2.3.</u> If $K$ is a simplicial complex, $Y$ is a topological space, $S \subset Y$ is a subset, and $f, g\colon |K| \to Y$ are maps, we say that <u>$f$ and $g$ agree on $S$</u> if there is a subdivision $\sigma(K)$ of $K$ such that

$N(f^{-1}(S), \sigma(K)) = N(g^{-1}(S), \sigma(K)) = N$, and $f|_N = g|_N$.

It is well known that a topological manifold is an absolute neighborhood retract, see, for instance, [4], p. 98. In proving the $C^\infty$-engulfing theorem, we shall need the following result from homotopy theory:

<u>Lemma 2.1.</u> Let $Y$ be a metrizable absolute neighborhood retract with metric $d$, and let $\epsilon > 0$. Then there is $\delta > 0$ such that for every closed subset $A$ of a metric space $X$ and for all maps $f_1, f_2\colon A \to Y$ with $d(f_1, f_2) < \delta$, if $f_1$ has an extension $\hat{f}_1\colon X \to Y$, then $f_2$ has an extension $\hat{f}_2\colon X \to Y$ such that $d(\hat{f}_1, \hat{f}_2) < \epsilon$.

<u>Proof:</u> This is Theorem V.3.1 of [1], p. 103.

<u>Theorem 2.1.</u> Let $M$ be a $C^\infty$-n-manifold without boundary, $V$ an open subset of $M$ such that $(M,V)$ is k-connected, $X \subset M$ a closed and k-dominated subset such that $X-V$ is compact, $k \leq n-3$. Let $K$ be a finite simplicial k-complex,

$f\colon |K| \to M$ continuous, $L \subset K$ a subcomplex such that $f|_{|L|}$ is a locally linearizable imbedding with linearizing system $\Sigma = \{\psi_j\}$ such that each $\psi_j$ is also a k-dominating coordinate map for $X$. Let $\epsilon > 0$. Then there is a map $g\colon |K| \to M$, a compact set $C \subset M$, and a $C^\infty$-isotopy $h_t\colon M \to M$ such that:

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and $h_1(V) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f|_{|L|}$

(3) $d(f,g) < \epsilon$ for some fixed metric $d$ on $M$.

<u>Corollary 2.1.</u> ($C^\infty$-Engulfing Theorem) If $M$ is a $C^\infty$-n-manifold without boundary, $V$ is an open subset of $M$ such that $(M,V)$ is k-connected, $X \subset M$ is closed and k-dominated, $X-V$ is compact and $k \leq n-3$, then there is a compact set $C \subset M$ and a $C^\infty$-isotopy $h_t\colon M \to M$ such that $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and $h_1(V) \supset X$.

<u>Proof:</u> Let $K = \emptyset$ in Theorem 2.1.

<u>Proof of Theorem 2.1.</u> We follow Newman's proof of the topological engulfing theorem, [10], allowing for differentiability and using the usual method of simplicial collapsing, instead of collapsing through principal simplices. We divide the proof into three steps.

For each $x \in M$, we choose a $C^\infty$-coordinate map $\mu_x\colon C_1^n \to M$ such that $x \in \mu_x(\mathrm{Int}\, C_1^n)$, and

(1) if $x \in f(|L|)$, $\mu_x \in \Sigma$

(2) if $x \in X - f(|L|)$, $\mu_x$ is a k-dominating coordinate map for $X$ such that $\mu_x(C_1^n) \cap f(|L|) = \emptyset$

(3) if $x \notin X \cup f(|L|)$, then $\mu_x(C_1^n) \cap (X \cup f(|L|)) = \emptyset$.

Step I: Reduction to the case $X \subset V$

Let $A(m)$ denote the theorem with the added hypothesis: $X - V \subset \mu_{x_1}(\mathrm{Int}\ C_1^n) \cup \ldots \cup \mu_{x_m}(\mathrm{Int}\ C_1^n)$ for some $x_1, \ldots, x_m \in M$.

(a) $A(1)$ implies $A(m)$.

Proof: Let $\mu_i = \mu_{x_i}$, $1 \leq i \leq m$. We use induction on $m$, $m \geq 2$. Let $X_m = X - \mu_m(\mathrm{Int}\ C_1^n)$. Then $X_m - V \subset \mu_1(\mathrm{Int}\ C_1^n) \cup \ldots \cup \mu_{m-1}(\mathrm{Int}\ C_1^n)$, so the hypotheses of $A(m-1)$ are satisfied. Thus there is a map $g_m: |K| \to M$, a compact set $C_m \subset M$, and a $C^\infty$-isotopy $h_t^m: M \to M$ such that

(1) $h_0^m = id_M$, $h_t^m(x) = x$ if $x \notin C_m$, and $h_1^m(V) \supset X_m \cup g_m(|K|)$

(2) $g_m|_{|L|} = f|_{|L|}$

(3) $d(g_m, f) < \varepsilon/2$

Now let $f' = g_m$, $V' = h_1^m(V)$. Then $X - V' \subset \mu_m(\mathrm{Int}\ C_1^n)$, so $A(1)$ may be applied: there is a map $g: |K| \to M$, a compact set $C' \subset M$, and a $C^\infty$-isotopy $h_t': M \to M$ such that:

(1) $h'_o = id_M$, $h'_t(x) = x$ if $x \notin C'$, and

$h'_1(V') \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f'|_{|L|}$

(3) $d(g,f') < \varepsilon/2$.

Let $C = C' \cup C_m$, and let $h_t = h'_t \circ h^m_t$. Then

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and $h_1(V) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f'|_{|L|} = g_m|_{|L|} = f|_{|L|}$.

(3) $d(g,f) \leq d(g,g_m) + d(g_m,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

(b) $A(0)$ implies $A(1)$.

Proof: Let $\mu = \mu_{x_1}$, where $X\text{-}V \subset \mu_{x_1}(\operatorname{Int} C^n_1)$. Since $\mu$ is a k-dominating coordinate map for $X$, $\mu^{-1}(X \cap \mu(C^n_1)) \subset P$, where $P$ is a k-dimensional subpolyhedron of $C^n_1$. If $x_1 \in f(|L|)$, let $\tilde{f} = f \cup \mu|_P\colon |K| \cup P \to M$. Then there is a subdivision $\sigma_1(K)$ of $K$ and a subdivision $\sigma_2(C^n_1)$ of $C^n_1$ with a subcomplex $H \subset \sigma_2(C^n_1)$ such that $|H| = P$, and $\mu^{-1} \circ \tilde{f}|_{\sigma_1(L)}\colon \sigma_1(L) \to \sigma_2(C^n_1)$ is a simplicial imbedding. If $\Delta_1 \in \sigma_1(L)$ and $\Delta_2 \in H$ are such that $f(\Delta_1) = \mu(\Delta_2)$, identify $\Delta_1$ and $\Delta_2$, and let $K^*$ be the simplicial complex obtained from $\sigma_2(K) \cup H$ by this identification. Let $p\colon \sigma_1(K) \cup H \to K^*$ be the projection. If $x_1 \notin f(|L|)$, let $H$ be a simplicial complex in $C_1^n$ such that $|H| = P$, and let $K^* = K \cup H$. Let $p\colon K \cup H \to K^*$ be the identity, and let $\tilde{f} = f \cup \mu|_P$. Let $f^*\colon |K^*| \to M$ be defined by $\tilde{f} = f^* \circ p$, and let $L^* = p(\sigma_1(L) \cup H)$, $X^* = X - \mu(\text{Int } C_1^n)$. Then $X^* \subset V$, and $f^*|_{|L^*|}$ is a locally linearizable imbedding, so we may apply A(0): There is a map $g^*\colon |K^*| \to M$, a compact set $C \subset M$, and a $C^\infty$-isotopy $h_t\colon M \to M$ such that:

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and

$h_1(V) \supset X^* \cup g^*(|K^*|)$.

(2) $g^*|_{|L^*|} = f^*|_{|L^*|}$.

(3) $d(f^*, g^*) < \varepsilon$.

Let $g = g^* \circ p|_{|K|}$. Then:

(1) $h_1(V) \supset X^* \cup g^*(|K^*|) =$

$X - \mu(\text{Int } C_1^n) \cup g(|K|) \cup \mu(P) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f|_{|L|}$: if $x \in |L|$, $g(x) = g^* \circ p(x) = f^* \circ p(x)$

$= \tilde{f}(x) = f(x)$.

(3) $d(g, f) < \varepsilon$.

<u>Step II: Reduction to the case $X \subset V$, and $f^{-1}(V) \subset \text{Int } \Delta^\ell$, for some $\Delta^\ell \in K$</u>

Let $B(\ell)$ denote the theorem with the added hypotheses: $X \subset V$ and $\dim N(|K| - f^{-1}(V), K) \leq \ell$, i.e. if

$\Delta \in K$ and $f(\Delta) \cap (M-V) \neq \emptyset$, then $\dim \Delta \subseteq \ell$.

Let $B(\ell,m)$ denote $B(\ell)$ with the added hypothesis: $f(|K^{(\ell-1)}|) \subset V$, and there are at most $m$ $\ell$-simplices $\Delta_1^\ell, \ldots, \Delta_m^\ell \in K$ such that $f(\Delta_1^\ell) \not\subseteq V$, $1 \leq i \leq m$, and, for each $\ell$-simplex $\Delta^\ell \in K$, if $f(\Delta^\ell) \cap X \neq \emptyset$, then $f(\Delta^\ell) \subset V$. Note that $\Delta_1^\ell, \ldots, \Delta_m^\ell$ must be principal in $K$.

(a) $B(\ell,1)$ implies $B(\ell,m)$ for all $m$.

Proof: We use induction on $m$. Suppose $B(\ell,1)$ and $B(\ell,m-1)$ are true, and the hypotheses of $B(\ell,m)$ hold, for some $m \geq 2$. Without loss of generality, we assume that $\epsilon$ is so small that for any $\epsilon$-approximation $g: |K| \to M$ to $f$, if $\Delta^\ell \in K$ is an $\ell$-simplex such that $g(\Delta^\ell) \cap X \neq \emptyset$, then $g(\Delta^\ell) \subset V$.

(i) Let $K' = K - \{\Delta_m^\ell\}$, $L' = L - \{\Delta_m^\ell\}$, $f' = f|_{|K'|}$. By Lemma 2.1, there is $\epsilon' > 0$ such that if $g': \partial\Delta^\ell_m \to M$ is an $\epsilon'$-approximation to $f'|_{\partial\Delta_m^\ell}$, then there is an extension $\hat{g}: \Delta_m^\ell \to M$ of $g'$ such that $d(f|_{\Delta_m^\ell}, \hat{g}) < \frac{\epsilon}{2}$. Now $\dim N(|K'| - (f')^{-1}(V), K') \leq \ell$, $f'(|(K')^{(\ell-1)}|) \subset V$, and there at most $(m-1)$ $\ell$-simplices $\Delta_1^\ell, \ldots, \Delta_{m-1}^\ell \in K'$ such that $f'(\Delta_i^\ell) \not\subseteq V$, $1 \leq i \leq m-1$, so the hypotheses of $B(\ell,m-1)$ are satisfied. Thus there is a map $g': |K'| \to M$, a compact set $C' \subset M$, and a $C^\infty$-isotopy $h_t^!: M \to M$ such that:

(1) $h'_o = id_M$, $h'_t(x) = x$ if $x \in C'$, and

$h'_1(V) \supset X \cup g'(|K'|)$.

(2) $g'|_{|L'|} = f'|_{|L'|}$.

(3) $d(f',g') < \epsilon'$.

Let $\tilde{f}\colon |K| \to M$ be defined as follows: $\tilde{f}|_{|K'|} = g'$; if $\Delta^\ell_m \in L$, $\tilde{f}|_{\Delta^\ell_m} = f|_{\Delta^\ell_m}$; if $\Delta^\ell_m \notin L$, let $\tilde{f}|_{\Delta^\ell_m}$ be an $\frac{\epsilon}{2}$ - approximation to $f|\Delta^\ell_m$ which extends $g'|_{\partial\Delta^\ell_m}$.

(ii) Let $\tilde{V} = h'_1(V)$. Now $\dim N(|K| - \tilde{f}^{-1}(\tilde{V}),K) \leq \ell$, $\tilde{f}(|K^{(\ell-1)}|) \subset \tilde{V}$, and there is only one $\ell$-simplex, $\Delta^\ell_m$, in $K$ such that $\tilde{f}(\Delta^\ell_m) \not\subset \tilde{V}$. Thus the hypotheses of $B(\ell,1)$ are satisfied, so there is a map $g\colon |K| \to M$, a compact set $\tilde{C} \subset M$, and a $C^\infty$-isotopy $\tilde{h}_t\colon M \to M$ such that:

(1) $\tilde{h}_o = id_M$, $\tilde{h}_t(x) = x$ if $x \notin \tilde{C}$, and

$\tilde{h}_1(\tilde{V}) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = \tilde{f}|_{|L|}$.

(3) $d(\tilde{f},g) < \frac{\epsilon}{2}$.

Let $C = C' \cup \tilde{C}$, $h_t = \tilde{h}_t \circ h'_t$. Then

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and

$h_1(V) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f|_{|L|}$.

(3) $d(f,g) < \epsilon$.

(b) $B(\ell-1)$ and $B(\ell,m)$, for all $m$, imply $B(\ell)$.

