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ON THE INTEGRALS OF PERRON TYPE

by

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ABSTRACT

Perron's method of defining a process of integration is through the use of major and minor functions. Many authors have adopted this method to define various integrals. In Chapter I, we give a very general abstract theory by first defining an abstract "derivate system" and then the corresponding Perron integral. We show that this unifies all the integral theories of Perron type (of first order) known to us, in addition the abstract theories of Pfeffer [26] and of Romanovski [29] are contained in our theory as particular cases.

Chapter II is devoted mainly to the study of Burkill's C_nP - integral. We know that the C_nP - integral is based on the theorem that if M is C_n - continuous in $[a,b]$, $\underline{C_n}DM(x) \geq 0$ almost everywhere and $\underline{C_n}DM(x) > -\infty$ nearly everywhere in $[a,b]$, then M is monotone increasing in $[a,b]$. Burkill's original proof of this, [6] , contains an error and we give it a new and correct proof. We also give a correct proof of Sargent's theorem that if a function is C_nP - integrable, then it is C_nD - integrable, [32] ; the original proof contains a gap.

A scale of symmetric CP - integrals and a scale of approximately mean-continuous integrals are obtained in Chapter III and in Chapter IV, respectively. The first one is more general than Burkill's CP - scale, while the second one is more general than the GM - scale defined by Ellis. Some other comparisons of various integrals are also given.

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INTRODUCTION

Various integrals defined for functions with domains in the real line have been generalized so as to apply to functions with domain in some abstract space. For example, the Lebesgue integral has been defined on a abstract measure space (see Saks [30]); the integral of Riemann type on the division space (see Henstock [11] and McShane [24]); the integral of Denjoy type on the Romanovski space (see Solomon [37]); the integral of Perron on certain topological spaces [26]. One of our purposes is to give a very general setting for Perron integrals. A so-called derivate system is defined in section 1, Chapter I, and then an integral theory of Perron type is obtained in the following sections. Doing this, we unify all the integral theories of Perron type, eg. the classical Perron integral, the C_nP - , SCP - , AP - integral of Burkill's [4] - [7], the MZ - integral defined by Marchinkiewicz and Zygmund [21], Kubota's AP - integral [19], and also the GM_n - integral defined by Ellis in [9] . For a good review of these integrals, we refer to James [14] and Jeffery [16]. In addition, we show that the P - and R - integral of Romanovski's in [29] and the integral of Pfeffer in [26] can also be obtained from our theory.

In the theory of integrals of Perron type it is of interest to define more general concrete integrals. Thus the SCP - integral is more general than the CP - integral; Kubota's AP - integral is more general than Burkill's AP - integral, the GM_n - integral is more general than C_nP - integral.

We obtain via our general theory a scale of symmetric Cesaro-Perron integrals (SCP - scale) and a scale of approximately mean-continuous integrals (AMP - scale); this we do in Chapter III and IV, respectively. The SCP - scale is more general than the CP - scale of Burkill, while the AMP - scale is more general than the mean-continuous scale due to Ellis.

The comparibility of these integrals is then studied. We prove that the SCP - integral and the MZ - integral are in fact equivalent, and in section 5 of Chapter III the relation of the symmetric P^{n+1} - integral (James [13]) and our SC_nP - integral is investigated. An ACP - integral and an AP^2 - integral are defined and proved to be equivalent in section 3, Chapter IV. This generalizes the result [3] for $n = 1$ that the C_nP - integral and the P^{n+1} - integral are equivalent.

Chapter II is devoted to the C_nP - integral. A gap in Burkill's original paper [6] is filled, and so is one in Sargent's paper [32]. We know that the theory of the C_nP - integral is based on theorem 2.2 in [6]. However, the proof there is defective; see line 9 on page 546. We supply a proof of this theorem based on some concepts in [32]. Sargent has defined a C_nD - integral and proved that it is equivalent to the C_nP - integral. But there is a defect in her proof that a C_nP - primitive is ACG^* (in C_n - sense). We give a complete proof, which is simpler than the one given recently and independently by Verblunsky in [38].

We close this introduction with some remarks about the notations used;

$A \sim B$ denotes the relative difference when A and B are sets;

the symbol $\subset$ to indicate inclusion, not necessarily proper;

for real numbers a, b with $a < b$, we denote by $[a,b]$, $]a,b[$ the closed and open interval, respectively; and $[a,b[$, $]a,b]$ denote the half-open intervals;

by Theorem II. 3, we mean Theorem 3 in Chapter II, and similarly section I.5, etc. If only Theorem 3 is quoted, we mean Theorem 3 of the same chapter.

CHAPTER I. THE GENERAL THEORY

Perron's method of defining a process of integration is through the use of majorants and minorants (see Saks [30]). Many authors have adopted this method to define various integrals. As a typical example for our general theory in this chapter, we quote Burkill's definition of major functions for his SCP - integral in [7] .

