SHEAF METHODS APPLIED TO COHERENT RINGS

by

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ABSTRACT

A commutative ring is called coherent if the intersection of any two finitely generated ideals is finitely generated and the annihilator ideal of an arbitrary element of the ring is finitely generated.

Pierce's representation of a ring R as the ring of all global sections of an appropriate sheaf of rings, k , is described. Some theorems are deduced relating the coherence of the ring R to certain properties of the sheaf k . The sheaves from the above representation for $R[X]$ and $R[[G^+]]$, where R is a commutative von Neumann regular ring and G is a linearly ordered abelian group, are calculated. Applications of the above theorems now show that $R[X]$ is coherent and yield necessary and sufficient conditions for $R[[G^+]]$ to be coherent.

TABLE OF CONTENTS

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Introduction

In this thesis all ring have unity, all modules are unital, and all non-zero ring homomorphisms preserve the identity. Except in the introduction all rings are commutative. Some terminology is now introduced.

(Definition) Let R be a ring. A left R-module M is finitely presented iff there exists an exact sequence of left R-modules $0 \to K \to F \to M \to 0$ where F and K are finitely generated and F is free.

(Definition) A ring R is left coherent iff every finitely generated left ideal is finitely presented.

Corresponding definitions may be made with respect to right modules and right ideals. A left coherent commutative ring is said to be coherent. The following definition allows an internal description of coherent rings to be given.

(Definition) Let R be a ring.

a) R has property a) iff the intersection of any two finitely generated left ideals in R is finitely generated.

b) R has property b) iff the intersection of any two principal ideals is principal.

c) R has property c) iff for any $r \in R$ the left-annihilator ideal of r ($\text{l.ann}(r) = \{s \in R : sr = 0\}$) is finitely generated.

d) R has property d) iff for any $r \in R$ $\text{l.ann}(r)$ is generated by an idempotent.

e) R has property e) iff any finitely generated left ideal in R is principal.

The concept of a coherent ring was introduced by Chase in [4]. He showed that for any ring R direct products of families of flat <u>right</u> R-modules are flat iff R is <u>left</u> coherent. The following is part of theorem 2.1 from that paper:

<u>Theorem</u>: A ring R is left coherent iff it has properties a) and c).

In view of this various combinations of properties a) - e) can be used either to generalize, define, or specialize the concept of coherence. Some elementary properties of finitely presented modules and (left) coherent rings appear as exercises in Bourbaki [1].

The following are examples of left coherent rings:

i) Any left Noetherian ring.

ii) Any left semi-hereditary ring.

iii) As a particular case of ii), any von Neumann-regular ring. (A ring R is von Neumann regular iff for each $r \in R$ there exists $r' \in R$ such that $rr'r = r$.)

iv) Let $\{R_\alpha\}$ be a directed system of <u>left</u> coherent rings such that if $\alpha \leq \alpha'$ then $R_{\alpha'}$ is a <u>right</u> flat R_α-module. Then $\varinjlim (R_\alpha) = R$ is a <u>left</u> coherent ring.

v) Let R be a left Noetherian ring and $\{X_\alpha\}$ be a finite or infinite set of indeterminates commuting with themselves and elements of R . Then $R[\{X_\alpha\}]$, the ring of polynomials in the indeterminates X_α with coefficients from the ring R , is left coherent.

vi) Let the notation be as in v). Then $R[[\{X_\alpha\}]]$, the ring of formal power series in the indeterminates X with coefficients from the ring R , is left coherent.

Examples i), ii), and iii) are easily verified. In addition, any semi-hereditary ring has property d). Example iv) is from Bourbaki (ex 11, p. 63 of [1]). Example v) follows from iv) and the Hilbert basis theorem since

$R[\{X_\alpha\}] = \varinjlim\, (\{R[X_1,\ldots X_n] : n$ is a natural number and $\{X_1,\ldots X_n\} \subseteq \{X_\alpha\}\})$.

Examples v) and vi) are similar.

There is some evidence to indicate that, from a homological point of view, left coherent rings are a reasonable generalization of left Noetherian rings in the sense that they, rather than left Noetherian rings, are the appropriate concept. For example Chase's characterization of left coherent rings in terms of right flat modules may be viewed as a generalization of exercise 4 p. 122 of [3] which states that the direct product of a family of right flat modules over a left Noetherian ring is right flat. The fact that l.gl.dim (R) = r.gl.dim (R) where R is left and right Noetherian is a special case of the following result about coherent rings: if R is a left (right) coherent ring and M is a finitely presented left (right) R-module then Pd(M) = w.dim (M) . (l.gl.dim (R) denotes the left global dimension of R , r.gl.dim (R) denotes the right global dimension of R , Pd(M) denotes the projective dimension of M and w.dim (M) denotes the weak dimension of M .) In particular cylic left (right) modules of a left (right) Noetherian ring are finitely presented. To specialize this to the Noetherian case the following result is used: for any ring R $l.gl.dim\ (R) = \sup\{Pd(M) : M$ is a cyclic left R-module$\}$. A similar result holds on the right. Thus if R is left and right Noetherian we have

$$l.gl.dim\ (R) = \sup\{Pd(M) : M \text{ is a cyclic left } R\text{-module}\}$$
$$= \sup\{w.dim\ (M) : M \text{ is a cyclic left } R\text{-module}\} \leq w.dim\ (R) \leq r.gl.dim\ (R)\ .$$

By symmetry we also have $r.gl.dim\ (R) \leq l.gl.dim\ (R)$.

In view of the proceeding it is reasonable to enquire as to whether or not coherent rings are a generalization of Noetherian rings with respect to non-homological properties. It is known that if R is left Noetherian so are $R[X]$ and $R[[X]]$. This suggests the following two questions. First, for what coherent rings R is $R[X]$ coherent? Second, for what coherent rings R is $R[[G^+]]$ coherent where $R[[G^+]]$ denotes the ring of formal power series with coefficients from R and positive indices from the linearly ordered abelian group G ? These questions are answered in this thesis for R in the category of commutative von Neumann regular rings.

For the rest of the introduction R denotes a commutative von Neumann regular ring. Let S be a commutative ring. The following fact is crucial in proving all theorems in this thesis. There exists a Boolean space X and a sheaf K over X of indecomposable rings such that S is isomorphic to the ring of all global sections of K over X . This representation of S is described in more detail in §0. In §1 (1.7 and 1.13 in particular) properties a), b), c), d), and e) for S and the weak global dimension are related to certain algebraic properties of the stalks of the sheaf K and to certain properties of the global sections of K over X . In §2 the sheaf associated with

$R\lceil X\rceil$ is calculated. It then follows from §1 that $R\lceil X\rceil$ has properties a), b), c), d), and e) (and thus is coherent) and that w.gl.dim $(R\lceil X\rceil) = 1$. Now let $S = R\lceil\lceil G^{+}\rceil\rceil$. In §3 a preliminary study of the structure of K is made and used to show that S has property c) (or d)) iff the Boolean ring of idempotents of R is λ - complete (see 3.3 of this thesis) where λ - is an ordinal depending on G.

In sections 4,5, and 6 the stalks of K are calculated for various R and G in sufficient detail to establish (via 1.7 and 1.13) that the following conditions are equivalent:

i) $R\lceil\lceil G^{+}\rceil\rceil$ has property a).

ii) $R\lceil\lceil G^{+}\rceil\rceil$ has property b).

iii) $R\lceil\lceil G^{+}\rceil\rceil$ has property e).

iv) w.gl.dim $(R\lceil\lceil G^{+}\rceil\rceil) \leq 1$.

If G is not isomorphic to the integers the above conditions are satisfied iff R is a finite direct sum of fields, (See §4). If G is isomorphic to the integers then the above conditions are satisfied iff R has a partial form of self-injectivity called $\aleph_1$ self-injectivity. (See sections 5 and 6.) In sections 6 and 7 it is shown by examples that both of the following implications are false: first, $R\lceil\lceil X\rceil\rceil$ has property a) (or an equivalent) $\rightarrow$ $R\lceil\lceil X\rceil\rceil$ has property c). Second, $R\lceil\lceil X\rceil\rceil$ has property c) $\rightarrow$ $R\lceil\lceil X\rceil\rceil$ has property a). If R is a Boolean ring the second implication does hold while the first does not.

It should be noted that these results are not in total agreement with the following theorem of Jensen (pp. 238 and 239 of [6]):

For a Boolean ring R the following are equivalent:

1) R is self-injective.

2) w.gl.dim $(R[[G^+]]) = 1$ and $R[[G^+]]$ is coherent, for any linearly ordered group G.

3) $R[[G^+]]$ is coherent for any linearly ordered group G.

If R is atomic (as a Boolean algebra) then 4) is equivalent to the preceeding conditions.

4) w.gl.dim $(R[[G^+]]) = 1$ for any linearly ordered group G.

Soublin (in [10]) gives an example of a commutative coherent ring T such $T[X]$ fails to have property a) and thus fails to be coherent. Soublin's example and the results of this thesis on $R[[X]]$ suggest that in non-homological settings it is misleading to think of coherent rings as being a generalization of Noetherian rings.

§0

Representation of rings by sections of sheaves

All results in this thesis (except for 0.11, 0.17, and 0.18) are paraphrased from part one of [8].

In this section R denotes a commutative ring with unity. It should be noted that in this thesis all rings are commutative with unity. Let X be an index set and $\{M_x : x \in X\}$ be a family of ideals in R such that $\bigcap_{x\in X} (M_x) = 0$. Then R may be represented as a subdirect product of the k_x where $k_x = R/M_x$. However such a representation gives little information about R unless there is a reasonable way of determining which subring of $(\bigoplus_{x\in X}\pi)(k_x)$ is isomorphic to R. Pierce in [8] shows that if X and $\{M_x : x \in X\}$ are appropriately chosen this can be accomplished by topologizing X and $k = \bigcup_{x\in X}(k_x)$ in such a way that $\{\sigma \in (\bigoplus_{x\in X}\pi)(k_x) : X \overset{\sigma}{\to} k$ is continuous$\}$ is a subring of $(\bigoplus_{x\in X}\pi)(k_x)$ isomorphic to R. In this situation the k_x are all indecomposable rings. This can also be expressed by saying that there exists a sheaf $k(R)$ of indecomposable rings over a topological space $X(R)$ such that R is isomorphic to the ring of all global sections of $k(R)$ over $X(R)$. The construction of the topological space $X(R)$ and the sheaf $k(R)$, along with some basic definitions and results, is outlined below.

0.1) (Definition) Sheaves of Rings: Let X be a topological space. Suppose that for each $x \varepsilon X$ a ring k_x with zero 0_x and identity 1_x is given. Assume $k_x \cap k_y = \phi$ for $x \neq y$. Let $k = \bigcup_{x \varepsilon X} (k_x)$. Let $\pi: k \to X$ denote the map such that if $r \varepsilon k_x$ then $\pi(r) = x$. Assume that k is topologized in such a way that the following three axioms are satisfied.

i) If $r \varepsilon k$ there exist open sets U in k and N in X such that $r \varepsilon U$ and π maps U homeomorphically onto N.

ii) Let $k + k$, denoting $\{(r,s) \varepsilon k \times k : \pi(r) = \pi(s)\}$, have the topology induced by the product topology in $k \times k$. Then the mapping $r \to -r$ on k to k and the mappings $(r,s) \to r \cdot s$ and $(r,s) \to r + s$ on $k + k$ to k are continuous.

iii) The mapping $x \to 1_x$ on X to k is continuous.

When these axioms are satisfied k is called a sheaf of rings over X. The rings k_x are called the stalks of the sheaf k. The pair (X,k) is called a ringed space.

0.2) (Definition) Isomorphisms of Ringed Spaces: Let (X,k) and (Y,S) be ringed spaces. An isomorphism of (X,k) onto (Y,S) is a pair (λ,μ) where λ is a homeomorphism of X onto Y and μ is a homeomorphism of S onto k such that μ maps $S_{\lambda(x)}$ isomorphically onto k_x for each $x \varepsilon X$. (X,k) and (Y,S) are isomorphic iff there exists an isomorphism of (X,k) onto (Y,S).

0.3) (Definition) Subsheaves: Let k be a sheaf of rings over the topological space X. A subset k' of k is called a subsheaf of k if k' is open in X and for each $x \varepsilon X$

$k' \cap k_x$ is a subring of k_x. It is easily verified that k' is also a sheaf of rings over X when given the topology induced by k.

The concepts required to determine a subring of $(\bigoplus_{x \varepsilon X} \pi)(k_x)$ using the topologies on X and k are now introduced. It should be noted that $(\bigoplus_{x \varepsilon X} \pi)(k_x) = \{\sigma : X \overset{\sigma}{\to} k$ and for each $x \varepsilon X$ $\pi(\sigma(x)) = x\}$.

0.4) (Definition) Sections: Let (X,k) be a ringed space and let Y be a subspace of X.

i) A section of k over Y is a continuous map $\sigma : Y \to k$ such that $\pi(\sigma(x)) = x$ for all $x \varepsilon Y$. The set of all sections of k over Y is denoted $\Gamma(Y,k)$. The elements of $\Gamma(X,k)$ are called the global sections of k over X. To say that an element $\sigma \varepsilon \Gamma(Y,k)$ can be extended to a global section means that there exists $\sigma' \varepsilon \Gamma(X,k)$ such that $\sigma'|_Y = \sigma$.

ii) Define pointwise addition and multiplication on $\Gamma(Y,k)$ using the addition and multiplication in the stalks. Then $\Gamma(Y,k)$ is a ring.

iii) For any $\sigma \varepsilon \Gamma(X,k)$ let $S(\sigma) = \{x \varepsilon X : \sigma(x) \neq 0_x\}$ and let $Z(\sigma) = \{x \varepsilon X : \sigma(x) = 0_x\}$.

iv) Let U be a subset of X that is both open and closed in X. Define $\psi_U : X \to k$ by $\psi_U(x) = 1_x$ when $x \varepsilon U$ and $\psi_U(x) = 0_x$ when $x \varepsilon X - U$. Then $\psi_U \varepsilon \Gamma(X,k)$. This notation will be used frequently in this thesis.

The topologies on X and k allow the following relationship to be established amongst elements of $\Gamma(X,k)$.

0.5) (lemma) Let (X,k) be a ringed space. Suppose that $x \varepsilon X$ and $\sigma, \tau \varepsilon \Gamma(X,k)$ are such that $\sigma(x) = \tau(x)$. Then there exists N_x, a neighborhood of x, such that

$$\sigma\big|_{N_x} = \tau\big|_{N_x} .$$

This result is used to show that certain properties of a stalk k_x hold "locally" as well.

0.6) (Corollary): Let (X,k) be a ringed space and let $\sigma \varepsilon \Gamma(X,k)$. Then $Z(\sigma)$ is open and $S(\sigma)$ is closed in X.

0.7) (Definition) Boolean Spaces: Let X be a topological space.

i) A subset U of X is clopen iff it is both open and closed in X.

ii) X is totally disconnected iff it has a basis consisting of clopen sets.

iii) X is a Boolean space iff it is compact, Hausdorff, and totally disconnected.

Sheaves used in this thesis will be over Boolean spaces. The next proposition asserts that Boolean spaces have a very special form of compactness.

0.8) (Proposition) Let X be a Boolean space. Then X has the partition property. That is to say if $\{N_\alpha\}$ is a covering of X by open sets there exists $\{P_1,\ldots P_n\}$, a finite collection of clopen subsets of X, such that:

i) For $1 \leq i \leq n$ there exists an α_i such that $P_i \subseteq N_{\alpha_i}$.

ii) $P_i \cap P_j = \phi$ for $i \neq j$.

iii) $\bigcup_{i=1}^{n} (P_i) = X$.

The collection $\{P_1, \ldots P_n\}$ is called a partition of X refining the cover $\{N_\alpha\}$.

0.9) (Definition) Reduced Ringed Spaces: A ringed space (X,k) is a reduced ringed space iff

i) X is a Boolean space

ii) For each $x \in X$ k_x is an indecomposable ring.

It should be noted that if (X,k) is a reduced ringed space and $\{P_1, \ldots, P_n\}$ is a partition of X then the map

$$\Gamma(X,k) \rightarrow \oplus \sum_{i=1}^{n} \Gamma(P_i,k) \qquad \text{such that}$$

$$\sigma \rightarrow \sum_{i=1}^{n} \sigma \Big|_{P_i} \qquad \text{is an isomorphism.}$$

This, along with 0.8, allows one to show that certain "local" properties are actually "global" in the sense that they are also properties of $\Gamma(X,k)$. We will frequently derive properties of $\Gamma(X,k)$ from those of the k_x using 0.5 and 0.8.

The following two results will also be useful.

0.10) (lemma) Let (X,k) be a reduced ringed space and let Y be a closed subset of X . Then each element of $\Gamma(Y,k)$ can be extended to a global section.

In §3 we shall see that certain partial forms of self-injectivity for the ring $\Gamma(X,k)$ may be defined by asserting that an analogue of 0.10 holds for certain open subsets Y of X.

0.11) (lemma) Let (X,k) be a reduced ringed space and $x \in X$. Then $k_x \cong \left(\varinjlim_{x \in U \text{ and } U \text{ is clopen in } X} (\Gamma(U,k))\right)$.

Proof: It is a standard fact about sheaves that $k_x \cong \left(\varinjlim_{x \in U \text{ and } U \text{ is open in } X} (\Gamma(U,k))\right)$. The lemma follows from this since X is a Boolean space and thus the set of clopen neighborhoods of x is cofinal in the set of all neighborhoods of x.

0.12) (Definition) Regular Ringed Spaces: A reduced ringed space (X,k) is regular iff for each $x \in X$ k_x is a field.

0.13) (lemma) Let (X,k) be a regular ringed space and let $\sigma \in \Gamma(X,k)$. Then $S(\sigma)$ and $Z(\sigma)$ are both clopen in X.

Pierce's construction of the reduced ringed space $(X(R),k(R))$ such that R is isomorphic to $\Gamma(X(R),k(R))$ is now outlined.

0.14) (Definition) i) Define $B(R) = \{e \in R : e^2 = e\}$. Note that $\langle B(R), +', \cdot \rangle$ is a Boolean ring where $e +' f = e + f - 2ef$.
ii) Let $X(R)$ be the set of maximal ideals in the Boolean ring $B(R)$. For any $e \in B(R)$ let $X(e) = \{M \in B(R) : e \notin M\}$.

Let $X(R)$ have the topology induced by $\{X(e) : e \in B(R)\}$. It is well known that $X(R)$ is a Boolean space. In fact $\{X(e) : e \in B(R)\}$ is a clopen basis for the open sets in $X(R)$.

iii) If $M \in X(R)$ let $\overline{M} = \{re : r \in R \text{ and } e \in M\}$. It may be verified that $\overline{M}$ is an ideal in R . Thus $R/\overline{M}$ is a ring. Further, $(R/\overline{M}) \cap (R/\overline{N}) = \phi$ if $M \neq N$.

iv) For any $M \in X(R)$ let $k_M(R) = (R/\overline{M})$. Let $k(R) = \bigcup_{M \in X(R)} (k_M(R))$ and let $\pi : k(R) \to X(R)$ be defined such that $\pi^{-1}(M) = k_M(R)$.

v) For any $r \in R$ let $\sigma_r : X(R) \to k(R)$ be defined by $\sigma_r(M) = r + \overline{M} \in k_M(R)$ for any $M \in X(R)$. Topologize $k(R)$ by letting $\{\sigma_r(U) : r \in R \text{ and } U \text{ is open in } X\}$ be a basis for the open sets. It is readily seen that for each $r \in R$ $\sigma_r \in \Gamma(X(R),k(R))$.

vi) Let $R°$ denote $(X(R),k(R))$.

vii) For any reduced ringed space, (X,k) , let $(X,k)^*$ denote $\Gamma(X,k)$.

viii) Define $\lambda_R : R \to \Gamma(X(R),k(R))$ by $\lambda_R(r) = \sigma_r$.

