REPRESENTATION OF ADDITIVE AND BIADDITIVE NONLINEAR FUNCTIONALS

by

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ABSTRACT

In this thesis we are concerned with obtaining an integral representation of a class of nonlinear additive and biadditive functionals on function spaces of measurable functions and on L^p-spaces, $p > 0$. The associated measure space is essentially atom-free finite and σ-finite.

Also we are concerned to the extend the presence of atoms in a measure space complicates the representation theory for functionals of the type under consideration here.

A class of nonlinear transformations on L^p-spaces, $1 \leq p \leq \infty$, called Urysohn operators [11] taking measurable functions to measurable functions is studied and we describe an integral representation for this class when the associated measure space is an arbitrary σ-finite measure space and this characterization extends our previous results where the measure space considered was atom-free.

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SECTION 1

INTRODUCTION

Some results of A. D. Martin and V. J. Mizel [1] on an integral representation of non-linear additive functionals defined on vector spaces of real-valued measurable functions have been extended by R. V. Chacon and Friedman [4]; Friedman and M. Katze [3]; and by V. J. Mizel [8]. We shall give a unified account of these results and shall make precise some of the results and shall make some generalizations.

In Section 2 we give the terminology and notations of measure theory. An atom in a measure space is also defined and we prove some theorems that we will need later on.

The essential part of Section 3 obtains the results of [1] in this more general setting and we also prove necessary and sufficient conditions for the integral representation of nonlinear additive functionals under various continuity conditions.

In Section 4, we construct the integral representation of an additive functional on $L_p(\mu)$-spaces for $p > 0$ and then we establish some necessary and sufficient conditions for the integral representation of non-linear additive functionals on L_p-spaces for $1 \leq p \leq \infty$.

In Section 5, we give some examples on integral representations and in Section 6 we establish analogous integral representations of nonlinear biadditive functionals. More precisely we establish necessary and sufficient conditions for a biadditive functional F defined on the Product $X_1 \times X_2$ of prescribed subspaces $X_1 \subset M_1$, $X_2 \subset M_2$ to permit an integral representation where M_1 and M_2 denote vector spaces of real-valued essentially bounded measurable functions on (X,Σ,μ).

In Section 7 we describe integral representations for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(X)$, $(1 \leq p \leq \infty)$ associated with an arbitrary σ-finite measure space (X,Σ,μ). The class of functionals considered here differs from those considered in [1], [2], [3] and [4] and its study is mainly motivated by its close connection with nonlinear integral equations in [11]. Our characterization extends earlier results in [1] and [2].

SECTION 2

PRELIMINARIES

We assume here some of the usual axioms of set theory and we use [1], [2], [3], [4] and [8] as standard references except for some notations and definitions which we give below. Throughout this thesis we will use the following abbreviations

"iff"	for the phrase	"if and only if"
"$\forall$"	instead of	"For every"
"$\exists$"	for	"There exists"
"$\in$"	for	"belongs to"
"w.l.o.g."	for	"without loss of generality"
"w.r.t."	for	"with respect to" and
"s.t."	for	"such that" .

We will use the following convections

2.1 ϕ denotes the empty set.

2.2 R denotes the real line.

2.3 $R^* = R \cup \{\infty\} \cup \{-\infty\}$ the extended real line.

2.4 $I = \{1,2,3,\ldots\ldots\ldots\}$

2.5 $A \sim B = \{x \in A \text{ and } x \notin B\}$

2.6 $\{x_i\}_{i=1}^{n} = \{x_1, x_2, \ldots, x_n\}$

2.7 $\{x_i\}_{i \in J} = \{x_i\ ,\ i \in J\}$ where J denotes an index set.

Let X be any set.

2.8 $P(X)$ = all subsets of X .

2.9 $\bar{A}$ denotes the closure of A , $A \subset X$.

2.10 $A^{\sim} = X \sim A$.

Let f be a function and A be a set.

2.11 $f[A] = \{y : y = f(x)$ for some $x \in A\}$

2.12 $f^{-1}[A] = \{x : y = f(x)$ for some $y \in A\}$

Unless otherwise mentioned, in this section, X will be an arbitrary nonempty class with elements $x, y, \ldots.$ and $E, F, A, B, \ldots.$ will be subsets of X .

2.13 A nonempty class Σ of subsets of X is called a ring if $E \cup F$, $E - F \in \Sigma$ whenever $E, F \in \Sigma$ and is called a σ-ring if $\bigcup_{n=1}^{\infty} E_n \in \Sigma$ whenever $E_n \in \Sigma$ $\forall\, n \in I$.

2.14 <u>Definition</u>. A ring Σ of subsets of X is called an algebra if $X \in \Sigma$ and a σ-ring Σ of subsets of X is called a σ-algebra if $X \in \Sigma$.

2.15 <u>Remark</u>. Let $\mathcal{E}$ be an arbitrary family of subsets of X . Let $\mathcal{S}(\mathcal{E})$ denote the intersection of all σ-algebras of subsets of X that contain $\mathcal{E}$.

$\mathcal{S}(\mathcal{E})$ is also a σ-algebra and is the smallest σ-algebra of subsets of X containing $\mathcal{E}$.

2.16 <u>Definition</u>. If X is a topological space, let $\mathcal{B}(X)$ be the smallest σ-algebra of subsets of X that contains every open set. Then the members of $\mathcal{B}(X)$ are called the Borel sets of X.

2.17 <u>Definition</u>. If $\mathcal{H} \subset \mathcal{P}(X)$, a set function φ on $\mathcal{H}$ is a function on $\mathcal{H}$ to R^* written as $\varphi : \mathcal{H} \to R^*$ is additive if $\forall\ A,B \in \mathcal{H}$, $A \cap B = \phi$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$ where $\mathcal{H}$ is any family of subsets of X.

2.19 <u>Definition</u>. An additive set function $\mu : \mathcal{P}(X) \to [0,\infty]$ is called an outer measure if the following hold:

(i) $\mu(\phi) = 0$ and

(ii) $\mu(E) \leq \sum_{i \in I} \mu(E_i)$ whenever $E \subset \bigcup_{i \in I} E_i \subset X$,

and μ is called a measure iff

(i) $\mu(\phi) = 0$ and

(ii) $\mu(E) = \sum_{i \in I} \mu(E_i)$ whenever $E = \bigcup_{i \in I} E_i \subset X$ and

$E_i \cap E_j = \phi$, $i \neq j$.

2.20 <u>Definition</u>. We will use caratheodory outer measure μ for which we have:

A set $A \subset X$ is μ-measurable iff $\forall\ E \subset X$,
$\mu(E) = \mu(E \cap A) + \mu(E \cap A^{\sim})$.

2.21 <u>Definition.</u> If μ is a measure on X, then we denote by M_μ the set $\{A : A \subset X$ and A is μ-measurable$\}$.

2.22 <u>Definition</u>. If for every $A \in M_\mu$, $\mu(A) < \infty$ then μ is called a finite measure.

2.23 <u>Definition</u>. μ is called σ-finite if $\forall\ A \in M_\mu$, $\exists$ a sequence $\{A_n\}_{n\in I}$ of sets in M_μ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty\ \forall\ n \in I$.

2.24 <u>Definition</u>. Let $A \subset X$ then the measure $\mu_A(E) = \mu(A\cap E)$ on X is called the restriction of μ by A.

In future we will take $X = (X,\Sigma,\mu)$ as a measure space where X is a nonempty class of elements and Σ is a nonempty class of measurable subsets of X which is a σ-algebra and μ is a measure on (X,Σ).

2.25 <u>Definition</u>. Let (X,Σ,μ) be a measure space. $A \in \Sigma$ is called an atom if $\mu(A) \neq 0$ and if $B \in \Sigma$, $B \subset A$ then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

A space (X,Σ,μ) will be called an atomic space if every subset of X that belongs to Σ is an atom.

2.26 <u>Lemma</u>. Let (X,Σ,μ) be a finite measure space and $E \in \Sigma$,

$0 < \mu(E) < \infty$ s.t. neither E nor any of its μ-measurable subsets is an atom, then E contains subsets of arbitrarily small positive measure.

<u>Proof</u>: Since E does not contain any atoms, there exists $F \subset E$, $F \in \Sigma$ s.t. $E = F \cup (E-F)$ and $\mu(F) > 0$, $\mu(E-F) > 0$. Hence one of F and $E-F$, call it F_1, satisfies $\mu(F_1) \leq \frac{1}{2}\mu(E)$. Now since F_1 is not an atom, continuing with the above method fo decomposition, it follows that E has subsets of arbitrarily small positive measure.

2.27 <u>Lemma</u>. If (X,Σ,μ) is a finite measure space then there is a countable family $\{A_i\}$ of atoms of X s.t. if $A = \bigcup_{i=1}^{\infty} A_i$ then $C = X\sim A$ is atom free and $X = A \cup C$ is a decomposition of X into atomic and atom-free parts.

<u>Proof</u>: Since X is a finite measure space, any atom $A \subset X$ is of finite measure. By identifying the μ-almost equal atoms, we get that the different atoms are disjoint. So the total number of different atoms contained in X is at most countable say $A_1, A_2, \ldots$. Let $A = \bigcup_{i=1}^{\infty} A_i$ and $C = X\sim A$ then clearly C is atom-free and the decomposition is μ-almost unique.

2.28 <u>Definition</u>. $X = (X,\Sigma,\mu)$ has the strong interdediate value property if $\forall$ $S \in \Sigma$ and $\forall$ real number a, $0 \leq a \leq \mu(S)$ $\exists$ $A \in \Sigma$, $A \subset S$ s.t. $\mu(A) = a$.

2.29 <u>Theorem</u>. A measure space X has the strong intermediate value property iff it is atom free.

[For Proof, see "Set Functions", by Hahn and Rosenthal Chapter 1, §5.6.]

2.30 <u>Definition</u>. (X,Σ,μ) has the weak intermediate value property iff $\forall$ real number a, $0 \leq a \leq \mu(X)$, $\exists$ $A \in \Sigma$, $A \subset X$, s.t. $\mu(A) = a$.

2.31 <u>Theorem</u>. (Z. Nehari). Let (X,Σ,μ) be a countable discrete measure space. Let $m_1 \geq m_2 \geq \ldots$ be the measures of atoms of X. Then (X,Σ,μ) has the weak intermediate value property iff $m_n \leq \sum_{k=n+1}^{\infty} m_k$, $n = 1,2,3,\ldots$.

2.32 <u>Definition</u>. Let (X,Σ,μ) be a measure space, then $f : X \to R^*$ is measurable iff $\forall$ $c \in R^*$, $\{x : f(x) < c\} \in \Sigma$.

2.33 <u>Definition</u>. $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$ is called the characteristic function.

2.34 <u>Theorem</u>. χ_E is measurable iff $E \in \Sigma$.

<u>Definition</u>. A function $f : X \to R^*$ which has only a finite set $c_1, c_2, \ldots, c_n$ of values and for which $f^{-1}(c_i) = \{s : s \in X, f(s) = c_i\} \in \Sigma$ is called a simple function.

2.35 <u>Lemma</u>. Let $f : X \to R^*$ be measurable, then there exists a sequence $\{f_n\}$ of simple functions s.t. $f_n \to f$. If $f \geq 0$, the sequence $\{f_n\}$ can be constructed s.t. $0 \leq f_n \leq f_{n+1}$ $\forall\, n$.

[For proof see Taylor]

2.36 <u>Definition</u>. Two real valued functions f, g are equi-measurable if $\forall$ Borel set S on $(-\infty,\infty)$, $f^{-1}(S)$ and $g^{-1}(S)$ are measurable and have equal measure.

2.37 <u>Definition</u>. Let (X,Σ,μ) be a finite measure space. Let f be a bounded measurable function with $\inf_x f(x) = m$, $\sup_x f(x) = M$.

Let $M_o > M$. If $\Delta = \{y_o, y_1, \ldots, y_n\}$ is a partition of $[m, M_o]$; and $E_j = \{x : y_{j-1} \leq f(x) < y_j\}$, $j = 1,2,\ldots,n$. Define

$$s(\Delta) = \sum_{j=1}^{n} y_{j-1}\, \mu(E_j)$$

$$S(\Delta) = \sum_{j=1}^{n} y_j\, \mu(E_j)$$

$m\mu(X) \leq s(\Delta) \leq S(\Delta) \leq M_o\mu(X)$. Then $\sup_\Delta s(\Delta) = \inf_\Delta S(\Delta) = \int_X f\, d\mu = \int f\, d\mu$.

Before we finish this section, we prove some lemmas which we shall need later on in Chapter 5 and for proofs we shall follow [1].

2.38 Lemma. Let $m_1, m_2, m_3, \ldots$ be a sequence of positive numbers s.t. $m_n > 2 \sum_{k=n+1}^{\infty} m_k$, $n = 1,2,\ldots$

Let $c_1, c_2, \ldots$ and $d_1, d_2, d_3, \ldots$ be two sequences of real numbers having values -1, 0 or 1. Then $\sum_{i=1}^{\infty} c_i m_i = \sum_{i=1}^{\infty} d_i m_i$ only if $c_i = d_i$, $i = 1,2,\ldots$

Proof: Let N_1 be the set of integers s.t. $c_i - d_i > 0$ for $i \in N_1$ and N_2 be the set of integers s.t. $d_i - c_i > 0$ for $i \in N_2$. We claim that $N_1 = N_2 = \phi$.

Suppose not. i.e. $N_1 \cup N_2 \neq \phi$. Let i_o be the smallest integer in $N_1 \cup N_2$. Thus $i_o \in N_1$ or $i_o \in N_2$ but not to both.

Suppose $i_o \in N_1$. Now if $i \in N_1$ and $j \in N_2$, then $c_i - d_i$ and $d_j - c_j$ are either 2 or 1.

Thus $\sum_{j \in N_1} (c_i - d_i) m_i \geq m_{i_o} > 2 \sum_{j=i_o+1}^{\infty} m_j \geq 2 \sum_{j \in N_2} (d_j - c_j) m_j$

i.e. $\sum_{i \in N_1} (c_i - d_i) m_i > \sum_{j \in N_2} (d_j - c_j) m_j$.

But by hypothesis we have that $\sum_{i=1}^{\infty} c_i m_i = \sum_{i=1}^{\infty} d_i m_i$ which implies that $\sum_{i \in N_1} (c_i - d_i) m_i = \sum_{j \in N_2} (d_j - c_j) m_j$ which is a contradiction to $N_1 \cup N_2 \neq \phi$.

Hence $N_1 = N_2 = \phi$.

2.39 Lemma. Let $\{m_n\}_{n\geq 1}$ be a sequence of positive numbers s.t. for $n \geq n_o \geq 1$, $m_n > 2 \sum_{i=n+1}^{\infty} m_i$. Let V be the vector space of all real sequences $s : s_1, s_2, \ldots$ such that $\sum_{i=1}^{\infty} s_i m_i < \infty$. Let S be the subspace of V consisting of those sequences s s.t. $\sum_{i\in I} s_i m_i = \sum_{j\in J} s_j m_j$ whenever $\sum_{i\in I} m_i = \sum_{j\in J} m_j$.

Then the algebraic dimension of S is infinite.

Proof: Let $H = \{(I,J),\ I \cap J = \phi$, I and J are subsets of the integers$\}$.

(a) s.t. $\sum_{i\in I} m_i = \sum_{j\in J} m_j$.

For a positive number n_o s.t. $1 \leq n_o \leq n$, (a) can be written uniquely as

(b) $\sum_{i=1}^{n_o} c_i m_i = \sum_{i=n_o+1}^{\infty} c_i m_i$ where c_i has the value $-1, 0$ or 1, $i = 1, 2, \ldots$.

The right hand side of (b) is uniquely determined, if it is determined, when the left hand side is given.

Thus there are a finite number of relations (b) which are valid as is the number of relations of (a) which proves that H is finite.

(c) Consider the system of equations $\sum_{i\in I} x_i m_i = \sum_{j\in J} x_j m_j$

$(I,J) \in H$.

where the unknown sequence $x : x_1, x_2, \ldots$ is a member of S . The system (c) is finite since H is finite. So suppose that (c) has exactly k equations where k can be zero also.

In any case if K is a subset of the integers containing exactly $k+1$ members then there is a nonzero solution S_K of (c) whose support is a subset of K .

Now if $\{K_i\}_{i \geq 1}$ is a sequence of disjoint subsets of $k+1$ integers each, then $S_{K_1}, S_{K_2}, \ldots$ is an infinite family of independent solutions of (c) . Hence $\dim S = \infty$.

SECTION 3

REPRESENTATION OF ADDITIVE FUNCTIONALS ON THE VECTOR SPACES OF REAL-VALUED MEASURABLE FUNCTIONS

Let (X,Σ,μ) be a measurable space with Σ a σ-algebra of subsets of X .

3.1 <u>Definition</u>. Let $f,g : X \to R^*$, then we say $f(x) = g(x)$ a.e. if $\{x : f(x) \neq g(x)\} \in \Sigma$ and has μ-measure zero.

3.2 <u>Definition</u>. A function f is said to be essentially bounded if $\exists$ a finite positive constant C s.t. $\mu\{x : |f(x)| > C\} = 0$ i.e. f is bounded a.e .

3.3 <u>Definition</u>. A sequence $\{f_n\}_{n=1}^{\infty}$ of functions which are finite a.e are said to converge a.e to a function $f(x)$ which is finite a.e. if $E = \{x : f_n(x) \not\to f(x)\} \in \Sigma$ and $\mu(E) = 0$.

3.4 <u>Definition</u>. A sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions is said to converge in measure to a measurable function f if $\forall\ \delta > 0$ we have $\lim\limits_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \delta\}) = 0$ and we write $f_n \xrightarrow{\mu} f$.