Proof: Suppose $B(\ell-1)$ and $B(\ell,m)$, for all $m$, are true, and the hypotheses of $B(\ell)$ are satisfied. Let $\sigma(K)$ be a subdivision of $K$ so fine that if $\Delta^{\ell} \in \sigma(K)$ is an $\ell$-simplex such that $f(\Delta^{\ell}) \cap X \neq \emptyset$, then $f(\Delta^{\ell}) \subset V$. Let $K_o = \{\Delta \in \sigma(K): f(\Delta) \subset V\}$, $K_o' = K_o \cup (\sigma(K))^{(\ell-1)}$, $L_o' = \sigma(L) \cap K_o'$, $f' = f|_{|K_o'|}$. Then $\dim N(|K_o'| - (f')^{-1}(V), K_o') \leq \ell-1$. Let $\varepsilon' > 0$ be such that any $\varepsilon'$-approximation $g': |K_o'| \cup |L_o| \to M$ to $f|_{|K_o'| \cup |L_o|}$ can be extended to an $\frac{\varepsilon}{2}$-approximation $g: |K| \to M$ to $f$. The hypotheses of $B(\ell-1)$ are satisfied, so there is a map $g'': |K_o'| \to M$, a compact set $C' \subset M$, and a $C^{\infty}$-isotopy $h_t': M \to M$ such that:

(1) $h_o' = id_M$, $h_t'(x) = x$ if $x \notin C'$, and
$h_1'(V) \supset X \cup g''(|K_o'|)$.

(2) $g''|_{|L_o'|} = f'|_{|L_o'|}$.

(3) $d(f',g'') < \varepsilon'$.

Let $g': |K_o'| \cup |L| \to M$ be defined by $g'|_{|K_o'|} = g''$, $g'|_{|L|} = f|_{|L|}$, and let $\tilde{f}: |K| \to M$ be an extension of $g'$ such that $d(\tilde{f},f) < \frac{\varepsilon}{2}$, and if $\Delta \in \sigma(K)-K_o'$, then $\tilde{f}(\Delta) \cap X = \emptyset$. Let $\tilde{V} = h_1'(V)$.

For some $m$, the hypotheses of $B(\ell,m)$ are satisfied, so there is a map $g\colon |K| \to M$, a compact set $\tilde{C} \subset M$, and a $C^\infty$-isotopy $\tilde{h}_t\colon M \to M$ such that

(1) $\tilde{h}_o = id_M$, $\tilde{h}_t(x) = x$ if $x \notin \tilde{C}$, and

$\tilde{h}_1(\tilde{V}) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = \tilde{f}|_{|L|}$.

(3) $d(g,\tilde{f}) < \frac{\varepsilon}{2}$.

Let $C = \tilde{C} \cup C'$, $h_t = \tilde{h}_t \circ h_t'$. Then

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and

$h_1(V) \supset X \cup g(|K|)$.

(2) $g|_{|L|} = f|_{|L|}$.

(3) $d(g,f) \leq d(g,\tilde{f}) + d(\tilde{f},f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

<u>Step III: Proof of $B(\ell,1)$</u>

In view of Step II, we need only show that $B(\ell-1)$ implies $B(\ell,1)$, $\ell \leq k$, since this proves $B(k)$, and hence the theorem. Thus we may assume:

(1) $X \subset V$.

(2) there is an $\ell$-simplex $\Delta^\ell \in K$ such that

$|K|-f^{-1}(V) \subset Int(\Delta^\ell)$.

(3) $f(\Delta^\ell) \cap X = \emptyset$.

(4) $B(\ell-1)$ is true.

Let $G = K \cup v*\Delta^\ell$, where $v*\Delta^\ell = \Delta^{\ell+1}$ is an $(\ell+1)$-simplex not in $K$. Since $(M,V)$ is $k$-connected, there is a map $\hat{f}\colon |G| \to M$ such that $\hat{f}|_{|K|} = f$, and $\hat{f}(v*\partial\Delta^{\ell}) \subset V$. Let $H = (K-\{\Delta^{\ell}\}) \cup v*\partial\Delta^{\ell}$. Then $G = H \cup v*\Delta^{\ell}$, $X \cup \hat{f}(|H|) \subset V$, and $X \cap \hat{f}(|L \cap v*\Delta^{\ell}|) = \emptyset$. There are points $x_1,\ldots,x_N \in M$ such that $\hat{f}(\Delta^{\ell+1}) \subset \mu_{x_1}(\operatorname{Int} C_1^n) \cup \ldots \cup \mu_{x_N}(\operatorname{Int} C_1^n)$. Let $\mu_i = \mu_{x_i}$, $1 \leq i \leq N$.

<u>Case A</u>

Suppose $\hat{f}(\Delta^{\ell+1}) \subset \mu_{x_1}(\operatorname{Int} C_1^n)$. Let $\mu = \mu_{x_1}$. There is a number $\alpha$ such that $0 < \alpha < 1$ and $\hat{f}(\Delta^{\ell+1}) \subset \mu(\operatorname{Int} C_\alpha^n)$.

(1) $X \cap \mu(C_1^n) \subset \mu(P)$, where $P$ is a k-dimensional subpolyhedron of $C_1^n$. There is a subdivision $\sigma_o(G)$ of $G$ and a simplicial complex $R$ lying in $C_1^n$ such that $|R| = P$, $(f|_{|L|})^{-1}(\mu(R))$ is a subcomplex of $\sigma_o(G)$ and $\mu^{-1}\circ f|_{(f|_{|L|})^{-1}(\mu(R))}\colon (f|_{|L|})^{-1}(\mu(R)) \to R$ is simplicial. If $\Delta_1 \in \sigma_o(L)$, $\Delta_2 \in R$ are such that $f(\Delta_1) = \mu(\Delta_2)$, identify $\Delta_1$ and $\Delta_2$ and let $G*$ be the simplicial complex obtained from $\sigma_o(G) \cup R$ by this identification. Let $p\colon \sigma_o(G) \cup R \to G*$ be the projection, let $L* = p(\sigma_o(L) \cup R)$, and let $f*\colon |G*| \to M$ be defined by $f*\circ p = f \cup \mu|_P$. (If $f(|\sigma_o(L)|) \cap \mu(P) = \emptyset$ then

$G^* = \sigma_o(G) \cup R)$.

(ii) By Theorem 4 of [10], there is a map $f^{**}: |G^*| \to M$ such that:

(1) $f^{**}$ and $f^*$ agree on $M - \mu(\mathrm{Int}\ C_1^n)$.

(2) $\mu^{-1} \circ f^{**}: |G^*| \to C_1^n$ agrees with a PL-optimal map in $C_\alpha^n$.

(3) $f^{**}|_{|L^*|} = f^*|_{|L^*|}$.

(4) $f^{**} \circ p(|H|) \subset V$, and $d(f^*, f^{**}) < \frac{\varepsilon}{2}$.

Let $f' = f^{**} \circ p|_{|G|}: |G| \to M$. Then:

(1) $f'$ and $\hat{f}$ agree on $M - \mu(\mathrm{Int}\ C_1^n)$.

(2) $\mu^{-1} \circ f'$ agrees with a PL-optimal map in $C_\alpha^n$ which is "in general position" with respect to $P \cap C_\alpha^n$.

(3) $f'|_{|L|} = \hat{f}|_{|L|} = f|_{|L|}$.

(4) $f'(|H|) \subset V$, and $d(f', f) < \frac{\varepsilon}{2}$.

(iii) Let $\sigma_1(G)$ be a subdivision of $\sigma_o(G)$ such that

$$\mu^{-1} \circ f'|_{N(f'^{-1}(\mu(C_\alpha^n)),\ \sigma_1(G))} : N(f'^{-1}(\mu(C_\alpha^n)),\ \sigma_1(G)) \to C_1^n$$

is optimal and "in general position" with respect to $P \cap C_\alpha^n$ in $\mathrm{Int}\ C_1^n$, and such that $\sigma_1(G) \searrow^s \sigma_1(H)$: there is a sequence $\{E_{1,i}^{\ell+1}\}_{i=0}^s$ of $(\ell + 1)$-complexes such that $E_{1,0}^{\ell+1} = \emptyset$, $E_{1,s}^{\ell+1} = \sigma_1(\Delta^{\ell+1})$, $E_{1,i+1}^{\ell+1} = E_{1,i}^{\ell+1} \cup \Delta_i^{n_i}$,

$\Delta_i^{n_i} = x_i * \Delta_i^{n_i-1}$, and $(\sigma_1(H) \cup E_{1,i}^{\ell+1}) \cap \Delta_i^{n_i} = x_i * \partial\Delta_i^{n_i-1}$, $1 \leq i \leq s-1$.

(iv) Induction Hypothesis: There is a map $g_i\colon |G| \to M$, a compact set $C_i \subset M$, and a $C^\infty$-isotopy $h_{i,t}\colon M \to M$ such that

(1) $h_{i,0} = id_M$, $h_{i,t}(x) = x$ if $x \notin C_i$, and

$h_{i,1}(V) \supset X \cup g_i(|\sigma_1(H) \cup (E_{1,i}^{\ell+1})^{(\ell)}|)$.

(2) $g_i \big| |\sigma_1(L) \cup N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G))| =$

$= f' \big| |\sigma_1(L) \cup N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G))|$.

(3) $d(g_i, f') < \frac{\varepsilon}{2}(1-2^{-i})$.

This is clearly true if $i = 0$.

(v) Induction Step: We have $n_i \leq \ell + 1$ for $i \leq s-1$. Thus $\mu^{-1}(f'(\Delta_i^{n_i}) \cap (X \cup f'(|\sigma_1(H) \cup (\sigma_1(\Delta^{\ell+1}))^{(\ell)}|))) \subseteq$ $\mu^{-1}(f'(\Delta_i^{n_i}) \cap \mu(P)) \cup \mu^{-1}(f'(\Delta_i^{n_i}) \cap f'(|\sigma_1(H) \cup (\sigma_1(\Delta^{\ell+1}))^{(\ell)}|)) =$ $\mu^{-1} \circ f'(\Delta_i^{n_i} \cap (\sigma_1(H) \cup (\sigma_1(\Delta^{\ell+1}))^{(\ell)}) \cup Q_i$, where $Q_i$ is a subpolyhedron of $\mu^{-1} \circ f'(\Delta_i^{n_i})$ such that $\dim Q_i \leq (\ell+1) + k-n \leq \ell-2$.

Consider now $\Delta^{n_i} = \mu^{-1} \circ f'(\Delta_i^{n_i})$, $\Delta^{n_i-1} = \mu^{-1} \circ f'(\Delta_i^{n_i-1})$, and $v = \mu^{-1} \circ f'(x_i)$, let $T^{n_i - 1}$ be the plane in $R^n$ determined by $\Delta_i^{n_i - 1}$, let $T^{n_i}$ be the plane in $R^n$ determined by $\Delta^{n_i}$, and let $\pi\colon T^{n_i} \to T^{n_i - 1}$ be the projection with $\pi(v) = b_{\Delta^{n_i - 1}}$. Let $A_i = \Delta^{n_i} \cap \pi^{-1}(\pi(Q_i))$, a subpolyhedron of $\Delta^{n_i}$ of dimension $\leq \ell - 1$. Let $P_i = (f')^{-1}(\mu(A_i)) = g_i^{-1}(\mu(A_i))$, $D_i = |\sigma_1(H) \cup (E_{1,i}^{\ell+1})^{(\ell)}| \cup P_i$, and let $D_i^{\alpha} = D_i \cap (f')^{-1}(\mu(C_\alpha^n))$. Then $D_i^\alpha$ is a polyhedral subset of $|G|$.

(vi) We now show that there is a continuous map $g_{i+1}\colon |G| \to M$, a compact set $C^* \subset M$, and a $C^\infty$-isotopy $h_t^*\colon M \to M$ such that

(1) $h_0^* = id_M, h_t^*(x) = x$ if $x \notin C^*$, and

$h_1^*(h_{i,1}(V)) \supset X \cup g_{i+1}(D_i)$.

(2) $g_{i+1}\big|\, |L| \cup |N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G))| =$

$$= f'\big|\, |L| \cup |N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G))|.$$

(3) $d(g_i, g_{i+1}) < \frac{\varepsilon}{2^{i+2}}$.

<u>Proof:</u> There is a subdivision $\sigma_2(G)$ of $\sigma_1(G)$ such that $\sigma_2(D_i)$ and $\sigma_2(D_i^\alpha)$ are subcomplexes of $\sigma_2(G)$, and

$f'|_{D_i^\alpha}: \sigma_2(D_i^{\alpha}) \to C_1^n$ is a simplicial map onto a simplicial complex in $C_\alpha^n$. Identify $\Delta_1$ and $\Delta_2$ if $\Delta_1, \Delta_2 \in \beta(\sigma_2(D_i^{\alpha}))$ and $f'(\Delta_2) = f'(\Delta_2)$, and let $K^*$ be the simplicial complex obtained from $\beta(\sigma_2(D_i))$ by this identification. Let $p: \beta(\sigma_2(D_i)) \to K^*$ be the projection, let $f^*: |K^*| \to M$ be defined by $f^* \circ p = g_i$, and let

$L^* = p(\beta(\sigma_2(L) \cap \sigma_2(D_i) \cup \sigma_2(D_i^{\alpha})))$. Then $f^*|_{|L^*|}$ is a locally linearizable imbedding. Let $V^* = h_{i,1}(V)$.