<u>0.15)</u> (Proposition) i) $\lambda_R : R \to \Gamma(X(R),k(R)) = (R°)^*$ is a ring isomorphism.

ii) The correspondances ° and * are inverse, one-one, and onto between the family of isomorphism classes of rings and the family of isomorphism classes of reduced ringed spaces.

iii) The ring R is von Neumann regular iff $(X(R),k(R))$ is a regular ringed space. Similarly a reduced ringed space (X,k)

is regular iff $(X,k)^*$ is a von Neumann regular ring.

iv) Let (X,k) be a reduced ringed space. For each $x \in X$ let $M_x = \{\sigma \in B((X,k)^*) : \sigma(x) = 0\}$. The map $X \to X((X,k)^*)$ given by $x \to M_x$ is a homeomorphism.

It is possible to define morphisms of ringed spaces. The correspondances $\circ$ and $*$ may then be extended to categorical equivalences. However these morphisms are in general somewhat complicated. They may be avoided in this thesis through use of the following proposition.

<u>0.16</u>) (Proposition) Let R be a subring of the ring S such that $B(R) = B(S)$. Then clearly $X(R) = X(S)$. Let $k = \{\eta_S(r) : r \in R\}$. Then k is a subsheaf of $k(S)$ such that $\eta_S|_R : R \to (X,k)^*$ is an isomorphism. Thus R° and (X,k) may be identified.

<u>0.17</u>) Let $(X,k) = R^\circ$. Let $x \in X$.

i) The map $R \to k_x$ given by $r \to \sigma_r(x)$ is an epimorphism.

ii) As an R-module under this epimorphism k_x is flat.

Proof: i) This is easily checked.

ii) It follows from the remark immediately after 0.9 that $\Gamma(U,k)$ is projective if U is clopen. The result now follows from 0.11.

Q.E.D.

<u>0.18</u>) (Theorem) Let $(X,k) = R^\circ$. Then

$$\text{w.gl.dim}\,(R) = \sup\,\{\text{w.gl.dim}\,(k_x) : x \in X\}\ .$$

Proof: To show that $\sup\{w.gl.dim\,(k_x) : x \in X\} \leq w.gl.dim\,(R)$ the following fact (paraphrased from ex. 10 p. 123 of [3]) will be used: Let $S \to T$ be a ring homomorphism such that T is flat as an S-module and let A be an S-module. Then $w.dim_T(A \otimes_S T) \leq w.dim_S(A)$ where any T-module is given the S-module structure induced by the homomorphism $S \to T$. When this is applied in this proof A will be a T-module and $S \to T$ will be onto so that we will have $A \otimes_S T \cong A$ as T-modules. Thus $w.dim_T(A) = w.dim_T(A \otimes_S T) \leq w.dim_S(A)$. It follows from this that $w.gl.dim(T) \leq w.gl.dim(S)$. By 0.17 and the preceeding comment it thus follows that $w.gl.dim(k_x) \leq w.gl.dim(R)$ for each $x \in X$.

To complete the proof suppose that n is an integer such that $w.gl.dim(k_x) \leq n$ for each $x \in X$. Let A and B be arbitrary R-modules and $m > n$ an integer.

Two facts will be used. First, for any $x \in X$ $(Tor_M^R(A,B)) \otimes k_x \cong Tor_m^{k_x}(A \otimes k_x, B \otimes k_x) = 0$. Second if C is an R-module such that $C \otimes k_x = 0$ for all $x \in X$ then $C = 0$. The lemma is completed by using these two facts with $C = Tor_M^R(A,B)$. The first fact follows from the hypothesis on m and a paraphrase of exercise 11 p. 123 of [3]. The paraphrase is true essentially because each k_x is flat as an R-module. The second fact is actually just a restatement of the proposition (1.7 in [8]) that $\bigcap_{M \in X(R)} (\overline{M}C) = 0$. This implies that the natural map

$C \to (\bigoplus_{M \in X(R)} \pi)(C/\overline{M}C)$ is a monomorphism.

Let $M \varepsilon X(R)$. Then

$$C \otimes k_M = C \otimes (\underset{\substack{\rightarrow \\ (x\varepsilon U \text{ and } U \text{ is clopen})}}{\lim} (\Gamma(U,k))) = \underset{\substack{\rightarrow \\ (x\varepsilon U \text{ and } U \text{ is clopen})}}{\lim} (C \otimes \Gamma(U,k)) \cong (C/\overline{M}C) \quad .$$

Thus if $C \otimes k_x = 0$ for each $x \varepsilon X$ we must have $C = 0$.

Q.E.D.

The preceeding theorem is true essentially because each stalk k_x is a direct limit of direct summands of R <u>and</u> the functors $\mathrm{Tor}_M^R(\ ,\ )$ commute with direct limits. Since the functors $\mathrm{Ext}_R^M(\ ,\ )$ do not necessarily commute with direct limits it is not suprising that there is no analogue of theorem 0.18 for the ordinary global dimension of R .

Theorem 0.18 provides a means of calculating the weak global dimension of R from those of the stalks. No analogue of this holds for arbitrary subdirect products. For instance the ring I of integers is a subdirect product of the fields I/M such that M is a maximal ideal in I . However the I/M all have weak global dimension zero yet $\mathrm{w.gl.dim}(I) = 1$.

An elementary method for constructing examples of ringed spaces is now given.

<u>0.19</u>) (Definition) <u>Simple Sheaves</u>: Let X be a topological space and let S be a ring. Let S have the discrete topology. Let $k = X \times S$ have the product topology. For each $x \varepsilon X$ let $k_x = \{x\} \times S$. Then k is a sheaf of rings over X called the <u>simple S sheaf over X</u> .

§1

Coherent rings

In this section R will denote a commutative Ring with unity and (X,k) will denote R° . Some results will be obtained relating properties a), b), c), d), and e) for the ring R with certain properties of the reduced ringed space (X,k) .

1.1) (Proposition): Consider the following conditions:

i) R has property c).

ii) R has property d).

iii) For any $\tau \in \Gamma(X,k)$, $S(\tau)$ is clopen.

iv) k is a Hausdorff space.

Then ii) → i) and iii)⟷iv). If each k_x is an integral domain then each of i), ii), iii), and iv) are equivalent.

Proof: ii) → i): This is obvious.

iii) →iv): By 0.14.v) for any $\sigma \in \Gamma(X,k)$ and $x \in X$, sets of the form $\sigma(N_x)$ where N_x is a clopen neighborhood of x form a basis for the neighborhoods of $\sigma(x)$ in k . Suppose that k is not Hausdorff. Thus, since X is Hausdorff, there exist two points on the same stalk, say k_x , that can not be separated by open sets. Thus there exist σ and τ (see 0.17) in $\Gamma(X,k)$ and $x \in X$ such that $\sigma(x) \neq \tau(x)$ yet for any clopen neighborhood of x, N_x , there exists $y \in N_x$ such that $\sigma(y) = \tau(y)$. Thus $S(\sigma - \tau)$ is not open.

iv) → iii): Suppose there exists $\tau \in \Gamma(X,k)$ such that $S(\tau)$ is not clopen. Then $S(\tau)$ is not open (see 0.6) so there exists $x \in S(\tau)$ such that for each clopen neighborhood of x, N_x, $N_x \cap (X - S(\tau)) \neq \phi$. Then $\sigma_{o_R}(x) = 0_x \neq \tau(x)$ yet for each N_x there exists $y \in (\sigma_{0_R}(N_x) \cap \tau(N_x))$. Thus k is not Hausdorff.

Now suppose that each k_x is an integral domain. Since $R \cong \Gamma(X,k)$ R can be replaced with $\Gamma(X,k)$ for the rest of this proof.

i) $\rightarrow$ iii): Let $\tau \in \Gamma(X,k)$ be such that $1.\mathrm{ann}(\tau) = \sum_{i=1}^{n} \Gamma(X,k)\tau_i$. $S(\tau) \subseteq X - \bigcup_{i=1}^{n} (S(\tau_i))$ since the k_x are domains. Let $x \in (X - \bigcup_{i=1}^{n} (S(\tau_i)))$ and suppose $\tau(x) = 0_x$. Find N_x , a clopen neighborhood of x , such that $N_x \subseteq (X - \bigcup_{i=1}^{n} S(\tau_i))$ (see 0.6) and $y \in N_x$ implies $\tau(y) = 0_y$. Then $\psi_{N_x} \in 1.\mathrm{ann}(\tau)$ yet $\psi_{N_x} \notin \sum_{i=1}^{n} \Gamma(X,k)\tau_i$. This contradicts the choice of the τ_i and yields $X - \bigcup_{i=1}^{n} (S(\tau_i)) \subseteq S(\tau)$. Thus $S(\tau) = X - \bigcup_{i=1}^{n} (S(\tau_i))$. Since the support of a section is always closed this establishes that $S(\tau)$ is clopen.

iii) $\rightarrow$ ii): Now let $\tau \in \Gamma(X,k)$ be such that $S(\tau)$ is clopen. Since the k_x are integral domains, $\sigma \in 1.\mathrm{ann}(\tau) \Longleftrightarrow S(\sigma) \subseteq X - S(\tau)$. Thus, if $\sigma \in 1.\mathrm{ann}(\tau)$, $\sigma = \psi_{(X-S(\frac{\sigma}{\tau}))} \in \Gamma(X,k)\cdot\psi_{(X-S(\tau))}\cdot$ $\psi_{(X-S(\tau))} \in \Gamma(X,k)$ since $(X - S(\tau))$ is clopen. Hence $1.\mathrm{ann}(\tau)$ is generated by an idempotent.

Q.E.D.

In order to relate properties a), b), and e) for the ring R with certain properties of the sheaf (X,k) it is necessary to have some lemmas allowing ideals in R to be studies in terms of the k_x .

1.2) (Definition): For I an ideal in R and $x \in X$ let

$$I_x = \{\sigma_a(x) : a \in I\} .$$

It is clear that each I_x is an ideal in k_x and that if I is finitely generated as an R-ideal then each I_x is finitely generated as a k_x-ideal.

(1.3) (lemma): Let I and J be two ideals in R and $x \in X$. Then $(I \cap J)_x = I_x \cap J_x$.

Proof: Clearly $(I \cap J)_x \subseteq I_x \cap J_x$. Now let $\sigma_c(x) \in I_x \cap J_x$. Thus there exist $a \in I$ and $b \in J$ such that $\sigma_c(x) = \sigma_a(x) = \sigma_b(x)$. Find N_x, a clopen neighborhood of x such that $\sigma_c\big|_{N_x} = \sigma_a\big|_{N_x} = \sigma_b\big|_{N_x}$. Find $e \in R$ such $\sigma_e = \psi_{N_x}$. Then

$$\sigma_{ea} = \psi_{N_x}\sigma_a = \psi_{N_x}\sigma_b = \sigma_{eb} \quad \text{so} \quad ea = eb \in I \cap J .$$

Thus $\sigma_c(x) = \psi_{N_x}(x)\sigma_a(x) = \sigma_e(x)\sigma_a(x) = \sigma_{ea}(x) \in (I \cap J)_x$.

Thus $I_x \cap J_x \subseteq (I \cap J_x)$. Hence $I_x \cap J_x = (I \cap J)_x$.

Q.E.D.

1.4) (lemma): Let I and J be two ideals in R.

i) Let σ_r, for some $r \in R$, be such that $\sigma_r(x) \in I_x$ for all $x \in X$. Then $r \in I$.

ii) If $I_x = J_x$ for all $x \in X$ then $I = J$.

Proof: i) For each $x \in X$ find $a_x \in I$ such that

$$\sigma a_x(x) = \sigma_r(x) \text{ and find } N_x \text{ , a clopen}$$

neighborhood of x such that $\sigma a_x\Big|_{N_x} = \sigma_r\Big|_{N_x}$. $\{N_x : x \in X\}$ is an open cover of X so that by the partition property there exists $\{P_1,\ldots P_n\}$, a partition of X into clopen sets, such that each $P_j \subseteq$ some N_{x_j} from the cover. For each i $(1 \leq i \leq n)$ let $e_i \in R$ be such that $\sigma_{e_i} = \psi_{P_i}$. For any $x \in X$ $x \in$ some P_j so

$$(\sigma \sum_{i=1}^{n} e_i \cdot a_{x_i})(x) = \sum_{i=1}^{n} \psi_{P_i}(x) \cdot \sigma_{a_{x_i}}(x) = \sigma_{a_{x_j}}(x) = \sigma_r(x) .$$

Thus $r = \sum_{i=1}^{n} e_i \cdot a_{x_i} \in I$.

ii) Suppose that $I_x = J_x$ for all $x \in X$. Let $a \in I$. Then for all $x \in X$ $\sigma_a(x) \in I_x = J_x$ so by i) $a \in I$. Thus $I \subseteq J$. Similarly $J \subseteq I$.

Q.E.D.

It is clear that if I is a finitely generated R-ideal then each I_x is a finitely generated k_x-ideal. The converse is not in general true. For example suppose that X contains a non-isolated point, y , and that each k_x is a field. Let $I = \{r \in R : \sigma_r(y) = 0_y\}$. Then each I_x is a principal ideal in k_x yet I is not finitely generated as an R-ideal. The next two lemmas are partial converses.

1.5) (lemma) Let n be a positive integer and I an ideal in R such that for each $x \varepsilon X$ there exists $\tau_{1,x}, \dots \tau_{n,x} \varepsilon \Gamma(X,k)$ and N_x a neighborhood of x such that for any $y \varepsilon N_x$ $\{\tau_{1,x}(y), \dots \tau_{n,x}(y)\}$ generates I_y . Then there exists a subset of R containing n or fewer elements that generates I .

Proof: $\{N_x : x \varepsilon X\}$ where the N_x are as described in the hypothesis is an open cover of X so that by the partition property there exists $\{P_1, \dots P_m\}$, a partition of X into clopen sets such that each $P_i \subseteq$ some N_{x_i} from the cover. Find $a_i \varepsilon R$ $(1 \leq i \leq n)$ such that $\sigma_{a_i} = \sum_{j=1}^{m} \tau_{i,x_j} \psi_{P_j}$. Let $J = \sum_{i=1}^{n} R \cdot a_i$. for any $x \varepsilon X$, $x \varepsilon$ some P_j, so that

$$J_x = \sum_{i=1}^{n} (k_x(\sum_{j=1}^{m} \tau_{i,x_j} \cdot \psi_{P_j})(x))$$

$$= \sum_{i=1}^{n} (k_x \cdot \tau_{i,x_{j'}}(x)) = I_x \quad .$$

Thus by 1.14 $I = J = \sum_{i=1}^{n} R \cdot a_i$.

Q.E.D.

1.6) (lemma) Let I be an ideal in R such that for each $x \varepsilon X$ there exists a positive integer $n(x)$, $\tau_{1,x}, \dots \tau_{n(x),x} \varepsilon \Gamma(X,k)$, and N_x , a neighborhood of x , such that for any $y \varepsilon N_x$ $\{\tau_{1,x}(y), \dots \tau_{n(x),x}(y)\}$ generates I_x . Then I is finitely generated as an ideal in R .

Proof: $\{N_x : x \in X\}$ where the N_x are as described in the hypothesis forms an open cover of X so that by the partition property there exists $\{P_1, \ldots P_m\}$, a partition of X into clopen sets, such that $P_j \subseteq$ some N_{x_j} from the cover. Let $n = \sup \{n(x_i) : 1 \leq i \leq m\}$. Define $\tau_i = \sum_{j=1}^{m} \tau_{i,x_j} \cdot \psi_{P_j}$ $(1 \leq i \leq n)$ where it is understood that $\tau_{i,x_j} = 0$ for $n(x_j) \lneq i \leqq n$. Then for each $x \in X$ $\{\tau_1(x), \ldots \tau_n(x)\}$ generates I_x so by 1.5 I is generated by some subset of R containing n or fewer elements.

Q.E.D.

The next proposition applies the foregoing to obtain various algebraic conditions on the k_x that are sufficient to imply that R has some of properties a), b) and e).

1.7) (Proposition) Suppose that for each $x \in X$ k_x has property e. Then: i) R has property e).

ii) If each k_x is an integral domain then R also has properties a) and b) and $\text{w.gl.dim}(R) \leq 1$.

iii) If R has property d) then R is semi-hereditary.

Proof: i) It suffices to show that the ideal $I = Ra + Rb$ is principal where a and $b \in R$ are arbitrary. Fix arbitrary $x \in X$. Then $I_x = k_x\sigma_a(x) + k_x\sigma_b(x)$ is a finitely generated ideal in k_x so there exists $c_x \in I$ (for simplicity c will denote c_x) such that $k_x\sigma_a(x) + k_x\sigma_b(x) = k_x\sigma_c(x)$. Thus there exists r, s, t_1, and $t_2 \in R$ such that

$$\text{I} \quad \sigma_r(x)\sigma_a(x) + \sigma_s(x)\sigma_b(x) = \sigma_c(x) \quad ; \text{ and}$$

$$\text{II} \quad \sigma_a(x) = \sigma_{t_1}(x)\sigma_c(x) \quad \text{and} \quad \sigma_b(x) = \sigma_{t_2}(x)\sigma_c(x) \quad .$$

Since there are only finitely many equations above there exists N_x , a neighborhood of x such that for any $y \in N_x$

$$\text{I}' \quad \sigma_r(y)\sigma_a(y) + \sigma_s(y)\sigma_b(y) = \sigma_c(y) \quad ; \text{ and}$$

$$\text{II}' \quad \sigma_a(y) = \sigma_{t_1}(y)\sigma_c(y) \quad \text{and} \quad \sigma_b(y) = \sigma_{t_2}(y)\sigma_c(y) \quad .$$

Conditions I' and II' combine to show that whenever $y \in N_x$, $k_y\sigma_a(y) + k_y\sigma_b(y) = k_y\sigma_c(y)$. Thus, by lemma 1.5, I is principal.

ii) The following fact will be used in this proof: If D is an integral domain with property e) and $x,y,w \in D$ are such that $0 = D_x + D_y = D_w$ then $\frac{x}{w}$, $\frac{y}{w}$, and $\frac{xy}{w} \in D$ and $D_x \cap D_y = D_{(\frac{xy}{w})}$. ($\frac{x}{w}$ etcetera denotes the element $x\cdot(w^{-1})$ in the classical ring of quotients for D .) This is analogous to the situation in Euclidean domains.

Let R have property e). Then properties a) and b) are equivalent for R . Therefore it suffices to show that for any $a,b \in R$, $I \cap J$ is principal where $I = Ra$ and $J = Rb$. Let $C = S(\sigma_a) \cap S(\sigma_b)$. C is closed in X . By i) find $c \in R$ such that $Ra + Rb = Rc$. Since $k_x\sigma_a(x) + k_x\sigma_b(x) = k_x\sigma_c(x)$ for all $x \in X$ it follows from the opening remark that $x \in C$ implies that

$$\frac{\sigma_a(x)}{\sigma_c(x)}\ ,\ \frac{\sigma_b(x)}{\sigma_c(x)}\ ,\ \text{and}\ \frac{\sigma_a(x)\sigma_b(x)}{\sigma_c(x)} \in k_x \quad \text{and}$$

$$(k_x\sigma_a(x)) \cap (k_x\sigma_b(x)) = k_x\left(\frac{\sigma_a(x)\sigma_b(x)}{\sigma_c(x)}\right) \quad .$$

Now I shall show that the maps $\sigma_\alpha, \sigma_\beta \colon C \to k$ given by $\sigma_\alpha(x) = \dfrac{\sigma_a(x)}{\sigma_c(x)}$ and $\sigma_\beta(x) = \dfrac{\sigma_b(x)}{\sigma_c(x)}$ are continuous. For arbitrary $x \in C$ find $r \in R$ such that $\sigma_\alpha(x) = \dfrac{\sigma_a(x)}{\sigma_c(x)} = \sigma_r(x)$. Thus $\sigma_a(x) = \sigma_r(x)\cdot\sigma_c(x)$. Pick a neighborhood of x, N_x , such that $y \in N_x$ implies $\sigma_a(y) = \sigma_r(y)\sigma_c(y)$. Hence $y \in C \cap N_x$ implies $\sigma_\alpha(y) = \dfrac{\sigma_a(y)}{\sigma_c(y)} = \sigma_r(y)$. Thus $\sigma_\alpha\big|_{N_x \cap C} = \sigma_r\big|_{N_x \cap C}$. Since σ_r is continuous at x so is σ_α . Since C is closed in X we may assume without loss of generality that $\sigma_\alpha \in \Gamma(X,k)$. (See 0.10). Similarly σ_β is continuous and may be viewed as a member of $\Gamma(X,k)$. Observe that $\sigma_\alpha\sigma_b\big|_C = \sigma_\beta\sigma_a\big|_C$.