<u>Example</u>: There exist sequences of functions that converge in measure and do not converge a.e. Let $X = [0,1]$ and Σ be all Lebesgue measurable subsets of X . For each integer

$n \in I$, define

$$f_n = \chi_{[\frac{j}{2^k}, \frac{j+1}{2^k}]} \quad \text{where } n = 2^k + j , \ 0 \leq j \leq 2^k$$

Then $\lambda(\{x : |f_n(x)| \geq \delta\}) \leq \frac{1}{2^k} \to 0$ as $n \to \infty$ $\forall \ \delta > 0$.

Thus $f_n \overset{\mu}{\to} 0$. But on the other hand, if $x \in [0,1]$, the sequence $\{f_n(x)\}$ converges nowhere on $[0,1]$.

Throughout this section, we denote by M the vector space of real-valued measurable functions on (X,Σ,μ) where two functions are identical when they are equal a.e.

3.6 Definition. A real-valued function F on a subspace of M is called an additive functional if

(i) $F(x+y) = F(x) + F(y)$ for $x,y \in M$ s.t.

$$\mu\{\text{supp } x \cap \text{supp } y\} = 0$$

(ii) $F(x) = F(y)$ if x,y are equimeasurable functions i.e. if for every Borel set B in R , $\mu(x^{-1}(B)) = \mu(y^{-1}(B))$.

In this section we will construct an integral representation of a nonlinear additive functional F defined on different subspaces of M under various continuity conditions on F when the underlying measure space (X,Σ,μ) is atom free.

Let $B = B(X,\Sigma) = L_\infty(\mu) = \{$all essentially bounded real-valued measurable functions$\}$.

3.7 <u>Definition</u>. A set function ϕ is said to be μ-absolutely continuous if given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t. for $E \in \Sigma$, $\mu(E) < \delta$ $|\varphi(E)| < \epsilon$.

For the following theorem, we will follow the methods given [1] and we will use "$x_n \to x$ boundedly a.e" instead of sayingthat $x_n \to x$ a.e and there exists a positive constant c s.t. $|x_n| \leq c$ and $|x| \leq c$.

3.8 <u>Theorem</u>. Let (X,Σ,μ) be a finite atom-free measure space for which $\mu(X) \neq 0$. If an additive functional $F : B \to R$ satisfies the condition: (1) $x_n \to x$ boundedly a.e implies $F(x_n) \to F(x)$. Then there exists a unique continuous function $f : R \to R$ s.t. $f(0) = 0$ and $\forall$ $x \in B$

$$F(x) = \int_X (f \circ x) d\mu \qquad (*)$$

<u>Proof</u>: (a) Let $C_a \in B$ be the constant function a for $a \in (-\infty,\infty)$.

Define $f(a) = \dfrac{F(C_a)}{\mu(X)}$, $a \in (-\infty,\infty)$. If $a_n \to a$ (reals) then $C_{a_n} \to C_a$ boundedly, so $F(C_{a_n}) \to F(C_a)$ by continuity of F . Thus $f(a_n) \to f(a)$ i.e.

f is continuous.

Now supp $C_o = \phi$. Thus if $x \in B$ then supp $x \cap$ supp $C_o = \phi$ which implies that

$$F(x) = F(C_o + x) = F(C_o) + F(x)$$

i.e. $F(C_o) = 0$ and thus $f(0) = 0$.

If there is an f satisfying this theorem then it is unique since for $a \neq 0$ we have $F(C_a) = \int_X f \circ C_a \, d\mu = f(a) \cdot \mu(X)$.

Now it remains to show that

$$F(x) = \int_X f(x(t)) d(\mu t) \quad \forall \ x \in B .$$

If $S \in \Sigma$, for fixed $a \in (-\infty, \infty)$, define

$$\varphi(S) = \varphi_a(S) = F(a\chi_S)$$

φ is finite valued set function $\forall$ $S \in \Sigma$ and satisfies:

(b) φ is additive on Σ .

For if $S_1, S_2 \in \Sigma$ and $S_1 \cap S_2 = \phi$ then

$$\begin{aligned} \varphi(S_1 \cup S_2) &= F(a\chi_{S_1 \cup S_2}) = F(a\chi_{S_1} + a\chi_{S_2}) \\ &= F(a\chi_{S_1}) + F(a\chi_{S_2}) \quad (\text{supp } a\chi_{S_1} \cap \text{supp } a\chi_{S_2} = \phi) \\ &= \varphi(S_1) + \varphi(S_2) . \end{aligned}$$

(c) φ is countably additive on Σ .

If $S_n \in \Sigma$ s.t. $S_n \downarrow \phi$ then $a\chi_{S_n} \to 0$ boundedly so $\varphi(S_n) = F(a\chi_{S_n}) \to F(C_o) = 0$ which implies that φ is countably additive on Σ .

(d) φ is μ-absolutely continuous.

Since $\mu(X) < \infty$, it suffices to show that $\varphi(S) = 0$ whenever $\mu(S) = 0$.

If $\mu(S) = 0$ then since $a\chi_S$ and C_o are equimeasurable, it follows that $\varphi(S) = F(a\chi_S) = F(C_o) = 0$.

(e) $\dfrac{\varphi(S)}{\mu(S)} = \dfrac{F(a\chi_S)}{\mu(S)}$ is defined $\forall$ $S \in \Sigma$ with $\mu(S) \neq 0$ and is a real-valued function ρ on $(0,\mu(X)]$ s.t.

$$\varphi(S) = \rho(\mu(S))\mu(S) \quad \text{and} \quad s\rho(s) + r\rho(r) = (s+r)\rho(s+r)$$

for r,s , $r+s \in (0,\mu(X)]$.

In fact let $0 < s \leq \mu(X)$ and $C(s) = \{S \in \Sigma : \mu(S) = s\}$.
Since X is atom-free, $C(s) \neq \phi$.

If $S,R \in C(s)$, then $a\chi_S$ and $a\chi_R$ are equimeasurable, so $F(a\chi_S) = F(a\chi_R)$. Hence

$\dfrac{\varphi(S)}{\mu(S)} = \dfrac{F(a\chi_S)}{\mu(S)} = \dfrac{F(a\chi_R)}{\mu(R)} = \dfrac{\varphi(R)}{\mu(R)}$ which implies that $\dfrac{\varphi(S)}{\mu(R)}$ is independent of $S \in C(s)$, and hence

$f(a) = \dfrac{F(a\chi_S)}{\mu(S)} = \dfrac{\varphi_a(S)}{\mu(S)} = \dfrac{\varphi(S)}{\mu(S)} = \rho(s)$, $S \in C(s)$ which proves that $\frac{\varphi(S)}{\mu(S)}$ is a real-valued function of ρ on $(0,\mu(X)]$.

To prove the last equality, we have that X is atom-free, there is an $S \in \Sigma$ with $\mu(S) = s$. Since $0 < s+r \leq \mu(X)$, $r \leq \mu(X) - s = \mu(X-S)$. Again since the space is atom-free, $\exists$ $R \in \Sigma$, $R \subset X-S$ s.t. $\mu(R) = r$ and since $S \cap R = \phi$, it follows that $\varphi(S \cup R) = \varphi(S) + \varphi(R)$ and since $\varphi(S) = \rho(\mu(S))\mu(S)$ and $\mu(R \cup S) = \mu(R) + \mu(S) = r+s$ we have $\rho(\mu(S \cup R))\mu(S \cup R) = \rho(\mu(S))\mu(S) + \rho(\mu(R))\mu(R)$, and so $(r+s)\rho(r+s) = s\rho(s) + r\rho(r)$.

(f) ρ is continuous.

For this let $\{s_n\}$ be a monotonic decreasing sequence of real numbers in $(0,\mu(X)]$ with $s_n \to s$. Since X is atom-free $\exists$ $S_1 \in \Sigma$ with $\mu(S_1) = s_1$ and $S_2 \in \Sigma$, $S_2 \subset S_1$ with $\mu(S_2) = s_2 \leq s_1 \ldots$.

Therefore there exists a monotonic sequence S_n, $S_n \in \Sigma$ with $\mu(S_n) = s_n$ and $\ldots S_n \subset S_{n-1} \subset \ldots \subset S_1$.

Let $S = \bigcap_{n=1}^{\infty} S_n$. Since $\mu(S_1) < \infty$, it follows that $\mu(S) = \lim_n \mu(S_n) = \lim_n s_n = s > 0$ and since φ is countably additive,

$$\varphi(S_n) \to \varphi(S)$$

so $\lim_n \rho(s_n) = \lim_n \frac{\varphi(S_n)}{\mu(S_n)} = \frac{\varphi(S)}{s} = \frac{\varphi(S)}{\mu(S)} = \rho(s)$.

A similar argument shows that $\rho(s_n) \to \rho(s)$ when s_n increases to s. Thus ρ is continuous.

(g) ρ is constant.

Let $s \in (0,\mu(X)]$. Since X is atom-free $\exists$ $X_o \subset X$ s.t. $\mu(X_o) = s$. Furthermore there is $X_1 \subset X_o$ s.t. $\mu(X_1) = \frac{n-1}{n}s$ and so on, $\exists$ $X_i \subset X_{i-1}$ with $\mu(X_i) = \frac{n-i}{n}s$.

It follows that if $S_i = X_{i-1} - X_i$ then $S_i \in \Sigma$ are disjoint and $\mu(S_i) = \mu(X_{i-1} - X_i) = \mu(X_{i-1}) - \mu(X_i) = \frac{n-i+1}{n}s - \frac{n-i}{n}s = \frac{s}{n}$, i.e. $\mu(S_i) = \frac{s}{n}$ for $i \in I$ and $S_i \cap S_j = \phi$ if $i \neq j$.

But since $(r+s)\rho(r+s) = r\rho(r) + s\rho(s)$ we have

$$s\rho(s) = (\frac{s}{n} + \frac{s}{n} + \dots \text{ n times})\rho(\frac{s}{n} + \frac{s}{n} + \frac{s}{n} + \dots + \frac{s}{n})$$
$$= \frac{s}{n}\rho(\frac{s}{n}) + \frac{s}{n}\rho(\frac{s}{n}) + \dots + \frac{s}{n}\rho(\frac{s}{n})$$

i.e. $s\rho(s) = n \cdot \frac{s}{n}\rho(\frac{s}{n}) = s\rho(\frac{s}{n})$ and since $s \neq 0$ we have $\rho(s) = \rho(\frac{s}{n})$.

If now $\frac{m}{n}s \in (0,\mu(X)]$ where m and n are non-negative integers, $(n \neq 0)$ then

$$\frac{m}{n}s\rho(\frac{m}{n}s) = \frac{s}{n}\rho(\frac{s}{n}) + \frac{s}{n}\rho(\frac{s}{n}) + \dots + \frac{s}{n}\rho(\frac{s}{n}), \quad m \text{ terms}.$$

i.e. $\rho(\frac{m}{n}s) = \rho(\frac{s}{n}) = \rho(s)$, and since ρ is continuous, we get that ρ is constant.

(h) $\rho = \rho_a = f(a) \ \forall \ a \in (-\infty,\infty)$

Since $f(a) = \dfrac{F(a\chi_X)}{\mu(X)} = \rho_a(\mu(X)) = \rho_a(\mu(S))$, $s \in \Sigma$. We have $\rho = \rho_a = f(a) \ \forall \ a \in (-\infty,\infty)$.

(i) If $x \in B$ and if x is a simple function then

$$F(x) = \int_X f(x(t))d\ \mu(t) .$$

Let $S_1, S_2, \ldots, S_n$ be a partition of X into measurable subsets and $x_1, x_2, \ldots, x_n$ be the values which x assumes on these subsets. Thus $x = \Sigma\, x_i \chi_{S_i}$ and since $x_i \chi_{S_i}$ have mutually disjoint supports, this implies that

$$F(x) = \sum_{i=1}^{n} F(x_i \chi_{S_i}) = \sum_{i=1}^{n} \varphi_{x_i}(S_i) = \sum_{\mu(S_i)\neq 0} \rho_{x_i}(\mu(S_i))\mu(S_i)$$

$$= \sum_{1}^{n} f(x_i)\mu(S_i) = \int f(x(t))d\ \mu(t) .$$

Thus the theorem is true for F defined on simple functions.

(j) Now $\forall \ x \in B$, we show that $F(x) = \int_X f(x(t))d\ \mu(t)$. Since $\forall \ x \in B$, there exists a sequence $\{x_n\}$ of simple functions s.t. $x_n \to x$ boundedly a.e. and since F is continuous $F(x_n) \to F(x)$. Thus by Lebesgue dominated convergence theorem, we have $F(x) = \int f(x_n(t))d\ \mu(t) = \int f(x(t))d\mu(t)$.

Q.E.D.

From now onward we will prove the necessary and sufficient conditions that an additive functional F defined on a prescribed subspace $B \subset M$ permits a representation of the form, $F(x) = \int_X (f \cdot x) d\mu \quad \forall \quad x \in X$ where $f : R \to R$ is uniquely determined by F. We will associate these theorems with finite or σ-finite atom-free measure space (X, Σ, μ) and we shall follow very closely [2] for proofs.

3.9 Definition. Let (X, Σ, μ) be the measure space. For every $E \in \Sigma$, we denote the total variation of μ on E by $\nu_\mu(E)$, defined as $\nu_\mu(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|$ where the supremum is taken over all finite sequences $\{E_i\}$ of disjoint sets in Σ with $E_i \subseteq E$. μ is said to be of bounded variation if $\nu_\mu(X) < \infty$ and μ is said to be of bounded variation on a set $E \in \Sigma$ if $\nu_\mu(E) < \infty$.

Let $f, g : X \to R$, the relation "$f-g$ is a null function" is an equivalence relation. Let $[f]$ denote the class of functions from $X \to R$ which are equivalent to f and let $P[X] = P(X, \Sigma, \mu)$ denote the set of all such sets $[f]$.

3.10 Definition. Let X be any topological space. The totally measurable functions on X are the functions in the closure of simple functions in $F(X)$ and we shall denote them by $TM(X)$.

It is proved in Dunford and Schwartz in Lemma III.2.12 that if x is totally measurable function and if f is a continuous function on R then $f \circ x$ is totally measurable function and also in III.2.11 that the totally measurable functions form a closed linear subspace of $P(X)$.

3.11 Theorem. Let (X,Σ,μ) be a finite atom-free measure space and let F be an additive functional on $B = L_\infty(\mu)$. Then the following conditions are equivalent:

(2) $\quad x_n \to x$ boundedly in measure $\Rightarrow F(x_n) \to F(x)$ and

(2′) $\quad F(x) = \int_X (f \circ x) d\mu \quad \forall \; x \in B \qquad (*)$.

with the representing function f satisfying the conditions that f is continuous and $f(o) = 0$.

Proof: Let $x_n \to x$ boundedly a.e. Then since (X,Σ,μ) is a finite measure space, $x_n \to x$ in measure so $F(x_n) \to F(x)$, therefore by Theorem 3.8, $\exists$ a unique continuous function f, $f(o) = 0$ s.t.

$$F(x) = \int_X f \circ x \, d\mu \quad \forall \; x \in B .$$

Conversely, suppose $f : R \to R$ is continuous and $f(o) = 0$.

Define F on $B = L_\infty(\mu)$ by $F(x) = \int_X (f \circ x) d\mu$.

Since $f(o) = 0$, $\forall$ $x,y \in B$ s.t. $\mu\{\text{supp } x \cap \text{supp } y\} = 0$ we have $F(x+y) = F(x) + F(y)$.

Now claim that: $x_n \to x$ boundedly in measure implies that $F(x_n) \to F(x)$ $\forall$ $x_n, x \in B$.

Let $\{x_n\}$ be a sequence in B s.t. $x_n \to x$ in measure and $|x_n| \leq c$, $|x| \leq c$ for some positive constant c . Since x_n, x are totally measurable and f is continuous, it follows that $f \circ x_n$ and $f \circ x$ are totally measurable and $f \circ x_n \to f \circ x$ in measure and $f \circ x_n$, $f \circ x$ are bounded. Therefore by Lebesgue dominated convergence theorem

$$\lim_n F(x_n) = \lim_n \int (f \circ x_n(t)) d\mu(t) = \int f \circ x(t) d\mu(t) = F(x) .$$

Now we take the case of a.e. convergence which is not necessarily bounded and convergence in measure.

3.12 <u>Theorem</u>. Suppose (X, Σ, μ) is a finite atom-free measure space and F is an additive functional on M . Then the following conditions are equivalent:

(3) $x_n \to x$ a.e. $\Rightarrow$ $F(x_n) \to F(x)$ and

$$F(x) = \int_X f \circ x \, d\mu \qquad (*)$$

$\forall$ $x \in B$, where f satisfies the conditions

(a) f is continuous and $f(o) = 0$ and

(b) Range (f) is bounded.

Proof: Suppose F is an additive functional on M which satisfies the given condition and let $F_1 = F/L_\infty(\mu)$, $L_\infty(\mu) \subset M$ and since for any sequence $\{x_n\}_{n\geq 1}$ in $L_\infty(\mu)$ with $x_n \to x$ a.e. and $|x_n| \leq c$ $\forall$ n and $|x| \leq c$ where c is some positive constant, we have by Theorem 3.8 that there exists a unique continuous function $f : R \to R$ s.t. $f(o) = 0$ and $F_1(x) = F(x) = \int_X (f\circ x)d\mu$ $\forall$ $x \in L_\infty(\mu)$.

To prove (b), suppose that range (f) is unbounded. There exists a sequence $\{r_n\}$ in R s.t. $|r_n| \to \infty$ and $1 \leq |f(r_n)| \cdot \infty$. Since the measure space is atom-free, there exists a sequence $\{B_n\}$, $B_n \in \Sigma$ and $B_{n+1} \subseteq B_n \downarrow \phi$ s.t. $\mu(B_n) = \frac{\mu(x)}{|f(r_n)|}$.