The hypotheses of $B(\ell-1)$ are satisfied by $K^*$, $L^*$, $f^*$, $X$ and $V^*$ since $N(|K^*| - (f^*)^{-1}(V^*), K^*) \subset P_i$, and $\dim P_i \leq \ell-1$. Thus there is a map $g^*: |K^*| \to M$, a compact set $C^* \subset M$, and a $C^\infty$-isotopy $h_t^*: M \to M$ such that:

(1) $h_0^* = id_M$, $h_t^*(x) = x$ if $x \notin C^*$, and

$h_1^*(V^*) \supset X \cup g^*(|K^*|)$.

(2) $g^*|_{|L^*|} = f^*|_{|L^*|}$.

(3) $d(g^*, f^*) < \frac{\varepsilon}{2^{i+2}}$.

Let $g_{i+1}: |G| \to M$ be defined by $g_{i+1}|_{|D_i|} = g^* \circ p$, and $g_{i+1}|_{\Delta^{\ell+1}} = g_i|_{\Delta^{\ell+1}} = f'|_{\Delta^{\ell+1}}$. Then

$g_{i+1}|_{|L| \cup |N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G)|} =$

$= f'|_{|L| \cup |N(f'^{-1}(\mu(C_\alpha^n)), \sigma_1(G))|}$, and

$X \cup g_{i+1}(D_i) = X \cup g^*(|K^*|) \subset h_1^*(h_{i,1}(V))$.

(vii) Let $U = \mu^{-1}(h_1^* \circ h_{i,1}(V) \cap \mu(\operatorname{Int} C_1^*))$, and let $F = A_i \cup \mu^{-1}(X \cup f'(|\sigma_2(H) \cup (E_{1,i}^{\ell+1})^{(\ell)}|))$. Then $F \cap \Delta^{n_i} = A_i \cup v * \partial\Delta^{n_i-1}$ and $A_i \cup v*\partial\Delta^{n_i-1} \subset U$, so we may apply Corollary 1.1. There is a compact set $C \subset \mathbb{R}^n - F$ and a $C^\infty$-isotopy $h_t'\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $h_0' = \mathrm{id}_{\mathbb{R}^n}$, $h_t'(x) = x$ if $x \notin C$, and $h_1'(U) \supset \Delta^{n_i}$.

Let $C_{i+1} = \mu(C_\alpha^n) \cup C_i \cup C^*$, and let $h_t\colon M \to M$ be defined by $h_t|_{\mu(C_1^n)} = \mu \circ h_t' \circ \mu^{-1}$, and $h_t|_{M-\mu(C_\alpha^n)} = \mathrm{id}_{M-\mu(C_\alpha^n)}$. Let $h_{i+1,t} = h_t \circ (h_t^* \circ h_{i,t})$. Then

(1) $h_{i+1,0} = \mathrm{id}_M$, $h_{i+1,t}(x) = x$ if $x \notin C_{i+1}$, and

$$h_{i+1,1}(V) \supset X \cup g_{i+1}(|\sigma_2(D_i)| \cup \Delta_i^{n_i}) \supset$$
$$X \cup g_{i+1}(|\sigma_1(H) \cup (E_{1,i+1}^{\ell+1})^{(\ell)}|).$$

(2) $$g_{i+1}|_{|L| \cup |N(f'^{-1}(\mu(C_\alpha^n)),\ \sigma_1(G))|} = f'|_{|L| \cup |N(f'^{-1}(\mu(C_\alpha^n)),\ \sigma_1(G))|}.$$

(3) $$d(g_{i+1}, f') \leq d(g_{i+1}, g_i) + d(g_i, f') < \frac{\epsilon}{2^{i+2}} + \frac{\epsilon}{2}(1 - 2^{-i}) = \frac{\epsilon}{2}(1 - 2^{-(i+1)}).$$

Thus we have completed the induction step.

(viii) Let $g = g_s|_{|K|}: |K| \to M$, $h_t = h_{s,t}$, $C = C_s$. Then

(1) $h_o = id_M$, $h_t(x) = x$ if $x \notin C$, and

$$h_1(V) = h_{s,1}(V) \supset X \cup g_s(|\sigma_1(H) \cup (E^{\ell+1}_{1,s})^{(\ell)}|) \supset X \cup g(|K|).$$

(2) $g|_{|L|} = g_s|_{|L|} = f'|_{|L|} = f|_{|L|}$.

(3) $d(g,f) \leq d(g,f') + d(f',f) < \frac{\varepsilon}{2}(1 - 2^{-s}) + \frac{\varepsilon}{2} < \varepsilon$.

Thus Case A is proved.

Case B

For some $N$, $\hat{f}(\Delta^{\ell+1}) \subset \mu_1(\text{Int } C^n_1) \cup \ldots \cup \mu_N(\text{Int } C^n_1)$.

(i) There is a number $\alpha$ such that $0 < \alpha < 1$ and

$\hat{f}(\Delta^{\ell+1}) \subset \mu_1(\text{Int } C^n_\alpha) \cup \ldots \cup \mu_N(\text{Int } C^n_\alpha)$.

Let $\sigma_1(G)$ be a subdivision of $G$ such that for each $\Delta \in \sigma_1(\Delta^{\ell+1})$, there is an integer $j(\Delta)$ such that $\hat{f}(\Delta) \subset \mu_{j(\Delta)}(\text{Int } C^n_\alpha)$, and $\sigma_1(G) \searrow^{s} \sigma_1(H)$: there is a sequence $\{E^{\ell+1}_{1,i}\}^s_{i=0}$ of $(\ell+1)$-complexes such that $E^{\ell+1}_{1,0} = \emptyset$, $E^{\ell+1}_{1,s} = \sigma_1(\Delta^{\ell+1})$, $E^{\ell+1}_{1,i+1} = E^{\ell+1}_{1,i} \cup \Delta_i^{n_i}$, where $\Delta_i^{n_i} = x_i * \Delta_i^{n_i - 1}$, and $(\sigma_1(H) \cup E^{\ell+1}_{1,i}) \cap \Delta_i^{n_i} = x_i * \partial\Delta_i^{n_i - 1}$

$1 \leq i \leq s-1$.

(ii) Let $\epsilon' > 0$ be so small that if $g: |G| \to M$ is an $\epsilon'$-approximation to $\hat{f}$, then $g(\Delta) \subset \mu_{j(\Delta)}(\mathrm{Int}\ C_\alpha^n)$, for all $\Delta \in \sigma_1(\Delta^{\ell+1})$, and, for any subcomplex $Q$ of $\sigma_1(\Delta^{\ell+1})$, any $\epsilon'$-approximation $g: |Q| \to M$ to $\hat{f}|_{|Q|}$ may be extended to an $\frac{\epsilon}{2^s}$-approximation to $\hat{f}$.

(iii) <u>Induction Hypothesis:</u> There is a map $g_i: |G| \to M$, a compact set $C_i \subset M$, and a $C^\infty$-isotopy $h_{i,t}: M \to M$ such that:

(1) $h_{i,0} = \mathrm{id}_M$, $h_{i,t}(x) = x$ if $x \notin C_i$, and

$h_{i,1}(V) \supset X \cup g_i(|\sigma_1(H) \cup (E_{1,i}^{\ell+1})^{(\ell)}|)$.

(2) $g_i|_{|L|} = f|_{|L|}$.

(3) $d(g_i, \hat{f}) < \epsilon(1-2^{-i})$.

(iv) <u>Induction Step:</u> We reduce Case B to Case A:

Let $V' = h_{i,1}(V)$, $H' = \sigma_1(H) \cup (E_{1,i}^{\ell+1})^{(\ell)}$,

$K' = H' \cup \Delta_i^{n_i-1}$, $G' = K' \cup \Delta_i^{n_i}$, $f' = g_i|_{|G'|}$,

$L' = \sigma_1(L) \cap G'$. Then $H' \cap \Delta_i^{n_i} = x_i * \partial\Delta_i^{n_i-1}$,

$X \cup f'(|H'|) \subset V'$, and

$X \cap f'(|L' \cap \Delta_i^{n_i}|) \subset X \cap f'(|L \cap \Delta^\ell|) = \emptyset$, so the hypotheses of Case A are satisfied. Thus there is a map

$g'\colon |G'| \to M$, a compact set $C' \subset M$, and a $C^\infty$-isotopy $h'_t\colon M \to M$ such that

(1) $h'_0 = \mathrm{id}_M$, $h'_t(x) = x$ if $x \notin C'$, and

$X \cup g'(|K'|) \subset h'_1(V')$.

(2) $g'|_{|L'|} = f'|_{|L'|}$.

(3) $d(f', g') < \epsilon'$.

Let $g_{i+1}\colon |G| \to M$ be defined by $g_{i+1}|_{|G'|} = g'$, $g_{i+1}|_{|L|} = f|_{|L|}$, and $d(g_{i+1}, \hat{f}) < \epsilon(1 - 2^{-(i+1)})$.

Let $h_{i+1,t} = h'_t \circ h_{i,t}$, $C_{i+1} = C' \cup C_i$. Then:

(1) $h_{i+1,0} = \mathrm{id}_M$, $h_{i+1,t}(x) = x$ if $x \notin C_{i+1}$, and

$X \cup g_{i+1}(|\sigma_1(H) \cup (E^{\ell+1}_{1,i+1})^{(\ell)}|) \subset X \cup g'(|K'|) \subset h_{i+1,1}(V)$.

(2) $g_{i+1}|_{|L|} = f|_{|L|}$.

(3) $d(g_{i+1}, \hat{f}) < \epsilon(1 - 2^{-(i+1)})$.

This completes the induction.

(v) Let $g = g_s$, $C = C_s$, $h_t = h_{s,t}$. Then

(1) $h_0 = \mathrm{id}_M$, $h_t(x) = x$ if $x \notin C$, and

$X \cup g(|K|) \subset X \cup g_s(|\sigma_1(H) \cup (E^{\ell+1}_{1,s})^{(\ell)}|) \subset h_1(V)$.

(2) $g|_{|L|} = f|_{|L|}$.

(3) $d(g, f) < \epsilon$.

This proves Case B, and hence $B(\ell, 1)$. Q.E.D.

We note that Corollary 2.1 holds for $k = 0$ with no restriction on $n$.

# CHAPTER THREE

## A $C^\infty$-Stretching Diffeomorphism

Let $\Delta^m = \Delta^k * \Delta^\ell \subset \mathbb{R}^m \subset \mathbb{R}^n$, where $m = k + \ell + 1 \leq n$. We let $E^\ell = \{(0,\ldots,0,x_{k+1},\ldots,x_{k+\ell},1) \in \mathbb{R}^m\}$, and let $L \subset \mathbb{R}^m$ denote the $x_m$-axis. We assume that $\Delta^m$ is situated in $\mathbb{R}^m$ such that $\Delta^k \subset \mathbb{R}^k$, $\Delta^\ell \subset E^\ell$, $b_{\Delta^k} = 0$, and $b_{\Delta^\ell} = (0,\ldots,0,1) \in L$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^m$, $p\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$, and $q\colon \mathbb{R}^m \to L$ be the orthogonal projections, and let $r = p \circ \pi$ and $s = q \circ \pi$.

For $\lambda \in \mathbb{R}$, let $H_o{}^m(\lambda) = \{(x_1,\ldots,x_m) \in \mathbb{R}^m\colon x_m > \lambda\}$, and let $H^m(\lambda) = \{(x_1,\ldots,x_m) \in \mathbb{R}^m\colon x_m \geq \lambda\}$.

<u>Lemma 3.1.</u> Let $U \subset \mathbb{R}^n$ be an open set and let $\varepsilon > 0$ be such that $\partial\Delta^m \subset U \cup H_o{}^m(1-2\varepsilon)$. Let $F$ be a closed set in $\mathbb{R}^n$ such that $F \cap \Delta^m \subset \partial\Delta^m$. Then there is a compact set $C \subset \mathbb{R}^n - F$ and a $C^\infty$-isotopy $h_t\colon \mathbb{R}^n \to \mathbb{R}^n$ such that:

(1) $h_o = \mathrm{id}_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and

$\Delta^m \subset h_1(U) \cup H_o{}^m(1-2\varepsilon)$.

(2) If $T$ is any linear subspace of $\mathbb{R}^n$ which contains $\Delta^m$, then $h_t(T) = T$ for all $t \in I$.

<u>Proof:</u> <u>Step A:</u> We first construct a "horizontal $C^\infty$-stretching diffeomorphism" $h_t^!: \mathbb{R}^n \to \mathbb{R}^n$. If $m = 1$, let $h_t^! = id_{\mathbb{R}^n}$. We assume that $m \geq 2$ for the rest of Step A and further that $1 - 2\epsilon > 0$.

(1) Let $\epsilon_o > 0$ be such that $\Delta^m \cap (\mathbb{R}^m - H_o^m(2\epsilon_o)) \subset U$. If $\delta > 0$, let $B_\delta^{m-1} = \{x \in R^{m-1}: \|x\| \leq \delta\}$. We choose a fixed $\delta > 0$ such that if $D_o^m = p^{-1}(B_\delta^{m-1}) \cap (H^m(\epsilon_o) - H_o^m(1-\epsilon))$, then $D_o^m \subset \text{Int } \Delta^m$. Let $D^m = p^{-1}(B_\delta^{m-1}) \cap (H^m(2\epsilon_o) - H_o^m(1-2\epsilon)) \subset D_o^m$.