Since C and $S(\sigma_\alpha\sigma_b - \sigma_\beta\sigma_a)$ are disjoint closed sets in the Boolean space X there exists a clopen set P such that $C \subseteq P$ and $S(\sigma_\alpha\sigma_b - \sigma_\beta\sigma_a) \subseteq X - P$.

Thus $\sigma_\alpha\sigma_b\big|_P = \sigma_\beta\sigma_a\big|_P$.

Define $\sigma_m \in \Gamma(X,k)$ by $\sigma_m(x) = \sigma_\alpha(x)\sigma_b(x) = \sigma_\beta(x)\sigma_a(x)$ for $x \in P$, and $\sigma_m(x) = 0$ for $x \notin P$. This is continuous since P is clopen. Observe that by its construction

$$\sigma_m(x) = \frac{\sigma_a(x)\sigma_b(x)}{\sigma_c(x)} \quad \text{for } x \in C \text{ and } \sigma_m(x) = 0 \text{ for } x \notin C \ .$$

Thus, using the opening discussion and 1.3, for $x \in C$

$$(R_m)_x = k_x\sigma_m(x) = k_x\left(\frac{\sigma_a(x)\sigma_b(x)}{\sigma_c(x)}\right)$$

$$= k_x\sigma_a(x) \cap k_x\sigma_b(x) = I_x \cap J_x = (I \cap J)_x \quad .$$

For $x \notin C$ $(R_m)_x = 0 = I_x \cap J_x = (I \cap J)_x$. Thus by 1.4 $R_m = I \cap J = R_a \cap R_b$.

To see that $w.gl.dim(R) \leq 1$ the following fact (from thm. 4.1 of [4] will be used: If S is a semi-hereditary ring then $w.gl.dim(S) \leq 1$. Since for each $x \in X$ k_x is an integral domain it follows that principal ideals in a k_x are projective so each k_x is semi-hereditary. Hence $w.gl.dim(R) = \sup \{w.gl.dim(k_x) : x \in X\} \leq 1$.

iii) Suppose that R has property d). Since by i) it has property e) it suffices to show that the principal ideals of R are projective. Let $a \in R$ be arbitrary and find $e \in B(R)$ such that $l.ann(a) = Re$. Then

$$0 \to Re \to Re \oplus R_{(1-e)} \cong R \to Ra \to 0$$

(where $r \to r.a$) is a split exact sequence so that Ra is projective.

Q.E.D.

1.8) (Definition) A ring S is local iff it has a unique maximal ideal.

In proposition 1.13 we use 1.7 to find a homological condition on R that, if each k_x is a local ring, is sufficient for R to have conditions a), b) and e).

First three lemmas about local rings are needed.

1.9) (lemma) Let S be a commutative ring. The following conditions are equivalent: i) S is a local ring.

ii) All nonunits of S are contained in a proper ideal M.

iii) The nonunits of S form an ideal.

iv) For r and $s \in S$ $r + s$ is a unit implies that either r is a unit or s is a unit.

Proof: i) <—> ii) <—> iii) is proposition 5 from §2.2 of [7]. i) <—> iv) is essentially the same as exercise 7 from §2.2 of [7].

Q.E.D.

1.10) (Corollary) Let S be a commutative local ring. Let $0 = a \in S$ and $x \in S$ be such that $xa = a$. Then x is a unit.

Proof: Rewrite $xa = a$ as $(1-x)\cdot a = 0$. Since $a \neq 0$ this shows that $1-x$ is not a unit. Then by 1.9 iv) x is a unit since $x + (1-x) = 1$.

Q.E.D.

1.11) (lemma) Let S be a commutative local ring with zero divisors. Then $\mathrm{w.gl.dim}(S) > 1$.

Proof: Find a, b non-zero in S such that $a.b = 0$.
I shall use exercise 5 from chapter 6 of [3] to show that the canonical map $S_a \otimes_S S_b \longrightarrow S_a \otimes_S S \cong S_a$ is not a monomorphism so $w.\dim_S(S_a) > 0$. Since $0 \to S_a \to S \to (S/S_a) \to 0$ is an exact sequence of modules this would show $w.\dim_S(S/S_a) > 1$ and hence $w.gl.\dim(S) > 1$.

Suppose that $S_a \otimes_S S_b \to S_a \otimes_S S$ is a monomorphism. Thus, since $a.b = 0$, $a \in S_a$ and $b \in S_b$, there exists (by the exercise) $r_j a \in S_a$ and $s_j \in S$ ($1 \leq j \leq$ some integer n) such that: i) $a = \sum_{j=1}^{n} (r_j a\, s_j)$; and

ii) $s_j b = 0$ for $1 \leq j \leq n$.

from i) get $a = (\sum_{j=1}^{n} r_j s_j)\, a$ so that by 1.10 $(\sum_{j=1}^{n} r_j s_j)$ is a unit. Thus by 1.9 iv) some $r_{j'} s_{j'}$ is a unit and hence $s_{j'}$ is a unit. This is a contradiction since $s_{j'} b = 0$ yet $b \neq 0$. This contradiction establishes that $S_a \otimes_S S_b \to S_a \otimes_S S$ is not a monomorphism so $w.gl.\dim(S) > 1$.

Q.E.D.

1.12) (lemma) Let S be a commutative local ring. Then:

i) $w.gl.\dim(S) \leq 1$ implies that S is an integral domain.

ii) $w.gl.\dim(S) \leq 1$ implies that S has property e).

Now suppose that S is also semi-prime. That is to say $s^n = 0 \to s = 0$ for each positive integer n and $s \in S$.
Then:

iii) S has property e) implies that $w.gl.\dim(S) \leq 1$.

Proof: i) This is just a restatement of 1.11.

ii) Let $w.gl.\dim(S) \leq 1$. In theorem 4.2 of [4] Chase has shown that if D is an integral domain then D is semi-hereditary iff $w.gl.dim(S) \leq 1$. It thus follows from i) that S is semi-hereditary. It is well known that if a ring is local all its finitely generated projective modules are free. Let $0 \neq I$ be a finitely generated ideal in S. Then I is projective and hence free. Thus there exists $\{a_1,\ldots a_n\}$, a free basis for I. If $n = 1$ we are done. Suppose $n > 1$. Then $a_2 \cdot a_1 - a_1 \cdot a_2 = 0$. This contradicts our choice of $\{a_1,\ldots a_n\}$. This contradiction establishes $n = 1$. Thus I is principal so that S has property e).

iii) Let S be semi-prime with property e). It suffices to show that S is an integral domain since then every principal ideal would be projective. Since S has property e) this would establish that S is semi-hereditary and thus $w.gl.dim(S) \leq 1$. Let $a,b \in S$ be such that $a \neq 0$, $b \neq 0$, yet $ab = 0$. Find $c \in S$ such that $S_a + S_b = S_c$. Hence find $x,y,r,s \in S$ such that $ra + sb = c$, $a = xc$, and $b = yc$. Hence $(rx + sy)c = ra + sb = c$ so by 1.10 $rx + sy$ is a unit. Thus by 1.9 either rx or sy is a unit. Suppose without loss of generality that rx is a unit. Then so is x. $xb^2 = xycb = yxcb = yab = 0$. Thus, since x is a unit, $b^2 = 0$ so $b = 0$. This contradiction establishes that S is an integral domain.

Q,E.D.

1.13) (Proposition) Let each k_x be a local ring. Then:

i) $w.gl.dim(R) \leq 1$ implies that R has properties a), b) and e).

ii) If R has property d) then $w.gl.dim(R) \leq 1$ implies that R is semi-hereditary.

Now suppose that R is semi-prime. Then:

iii) If R has property e) then $w.gl.dim(R) \leq 1$.

iv) If R has property e) then R also has properties a) and b).

Proof: i) Suppose that $w.gl.dim(R) \leq 1$. By 0.18 $w.gl.dim(R) = \sup \{w.gl.dim(k_x) : x \in X\}$. Thus for each $x \in X$ $w.gl.dim(k_x) \leq 1$ so that by 1.12 k_x is an integral domain with property e). By 1.7 this establishes that R has properties a), b) and e).

ii) This follows from i) and 1.7. iii).

iii) Fix an arbitrary $x \in X$. First note that k_x is semi-prime. To see this suppose there exists $0 \neq \sigma_a(x) \in k_x$ and a positive integer n such that $(\sigma_a(x))^n = 0$. Choose N_x , a clopen neighborhood of x , such that $(\sigma_a(y))^n = 0$ for any $y \in N_x$. Choose $e \in R$ such that $\sigma_e = \psi_{N_x}$. Then $(\sigma_{ea}(y))^n = (\psi_{N_x}(y)\sigma_a(y))^n = 0$ for any $y \in X$ and $\sigma_{ea}(x) = \psi_{N_x}(x)\sigma_a(x) = \sigma_a(x)$ so that $0 \neq ea \in R$ yet $(ea)^n = 0$. This contradicts the hypothesis that R is semi-prime and thus establishes that k_x is semi-prime. Now let R have property e). Then for each $x \in X$ k_x also has property e) and hence, by 1.12 iii), $w.gl.dim(k_x) \leq 1$. Thus $w.gl.dim(R) = \sup \{w.gl.dim k_x : x \in X\} \leq 1$.

iv) This follows from iii) and i).

Q.E.D.

In order to apply 1.7 and 1.13 to a particular R it will be necessary to compute the sheaf (X,k) . The following lemmas will be used for such a computation in §5.

1.14) (Definition:) Let λ be a cardinal. i) A set $U \subseteq X$ is a λ-set iff there exists a family of clopen subsets of X $\{U_\alpha\}$ such that $U = \bigcup_\alpha (U_\alpha)$ and $|\{U_\alpha\}| < \lambda$.

ii) (X,k) has the λ-extension property iff for any λ-set U and $\sigma \in \Gamma(U,k)$ there exists $\sigma' \in \Gamma(X,k)$ such that $\sigma'|_U = \sigma$.

iii) (X,k) has the unique λ-extension property iff it has the λ-extension property and for any λ-set U and $\sigma, \sigma' \in \Gamma(X,k)$, $\sigma|_U = \sigma'|_U \to \sigma|_{\overline{U}} = \sigma'|_{\overline{U}}$.

1.15) (lemma) Let (X,k) have the unique λ-extension property and $U = \underset{\alpha}{U}(U_\alpha)$ where $\{U_\alpha\}$ is a disjoint family of clopen sets such that $|\{U_\alpha\}| < \lambda$. Let I be a finitely generated ideal in R and $r \in R$ be such that $\sigma_r(U) \in I_u$ for all $u \in U$. Then $\sigma_r(x) \in I_x$ for all $x \in \overline{U}$.

Proof: Let $a_1,\dots a_n$ generate I . Then for each $x \in X$ $\sigma_{a_1}(x),\dots\sigma_{a_n}(x)$ generates I_x . For each α let $e_\alpha \in R$ be such that $\sigma_{e_\alpha} = \psi_{U_\alpha}$. For each α I_{e_α} is generated by $a_1e_\alpha,\dots a_ne_\alpha$. Also $\sigma_{re_\alpha}(x) = \sigma_r(x)\sigma_{e_\alpha}(x) = \sigma_r(x)\psi_{U_\alpha}(x)$ $\in (I_x)\sigma_{e_\alpha}(x) = (I_{e_\alpha})x$ for any $x \in X$ so by 1.4 $re_\alpha \in I_{e_\alpha}$.

Thus there exists $s_{1,\alpha},\ldots s_{n,\alpha} \in R$ such that

$re_\alpha = \sum_{i=1}^{n} s_{i,\alpha} a_i e_\alpha$. For each i $(1 \le i \le n)$ define $\tau_i : U \to k$

by $\tau_i(x) = \sigma_{s_{i,\alpha}}(x)$ where α is chosen such that $x \in U_\alpha$.

Since the U_α are disjoint the τ_i are well-defined. Since the U_α are open, for arbitrary $x \in U$, $x \in$ some U_α and U_α is a neighborhood of x such that $\tau_i\big|_{U_\alpha} = \sigma_{s_{i,\alpha}}\big|_{U_\alpha}$, so τ_i is continuous at x . Thus $\tau_i \in \Gamma(U,k)$.

For arbitrary $x \in U$ find U_α such that $x \in U_\alpha$. Then $\sigma_r(x) = \sigma_r(x)\psi_{U_\alpha}(x) = \sigma_r(x)\sigma_{e_\alpha}(x) = \sigma_{re_\alpha}(x) = \sum_{i=1}^{n} \sigma_{s_{i,\alpha}}(x) \cdot \sigma_{a_i e_\alpha}(x)$

$= \sum_{i=1}^{n} \sigma_{s_{i,\alpha}}(x)\sigma_{a_i}(x)\psi_{U_\alpha}(x) = \sum_{i=1}^{n} \tau_i(x)\sigma_{a_i}(x)$. Thus

$\sigma_r\big|_U = \left(\sum_{i=1}^{n} \tau_i \sigma_{a_i}\right)\Big|_U$. Since (X,k) has the λ-extension property we may assume that all the sections in the above equation are global.

Since (X,k) has the unique λ-extension property

$\sigma_r\big|_{\overline{U}} = \left(\sum_{i=1}^{n} \tau_i \sigma_{a_i}\right)\Big|_{\overline{U}}$, so for $x \in \overline{U}$ $\sigma_r(x) = \left(\sum_{i=1}^{n} \tau_i(x)\sigma_{a_i}(x)\right) \in I_x$.

Q.E.D.

1.16) (lemma) Let (X,k) , I , U , and $\{U_\alpha\}$ be as in the hypothesis of 1.15. Let $b_1,\ldots b_m \in R$ be such that $\sigma_{b_1}(u),\ldots\sigma_{b_m}(u)$ generates I_u for all $u \in U$. Then $\sigma_{b_1}(x),\ldots,\sigma_{b_m}(x)$ generates I_x for all $x \in \overline{U}$.

Proof: Let $J = \sum_{i=1}^{m} R_{b_i}$. Then by hypothesis $J_u = I_u$ for all $u \in U$. Let $a \in J$ be arbitrary. Then for all $u \in U$

$\sigma_a(U) \varepsilon I_u$. Since I is finitely generated we have, by 1.15, $\sigma_a(x) \varepsilon I_x$ for all $x \varepsilon \overline{U}$. Thus $J_x \cap I_x$ for all $x \varepsilon \overline{U}$. Since J is finitely generated we also have $I_x \cap J_x$ for all $x \varepsilon \overline{U}$.

Q.E.D.

In order to apply 1.16 it is necessary to obtain some information about the existence of disjoint families of clopen sets.

1.17) (lemma) i) Let U be an $\aleph_1$ - set in X . Then U can be expressed as the union of a coutable disjoint family of clopen sets.

ii) Let U be open in X . Then there exists a disjoint family of clopen subsets of X , $\{V_\alpha\}$, such that $\overline{V} = \overline{U}$ where $V = \bigcup_\alpha (V_\alpha)$.

Proof: i) This is a standard construction. Since U is an $\aleph_1$ - set we have $U = \bigcup_i (U_i)$ where $\{U_i\}$ is an appropriate countable family of clopen sets. Define $U'_o = U_o$ and for $0 < n \varepsilon I$ let $U'_n = U_n - \bigcup_{0<i<n} (U'_i)$. Then $U = \bigcup_i (U'_i)$ and $\{U'_i\}$ is a disjoint countable family of clopen sets.

ii) Let $S = \{F : F$ is a disjoint family of clopen sets such that $W \varepsilon F \to W \quad U\}$. S is partially ordered by inclusion, $\subseteq$. If C is a chain in S it is easily checked that $\bigcup_{F\varepsilon C} (F) \varepsilon S$ so that by Zorn's lemma S has a maximal element, say $M = \{V_\alpha\}$. Then $\overline{V} \supseteq U$ for if there existed $x \varepsilon U - \overline{V}$ there would exist, since U is open, $N_x \subseteq U - \overline{V} \subseteq U - V$ a neighborhood of x . Thus $M \cup \{N_x\}$ would belong to S , contradicting the maximality of M . Thus we have $V \subseteq U \subseteq \overline{V}$. Hence $\overline{V} = \overline{U}$.

Q.E.D.

It should be noted that if U is a λ-set the set V obtained in 1.17 ii) need not be a λ-set. Since X is clopen it is a 2-set. I shall show that if X is infinite there exists an open set V of the form $V = \underset{\alpha}{\cup}(V_\alpha)$ (where $\{V_\alpha\}$ is a disjoint family of clopen sets) such that $\overline{V} = X = \overline{X}$ yet V is not a 2-set. Suppose X is infinite. Since X is compact there exists $x \in X$ such that $\{x\}$ is not open. Let $U = X - \{x\}$. Then $\overline{U} = X$. Obtain $\{V_\alpha\}$ a disjoint family of clopen sets and $V = \underset{\alpha}{\cup}(V_\alpha) \subset U$ such that $\overline{V} = \overline{U} = X = \overline{X}$. V is not a 2-set for if it were it would be closed and we would have $V = \overline{V} = \overline{U} = X$, a contradiction.

§2

Coherence of $R\lceil X\rceil$

Throughout this section R will denote a commutative von Neumann regular ring and S will denote the polynomial ring $R\lceil X\rceil$. (X,K) will denote S°. Clearly $B(R) = B(S)$. Thus by 0.16) R° may be identified with (X,k) where k is a sub-sheaf of K such that $\lambda_S\big|_R : R \to \Gamma(X,k)$ is an isomorphism. Since R is von Neumann regular (X,k) is a regular ringed space. (see 0.15).

2.1) (lemma) i) For each $x \in X$, $K_x \cong k_x\lceil X\rceil$.

ii) For each $x \in X$ K_x is an integral domain with property e).

iii) For $(\sum_{i=1}^{n} a_i X^i) \in S$

$$(\sigma_{\sum_{i=1}^{n} a_i \cdot X^i})(x) = \sum_{i=1}^{n} \sigma_{a_i}(x)\cdot X^i \quad \text{, where}$$

K_x and $k_x\lceil X\rceil$ are identified.

Proof: i) Refer to the basic definitions given in 0.14. An arbitrary point in X is actually a maximal proper ideal in $B(R)$. Then $K_M = R\lceil X\rceil/R\lceil X\rceil\cdot M \cong (R/R\cdot M)\lceil X\rceil = k_M(R)\lceil X\rceil = k_M\lceil X\rceil$.

ii) Since each k_x is a field (R is von-Neumann regular) this result is immediate from i).

iii) It is clear that under the identification $K_x = k_x\lceil X\rceil$ $\sigma_{X^i}(x) = \overline{X^i}$ is identified with $\overline{1}\cdot X^i = 1_x \cdot X^i$. The result now follows from the fact that $\sigma_{(*)}(x) : S \to K_x = k_x\lceil X\rceil$ is a homomorphism.

Q.E.D.

2.2) (Proposition) $w.gl.dim(S) = 1$.

Proof: It is well known that if T is a Noetherian ring then $w.gl.dim(T) = gl.dim(T)$. (see ex. 3 p. 122 of [3].) It is also well known (Chapter IX Theorem 7.11 of [3]) that if F is a field then $gl.dim(F[X]) = 1$, and $F[X]$ is Noetherian. Thus, by 2.1. i) and 0.18

$$w.gl.dim(S) = \sup\{w.gl.dim(K_x) : x \in X\} = 1 \quad .$$

2.3) (Theorem) $R[X] = S$ is a semi-hereditary ring with properties a), b), c), d), and e).