Let $x_n = r_n \chi_{B_n}$, x_n is measurable and hence $x_n \in M$ and since $B_n \downarrow \phi$, $x_n \to 0$ a.e. and since $x_n \in L_\infty(\mu)$, we have $F(x_n) = \int (f\circ x_n)d\mu = f(r_n)\,\frac{\mu(x)}{|f(r_n)|} = \pm\,\mu(X) \nrightarrow 0$

$F(x_n) \nrightarrow F(o) = 0$ which is a contradiction. Hence range (f) is bounded.

Now let $x \in M$ be any function. There exists a sequence $\{s_n\}$ of simple functions s.t. $s_n \to x$ a.e. and since f is continuous, $f\circ s_n \to f\circ x$ a.e. where $|f\circ s_n|$, $|f\circ x| \leq c$ for some positive constant c. Thus by Lebesgue convergence theorem, $\int f\circ x\, d\mu = \lim_n \int (f\circ s_n)d\mu = \lim_n F(s_n) = F(x)$.

Conversely, let f satisfy (a) and (b). Define

$F : M \to R$ by $F(x) = \int (f \circ x) d\mu$.

F exists and is real-valued because f is continuous and range (f) is bounded; and since $f(o) = 0$, we have that F is additive.

Let $\{x_n\} \in M$ be a sequence s.t. $x_n \to x$ a.e., $x \in M$. Since f is continuous, $f \circ x_n$, $f \circ x \in M$ and $f \circ x_n \to f \circ x$ a.e., by Dominated Convergence Theorem it follows that $F(x_n) \to F(x)$. Q.E.D.

Since convergence a.e. in a finite measure space implies convergence in measure, we have

3.13 <u>Corollary</u>. Let F be an additive functional on M with (X,Σ,μ) a finite atom-free measure space. Then F satisfies the continuity condition (4): $x_n \to x$ in measure implies $F(x_n) \to F(x)$ iff (*) holds with f satisfying (a) and (b) in Theorem 3.12.

(The Proof is similar to the above theorem.)

If the space (X,Σ,μ) is σ-finite and $\mu(X) = \infty$ then the above theorems become:

3.14 <u>Theorem</u>. The additive functional F on $L_\infty(\mu)$ or M satisfies condition (3) in 3.12 or condition (4) in 3.13 respectively iff $F \equiv 0$.

<u>Proof</u>: Case (i). Let $A \in \Sigma$ be s.t. $\mu(A) = \mu(X - A) = \infty$. Since (X, Σ, μ) is atom-free $\exists$ $A_1, A_2 \in \Sigma$, $A_1 \cap A_2 = \phi$ s.t. $A_1 \cup A_2 = A$ and $\mu(A_1) = \mu(A_2) = \infty$.

If r is any constant then $r\chi_A$, $r\chi_{A_1}$ and r_{A_2} are equimeasurable. Since $r\chi_A = r\chi_{A_1 \cup A_2} = r\chi_{A_1} + r\chi_{A_2}$ and $\mu\{\text{supp } r\chi_{A_1} \cap \text{supp } r\chi_{A_2}\} = 0$ we have, by the additivity of F, that $F(r\chi_A) = F(r\chi_{A_1}) + F(r\chi_{A_2}) = 2F(r\chi_A)$ (by equimeasurability) and which implies that $F(r\chi_A) = 0$.

Case (ii). Assume that for any $A \in \Sigma$, $\mu(A) < \infty$, there exists A_1, A_2, $A_1 \supset A_2$ which together with their complements are of infinite measure and $A = A_1 \sim A_2$, then again by the additivity of F, we have $F(r\chi_A) = 0$ when $\mu(A) < \infty$. Hence if x is a simple function, then $F(x) = 0$. Now if X is a countable valued function for which each value has support and complement of infinite measure, then as above we get that $F(x) = 0$.

SECTION 4

REPRESENTATION OF ADDITIVE FUNCTIONALS ON L^P SPACES

4.1 Definition. For $0 < p < \infty$, $L^p_\mu(X) = \{$all real-valued measurable functions f on X s.t. $\| f \|_p = (\int_X |f|^p d\mu < \infty\}$.

For $1 \leq p < \infty$, L^p is a Banach space but this is no longer true for $0 < p < 1$, for the triangle inequality does not hold for $0 < p < 1$. But instead we have:

4.2 Proposition. Let $0 < p < 1$ and let $X = L_p[0,1]$ then

(a) $$\| f + g \|_p \leq 2^{\frac{1}{p}-1} (\| f \|_p + \| g \|_p) .$$

(b) If for any two functions $f,g \in X$, $d(f,g) = \int |f - g|^p d\mu$ then d is a metric where $d(f,g) = 0 \Rightarrow f = g$ a.e.

Proof: Firstly we show

(1) $$2^{p-1}(1 + x^p) \geq (1 + x)^p \quad \text{for } 0 < p < 1$$

Let $f(x) = 2^{p-1}(1 + x^p) = (1 + x)^p$

and $f'(x) = 2^{p-1}.p\, x^{p-1} - p(1 + x)^{p-1} = 0 \Rightarrow 2x = 1 + x$

i.e. $x = 1$

and $f''(x) = 2^{p-1}.p(p - 1)x^{p-2} - p(p - 1)(1 + x)^{p-2}$

$f''(1) = p(p - 1)[2^{p-1} - 2^{p-2}] < 0$

Hence $x = 1$ is a maximum point for $f(x)$ and $f(1) = 0$.

Thus we have $2^{p-1}(1+x^p) \geq (1+x)^p$.

(2) We show that $2^{2(\frac{1}{p}-1)}(1+x) > (1+x^p)^{1/p}$ for $0 < p < 1$.

Let $g(x) = 2^{2(\frac{1}{p}-1)}(1+x) - (1+x^p)^{1/p}$

and $g'(X) = 2^{2(\frac{1}{p}-1)} - \frac{1}{p}(1+x^p)^{\frac{1}{p}-1} .p\ x^{p-1} = 0$

i.e. $4 = (1+x^p)x^{-p}$ i.e. $4 = x^{-p}+1$ or $x = (\frac{1}{3})^{1/p}$.

Also $g''(x) = -[(\frac{1}{p}-1)(1+x^p)^{\frac{1}{p}-2} .x^{p-1} + (p-1)x^{p-2}(1+x^p)^{\frac{1}{p}-1}]$

and $g''((\frac{1}{3})^{1/p}) > 0$.

Hence $x = (\frac{1}{3})^{1/p}$ is a minimum point for $g(x)$ and we have (2).

Now for the proof of (a). Assume that $|f(x)| \neq 0$, then

$$|f(x) + g(x)|^p \leq |f(x)|^p . (1 + \frac{|g(x)|}{|f(x)|})^p$$

$$\leq |f(x)|^p . 2^{p-1} (1 + \frac{|g(x)|^p}{|f(x)|^p}) \quad \text{by (1)}$$

$$\leq 2^{p-1}(|f(x)|^p + |g(x)|^p)$$

$$\| f+g \|^p = \int |f(x) + g(x)|^p d\mu \leq 2^{p-1}[\int |f(x)|^p d\mu + \int |f(x)|^p d\mu]$$

$$= 2^{p-1}(\| f \|^p + \| g \|^p) .$$

Suppose that $\| f \| \neq 0$, then

$$\| f+g \|_p < 2^{\frac{p-1}{p}} (\| f \|_p^p + \| g \|_p^p)^{1/p} = 2^{1-\frac{1}{p}} \| f \|_p (1 + \frac{\| g \|^p}{\| f \|^p})^{1/p}$$

$$\leq 2^{1-\frac{1}{p}} \| f \|_p \cdot 2^{2(\frac{1}{p}-1)} \cdot (1 + \frac{\| g \|_p}{\| f \|_p}) \quad \text{by (2)} .$$

$$= 2^{\frac{1}{p}-1} (\| f \|_p + \| g \|_p)$$

$$\| f+g \|_p \leq 2^{\frac{1}{p}-1} (\| f \|_p + \| g \|_p) .$$

(b) If $d(f,g) = \int |f-g|^p d\mu$, then

(i) $d(f,g) \geq 0$ and $d(f,f) = 0$.

(ii) $d(f,g) = 0 \;\therefore\; f = g$ a.e.

(iii) $d(f,g) = d(g,f)$.

(iv) Let $f,g,h \in X$, then $d(f,g) = \int |f-g|^p d\mu = \int |f-h+h-g|^p d\mu$

$$\leq \int 2^{p-1}(|f-h|^p + |h-g|^p) d\mu \qquad \text{(by (1)}$$

$$\leq \int |f-h|^p d\mu + \int |h-g|^p d\mu = d(f,h) + d(h,g)$$

Hence d is a metric on X .

So, for $0 < p < 1$, we have somewhat weaker condition $\| f_1 + f_2 \|_p \leq 2^{\frac{1}{p}-1} [\| f_1 \|_p + \| f_2 \|_p]$ and we have that X is a linear topological space. It follows from the theorems of D. H. Hyers [9] and J. V. Wehausen [10] that such a linear topological space L^p in which the neighbourhoods of a point f_o are spheres of radius $\epsilon > 0$, can be given an equivalent Frechet metric. This suggests that while many theorems on

Banach spaces which can be applied to the space $L_\mu^p(X)$ with $p \geq 1$ may fail to hold in spaces where $0 < p < 1$, there may still remain many theorems on Frechet spaces and pseudo-normed spaces which may be applicable.

In general it appears that the class of linear functionals is a subclass of the class of additive functionals. But M.M. Day [7] has shown that if the underlying measure space is atom-free then any linear functional on L^p, $0 < p < 1$ is identically zero, which proves that almost no results depending on the use of linear functionals can be usefully applied to these spaces.

In this section we firstly give the proof of the theorem of N. Friedman and M. Katz [3] which is the general representation theorem for an additive functional defined on L_p, $p > 0$, which reduces to the standard representation theorem for linear functionals when $p \geq 1$.

The methods used in the proofs will be the same as in [4].

Let M be the vector space of all real-valued functions defined on X and for each $f \in M$ there is defined a number $\| f \| \geq 0$ which may be regarded as a generalized norm. We consider a corresponding space M^1 and say that:

4.3 <u>Definition.</u> $F : M \to M^1$ is an ADDITIVE TRANSFORMATION if

(1) Continuity: $\forall\ \epsilon > 0$ and $k > 0\ \exists\ \delta = \delta(k,\epsilon)$ s.t. $\| f \| \leq k$, $\| g \| \leq k$ and $\| f - g \| \leq \delta$ $\| F(f) - F(g) \| \leq \epsilon$ for $f,g \in M$.

(2) Boundedness: $\forall\ k > 0\ \exists\ H = H(k)$ s.t. $\| f \| \leq k$ $\| F(f) \| \leq H$.

(3) Additivity: $F(f+g) = F(f) + F(g)$ if $f(s).g(s) = 0$ for $s \in X$.

Let (X,Σ,μ) be a finite atom-free measure space and $M = L_p(X,\Sigma,\mu)$, $p > 0$.

4.4 <u>Theorem.</u> F is an additive functional on L_p iff

$$F(f) = \int_X K(f(s),s)\alpha(s)d\mu \quad , \qquad f \in L_p$$

where (i) $K(0,s) = 0$

(ii) $K(x,s)$ is a measurable function of $s\ \forall\ x$.

(iii) $K(x,s)$ is a continuous function of x for $\alpha d\mu$ - a.a.s .

(iv) $\forall\ k > 0\ \exists\ H = H(k)$ s.t. $|x| \leq b$ implies $|k(x,s)| \leq H$ for $\alpha d\mu$ - a.a.s .

(v) if $F(f(s)) = K(f(s),s)\alpha(s)$ then F is a transformation from L_p to L_1 .

Proof of the Lemma is given in Lemma 4.12 next and condition (v) follows by utilizing the conditions (1) and (2) of Definition 4.3 for F .

4.5 Definition. Let (X,Σ) be a measurable space. An extended real-valued set function μ defined on Σ is called a signed measure if it satisfies the following:

(i) μ assumes at most one of the values $+\infty$, $-\infty$

(ii) $\mu(\phi) = 0$

(iii) $\mu(\bigcup_{1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for any sequence E_i of disjoint measurable sets where the equality taken means that the series on the right converges absolutely if $\mu(\bigcup E_i) < \infty$ and that it properly diverges otherwise.

4.6 Definition. If μ is a signed measure then $|\mu|E = \mu^+(E) + \mu^-(E)$ is called the total variation of μ where μ^+ and μ^- are called the positive and negative variations of μ.

4.7 Definition. A measure ν is said to be absolutely continuous with respect to a measure μ if $\nu(A) = 0$ $\forall$ set A for which $\mu(A) = 0$.

4.8 Lemma. $\forall$ $h \in (-\infty,\infty)$ there exists a function $K_h(s)$ which is a measurable function of s and is uniquely defined up to μ-null sets s.t.

(a) $K_o(s) = 0$, $s \in X$

(b) $F(h\chi_B) = \int_B K_h(s)d\mu$, $B \in \Sigma$

Proof: Define $\mu_h(B) = F(h\chi_B)$. Clearly $\mu_h(\phi) = 0$.

Conditions (1), (2) and (3) in 4.3 imply that μ_h is a signed measure and since $|\mu_h|(E) = \mu_h^+(E) + \mu_h^-(E) < \infty$ for each set $E \in \Sigma$ μ_h is of finite variation on Σ .

Finally if $\mu(B) = 0$, $B \in \Sigma$ $\mu_h(B) = \int_B K_h(s)d\mu = 0$ μ_h is absolutely continuous w.r.t. μ . Hence by Radon-Nikodym Theorem, there exists a function K_h that satisfies (a) and (b) .

4.9 Remarks. (1) If F is linear then we have

$$\mu_h(B) = F(h\chi_B) = hF(\chi_B) = h.\mu_1(B) \ , \quad B \in \Sigma$$

and hence $K_h(s) = hK_1(s)$, $s \in X$.

(2) Let $K_h(s) = K_*(h,s)$, then we have

$$F(h\chi_B) = \int_X K_*(h\chi_B^{(s)},s)d\mu \ , \quad B \in \Sigma \ .$$

4.10 Lemma. There exists a kernel $K(x,s)$ and d satisfying (i) - (iv) of Theorem 3.4.

Proof: By Lemma 11 in [4], it can be shown that $K_*(x,s)$ in 4.9 (2) is continuous in x for μ - a.a.s. Next by the proof of Lemma 12 in [4], we can obtain K and α to fulfil the requirements.

4.11 Remark. If F is linear, then

$$F(h\chi_B) = hF(\chi_B) \quad \text{and we have that}$$

$$K(x,s) = x \quad \text{and} \quad \alpha(s) = K_1(s) \ .$$

Now the following lemma yields the proof of the Theorem.

For each $f \in L_p$ define

$$F_1(f) = \int_X K(f(s),s)\alpha s d\mu \ .$$

4.12 Lemma. $F_1(f) = F(f)$, $f \in L_p$.

Proof: Since a simple function f is a finite linear combination of characteristic functions, by the additivity of F, we have that, if f is a simple function, then

$$F(f) = \int_X K(f(s),s)\alpha(s)d\mu = F_1(f)$$

Now suppose that $\exists$ $b > 0$ s.t. $|f(s)| \le b$. There exists a sequence $\{f_n\}$ of simple functions, $|f_n(s)| \le b$, s.t. $f_n \to f$ a.e. and $\lim_n \| f_n - f \|_p = 0$. Hence by the continuity of F , we have $\lim_n F(f_n) = F(f)$ and since $K(x,s)$ is a continuous function of x for $\alpha d\mu$ - a.a.s , we have that $\lim_n K(f_n(s),s)\alpha(s) = K(f(s),s)\alpha(s)$ for μ - a.a.s. and since $|f_n| \le b$ $|K(f_n(s),s)| \le H = H(b)$. Hence by bounded convergence theorem, we get that

$$\lim_n F_1(f_n) = \lim_n \int_X K(f_n(s),s)\alpha(s)d\mu = \int_X K(f(s),s)\alpha(s)d\mu = F(f)$$

and since $F_1(f_n) = F(f)$, it follows that $F_1(f) = F(f)$ for bounded f .

Finally consider $f \in L_p$ and let $E = \{s : K(f(s),s)\alpha(s) > 0\}$ and $G = \{s : K(f(s),s)\alpha(s) < 0\}$.

To make f_n bounded, we define

$$f_n(s) = f(s) \qquad \text{if} \quad |f(s)| \leq n \quad \text{and}$$

$$f_n(o) = 0 \qquad \text{if} \quad |f(s)| > n$$

and hence $\lim_n \| f_n - f \|_p = 0$.

Hence by condition (1) in 3.3, we have

$$\lim F(f_n) = F(f) \text{ ,} \quad \text{and since } f_n \text{ is bounded,}$$

$$F_1(f_n) = F(f_n) \text{ .}$$

Let $A_n = \{s : |f(s)| \leq n\}$, $E_n = E \cap A_n$, $F_n = G \cap A_n$

$$f_{n,1} = \chi_{E_n} f_n \quad \text{and} \quad f_{n,2} = \chi_{F_n} f_n \text{ .}$$

We have $\| f_n \|_p \leq \| f \|_p$, hence $\| f_{n,i} \|_p \leq \| f \|_p$, $i = 1,2$. By condition (2) in 4.3, we have

$$F(f_{n,i}) = \int_X K(f_{n,i}) \leq B(\| f \|_p) \qquad i = 1,2 \text{ .}$$

Therefore $F(f_{n,i}) = \int_X K(f_{n,i}(s),s)\alpha(s)d\mu$ $i = 1,2$ are uniformly bounded in n and we can write

$$F(f_{n,1}) = \int_X K(f_{n,1}(s),s)\alpha(s)d\mu = \int_{E_n} K(f(s),s)\alpha(s)d\mu$$

and by Lebesgue monotone convergence theorem, we have

$$\lim_n F(f_{n,i}) = \int_E K(f(s),s)\alpha(s)d\mu \ .$$

Similarly $$\lim_n F(f_{n,2}) = \int_G K(f(s),s)\alpha(s)d\mu$$

Therefore $F(f) = \lim_n F(f_n) = \lim_n \{F(f_{n,1}) + F(f_{n,2})\} = F_1(f)$.