See Figure 5. Finally, let

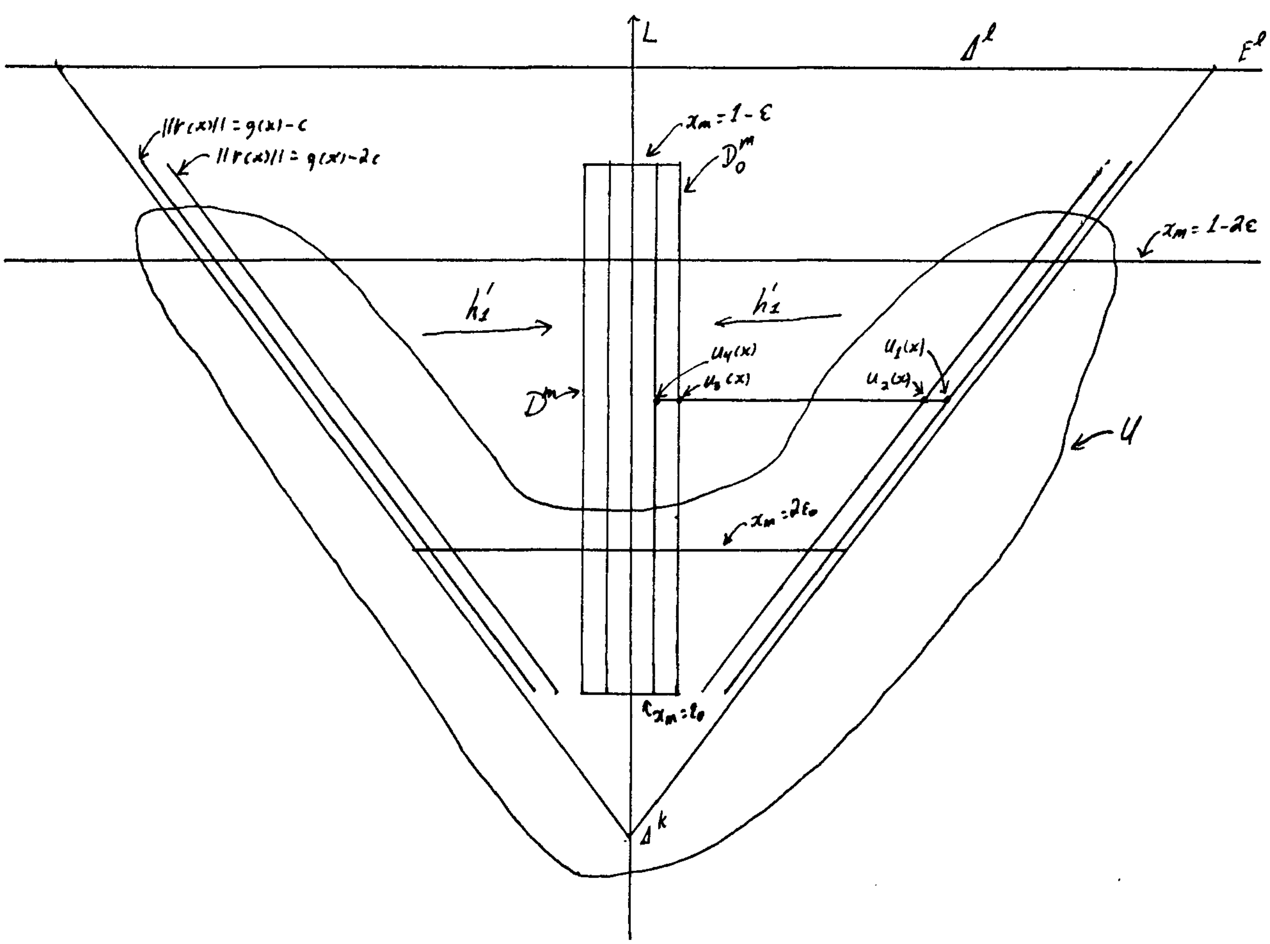


FIGURE 5

$$c = \tfrac{1}{4}\min\{\mathrm{dist}(\partial\Delta^m - H_o^m(1-2\varepsilon), \mathbb{R}^n - U),\ \mathrm{dist}(D_o^m, \partial\Delta^m)\} > 0.$$

We wish to construct a $C^\infty$-isotopy $h_t^!: \mathbb{R}^n \to \mathbb{R}^n$ and a compact set $C \subset \mathbb{R}^n - F$ such that:

(1) $h_o^! = \mathrm{id}_{\mathbb{R}^n}$, $h_t^!(x) = x$ if $x \in C$, and

$(\Delta^m - D^m) \cup \mathrm{Cl}(\partial D^m - H^m(1-2\varepsilon)) \subset h_1^!(U) \cup H_o^m(1-2\varepsilon)$.

(2) $h_t^!(x) - x \in \mathbb{R}^{m-1}$ for all $x \in \mathbb{R}^{n-1}$ and all $t \in I$.

(2) Consider the continuous map $\hat{g}: \pi^{-1}(H_o^m(0) - H^m(1) - L) \to \mathbb{R}$ defined by $\hat{g}(x) = \|f(x) - s(x)\|$, where $f(x)$ is the point of $\partial\Delta^m$ lying on the ray from $s(x)$ through $\pi(x)$. Note that $\hat{g}(x) = \hat{g}(s(x) + \lambda\cdot r(x))$ for all $\lambda > 0$ and all $x \in \pi^{-1}(H_o^m(0) - H^m(1) - L)$. We construct a $C^\infty$-c-approximation to $\hat{g}|_{\pi^{-1}(H^m(\varepsilon_o) - H_o^m(1-\varepsilon) - L)}$ with the same property as follows:

Consider

$S^{m-2}\times[\varepsilon_o, 1-\varepsilon] = p^{-1}(\{x \in \mathbb{R}^{m-1}: \|x\| = 1\}) \cap (H^m(\varepsilon_o) - H_o^m(1-\varepsilon))$,

and $\hat{g}|_{S^{m-2}\times[\varepsilon_o,1-\varepsilon]}: S^{m-2}\times[\varepsilon_o, 1-\varepsilon] \to \mathbb{R}$. Let

$\tilde{g}: S^{m-2}\times[\varepsilon_o, 1-\varepsilon] \to \mathbb{R}$ be a $C^\infty$-c-approximation to $\hat{g}|_{S^{m-2}\times[\varepsilon_o,1-\varepsilon]}$. Let $g(x) = \tilde{g}(s(x) + \frac{r(x)}{\|r(x)\|})$ for all

$x \in \pi^{-1}(H^m(\varepsilon_o) - H_o^m(1-\varepsilon) - L)$.

(3) Now we may construct a "horizontal stretching interval" for each $x \in \pi^{-1}(H^m(\varepsilon_o) - H_o^m(1 - \varepsilon) - L)$. Let

$$u_1(x) = s(x) + (g(x) - c)\cdot\frac{r(x)}{\|r(x)\|},$$

$$u_2(x) = s(x) + (g(x) - 2c)\cdot\frac{r(x)}{\|r(x)\|},$$

$$u_3(x) = s(x) + \delta\cdot\frac{r(x)}{\|r(x)\|},$$

$$u_4(x) = s(x) + \frac{\delta}{2}\cdot\frac{r(x)}{\|r(x)\|}.$$

The stretching interval will be $[u_1(x),u_4(x)]$ and by "stretching" we will map $[u_1(x),u_2(x)]$ onto $[u_1(x),u_3(x)]$. The length of $[u_1(x),u_4(x)]$ is $\gamma(x) = g(x)-c-\frac{\delta}{2} \geq \hat{g}(x)-2c-\frac{\delta}{2} \geq$ $\geq (4c+\delta) - 2c - \frac{\delta}{2} = 2c + \frac{\delta}{2} > 0$. Notice that $[u_1(x),u_2(x)] \subset U$ if $x \in H^m(2\varepsilon_o) - H_o^m(1-2\varepsilon) - L$, since $\mathrm{dist}(u_2(x), \partial\Delta^m - H_o^m(1-2\varepsilon)) \leq \|f(x) - u_2(x)\| \leq 3c$.

To apply Lemma 1.1, we map the interval $[u_1(x),u_4(x)]$ linearly onto the interval $[0,1]$ by a transformation such that $u_1(x)$ is mapped onto $0$ and $u_4(x)$ is mapped onto $1$. Then $u_2(x)$ is mapped onto $\alpha(x) = \frac{\|u_2(x)-u_1(x)\|}{\|u_4(x)-u_1(x)\|}$ $= \frac{c}{\gamma(x)}$ and $u_3(x)$ is mapped onto $\beta(x) = \frac{\|u_3(x)-u_1(x)\|}{\|u_4(x)-u_1(x)\|} =$ $\frac{\gamma(x)-\frac{\delta}{2}}{\gamma(x)}$. Of course $\alpha(x) < \beta(x)$.

(4) Before defining $h_t^!$, we must construct a $C^\infty$-function $\varphi \colon R^n \to \mathbb{R}$ with the proper support.

Let $\rho_1 \colon \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $0 \leq \rho_1(t) \leq 1$ for all $t \in \mathbb{R}$, $\rho_2(t) = 1$ if $2\varepsilon_o \leq t \leq 1 - 2\varepsilon$, and $\rho_2(t) = 0$ if $t \leq \varepsilon_o$, or $t \geq 1-\varepsilon$.

Let $C_o = \{x \epsilon H^m(\varepsilon_o) - H_o^m(1-\varepsilon) - L \colon \|p(x)\| \leq g(x) - c\} \cup (L \cap (H^m(\varepsilon_o) - H_o^m(1-\varepsilon)))$. Then $C_o \subset \operatorname{Int} \Delta^m$. Let $\eta = \operatorname{dist}(C_o, F \cup \partial\Delta^m)$.

Let $\rho_2 \colon \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $0 \leq \rho_2(t) \leq 1$ for all $t \in \mathbb{R}$, $\rho_2(t) = 1$ if $t \leq 0$, and $\rho_2(t) = 0$ if $t \geq \eta$.

Let $\varphi(x) = \rho_1(s(x)) \cdot \rho_2(2\|x - \pi(x)\|)$ for $x \in \mathbb{R}^n$. Let $C = \operatorname{Cl}(\varphi^{-1}((0,1])) \cap \pi^{-1}(\operatorname{Cl}(C_o))$. Then $C$ is compact, $C_o \subset C$, and $C \cap F = \emptyset$. Note that $\varphi(x)$ does not depend on $\|r(x)\|$.

(5) If $x \in \pi^{-1}(H^m(\varepsilon_o) - H^m(1-\varepsilon) - L)$, we let $h_t^!(x) = x + t \cdot \varphi(x)$. [{stretching diffeomorphism with respect to the interval $[u_1(x), u_4(x)]$ applied to $x$}-$x$] =

$$x + t \cdot \varphi(x)\left[\theta^{\beta(x)}_{\alpha(x)}\left(\frac{\|p(u_1(x))\| - \|r(x)\|}{\gamma(x)}\right)(u_4(x) - u_1(x)) + (\pi(x) - u_1(x))\right],$$

otherwise, we let $h_t^!(x) = x$.

Obviously, $\lim_{\|x\|\to\infty} \|h_t^!(x)\| = \infty$, so to apply Lemma 1.2, we need only show that the Jacobian matrix of $h_t^!$ is always non-singular. By the definition of $h_t^!$, if $\|r(x)\| \leq \frac{\delta}{2}$, then $h_t^!(x) = x$, that is, $h_t^!$ is the identity on a neighborhood of $\pi^{-1}(L)$. Thus we need only show that $h_t^!|_{\mathbb{R}^n-\pi^{-1}(L)}$ has a non-singular Jacobian matrix.

We perform a coordinate transformation. We define a $C^\infty$-diffeomorphism

$$e\colon \mathbb{R}^n - \pi^{-1}(L) \to \mathbb{R}_+ \times S^{m-2} \times (\mathbb{R}^{m-1})\perp$$

where $\mathbb{R}_+ = \{t \in \mathbb{R}\colon t > 0\}$, $S^{m-2} = \{x \in \mathbb{R}^{m-1}\colon \|x\| = 1\}$, and $\perp$ denotes the orthogonal complement, as follows: if $x \in \mathbb{R}^n - \pi^{-1}(L)$, let $e(x) = (\|r(x)\|, \frac{r(x)}{\|r(x)\|}, x - r(x))$. Consider

$$H_t = e\circ(h_t|_{\mathbb{R}^n-\pi^{-1}(L)})\circ e^{-1}\colon \mathbb{R}_+ \times S^{m-2} \times (\mathbb{R}^{m-1})\perp \to \mathbb{R}_+ \times S^{m-2} \times (\mathbb{R}^{m-1})\perp.$$

Let $f_{t,(u,y)}\colon \mathbb{R}_+ \to \mathbb{R}_+$ for $(u,y) \in S^{m-2} \times (\mathbb{R}^{m-1})\perp$, and $t \in I$, be defined by $f_{t,(u,y)}(w) = \|r(h_t^!(e^{-1}(w,u,y)))\|$. Then $H_t(w,u,y) = (f_{t,(u,y)}(w),u,y)$, and if $e^{-1}(w,u,y) \in \pi^{-1}(H^m(\varepsilon_0) - H_0^m(1-\varepsilon) - L)$, then $f_{t,(u,y)}(w) =$

$$w - t\circ\varphi(e^{-1}(w,u,y))[\theta^{\frac{\beta(e^{-1}(w,u,y))}{\alpha(e^{-1}(w,u,y))}}(\frac{g(e^{-1}(w,u,y)) - c - w}{\gamma(e^{-1}(w,u,y))})\gamma(e^{-1}(w,u,y)) -$$

$g(e^{-1}(w,u,y)) + c + w]$, otherwise $f_{t,(u,y)}(w) = w$.