Proof: It follows from 2.1 ii) and 1.7 that S has properties a), b), and e). Let $(\sum_{i=1}^{n} a_i \cdot X^i) = s \in S$ be arbitrary. Since R is von Neumann regular and the $a_i \in R$, therefore each $S(\sigma_{a_i})$ is clopen. Thus $S(\sigma_s) = S(\sum_{i=1}^{n} \sigma_{a_i} . X^i) = \bigcap_{i=1}^{n} (S(\sigma_{a_i}))$ is clopen so by 1.1 S has property d) (and therefore c)). The result now follows from 1.7 iii).

§3

Representation of $R\ulcorner\ulcorner G^+\urcorner\urcorner$ by sections of sheaves

In this section R will denote a commutative von Neumann regular ring and G an abelian linearly ordered group. The next lemma will be needed to define and work with $R\ulcorner\ulcorner G^+\urcorner\urcorner$, the ring of formal power series with coefficients from R and indices from $G^+ = \{g \in G : g \geq 0_G\}$.

(lemma) Let U and V be well-ordered as subsets of G^+ .
Then: i) $U + V$ is well-ordered as a subset of G^+ .
ii) $U \cup V$ is well-ordered as a subset of G^+ .
iii) For any $g \in G$ there are only finitely many $u \in U$ and $v \in V$ such that $u + v = g$.

Proof: i) Let $\phi \neq S \subseteq U + V$. Let u_1 = the least element of $\{u \in U ; u + v \in S$ for some $v \in V\}$. Let v_1 = the least element of $\{v \in V : u_1 + v \in S\}$. If $u_1 + v_1$ is the least element of S we are done. Otherwise let u_2 = the least element of $\{u \in U : u + v \in S$ and $u + v \quad u_1 + v_1\}$ and let v_2 = the least element of $\{v \in V : u_2 + v \in S\}$. By the construction of u_1, u_2, v_1 , and v_2 we have $u_1 < u_2$ and $v_2 < v_1$. If $u_2 + v_2$ is not the least element of S we continue as above and either obtain $u_n + v_n$ as the least element of S for some natural number n or $v_{i+1} < v_i$ for all natural numbers i . The latter would yield a subset $\{v_i\}$ of V with no least element contradicting the fact that V is well-ordered. Thus S has a least element so $U + V$ is well-ordered.

ii) Let $U' = U \times \{0\}$ and $V' = V \times \{0\}$. Then $U \cup V \subseteq U' + V'$. Since U' and V' are obviously well-ordered the result follows from i). A direct proof without reference to G^+ is also easy.

iii) Suppose that there are infinitely many $u \varepsilon U$ such that $u + v = g$ for some $v \varepsilon V$. Let u_1 = the least element of $\{u \varepsilon U : u + v = g$ for some $v \varepsilon V\}$. Let v_1 be the element of V such that $u_1 + v_1 = g$. By supposition $\{u \varepsilon U : u > u_1$ and $u + v = g$ for some $v \varepsilon V\} \neq \phi$ so we may choose u_2 its least element and $v_2 \varepsilon V$ such that $u_2 + v_2 = g$. By construction $u_2 > u_1$ so $v_2 < v_1$. Continuing this way u_i and v_i are defined for each positive integer i such that $u_{i+1} > u_i$ $v_{i+1} < v_i$. Thus $\{v_i\}$ V contains no least element. This contradicts the fact that V is well-ordered.

Q.E.D.

(Definition): <u>The ring $R[[G^+]]$ of formal power series with coefficients from R and indices from G^+</u> .

a) $R[[G^+]]$ consists of all formal sums of the form $\Sigma\, r_\alpha \cdot X^{g_\alpha}$ where the $r_\alpha \varepsilon R$, the $g_\alpha \varepsilon G^+$, no g_α appears twice, and $\{g_\alpha\}$ is a well-ordered subset of G^+ .

b) Two elements, $\Sigma\, r_\alpha \cdot X^{g_\alpha}$ and $\Sigma\, s_\alpha \cdot X^{h_\alpha} \varepsilon R[[G^+]]$, are equal iff

i) $r_\alpha \neq 0 \rightarrow$ there exists h_β such that $h_\beta = g_\alpha$ and $r_\alpha = s_\beta$.

ii) $s_\beta \neq 0 \rightarrow$ there exists g_α such that $g_\alpha = h_\beta$ and $r_\alpha = s_\beta$.

c) (Remark) In view of 3.1 ii) any two elements of $R\ulcorner\ulcorner G^+\urcorner\urcorner$, $\Sigma\, r_\alpha X^{g_\alpha}$ and $\Sigma\, s_\beta X^{h_\beta}$, can be rewritten as $\Sigma\, r'_\mu X^{g'_\mu}$ and $\Sigma\, s'_\mu X^{g'_\mu}$ respectively, where $\{g'_\mu\}$ G^+ is a common set of indices that is well-ordered. Thus, given any finite subset F of $R\ulcorner\ulcorner G^+\urcorner\urcorner$ we may choose $\{g_\alpha\}$, a well-ordered subset of G^+ such that each element of F has the form $\Sigma\, r_\alpha \cdot X^{g_\alpha}$ for appropriate $r_\alpha \varepsilon R$. This will be done without comment when required. Since a union of infinitely many well-ordered subsets of G^+ need not be well-ordered no similar simplification is possible when F is infinite.

In d), e), and f) let $\Sigma\, r_\alpha \cdot X^{g_\alpha}$ and $\Sigma\, s_\alpha X^{g_\alpha} \varepsilon R\ulcorner\ulcorner G^+\urcorner\urcorner$.

d) <u>Addition</u>: Let $(\Sigma\, r_\alpha X^{g_\alpha}) + (\Sigma\, s_\alpha X^{g_\alpha}) = \Sigma(r_\alpha + s_\alpha)X^{g_\alpha}$.

e) <u>Multiplication:</u> Let $\{h_\beta\} = \{g_\alpha\} + \{g_\alpha\}$. Then let $(\Sigma\, r_\alpha X^{g_\alpha})(\Sigma\, s_\alpha X^{g_\alpha}) = \Sigma\, t_\beta \cdot X^{h_\beta}$ where

$$\text{each} \quad t_\beta = \left(\sum_{g_\alpha + g_\mu = h_\beta} r_\alpha \cdot s_\mu\right).$$

By 3.1 iii) the definition of any t_β involves only a finite sum. By 3.1 i) $\{h_\beta\}$ is well-ordered as a subset of G^+ so $\Sigma\, t_\beta X^{h_\beta} \varepsilon R\ulcorner\ulcorner G^+\urcorner\urcorner$.

f) <u>Notation</u>: i) Since $\{g_\alpha\}$ is well-ordered we may assume without loss of generality that it is indexed by an ordinal in an order preserving way. That is to say there exists an ordinal λ such that $g : \lambda \to G^+$ given by $\alpha \to g_\alpha$ for $\alpha < \lambda$ is a strictly monotonically increasing map. It will often be assumed without comment that $\{g_\alpha\}$ is indexed this way.

ii) If G is the group of integers then $G^+ = \aleph_0$ is already well-ordered. Thus any element of $R\lceil\lceil G^+\rceil\rceil$ can be written in the form $\sum_{0\leq i<\aleph_0} r_i \cdot X^i$.

iii) To conform with tradition, if G is the group of integers then $R\lceil\lceil G^+\rceil\rceil$ will be denoted $R\lceil\lceil X\rceil\rceil$.

Throughout the section S will denote $R\lceil\lceil G^+\rceil\rceil$ and (X,K) will denote S° . Clearly $B(R) = B(S)$. Thus by 0.16 R may be identified with $(X,k)^*$ where k is the subsheaf of K such that $\lambda_S\big|_R : R \to \Gamma(X,k)$ is an isomorphism. Since R is von Neumann regular (X,k) is a regular ringed space.

The analogue of 2.1 i) for (X,K) does not hold. In fact it will be shown that there is a canonical epimorphism $K_x \to k_x\lceil\lceil G^+\rceil\rceil$ and that this is an isomorphism if x is an isolated point in X . Lemmas 3.1 and 3.2 are working lemmas for studying this and other structural facts about the K_x .

3.1) (Lemma) For $(\Sigma\, r_\alpha X^{g_\alpha}) \in S$, $S({}^\sigma\Sigma\; r_\alpha X^{g_\alpha}) = \overline{\bigcup_\alpha(S(\sigma_{r_\alpha}))}$.

<u>Proof</u>: I shall prove the equivalent fact:

$$Z({}^\sigma\Sigma\, r_\alpha X^{g_\alpha}) = (\bigcap_\alpha(Z(\sigma_{r_\alpha})))^{\text{int.}} \quad .$$

From the definitions (see 0.14) a point in X is actually a maximal proper ideal in $B(R) = B(S)$ and for $M \in X$ $K_M = R\lceil\lceil G^+\rceil\rceil/R\lceil\lceil G^+\rceil\rceil.M$ and $k_M = R/R.M$. Let $M \in Z({}^\sigma\Sigma\, r_\alpha X^{g_\alpha})$. Then $\Sigma\; r_\alpha X^{g_\alpha} \in R\lceil\lceil G^+\rceil\rceil.M$ so $\Sigma\; r_\alpha X^{g_\alpha} = (\Sigma\, r_\alpha X^{g_\alpha})\cdot e =$ $\Sigma(r_\alpha e)X^{g_\alpha}$ for some $e \in M$. Thus each $r_\alpha (= r_\alpha e) \in RM$.

$X(1-e)$ (see 0.14) is a neighborhood of M such that $M' \varepsilon X(1-e) \Longleftrightarrow e \varepsilon M'$. Thus $M' \varepsilon X(1-e) \rightarrow$ each $r_\alpha = r_\alpha e \varepsilon R \cdot M'$ $\rightarrow$ each $\sigma_{r_\alpha}(M') = 0 \rightarrow M' \varepsilon \bigcap_\alpha (Z(\sigma_{r_\alpha}))$. Thus $M \varepsilon (\bigcap_\alpha (Z(\sigma_{r_\alpha})))^{\text{int.}}$.

Conversely let $M \varepsilon (\bigcap_\alpha (Z(\sigma_{r_\alpha})))^{\text{int.}}$. Then there exists a neighborhood $X(1-e)$ of M (for appropriate $e \varepsilon B(R)$) such that $M' \varepsilon X(1-e) \rightarrow M' \varepsilon \bigcap_\alpha (Z(\sigma_{r_\alpha}))$. In terms of sections this says that $\sigma_e(M') = 0 \rightarrow \sigma_{r_\alpha}(M') = 0$ for each r_α. Thus for any $M' \varepsilon X$, $\sigma_r(M') = \sigma_{r_\alpha}(M')\, \sigma_e(M') = \sigma_{r_\alpha e}(M')$ for each r_α. Thus $r_\alpha = r_\alpha e$ for each r_α. Hence $\Sigma\, r_\alpha \cdot X^{g_\alpha} = \Sigma (r_\alpha e) X^{g_\alpha} = (\Sigma\, r_\alpha X^{g_\alpha}) \cdot e \varepsilon R[[G^+]] \cdot M$. Thus $M \varepsilon Z(\sigma_{\Sigma r_\alpha \cdot X^{g_\alpha}})$.

Q.E.D.

3.2) (Lemma) Let $\Sigma\, r_\alpha X^{g_\alpha}$ and $\Sigma\, s_\alpha X^{g_\alpha} \varepsilon S$ and let $x \varepsilon X$ be arbitrary but fixed. Then $(\sigma_{\Sigma r_\alpha \cdot X^{g_\alpha}})(x) = (\sigma_{\Sigma s_\alpha X^{g_\alpha}})(x)$ iff there exists N_x, a neighborhood of x, such that $\sigma_{r_\alpha}\big|_{N_x} = \sigma s_\alpha\big|_{N_x}$ for each α.

Proof: Since $\lambda_S : S \rightarrow \Gamma(X,K)$ is an isomorphism it suffices to show that $(\sigma_{\Sigma r_\alpha X^{g_\alpha}})(x) = 0$ iff there exists N_x, a neighborhood of x, such that $\sigma_{r_\alpha}\big|_{N_x} = 0$ for each r_α. The lemma now follows from 3.1.

Q.E.D.

The algebraic structure of the K_{x}'s is related to the topological structures of X and K and these are related to the algebraic structure of R. To establish these relationships more precisely some concepts, related to 1.14, must be introduced. From 3.3 to 3.8 T will denote a commutative ring, (Z,η) will denote T°, Z' an arbitrary Boolean space, and λ an arbitrary cardinal.

3.3) (Definition) i) Z' is λ-extremally disconnected iff $\overline{U}$ is clopen for each λ-set U .

ii) Z' is extremally disconnected iff it is μ-extremally disconnected for each cardinal μ . That is to say Z' is extremally disconnected iff $\overline{U}$ is clopen for each open $U \subseteq Z'$.

iii) Z' has the λ-disjointness property iff for U and V λ-sets in Z' , $U \cap V = \phi \rightarrow \overline{U} \cap \overline{V} = \phi$.

iv) Z' has the disjointness property iff it has the μ-disjointness property for each cardinal μ . That is to say, Z' has the disjointness property iff for any open subsets of Z' , U and V , $U \cap V = \phi \rightarrow \overline{U} \cap \overline{V} = \phi$.

v) An ideal I in T is a λ-ideal iff there exists $\{t_\alpha\} \subseteq T$ generating I such that $|\{t_\alpha\}| < \lambda$.

vi) T is λ-self-injective iff for each λ-ideal I and $f \in \mathrm{Hom}_T(I,T)$ there exists $f' \in \mathrm{Hom}_T(T,T)$ such that $f'|_I = f$.

vii) (Remark) T is self-injective iff it is μ-self-injective for each cardinal μ .

viii) Suppose that T is a Boolean ring and that T has the partial ordering induced by its Boolean ring structure. That is to say for $s,t \in T$ $s \leq t \Longleftrightarrow s.t = s$. Then T is λ-complete iff $\sup \{t_\alpha\} \in T$ (i.e. exists) for any $\{t_\alpha\} \subseteq T$ such that $|\{t_\alpha\}| < \lambda$.

The elementary relationships amongst the above concepts will be found in lemmas 3.4 - 3.9.

3.4) (Lemma) Let Z' be λ-extremally disconnected. Then Z' has the λ-disjointness property.

Proof: Suppose that U and V are open subsets of Z' such that U is a λ-set, $U \cap V = \phi$, yet there exists $x \varepsilon \overline{U} \cap \overline{V}$. Since $\overline{U}$ is open find N_x, a neighborhood of x, such that $N_x \subseteq \overline{U}$. Since $x \varepsilon \overline{V}$ there exists $v \varepsilon N_x \cap V \subseteq \overline{U} \cap V$. Since $v \varepsilon V$ and N_x and V are open there exists $N_v \subseteq N_x \cap V$, an open neighborhood of v. Since $v \varepsilon \overline{U}$ there exists $u \varepsilon U \cap N_v$. This contradicts $U \cap V = \phi$. Thus $\overline{U} \cap \overline{V} = \phi$. In particular this will be true if V is a λ-set.

Q.E.D.

3.5) (Lemma) Let Z' have the disjointness property. Then Z' is extremally disconnected.

Proof: Let U be an open set in Z'. Then U and $(Z'-\overline{U})$ are open sets in Z' such that $U \cap (Z' - \overline{U}) = \phi$. Hence $\overline{U} \cap (X - \overline{U}) = \phi$. Thus, given $x \varepsilon \overline{U}$ there exists a neighborhood N_x of x such that $N_x \cap (X - \overline{U}) = \phi$. Thus $N_x \subseteq \overline{U}$ so $\overline{U}$ is open and hence clopen.

Q.E.D.

The above proof can not be generalized to show that if Z' has the λ-disjointness property then Z' is λ-extremally disconnected since a set $U \subseteq Z'$ may be a λ-set while $(Z' - \overline{U})$

is not. In §7 it will be shown that for each cardinal μ there exists a Boolean space with the μ-disjointness property that is not $\aleph_1$-extremally disconnected.

3.6) (Lemma) i) Let (Z,η) have the λ-extension property. Then Z has the λ-disjointness property.

ii) Let (Z,η) have the extension property. Then Z is extremally disconnected.

Proof: i) Suppose that U and V are λ-sets in Z such that $U \cap V = \phi$ yet there exists $x \in \overline{U} \cap \overline{V}$. Define $\tau \in \Gamma(U \cup V,\eta)$ by $\tau(x') = 0$ for $x' \in U$ and $\tau(x') = 1$ for $x' \in V$. Since (Z,η) has the λ-extension property find $\tau' \in \Gamma(Z,\eta)$ such that $\tau'|_{U \cup V} = \tau$. $\tau(x) = 0$ since $x \in \overline{U}$. Also $\tau(x) = 1$ since $x \in \overline{V}$. This contradiction establishes that $\overline{U} \cap \overline{V} = \phi$.

ii) This follows from i) and 3.5.

Q.E.D.

3.7) (Lemma) i) T is λ-self-injective $\rightarrow$ (Z,η) has the λ-extension property.

ii) Let T be von Neumann regular. Then T is λ-self-injective $\Longleftrightarrow$ (Z,η) has the λ-extension property.

iii) Let T be a Boolean ring. Then Z has the λ-disjointness property $\Longleftrightarrow$ (Z,η) has the λ-extension property $\Longleftrightarrow$ T is λ-self-injective.

Proof: i) Suppose that T is λ-self-injective. Let $U = \bigcup_\alpha (U_\alpha)$ where the U_α are clopen in Z' and $|\{U_\alpha\}| < \lambda$. Let $\tau \in \Gamma(U,\eta)$ be given. Let $I = \{\sigma \in \Gamma(Z,\eta) : S(\sigma) \subseteq$ some union of finitely many $U_\alpha\}$. Then I is a λ-ideal in $\Gamma(Z,\eta)$ generated by $\{\psi_{U_\alpha}\}$. Define $f \in \mathrm{Hom}_{\Gamma(Z,\eta)}(I, \Gamma(Z,\eta))$ as follows: for any $\sigma \in I$ let $f(\sigma)(x) = \sigma(x)\tau(x)$ when $x \in U$ and $f(\sigma)(x) = 0$ when $x \notin U$. Since $S(\sigma)$ is a closed subset of the open set U there exists a clopen set V such that $S(\sigma) \subseteq V \subseteq U$. $f(\sigma)\big|_V$ is continuous because $\sigma\big|_V$ and $\tau\big|_V$ are. $f(\sigma)\big|_{(Z-V)} = 0$

so that is also continuous. Since V is clopen this establishes $f(\sigma) \in \Gamma(Z,\eta)$. It is easily checked that f is a homomorphism. Since $T \cong \Gamma(Z,\eta)$ is λ-self-injective find $f' \in \mathrm{Hom}_{\Gamma(Z,\eta)}(\Gamma(Z,\eta), \Gamma(Z,\eta))$ such that $f'\big|_I = f$. Then $f'(1) \in \Gamma(Z,\eta)$. For any $x \in U$ find α such that $x \in U_\alpha$. Then $\psi_{U_\alpha} \in I$ and $f'(1)(x) = f'(1)(x) \cdot \psi_{U_\alpha}(x) = (f'(1) \cdot \psi_{U_\alpha})(x) = (f'(1 \cdot \psi_{U_\alpha}))(x) = \psi_{U_\alpha}(x)\tau(x) = \tau(x)$. Thus $f'(1)\big|_U = \tau$.

Let (Z,η) have the λ-extension property. Let I be an ideal in T generated by $\{e_\alpha\} \subseteq T$ where $|\{e_\alpha\}| < \lambda$. Since R is von Neumann regular we may assume without loss of generality that the e_α are idempotents. Let $f \in \mathrm{Hom}_T(I,T)$ be given. Let $U = \bigcup_\alpha (S(\sigma e_\alpha))$. U is a λ-set. Define $\tau : U \to \eta$ by $\tau(x) = \sigma_{f(e_\alpha)}(x)$ where $x \in S(\sigma e_\alpha)$. To see that τ is well-defined suppose that $x \in S(\sigma e_\alpha) \cap S(\sigma e_\beta)$. Then $\sigma_{f(e_\alpha)}(x) = \sigma_{f(e_\alpha)}(x) \cdot \sigma_{e_\beta}(x) = \sigma_{f(e_\alpha) \cdot e_\beta}(x) = \sigma_{f(e_\alpha \cdot e_\beta)}(x) = \sigma_{f(e_\beta) \cdot e_\alpha}(x) = \sigma_{f(e_\beta)}(x) \cdot \sigma_{e_\alpha}(x) = \sigma_{f(e_\beta)}$. To see that τ is continuous on

U note that if $x \in U$ then $x \in$ some $S(\sigma e_\alpha)$ and $\tau\big|_{S(\sigma e_\alpha)} = \sigma_{f(e_\alpha)}\big|_{S(\sigma e_\alpha)}$, so τ is continuous at x and $\tau \in \Gamma(U,\eta)$.