Q.E.D.

Now in the following theorems we again prove the integral representation of an additive functional on L_p-spaces, $1 \leq p < \infty$, under different continuity conditions on F , when the underlying measure space is atom-free and finite or σ-finite. For this purpose we shall follow [2].

4.13 <u>Theorem</u>. Let (X,Σ,μ) be a finite atom-free measure space. F is an additive functional on $L_p(\mu)$, $1 \leq p < \infty$, then the following are equivalent: (3): $x_n \to x$ a.e. $\Rightarrow F(x_n) \to F(x)$ and

(*) $$F(x) = \int_X (f\circ x)d\mu \quad \forall \quad x \in L_p(\mu)$$

where f satisfies the conditions:

(a) f is continuous and $f(o) = 0$

(b) range (f) is bounded.

Proof: Let f satisfy the continuity condition (a) and also condition (b). Then as in Section 2, if $F : L_p(\mu) \to R$ is a functional defined by $F(x) = \int_X (f \circ x) d\mu \;\; \forall \;\; x \in L_p(\mu)$, it is a well-defined additive functional and it satisfies condition (3): $x_n \to x$ a.e. $F(x_n) \to F(x)$. Conversely, if F is an additive functional on $L_p(\mu)$ which satisfies the continuity condition (3) then for a sequence $\{x_n\} \subset L_\infty(\mu)$ s.t. $x_n \to x$ a.e., $x \in L_\infty(\mu)$ and $|x_n|, |x| \leq b$ for some constant $b > 0$, we have that $F_1 = F/_{L_\infty(\mu)}$ also satisfies (3). Then there exists a sequence $\{r_n\}$ s.t. $|r_n| \to \infty$ and $1 \leq |f(r_n)| \nearrow \infty$. By the strong intermediate value property $\exists$ a decreasing sequence $\{B_n\}$ of measurable sets s.t. $\mu(B_n) = \frac{\mu(X)}{|f(r_n)|}$. Let $x_n = r_n \chi_{B_n}$. $x_n \in M$ and $x_n \to 0$ a.e. However, since $x_n \in L_\infty(\mu)$, we have $F(x_n) = \int (f \circ x_n) d\mu = \pm \mu(X)$ which contradicts that $F(x_n \to 0/$. Thus range (f) is bounded.

Now as usual, for $x \in M$, there exists a sequence $\{S_n\}$ of simple functions s.t. $S_n \to x$ a.e. and by the continuity of f, $f \circ S_n \to f \circ x$ boundedly a.e. Thus $f \circ x \in L_1(\mu)$ and $\int (f \circ x) d\mu = \lim_n \int (f \circ S_n) d\mu = \lim F(S_n) = F(x)$. The uniqueness of f follows from Theorem 3.8 by applying it to F_1.

Q.E.D.

Since (X,Σ,μ) is a finite measure space, convergence a.e. implies convergence in measure, we have

4.14 Corollary. If Condition (3) in Theorem 4.13 is replaced by a condition (4) $x_n \to x$ in measure $\Rightarrow F(x_n) \to F(x)$, then the above theorem is still true.

If the underlying measure space (X,Σ,μ) is atom-free σ-finite and $\mu(X) = \infty$ then we prove that $F \equiv 0$.

4.15 Theorem. If (X,Σ,μ) is σ-finite, $\mu(X) = \infty$ and F is an additive functional on $L_p(\mu)$, $1 \leq p < \infty$, then F satisfies condition (3) in Theorem 4.13 iff $F \equiv 0$.

Proof: Let $x_1 = x_1\chi_{A_1}$, if $A_1 \in \Sigma$ and $0 < \mu(A) < \infty$.

$$x_1 \in L_p(\mu) .$$

Since (X,Σ,μ) is atom-free, we can find a sequence $\{A_n\}_{n\geq 1}$, $A_i \cap A_j = \phi$, $i \neq j$ and $A_i \in \Sigma$ $\forall$ $i \geq 1$, s.t.

$$\mu(A_n) = \mu(A_1) \qquad \forall \ n \geq 1 .$$

Thus $x_n = x_n\chi_{A_n}$ for $n \geq 2$ and x_1 are equimeasurable and for this sequence $\{x_n\}$ we have that $x_n \to 0$ a.e.

However since F is additive, we have that

$$F(x_n) = F(x_1) \quad \text{is constant.}$$

$F(x) = 0$ for functions x where $\mu(\operatorname{supp} x) < \infty$

By (3) of 3.12, it follows that $F(x) = 0 \;\; \forall \;\; x \in L_p(\mu)$.

The converse is vacuously true. Q.E.D.

4.16 Remark. M. M. Day has proved that any linear functional on $L^p_\mu(X)$, $0 < p < 1$ is identically zero, where μ is Lebesgue measure and (X,Σ,μ) is atom-free, but from Theorems 3.14 and 4.15 we have seen that for $1 \leq p \leq \infty$, any nonlinear additive functional F on $L_p(\mu)$ that satisfies (3) of Theorem 3.12 is identically zero when the underlying measure space (X,Σ,μ) is atom-free σ-finite and $\mu(X) = \infty$.

Now for analogue of Corollary 4.14 in the σ-finite case, we have

4.17 Theorem. Let (X,Σ,μ) be a σ-finite atom-free measure space and suppose $\mu(X) = \infty$. Let F be an additive functional on $L_p(\mu)$, $1 \leq p < \infty$. Then the following conditions are equivalent;

(4) $x_n \to x$ in measure $\Longrightarrow F(x_n) \to F(x)$ and

(*) $F(x) = \int_X (f \circ x) d\mu \;\; \forall \;\; x \in L_p(\mu)$ with an f satisfying:

(a*) f is continuous and $f[-h,h] = 0$ for some $h > 0$.

Proof: Let $B \in \Sigma$, $0 < \mu(B) < \infty$ and let μ_B be the restriction of μ to B. Let $y' = y\chi_B$, $y' \in L_p(\mu)$. Define F_B on $L_p(\mu_B)$ by $F_B(y) = F(y')$ where F is an additive functional on $L_p(\mu)$, $1 \leq p \leq \infty$. F_B is a well-defined functional and satisfies (4). Hence by Corollary 4.14 we have

$$F_B(y) = \int_X f \circ y \, d\mu \quad \text{where} \quad f : R \to R \quad \text{is continuous,}$$

$f(o) = 0$ and range (f) is bounded.

Now we claim that this f determined by F_B is independent of $B \in \Sigma$. For if $C \in \Sigma$ and $0 < \mu(C) \leq \mu(B) < \infty$, we have by the strong intermediate value property that, there exists $B_1 \in \Sigma$, $B_1 \subset B$ s.t. $\mu(B_1) = \mu(C)$. Since for any real number r, $r\chi_{B_1}$, $r\chi_C$ are equimeasurable, we have that if f,g represent F_B and F_C then

$$F(r\chi_{B_1}) = F(r\chi_C) \Longrightarrow F_B(r\chi_{B_1}) = F_C(r\chi_C)$$

$$f(r)\mu(B_1) = g(r)\mu(C)$$

$$f(r) = g(r) \quad \text{i.e.} \quad f = g \text{ since } \mu(C) = \mu(B_1) \neq 0$$

Hence if $x \in L_p(\mu)$, $\mu(\text{supp } x) < \infty$ then with f determined above we have

$$F(x) = \int_{\mathrm{supp}(x)} (f \circ x)\,d\mu = \int (f \circ x)\,d\mu \quad \forall \quad x \in L_p(\mu)\ ,$$
$$\mu(\mathrm{supp}\ x) < \infty\ .$$

Now if f does not satisfy condition (a)*, then there exists a null sequence $\{\alpha_n\}$ of reals s.t. $f(\alpha_n) \neq 0 \quad \forall\ n$, where a sequence $\{\alpha_n\}$ is called null if it converges to zero.

As in Theorem 4.15, let $\{A_i\}_{i \geq 1}$ be a sequence of pairwise disjoint sets $A_i \in \Sigma$ s.t.

$$\mu(A_i) = \mu(A_1) < \infty \quad , \quad \forall \quad i \geq 1\ .$$

Therefore by the addivity of F, for any integer m, we have $F(\alpha_n \chi_{\bigcup_1^m A_i}) = m\ F(\alpha_n \chi_{A_1}) = m\ f(d_n)\mu(A_1)$.

Let $x_n = \alpha_n \chi_{\bigcup_1^m A_i}$ where we choose m s.t.

(i) $|F(x_n)| \geq 1$

(ii) ess. sup $|x_n| = |\alpha_n|$

where $\mathrm{ess.sup}\ |x_n(t)| = \inf\ \{M : \mu(t : |x_n(t)| > M) = 0\}$ and since $x_n \to 0$ in measure we have contradiction, for $f(x_n) \not\to 0 = F(o)$. Hence f satisfies (a)* .

Let $E_{\frac{1}{n}} = \{t \in X : |s(t)| \geq \frac{1}{n}\}$ and for arbitrary $x \in L_p(\mu)$ let $x_n = x\chi_{E_{\frac{1}{n}}}$. Thus $\mu[\mathrm{supp}\ (x_n)] = \mu(E_{\frac{1}{n}}) < \infty$ and $x_n \to x$ in measure by Lebesgue's limit theorem, we have

$$F(x) = \lim F(x_n) = \int f\bullet(x\chi_{E_{\frac{1}{n}}})d\mu$$

and since for h chosen in (a)* we have for all n with $\frac{1}{n} < h$ by (a)* that $f\circ x\chi_{E_{\frac{1}{n}}} = f\circ x$, it follows that

$F(x) = \int(f\circ x)d\mu$.

Conversely, let f satisfy (a)* and (b). For $x \in L_p(\mu)$ we have as above that $\forall$ integer m, $\mu(E_{\frac{1}{n}}) < \infty$ and (a)* gives that $\forall$ m, $\frac{1}{m} < h$, $f\circ x = f\circ(x\chi_{E_{\frac{1}{m}}})$ which together with (b) implies that $f\circ x$ is dominated by a bounded function with supp. $\subset E_{\frac{1}{m}}$.

$f(x) \in L_1(\mu)$ and so $F(x) = \int(f\circ x)d\mu$ is defined $\forall$ $x \in L_p(\mu)$ and is additive. Now suppose $x_n \to x$ in measure, we have $\forall$ $m > 0$

(i) $x_n\chi_{E_{\frac{1}{m}}} \xrightarrow{\mu} x\chi_{E_{\frac{1}{m}}}$

(ii) $x_n\chi_{E_{\frac{1}{m}}} \xrightarrow{\mu} x\chi_{E_{\frac{1}{m}}}$ where $\tilde{E}_{\frac{1}{m}} = X - E_{\frac{1}{m}}$

Fix m s.t. $\frac{1}{m} < \frac{h}{2}$.

Also we see that conditions (a)* and (b) imply conditions (a) and condition (b) that range (f) is bounded.

We have by Corollary 4.14 and with the choice of m that (i) implies (iii)

$$\int (f\circ x_n \chi_{E_{\frac{1}{m}}})d\mu \to \int (f\circ x\chi_{E_{\frac{1}{m}}})d\mu = \int (f\circ x)d\mu = F(x) ,$$

and if we take $C_{n,h} = \{t \in E_{\frac{1}{m}} : |x_n(t)| \geq h\} \subset \{t : |x_n(t)-x(t)| \geq \frac{h}{2}\}$ we have by (a)* that $\operatorname{supp}(f\circ x_n\chi_{E_{\frac{1}{m}}}) \subset C_{n,h}$ and by (ii) we have that $\mu(C_{n,h}) \to 0$.

Hence the boundedness of range (f) implies that for $n \geq 1$, the functions $f\circ x_n\chi_{E_{\frac{1}{m}}}$ are dominated uniformly by bounded functions with support $\subset C_{n,h}$ which implies

(iv) $$\int f\circ(x\chi_{E_{\frac{1}{m}}})d\mu \to 0$$

Thus by (iii) and (iv) we have that

$$\begin{aligned}\lim_n F(x_n) &= \int [f\circ(x_n\chi_{E_{\frac{1}{m}}}) + f\circ(x_n\chi_{E_{\frac{1}{m}}})]d\mu \\ &= \lim_n \int [f\circ(x_n\chi_{E_{\frac{1}{m}}})]d\mu + \lim_n \int f\circ(x_n\chi_{E_{\frac{1}{m}}})d\mu \\ &= F(x) .\end{aligned}$$

Q.E.D.

4.18 <u>Theorem</u>. Let (X,Σ,μ) be a finite atom-free measure space and F an additive functional on $L_p(\mu)$, $1 \leq p < \infty$. Then F satisfies (5): $x_n \to x$ in L_p norm $\Longrightarrow F(x_n) \to F(x)$,

iff (*) holds with an f satisfying conditions

(a) f is continuous and $f(o) = 0$

(c) $|f(r)| \leq k(1 + |r|)^p \quad \forall \; r \in R$ and some $K \geq 0$.

<u>Proof</u>: Since convergence in L_p-norm implies convergence in measure in a finite measure space, for F an additive functional on $L_p(\mu)$ and $F_1 = F/_{L_\infty(\mu)}$ we have as in Theorem 3.8 that $F(x) = F_1(x) = \int(f\circ x)d\mu \quad \forall \quad x \in L_\infty(\mu)$ where f satisfies (a). Now suppose f does not satisfy (c). Then there exists a sequence $\{r_n\} \subset R$ s.t. $|f(r_n)| \geq n(1 + |r_n|)^p$. Let $\{B_n\}$ be a sequence of sets in X s.t. $B_n \in \Sigma \;\forall\; n$ and $\mu(B_n) = \dfrac{1}{|f(r_n)|}\mu(X)$.

Since $\int|r_n\chi_{B_n}|^p d\mu = |r_n|^p\mu(B_n) = |r_n|^p \dfrac{1}{|f(r_n)|}\mu(X)$

$$\leq \frac{r_n^p \mu(X)}{n(1+|r_n|)^p} \leq \frac{1}{n}\mu(X) \to 0$$

we have that $r_n\chi_{B_n} \to 0$ in L_p-norm.

But $F(r_n\chi_{B_n}) = \int(f\circ r_n\chi_{B_n})d\mu = f(r_n)\mu B_n) = \pm\mu(X) \neq 0$ which contradicts the continuity of F. Thus f satisfies (c).

Now if $B = \{t : |x(t)| < c_1\}$ and $\tilde{B} = X \sim B$ condition (c) implies that there exists constants c_1 and k_1 s.t. for $|r| \geq c_1$, $|f(r)| \leq k|r|^p$. Thus $|f \circ x|$ is bounded on B and $|f \circ (x\chi_{\tilde{B}})| \leq K|x\chi_{\tilde{B}}|^p$, and since $x \in L_p(\mu)$ we can select a sequence $\{x_n\} \subset L_\infty(\mu)$ s.t. $x_n \to x$ a.e. because $L_\infty(\mu)$ is a dense subset of $L_p(\mu)$.

Thus by the continuity of F and by previous theorems we have that $F(x) = \lim_n F(x_n) = \lim_n \int (f \circ x_n) d\mu$ if

$$f \circ x_n \to f \circ x \in L_1(\mu) .$$

Now by the continuity of f and the fact that $x_n \to x$ a.e. we have $f \circ x_n \to f \circ x$ a.e. Also by (c) $\exists$ constants K and M_1 s.t. for $|t| > K$, $|f(t)| < M_1|t|^p$.

Thus from (i) we have (i)$'$: $\forall\ \epsilon > 0\ \exists\ \delta'(\epsilon) > 0$ s.t. for $\mu(B) < \delta'(\epsilon)$ we get $\int_B |f \circ x_n| d\mu < \epsilon\ \ \forall\ n \geq 1$ which is valid for $\delta'(\epsilon) = \min\{\delta(\frac{\epsilon}{2M_1}), \frac{\epsilon}{2K_1}\}$, where we let $K_1 = \sup_{|t| \leq K} |f(t)|$.

Thus for the case $p = 1$, by Vitali's Theorem, we have, for finite measure space (X, Σ, μ) that $f \circ x_n \to f \circ x \in L_1(\mu)$.

Hence we have the required representation (*).

Conversely if $f : R \to R$ satisfies (a) and (c) then the functional $F(x) = \int_X (f \circ x) d\mu$ is well-defined and has the additive property.

Now if a sequence $\{x_n\}_{n\geq 1} \subset L_p(\mu)$ is such that $\| x_n - x \|_p \to 0$, then as above, for every subsequence $\{x_m\}$ of $\{x_n\}$ which converges pointwise as well as in norm, we have that $f \circ x_m \to f \circ x \in L_1$ and since every norm convergent sequence in $L_p(\mu)$ converges in measure, it contains an a.e. convergent subsequence.

It follows that every subsequence of $\{f \circ x_n\}_{n\geq 1}$ contains a subsequence which converges in L_1 norm to $f \circ x$. Hence $\{f \circ x_n\}_{n\geq 1}$ itself converges to $f \circ x$ in L_1 norm and hence

$$\lim_n F(x_n) = \lim_n \int_X f \circ x_n \, d\mu = \int_X f \circ x \, d\mu = F(x) .$$

An analogue to the above theorem, when the space is σ-finite, is

4.19 <u>Theorem.</u> Let (X, Σ, μ) be a σ-finite atom-free measure space with $\mu(X) = \infty$ and let F be an additive functional on $L_p(\mu)$. Then the following conditions are equivalent:

5: $x_n \to x$ in L_p norm $\Longrightarrow F(x_n) \to F(x)$ and

(*) $F(x) = \int_X (f \circ x) d\mu \quad \forall \; x \in L_p(\mu)$ with f satisfying conditions

(a) f is continuous and $f(o) = 0$

(d) $|f(r)| \leq k|r|^p \quad \forall \; r \in R$ and some $K \geq 0$.