We must show that the differential of $H_t$ is everywhere nonsingular. To prove this we have only to show that $\frac{df_{t,(u,y)}(w)}{dw} > 0$ for all $(w,u,y) \in \mathbb{R}_+ \times S^{m-2} \times (\mathbb{R}^{m-1})^\perp$, and all $t \in I$. If $e^{-1}(w,u,y) \notin \pi^{-1}(H^m(\epsilon_o) - H^m_o(1-\epsilon) - L)$, then $\frac{df_{t,(u,y)}(w)}{dw} = 1$. Suppose $e^{-1}(w,u,y) \in \pi^{-1}(H^m(\epsilon_o) - H^m_o(1-\epsilon) - L)$. First we note that $g(e^{-1}(w,u,y))$ does not depend on $w$ by the construction of $g$. Hence $\alpha(e^{-1}(w,u,y))$, $\beta(e^{-1}(w,u,y))$, and $\gamma(e^{-1}(w,u,y))$ do not depend on $w$. Further $\frac{d\theta^\beta_\alpha(z)}{dz} > 0$ for all $z \in R$. We differentiate:

$$\frac{df_{t,(u,y)}(w)}{dw} = 1 - t \cdot \varphi(e^{-1}(w,u,y))\left[-\theta'{}^{\beta(e^{-1}(w,u,y))}_{\alpha(e^{-1}(w,u,y))}\left(\frac{g(e^{-1}(w,u,y)) - c - w}{\gamma(e^{-1}(w,u,y))}\right) + 1\right] > 0.$$

Thus the rank of the Jacobian matrix of $h'_t$ is $n$, so by Lemma 1.2, $h'_t$ is a $C^\infty$-diffeomorphism. It satisfies the required properties (1) and (2).

<u>Step B:</u> Next we construct a "vertical $C^\infty$-stretching diffeomorphism" $h''_t : \mathbb{R}^n \to \mathbb{R}^n$ such that

(1) $h''_o = id_{\mathbb{R}^n}$, $h''_t(x) = x$ if $x \notin C$, and

$\Delta^m \subset h_1''(h_1'(U)) \cup H_o^m(1-2\epsilon)$.

(2) $h_t''(x) - x \in L$ for all $x \in \mathbb{R}^n$ and all $t \in I$.

(1) Since $Cl(\partial D^m - H^m(1-2\epsilon)) \subset h_1'(U)$, we may let

$$d = \mathrm{dist}(\mathbb{R}^n - h_1'(U), Cl(\partial D^m - H^m(1-2\epsilon))) > 0.$$

(If $m = 1$, let $D^1 = \partial D^1 = L \cap (H^1(2\epsilon_o) - H_o^1(1-2\epsilon)))$ and $D_o^1 = L \cap (H^1(\epsilon_o) - H_o^1(1-\epsilon_o))$. If $m \geq 2$, we assume that $d < \delta$. We notice that $\Delta^m - H_o^m(2\epsilon_o) \subset h_1'(U)$ by the construction of $h_t'$. Let $v = (0,\ldots,0,1) \in L$.

For each $x \in \mathbb{R}^n$ we define a "vertical stretching interval". Let

$v_1(x) = r(x) + \epsilon_o \cdot v$,

$v_2(x) = r(x) + 2\epsilon_o \cdot v$,

$v_3(x) = r(x) + (1-2\epsilon) \cdot v$,

$v_4(x) = r(x) + (1-\epsilon) \cdot v$.

The "stretching interval" will be $[v_1(x), v_4(x)]$, and by "stretching" we will map $[v_1(x), v_2(x)]$ onto $[v_1(x), v_3(x)]$. The interval $[v_1(x), v_4(x)]$ has length $\gamma = 1-\epsilon-\epsilon_o$. To apply Lemma 1.1, we map the interval $[v_1(x), v_4(x)]$ linearly onto $[0,1]$ such that $v_1(x)$

is mapped onto $0$ and $v_4(x)$ is mapped onto 1. Then $v_2(x)$ is mapped onto $\alpha = \frac{\epsilon_o}{\gamma}$, and $v_3(x)$ is mapped onto $\beta = \frac{\gamma - \epsilon}{\gamma}$. We note that $\alpha < \beta$. See Figure 6.

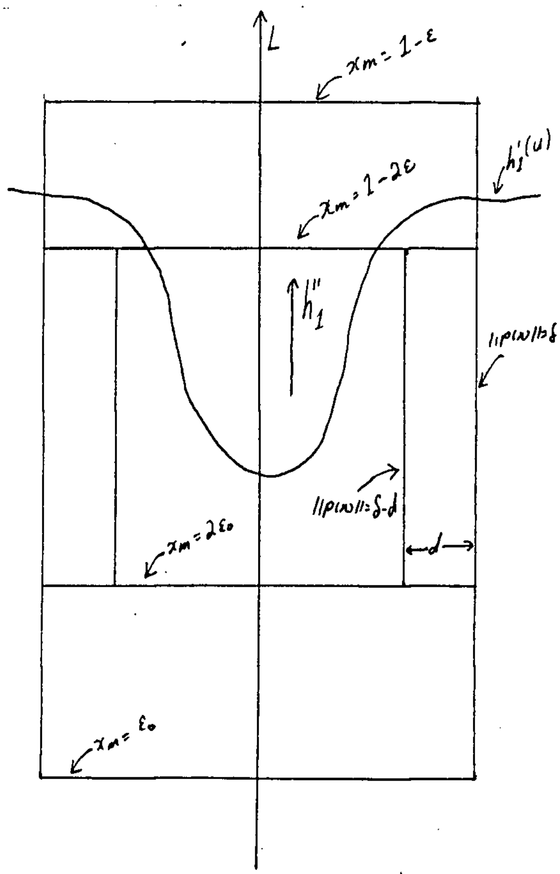


FIGURE 6

(2) Before defining $h_t''$ we must construct a $C^\infty$-function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ with the proper support.

If $m = 1$, let $\eta = \text{dist}(D_o^1, F)$. If $m \geq 2$, note that $D_o^m \subset C_o$, and hence $\text{dist}(D_o^m, F) \leq \text{dist}(C_o, F) = \eta$. Let $\lambda_1\colon \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $0 \leq \lambda_1(t) \leq 1$ for all $t \in \mathbb{R}$, $\lambda_1(t) = 1$ if $t \leq 0$ and $\lambda_1(t) = 0$ if

$t \geq \eta$. If $m = 1$, let $\psi(x) = \lambda_1(2\|x-\pi(x)\|)$, and let $C = \pi^{-1}(D_o) \cap Cl(\psi^{-1}((0,1]))$.

If $m \geq 2$, let $\lambda_2 \colon \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $0 \leq \lambda_2(t) \leq 1$ for all $t \in \mathbb{R}$, $\lambda_2(t) = 1$ if $t \leq \delta - d$, and $\lambda_2(t) = 0$ if $t \geq \delta$. Let $\psi(x) = \lambda_1(2\|x-\pi(x)\|)\cdot\lambda_2(\|r(x)\|)$ for all $x \in \mathbb{R}^n$. Note that $\frac{\partial\psi}{\partial x_m}(x) = 0$, and

$$Cl(\psi^{-1}((0,1])) \cap \pi^{-1}(H^m(\epsilon_o) - H^m_o(1-\epsilon)) \subset C.$$

(3) Let $x \in \mathbb{R}^n$. Similarly as in Step A, we define

$h_t''^m(x) = x_m + t\cdot\psi(x)[\gamma\cdot\theta^\beta_\alpha(\frac{x_m-\epsilon_o}{\gamma}) + \epsilon_o - x_m]$, and then

$h_t''(x) = (x_1,\ldots,x_{m-1},h_t''^m(x),x_{m+1},\ldots,x_n)$. We compute

$$\frac{\partial h_t''^m}{\partial x_m}(x) \colon \frac{\partial h_t''^m(x)}{\partial x_m} = (1-t\cdot\psi(x)) + t\cdot\psi(x)\theta'^\beta_\alpha(\frac{x_m-\epsilon_o}{\gamma}) > 0.$$

Hence the rank of the Jacobian matrix of $h_t''$ is $n$, and again $\lim_{\|x\|\to\infty} \|h_t''(x)\| = \infty$. By Lemma 1.2, $h_t'' \colon \mathbb{R}^n \to \mathbb{R}^n$ is a $C^\infty$-diffeomorphism onto $\mathbb{R}^n$ which satisfies properties (1) and (2) by construction.

Combining Step A and Step B, we let $h_t = h_t''\circ h_t'$. Then $h_t(x) - x \in \mathbb{R}^m$ for all $x \in \mathbb{R}^n$, so $h_t(T) = T$ if $T$ is a linear subspace of $\mathbb{R}^n$ which contains $\Delta^m$. Q.E.D.

If $\Delta^m = \Delta^k * \Delta^\ell \subset \mathbb{R}^n$ is an arbitrary m-simplex in $\mathbb{R}^n$, let $E(\Delta^m)$ be the m-dimensional plane determined by $\Delta^m$, let $E(\Delta^k, \Delta^\ell)$ be an (m-1)-dimensional plane in $E(\Delta^m)$ parallel to the planes $E(\Delta^k)$ and $E(\Delta^\ell)$ determined by the simplices $\Delta^k$ and $\Delta^\ell$ respectively, and with $E(\Delta^k, \Delta^\ell) \cap \operatorname{Int}\Delta^m \neq \emptyset$. Let $H_0^m(\Delta^m, \Delta^\ell)$ be the component of $E(\Delta^m) - E(\Delta^k, \Delta^\ell)$ which contains $\Delta^\ell$.

Corollary 3.1. If $\Delta^m = \Delta^k * \Delta^\ell$ is an arbitrary m-simplex in $\mathbb{R}^n$, U is an open set in $\mathbb{R}^n$ such that $\partial\Delta^m \subset U \cup H_0^m(\Delta^m, \Delta^\ell)$, and F is a closed subset of $\mathbb{R}^n$ such that $F \cap \Delta^m \subset \partial\Delta^m$, then there is a compact set $C \subset \mathbb{R}^n - F$, and a $C^\infty$-isotopy $h_t\colon \mathbb{R}^n \to \mathbb{R}^n$ such that:

(1) $h_0 = \mathrm{id}_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and

$\Delta^m \subset h_1(U) \cup H_0^m(\Delta^m, \Delta^\ell)$.

(2) If $T \subset \mathbb{R}^n$ is a hyperplane containing $\Delta^m$, then $h_t(T) = T$ for all $t \in I$.

Theorem 3.1. Let K be a simplicial complex in $\mathbb{R}^n$, L a full finite subcomplex of K, and $L^c = \{\Delta \in K\colon \Delta \cap L = \emptyset\}$ the subcomplex complementary to L. Let U and V be open sets in $\mathbb{R}^n$ such that $|L| \subset U$ and $|L^c| \subset V$. Let $F \subset \mathbb{R}^n$ be a closed set such that $F \cap |K| \subset |L| \cup |L^c|$. Then there is a compact set $C \subset \mathbb{R}^n - F$ and a $C^\infty$-isotopy $h_t\colon \mathbb{R}^n \to \mathbb{R}^n$ such that:

(1) $h_o = id_{\mathbb{R}^n}$, $h_t(x) = x$ if $x \notin C$, and $|K| \subset h_1(U) \cup V$.

(2) $h_t(\Delta) = \Delta$ for all $\Delta \in K$ and $t \in I$.

Proof: If $\Delta \in K - (L \cup L^c)$, then $\Delta = \Delta^k * \Delta^\ell$ where $\Delta^k \in L$ and $\Delta^\ell \in L^c$. For each principlal simplex $\Delta \in K - (L \cup L^c)$ let $H_o^m(\Delta, \Delta^\ell)$ be chosen so that $\Delta \cap H_o^m(\Delta, \Delta^\ell) \subset V$. If $\Delta \in K - (L \cup L^c)$ is not a principal simplex, then let

$$H_o^m(\Delta, \Delta^\ell) = \bigcap_{\tilde{\Delta} \text{ is a principal simplex in } K-(L \cup L^c) \text{ with } \Delta < \tilde{\Delta}} H_o^m(\tilde{\Delta}, \tilde{\Delta}^\ell).$$

Let $F' = F \cup |L| \cup |L^c|$.

Induction Hypothesis: There is a $C^\infty$-isotopy $h_t^{m-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ and a compact set $C_{m-1} \subset \mathbb{R}^n - F'$ such that

(1) $h_o^{m-1} = id_{\mathbb{R}^n}$, $h_t^{m-1}(x) = x$ if $x \notin C_{m-1}$, and for all

$\Delta \in K^{(m-1)} - (L \cup L^c)$, $\Delta \subset h_1^{m-1}(U) \cup H_o^m(\Delta, \Delta^\ell)$.

(2) $h_t^{m-1}(\Delta) = \Delta$ for all $\Delta \in K$ and $t \in I$.

This is clearly true for $m = 1$($h_t^o = id_{\mathbb{R}^n}$ for all $t \in I$).

Induction Step: Let there be $k_m$ m-simplices $\Delta_1^m, \ldots, \Delta_{k_m}^m \in K^{(m)} - (L \cup L^c)$. First note that if $\Delta'$ is a face of $\Delta$, then

$H^m(\Delta', \Delta'^{\ell}) \subset H_o^m(\Delta, \Delta^{\ell})$. Hence $\partial\Delta_j^m \subset h_1^{m-1}(U) \cup H_o^m(\Delta_j^m; \Delta_j^{\ell})$ for $1 \leq j \leq k_m$. Let $F_j = F' \cup \{\Delta \in K\colon \Delta \cap \operatorname{Int}\Delta_j^m = \emptyset\}$, $1 \leq j \leq k_m$. Then $F_j \cap \Delta_j^m = \partial\Delta_j^m$.