Choose $\tau' \in \Gamma(Z,\eta)$ such that $\tau'\big|_U = \tau$. Choose $t \in T$ such that $\sigma_t = \tau$. Define $f' \in \mathrm{Hom}_T(T,T)$ by $f'(s) = s.t$ for any $s \in T$. Then for any e_α, $\sigma_{f'(e_\alpha)} = \sigma_{e_\alpha t} = \sigma_{e_\alpha}\tau$. Thus if $x \in S(\sigma_{e_\alpha})$, $\sigma_{f'(e_\alpha)}(x) = \sigma_{e_\alpha}(x)\tau(x) = \sigma_{e_\alpha}(x)\sigma_{f(e_\alpha)}(x) = \sigma_{f(e_\alpha)}(x)$ and if $x \notin S(\sigma_{e_\alpha})$, $\sigma_{f'(e_\alpha)}(x) = \sigma_{e_\alpha}(x)\tau(x) = 0 = \sigma_{e_\alpha}(x)\cdot\sigma_{f(e_\alpha)}(x) = \sigma_{f(e_\alpha)}(x)$. Therefore $\sigma_{f'(e_\alpha)} = \sigma_{f(e_\alpha)}$ so $f'(e_\alpha) = f(e_\alpha)$. Since $\{e_\alpha\}$ generates I this establishes $f'\big|_I = f$. Thus T is λ-self-injective.

The converse was established in i).

iii) Since T is a Boolean ring each η_x for $x \in Z$ is a field satisfying the polynomial identity $X^2 - X = 0$. Thus each η_x is the two element field $\{0_x, 1_x\}$. In view of ii) and 3.6.i) it suffices to show that if Z has the λ-disjointness property then (Z,η) has the λ-extension property. Let Z have the λ-disjointness property.

Let $U = \bigcup_\alpha(U_\alpha)$ where the U_α are clopen in Z and $|\{U_\alpha\}| < \lambda$. Let $\tau \in \Gamma(U,\eta)$ be given. By the introductory remark, $x \in U$ implies that $\tau(x) = 0$ or 1 . For each α let $V_\alpha = \{x \in U_\alpha : \tau(x) = 0\}$ and $W_\alpha = \{x \in U_\alpha : \tau(x) = 1\}$. Then V_α and W_α are open complements in U_α . Thus they are clopen subsets of the space U_α where U_α has the relative Topology induced by Z . Thus, since the U_α are clopen in Z , so are the V_α and W_α . Let $V = \bigcup_\alpha(V_\alpha)$ and $W = \bigcup_\alpha(W_\alpha)$. Then V and W are λ-sets in Z such that $V \cap W = \phi$.

Thus, by hypothesis, $\overline{V} \cap \overline{W} = \phi$. Hence there exists a clopen set C such that $\overline{V} \subseteq C$ and $\overline{W} \cap C = \phi$. Let $\tau' = \psi_C$. Then $\tau' \in \Gamma(Z,\eta)$ is such that $\tau'|_U = \tau$.

Q.E.D.

The following lemma is crucial in applying the concept of the λ-disjointness property to investigate the structure of (X,K).

3.8) (Lemma) Let Z' have the λ-disjointness property. Then for any λ-sets U and V, $\overline{U \cap V} = \overline{U} \cap \overline{V}$.

<u>Proof:</u> Let $U = \bigcup_{\alpha}(U_\alpha)$ and $V = \bigcup_{\beta}(V_\beta)$ where the U_α and V_β are clopen in Z' and $|\{U_\alpha\}| < \lambda$ and $|\{V_\beta\}| < \lambda$. $\overline{U \cap V} \subseteq \overline{U} \cap \overline{V}$ is true in general. If $\overline{U} \cap \overline{V} = \phi$ there is nothing to prove. Suppose $x \in \overline{U} \cap \overline{V}$. Let F_x denote the set of all clopen neighborhoods of x. F_x is directed under $\supseteq$ and is a basis for the neighborhoods of x. Let $d = N_x \in F_x$. Observe that $U \cap N_x = \bigcup_{\alpha}(U_\alpha \cap N_x)$ and $V \cap N_x = \bigcup_{\beta}(V_\beta \cap N_x)$ so that $U \cap N_x$ and $V \cap N_x$ are λ-sets. Clearly $x \in \overline{(U \cap N_x)} \cap \overline{(V \cap N_x)}$ so that by the hypothesis there exists an element, say x_d, in $(U \cap N_x) \cap (V \cap N_x)$. $\{x_d : d \in F_x\}$ is a net on $U \cap V$ converging to x. Thus $x \in \overline{U \cap V}$. Hence $\overline{U \cap V} \supseteq \overline{U} \cap \overline{V}$ so

$$\overline{U \cap V} = \overline{U} \cap \overline{V} .$$

Q.E.D.

The foregoing lemmas are summed up in the following proposition about R and (X,k).

3.9) (Proposition) i) R is λ-self-injective $\Longleftrightarrow$ (X,k) has the λ-extension property.

ii) R is λ-self-injective $\rightarrow B(R)$ is λ-self-injective.

iii) $B(R)$ is λ-self-injective $\Longleftrightarrow$ X has the λ-disjointness property.

iv) $B(R)$ is λ-self-injective $\rightarrow \overline{U \cap V} = \bar{U} \cap \bar{V}$ whenever U and V are λ-sets in X.

v) If R is self-injective so is $B(R)$.

vi) $B(R)$ (as a Boolean ring) is λ-complete $\Longleftrightarrow$ X is λ-extremally disconnected.

vii) $B(R)$ is λ-complete $\rightarrow B(R)$ is λ-self-injective.

Proof: i) This is just a restatement of 3.7.ii).

ii) Let R λ-self-injective. Then by 3.7.i) (X,k) has the λ-extension property so that by 3.6.i) X has the λ-disjointness property. Since $B(B(R)) = B(R)$ it follows from the basic definitions (see 0.14) that $X = X(R) = X(B(R))$. Thus by 3.7.iii) $B(R)$ is λ-self-injective.

iii) As in ii), $X(B(R)) = X$. Thus this follows from 3.7.iii).

iv) As in ii), $X(B(R)) = X$. Let $B(R)$ be λ-self-injective. Then by iii) X has the λ-disjointness property. The result now follows from 3.8.

v) This follows from ii) since a ring is self-injective iff it is μ-self-injective for each cardinal μ.

vi) This result is standard. It follows via a trivial alteration of 22.4 in [9]. This alteration is required because the definition of λ-completeness is slightly different in this thesis.

vii) This follows from vi), 3.4, and 3.9.iii).

Q.E.D.

The next lemma gives some insight into the difficulties involved in studying the algebraic structure of the K_x's .

3.10) (Lemma) i) There is a (canonical) epimorphism $f_x : K_x \to k_x[[G^+]]$ for each $x \in X$.

ii) For $x \in X$ the epimorphism f_x is an isomorphism if x is an isolated point in X .

iii) If X is infinite there exists $x \in X$ such that f_x is not an isomorphism.

Proof: i) For each $x \in X$ define f_x by $f_x(\sigma_{\Sigma a_\alpha \cdot X^{g_\alpha}}) = \Sigma\sigma_{a_\alpha}(x) \cdot X^{g_\alpha}$. It follows from 3.2 that each f_x is well-defined.

ii) Let x be an isolated point in X . Then $\{x\}$ is a neighborhood of x . Suppose $f_x(\sigma_{\Sigma a_\alpha \cdot X^{g_\alpha}}) = 0$. Then each $\sigma_{a_\alpha}(x) = 0$ and, since $\{x\}$ is open, $x \in (\bigcap_\alpha(Z(\sigma_{a_\alpha})))^{\text{int.}}$. Thus by 3.1 $(\sigma_{\Sigma a_\alpha \cdot X^{g_\alpha}})(x) = 0$.

iii) Let X be infinite. First I must show that since X is infinite there exists an $\aleph_1$-set that is not closed. Since X is compact and infinite there exists $y \in X$ such that $\{y\}$ is not open. Thus $X - \{y\}$ is not closed. By 1.17.ii) find $\{V_\alpha\}$, a disjoint family of clopen sets such that $\overline{\bigcup_\alpha(V_\alpha)} = \overline{X - \{y\}} = X$. Since $X - \{y\}$ is not closed there are infinitely many $\{V_\alpha\}$. Pick $\{U_i\}$, a countably infinite sub-family of $\{V_\alpha\}$. Let $U = \bigcup_i(U_i)$. Then U is an $\aleph_1$-set. Since the U_i are open U is not compact and thus is not closed in X. Thus there exists $x \in X$ such that $x \in \overline{U}$ yet $x \notin$ any U_i. Let $g_{(\ )} : \aleph_1 \to G^+$ be a strictly monotonically increasing function and for each i let $e_i \in R$ be such that $\sigma_{e_i} = \psi_{U_i}$. Then let $s = (\Sigma e_i \cdot X^{g_i}) \in S$. Hence $f_x(x) = \Sigma \psi_{U_i}(x) \cdot X^{g_i} = 0$ yet $\sigma_s(x) \neq 0$ (by 3.1) since $x \in \overline{\bigcup_i U_i} = \overline{\bigcup_i (S(\sigma e_i))}$. Thus f_x is not an isomorphism.

Q.E.D.

If each f_x in 3.10 had been an isomorphism then each $k_x \ulcorner\ulcorner G^+ \urcorner\urcorner \cong K_x$ would immediately be an integral domain for which the lattice of ideals was linearly ordered under inclusion. (Each k_x is a field.) Thus each K_x would have been an integral domain with property e) so that S, by 1.17.ii), would have been a ring with properties a), b), and e). However, it will be possible to show that under certain circumstances each K_x is an integral domain (even though some f_x are not isomorphisms) for which the lattice of ideals is linearly ordered, and that under other circumstances S has none of properties a), b) and e).

The following concept will be used to investigate the algebraic structure of the K_x's.

3.11) (Definition) Let $s = \Sigma a_\alpha X^{g_\alpha} \in S$. For any $x \in \bigcup_\alpha (S(\sigma a_\alpha))$ let $\mathrm{val}_x(\sigma_s) =$ the least element in $\{g_\alpha : \sigma a_\alpha(x) \neq 0\}$.

3.12) (Lemma) Let $s = (\Sigma a_\alpha X^{g_\alpha}) \varepsilon S$, $t = (\Sigma b_\alpha X^{g_\alpha}) \varepsilon S$, $U = \underset{\alpha}{U}(S(\sigma_{a_\alpha}))$, $V = \underset{\alpha}{U}(S(\sigma_{b_\alpha}))$, $\sigma = \sigma_s$, $\tau = \sigma_t$, and $x \varepsilon U$.

i) $val_x(\sigma) = g_\beta$ implies that there exists N_x , an open neighborhood of x , such that $y \varepsilon N_x \to val_y(\sigma) \leq g_\beta$.

ii) If G is the group of integers then $val_x(\sigma) = g_\beta$ implies that there exists N_x , an open neighborhood of x , such that $y \varepsilon N_x \to val_y(\sigma) = g_\beta$.

iii) Let $z \varepsilon X$ and $g \varepsilon G^+$ be such that for each open neighborhood of z, N_z , there exists $u \varepsilon N_z \cap U$ such that $val_u(\sigma) = g$. Then $z \varepsilon U$ and $val_z(\sigma) = g$.

iv) Suppose that $x \varepsilon U \cap V$. Then

$$val_x(\sigma\tau) = val_x(\sigma) + val_x(\tau) \quad .$$

v) Let $z \varepsilon S(\tau)$. Then for any N_z , an open neighborhood of z , there exists $v \varepsilon N_z \cap V$.

vi) Let $z \varepsilon U$. Then $\sigma(z)$ is not a zero-divisor in K_z .

vii) Let $z \varepsilon S(\sigma)$. Then for each N_z , an open neighborhood of z , there exists $u \varepsilon N_z$ such that $\sigma(u)$ is not a zero divisor in K_u .

Proof: i) Let $val_x(\sigma) = g_\beta$. Then $\sigma a_\beta(x) \neq 0$. Since R is von Neumann regular find N_x , an open neighborhood of x , such that $y \varepsilon N_x \to \sigma a_\beta(x) \neq 0$. By the definition of $val_y(\sigma)$ this shows that $y \varepsilon N_x \to val_y(\sigma) \leq g_\beta$.

ii) Let G be the group of integers. Then s may be written in the form $s = \Sigma a_i \cdot X^i$ where the $i \varepsilon G^+$. Let $val_x(\sigma) = n$. Let $N_x = S(\sigma a_n) - (\underset{i<n}{U}(S(\sigma a_i)))$. Clearly N_x is a neighborhood of x such that $y \varepsilon N_x \to val_y(\sigma) = n$.

iii) By hypothesis $z \varepsilon \overline{S(\sigma a_\beta)}$. Since $S(\sigma a_\beta)$ is closed this yields $\sigma a_\beta(z) \neq 0$ so $z \varepsilon U$ and $val_z(\sigma) \leq g_\beta$. Suppose $val_z(\sigma) < g$. Then by i) obtain N_z , an open neighborhood of z , such that $y \varepsilon N_z \rightarrow val_y(\sigma) \leq val_z(\sigma) < g_\beta$. This contradicts the hypothesis that there exists $y' \varepsilon N_z$ such that $val_y(\sigma) = g_\beta$.

iv) Let $s \cdot t = \Sigma\, c_\mu \cdot X^{h_\mu}$. Let $val_x(\sigma) = g_{\alpha'}$ and $val_x(\tau) = g_{\beta'}$. Then

$$g_\lambda < g_{\alpha'} \rightarrow \sigma a_\lambda(x) = 0 \text{ and}$$

$$g_\lambda < g_{\beta'} \rightarrow \sigma b_\lambda(x) = 0 \text{ . Also,}$$

$$\sigma a_{\alpha'}(x) \neq 0 \text{ and } \sigma b_{\beta'}(x) \neq 0 \text{ . Note that}$$

$$\sigma c_\mu(x) = (\sigma_{\sum\limits_{g_\alpha+g_\beta=h_\mu} a\alpha\cdot b\beta})(x) = \sum_{g_\alpha+g_\beta=h_\mu} \sigma a_\alpha(x)\cdot\sigma b_\beta(x) \text{ .}$$

Thus $\sigma c_\mu(x) = 0$ for $h_\mu < g_{\alpha'} + g_{\beta'}$ and

$\sigma c_\mu(x) \neq 0$ for $h_\mu = g_{\alpha'} + g_{\beta'}$. Consequently

$val_x(\sigma\tau) = g_{\alpha'} + g_{\beta'} = val_x(\sigma) + val_x(\tau)$.

v) This is trivial since by 3.1) $S(\tau) = \overline{V}$.

vi) Since τ is aribtrary it suffices to show that if $\tau(z) \neq 0$ then $\sigma(z)\,\tau(z) \neq 0$.

Let $z \varepsilon S(\tau)$. Since $z \varepsilon U$ find $g \varepsilon G^+$ such that $val_z(\sigma) = g$. By i) find N_z , a neighborhood of z , such that $y \varepsilon N_z \rightarrow val_y(\sigma) \leq g$. Let N'_z be an arbitrary neighborhood of z . Then by v) there exists $y \varepsilon N_z \cap N'_z$ and $g' \varepsilon G^+$ such that $val_y(\tau) = g'$. Since $y \varepsilon N_z$ find $g'' \varepsilon G^+$ such that $val_y(\sigma) = g''$. Hence (by iv)

$\mathrm{val}_y(\sigma\tau) = g' + g''$ so that $(\sigma\cdot\tau)(y) \neq 0$. Thus $z \in \overline{S(\sigma\cdot\tau)}$ so $0 \neq (\sigma\cdot\tau)(z) = \sigma(z)\ \tau(z)$.

vii) Since $S(\sigma) = \overline{U}$ this follows immediately from vi).

Q.E.D.

3.13) (Definition) Let λ be a cardinal. G is a λ-group iff there does not exist a strictly monotonically increasing function $g_{(\ )} : \lambda \to G^{+}$.

Remark: The additive group of integers is an $\aleph_1$-group but not an $\aleph_0$-group.

The foregoing concepts will now be used to determine exactly when the $K_{x's}$ are integral domains. This will be used immediately to determine exactly when S has properties c) or d).

3.14) (Proposition) Suppose λ is the smallest cardinal such that G is a λ-group. Then each K_x is an integral domain iff $B(R)$ is λ-self-injective.

Proof: In view of 3.9.iii) it suffices to show that each K_x is an integral domain iff X has the λ-disjointness property.

Suppose that X has the λ-disjointness property. Let $s = (\Sigma a_\alpha X^{g_\alpha})$ and [illegible] X^{g_α} [illegible]. Let $s\cdot t = \Sigma c_\mu X^{h_\mu}$. To show that each K_x is an integral domain it suffices to show $S(\sigma_s) \cap S(\sigma_t) \subseteq S($ [illegible] $)$. [illegible] it suffices to show that $\bigcup_\alpha \overline{U(S(\sigma a_\alpha))} \cap \bigcup_\alpha \overline{U(S(\sigma b_\alpha))} \subseteq \bigcup_\mu \overline{U(S(\sigma c_\mu))}$. However, by remark f) and the hypothesis [illegible] λ $\bigcup_\alpha U($ [illegible] $)$ and $\bigcup_\alpha U(S(\sigma b_\alpha))$ are λ-sets so that, by 3.8, $\overline{\bigcup_\alpha U(S(\sigma a_\alpha)) \cap \bigcup_\alpha U(S(\sigma b_\alpha))} = \overline{\bigcup_\alpha U(S(\sigma a_\alpha))} \cap \overline{\bigcup_\alpha U(S(\sigma b_\alpha))}$. Thus it suffices to show that $U(S(\sigma a_\alpha)) \cap U(S(\sigma b_\alpha)) \subseteq U(S(\sigma c_\mu))$. But this follows immediately from 3.12.iv).

Conversely, suppose that each K_x is an integral domain yet X does not have the λ-disjointness property. That is to say there exists a cardinal $\lambda' < \lambda$ and families of clopen sets, $\{U_\alpha : \alpha < \lambda'\}$ and $\{V_\alpha : \alpha < \lambda'\}$ such $U \cap V = \phi$ yet $\overline{U} \cap \overline{V} \neq \phi$ where $U = \bigcup_{\alpha<\lambda'} (U_\alpha)$ and $V = \bigcup_{\alpha<\lambda'} (V_\alpha)$. In view of the hypothesis on G there exists a strictly monotonically increasing function $g_{(\ )} : \lambda' \to G^+$. For each $\alpha < \lambda'$ let e_α and $f_\alpha \in R$ be such that $\psi_{U_\alpha} = \sigma_{e_\alpha}$ and $\psi_{V_\alpha} = \sigma_{f_\alpha}$. Let $s = \Sigma\, e_\alpha X^{g_\alpha}$ and $t = \Sigma\, f_\alpha X^{g_\alpha}$. Let $x \in \overline{U} \cap \overline{V}$. Since $U \cap V = \phi$ it is easily seen that $\sigma_s(x)\, \sigma_t(x) = 0$. Since $x \in \overline{U}$ and $x \in \overline{V}$ it follows from 3.2 that $\sigma_s(x) \neq 0$ and $\sigma_t(x) \neq 0$. Hence K_x is not an integral domain. This contradicts the assumption that K_x is an integral domain. Thus X has the λ-disjointness property.

Q.E.D.

3.15) (Proposition) Let λ be the smallest cardinal such that G is a λ-group. Then the following are equivalent:

i) S has property c).

ii) S has property d).

iii) $B(R)$ is λ-complete.