Proof: If F is an additive functional on $L_p(\mu)$ which satisfies condition (5) then by Theorem 4.17 for all functionals F_B obtained from F defined by $F_B(y) = F(y\chi_B)$, $0 < \mu(B) < \infty$, there exists a unique continuous function $f : R \to R$ which satisfies (a) and (c) by Theorem 4.18. We claim that f satisfies (d).

Suppose not. Then there exists a null sequence $\{\alpha_n\}$ of non-zero reals s.t. $|f(\alpha_n)| > n|\alpha_n|^p$.

Let $\{B_n\}$ be a sequence in X s.t. $B_n \in \Sigma$ and since (X,Σ,μ) is atom-free, we have $\mu(B_n) = \dfrac{1}{|f(\alpha_n)|}$.

Since $\int|\alpha_n\chi_{B_n}|^p d\mu = |\alpha_n|^p\mu(B_n) = |\alpha_n|^p \dfrac{1}{|f(\alpha_n)|} < \dfrac{|\alpha_n|^p}{n|\alpha_n|^p} \to 0$

we have that $\{\alpha_n\chi_{B_n}\} \subset L_p(\mu)$ and $\|\alpha_n\chi_{B_n}\|_p \to 0$.

However $F(\alpha_n\chi_{B_n}) \not\to 0 = F(o)$ which contradicts (5). This establishes (d).

Now for the representation of F , we have for $x \in L_p(\mu)$ and for E_c defined in Theorem 4.17 that if $x_n = x\chi_{E_n}$ then $\|x_n - x\|_p \to 0$ and $|x_n| \le |x|$. Thus for this sequence $\{x_n\}$, $f\circ x_n \to f\circ x$ a.e. and by (d) , $|f\circ x_n| \le K|x|^p \in L_1(\mu)$.

By Lebesgue dominated convergence theorem, we have that

$$F(x) = \lim_n F(x_n) = \lim_n \int f\circ(x\chi_{E_n})d\mu = \int(f\circ x)d\mu \ .$$

For the converse we need only show the continuity Property (5) of F, for the conditions (a) and (d) on F define $F(x)$ to be an additive functional on $L_p(\mu)$.

Appealing again to the Vitali's Theorem, if $x_n \to x$ in L_p norm, then there exist subsequences $\{x_m\}$ s.t. $\| x_m - x \|_p \to 0$ as well as $x_m \to x$ a.e. and for such subsequences, by the continuity of f we have that $f \circ x_m \to f \circ x$ a.e. Thus Vitali's Theorem ensures that every subsequence of $\{f \circ x_n\}$ contains a subsequence $\{f \circ x_m\}$ which converges to $f \circ x$ in L_1 norm and hence the sequence $\{f \circ x_n\}_{n \geq 1}$ itself converges to $f \circ x$ in L_1 norm.

Hence $\lim\limits_n F(x_n) = \lim\limits_n \int (f \circ x_n) d\mu = \int (f \circ x) d\mu = F(x)$.

SECTION 5

EXAMPLES AND COUNTER EXAMPLES ON REPRESENTATION OF ADDITIVE FUNCTIONALS

Let $B = L_\infty(\mu)$ be the set of all essentially bounded real-valued measurable functions on X. Let $F : R \to R$ be continuous for which $f(o) = 0$. For every $x \in B$, consider

$$F(x) = \int_X (f\ x(t)d\mu(t) \qquad \text{-----------} \quad (*).$$

F satisfies

5.1 (a) If x,y have disjoint support, then since $\mu(\operatorname{supp} x \cap \operatorname{supp} y) = 0$ and $f(o) = 0$, we have $F(x+y) = F(x) + F(y)$.

5.1 (b) If $\{x_n\}$ is a sequence in B s.t. $x_n \to x$ a.e. and $|x_n|$, $|x| \leq M_1$ for some positive constant M_1 then $F(x_n) \to F(x)$ since $f(x_n) \to f(x)$ a.e. and $|f(x_n)| \leq \sup_{|y| \leq C} f(y) \Rightarrow$ by Lebesgue'd dominated convergence theorem that

$$\lim F(x_n) = \lim_n \int f\circ(x_n(t))d\mu(t) = \int f(x(t))d\mu(t) = F(x)$$

$\Longrightarrow$ F is continuous.

5.1 (c) If $x,y \in B$ s.t. x,y are equimeasurable, then

$F(x) = F(y)$. For if x,y are equimeasurable then so are $f(x)$ and $f(y)$ and hence

$$F(x) = \int_X f(x(t))d\mu(t) = \int_X f(y(t))d\mu(t) = F(y) .$$

In this section we will be concerned to the extent to which Properties 5.1 {(a), (b), (c)} characterize functionals of the type (*).

Our first example shows that Theorem 3.8 is false if the underlying measure space (X,Σ,μ) is atomic and we shall follow mainly V. J. Mizel and A. D. Martin [1] in this section.

5.2. Example. Let $X = \{1,2\}$, $\Sigma = \{$all measurable subsets of $X\}$. Let μ be the measure on (X,Σ) defined by $\mu(1) = m_1$, $\mu(2) = m_2$, $m_1 \neq m_2$.

For $x \in B(X,\Sigma)$, let $x(i) = x_i$, $i = 1,2$,

For each $x \in B(X,\Sigma)$ define the functional F by $F(x) = f_1(x_1)m_1 + f_2(x_2)m_2$ where $f_i : R \to R$ are continuous and $f_i(0) = 0$ for $i = 1,2$.

Thus by the continuity condition of R , if $x^{(n)} \to x^{(o)}$ boundedly then $x_1^{(n)} \to x_1^{(o)}$ and $x_2^{(n)} \to x_2^{(o)}$ implies that $F(x^{(n)}) = f_1(x_1^{(n)})m_1 + f_2(x_2^{(n)})m_2 \to f_1(x_1^{(o)})m_1 + f_2(x_2^{(o)})m_2 = F(x^{(o)})$.

For additivity of F :

(i) if $x_1, x_2 \in B$ s.t. $\text{supp } x_1 \cap \text{supp } x_2 = \phi$ then if $x_1 \neq 0$, and $x_2 \neq 0$ then $x_1 = x_1^{(1)} \chi_{(1)}$ and $x_2 = x_2^{(2)} \chi_{(2)}$ where $\mu(1) = m_1$, $\mu(2) = m_2$, implies that $F(x_1 + x_2) = F(x_1) + F(x_2)$.

(ii) Since $m_1 \neq m_2$, two proper subsets of X have equal measure iff they are equal, which implies that $x, y \in B$ are equimeasurable iff $x = y$.

And since $m_1 = \mu\{w : x(w) = x_1\} = \mu\{w : y(w) = x_1\}$ i.e. $\{w : x(w) = x_1\} = \{w : y(w) = x_1\}$ implies that $x = y$.
It follows that x, y are equimeasurable iff $x = y \Rightarrow F(x) = F(y)$.
Now if the theorem is to hold true we would have for some continuous function $f : R \to R$, $f(o) = 0$ that

$$F(x) = f_1(x_1)m_1 + f_2(x_2)m_2 = f(x_1)m_1 + f(x_2)m_2 .$$

But if $x_1 = 0$ for some x then

$$F(x) = f_2(x_2)m_2 = f(x_2)m_2 \quad \text{i.e.} \quad f_2 = f .$$

Similarly $f_1 = f$. But it is not necessary that $f_1 = f_2$ always. Q.E.D.

So we have seen in the above example that, presence of atoms in the measure space makes the Representation (*) false.

5.3 Theorem. If $X = (X,\Sigma,\mu)$ is a finite measure space and $X = A \cup C$ is the decomposition of X into atomic and atom free parts then the sufficient condition for Theorem 3.8 to hold true is that, for every atom A_i of X, $\mu(A_i) \leq \mu(C)$.

Proof: If $x \in B(X,\Sigma)$, let $x_a = x\chi_A$ and $x_c = x\chi_C$ and hence $x = x_a + x_c$ which by the additivity of F implies that

$$F(x) = F(x_a) + F(x_c) \qquad [\quad \text{supp } x_a \cap \text{supp } x_c = \phi]$$

Now $C = (C,\Sigma,\mu)$ becomes an atom-free measure space when μ is restricted to C and its measurable subsets, and in the same way we may take x_c as a member of $B_c = B(C,\Sigma,\mu)$. Hence as in Theorem 3.8, $\exists$ $f : R \to R$ continuous s.t.

$$F(x_c) = \int_C f(x_c(t))d\mu(t) = \int_C f(x(t))d\mu(t) .$$

Now $A_1, A_2, \ldots$ are atoms of X and any measurable function is constant on each A_i with value x_i, $i = 1,2,\ldots$.

If $\mu(A_i) = m_i$, $i = 1,2,\ldots$, then $\forall$ $x \in B(X,\Sigma)$, $x_a \in B(A,\Sigma,\mu)$ and since $A = \bigcup_1^\infty A_i$

$$F(x_a) = F(\sum_1^\infty x_i\chi_{A_i}) = F(\lim_{n\to\infty} \sum_1^\infty x_i\chi_{A_i}) = \lim_{n\to\infty} F(\sum_1^\infty x_i\chi_{A_i})$$

and since $\sum_1^n x_i\chi_{A_i} \to \sum_1^\infty x_i\chi_{A_i}$ boundedly we have

$$\lim_{n\to\infty} F(\sum_1^n x_i\chi_{A_i}) = \lim_{n\to\infty} \sum_1^n F(x_i\chi_{A_i}) = \sum_1^\infty f_i(x_i)m_i$$

where $f_i(y) = \dfrac{F(y\chi_{A_i})}{m_i}$ and $f_i(o) = \dfrac{F(0\chi_{A_i})}{m_i} = 0$ as in Theorem 3.8 .

Now if we can show that $f_i = f \quad \forall \quad i = 1,2,\ldots$, then

$$F(x_a) = \sum_1^\infty f(x_i)\, m_i = \int_A f(x(t))d\mu(t)$$

i.e. if $x \in B = B(X,\Sigma,\mu)$

then $$F(x) = F(x_a) + F(x_c) = \int_A f(x(t))d\mu(t) + \int f(x(t))d\mu(t)$$

$$= \int_X f(x(t))d\mu(t)$$

and hence the theorem follows .'

So we now show that $f = f_i$, $i = 1,2,\ldots$.
Since $\mu(A_i) \leq \mu(C)$, by nonatomicity of C , $\exists \; S_i \in \Sigma$, $S_i \subset C$ s.t. $\mu(S_i) = \mu(A_i)$, $i = 1,2,\ldots$, and hence for every real number a , $a\chi_{S_i}$ and $a\chi_{A_i}$ are equimeasurable and it follows that

$$F(a\chi_{S_i}) = F(a\chi_{A_i}) = f_i(a)m_i \qquad i = 1,2,\ldots \; .$$

So $\int_C f(a\chi_{S_i})d\mu = \int_C f(a)\chi_{S_i}d\mu = f(a)\mu(S_i) = f(a)\mu(A_i) = f(a)m_i$

and since $m_i > 0$,

$$f_i(a)m_i = f(a)m_i \qquad f_i(a) = f(a) \quad \forall \text{ real } a \; .$$

Q.E.D.

We give an example to show that the condition given in Theorem 5.3 is not necessary for the theorem to hold true. It may be possible that the atom-free part of X is empty while its atomic part is nonempty.

5.4 <u>Example</u>. Let $X = \{1,2,3,\dots\}$, Σ = all measurable subsets of X and let the measure μ be defined on Σ by $\mu(n) = \frac{1}{2^n}$, $n = 1,2,3,\dots$.

So here C = the atom-free part is empty but we show that the theorem still holds true.

For $x \in B$, take $x(n) = x_n$. The functional F which satisfies 5.1 (a) and 5.1 (b) can be defined by

$$F(x) = \sum_{n=1}^{\infty} \frac{f_n(x_n)}{2^n}$$

Now since $\frac{1}{2^n} = \sum_{k=n+1}^{\infty} \frac{1}{2^k}$, the sets $S_n = \{n\}$ and $T_n = \{n+1, n+2, \dots\}$ have the same measure and hence for any real $a \in (-\infty,\infty)$ $a\chi_{S_n}$ and $a\chi_{T_n}$ are equimeasurable and $\forall$ n

$$F(a\chi_{S_n}) = F(a\chi_{T_n}) \qquad \frac{f_n}{2^n} = \sum_{k=n+1}^{\infty} \frac{f_k}{2^k}$$

Also then $\frac{f_{n-1}}{2^{n-1}} = \sum_{k=n}^{\infty} \frac{f_k}{2^k}$ which implies by subtraction that

$$\frac{f_n}{2^n} - \frac{f_{n-1}}{2^{n-1}} = \frac{f_n}{2^n} \quad \text{i.e.} \quad f_n = f_{n-1} \ \forall \ n \geq 2$$

and hence the Theorem 5.3 is still true.

5.5 <u>Lemma</u>. Let $X = \{1,2,3,\ldots\}$ and define $\mu(n) = m_n$ where m_n is a positive number s.t. for every $n = 1,2,\ldots$

$m_n > \sum_{k=n+1}^{\infty} m_k$, then $\sum_{i\in I} m_i = \sum_{j\in J} m_j \implies I = J$ and hence two subsets of $X = (X,\Sigma,\mu)$ have the same measure only if they are equal.

<u>Proof</u>: Assume that $I \cap J = \phi$ and claim that $I \cup J = \phi$. Suppose $I \cup J \neq \phi$, let $n_o = \min(I,J)$ and thus $n_o \in I$ or $n_o \in J$ and not to both. Let $n_o \in I$, Then

$\sum_{i\in I} m_i \geq m_{n_o} > \sum_{k=n_o+1}^{\infty} m_k \geq \sum_{j\in J} m_j$ and therefore $\sum_{i\in I} m_i > \sum_{j\in J} m_j$ which is a contradiction.

Hence $I = J = \phi$.

So two subsets of X have equal measure only if they are equal. It follows that two real measurable functions are equimeasurable only if they are equal and as in Example 5.2, Condition 5.1 (c) is vacuously satisfied and hence every functional F s.t.

$$F(x) = \sum_{i=1}^{\infty} f_i(x_i)m_i = \int f(x(t))d\mu(t) \qquad \text{---------} \quad (1)$$

satisfies Theorem 3.8 provided $f_1(x) = f(x,t)$ is continuous in $x \ \forall \ t \in T$ and $f(o,t) = 0$ and the series (1) converges uniformly and absolutely, for $|x_n|$, $\forall$ n, to be uniformly bounded.

Hence for this (X,Σ,μ), Theorem 3.8 is not true.

The next theorem, 5.6, gives a necessary condition for Theorem 3.8 to hold true for a countably infinite discrete measure space.

5.6. Theorem. Let $X = (X,\Sigma,\mu)$ be the countably infinite, discrete measure space and $m_1 \geq m_2 \geq m_3 \geq \ldots$ be the measures of the atoms of X. Then a necessary condition for Theorem 3.8 to be true is that for infinitely many n, $m_n \leq 2 \sum_{k=n+1}^{\infty} m_k$

------(2)

Proof: Suppose condition (2) is false for all but a finite number of n and take $n_o > 1$ (where n_o chosen is the same as in Lemma 2.39) s.t. $\forall$ $n > n_o$, $m_n > 2 \sum_{k=n+1}^{\infty} m_k$ -------(3)

Let $\mathcal{F}$ = {all functionals F on $B(X,\Sigma,\mu)$ which satisfy 5.1 (a), 5.1 (b), 5.1 (c)} .

Then for every real-valued function $f_i : R \to R$

$$F = \sum_{i=1}^{\infty} f_i \, m_i \qquad \text{-------(4)}$$

satisfies 5.1 (a) iff $f_i(o) = 0$ $\forall$ $i = 1,2,\ldots$ and satisfies 5.1 (b) iff $\sum_{i=1}^{\infty} f_i(a_i)m_i$ converges uniformly and absolutely for a_i in any compact set of R. And lastly F satisfies 5.1 (c) iff $\forall\ a \in (-\infty,\infty)$, the sequence $\{f_i(a)\}_{i\geq 1}$ satisfies that $\sum_{i\in I} f_i m_i = \sum_{j\in J} f_j m_j$ for $(I,J) \in H$ ------(5)

as in Lemma 2.39.

Let $\mathcal{G} \subset \mathcal{F}$ be the set of those functionals F for which $f_1 = f_2 = \ldots$.

We claim that $\mathcal{F} = \mathcal{G}$ for the above theorem to be true.

Let K be a finite subset of the positive integers containing $k+1$ integers where k is the number of equations in 2.39 (c). Let $\{S_i\}_{i\geq 1}$ be the solution of 2.39 (c) whose support is K and let $g : R \to R$ be a continuous function which vanishes only at zero.

$$\text{Let} \quad f_i = S_i g \quad \text{for} \quad i = 1,2,\ldots$$
$$\text{so} \quad f_i \neq f_j \quad \text{for} \quad i \neq j \qquad \text{------(6)}$$

Then $F = \sum_1^\infty f_i m_i$ satisfies (5) above and also 5.1 (a), 5.1 (b) and 5.1 (c) and hence $F \in \mathcal{F}$ and by (6), $F \notin \mathcal{G}$ which contradicts our assumption. Q.E.D.

The following example shows that the condition in Theorem 5.6 is not sufficient. Even the strong condition of Theorem 2.31 which is equivalent to the weak intermediate value property, is sufficient iff $r \geq 1$ where $0 < r < 1$ and

$m_i = r^i$ for $i = 1,2,\ldots$.