For each $j$, $j = 1, \ldots, k_m$, we apply Corollary 3.1 with $\Delta_j^m = \Delta_j^k * \Delta_j^{\ell}$, where $\Delta_j^k \in L$, $\Delta_j^{\ell} \in L^c$, $F_j$ is the closed subset, $h_1^{m-1}(U)$ is the open subset, and with respect to $H_o^m(\Delta_j^m, \Delta_j^{\ell})$. There are isotopies $h_t^{m,j}\colon \mathbb{R}^n \to \mathbb{R}^n$ and compact subsets $C_{m,j} \subset \mathbb{R}^n - F_j$ such that:

(1) $h_o^{m,j} = id_{\mathbb{R}^n}$, $h_t^{m,j}(x) = x$ if $x \notin C_{m,j}$, and

$$\Delta_j^m \subset h_1^{m,j}(h_1^{m-1}(U)) \cup H_o^m(\Delta_j^m, \Delta_j^{\ell}),\ j = 1, \ldots, k_m.$$

(2) If $T \subset \mathbb{R}^n$ is a hyperplane containing $\Delta_j^m$, then $h_t^{n,j}(T) = T$ for all $t \in I$.

We conclude that $h_t^{m,j}(\Delta) = \Delta$ for all $\Delta \in K$ and $t \in I$. Let $h_t^m = h_t^{m,k_m} \circ \cdots \circ h_t^{m,1} \circ h_t^{m-1}$ and $C_m = C_{m-1} \cup C_{m,1} \cup \cdots \cup C_{m,k_m}$. Then

(1) $h_o^m = id_{\mathbb{R}^n}$, $h_t^m(x) = x$ if $x \notin C_m$, and for all

$$\Delta \in K^{(m)} - (L \cup L^c),\ \Delta \subset h_1^m(U) \cup H_o^m(\Delta, \Delta^{\ell}).$$

(2) $h_t^m(\Delta) = \Delta$ if $\Delta \in K$ and $t \in I$.

If $\dim K = k$, let $h_t = h_t^k$ and $C = C_k$. Q.E.D.

# CHAPTER FOUR

## Open Cylinders

Theorem 4.1. Let $M_1$ and $M_2$ be compact connected $C^\infty$-manifolds and let $f: M_1 \times \mathbb{R} \to M_2 \times \mathbb{R}$ be a $C^\infty$-diffeomorphism such that $M_2 \times \{0\} \subset f(M_1 \times \mathbb{R})$. Then, for any number $\rho > 0$, there is a $C^\infty$-diffeomorphism $\hat{f}$ of $M_1 \times \mathbb{R}$ onto $M_2 \times \mathbb{R}$ such that $\hat{f}|_{M_1 \times [-\rho,\rho]} = f|_{M_1 \times [-\rho,\rho]}$. Further, if $f(M_1 \times (-\infty,\rho)) \supset M_2 \times (-\infty,0]$, we may require that $\hat{f}|_{M_1 \times (-\infty,\rho]} = f|_{M_1 \times (-\infty,\rho]}$.

Proof: Our proof is similar to that used by K. W. Kwun in [5].

(1) There are positive numbers a and b, with $a > \rho$, such that $M_2 \times [-b,b] \subset f(M_1 \times (-a,a))$. Without loss of generality, we may assume that $f(M_1 \times (a+1,\infty)) \cap M_2 \times (-\infty,b) = \emptyset$ (otherwise $f$ is replaced by its reflection). Let $g_0$ be a $C^\infty$-diffeomorphism of $M_1 \times \mathbb{R}$ onto itself such that

$g_0|_{M_1 \times (\mathbb{R}-(-a,a+1))} = id_{M_1 \times (\mathbb{R}-(-a,a+1))}$, and

$f \circ g_0(M_1 \times (-\infty,a]) \subset M_2 \times (-\infty,b)$. Let $f_1 = f \circ g_0$. Then

$f_1(M_1 \times (-\infty,a]) \subset M_2 \times (-\infty,b)$ and

$M_2 \times [-b,b] \subset f_1(M_1 \times (-\infty,a+1))$. $g_0$ and all other $C^\infty$-diffeomorphisms used in this proof may be constructed by using Lemma 1.1.

(2) Suppose we have constructed a sequence $f_1, \ldots, f_k$ of $C^\infty$-diffeomorphisms of $M_1 \times \mathbb{R}$ into $M_2 \times \mathbb{R}$ such that

$f_i(M_1 \times (-\infty, a+i-1)) \subset M_2 \times (-\infty, b+i-1)$,

$M_2 \times [-b, b+i-1] \subset f_i(M_1 \times (-\infty, a+i))$, and

$f_i|_{M_1 \times (-\infty, a+i-2]} = f_{i-1}|_{M_1 \times (-\infty, a+i-2]}$, $i \geq 2$. Let $h_k$ be a $C^\infty$-diffeomorphism of $M_2 \times \mathbb{R}$ onto itself such that $h_k|_{M_2 \times (-\infty, b+k-1]} = id_{M_2 \times (-\infty, b+k-1]}$, and $h_k \circ f_k(M_1 \times (-\infty, a+k)) \supset M_2 \times [-b, b+k]$. Let $g_k$ be a $C^\infty$-diffeomorphism of $M_1 \times \mathbb{R}$ onto itself such that

$g_k|_{M_1 \times (\mathbb{R} - (a+k-1, a+k+1))} = id_{M_1 \times (\mathbb{R} - (a+k-1, a+k+1))}$, and

$h_k \circ f_k \circ g_k(M_1 \times \{a+k\}) \subset M_2 \times (b+k-1, b+k)$. Let $f_{k+1} = h_k \circ f_k \circ g_k$.

(3) Let $\tilde{f} = \lim\limits_{i \to \infty} f_i$. Then $M_2 \times (-b, \infty) \subset \tilde{f}(M_1 \times \mathbb{R})$. Note that

$\tilde{f}|_{M_1 \times (-\infty, a+1]} = h_1 \circ g_0 \circ g_1|_{M_1 \times (-\infty, a+1]}$. Hence

$h_1^{-1} \circ \tilde{f} \circ g_0^{-1}|_{g_1 \circ g_0(M_1 \times (-\infty, a+1])} = f|_{g_1 \circ g_0(M_1 \times (-\infty, a+1])}$.

Note that $g_1 \circ g_0(M_1 \times (-\infty, a+1)) \supset M_1 \times (-\infty, a]$. Let

$f^* = h_1^{-1} \circ \tilde{f} \circ g_1^{-1} \circ g_0^{-1}$. Then $f^*|_{M_1 \times (-\rho, \rho)} = f|_{M_1 \times (-\rho, \rho)}$,

and

$M_2 \times (-b, \infty) \subset f^*(M_1 \times \mathbb{R})$. If $f(M_1 \times (-\infty, \rho)) \supset M_2 \times (-\infty, 0]$,

$$\text{let} \quad \hat{f}(x) = \begin{cases} f(x), & x \in M_1 \times (-\infty, \rho) \\ f^*(x), & x \in M_1 \times (-\rho, \infty). \end{cases}$$

If $f(M_1 \times (-\infty, \rho)) \not\supset M_2 \times (-\infty, 0]$, $f^*$ may be extended in a manner symmetrical to the methods of (2) to obtain the required $\hat{f}$. Q.E.D.

Lemma 4.1. There is a $C^\infty$-diffeomorphism $h$ of $[0,1] \times (0,1)$ onto $(-1,1] \times [0,1) - [0,1] \times \{0\}$ which leaves a neighborhood of $\{1\} \times (0,1)$ fixed.

Proof: Let $f: [-1,0] \to [0,1]$ be a $C^\infty$-function such that $f|_{(-1,0)}$ is a $C^\infty$-imbedding, $f(-1) = 1$, $f(0) = 0$, $\frac{df}{dx}(x) < 0$ and $f(x) < -x$ for $-1 < x < 0$. See Figure 7. We move $\{0\} \times (0,1)$ onto the graph of $f|_{(-1,0)}$ by means of a horizontal stretching diffeomorphism $h_1$. The obvious linear transformation which carries $\frac{1}{2}$ onto $0$ and $-1$ onto $1$ carries $0$ onto $\alpha = \frac{1}{3}$ and carries $f^{-1}(y)$ onto $\beta(y) = \dfrac{\frac{1}{2} - f^{-1}(y)}{\frac{3}{2}}$ for all $y \in (0,1)$. If $(x,y) \in [0,1] \times (0,1)$, let $h_1(x,y) = (\frac{1}{2} - \frac{3}{2}\, \theta_\alpha^{\beta(y)}(-\frac{2}{3}(x - \frac{1}{2})), y)$. Then

$$h_1([0,1] \times (0,1)) = \{(x,y) : f^{-1}(y) \leq x \leq 1,\ 0 < y < 1\}.$$

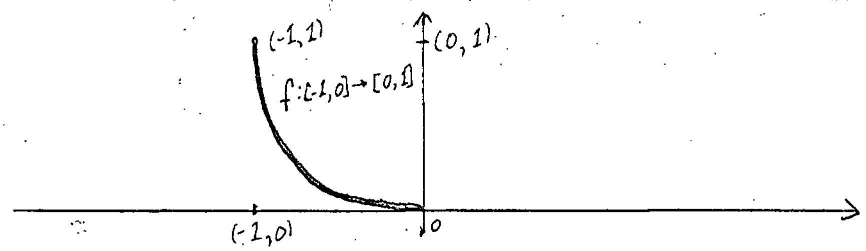


FIGURE 7

Next we construct a vertical $C^\infty$-stretching diffeomorphism $h_2$ which carries $h_1([0,1]\times(0,1))$ onto $(-1,1]\times[0,1) - [0,1]\times(0)$ by moving the graph of $f|_{(-1,0)}$ onto $(-1,0)\times\{0\}$. The obvious linear transformation which carries $-x$ onto $0$ and $-\frac{1}{2}$ onto $1$ carries $f(x)$ onto $\alpha(x) = \frac{f(x)+x}{x-\frac{1}{2}}$ and carries $0$ onto $\beta(x) = \frac{x}{x-\frac{1}{2}}$, for $x < 0$. Clearly, $\alpha(x) < \beta(x)$, since $f(x) > 0$. If $(x,y)\in h_1([0,1]\times(0,1))$, and $x < 0$, let $h_2(x,y) = (x,(x-\frac{1}{2})\theta^{\beta(x)}_{\alpha(x)}(\frac{y+x}{x-\frac{1}{2}})-x)$. If $(x,y)\in[0,1]\times(0,1)$, let $h_2(x,y) = (x,y)$. Then $h_2|_{h_1([0,1]\times(0,1))}$ is $C^\infty$, and is a diffeomorphism since $\frac{\partial}{\partial y}((x-\frac{1}{2})\theta^{\beta(x)}_{\alpha(x)}(\frac{y+x}{x-\frac{1}{2}})-x) = (x-\frac{1}{2})\theta'^{\beta(x)}_{\alpha(x)}(\frac{y+x}{x-\frac{1}{2}})\frac{1}{x-\frac{1}{2}} > 0$ for all $x < 0$ and all $y \in \mathbb{R}$. Further,

$$h_2\circ h_1([0,1]\times(0,1)) = (-1,1]\times[0,1) - [0,1]\times\{0\}.$$

Let $h = h_2\circ h_1$. Q.E.D.

<u>Corollary 4.1.</u> Let $M_1$ and $M_2$ be compact $C^\infty$-manifolds such that Int $M_1$ and Int $M_2$ are $C^\infty$-diffeomorphic. Then $M_1\times\mathbb{R}$ and $M_2\times\mathbb{R}$ are $C^\infty$-diffeomorphic.

<u>Proof:</u> Let $f_i\colon \partial M_i\times[-1,\infty) \to M_i$ be a $C^\infty$-collaring of $\partial M_i$ in $M_i$ (see [8], p. 56), and let $M_i' = M_i - f_i(\partial M_i\times[-1,0))$, $M_i'' = M_i - f_i(\partial M_i\times[-1,1))$, $i = 1,2$. We construct a $C^\infty$-diffeomorphism $h_i'$ of $M_i'\times(0,1)$ onto Int $M_i\times[0,1) - M_i'\times\{0\}$. Let $h_i$ be the $C^\infty$-diffeomorphism of $(M_i - \text{Int } M_i'')\times(0,1)$ onto $\partial M_i\times[-1,1)\times(0,1)$

defined by $h_i(m,t) = (f_i^{-1}(m),t)$. Let $h$ be as in Lemma 4.1, and let $\hat{h}_i\colon \partial M_i \times [0,1] \times (0,1) \to \partial M_i \times ((-1,1] \times [0,1) - [0,1] \times \{0\})$ be defined by $\hat{h}_i(m,x,y) = (m,h(x,y))$. $\hat{h}_i$ leaves a neighborhood of $\partial M_i \times \{1\} \times (0,1)$ fixed. Let $h_i'\colon M_i' \times (0,1) \to \mathrm{Int} M_i \times [0,1) - M_i' \times \{0\}$ be defined by

$$h_i'|_{(M_i - \mathrm{Int}\, M_i'') \times (0,1)} = h_i^{-1} \circ \hat{h}_i \circ h_i, \quad \text{and} \quad h_i'|_{M_i'' \times (0,1)} = \mathrm{id}_{M_i'' \times (0,1)}.$$

Let $g$ be a $C^\infty$-diffeomorphism of $\mathrm{Int}\, M_1$ onto $\mathrm{Int}\, M_2$. Let $D = (h_1')^{-1} \circ (g \times \mathrm{id}_{[0,1]})^{-1} \circ h_2'(M_2' \times (0,1)) \subset M_1' \times (0,1)$. Since $M_1$ and $M_2$ are compact, there is a number $a \in (0,1)$ such that $M_1' \times (a,1) \subset D$, and if we let $\hat{f} = (h_2')^{-1} \circ (g \times \mathrm{id}_{[0,1)}) \circ h_1'|_D$, then for some $b \in (0,1)$, $\hat{f}(M_1' \times (a,1)) \supset M_2' \times \{b\}$. Then Theorem 4.1 implies that $M' \times (a,1)$ is $C^\infty$-diffeomorphic to $M_2' \times \mathbb{R}$, and therefore that $M_1 \times \mathbb{R}$ is $C^\infty$-diffeomorphic to $M_2 \times \mathbb{R}$. Q.E.D.