Proof: ii) $\to$ i): This is obvious.

iii) $\to$ ii): Suppose $B(R)$ is λ-complete. Then by 3.9.vi) X is λ-extremally disconnected. Hence by 3.4 X has the λ-disjointness property so that by 3.14 each K_x is an integral domain. Let $s = (\Sigma\, a_\alpha \cdot X^{g_\alpha}) \in S$. Then by the hypothesis on λ $|\{g_\alpha\}| < \lambda$ so that $U = \underset{\alpha}{U}(S(\sigma_{a_\alpha}))$ is a λ-set. Hence $S(\sigma_s) = \overline{U}$ is clopen in X. Thus, by 1.1, S has property d).

i) $\to$ iii): Suppose that X is not λ-extremally disconnected (i.e. $B(R)$ is not λ-complete) yet S has property c). Since X is not extremally disconnected find $\{U_\alpha : \alpha < \lambda'\}$, a family of clopen sets indexed by some cardinal $\lambda' < \lambda$, such that there exists $x \in \overline{U} \cap (X - \overline{U})$ where $U = \underset{\alpha<\lambda'}{U} (U_\alpha)$. For each $\alpha < \lambda'$ let $e_\alpha \in S$ be such that $\psi_{U_\alpha} = \sigma_{e_\alpha}$. By the hypothesis on λ there exists a strictly monotonically increasing function $g_{(\ )} : \lambda' \to G^+$. Let $s = \sum_{\alpha<\lambda'} e_\alpha X^{g_\alpha} \in S$. I shall show that l.ann (s) is not finitely generated as an ideal in S. Suppose that there exists $t_1, \ldots t_n \in S$ such that l.ann (s) $= \sum_{i=1}^{n} S \cdot t_i$. Let N_x be a neighborhood of x. Since $x \in (X - \overline{U})$ there exists $y \in N_x$ such that $y \in (X - \overline{U})$. That is to say there exists $y \in N_x$ such that $\sigma_s(y) = 0$. Thus $w \in$ l.ann (s) where W is a clopen set chosen such that $y \in W$ and $\sigma_s\big|_W = 0$ and w is chosen such that $\psi_W = \sigma_w$. Hence, there exists $i'(1 \le i' \le n)$ such that $\mathrm{val}_y(\sigma_{t_{i'}}) = 0$. Since there are only finitely many $t_{i'}$s there exists $j(1 \le j \le n)$ such that for each neighborhood, N'_x, there exists

$y' \in N'_x$ such that $\mathrm{val}_{y'}(\sigma_{t_j}) = 0$. Thus by 3.12.iii) $\mathrm{val}_x(\sigma_{t_j}) = 0$. Hence (by 3.12.vi) $\sigma_{t_j}(x)$ is not a zero divisor in K_x . Since $x \in \overline{U}$, $\sigma_s(x) \neq 0$. This is a contradiction since $\sigma_{o_s}(x) = \sigma_{t_j \cdot s}(x) = \sigma_{t_j}(x) \cdot \sigma_s(x)$. Therefore 1.ann (s) is not finitely generated.

Q.E.D.

The following lemma will be used to investigate the ideal structure of the K_x's .

<u>3.16</u>) (Lemma) Let λ be an ordinal and let $s = (\Sigma\, a_\alpha X^{g_\alpha}) \in S$ and $x \in X$ be such that $\mathrm{val}_x(\sigma_s) = 0_G$. That is to say $g_o = 0_G$ and $\sigma_{a_o}(x) \neq 0_x$. Then there exists $t = (\Sigma\, b_\mu X^{h_\mu}) \in S$ such that

$$\sigma_s(y) \cdot \sigma_t(y) = 1y \quad \text{for all } y \in S(\sigma_{s_o}) .$$

<u>Proof</u>: Let $N = S(\sigma_{a_o})$. Since R is von Neumann regular this is clopen in X . Define $t = (\Sigma\, b_\mu X^{h_\mu}) \in S$ as follows: Let $\sigma_{b_o}(y) = (\sigma_{a_o}(y))^{-1}$ for $y \in N$ and $\sigma_{b_o}(y) = 0_y$ for $y \notin N$. Let $h_o = 0_G$. For some ordinal δ assume that σ_{b_μ} and h_μ are defined for all ordinals $\mu < \delta$ in such a way that:

i) $(\sum_{\mu<\delta} b_\mu X^{h_\mu}) \in S$.

ii) The smallest $g \in G^+ - \{0\}$ such that $\sum_{h_\mu+g_\alpha=g} \sigma_{b_\mu}(y')\sigma a_\alpha(y') \neq 0$ for some $y' \in N$ is an upper bound to $\{h_\mu : \mu < \delta\}$.

If no such g exists, since $\{h_\mu : \mu < \delta\} + \{g_\alpha : \alpha < \lambda\}$ is a well-ordered subset of G, it follows that:

iii) $\sigma_{b_o}(y)\sigma_{a_o}(y) = 1$ for all $y \varepsilon N$; and

iv) $\Sigma\, \sigma_{b_\mu}(y)\cdot\sigma_{a_\alpha}(y) = 0$ for all $y \varepsilon N$ and any $g > 0$.

In this case let $t = \sum_{\mu<\delta} b_\mu \cdot X^{h_\mu}$. It follows from iii), iv), 3.2 and the fact that N is open that

$$\sigma_t(y)\cdot\sigma_s(y) = 1y \quad \text{for all} \quad y \varepsilon N \ .$$

If such a g does exist let

$$\sigma_{b\delta}(y) = \Big(- \sum_{\substack{h_\mu+g_\alpha=g \\ \text{and } \mu<\delta}} \sigma_{b_\mu}(y)\sigma_{a_\alpha}(y)\Big)\cdot(\sigma_{a_o}(y))^{-1}$$

for $y \varepsilon N$ and $\sigma_b(y) = 0_y$ for $y \notin N$. Then $\sum_{\mu<\delta+1} b_\mu X^{h_\mu}$ satisfies i) and ii) above with $\delta + 1$ in place of δ . Continue by transfinite induction until the g of ii) does not exist.

3.17) (Corollary) Let $s = (\sum_\alpha a X^{g_\alpha})$, $t = (\sum_\alpha b X^{g_\alpha}) \varepsilon S$, g_{α_1}, $g_{\alpha_2} \varepsilon G$, $x \varepsilon X$, and the clopen neighborhood of x, N_x , be such that $y \varepsilon N_x \rightarrow \mathrm{val}_y(\sigma_s) = g_{\alpha_1} \cdot g_{\alpha_2} = \mathrm{val}(\sigma_t)$. Then there exists $s' \varepsilon S$ such that $\sigma_{s'}(x)\sigma_s(x) = \sigma_t(x)$.

Proof: Since $\alpha < \alpha_1 \to \sigma_{a_\alpha}(y) = 0$ for $y \in N_x$ we may (by 3.1) assume without loss of generality that $a_\alpha = 0$ for $\alpha < \alpha_1$. Similarly we may assume that $b_\alpha = 0$ whenever $\alpha < \alpha_1$. Let $s'' = \Sigma\, a_\alpha \cdot X^{(g_\alpha - g_{\alpha_1})}$. Then $val_x(\sigma_{s''}) = 0$. Thus, by 3.16, find $t' \in S$ such that $\sigma_{t'}(x)\sigma_{s''}(x) = 1_x$. Let $s' = (\Sigma\, b_\alpha \cdot X^{g_\alpha - g_{\alpha_1}}) \cdot t'$. It may then be verified that $\sigma_{s'}(x)\sigma_s(x) = \sigma_t(x)$.

Q.E.D.

The following result is standard.

3.18) (Corollary) Let R be a field. Then S is an integral domain and the lattice of ideals in S is linearly ordered. That is to say, if I and J are ideals in S then either $I \subseteq J$ or $J \subseteq I$.

Proof: Since R is a field $B(R) = \{0,1\}$ so $X(R) = \{x\}$ where $x = \{0\}$. Then $k_x = (R/R\cdot 0) = R$ and $K_x = (S/S\cdot 0) \cong S$. We shall show that K_x is an integral domain with a linearly ordered lattice of ideals. Since X contains only one point it clearly has the disjointness property so that by 3.14 and 3.9 it is an integral domain. To show that the lattice of ideals in K_x is linearly ordered it suffices to show that for any $s,t \in S$ such that $\sigma_s(x)$ and $\sigma_t(x) \neq 0$, either: i) $K_x\sigma_s(x) \subseteq K_x\sigma_t(x)$ or

ii) $K_x\sigma_t(x) \subseteq K_x\sigma_s(x)$.

Since X is discrete there exists g_α and $g_\beta \varepsilon G$ such that $val_x(\sigma_s) = g_\alpha$ and $val_x(\sigma_t) = g_\beta$. Suppose $g_\alpha \leq g_\beta$. Then by 3.17, since $\{x\}$ is open, there exists $s' \varepsilon S$ such that $\sigma_{s'}(x) \cdot \sigma_s(x) = \sigma_t(x)$. Thus i) holds. If $g_\beta \leq g_\alpha$ then similarly ii) would hold.

Q.E.D.

3.19) (Theorem) Let R be a finite direct sum of fields. Then S has properties a), b), c), d), and e), S is semi-hereditary and $w.gl.dim(S) \leq 1$.

Proof: Since R is a finite direct sum of fields $B(R)$ and hence $X = X(R)$ are finite. Since X is Hausdorff this implies that each $x \varepsilon X$ is isolated. Thus by 3.10 each $K_x \cong k_x[[G^+]]$. Since R is von Neumann regular each k_x is a field so by 3.18 each K_x is an integral domain for which the lattice of ideals is linearly ordered. Such domains have property e). Since X is finite it is extremally disconnected. The theorem now follows from 3.9, 3.15, and 1.7.

Q.E.D.

The next lemma will allow 1.13 to be used to investigate the algebraic structure of S .

3.20) (Lemma) i) S is semi-prime.

ii) For each $x \varepsilon X$ K_x is a local ring.

Proof: i) This is obvious since R is semi-prime.

ii) Let $x \varepsilon X$ and $s = (\Sigma\, a_\alpha X^{g_\alpha})$ and $t = (\Sigma\, b_\alpha X^{g_\alpha}) \varepsilon S$ be such that $\sigma_s(x) + \sigma_t(x) = 1_x$. Then $\sigma_{a_o}(x) + \sigma_{b_o}(x) = 1_x$ where $g_o = 0_G$ so either $\sigma_{a_o}(x) \neq 0_x$ or $\sigma_{b_o}(x) \neq 0_x$. Suppose without loss of generality that $\sigma_{a_o}(x) \neq 0$. Then $val_x(\sigma_a) = 0_G$ so that by 3.16 $\sigma_a(x)$ is a unit in K_x . Thus K_x is a local ring.

Q.E.D.

§4 The structure of R when $R[[G^+]]$ is not coherent and $G \neq I$, the integers.

In this section R will denote a commutative von Neumann regular ring, G a linearly ordered abelian group that is not isomorphic to the additive group of integers, and S the ring $R[[G^+]]$. (X,K) will denote S° and as in §3 k , a subsheaf of K , is chosen such that $\eta_S|_R : R \to \Gamma(X,k)$ is an isomorphism.

4.1) (Lemma) R is a finite direct sum of fields iff X is finite.

Proof: Let R be a finite direct sum of fields. Then clearly B(R) is finite so that X = X(R) = the set of maximal proper ideals of B(R) is also finite.

Now let $X = \{x_1, \ldots x_n\}$ be finite. Thus each $\{x_i\}$ is clopen in X so that it is easily checked that the map $R \to \sum_{i=1}^{n} k_{x_i}$ given by $r \to \sum_{i=1}^{n} \sigma_r(x_i)$ is a ring isomorphism.

Q.E.D.

The structure of S when R is a finite direct sum of fields was discussed in 3.19. Therefore it will be assumed for the rest of this section that X is infinite. It will be shown that S is not semi-hereditary, w.gl.dim(S) > 1 , and S has none of properties a), b), and e).

4.2) (Lemma) There exists $\{U_i : i \in \aleph_0\}$, a disjoint family of clopen sets in X , such that there exists $x \in \overline{U} - U$ where $U = \bigcup_i (U_i)$.

Proof: This was established in the proof of 3.10.iii).

Q.E.D.

4.3) (Lemma) There exists $g_{( )} : \aleph_0 \to G^+$, a strictly monotonically increasing function, and $g \in G$ such that each $g_i < g$.

Proof: Either $G^+ - \{0\}$ contains a least element or it does not. First suppose that it does contain a least element, say g_0. Then, by the group structure of G, $T_h = \{g \in G : g > h\}$ contains a least element for any $h \in G$. For any $i \in \aleph_0$ let g_{i+1} be the least element in T_{g_i}. The map $g_{( )} : \aleph_0 \to G^+$ is not cofinal for if it were then $\pm i \to \pm g_i$ would be an isomorphism between the additive group of integers and G. By construction the map $i \to g_i$ is strictly monotonically increasing.

Now suppose that $G^+ - \{0\}$ does not contain a least element. Pick g and $g'_0 \in G$ such that $0 < g'_0 < g$. For any $i \in \aleph_0$ assume that g'_i is defined and pick g'_{i+1} such that $0 < g'_{i+1} < g'_i$. Let $g_i = g - g'_i$. Then each $g_i < g$ and the map $i \to g_i$ is strictly monotonically increasing.

Q.E.D.

For the rest of this section let g, the g_i, U, the U_i, and x be as described in 4.2 and 4.3. For each $i \in \aleph_0$ let $e_i \in R$ be such that $\sigma_{e_i} = \psi_{U_i}$. Let $s_1 = \Sigma\, e_i \cdot X^{g_i}$ and $s_2 = X^g$. Let $I = S.s_1$, and $J = S.s_2$. Then I and J are principal ideals in S. We shall show that $I \cap J$ is not finitely generated. Lemma 4.2 asserts that since X is infinite it contains a point x that, in a fairly particular way, fails to be isolated. I shall use this fact to show that if $I \cap J$ is finitely generated then there exists an element $t \in S$ such that some infinite subset of $\{g - g_i\}$ is a subset of its set of indices. Since such a subset contains no least element this would contradict $t \varepsilon S$.

4.4) (Lemma) $I \cap J$ is not finitely generated.

Proof: Suppose that $I \cap J$ is generated by $\{t_1, \ldots t_n\} \subseteq S$. By 1.3 for each $x \varepsilon X$ $(I \cap J)_x = I_x \cap J_x$ so that $\{\sigma_{t_1}(x), \ldots \sigma_{t_n}(x)\}$ generates $I_x \cap J_x$. Suppose that $y \varepsilon$ some U_i. Then $\mathrm{val}_y(\sigma s_1) = g_i$ $g = \mathrm{val}_y(\sigma s_2)$ and U_i is a neighborhood of y so that by 3.1 and 3.17

$$I_y \cap J_y = (K_y \sigma_{s_1}(y)) \cap (K_y \sigma_{s_2}(y)) = K_y \sigma_{s_2}(y) .$$

Hence there exists an i' $(1 \leq i' \leq n)$ such that $\mathrm{val}_y(\sigma_{t_{i'}}) = g$. Since $x \varepsilon \overline{U}$ and there are only finitely many $t_{i's}$ there exists j $(1 \leq j \leq n)$ such that for each neighborhood N_x of x there exists $y \varepsilon N_x$ such that $\mathrm{val}_y(\sigma_{t_j}) = g$. For each positive integer m $\bigcap_{i=1}^{m} (X-U_i)$ is a neighborhood of x so there exists $i_m \geq m$ and $y_m \varepsilon U_{i_m}$ such that $\mathrm{val}_{y_m}(\sigma_{t_j}) = g$. But $t_j = t.s_1$ for some $t \varepsilon S$ since $t_j \varepsilon I = S.s_1$. Then

$$\mathrm{val}_{y_m}(\sigma_t) = \mathrm{val}_{y_m}(\sigma_{t_j}) - \mathrm{val}_{y_m}(\sigma_{s_1}) = g - g_{i_m} \quad \text{for each } m .$$

Thus $\{g - g_{i_m} : m \varepsilon \aleph_0\}$ is a subset of the set of indices for t. Since it has the inverse order of an infinite well ordered-set it is not well-ordered. This contradicts $t \varepsilon S$ and thus establishes that $I \cap J$ is not finitely generated.

Q.E.D.

4.5) (Theorem) S is not semi-hereditary, $\text{w.gl.dim}(S) > 1$, and S has none of properties a), b) and e).

Proof: It follows from 4.4 that S has neither property a) nor property b). The rest follows from 1.13.

Q,E.D.

§5

A condition for $R\lceil\lceil X\rceil\rceil$ to be coherent

In this section R will denote a commutative von Neumann regular ring that is $\aleph_1$-self-injective and S will denote $R\lceil\lceil X\rceil\rceil$. Let (X,K) denote S° and choose k, a subsheaf of K, such that $\lambda_S\big|_R : R \to \Gamma(X,k)$ is a ring isomorphism.

Since R is $\aleph_1$-self-injective it follows from 3.9 that (X,k) has the $\aleph_1$-extension property, X has the $\aleph_1$-disjointness property and $B(R)$ is $\aleph_1$-self-injective. Since the additive group of integers is an $\aleph_1$-group but not an $\aleph_0$-group it follows from 3.14 that each K_x is an integral domain. These facts and the results of §1 will be used to show that for each $x \in X$ the lattice of ideals of K_x is linearly ordered so that K_x has property e). It will then follow from 1.7 that w.gl.dim$(S) \leq 1$ and S has properties a), b), and e).

5.1) (Lemma) i) (X,K) has the $\aleph_1$-extension property.

ii) (X,K) has the unique $\aleph_1$-extension property.

Proof: i) Let U be an $\aleph_1$-set and let $\tau \in \Gamma(U,K)$ be given. Let $i \geq 0$ be a fixed but arbitrary integer. Define $\tau_i : U \to k$ as follows: For $x \in X$ find $s_x = (\Sigma\, a_{j,x}\cdot X^j) \in S$ such that $\tau(x) = \sigma_{s_x}(x)$ and let $\tau_i(x) = (\sigma_{a_{i,x}})(x)$. It follows from 3.1 that $\tau_i(x)$ is independant of the particular $s_x \in S$ such that $\tau(x) = \sigma_{s_x}(x)$.

Temporarily fix arbitrary $x \in U$. To see that each τ_i is continuous at x note that since σ_{s_x} is continuous at x

there exists $N_x \subseteq U$, a neighborhood of x , such that $y \in N_x \rightarrow \tau(y) = \sigma_{s_x}(y)$. Hence $y \in N_x \rightarrow \tau_i(y) = (\sigma_{a_{i,x}})(y)$. Since $\sigma_{a_{i,x}}$ is continuous at x so is τ_i . Since (X,k) has the $\aleph_1$-extension property find $a_i \in R$ such that $\sigma_{a_i}\big|_U = \tau_i$. Perform the above construction for each integer $i \geq 0$. Let $s = \Sigma\, a_i \cdot X^i$.

Now I shall show that $\sigma_s\big|_U = \tau$. Fix arbitrary $x \in X$ and choose $N_x \subseteq U$ a neighborhood of x such that $y \in N_x \rightarrow \tau(y) = \sigma_{s_x}(y)$. Thus $y \in N_x \rightarrow (\sigma_{a_{i,x}})(y) = (\sigma_{a_{i,y}})(y)$ for all i . Thus $\sigma_{a_i}\big|_{N_x} = \tau_i\big|_{N_x} = (\sigma_{a_{i,x}})\big|_{N_x}$ for all i . Hence by 3.2 $\tau(x) = \sigma_s(x)$. Since $x \in U$ was arbitrary this establishes $\sigma_s\big|_U = \tau$. Thus (X,K) has the $\aleph_1$-extension property.

ii) Note that for any $\sigma \in \Gamma(X,K)$ $S(\sigma) = \overline{V}$ for some $\aleph_1$-set V .