5.7 Example. Let $p > 1$ be an integer. $(\frac{1}{2})^p = r$ lies between $\frac{1}{2}$ and 1 .

Since r also satisfies $2x^p - 1 = 0$ ------(7)
it is an algebraic number of degree p .

$$\text{Since } \sum_{k=o}^{\infty} r^{p+kp} = \frac{r^p}{1-r^p} = 1 \text{ for } r^p = \tfrac{1}{2} \qquad \text{------(8)}$$

we have for any $i = 1,2,\ldots$ that

$$r^i = \sum_{k=o}^{\infty} r^{p+kp+i} \qquad \text{------(9)}$$

Let $X = \{1,2,\ldots\}$ and $\mu(i) = r^i = (\frac{1}{2})^{i/p} = m_i$, we have that $m_1 \geq m_2 \geq \ldots$ and also that

$$m_n \leq \sum_{k=n+1}^{\infty} m_k .$$

So X satisfies the weak intermediate value property provided that $p \geq 1$.

We claim that Theorem 3.8 does not hold for this $X = (X,\Sigma,\mu)$.

For if $F = \sum_{i=1}^{\infty} f_i r^i$ is a functional on B which satisfies conditions 5.1 (a), 5.1 (b) and 5.1 (c) then by (9) above, if $g_i = f_i r^i = f_i m_i$, $i = 1,2,\ldots$, then

$$g_i = \sum_{k=o}^{\infty} g_{p+kp+i} = \sum_{k=1}^{\infty} g_{kp+i} \qquad \text{-----(10)}$$

$$g_{i+p} = \sum_{k=2}^{\infty} g_{kp+i} \qquad \text{-----(11)}$$

and subtracting (11) from (10), we get

$$g_i - g_{i+p} = g_{i+p} \Longrightarrow g_{i+p} = \frac{g_i}{2}$$

i.e. $f_{i+p} = f_i$ for $i = 1,2,\ldots$.

Thus $f_{p+1}, f_{p+2}, \ldots$ and hence F can be completely determined if $f_1, f_2, \ldots, f_p$ are known.

We claim now that $f_1, f_2, \ldots, f_p$ are arbitrary continuous functions on R and $f_i(o) = 0$ for $i = 1,2,\ldots,p$. Now $\forall$ $(I,J) \in H$ as in Lemma 2.39,

$$\sum_{i\in I} r^i = \sum_{j\in J} r^j \qquad \text{-----(12)}$$

implies

$$\sum_{i\in I} f_i r^i = \sum_{j\in J} f_j r^j \qquad \text{-----(13)}$$

Dividing by the lowest power r^h of r , (12) and (13) can be written as

$$1 = \sum_{k=1}^{\infty} c_k r^k \qquad \text{-----(14)}$$

and

$$f_h = \sum_{k=1}^{\infty} c_k f_{h+k} r^k \qquad \text{-----(15)}$$

Furthermore (14) becomes $1 = \sum_{k=1}^{p} d_k r^k$ -----(16)

where $d_k = \sum_{n=o}^{\infty} \frac{c_{k+np}}{2^n} r^k$ for $k = 1,\ldots,p$ where we use the fact that $k = a_k \cdot p + b_k$ and in the same way by using the periodicity of function sequence $f_1, f_2, \ldots$, (15) can be written as

$$f_h = \sum_{k=1}^{p} d_k f_{h+k} r^k \qquad \text{-----(17)}$$

Since r satisfies a unique irreducible polynomial of degree p , by comparing (2) and (16) we get that

$d_1 = d_2 = \ldots = d_{k=1} = 0$ and

$$d_p = 2 ,$$

and since $2r^p = 1$ i.e. $d_p r^p = 1$, relation (12) reduces to $r^h = 2r^{h+p}$ and (17) reduces to

$$f_h = f_{h+p} , \; h = 1,2,\ldots, \qquad \text{Q.E.D.}$$

SECTION 6

REPRESENTATION OF BIADDITIVE FUNCTIONALS

6.1 <u>Definition</u>. If X,Y are two sets then $X \times Y = \{(x,y) : x \in X , y \in Y\}$ is called the cartesian product of X and Y.

6.2 <u>Definition</u>. If $A \subset X$ and $B \subset Y$ then $A \times B \subset X \times Y$ and any set of the form $A \times B$ is called a rectangle.

Let (X,S_1,μ_1) and (Y,S_2,μ_2) be measure spaces where S_1 and S_2 are σ-algebras of subsets of X and Y respectively.

6.3 <u>Definition</u>. A set of the form $A \times B$ where $A \in S_1$, $B \in S_2$ is called a measurable rectangle.

6.4 <u>Definition</u>. If $E \subset X \times Y$, for $x \in X$ and $y \in Y$, define

$$E_x = \{y : (x,y) \in E\} .$$

$$E^y = \{x : (x,y) \in E\} .$$

E_x and E^y are called the x-sections and y-sections respectively of E and $E_x \subset Y$, $E^y \subset X$.

6.5 <u>Theorem</u>. If $E \in S_1 \times S_2$, then $E_x \in S_2$ and $E^y \in S_1$ for $x \in X$ and $y \in Y$.

6.6 <u>Definition</u>. With each function f on $X \times Y$ and with each $x \in X$ we associate a function f_x defined on Y by $f_x(y) = f(x,y)$. Similarly f^y is the function defined on X by $f^y(x) = f(x,y)$.

6.7 <u>Theorem</u>. Let f be an $S_1 \times S_2$-measurable function on $X \times Y$. Then

(a) $\forall \ x \in X$, f_x is an S_2-measurable function.

(b) $\forall \ y \in Y$, f^y is an S_1-measurable function.

6.8 <u>Theorem</u>. Let (X, S_1, μ_1) and (Y, S_2, μ_2) be σ-finite measure spaces. Suppose $E \in S_1 \times S_2$. If $\varphi(x) = \mu_2(E_x)$, $\psi(y) = \mu_1(E^y)$ $\forall \ x \in X$, $y \in Y$, then φ is S_1-measurable and ψ is S_2-measurable and $\int_X \varphi d\mu_1 = \int_Y d\mu_2$ and since

$$\mu_2(E_x) = \int_Y \chi_E(x,y) d\mu_2(y) \quad \forall \ x \in X$$

we have that

$$\int_X d\mu_1(x) \int_Y \chi_E(x,y) d\mu_2(y) = \int_Y d\mu_2(y) \int_{X_1} \chi_E(x,y) d\mu_1(x)$$

5.9 <u>Definition</u>. If (X_i, S_i, μ_i) for $i = 1,2$, are σ-finite measure spaces and if $E \in S_1 \times S_2$ then define

$$(\mu_1 \times \mu_2)(E) = \int_X \mu_2(E_x)d\mu_1(x) = \int_Y \mu_1(E^y)d\mu_2(y)$$

The product $\mu_1 \times \mu_2$ of measures μ_1 and μ_2 is also a measure. (For Proof see P. R. Halmos) and also that $\mu_1 \times \mu_2$ is σ-finite.

In this section we prove the integral representation of biadditive functionals when the associated measure space is finite non-atomic and for this purpose we shall follow closely the proofs in [2].

Similar results hold when the underlying measure space is σ-finite non-atomic but we shall restrict ourselves to the finite non-atomic measure space only.

5.10 <u>Definition</u>. Let (X_i, S_i, μ_i), $i = 1,2$ be two measure spaces and let M_i be the space of measurable functions on X_i. If B_i is a vector subspace of M_i for $i = 1,2$, then a functional N on $B_1 \times B_2$ is said to be biadditive if $N(\cdot, y)$ and $N(x, \cdot)$ are additive for every function $y \in B_2$ and $x \in B_1$ respectively.

The results for biadditive functionals are analogous to the ones proved for additive functionals in Section 2. We shall establish the necessary and sufficient conditions that a biadditive functional N defined on the product $B_1 \times B_2$ of prescribed subspaces $B_1 \subset M_1$, $B_2 \subset M_2$ permit a representation of the form

$$(**) \qquad N(x_1,x_2) = \int_{X_1 \times X_2} \varphi(x_1,x_2)\,d(\mu_1 \times \mu_2) \quad \text{for all} \quad x_i \in B_i \,, \quad i = 1,2 \,.$$

where φ is a unique continuous real-valued function on R^2.

6.11 Definition. A function φ is said to be separately continuous if $\varphi(x,\cdot)$ and $\varphi(\cdot,y)$ are continuous $\forall \; x,y \in R$.

6.12 Remark. If $\varphi(x,y)$ is continuous in both variables separately then it may not be continuous in both variables jointly.

Example. Consider $\varphi(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \quad x^2+y^2 \neq 0 \\ 0 & , \quad x^2+y^2 = 0 \end{cases}$

Now $\varphi(o,y) = 0$, $y \neq 0$. Therefore

$$\lim_{y \to y_o} \varphi(o,y) = 0 = \varphi(o,y_o) \quad \text{and since} \quad \varphi(o,o) = 0$$

$$\lim_{y \to o} \varphi(o,y) = 0 = \varphi(o,o) \,.$$

Hence φ is continuous for all y and by symmetry, it is continuous for all x also.

But $$\lim_{\substack{x \to o \\ y \to o}} \varphi(x,y) \neq \varphi(o,o) = 0$$

For let $(x,y) \to (o,o)$ along the line $y = mx$. Then

$$\lim_{\substack{x\to o\\ y\to o}} \varphi(x,y) = \frac{m}{1+m^2} \neq 0$$

Hence the given function is discontinuous at (o,o).

5.13 <u>Theorem</u>. Let (X_i, S_i, μ_i) $i = 1,2$, be finite non-atomic measure spaces and let N be a biadditive functional on $L_\infty(\mu_1) \times L_\infty(\mu_2)$. Then N satisfies condition:

$$(1,1) \qquad \begin{cases} x_n \to x \text{ boundedly a.e.} \implies N(x_n,y) \to N(x,y) \ \forall\, y \in B_2 \\ y_n \to y \text{ boundedly a.e.} \implies N(x,y_n) \to N(x,y) \ \forall\, x \in B_1 \end{cases}$$

iff (**) holds with a representing function φ satisfying

(a,a): φ is separately continuous and $\varphi(c,o) = \varphi(o,d) = 0$ $\forall\, c,d \in R$,

and

(b,b): φ is bounded on every subset of R^2.

<u>Proof</u>: Let $x_1 \in L_\infty(\mu_1)$. Then $N(x_1,\cdot)$ is an additive functional which satisfies condition (1): $N(x_1,\cdot)$ is continuous and $N(x_1,o) = 0$ and hence by Theorem 3.8, there exists a unique continuous function $f_{x_1} : R \to R$, $f_{x_1}(o) = 0$, s.t.

$$N(x_1,x_2) = \int_{X_2} (f_{x_1} \circ x_2)\,d\mu_2 \quad \forall \quad x_2 \in L_\infty(\mu_2) \qquad \text{------}(1)$$

Let χ_i denote χ_{X_i}, $i = 1,2$.

Define $\varphi : R^2 \to R$ by

$$\varphi(c,d) = \frac{1}{\mu_1(X_1)} \int_{c\chi_1} (d\chi_2) = \frac{N(c\chi_1, d\chi_2)}{\mu_1(X_1)\mu_2(X_2)} \quad \text{------}(2)$$

Since N is biadditive, (1,1) implies that φ is separately continuous.

Let E_i be the disjoint measurable sets in X_1 . For a fixed $x_2 \in L_\infty(\mu_2)$ and for each simple function $x_1 = \sum_{i=1}^{k} c_i \chi_{E_i}$ we have by the biadditivity of N that

$$N(\sum_{i=1}^{k} c_i \chi_{E_i}, x_2) = \sum_{i=1}^{k} N(c_i \chi_{E_i}, x_2) \quad \text{------}(3)$$

If each $c_i = c_1$ and $\mu_1(E_i) = \mu_1(E_1)$ then we get

$$N(c_1 \chi_{\bigcup_{i=1}^{k} E_i}, x_2) = kN(c_1 \chi_{E_1}, x_2) \quad \text{------}(4)$$

and hence if $\{E_i\}_{i=1}^{k}$ is a partition of X_1 then

$$N(c_1\chi_{E_1}, x_2) = \frac{1}{k} N(c_1\chi_1, x_2) = \frac{\mu_1(E_1)}{\mu_1(X_1)} N(c_1\chi_1, x_2) \quad \text{-----}(5)$$

If $\mu_1(F)$ is an integral multiple of $\mu_1(X_1)$ then from (1) we have $N(c\chi_F, \cdot) = \int_{X_2} f_{c\chi_F} \circ (\cdot) d\mu_2$.

By (5) and (1)

$$N(c\chi_F,\cdot) = \frac{\mu_1(F)}{\mu_1(X_1)} N(c\chi_1,\cdot) = \frac{\mu_1(F)}{\mu_1(X_1)} \mu_1(X_1)\mu_2(X_2)\varphi(c,\cdot)$$

i.e. $$N(c\chi_F,\cdot) = \mu_1(F)\mu_2(X_2)\varphi(c,\cdot)$$

$$\mu_1(F)\mu_2(X_2)\varphi(c,\cdot) = \int_{X_2} f_{c\chi_F}\circ(\cdot)d\mu_2$$

i.e. $$f_{c\chi_F}(\cdot) = \mu_1(F)\varphi(c,\cdot) \qquad \text{------(6)}$$

for $F \in S_1$, s.t. $\mu_1(F)$ is an integral multiple of $\mu_1(X_1)$. As in 3.8(g), by applying the additivity again, (5) implies that (6) is true whenever $\frac{\mu_1(F)}{\mu_1(X)}$ is a rational number also and hence the continuity of N implies that (6) is true in general.

Now (3) can be written as

$$N\left(\sum_{i=1}^{k} c_i\chi_{E_i}, x_2\right) = \sum_1^k \frac{\mu_1(E_i)}{\mu_1(X_1)} N(c_i\chi_1, x_2) = \sum_1^k \frac{\mu_1(E_i)}{\mu_1(X_1)} \int_{X_2} (f_{c_i\chi_1}\circ x_2)d\mu_2$$

$$= \sum_{i=1}^{k} \mu_1(E_i) \int_{X_2} \varphi(c_i, x_2)d\mu_2 \qquad \text{------(7)}$$

$$= \int_{X_1 \times X_2} \varphi\left(\sum_{i=1}^{k} c_i\chi_{E_i}, x_2\right)d(\mu_1\times\mu_2)$$

which proves the representation (**) when $x_1 \in L_\infty(\mu_1)$ is a simple function.

To prove that (**) holds in general, we prove firstly that φ satisfies (b,b).

Suppose φ does not satisfy (b,b). There exists a rectangle $Q = \{(c,d) : |c| \leq K_1 , |d| \leq K_2\}$ s.t. $\varphi(c,d)$ is unbounded on Q. For fixed c^*,d^*, let $K_{c^*} = \max_{|d|\leq K_2} |\varphi(c^*,d)|$ and $\ell_{d^*} = \max_{|c|\leq K_1} |\varphi(c,d^*)|$. Since φ is separately continuous, K_{c^*}, ℓ_{d^*} are well-defined. But by assumption both $A_1 = \{K_c : |c| \leq K_1\}$ and $A_2 = \{\ell_d : |d| \leq K_2\}$ are unbounded.

Let $\{\theta_j\}$ be a sequence of positive numbers s.t. $\sum_{j\geq 1} \theta_j = 1$ and $\sum_{j\geq n+1} \theta_j \leq \frac{1}{2}\theta_n$ for $n \geq 1$.

Choose a sequence of points $\{(c_i,d_i)\}_{i\geq 1}$ in Q by induction as follows:

Choose G s.t. $K_{c_1} > 4\theta_1^{-1}$ and d_1 s.t. $|\varphi(c_1,d_1)| = K_{c_1}$. In general, having chosen (c_i,d_i) for $1 \leq i \leq n-1$, choose c_n so that
$K_{c_n} \geq 2\theta_n^{-1}\sum_{i=1}^{n-1} \ell_{d_i}\theta_i + 2^{n+1}\theta_n^{-1}\alpha_n$ where

$$d_n \geq \left(\tfrac{3}{2}\right) \sum_{i=1}^{n-1} K_{c_i} 2^{-i}\theta_i + \sum_{i=1}^{n-1} 2^{-i} \sum_{j=1}^{i-1} \ell_{d_j}\theta_j + n$$

and then choose d_n s.t. $|\varphi(c_n,d_n)| = K_{c_n}$.

Let $\{E_i\}_{i\geq 1}$ and $\{F_j\}_{j\geq 1}$ be sequence of disjoint measurable sets in X_1 and X_2 respectively s.t.

$\mu_1(E_i) = 2^{-i}\mu_1(X_1)$ and $\mu_2(F_j) = \theta_j\mu_2(X_2)$. Let

$$x_2 = \sum_{j\geq 1} d_j\chi_{F_j} \qquad x_2 \in L_\infty(\mu_2) \quad .$$

Also, the sequence of functions $x_1^n = \sum_{i=1}^{n} c_i\chi_{E_i}$ and the function $x_1 = \sum_{i\geq 1} c_i\chi_{E_i}$ are in $L_\infty(\mu_1)$ and $x_1^n \to x_1$ boundedly a.e. Thus $N(x_1^n,x_2) \to N(x_1,x_2)$ as $n \to \infty$.