<u>Corollary 4.2.</u> If $M_1$ and $M_2$ are compact $C^\infty$-manifolds such that $\mathrm{Int}\, M_1$ is $C^\infty$-diffeomorphic to $\mathrm{Int}\, M_2$, then $\partial M_1 \times \mathbb{R}$ is $C^\infty$-diffeomorphic to $\partial M_2 \times \mathbb{R}$.

<u>Theorem 4.2.</u> Let $M$ be a $C^\infty$-n-manifold such that $M = \bigcup_{i=1}^{\infty} O_i^n$, where $O_i^n$ is an open $C^\infty$-n-cell in $M$ with $O_i^n \subset O_{i+1}^n$, for all $i \geq 1$. Then $M$ is $C^\infty$-diffeomorphic to $\mathbb{R}^n$.

Proof: Let $f_i\colon \mathbb{R}^n \to M$ be a $C^\infty$-diffeomorphism such that $f_i(\mathbb{R}^n) = O_i^n$, $i \geq 1$. We may assume that $f_i(0) = p \in M$, $i \geq 1$. Since $M$ is the union of countably many compact sets, we may further assume that $M = \bigcup_{i=1}^{\infty} f_i(D_i^n)$ (where $D_i^n = \{x \in \mathbb{R}^n : \|x\| \leq i\}$). We construct a sequence of $C^\infty$-diffeomorphisms $g_i$ of $\mathbb{R}^n$ into $M$ such that $g_1 = f_1$, $g_{i+1}|_{D_i^n} = g_i|_{D_i^n}$, $g_i(\mathbb{R}^n) = O_i^n$, and $g_{i+1}(D_{i+1}^n) \supset f_{i+1}(D_{i+1}^n)$, $i \geq 1$. Suppose that $g_1, \ldots, g_k$ are constructed. Define $g_{k+1}$ as follows: consider $f_{k+1}^{-1} \circ g_k|_{\mathbb{R}^n - \{0\}}\colon \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$. By Theorem 4.1, there is a $C^\infty$-diffeomorphism $h_{k+1}\colon \mathbb{R}^n - \{0\}$ such that $h_{k+1}|_{D_k^n - \{0\}} = f_{k+1}^{-1} \circ g_k|_{D_k^n - \{0\}}$, $h_{k+1}(\mathbb{R}^n - \{0\}) = \mathbb{R}^n - \{0\}$, and $h_{k+1}(D_{k+1}^n) \supset D_{k+1}^n$. Let $g_{k+1}(x) = g_k(x)$ if $x \in D_k^n$, and $g_{k+1}(x) = f_{k+1} \circ h_{k+1}(x)$ if $x \in \mathbb{R}^n - \{0\}$. Then $g = \lim_{i\to\infty} f_i$ is a $C^\infty$-diffeomorphism of $\mathbb{R}^n$ onto $M$. Q.E.D.

The following theorem may be proved in a similar manner:

Theorem 4.3. Let $M$ be a $C^\infty$-manifold with compact connected boundary $\partial M$. If there are $C^\infty$-collarings $f_i\colon \partial M \times [0,\infty) \to M$ such that $M = \bigcup_{i=1}^{\infty} f_i(\partial M \times [0,\infty))$, and

$f_i(\partial M \times [0,\infty)) \subset f_{i+1}(\partial M \times [0,\infty))$, $i > 1$, then $M$ is $C^\infty$-diffeomorphic to $\partial M \times [0,\infty)$.

# CHAPTER FIVE

## Coverings of Manifolds

The following lemma is a consequence of Corollary 2.1:

Lemma 5.1. Let $M$ be a $C^\infty$-manifold, and let $g: \mathbb{R}^n \to M$ be a $C^\infty$-diffeomorphism. Let $P$ be a $k$-dimensional sub-polyhedron of $\mathbb{R}^n$, not necessarily compact, such that $g(P)$ is closed, and let $U \subset M$ be an open set such that $g(P)-U$ is compact. Let $E \supset \partial M$ be a closed set such that $E \subset U$, and $(M-E, U-E)$ is $k$-connected. If $k \leq n-3$, there is a compact set $C \subset M-E$, and a $C^\infty$-diffeomorphism $h: M \to M$ such that $h(U) \supset g(P)$, and $h(x) = x$ if $x \notin C$.

Note that $h|_E = \mathrm{id}_E$, and, in particular, that $h$ is the identity on a neighborhood of $\partial M$.

Lemma 5.2. Let $M$ be a $C^\infty$-$n$-manifold, let $U_1, \ldots, U_m$, $V_1, \ldots, V_m$ be open subsets of $M$ such that $\mathrm{Cl}\, V_i \subset U_i$ and $(M-\mathrm{Cl}\, V_i, U_i-\mathrm{Cl}\, V_i)$ is $k_i$-connected, if $k_i > 0$, let $k_i \leq n-3$, $1 \leq i \leq m$. Let $E_1, \ldots, E_m$ be closed subsets of $M$ such that $E_i \subset V_i$, $1 \leq i \leq m$, and $\partial M \subset \bigcup_{i=1}^{m} E_i$. Let $g: C_1^n \to M$ be a $C^\infty$-diffeomorphism and let $0 < \alpha < 1$. If $k_1 + \ldots + k_m + m \geq n+1$, there are compact sets $C_1, \ldots, C_m$ in $M$ such that $C_i \cap (E_i \cup \partial M) = \emptyset$, $1 \leq i \leq m$, and $C^\infty$-diffeomorphisms $h_i$ of $M$ onto itself such that $h_i(x) = x$, if $x \notin C_i$, $1 \leq i \leq m$, and $g(C_\alpha^n) \subset \bigcup_{i=1}^{m} h_i(U_i)$.

Proof: Let $G$ be the simplicial complex determined by a simplicial subdivision of $C_1^n$ such that $C_\alpha^n$ is the set of points of a subcomplex $K$ of $G$, $|N(K,G)| \subset \operatorname{Int} C_1^n$, and for any simplex $\Delta \in G$ such that $g(\Delta) \cap E_i \neq \emptyset$, we have $g(\Delta) \subset V_i$, i.e.: $g(|N(g^{-1}(E_i),G)|) \subset V_i$, $1 \leq i \leq m$.

Let $L_o = K$. We construct inductively two sequences $L_o,\ldots,L_{m-1}$ and $K_1,\ldots,K_{m-1}$ of simplicial complexes as follows: suppose $L_{i-1}$ is defined. Let $K_i = \beta(L_{i-1}^{(k_i)})$, and let $L_i$ be the complementary complex of $K_i$ in $\beta(L_{i-1})$, $1 \leq i \leq m-1$. Then $\dim L_i = n-i-(k_1+\ldots+k_i)$. Thus $\dim L_{m-1} = n-m+1-(k_1+\ldots+k_{m-1}) \leq k_m$. Let $K_m = L_{m-1}$.

We now apply Lemma 5.1 with respect to each $K_i$. Let $P_i = g^{-1}(g(|K_i|) - \operatorname{Cl} V_i)$. Then $P_i$ is a $k_i$-dimensional polyhedron in $\operatorname{Int} C_1^n - g^{-1}(\operatorname{Cl} V_i)$, $g(P_i)$ is closed in $M - \operatorname{Cl} V_i$, and $g(P_i) - U_i$ is compact, so there are $C^\infty$-diffeomorphisms $h_i^!: M \to M$ and compact sets $C_i'$, $1 \leq i \leq m$, such that $h_i^!(x) = x$ if $x \notin C_i'$, and $g(P_i) \subset h_i^!(U_i)$.

Let $W_i = g^{-1}(h_i^!(U_i))$, $1 \leq i \leq m$. Then $|K_i| \subset W_i$. The barycentric subdivisions used in the definitions of $K_i$ and $L_i$ imply that $K_i$ and $L_i$ are full subcomplexes of

$\beta(L_{i-1})$, $1 \leq i \leq m - 1$. Applying Theorem 3.1, we construct inductively a sequence of $C^\infty$-diffeomorphisms $S_{m-i}: C_1^n \to C_1^n$ such that $S_{m-i}$ is the identity on

$$C_1^n - |N(K,G)| \cup |N(g^{-1}(E_i),G)|, \; 1 \leq i \leq m - 1, \quad \text{and}$$

$$|L_{m-2}| \subset S_{m-1}(W_{m-1}) \cup W_m ,$$

$$|L_{m-3}| \subset S_{m-2}(W_{m-2}) \cup S_{m-1}(W_{m-1}) \cup W_m,$$

$$\vdots$$

$$|K| = |L_o| \subset S_1(W_1) \cup \ldots \cup S_{m-1}(W_{m-1}) \cup W_m.$$

For example, we construct $S_{m-1}$. In the notation of Theorem 3.1, let $U = W_{m-1}$, $V = W_m$,

$$L = K_{m-1} \cup \{\beta(\Delta): \Delta \in L_{m-2} \text{ and } g(\Delta) \cap E_{m-1} \neq \emptyset\}$$

$$L^c = \{\Delta \in \beta(L_{m-2}): \Delta \cap L = \emptyset\},$$

$$F = (\mathbb{R}^n - |N(K,G)|) \cup g^{-1}(E_{m-1})$$

Note that $|L| \subset W_m$, $L$ is full in $\beta(L_{m-2})$, and $F \cap |\beta(L_{m-2})| \subset |L| \cup |L^c|$. Let $S_{m-1}$ be the $h_1$ obtained in Theorem 3.1.

We lift the $C^\infty$-diffeomorphisms $S_i$ onto M: let $\hat{S}_i: M \to M$ be defined by $\hat{S}_i(p) = g \circ S_i \circ g^{-1}(p)$, if $p \in g(C_1^n)$, and $\hat{S}_i(p) = p$ if $p \notin g(|N(K,G)|)$, $1 \leq i \leq m - 1$. Note that $\hat{S}_i|_{E_i} = \mathrm{id}_{E_i}$. It follows that $g(C^n_\alpha) = g(|K|) \subset$ $\hat{S}_1 \circ h_1'(U_1) \cup \ldots \cup \hat{S}_{m-1} \circ h_{m-1}'(U_{m-1}) \cup h_m'(U_m)$. Let $h_i = \hat{S}_i \circ h_i'$, $1 \leq i \leq m-1$, and let $h_m = h_m'$. Q.E.D.

Theorem 5.1. Let $M$ be a $C^\infty$-n-manifold, and let $U_1,\ldots,U_m$ be open subsets of $M$ such that $U_i = \bigcup_{j=1}^{\infty} V_{i,j}$, where $V_{i,j}$ is open, $\mathrm{Cl}\, V_{i,j} \subset V_{i,j+1}$, $(M - \mathrm{Cl}\, V_{i,j}, V_{i,j+1} - \mathrm{Cl}\, V_{i,j})$ is $k_i$-connected, $k_i \leq n-3$, if $k_i > 0$, $j \geq 1$, $1 \leq i \leq m$, and $\partial M \subset \bigcup_{i=1}^{m} V_{i,1}$. Then, if $k_1 + \ldots + k_m + m \geq n+1$, there are $C^\infty$-diffeomorphisms $h_i\colon M \to M$ such that

$$h_i|_{\mathrm{Cl}\,V_{i,1}} = \mathrm{id}_{\mathrm{Cl}\,V_{i,1}}, \quad 1 \leq i \leq m, \quad \text{and} \quad M = \bigcup_{i=1}^{m} h_i(U_i).$$

Proof: Let $g_j\colon C^n_1 \to M$, $j = 1,2,\ldots$ be a sequence of $C^\infty$-diffeomorphisms such that $\mathrm{Int}\, M = \bigcup_{j=1}^{\infty} g_j(C^n_{\frac{1}{2}})$. Suppose we have constructed $m$ sequences $\{f_{i,o},\ldots,f_{i,k}\}$, $i = 1,\ldots,m$, of $C^\infty$-diffeomorphisms of $M$ onto itself such that

$\bigcup_{j=1}^{k} g_j(C^n_{\frac{1}{2}}) \subset \bigcup_{i=1}^{m} f_{i,k}(V_{i,2k})$, and

$f_{i,j}|_{V_{i,2j-2}} = f_{i,j-1}|_{V_{i,2j-2}}$, $1 \leq j \leq k$, where $f_{i,o} = \mathrm{id}_M$.