Let U be an $\aleph_1$-set in X and let $\sigma_s, \sigma_t \in \Gamma(X,K)$ be such that $\sigma_s\big|_U = \sigma_t\big|_U$. Find an $\aleph_1$-set V such that $S(\sigma_s - \sigma_t) = \overline{V}$. $U \cap V = \phi$ since $\sigma_s\big|_U = \sigma_t\big|_U$. Therefore $\overline{U} \cap \overline{V} = \phi$ since X has the $\aleph_1$-disjointness property. Thus $\sigma_s\big|_{\overline{U}} = \sigma_t\big|_{\overline{U}}$. In view of i) this shows that (X,K) has the unique $\aleph_1$-extension property.

Q.E.D.

5.2) (Lemma) i) Each K_x is an integral domain.

ii) For each $x \in X$ the lattice of ideals in K_x is linearly ordered.

Proof: i) This was established in the opening remarks of this section.

ii) It suffices to show that for any $s,t \in S$ and $x \in X$ either $K_x\sigma_s(x) \subseteq K_x\sigma_t(x)$ or $K_x\sigma_t(x) \subseteq K_x\sigma_s(x)$. Fix arbitrary $s,t \in S$ and let $I = S.s + S.t$. It clearly suffices to show that for any $x \in (S(\sigma_s) \cap S(\sigma_t))$ I_x is either generated by $\sigma_s(x)$ or by $\sigma_t(x)$.

Suppose $s = \Sigma\, a_i \cdot X^i$, $t = \Sigma\, b_i \cdot X^i$, $V = \bigcup_i (S(\sigma_{b_i}))$ and $U = \bigcup_i (S(\sigma_{a_i}))$. U and V are $\aleph_1$-sets. For any integers $m \geq 0$ and $n \geq 0$ let $C_{m,n} = (S(\sigma_{a_m}) - \bigcup_{0 \leq i < m} (S(\sigma_{a_i}))) \cap (S(\sigma_{b_n}) - \bigcup_{0 \leq j < n} (S(\sigma_{b_j})))$. Since R is von Neumann Regular each $C_{m,n}$ is clopen in X. Let $W_s = \{x \in U \cap V : val_x(\sigma_s) < val_x(\sigma_t)\}$ and $W_t = \{x \in U \cap V : val_x(\sigma_t) \leq val_x(\sigma_s)\}$. Then $W_s = \bigcup_{0 \leq m < n < \infty} (C_{m,n})$ and $W_t = \bigcup_{0 \leq n \leq m < \infty} (C_{m,n})$ so that W_s and W_t are $\aleph_1$-sets. Clearly $W_s \cap W_t = \phi$ and $W_s \cup W_t = U \cap V$. Thus $\overline{W}_s \cap \overline{W}_t = \phi$ (X has the $\aleph_1$ disjointness property) and by 3.8 $\overline{W}_s \cup \overline{W}_t = \overline{W_s \cup W_t} = \overline{U \cap V} = \overline{U} \cap \overline{V} = S(\sigma_s) \cap S(\sigma_t)$. Let $x \in S(\sigma_s) \cap S(\sigma_t)$. Then x belongs to exactly one of $\overline{W}_s$ or $\overline{W}_t$. First suppose $x \in \overline{W}_s$. For any $y \in W_s$ $y \in C_{m,n}$ for some integers m and n such that $0 \leq m < n$. $C_{m,n}$ is a neighborhood of y such that $y' \in C_{m,n}$ implies that $\mathrm{val}_{y'}(\sigma_s) = m < n = \mathrm{val}_{y'}(\sigma_t)$. Thus by 3.17 there exists $s' \in S$ such that $\sigma_{s'}(y)\sigma_s(y) = \sigma_t(y)$. Hence $\sigma_s(y)$ generates $I_y = K_y\sigma_s(y) + K_y\sigma_t(y)$ for any $y \in W_s$. Since W_s is an $\aleph_1$-set it can, by 1.17, be expressed as union of a countable disjoint family of clopen sets. Since $x \in \overline{W}_s$ it now follows from 1.16 that $\sigma_s(x)$ generates I_x . Now suppose $x \in \overline{W}_t$. It similarly follows that $\sigma_t(x)$ generates I_x .

Q.E.D.

5.3) (Theorem) S has properties a), b), and e), and $\text{w.gl.dim}(S) \leq 1$. Also, the following properties are equivalent:

i) $B(R)$ is $\aleph_1$-complete.

ii) S has property c)

iii) S has property d)

iv) S is semi-hereditary

v) S is coherent.

Proof: By 5.2 each K_x is an integral domain whose lattice of ideals is linearly ordered. Such rings have property e). Thus by 1.7 S has properties a), b), and e) and $\text{w.gl.dim}(S) \leq 1$.

i) <—> ii) <—> iii): Since the additive group of integers is an $\aleph_1$-group but not an $\aleph_1$-group this follows from 3.15.

iv) —> v): This is trivial since every semi-hereditary ring is coherent.

iii) —> iv): This follows from 1.7.iii).

v) —> ii): This is trivial since every coherent ring has property c).

Q.E.D.

§6 A necessary condition for $R[[X]]$ to be coherent

In this section R will denote a commutative von Neumann regular ring that is not $\aleph_1$ self-injective and S will denote $R[[X]]$. (X,K) will denote S° and k the subsheaf of K such that $\lambda_S\big|_R : R \to \Gamma(X,k)$ is a ring isomorphism. Since R is not $\aleph_1$-self-injective it follows from 3.9 that there exists U, an $\aleph_1$-set, and $\tau \in \Gamma(U,k)$ such that τ can not be extended to a global section. U and τ will retain this meaning throughout §6.

6.1) (Lemma) i) There does not exist $\tau' \in \Gamma(\overline{U},k)$ such that $\tau'\big|_U = \tau$.

ii) Suppose that $Y \subseteq \overline{U}$ is a set of points such that for any $x \in Y$ there exists $\tau_x \in \Gamma(U \cup \{x\},k)$ such that $\tau_x\big|_U = \tau$.

Then there exists $\tau' \in \Gamma(U \cup Y,k)$ such that $\tau'\big|_U = \tau$.

iii) There exists $x \in \overline{U} - U$ such that there does not exist $\tau_x \in \Gamma(U \cup \{x\},k)$ such that $\tau_x\big|_U = \tau$.

Proof: i) Suppose that such a τ' exists. Then since $\overline{U}$ is closed in X there exists $\tau'' \in \Gamma(X,k)$ such that $\tau''\big|_{\overline{U}} = \tau'$. Hence $\tau''\big|_U = \tau$. This contradicts our choice of τ.

ii) Note that if $x \in \overline{U}$ and $\tau_x, \tau'_x \in \Gamma(U \cup \{x\},k)$ are such that $\tau_x\big|_U = \tau'_x\big|_U$ then $\tau_x = \tau'_x$ since $x \in \overline{U}$ and k is Hausdorff. (It follows from 0.15 and 1.1) that k is Hausdorff).

Define $\tau' : U \cup Y \to k$ by $\tau'(u) = \tau(u)$ for $u \in U$ and

$\tau'(x) = \tau_x(x)$ for $x \in Y - U$. Clearly $\tau'|_U = \tau$. It must now be shown that τ' is continuous. If $x \in U$ then τ' is continuous at x because τ is and U is open in X . Let $x \in Y - U$. Find $a_x \in R$ such that $\tau'(x) = \sigma_{a_x}(x)$. Thus $\tau_x(x) = \tau'(x) = \sigma_{a_x}(x)$ has a neighborhood basis consisting of sets of the form $\sigma_{a_x}(U_x)$ where U_x is a clopen neighborhood of x . Thus, since τ_x is continuous, there is a clopen neighborhood of x, N_x , such that $\tau_x(N_x \cap (U \cup \{x\})) \subseteq \sigma_{a_x}(N_x)$. Hence $y \in N_x \cap U$ implies that $\tau(y) = \tau_x(y) = \sigma_{a_x}(y)$. Then for any $y \in Y \cap N_x$ $\sigma_{a_x}|_{((U \cup \{y\}) \cap N_x)}$ is a continuous map extending $\tau|_{U \cap N_x}$. Since $y \in \overline{U \cap N_x}$ and k is Hausdorff it follows that $\tau'(y) = \tau_y(y) = \sigma_{a_x}(y)$. Since σ_{a_x} is continuous this establishes that τ' is continuous at x . Thus $\tau' \in \Gamma(U \cup Y,k)$.

iii) Suppose that for each $x \in \overline{U}$ there exists $\tau_x \in \Gamma(U \cup \{x\},k)$ such that $\tau_x|_U = \tau$. Then apply ii) with $Y = \overline{U}$ and obtain $\tau' \in \Gamma(\overline{U},k)$ such that $\tau'|_U = \tau$. This contradicts i). Thus the lemma is established.

Q.E.D.

For the rest of this section let $x \in \overline{U}$ be such that τ can not be extended to an element in $\Gamma(U \cup \{x\},k)$. Since U is an $\aleph_1$-set find (by 1.17) $\{U_i : i \geq 0$ is an integer$\}$, a disjoint family of clopen sets, such that $U = \bigcup_i (U_i)$. For each i

find $e_i \in R$ such that $\sigma_{e_i} = \psi_{U_i}$ and $r_i \in R$ such that $\sigma_{r_i}(x) = \tau(x)$ when $x \in U_i$ and $\sigma_{r_i}(x) = 0$ when $x \notin U_i$. The e_i and r_i exist since the U_i are clopen. Let $s_1 = \Sigma\, r_i X^i$ and $s_2 = \Sigma\, e_i X^i$. Let $I = S.s_1$ and $J = S.s_2$. I and J are principal ideals. It will be shown that $I \cap J$ is not finitely generated by showing that $I_x \cap J_x = (I \cap J)_x$ is not.

6.2) (Lemma) Let $c_1, c_2 \in S$, the integer $n \geq 0$, and $y \in \overline{U}$ be such that $\sigma_{c_1}(y)\sigma_{s_1}(y) = \sigma_{c_2}(y)\sigma_{s_2}(y)$ and $\mathrm{val}_y(\sigma_{c_1}) = n$. Then there exists $\tau_y \in \Gamma(U \cup \{y\}, k)$ such that $\tau_y\big|_U = \tau$.

Proof: Find N_y, a neighborhood of y, such that $\sigma_{c_1}(y')\sigma_{s_1}(y') = \sigma_{c_2}(y')\sigma_{s_2}(y')$ and $\mathrm{val}_{y'}(\sigma_{c_1}) = n$ for any $y' \in N_y$. Let $c_1 = \Sigma\, a_i X^i$ and $c_2 = \Sigma\, b_i X^i$. Let $y' \in U \cap N_x$ be arbitrary but fixed. Find m such that $y' \in U_m$. Since U_m is a neighborhood of y' it follows from 3.1 and the definitions of the r_i and e_i that $\sigma_{s_1}(y') = (\sigma_{r_m X^m})(y')$ and

$\sigma_{s_2}(y') = (\sigma_{e_m X^m})(y') = \sigma_{X^m}(y')$. Hence

$$(\sigma_{\sum_i a_i r_m X^{i+m}})(y') = (\sigma_{c_1 s_1})(y') = (\sigma_{c_2 s_2})(y') = (\sigma_{\sum_i b_i \cdot X^{i+m}})(y') .$$

Thus $(\sigma_{a_n r_m})(y') = \sigma_{b_n}(y')$. Since $y' \in S(\sigma_{a_n})$ $(\mathrm{val}_{y'}(\sigma_{c_1}) = n)$ and $\tau(y') = \sigma_{r_m}(y')$ this establishes

$\tau(y') = \dfrac{\sigma_{b_n}(y')}{\sigma_{a_n}(y')}$. Since $y' \in U \cap N_y$ is arbitrary and

$y \in \overline{U \cap N_y}$ this implies that τ can be extended to a member

τ_y of $\Gamma(U \cup \{y\},k)$ by setting $\tau_y(u) = \tau(u)$ for $u \in U$

and $\tau_y(y) = \dfrac{\sigma_{b_n}(y)}{\sigma_{a_n}(y)}$.

Q.E.D.

6.3) (Lemma) $I_x \cap J_x \neq 0$.

Proof: Since $s_1 \cdot s_2 \in I \cap J$ therefore $\sigma_{s_1}(x)\sigma_{s_2}(x) = \sigma_{s_1 s_2}(x) \in (I \cap J)_x = I_x \cap J_x$. Thus it suffices to show that $\sigma_{s_1}(x)\sigma_{s_2}(x) \neq 0$. For any $y \in U$, $y \in$ some U_i so $\mathrm{val}_y(\sigma_{s_1}) = i = \mathrm{val}_y(\sigma_{s_2})$ and by 3.12.vi) $\sigma_{s_1}(y)\sigma_{s_2}(y) \neq 0$. Since $x \in \overline{U}$ and any section has closed support this establishes $\sigma_{s_1}(x)\sigma_{s_2}(x) \neq 0$.

Q.E.D.

Let $t_1 s_1, \ldots t_n s_1 \in I \cap J$ be such that each of $\sigma_{t_1}(x)\sigma_{s_1}(x), \ldots \sigma_{t_n}(x)\sigma_{s_1}(x)$ is none zero. It will be shown in lemmas 6.4 - 6.6 that the elements $\sigma_{t_1}(x)\sigma_{s_1}(x), \ldots \sigma_{t_n}(x)\sigma_{s_1}(x)$ do not generate $I_x \cap J_x$.

For each integer $m \geq 0$ let

$$V_m = \{y \in X : \text{for some } j (1 \leq j \leq n)\ \mathrm{val}_y(\sigma_{t_j}) = m + 1 \ \underline{\text{and}}\ \mathrm{val}_y(\sigma_{t_i}) \leq m \text{ holds for no } i\ (1 \leq i \leq n)\} .$$

Since there are only finitely many $t_{i's}$ it follows from 0.15 that each V_m is clopen.

Let $Y = U \cup (\overline{U} \cap (\underset{m}{U}(V_m)))$. It follows from 6.2 and 6.1.ii) that τ may be viewed as a member of $\Gamma(Y,k)$. Let $m \geq 0$ be an integer. Since $\overline{U} \cap V_m$ is closed $\tau|_{\overline{U} \cap V_m}$ may be viewed as a global section. Pick $f_m \in R$ such that $(\sigma_{f_m})(x) = (\tau|_{\overline{U} \cap V_m})(x)$ for $x \in V_m$ and $(\sigma_{f_m})(x) = 0$ for $x \notin V_m$. Pick $v_m \in R$ such that $\sigma_{v_m} = \psi_{V_m}$. It may be verified that for any integer $i \geq 0$ and $m \geq 0$

$$\sigma_{f_m}\sigma_{e_i} = (\tau\psi_{V_m})\psi_{u_i} = \psi_{V_m}(\tau\psi_{u_i}) = \sigma_{v_m}\cdot\sigma_{r_i}$$

so that $f_m e_i = v_m r_i$. This may be used to show that $(\Sigma f_m X^m)(\Sigma e_i X^i) = (\Sigma v_m X^m)(\Sigma r_i X^i)$. Let $s_3 = (\Sigma f_m X^m)(\Sigma e_i X^i)$

. Then $s_3 \in I \cap J$. Thus $\sigma_{s_3}(x) \in I_x \cap J_x$.

It will be shown in lemma 6.5 that $\sigma_{s_3}(x)$ is not in the ideal generated by the $\sigma_{t_1}(x)\sigma_{s_1}(x), \ldots \sigma_{t_1}(x)\sigma_{s_1}(x)$.

6.4) (Lemma) Let N_x be a neighborhood of x . Then there exists an integer $m \geq 0$ such that $N_x \cap U \cap V_m \neq \phi$.

Proof: Suppose that for each m $N_x \cap U \cap V_m = \phi$. Let $M = \{y \in X : \mathrm{val}_y(\sigma_{t_i}) = 0$ for some i $(1 \leq i \leq n)\}$. Since there are only finitely many t_i's , M is clopen. It follows from 6.2 and the choice of x that $x \notin M$. Let $N'_x = (X-M) \cap N_x$. Then N'_x is a neighborhood of x such that $\mathrm{val}_y(\sigma_{t_i}) = m$ fails, for each integer $m \geq 0$ and $i (1 \leq i \leq n)$, to hold. Since $N'_x \cap U$ is open it thus follows from 3.12.v) that $N'_x \cap U \subseteq Z(\sigma_{t_i})$ for each i $(1 \leq i \leq n)$. Clearly $(X - \overline{U}) \cap N_x \subseteq Z(\sigma_{s_1})$. Thus $U \cap (X - \overline{U}) \cap N_x \subseteq Z(\sigma_{t_i}\sigma_{s_1})$ so that by 3.1 $\sigma_{t_i}(x)\sigma_{s_1}(x) = 0$ for each i $(1 \leq i \leq n)$. This is contrary to the hypothesis so the lemma is established.

Q.E.D.

<u>6.5)</u> (Lemma) $\sigma_{s_3}(x) \notin \sum_{i=1}^{n} K_x\sigma_{t_i}(x)\sigma_{s_1}(x)$.

<u>Proof</u>: Suppose that the lemma is false. Then find $\lambda_1, \ldots \lambda_n \in S$ and N_x , a neighborhood of x , such that $y \in N_x$ implies that $\sigma_{s_3}(y) = \sum_{i=1}^{n} (\sigma_{\lambda_i}(y)\sigma_{t_i}(y)\sigma_{s_1}(y))$. By 6.4 find an integer m and a $y \in N_x \cap U \cap V_m$. Then $y \in U_j$ for some j so by 3.12.iv) and the definitions of s_3 and s_1 and the above equations it follows that

$$m + j = \mathrm{val}_y(\sigma_{s_3}) \geq \inf\{\mathrm{val}_y(\sigma_{t_i}\cdot\sigma_{s_1}) : 1 \leq i \leq n\} = m + 1 + j \ .$$

This contradiction establishes the lemma.

Q.E.D.

6.6) (Corollary) i) $I_x \cap J_x$ is not finitely generated

ii) K_x is not coherent.

iii) If $B(R)$ is $\aleph_1$-self-injective then K_x is an integral domain that is not coherent.

iv) $I \cap J$ is not finitely generated.

Proof: i) This follows immediately from 6.5.

ii) This follows immediately from i).

iii) This follows from ii) and 3.15.

iv) There is an epimorphism $S \to K_x$ given by $s \to \sigma_s(x)$. The image of $I \cap J$ under this map is $(I \cap J)_x = I_x \cap J_x$. The result now follows from i).

6.7) (Theorem) i) S has neither properties a) nor b).

ii) S does not have property e).

iii) S is not coherent.

iv) S is not semi-hereditary.

v) $w.gl.dim(S) > 1$.

Proof: i) This was established in 6.6iv).

ii), iii), iv), v): These now follow from i), either directly or via 1.13.

Q.E.D.

We now construct a von Neumann regular ring R'' such that $R''[[X]]$ has properties c) and d) but not property a). Let X' be an infinite Boolean space. Find $x \in X'$ and an $\aleph_1$-set U in X' such that $x \in \overline{U} - U$. Let S' be the simple GF(4) sheaf over X' where $GF(2^n)$ is the field with 2^n elements. Let $\mathcal{S}''$ be the subsheaf of $\mathcal{S}'$ such that $\mathcal{S}''_x = GF(2)$ and $\mathcal{S}''_y = \mathcal{S}'_y$ for $x \neq y$. Let $R'' = \Gamma(X',\mathcal{S}'')$. It is easily seen that $\mathcal{S}''$ does not have the $\aleph_1$-extension property so that by 3.9 and 6.7 $R''[[X]]$ does not have property a). If X' is chosen to be $\aleph_1$-extremally disconnected it follows from 3.15 that $R''[[X]]$ does have properties c) and d).

§ 7

Example of a Boolean ring R that is λ-self-injective but is not $\aleph_1$-complete.

In this section let λ be a fixed but arbitrary Cardinal such that $\lambda \geq \aleph_1$. Identify λ with the least ordinal of cardinality λ.

In this section we construct a Boolean space X with the λ-disjointness property that is not $\aleph_1$-extremally disconnected. R is constructed by letting $R = \Gamma(X,k)$ where $GF(2)$ is the two element field and k is the simple $GF(2)$ sheaf over X. It follows from 3.9, and 5.3 that R is the required Boolean ring, $\text{w.gl.dim}(R\llbracket X \rrbracket) \leq 1$, $R\llbracket X \rrbracket$ has properties a), b), and e), yet $R\llbracket X \rrbracket$ has neither property c) nor d).