Now consider the integral representation for $N(x_1,x_2)$ which we have established when x_1 or x_2 is a simple function, we have that

$$N(x_1^n,x_2) = \int\varphi(\sum_{i=1}^{n} c_i\chi_{E_i}, \sum_{j\geq 1} d_j\chi_{F_j})d(\mu_1\times\mu_2)$$

$$= \sum_{i=1}^{n}\sum_{j\geq 1} \varphi(c_i,d_j)\mu_1(E_i)\mu_2(F_j) \qquad \text{------(8)}$$

Also we have that for each $1 \leq i \leq n$,

$$\left|\sum_{j=1}^{i-1}\varphi(c_i,d_j)\mu_1(E_i)\mu_2(F_j)\right| \leq \sum_{j=1}^{i-1} \ell_{d_j} 2^{-i}\theta_j\mu_1(X_1)\mu_2(X_2)$$

$$\leq 2^{-i}(K_{c_i}\frac{\theta_i}{2} - 2^i d_i)\mu_1(X_1)\mu_2(X_2)$$

$$\text{------(9)}$$

and $$\left|\sum_{j\geq i+1}\varphi(c_i,d_j)\mu_1(E_i)\mu_2(F_j)\right| \leq \sum_{j\geq i+1} K_{c_i} 2^{-i}\theta_j\mu_1(X_1)\mu_2(X_2)$$

$$\leq 2^{-i}(\frac{\theta_i}{2})\, K_{c_i}\,\mu_1(X_1)\mu_2(X_2) \quad \text{---(10)}$$

Hence from (8), (9) and (10) we have that

$$|N(x_1^n,x_2)| \geq |\sum_{j\geq 1} \varphi(c_n,d_j)\theta_j\mu_1(E_n)\mu_2(X_2)| - \sum_{i=1}^{n-1} |\sum_{j\geq 1} \varphi(c_i,d_j)\theta_j\mu_1(E_i)\mu_2(X_2)|$$

$$\geq [K_{c_n}\theta_n - \sum_{j=1}^{n-1} \ell_{d_j}\theta_j - K_{c_n}\frac{\theta_n}{2}]2^{-n}\mu_1(X_1)\mu_2(X_2)$$
$$- \sum_{i=1}^{n-1} [K_{c_i}\theta_i + \sum_{j=1}^{i-1} \ell_{d_j}\theta_j + K_{c_i}\frac{\theta_i}{2}]2^{-i}\mu_1(X_1)\mu_2(X_2)$$

$$\geq n$$

which contradicts the fact that $N(x_1^n,x_2) \to N(x_1,x_2)$.

Hence φ has property (b,b).

Now for (**) to hold in general, let $x_1 \in L_\infty(\mu_1)$. There exists a sequence $\{x_1^n\} \subset L_\infty(\mu_1)$ of simple functions, s.t. $x_1^n \to x_1$ boundedly a.e.

Hence we have that

$$N(x_1,x_2) = \lim_{n\to\infty} N(x_1^n,x_2) = \lim_{n\to\infty} \int_{X_1\times X_2} \varphi(x_1^n,x_2)d(\mu_1\times\mu_2) \qquad \text{-----(11)}$$

$\forall \ x_2 \in L_\infty(\mu_2)$.

Now by the separate continuity of φ and the property (b,b) we have that the functions $h_n = \varphi(x_1^n,x_2) : X_1\times X_2 \to R$ converge boundedly pointwise to $h = \varphi(x_1,x_2)$ outside a set of the form $(N_1 \times X_2) \cup (X_1 \times N_2)$ where N_i are null sets in X_i.

Hence by the Lebesgue dominated convergence theorem we have from (11) that

$$N(x_1,x_2) = \int_{X_1 \times X_2} \varphi(x_1,x_2)\, d(\mu_1 \times \mu_2) \quad \forall \quad x_1 \in L_\infty(\mu_1) \text{ and } x_2 \in L_\infty(\mu_2)$$

and this proves (**).

Conversely, let $N(x_1,x_2) = \int_{X_1 \times X_2} \varphi(x_1,x_2)\, d(\mu_1 \times \mu_2)$

where φ satisfies (a,a) and (b,b).

From the steps leading to (11) we have that a finite valued N above is well-defined and that it satisfies (1,1) follows from the proofs in Section 2.

5.14 Remark. It is clear from the proof of the above theorem that it still holds if (1,1) is replaced by condition

(2,2) $\quad x_n \to x$ boundedly a.e., $y_n \to y$ boundedly a.e. $N(x_n,y_n) \to N(x,y)$ and (a,a) is replaced by condition:

$(a,a)_1$: φ is jointly continuous and $\varphi(c,o) = \varphi(o,d) = 0$ $\forall$ $c,d \in R$.

Since in a totally finite measure space, convergence boundedly a.e. implies convergence boundedly in measure, we have

5.15 Corollary. Let (X_i, S_i, μ_i) be as in Theorem 5.13, and let N be a biadditive functional on $L_\infty(\mu_1) \times L_\infty(\mu_2)$. Then N satisfies condition:

(3,3) $x_n \to x$ boundedly in measure $\Longrightarrow N(x_n, y) \to N(x,y)$ $\forall\ y \in B_2$

$y_n \to y$ boundedly in measure $\Longrightarrow N(x, y_n) \to N(x,y)$ $\forall\ x \in B_1$

or

(4,4) $x_n \to x$ boundedly in measure, $y_n \to y$ boundedly in measure $\Longrightarrow N(x_n, y_n) \to N(x,y)$, iff (**) holds with a $\varphi : R^2 \to R$ having the properties (a,a) or $(a,a)_1$ and (b,b).

Proof: We need only show that if φ satisfies (a,a) and (b,b) then N satisfies (3,3). Suppose $x_1^n \to x_1$ boundedly in measure then there exist subsequences $\{x_1^m\}$ of $\{x_1^n\}$ s.t. $x_1^m \to x_1$ boundedly a.e. and Theorem 5.13 implies that for all such subsequences $\varphi(x_1^m, x_2) \to \varphi(x_1, x_2)$ in $L_1(\mu_1 \times \mu_2)$-norm and by the same argument every subsequence of $\{\varphi(x_1^n, x_2)\}$ contains a subsequence converging to $\varphi(x_1, x_2)$ in $L_1(\mu_1 \times \mu_2)$ norm and this implies that $\{\varphi(x_1^n, x_2)\}$ itself converges to $\varphi(x_1, x_2)$ in $L_1(\mu_1 \times \mu_2)$ norm and hence by Lebesgue limit theorem $N(x_1^n, x_1) \to N(x_1, x_2)$.

Similarly $N(x_1,x_2^n) \to N(x_1,x_2)$. Q.E.D.

5.16 Theorem. Let (X_i,S_i,μ_i) be as in Theorem 5.13. Then a biadditive functional N on $L_\infty(\mu_1)\times L_\infty(\mu_2)$ satisfies condition:

$$(5,5) \qquad \begin{array}{l} x_n \to x \text{ a.e.} \implies N(x_n,y) \to N(x,y) \text{ for all } y \in X_2 \\ y_n \to y \text{ a.e.} \implies N(x,y_n) \to N(x,y) \text{ for all } x \in X_1 \end{array}$$

iff (**) holds with a $\varphi : R^2 \to R$ satisfying (a,a) and

$(b,b)_1$: φ is bounded on finite strips of the following types for all $h \geq 0$,

$$S_h^1 = \{(c,d) : |d| \leq h\}, \; S_h^2 = \{(c,d) : |c| \leq h\}$$

Proof: Let φ satisfy (a,a) and $(b,b)_1$ and let

$$N(x_1,x_2) = \int_{X_1\times X_2} \varphi(x_1,x_2)d(\mu_1\times\mu_2)$$

It is just a routine verification, that the bounded convergence theorem implies that N satisfies (5,5).

Conversely, let N be a biadditive functional on $L_\infty(\mu_1)\times L_\infty(\mu_2)$ satisfying (5,5). Then N also satisfies (1,1). Hence there exists a unique function $\varphi : R^2 \to R$ satisfying condition (a,a) and (b,b) s.t. (**) holds for all $(x_1,x_2) \in L_\infty(\mu)\times L_\infty(\mu_2)$.

Now we claim that φ satisfies $(b,b)_1$. Suppose not. Then there exists a strip

$$S_h^1 = \{(c,d) : |d| \leq h\} \text{ or a strip } S_{h'}^2 = \{(c,d) : |c| \leq h'\}$$

s.t. φ is unbounded on S_h^1 or $S_{h'}^2$. Suppose that φ is unbounded on S_h^1 . As before, let $\{\theta_i\}_{i\geq 1}$ be a sequence of positive reals s.t. $\sum_{i\geq 1} \theta_i = 1$ and $\sum_{i\geq n+1} \theta_i \leq \frac{1}{2}\theta_n$, $n \geq 1$.

We choose a sequence of points $\{(c_i,d_i)\}_{i\geq 1}$ in S_h^1 inductively as follows:

$$\text{Let } K_c = \max_{|d|\leq h} |\varphi(c,d)| \ , \ \ell_d = \sup_{-\infty<c<\infty} |\varphi(c,d)| \ .$$

Since φ is separately continuous, K_c is welldefined and by the continuity condition (3) of Theorem 4.13 on $N(\cdot,d\chi_2)$, ℓ_d is finite.

Since φ is unbounded on S_h^1 , K_c and ℓ_d are unbounded functions of c and d , $|d| \leq h$. Choose c_1 s.t. $K_{c_1} \geq 4\,\theta_1^{-1}$ and then take $d_1 \in [-h,h]$ s.t. $|\varphi(c_1,d_1)| = K_{c_1}$. Having chosen $\{(c_i,d_i)\}$ $1 \leq i \leq n-1$, select c_n s.t. $K_{c_1} \geq 2\,\theta_n^{-1} \sum_{i=1}^{n-1} \theta_i \ell_{d_i} + n\,2^{n+1}\theta_n^{-1}$ and then choose d_n s.t. $|\varphi(c_n,d_n)| = K_{c_n}$. For $\{F_i\}_{i\geq 1}$ a disjoint sequence of measurable sets such that $\mu_2(F_j) = \theta_j \mu_2(X_2)$, let $x_2 = \sum_{j\geq 1} d_j \chi_{F_j}$, $x_2 \in L_\infty(\mu_2)$.

Let $\{E_i\}$ be a nested sequence of measurable sets in X_1 s.t.

$$\mu_1(E_i) = 2^{-i}\mu_1(X_1) .$$

As in Theorem 5.13, if $x_1^n = c_n \chi_{E_n}$ then

$$N(x_1^n, x_2) \geq n\, \mu_1(X_1)\mu_2(X_2) .$$

And since $x_1^n \to 0$ a.e., the continuity of N implies that $N(x_1^n, x_2) \to 0$. Hence we have a contradiction which establishes the property $(b,b)_1$ for φ.

5.17 <u>Corollary</u>. Since in a totally finite measure space, convergence a.e. implies convergence in measure, the above theorem is true if condition (5,5) is replaced by condition:

(6,6) $\qquad x_n \to x$ a.e., $y_n \to y$ a.e. $\qquad N(x_n, y_n) \to N(x,y)$.

Since the above theorems are true for $n = 2$, these are true for n-additive functionals for n finite.

SECTION 7

REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON L^p-SPACES

Let T be a subset of the n-dimensional space R^n s.t. $\mu(T) < \infty$, where μ is the Lebesgue measure.

7.1 Definition. A real valued function $\varphi : T \times R \to R$ is said to be of Caratheodory type for T, denoted by $\varphi \in Car(T)$ if it satisfies.

(a) $\varphi(t,\cdot) : R \to R$ is continuous for a.a - $t \in T$.

(b) $\varphi(\cdot,c) : T \to R$ is measurarable for all $c \in R$.

7.2 Remark. If $T = (T,\Sigma,\mu)$ is a σ-finite measure space and if $M(T)$ denotes the class of real-valued measurable functions on X then since for each simple function x, the function $\varphi \circ x$ defined by $(\varphi \circ x)(t) = \varphi(t,x(t))$ belongs to $M(T)$ and since for each $x \in M(T)$, there exists a sequence $\{x_n\}$ of simple functions converging to x, it follows by using 7.1(a) that $\varphi \circ x \in M(T)$.

7.3 Definition. We denote by A, the operator defined on the set of real functions on T by

$$Au(s) = \varphi[s,u(s)] \quad \text{where} \quad \varphi \in CAR\ (T) .$$

In the beginning of this section, we shall be mainly interested in the properties of the operator A for the case when it acts from a space L^{p_1} to L^{p_2} for $p_1, p_2 \geq 1$ and for this purpose we shall follow M. A. Krasnoselskii [11].

Towards the end of this section we shall state two theorems by V. J. Mizel [8] which prove the integral representation of nonlinear transformations on L^p-spaces and also extend our earlier results of integral representation of nonlinear functionals defined essentially for atom-free finite or σ-finite measure spaces.

7.4 Lemma. (V. V. Nemytskii [12]).

Let G be a set of finite measure. Then the operator A transforms every sequence $\{u_n(s)\}$, $s \in G$ ----(1) which converges in measure into a sequence of functions which also converges in measure.

Proof: Suppose that the sequence $\{u_n(s)\}$ converges in measure to $u_o(s)$ for $s \in G$. Let

$G_K = \{s \in G$ s.t. for given $\epsilon > 0$, $|u_o(s)-u(s)| < \frac{1}{K} \Longrightarrow$

$|\varphi[s_o u(s_o)] - \varphi(s,u)| < \epsilon$. $G_1 \subset G_2 \subset G_3 \ldots$, and the continuity of the function $\varphi(s,u)$ w.r.t. u for a.a. - $s \in G$ implies that

$$\mu\left(\bigcup_{K=1}^{\infty} G_K\right) = \mu(G) \Longrightarrow \lim_{K\to\infty} \mu(G_K) = \mu(G) \quad \text{-----(2)}$$

and hence, given $\eta > 0$, $\exists$ k_1 s.t.

$$\mu(G_{k_o}) > \mu(G) - \eta/2 .$$

Let $F_n = \{s \in G : |u_o(s) - u_n(s) < \frac{1}{k_o}\}$. Choose N s.t. $\mu(F_n) > \mu(G) - \frac{\eta}{2}$ for all $n > N$. Consider the sequence of functions $\{Au_n(s)\}$ where

$$Au_n(s) = \varphi[s,u_n(s)]$$

and let

$$D_n = \{s \in G : |\varphi[s,u_o(a)] - \varphi[s,u_n(s)]| < \epsilon .$$

Then we have that $G_{k_o} \cap F_n \subset D_n$ and it follows that $\mu(D_n) > \mu(G) - \eta$ and since ϵ and η are arbitrary, this completes the proof.

7.5 Definition. Let $d(x,y) = \|x-y\|$ for $x,y \in L^p$, then a sequence $\{x_n\}$ in L^p converges strongly to $x \in L^p$ if $\lim_{n\to\infty} d(x_n,x) = \|x_n-x\| = 0$.

7.6 Theorem. If $A : L^{p_1} \to L^{p_2}$ transforms every function in L^{p_1} into a function in L^{p_2} $(p_1,p_2 \geq 1)$ then A is continuous.

Proof: Case (i). Suppose $\mu(G) < \infty$.

Let θ be the zero function in the space L^{p_1} . Assume that $A\theta = \theta$ and we show that the operator A is continuous at the zero θ . Suppose it is not continuous at θ . Then there exists a sequence $\varphi_n(s) \in L^{p_1}$ $(n = 1,2,\ldots)$ and $\varphi_n(s)$ converges strongly to θ s.t.

$$\int_G |A\varphi_n(s)|^{p_2} ds > a \quad (n = 1,2,\ldots) \qquad \text{------}(3)$$

for some positive number a .

Assume that $\sum_{n=1}^{\infty} \int_G |\varphi_n(s)|^{p_1} ds < \infty$ ------(4)

We construct by induction, a sequence of numbers ε_k , functions $\varphi_{n_k}(s)$ and sets $G_k \subset G$ $(k = 1,2,\ldots)$ s.t. the following conditions are satisfied:

(a) $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$

(b) $\mu(G_k) \leq \varepsilon_k$

(c) $\int_{G_k} |A\,\varphi_{n_k}(s)|^{p_2} > \frac{2}{3} a$

(d) For any set $D \subset G$, $\mu(D) \leq 2\,\varepsilon_{k+1}$

$$\int_D |A\,\varphi_{n_k}(s)|^{p_2} ds < \frac{a}{3} .$$

We suppose that $\varepsilon_1 = \mu(G)$, $\varphi_{n_1}(s) = \varphi_1(s)$, $G_1 = G$.

If ε_k , $\varphi_{n_k}(s)$ and G_k have been constructed, then for ε_{k+1} , we select a number s.t. condition (d) is satisfied, which is possible by virtue of the absolute convergence of the integral $\int_G |A\,\varphi_{n_k}(s)|^{p_2} ds$, and hence condition (a) is also satisfied, since the function $\varphi_{n_k}(s)$ satisfies condition (c).

By Lemma 7.4, it is possible to find a number n_{k+1} and a set $F_{k+1} \subset G$ s.t.

$$|A\,\varphi_{n_{k+1}}(s)| < [\tfrac{a}{3\mu(G)}]^{1/p_2} \quad \text{for} \quad s \in F_{k+1} \qquad \text{------(5)}$$

$$\text{With} \quad \mu(G) - \mu(F_{k+1}) < \varepsilon_{k+1} \qquad \text{------(6)}$$

Let $G_{k+1} = G - F_{k+1}$. Then condition (b) is satisfied by (6). Also by (3) and (5).

$$\int_{G_{k+1}} |A\,\varphi_{n_{k+1}}(s)|^{p_2} ds = \int_G |A\,\varphi_{n_{k+1}}(s)|^{p_2} ds - \int_{F_{k+1}} |A\,\varphi_{n_{k+1}}(s)|^{p_2} ds > a - \frac{a}{3\mu(G)}\,\mu(G)$$
$$= \frac{2}{3}\,a\,.$$

and this satisfies condition (c).

Consider the sets $D_k = G_k - \bigcup_{i=k+1}^{\infty} G_i \quad (k = 1,2,\ldots)$.