We apply Lemma 5.2 with $E_i = \mathrm{Cl}\, V_{i,2k}$, $V_i = V_{i,2k+1}$, $U_i = V_{i,2k+2}$, and $g = g_{k+1}$ to get $C^\infty$-diffeomorphisms $f_{i,k+1}$, $1 \leq i \leq m$, of $M$ onto itself such that

$$\bigcup_{j=1}^{k+1} g_j(C^n_{\frac{1}{2}}) \subset \bigcup_{i=1}^{m} f_{i,k+1}(V_{i,2k+2}), \quad \text{and} \quad f_{i,k+1}|V_{i,2k} = f_{i,k}|V_{i,2k}.$$

Let $h_i(x) = \lim_{k\to\infty} f_{i,k}(x)$ for all $x \in M$. Q.E.D.

<u>Corollary 5.1.</u> Let $M$ be a $k$-connected $C^\infty$-$n$-manifold without boundary, with $k \leq n - 3$ if $k > 0$. Then, if $m \geq \frac{n+1}{k+1}$, $M$ may be covered with $m$ open $C^\infty$-$n$-cells.

<u>Proof:</u> Let $U_1, \ldots, U_n$ be open $C^\infty$-$n$-cells in $M$. Then $(M, U_i)$ is $k$-connected, so if we let $k_i = k$, $1 \leq i \leq m$, we have $k_1 + \ldots + k_m + m = mk + m \geq n + 1$.

<u>Corollary 5.2.</u> Let $M$ be a $k$-connected $C^\infty$-$n$-manifold (with $k \leq n-3$ if $k > 0$) with $\ell$ boundary components $N_1, \ldots, N_\ell$, and let $f_i \colon N_i \times [0,\infty) \to M$ be $C^\infty$-collarings, $1 \leq i \leq \ell$. If $m \geq \frac{n-\ell+1}{k+1}$, there are $\ell$ $C^\infty$-diffeomorphisms $h_i$ of $M$ onto itself such that each $h_i$ is the identity on a neighborhood of $N_i$ and $m$ $C^\infty$-diffeomorphisms $g_i \colon \mathbb{R}^n \to M$ such that

$$M = \bigcup_{i=1}^{\ell} h_i \circ f_i(N_i \times [0,\infty)) \cup \bigcup_{i=1}^{m} g_i(\mathbb{R}^n).$$

Proof: $(M,N_i)$ is at least 0-connected, and $\ell + mk + m \geq n+1$.

Corollary 5.3. Let $M$ be a connected $C^\infty$-n-manifold, $n \geq 5$, with two connected boundary components $N_1$ and $N_2$ such that the inclusion of $N_i$ into $M$ is a homotopy equivalence, $i = 1,2$. Then there are $C^\infty$-diffeomorphisms $h_i\colon N_i \times [0,\infty) \to M$ such that $h_i(x,0) = x$ for all $x \in N_i$, $i = 1,2$, and $M = h_1(N_1 \times [0,\infty)) \cup h_2(N_2 \times [0,\infty))$.

Corollary 5.4. Let $M$ be a contractible $C^\infty$-n-manifold without boundary, $n \geq 5$. Then $M$ can be covered with two open $C^\infty$-n-cells.

Theorem 5.2. Let $M$ be a contractible $C^\infty$-n-manifold without boundary, $n \geq 5$, which is 1-connected at $\infty$. Then $M$ is $C^\infty$-diffeomorphic to $\mathbb{R}^n$.

Proof: By Theorem 4.2, we need only show that if $C \subset M$ is compact, there is a $C^\infty$-diffeomorphism $f\colon \mathbb{R}^n \to M$ such that $C \subset f(\mathbb{R}^n)$. Let $f_1, f_2\colon \mathbb{R}^n \to M$ be $C^\infty$-diffeomorphisms such that $M = f_1(\mathbb{R}^n) \cup f_2(\mathbb{R}^n)$. Since $M$ is a normal space, there are closed sets $A_1, A_2 \subset M$ with $A_1 \subset f_1(\mathbb{R}^n)$, $A_2 \subset f_2(\mathbb{R}^n)$ and $M = A_1 \cup A_2$. We consider a fixed simplicial subdivision of $\mathbb{R}^n$ into a simplicial complex $K$ such that

(a) $C_i^n$ is the set of points of a subcomplex of $K$, $i \geq 1$.

(b) If $\Delta \in K$ and $f_1(\Delta) \cap A_2 \neq \emptyset$, then $f_1(\Delta) \subset f_2(\mathbb{R}^n)$.

(c) If $\Delta \subset C^n_{i+1} - \operatorname{Int} C^n_i$, then $\operatorname{diam} f_1(\Delta) < \frac{1}{i}$, $i \geq 1$.

Let $L = N(f_1^{-1}(A_1), K)$, and, for all $i \geq 1$, let $K_i = N(f_1^{-1}(A_1) \cap C^n_i, K)$. Each $K_i$ is a subcomplex of $L$, and $L = \bigcup_{i=1}^{\infty} K_i$. Let $D \supset C$ be a compact set such that $M - D$ is simply connected. Then $(M, M-D)$ is 2-connected, $f_1(|L^{(2)}|)$ is closed (because of condition (c)) and 2-dominated, and $f_1(|L^{(2)}|) \cap D$ is compact, so by Lemma 5.1, there is a compact set $C_1 \subset M$ and a $C^\infty$-diffeomorphism $h_1 \colon M \to M$ with $f_1(|L^{(2)}|) \subset h_1(M-D)$, and $h_1(x) = x$ if $x \notin C_1$. Since $C_1 \cup D$ is compact, there is an integer $i > 1$ with $C_1 \cup D \subset f_1(|K_i|) \cup f_2(C^n_i) \cap A_2$. Let $H^{n-3}$ be the subcomplex of $\beta(K_i)$ complementary to $\beta(K_i{}^{(2)})$. We have $f_1(|K_i{}^{(2)}|) \subset h_1(M-D)$.

Let $E = f_1(|N(f_1^{-1}(f_2(C^n_i) \cap A_2), K)|)$. By condition (b), $E \subset f_2(\mathbb{R}^n)$. By condition (c), if $\Delta \subset C^n_{\ell+1} - \operatorname{Int} C^n_\ell$, for $\ell \geq 1$, then $\operatorname{diam}(f_1(\Delta) \cup f_2(C^n_i) \cap A_2) \leq \operatorname{diam} f_1(\Delta) + \operatorname{diam}(f_2(C^n_i) \cap A_2) \leq \frac{1}{\ell} + \operatorname{diam}(f_2(C^n_i) \cap A_2)$. Therefore $\operatorname{diam} E \leq 2 + \operatorname{diam}(f_2(C^n_i) \cap A_2)$. Since $E$ is bounded, there is an integer $j > i$ such that $f_2(\text{Int } C_j^n) \supset E$, that is, if $f_1(\Delta) \cap f_2(C_i^n) \cap A_2 \neq \emptyset$, then $f_1(\Delta) \subset f_2(\text{Int } C_j^n)$.

Let $M_2 = M - f_2(C_j^n)$, and let $V_2 = f_2(\text{Int } C_{j+1}^n - C_j^n)$. Then $(M_2, V_2)$ is $(n-2)$-connected, $f_1(|H^{n-3}|) \cap M_2$ is closed and $(n-3)$-dominated in $M_2$, and $f_1(|H^{n-3}|) \cap (M_2 - V_2)$ is compact. By Lemma 5.1, there is a compact set $C_2 \subset M_2$ and a $C^\infty$-diffeomorphism $h_2\colon M_2 \to M_2$ with $h_2(V_2) \supset f_1(|H^{n-3}|) \cap M_2$ and $h_2(x) = x$ if $x \notin C_2$. We may extend $h_2$ to all of $M$ by letting $h_2(x) = x$ if $x \in f_2(C_j^n)$. Then $f_1(|H^{n-3}|) \subset h_2 \circ f_2(\text{Int } C_{j+1}^n)$.

Next we consider two open subsets of $\mathbb{R}^n$, $U = f_1^{-1} \circ h_2 \circ f_2(\mathbb{R}^n)$, and $V = f_1^{-1} \circ h_1(M-D)$. We apply Theorem 3.1 with $L = H^{n-3} \cup \{\beta(\Delta)\colon \Delta \in K_i \text{ and } \Delta \subset U\}$ and $L^c = \{\Delta \in \beta(K_i)\colon \Delta \cap L = \emptyset\} \subset \beta(K_i^{(2)})$. We let $F = |L \cup L^c \cup N(f_1^{-1}(f_2(C_i^n) \cap A_2), K)| \cup (\mathbb{R}^n - \text{Int } C_{i+1}^n)$ and obtain a compact set $\tilde{C} \subset \mathbb{R}^n - F$ and a $C^\infty$-diffeomorphism $s\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $s(x) = x$ if $x \notin \tilde{C}$, $s(\Delta) = \Delta$ for all $\Delta \in \beta(K_i)$, and $s(U) \cup V \supset |K_i|$. Let $\tilde{s}\colon M \to M$ be defined $\tilde{s}(p) = f_1 \circ s \circ f_1^{-1}(p)$ if $p \in f_1(\mathbb{R}^n)$, and $\tilde{s}(p) = p$ otherwise.

Then $f_1(|K_i|) \cup f_2(C_1^n) \cap A_2 \subset h_1(M-D) \cup \tilde{s}\circ h_2\circ f_2(R^n)$.
(since $\tilde{s}\circ h_2|_E = id_E$). Consequently

$$C_1 \cup D \subset h_1(M-D) \cup \tilde{s}\circ h_2\circ f_2(\mathbb{R}^n).$$

Since $M - (C_1 \cup D) \subset M - C_1 \subset h_1(M-D)$, we have

$$M = h_1(M-D) \cup \tilde{s}\circ h_2\circ f_2(\mathbb{R}^n), \quad \text{or}$$

$$M = (M-D) \cup h_1^{-1}\circ\tilde{s}\circ h_2\circ f_2(\mathbb{R}^n).$$

Let $f = h_1^{-1}\circ\tilde{s}\circ h_2\circ f_2$. Then $f(\mathbb{R}^n) \supset D \supset C$. Q.E.D.

We can strengthen Corollary 5.3 as follows:

Theorem 5.3. Let $M$ be a connected $C^\infty$-n-manifold, $n \geq 5$, with two boundary components $N_1$ and $N_2$ such that the inclusion of $N_i$ into $M$ is a homotopy equivalence, $i = 1,2$. Then there is a $C^\infty$-diffeomorphism of $N_1 \times [0,\infty)$ onto $M - N_2$.

Proof: Let $g_j\colon C_1^n \to M$, $j = 1,2,\ldots$ be a sequence of $C^\infty$-diffeomorphisms such that $\operatorname{Int} M = \bigcup_{j=1}^{\infty} g_j(C_{\frac{1}{2}}^n)$. Let $f_o$ be the $C^\infty$-diffeomorphism $h_1$ of Corollary 5.3. We construct inductively a sequence $f_o, f_1, f_2, \ldots$ of $C^\infty$-diffeomorphisms of $N_1 \times [0,\infty)$ into $M$ such that for each $j \geq 1$,
$\bigcup_{i=1}^{j} g_i(C_{\frac{1}{2}}^n) \subset f_j(N_1 \times [0,j+1))$, and $f_j|_{N_1\times[0,j]} = f_{j-1}|_{N_1\times[0,j]}$.

Let $h: N_2 \times [0,\infty) \to M$ be a $C^\infty$-collaring such that $h(N_2 \times [0,\infty)) \cap (g_{j+1}(C^n_{\frac{1}{2}}) \cup f_j(N_1 \times [0,j+2])) = \emptyset$. Let $M_j = M - f_j(N_1 \times [0,j+1))$. By Theorem 5.1, there are $C^\infty$-diffeomorphisms $r_1$ and $r_2$ of $M_j$ onto itself which are the identity on a neighborhood of the boundary of $M_j$ such that

$$M_j \subset r_1(f_j(N_1 \times [j+1,j+2))) \cup r_2(h(N_2 \times [0,\infty)).$$

Let $f_{j+1}|_{N_1 \times [0,j+1]} = f_j|_{N_1 \times [0,j+1]}, f_{j+1}|_{N_1 \times [j+1,\infty)} =$

$$= r_2^{-1} \circ r_1 \circ f_j|_{N_1 \times [j+1,\infty)}.$$

Then $M = f_{j+1}(N_1 \times [0,j+2)) \cup h(N_2 \times [0,\infty))$. Since

$g_{j+1}(C^n_{\frac{1}{2}}) \cap h(N_2 \times [0,\infty)) = \emptyset$, we have

$\bigcup_{i=1}^{j+1} g_i(C^n_{\frac{1}{2}}) \subset f_{j+1}(N_1 \times [0,j+2))$. Let $f = \lim_{j\to\infty} f_j$. Then $M - N_2 = f(N_1 \times [0,\infty))$. Q.E.D.

<u>Corollary 5.5.</u> If $M$ is a $C^\infty$-n-manifold, $n \geq 5$, with two boundary components $N_1$ and $N_2$ whose inclusions into $M$ are homotopy equivalences, then $N_1 \times \mathbb{R}$, $N_2 \times \mathbb{R}$, and $\operatorname{Int} M$ are $C^\infty$-diffeomorphic. If $M$ is compact, then $M \times \mathbb{R}$ is $C^\infty$-diffeomorphic to $N_i \times [0,1] \times \mathbb{R}$, $i = 1,2$.

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