X is constructed to be a one point union of the form $X = (Y \cup W/_{p \sim q})$ where Y and W are Boolean spaces and the fixed points $p \in Y$ and $q \in W$ are identified. Y and W are constructed such that they have the λ-disjointness property and $q \notin (cl_W(V) - V)$ for any λ-set V in W. This results in X having the λ-disjointness property. Y and W are also constructed such that q is not isolated in W and there exists an $\aleph_1$-set N in Y such that $p \in (cl_Y(N) - N)$. Thus X can not be $\aleph_1$-extremally disconnected for N is an $\aleph_1$-set in X such that $q \in cl_X(X - cl_X(N)) \cap cl_X(N)$ so that $cl_X(N)$ is not open in X.

First the space W is constructed. If $W = (\lambda + 1)$ then $q = \lambda$ would be as described above. However $(\lambda + 1)$ does not even have the λ-disjointness property. Let $S = \Gamma((\lambda + 1),\mathcal{S})$ where $\mathcal{S}$ is the simple GF(2) sheaf over $(\lambda + 1)$. W is actually constructed to be $X(T)$ where T turns out to be the λ-completion of the Boolean ring S .

Let Q be the complete ring of quotients of S . Q is a self-injective Boolean ring so that by 2.4 of [6] it is complete. For any subset F of Q let $\bigvee F$ denote the least upper bound of F in Q and let $\bigwedge F$ denote the greatest lower bound of F in Q . For any, $q,r \in S$ let $q \vee r$ denote $\bigvee\{q,r\}$ and $q \wedge r$ denote $\bigwedge\{q,r\}$. The following facts will be used.

i) For $\tau_1,\tau_2 \in S$, $\tau_1 \leq \tau_2 \Longleftrightarrow \tau_1 \vee \tau_2 = \tau_2 \Longleftrightarrow \tau_1 \wedge \tau_2 = \tau_1$

$\Longleftrightarrow \tau_1 \cdot \tau_2 = \tau_1 \Longleftrightarrow S(\tau_1) \subseteq S(\tau_2) \Longleftrightarrow Z(\tau_2) \subseteq Z(\tau_1)$.

ii) Let I be an index set, $\{a_\alpha : \alpha \in I\}$ be a subset of Q and $b \in Q$. Then $\bigvee\{a_\alpha \vee b : \alpha \in I\} = (\bigvee\{a_\alpha : \alpha \in I) \vee b$. Also, $1 - \bigvee\{a_\alpha : \alpha \in I\}$ $= \bigwedge\{1 - a_\alpha : \alpha \in I\}$.

iii) Let I and J be index sets and $\{a_{\alpha,\beta} : \alpha \in I$ and $\beta \in J\}$ be a subset of Q . Then

$$\bigvee\{a_{\alpha,\beta} : \alpha \in I \text{ and } \beta \in J\} = \bigvee\{\bigvee\{a_{\alpha,\beta} : \alpha \in I\} : \beta \in J\} .$$

iv) Let $r \in Q$. Then there exists subsets of S,F and F' , such that $r = \bigvee F = \bigwedge F'$.

The first of the above facts is obvious and the second and third are from [8]. The fourth is from 2.4 of [6].

Let $T = \{t \in Q :$ there exists $\{\tau_\alpha : \alpha < \lambda\} \subseteq S$ such that $t = \bigvee\{\tau_\alpha : \alpha < \lambda\}\}$.

7.1) (Lemma) Let $\{t_\beta : \beta < \lambda\}$ be a subset of T. Then $\bigvee\{t_\beta : \beta < \lambda\} \in T$.

Proof: By the definition of T there exists $\{\tau_{\alpha,\beta}: \alpha < \lambda$ and $\beta < \lambda\}$, a subset of S, such that $t_\beta = \bigvee\{\tau_{\alpha,\beta}\ \alpha < \lambda\}$ for each $\beta < \lambda$. Thus $\bigvee\{t_\beta : \beta < \lambda\} = \bigvee\{\bigvee\{\tau_{\alpha,\beta}: \alpha < \lambda\} : \beta < \lambda\}$ $= \bigvee\{\tau_{\alpha,\beta}: \alpha < \lambda$ and $\beta < \lambda\} \in T$.

Q.E.D.

In particular if $s, t \in T$ then $s \vee t \in T$.

7.2) (Lemma) Let $\{C_\alpha : \alpha < \lambda\}$ be a family of closed subsets of $(\lambda + 1)$ such that, for each $\alpha < \lambda$, $\lambda \notin C_\alpha$. Then there exists C, a clopen subset of $(\lambda + 1)$, such that $C_\alpha \subseteq C$ for each $\alpha < \lambda$ yet $\lambda \notin C$.

Proof: For each $\alpha < \lambda$ let d_α be the least upper bound for C_α in the ordinal $\lambda + 1$. Since each C_α is closed we have each $d_\alpha < \lambda$. Let d be the least upper bound for $\{d_\alpha : \alpha < \lambda\}$. It follows from the choice of the ordinal λ that $d < \lambda$. Let $C = \{\beta \in (\lambda + 1) : 0 \leq \beta \leq d\}$. Then C is the required clopen subset of $(\lambda + 1)$.

Q.E.D.

7.3) (Lemma) Let $\{\tau_\alpha : \alpha < \lambda\}$ be a subset of S such that $\tau_{\alpha'}(\lambda) = 0$ for some $\alpha' < \lambda$. Then $(\bigwedge\{\tau_\alpha: \alpha < \lambda\}) \in T$.

Proof: Let $L = \{\rho \in S : \rho \leq \text{each } \tau_\alpha\}$.

It follows from fact iv) that $\bigvee L = \bigwedge\{\tau_\alpha : \alpha < \lambda\}$. It remains to show that $\bigvee L \in T$. Let $U = \bigcup_{\rho \in L} (S(\rho))$. It follows from fact i) in the opening remarks to this section that $U \subseteq S(\tau_{\alpha'})$. Since $S(\tau_{\alpha'})$ is closed and does not contain λ it follows that $|U| < \lambda$. Thus there exists $\{N_\alpha : \alpha < \lambda\}$, a family of clopen subsets of $(\lambda + 1)$, such that $U = \bigcup_{\alpha<\lambda} (N_\alpha)$. It thus follows from fact i) that $\{\sigma \in S : \sigma \geq \text{each } \psi_{N_\alpha}\} = \{\sigma \in S : \sigma \geq \text{each } \rho \in L\}$. This establishes via fact iv) and the definition of T that $\bigwedge\{\tau_\alpha : \alpha < \lambda\} = \bigvee L = \bigvee\{\psi_{N_\alpha} : \alpha < \lambda\} \in T$.

Q.E.D.

7.4 (Lemma) Let $\{\tau_\alpha : \alpha < \lambda\}$ be a subset of S such that $\tau_\alpha(\lambda) = 1$ for each $\alpha < \lambda$. Then $\bigwedge\{\tau_\alpha : \alpha < \lambda\} \in T$.

Proof: By hypothesis $\{Z(\tau_\alpha) : \alpha < \lambda\}$ is a family of closed subsets of $(\lambda + 1)$ such that for each $\alpha < \lambda$ $\lambda \notin Z(\tau_\alpha)$. By 7.1 find C , a clopen subset of $(\lambda + 1)$, such that each $Z(\tau_\alpha) \subseteq C$ yet $\lambda \notin C$. Let $D = (\lambda + 1) - C$. Then for each τ_α , $\tau_\alpha = (\tau_\alpha \cdot \psi_C) \vee \psi_D$. Each $(\tau_\alpha \cdot \psi_C)(\lambda) = 0$. Thus by 7.3 and the comment after 7.1

$$\bigwedge\{\tau_\alpha : \alpha < \lambda\} = \bigwedge\{(\tau_\alpha \cdot \psi_C) \vee \psi_D : \alpha < \lambda\} = (\bigwedge\{\tau_\alpha \cdot \psi : \alpha < \lambda\}) \vee \psi_D \in T .$$

7.5) (Lemma) Let $\{\tau_\alpha : \alpha < \lambda\}$ be a subset of S. Then $\bigwedge\{\tau_\alpha : \alpha < \lambda\} \in T$.

Proof: This follows immediately from 7.3 and 7.4.

Q.E.D.

7.6) (Lemma) If $t \in T$ then $1 - t \in T$.

Proof: Suppose that $\{\tau_\alpha : \alpha < \lambda\}$ is a subset of S such that $\bigvee\{\tau_\alpha : \alpha < \lambda\} = t$. Then

$$1 - t = 1 - (\bigvee\{\tau_\alpha : \alpha < \lambda\}) = \bigwedge\{1 - \tau_\alpha : \alpha < \lambda\} \in T .$$

Q.E.D.

7.7) T is a λ-complete Boolean ring. Clearly S is a subring of T.

Proof: It is well known that this follows from 7.1, the comment immediately after 7.1, and 7.6.

Q.E.D.

Let $W = X(T)$. Let $f : W \to X(S)$ be defined by $f(M) = M \cap S$ for each maximal proper ideal M in T. Then f is a continuous onto map. (This is a well known fact about Boolean algebras. The map f is actually the function $X(T) \to X(S)$ induced by the inclusion map $S \to T$. See §11 of [9].) Since T is λ-complete, W is λ-extremally disconnected, and thus W has the λ-disjointness property.

7.8) (Lemma) There exists exactly one $q \in W$ such that $f(q) = \lambda$.

Proof: Since f is onto it suffices to show that at most one such q exists. Under the identification $(\lambda + 1) = X(S)$ the point λ corresponds to the ideal $M_\lambda = \{\sigma \in S : \sigma(\lambda) = 0\}$. (See 0.15.iii.) Thus it suffices to show that the elements of M are uniquely determined where M is an ideal in T such that $M \cap S = M_\lambda$. Let $t = \bigvee\{\tau_\alpha : \alpha < \lambda\}$ where $\{\tau_\alpha : \alpha < \lambda\}$ is a subset of S .

First suppose that for each $\alpha < \lambda$ $\lambda \notin S(\tau_\alpha)$. Then by 7.2 find C , a clopen subset of $(\lambda + 1)$, such that each $S(\tau_\alpha) \subseteq C$ yet $\lambda \notin C$. Then $\psi_C \in M_\lambda \subseteq M$. Since $t = \bigvee\{\tau_\alpha : \alpha < \lambda\} \leq \psi_C$ we have $t = t \cdot \psi_C \in M$.

Now suppose that there exists $\alpha' < \lambda$ such that $\lambda \in S(\tau_\alpha)$. Then $t \notin M$ for if $t \in M$ we would have $\tau_{\alpha'} = (\tau_{\alpha'} \cdot t) \in M \cap S = M_\lambda$. This would contradict $\tau_{\alpha'}(\lambda) \neq 0$.

Q.E.D.

For the rest of this section let q denote the unique point in W such that $f(q) = \lambda$.

7.9) (Lemma) i) q is not isolated in W .

ii) There does not exist a λ-set V in W such that $q \in (\mathrm{cl}_W(V) - V)$.

Proof: i) Suppose that q is isolated in W . Then $W - \{q\}$ is compact. Since $f : W \to (\lambda + 1)$ is onto and $f^{-1}(\lambda) = \{q\}$ it follows that $f(W - \{q\}) = \lambda \quad (\lambda + 1)$. Thus λ is a compact subset of $(\lambda + 1)$. This is a contradiction since λ is a limit ordinal.

ii) Suppose that the lemma is false. Thus there exists $\{C_\alpha : \alpha < \lambda\}$, a family of clopen subsets of W , such that for each $\alpha < \lambda$ $q \notin C_\alpha$ yet $q \in cl_W(\bigcup_{\alpha<\lambda} (C_\alpha))$. Then each $f(C_\alpha)$ is compact in $(\lambda + 1)$ and thus is closed. It follows from 7.8 that for each $\alpha < \lambda$, $\lambda \notin f(C_\alpha)$. Then

$$\begin{aligned} \lambda = f(q) \in f(cl_W(\bigcup_{\alpha<\lambda} (C_\alpha))) \\ \subseteq cl_{(\lambda+1)}(f(\bigcup_{\alpha<\lambda}(C_\alpha)) \\ = cl_{(\lambda+1)}(\bigcup_{\alpha<\lambda}(f(C_\alpha))) \end{aligned}$$.

Since for each $\alpha < \lambda$ $f(C_\alpha)$ is closed and $\lambda \notin f(C_\alpha)$ this contradicts 7.2.

Q.E.D.

Let Y be the Stone-Czech compactification of the set N of natural numbers where N has the discrete topology. For the rest of this section let p denote a fixed but arbitrary element in $(cl_Y(N) - N)$. Let $X = (Y \cup W/_{p\sim q})$ be the one point union of Y and W in which p and q are identified. Y and W may each be topologically identified with a subspace of X in such a way that $Y \cup W = X$ and $Y \cap W = \{p\} = \{q\}$.

7.10) (Lemma) i) if $x \in (Y - \{p\})$ then $\{N_x : N_x$ is a neighborhood of x in $Y\}$ is a neighborhood basis for x in X . If $x \in (W - \{q\})$ then $\{N_x : N_x$ is a neighborhood of x in $W\}$

is a neighborhood basis for x in X. If $x = p = q$ then $\{N_p \cup N_q : N_p$ is a neighborhood of p in Y and N_q is a neighborhood of q in $W\}$ is a neighborhood basis for x in X.

ii) If $U \subseteq X$ then $cl_X(U) = cl_Y(U \cap Y) \cup cl_W(U \cap W)$.

iii) X is a Boolean space.

iv) The set of points isolated in X is dense in X.

Proof: i) This is straightforward to check.

ii) This can readily be checked using i).

iii) That X is Hausdorff and totally disconnected follows from i). X is compact because it is a continuous image of the disjoint union of Y and W. The disjoint union of two compact spaces is clearly compact.

iv) The following fact (paraphrased from p. 28 example A of [9]) will be used: a Boolean ring A is atomic if and only if $X(A)$ has a dense subset consisting only of isolated points. It follows from this that S is atomic for $\{\alpha + 1 : \alpha < \lambda\}$ consists only of isolated points and is dense in $(\lambda + 1)$. It follows readily from this and the definition of T that T also is atomic. Thus $cl_W(N') = W$ where N' denotes the set of points that are isolated in W. Since N consists only of points that are isolated in Y it follows from i) that $N \cup N'$ consists only of points that are isolated in X. Note that $cl_X(N \cup N') = cl_Y(N) \cup cl_W(N') = Y \cup W = X$.

Q.E.D.

7.11) (Lemma) X is not $\aleph_1^{\lambda}$-extremally disconnected.

Proof: It follows from 7.10i) that N is an $\aleph_1^{\lambda}$-set in X. It follows from 7.10ii) that $cl_X(N) = Y$. Since $p = q$ is not isolated in W it follows from 7.10i) that Y is not open in X.

Q.E.D.

7.12) (Lemma) X has the λ-disjointness property.

Proof: W has the λ-disjointness property since T is λ-complete and $W = X(T)$. It is established in [5] that the Stone-Czech compactification of an extremally disconnected space is extremally disconnected. Thus Y is extremally disconnected so that by 3.4 it has the λ-disjointness property.

Let $\{U_\alpha : \alpha < \lambda\}$ and $\{V_\alpha : \alpha < \lambda\}$ be families of clopen subsets of X such that $U \cap V = \phi$ where $U = \bigcup_{\alpha<\lambda} (U_\alpha)$ and $V = \bigcup_{\alpha<\lambda} (V_\alpha)$. For each $\alpha < \lambda$

let $A_\alpha = U_\alpha \cap Y$, $B_\alpha = V_\alpha \cap Y$,
$C_\alpha = U_\alpha \cap W$, and $D_\alpha = V_\alpha \cap W$.

Let $A = \bigcup_{\alpha<\lambda}(A_\alpha)$, $B = \bigcup_{\alpha<\lambda}(B_\alpha)$, $C = \bigcup_{\alpha<\lambda}(C_\alpha)$, and $D = \bigcup_{\alpha<\lambda}(D_\alpha)$. Then A and B are λ-sets in the space Y such that $A \cap B = \phi$ and C and D are λ-sets in the space W such that $C \cap D = \phi$. Thus $cl_Y(A) \cap cl_Y(B) = \phi$ and $cl_W(C) \cap cl_W(D) = \phi$. Note that by 7.10 ii) $cl_X(U) = cl_X(A \cup C) = cl_Y(A) \cup cl_W(C)$ and $cl_X(V) = cl_X(B \cup D) =$

$cl_Y(B) \cup cl_W(D)$. Thus $cl_X(U) \cap cl_X(V) =$

$$(cl_Y(A) \cap cl_Y(B)) \cup (cl_Y(A) \cap cl_W(D)) \cup (cl_W(C) \cap cl_Y(B)) \cup (cl_W(C) \cap cl_W(D)) \ .$$

Hence $cl_X(U) \cap cl_X(V) = (cl_Y(A) \cap cl_W(D)) \cup (cl_W(C) \cap cl_Y(B))$. It now suffices to show that $cl_Y(A) \cap cl_W(D) = \phi$ and $cl_W(C) \cap cl_Y(B) = \phi$. There are three cases to be considered.

For the first case suppose that $p \notin U$ and $p \notin V$. Then $p \notin C$ and $p \notin D$ so by 7.9 ii) $p \notin cl_W(C)$ and $p \notin cl_W(D)$. (Recall that Y and W are identified with subspaces of X and in X $p = q$.) Hence

$$cl_X(C) \cap Y = cl_W(C) \cap Y = \phi \text{ and}$$
$$cl_X(D) \cap Y = cl_W(D) \cap Y = \phi \text{ . Thus}$$
$$cl_Y(A) \cap cl_W(D) \subseteq Y \cap cl_W(D) = \phi \text{ and}$$
$$cl_W(C) \cap cl_Y(B) \subseteq cl_W(C) \cap Y = \phi \ .$$

For the second case suppose that $p \in U$. Then $p \notin V$ since $U \cap V = \phi$. Suppose without loss of generality that $p \in U_0$. Then $B \subseteq Y - U_0$ and $Y - U_0$ is clopen in Y so that $cl_Y(B) \subseteq Y - U_0$ and hence $cl_Y(B) \cap W = \phi$. Similarly $cl_W(D) \cap Y = \phi$. Thus $cl_Y(A) \cap cl_W(D) \subseteq Y \cap cl_W(D) = \phi$ and $cl_W(C) \cap cl_Y(B) \subseteq W \cap cl_Y(B) = \phi$

For the third case suppose that $p \in V$. This is similar to the first case.

Q.E.D.

7.13) (Theorem) i) There exists a Boolean space X that has the λ-disjointness property but is not $\aleph_1$-complete.

ii) There exists a Boolean ring R such that R is atomic, λ-self-injective, but not $\aleph_1$-complete.

iii) Let R be as in ii). Then $w.gl.dim(R\lceil\lceil X\rceil\rceil) \leq 1$, $R\lceil\lceil X\rceil\rceil$ has properties a), b), and e), yet $R\lceil\lceil X\rceil\rceil$ has neither property c) nor d).

Proof: i) X has already been constructed.

ii) The construction of R given X was described in the opening remarks to this section. That R is atomic follows from 7.10 iv) and the fact that $X(R) = X$. The rest of ii) follows from i) and 3.9.

iii) This follows from ii) and 5.3.

Q.E.D.

T. Crammer has observed in private communication that $(Y - N)$ in the relative topology as a subspace of Y is a Boolean space with the $\aleph_1$-disjointness property that is not $\aleph_1$-extremally disconnected. Since $(Y - N)$ has no isolated points it follows, similar to 7.13, that there exists an atomless Boolean ring R' such that R' is $\aleph_1$-self-injective but is not $\aleph_1$-complete. Again, $w.gl.dim(R'\lceil\lceil X\rceil\rceil) \leq 1$, $R'\lceil\lceil X\rceil\rceil$ has properties a), b), and e), yet $R'\lceil\lceil X\rceil\rceil$ has neither property c) nor d).

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