By (a) and (b) we have that

$$\mu(\bigcup_{i=k+1}^{\infty} G_i) \leq \sum_{i=k+1}^{\infty} \varepsilon_i < 2\varepsilon_{k+1} \quad . \ (k = 1,2,\ldots) \qquad \text{------(7)}$$

Define the function $\psi(s)$ by

$$\psi(s) = \begin{cases} \varphi_{n_k}(s) & \text{if } s \in D_k \quad (k=1,2,\ldots) \\ 0 & \text{if } s \notin \bigcup_{i=1}^{\infty} D_i \end{cases} \qquad \text{------(8)}$$

From (c), (d) and (7) we have for $k = 1,2,\ldots$ that

$$\int_{D_k} |A\ \psi(s)|^{p_2} ds = \int_{D_k} |A\ \varphi_{n_k}(s)|^{p_2} \geq \int_{G_k} |A\ \varphi_{n_k}(s)|^{p_2} ds -$$

$$\int_{G_k - D_k} |A\ \varphi_{n_k}(s)|^{p_2} ds > \frac{a}{3} \qquad \text{------(9)}$$

By (4), $\psi \in L^{p_1}$ and by hypothesis $A\ \psi \in L^{p_2}$. But (9) shows that $A\ \psi \notin L^{p_2}$. Since $D_i \cap D_j = \phi$, $i \neq j$, it follows that $\int_G |A\ \psi(s)|^{p_2} ds \geq \sum_{k=1}^{\infty} \int_{D_k} |A\ \psi(s)|^{p_2} ds = \infty$, and this contradiction proves that the operator A s.t. $A\theta = \theta$, is continuous at θ.

Now we prove in general that A is continuous at $u_o \in L^{p_1}$. Consider the function $g(s,u) = f[s,u_o(s)+u] - f[s,u_o(s)]$ for $s \in G$ and $u \in (-\infty,\infty)$.

The operator A_1 determined by the function $g(s,u)$ where $A_1 u(s) = g[s,u(s)]$ satisfies the condition that $A_1\theta = \theta$ and we have proved that it is continuous at the point $u_o \in L^{p_1}$.

Case (ii). Suppose that $\mu(G) = \infty$.

Assume w.l.o.g. that the operator A is discontinuous at θ and $A\theta = \theta$.

As in Case (i), assume that $\varphi_n(s) \in L^{p_1}$ $(n = 1,2,\ldots)$ is a sequence of functions s.t.

$$\int_G |A\, \varphi_n(s)|^{p_2} ds > a \qquad (n = 1,2,\ldots) \qquad \text{------}(10)$$

for some positive number a .

Assume also that $\sum_{n=1}^{\infty} \int_G |\varphi_n(s)|^{p_1} ds < \infty$ ------(11)

Again we construct by induction, a sequence of functions $\varphi_{n_k}(s)$ and sets $D_k \subset G$ $(k = 1,2,\ldots)$ s.t.

(a) $\mu(D_k) < \infty$, $D_i \cap D_j = \phi$ $i \neq j$.

(b) $\int_{D_k} |A\, \varphi_{n_k}(s)|^{p_2} ds > \frac{a}{2}$ $\qquad (k = 1,2,\ldots)$

We let $\varphi_{n_1}(s) = \varphi_1(s)$ and construct D_1 by virtue of (10).

Having constructed $\varphi_{n_k}(s)$ and D_k , we see by (a) that $\mu(\bigcup_{i=1}^{\infty} D_i) < \infty$ and by Case (i) in which we have proved the continuity of A for $\mu(G) < \infty$, we can find an integer n_{k+1} s.t.

$$\int_{\bigcup_{i=1}^{k} D_i} |A\,\varphi_{n_{k+1}}(s)|^{p_2} ds < \frac{a}{2} \qquad \text{-----(12)}$$

and then we can find a set G_{k+1} , $\mu(G_{k+1}) < \infty$, s.t.

$$\int_{G_{k+1}} |A\,\varphi_{n_{k+1}}(s)|^{p_2} ds > a \qquad \text{-----(13)}$$

Let $D_{k+1} = G_{k+1} - \bigcup_{i=1}^{k} D_i$, condition (a) is satisfied and since

$$\int_{D_{k+1}} |A\varphi_{n_{k+1}}(s)|^{p_2} ds = \int_{G_{k+1}} |A\varphi_{n_{k+1}}(s)|^{p_2} ds - \int_{\bigcup_{i=1}^{k} D_i} |A\varphi_{n_{k+1}}(s)|^{p_2} ds$$

$$\geq a - \frac{a}{2} = \frac{a}{2}$$

and it follows from this that the function $\varphi_{n_{k+1}}(s)$ satisfies condition (b).

Again define a function $\psi(s)$ as in (8), then by (11) we see that $\psi \in L^{p_1}$ and by hypothesis $A\psi \in L^{p_1}$. But by condition (b), we have that $A\psi \notin L^{p_2}$.

This contradiction proves the theorem.

An operator is said to be bounded if it transforms any set which is bounded (in the sense of norm) into another bounded set. We know that a linear operator A is continuous iff it is bounded. But for a nonlinear operator, the notions of continuity and boundedness are independent of one another.

7.7 Example. Consider the space ℓ^2 of numerical sequences $\varphi = \{\zeta_1, \zeta_2, \ldots\}$ with norm defined by

$$\|\varphi\| = \{\sum_{i=1}^{\infty} \xi_i^2\}^{\frac{1}{2}} .$$

Let F be a functional in ℓ^2 defined by

$$F(\varphi) = \Sigma(|\xi_i| - 1)\cdot i$$

where the sum extends over those values of the index i, depending on φ, for which $|\xi_i| \geq 1$. For each element $\varphi \in \ell^2$ there is a finite number of such values of the index. The function $F(\varphi)$ is continuous and it is bounded and in fact equal to zero on the sphere $\|\varphi\| \leq 1$ and it is not bounded on any sphere with radius larger than one.

The above example shows that the boundedness of a nonlinear operator does not in general follow from its continuity.

7.8 Theorem. Suppose that the operator A transforms every function in L^{p_1} into a function in L^{p_2} $(p_1, p_2 \geq 1)$. Then the operator A is bounded.

Proof: We can assume w.l.o.g. that $A\theta = \theta$. Since A is continuous at θ by Theorem 7.6, there exists an $r > 0$ s.t.

$$\int_G |\varphi(s)|^{p_1} ds \leq r^{p_1} \text{ implies } \int_G |A\varphi(s)|^{p_2} ds \leq 1 \quad \text{----(14)}$$

Suppose that $u(s) \in L^{p_1}$ and $nr^{p_1} \leq \| u \|^{p_1} \leq (n+1)r^{p_1}$, n is an integer. Let $G_1, G_2, \ldots, G_{n+1}$ be a partition of G s.t. $\int_{G_i} |u(s)|^{p_1} ds \leq r^{p_1}$ $(i = 1,2,\ldots,n+1)$.

So when $\| Au(s) \| = \{\int_G |Au(a)|^{p_2} ds\}^{1/p_2} \leq [(\frac{\| u \|}{r})^{p_1} + 1]^{1/p_2}$ we have by (14) that

$$\int_G |Au(s)|^{p_2} ds \leq \sum_{i=1}^{n+1} \int_{G_i} |Au(s)|^{p_2} ds \leq n+1$$

Q.E.D.

7.9 Definition. Let $T = (T, \Sigma, \mu)$ be a σ-finite measure space. A function $\varphi \in$ Car (T) is said to be in Caratheodory p-class for T, denoted by $\varphi \in \mathrm{Car}^p(T)$ for $1 \leq p \leq \infty$, if it satisfies $\varphi \circ x \in L^1(T)$ for $x \in L^p(T)$.

7.10 Theorem. If $A : L^{p_1} \to L^{p_2}$ is an operator $p_1, p_2 \geq 1$ defined on T by $Au(s) = f[s, u(s)]$ where $f \in$ Car (T), then

$$|f(s,u)| \leq a(s) + b|u|^{p_1/p_2} \quad \text{----(15)}$$

where b is a positive constant and $a(s) \in L^{p_2}$.

Proof: By Theorem 7.8, we can find a positive number b s.t. $\int_T |f(s,u(s)|^{p_2} ds \leq b^{p_2}$ whenever $\int_T |u(s)|^{p_1} ds \leq 1$.

Define $\varphi(s,u) = \begin{cases} |f(s,u) - b|u|^{p_1/p_2} & \text{if } |f(s,u)| \geq b|u|^{p_1/p_2} \\ 0 & \text{if } |f(s,u)| \leq b|u|^{p_1/p_2} \end{cases}$

We have that $|\varphi(s,u)|^{p_2} \leq |f(s,u)|^{p_2} - b^{p_2}|u|^{p_1}$ if $\varphi(s,u) \neq 0$

Consider an arbitrary function $u(s) \in L^{p_1}$ and let

$$T^+ = \{s \in T : \varphi[s,u(s)] > 0\}$$

Let $\int_T |u(s)|^{p_1} ds = n+\epsilon$ where n is an integer and $0 \leq \epsilon \leq 1$. The set T^+ can be partitioned into $n+1$ sets $T_1, T_2, \ldots, T_{n+1}$ s.t. $\int_{T_i} |u(s)|^{p_1} ds < 1 \quad i = 1,2,\ldots,n+1$.

Then
$$\int_T |f[s,u(s)]|^{p_2} ds \leq \int_{T^+} |f[s,u(s)]|^{p_2} ds$$
$$- b^{p_2} \int_{T^+} |u(s)|^{p_1} ds \leq (n+1)b^{p_2} - b^{p_2}(n+\epsilon)$$
$$\leq b^{p_2} \qquad \text{-----(16)}$$

Let $\{T_k\}$ be a sequence of sets of finite measure s.t. $T_1 \subset T_2 \subset \ldots$ and $T = \bigcup_{i=1}^{\infty} T_k$. Since $\varphi(s,u)$ is continuous w.r.t u at almost all $s \in T$, we can define a sequence $\{u_k(s)\}_{k=1}^{\infty}$ of functions defined on almost all T s.t. $u_k(s) = 0$ when $s \notin T_k$ and

$$\varphi[s,u_k(s)] = \max_{-k \leq u \leq k} \varphi(s,u)$$

So $u_k(s) \in L^{p_1}$ and we set

$$a(s) = \sup_{-\infty<u<\infty} \varphi(s,u) = \lim_{k\to\infty} \varphi[s,u_k(s)]$$

(16) and Faton's lemma implies that

$$\int_T |a(s)|^{p_2} ds \leq \sup_k \int_T |\varphi(s,u_k(s)|^{p_2} ds \leq b^{p_2}$$

$$a(s) \in L^{p_2} .$$

Since $a(s) = \sup_{-\infty<u<\infty} \varphi(s,u) \geq \sup_{-\infty<u<\infty} \{|f(s,u)| - b|u|^{p_1/p_2}\}$, we have that $|f(s,u)| \leq a(s) + b|u|^{p_1/p_2}$ for $s \in T$, $u \in (-\infty,\infty)$.

7.11 Remark. Let $T = (T,\Sigma,\mu)$ be a finite atom-free measure space. It follows from the above theorem that $\varphi \in Car^p(T)$ $1 \leq p < \infty$ iff $|\varphi(s,u)| \leq a(s) + b|u|^p$ for some $a \in L^1(T)$.

7.12 Definition. For $s \in S$, $t \in T$ and $u \in (-\infty,\infty)$, the nonlinear integral operator A defined by

$$A\varphi(s) = \int_T K[s,t,\varphi(t)]dt \qquad \text{-----(17)}$$

is called P. S. Uryson's operator and it takes measurable functions to measurable functions where S,T are Lebesgue measurable subsets of R^n and $K : SxRxT \to R$ is a real valued function which is measurable on SxT for each fixed value of its second argument and continuous on R for almost all arguments in $S \times T$.

7.13 Remark. Let $C(S)$ denote the class of continuous functions defined on S, then an important subclass of (17) is the class of Uryson's operators whose range is in $C(S)$ where S is compact. This subclass included the case in which the kernel φ is independent of its first argument so that the operator A reduces to a real-valued functional F defined by

$$F(x) = \int_T \varphi(x(t),t)dt \qquad \text{-----}(18)$$

In this section, we characterize for all σ-finite measure space $T = (T,\Sigma,\mu)$ and all compact Hausdorff spaces, the nonlinear transformations $A : L^p(T) \to C(S)$, $1 \leq p < \infty$, which have the form (17) and also in particular, we characterize functionals on $L^p(T)$ of the form given in (18). This later characterization extends our earlier results of A. D. Martin and V. J. Mizel [1] and V. J Mizel and K. Sundaresan [2] given in previous sections, concerning functionals of the form

$$F(x) = \int_T \varphi(x(t))d\mu(t) \qquad \text{-----}(19)$$

defined essentially on nonatomic σ-finite measure spaces.

Let $T = (T,\Sigma,\mu)$ be a finite measure space.

7.14 <u>Lemma</u>. Let F be a real-valued functional on $L^p(T)$, $1 \leq p < \infty$, which satisfies

(i) $F(x+y) - F(x) - F(y) = C_F =$ constant, whenever $x,y = 0$ a.e.

(ii) F is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$.

(iii) F is continuous relative to L^p norm, if $p < \infty$ and is continuous w.r.t. bounded a.e. convergence if $p = \infty$, then for every real number h, the set function ν_h defined by $\nu_h(E) = F(h\chi_E)$, for $E \in S$ is a μ-continuous measure.

<u>Proof</u>: By taking $F_1 = F + C_F$ which is a functional of the same type as F with $C_{F_1} = 0$, condition (i) reduces to the case where $C_F = 0$ i.e. $F(x+y) = F(x) + F(y)$ whenever $xy = 0$ a.e.

Now $\nu_h(\phi) = F(h\chi_\phi) = 0$. Let $E_n = \bigcup_1^n E_i$, and $E_i \cap E_j = \phi$ for $i \neq j$ then $E_n \nearrow E$ implies $h\chi_{E_n} \to h\chi_E$ and hence

$$\nu_h(E_n) = F(h\chi_{E_n}) \to F(h\chi_E) = \nu_h(E).$$

Hence ν_h is a μ-continuous measure on T.

Now we state the following theorems by V. J. Mizel [8] that prove the integral representation of nonlinear transformations on L^p-spaces , $1 \leq p \leq \infty$.

7.15 Theorem. Let $T = (T,\Sigma,\mu)$ be a finite measure space and let F be a real-valued functional on $L^p(T)$, $1 \leq p \leq \infty$, that satisfies

(i) $F(x+y) - F(x) - F(y) = \text{constant} = C_F$ whenever $xy = 0$ a.e.

(ii) F is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$.

(iii) F is continuous relative to L^p-norm, if $p < \infty$ and is continuous w.r.t. bounded a.e. convergence if $p = \infty$.

Then there exists a function $\varphi \in \mathrm{Car}^p(T)$ s.t.

$$(*) \qquad F(x) = -C_F + \int_T \varphi \circ x \, d\mu \quad \text{for} \quad x \in L^p(T)$$

where φ can be taken to satisfy

(a) $\varphi(0,\cdot) = 0$ a.e. and is unique up to sets of the form $R \times N$ with N a null set in T .

Conversely, $\forall \ \varphi \in \mathrm{Car}^p(T)$ satisfying (a) and for every $C_F \in R$, (*) defines a functional satisfying (i), (ii), and (iii).

The above results extend to σ-finite measure spaces and the proof for $p = \infty$ is as it is and for $p < \infty$ it is valid if the phrase "bounded subset of $L^\infty(T)$" is replaced by bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.

7.17 Theorem. Let $T = (T,\Sigma,\mu)$ be a finite measure space and let $A : L^p(T) \to C(S)$, $1 \le p \le \infty$ be a transformation where S is a compact Hausdorff space. Suppose A satisfies the conditions

(ia) $A(x+y) = A(x) + A(y)$ whenever $x.y = 0$ a.e.

(iia) A is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$.

(iiia) A is continuous relative to L^p norm if $p < \infty$ and is continuous w.r.t. bounded a.e. convergence if $p = \infty$. Then there exists a transformation $\varphi : S \to \mathrm{Car}^p(T)$ s.t.

$$(**) \qquad A(x)(s) = \int_T \varphi \circ x \, d\mu .$$

The transformation φ can be taken to satisfy

(a) $\varphi(s)\circ 0 = 0$ a.e. $\forall$ $s \in S$, in which case φ is unique for each s up to sets of the form $R\times N$ with N a null set in T. Moreover, φ has the following additional properties:

(b) The mapping $s \to \varphi(s)\circ x \in L^1(T)$ is weakly continuous $\forall$ $x \in L^p(T)$.

(c) The mapping $x \to \varphi(s)\circ x$ is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$, uniformly in s.

(d) The mapping $x \to \varphi(s)\circ x$ is weakly continuous on $L^p(T)$ uniformly in s, if $p \leq \infty$, if $x_n \to x$ boundedly a.e. then

$$\lim_{\mu(E)\to 0} \int_E (\varphi(s)\circ x_n)d\mu \to 0 \quad \text{uniformly in } s \text{ and } n$$

if $p = \infty$.

Conversely every transformation $\varphi : s \to \mathrm{Car}^p(T)$ $1 \leq p \leq \infty$, satisfying (a), (b), (c) and (d) determines, by means of (**) a transformation

$$A : L^p(T) \to C(S)$$

satisfying (ia), (iia) and (iiia).

The above result also extends to σ-finite measure spaces. For $p = \infty$, it is valid if the following condition is added.

(e) If $x_n \to x$ boundedly a.e., then for any sequence $E_i \downarrow \phi$, $\int_{E_i} (\varphi(s)\circ x_n)d\mu \to 0$ uniformly in s and n.

For $p < \infty$, it is valid if the phrase "bounded subset of $L^\infty(T)$" is replaced by "a set of finite measure".

The proof in Theorem 7.15 utilizes the Lemma 7.14 and the representation (*) is then established by use of the Vitali convergence.

The converse utilizes the Lemma 7.4 by Nemytskii in [12] and Banach-Saks Theorem.

The last theorem, 7.17, utilizes Theorem 7.15 and Vitali-Hahn-Saks Theorem on convergence of measures.

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