If f is a function defined on $[a,b]$, a function M is called a SCP - major function of f on $[a,b]$ with base B (where B is a subset of $[a,b]$ with measure $b - a$ and $a , b \in B$) if

(a) M is C - continuous in B , and SC - continuous in $]a,b[$;

(b) $\underline{SC}DM(x) \geq f(x)$ almost everywhere in $[a,b]$;

(c) $\underline{SC}DM(x) > -\infty$ except for a countable set of points;

(d) $M(a) = 0$.

For the definitions of C - and SC - continuity, and also $\underline{SC}DM(x)$, see section III 3, below.

Firstly, we generalize the domain of the functions to an abstract space X with a distinguished family σ of subsets of X . Secondly, we generalize the base B to the concept "base mapping $\mathcal{B}$ " , condition (a) to the "legitimate mapping", the derivate $\underline{SCD}$ to the abstract "derivate operator", condition (c) to the so-called "inequality property". Then a "derivate system" is defined and a naturally corresponding integral of Perron type arises. The finite additivity of the integral is established and a convergence theorem similar to that of Lebesgue dominated convergence theorem is obtained.

From this general theory, we obtain the integral theory of Romanovski [29] and of Pfeffer [26]. A differential property and a characterization of integrability is obtained for the abstract integral in the case that domain of the function is the real line.

§1. SETTINGS.

Suppose X is a given set, σ a given collection of subsets of X and $A \subset X$. Define σ_A by

$$\sigma_A = \{A' \mid A' \in \sigma , A' \subset A\} .$$

1.1 DERIVATE OPERATORS.

Let $A \in \sigma$, β be a subset of σ_A, $\overline{V}$ be a semi-vector space of (set) functions defined on β, where by a semi-vector space $\overline{V}$ we mean that $F_1 , F_2 \in \overline{V}$ implies $\alpha_1 F_1 + \alpha_2 F_2 \in \overline{V}$ for all real numbers $\alpha_1 , \alpha_2 \geq 0$. A lower derivate operator on $\overline{V}$ is a mapping $\underline{\mathcal{D}}$ with domain $\overline{V} \times A$ such that for each $v \in \overline{V}$, and for each $x \in A$, the image $\underline{\mathcal{D}}(v,x) = \underline{\mathcal{D}}v(x)$ is an extended real number, and satisfying the following axioms:

($\underline{\mathcal{D}}$1) for all $x \varepsilon A$, $\underline{\mathcal{D}}(0,x) = 0$;

($\underline{\mathcal{D}}$2) for all $x \varepsilon A$, v_1 , $v_2 \varepsilon \overline{V}$, $\underline{\mathcal{D}}(v_1+v_2,x) \geq \underline{\mathcal{D}}(v_1,x) + \underline{\mathcal{D}}(v_2,x)$ whenever the addition on the right hand side makes sense;

($\underline{\mathcal{D}}$3) for all $x \varepsilon A$, $v \varepsilon \overline{V}$, $\alpha > 0$, $\underline{\mathcal{D}}(\alpha v,x) = \alpha\underline{\mathcal{D}}(v,x)$;

($\underline{\mathcal{D}}$4) for all $x \varepsilon A$, $\underline{\mathcal{D}}(-v,x) \leq -\underline{\mathcal{D}}(v,x)$ whenever both v and $-v$ are in $\overline{V}$.

For each $v \varepsilon \overline{V}$, $x \varepsilon A$, defining $\overline{\mathcal{D}}(-v,x) = -\underline{\mathcal{D}}(v,x)$, we call $\overline{\mathcal{D}}$ the upper derivate operator corresponding to $\underline{\mathcal{D}}$. Letting $\underline{V} = \{v \mid -v \varepsilon \overline{V}\}$, we see that $\underline{V} \times A$ is the domain of $\overline{\mathcal{D}}$. It is easy to see that $\overline{\mathcal{D}}$ has properties ($\overline{\mathcal{D}}$1) - ($\overline{\mathcal{D}}$4) of which the meaning is immediate. Furthermore, for $v \varepsilon \overline{V} \cap \underline{V}$, $\overline{\mathcal{D}}(v,x) \geq \underline{\mathcal{D}}(v,x)$, $\overline{\mathcal{D}}(\alpha v,x) = \alpha\underline{\mathcal{D}}(v,x)$ for all $x \varepsilon A$ and $\alpha < 0$.

If $\underline{\mathcal{D}}(v,x)$ and $\overline{\mathcal{D}}(v,x)$ are equal, we say that v is $\mathcal{D}$ - differentiable at x , and the common value, denoted by $\mathcal{D}(v,x)$ or $\mathcal{D}v(x)$ is called the $\mathcal{D}$ - derivative of v at x ; for example, clearly $\mathcal{D}(0,x) = 0$ for all $x \varepsilon A$.

1.2. BASE MAPPINGS.

Let $A \varepsilon \sigma$. A subset β of σ_A will be called a base in A if $A \varepsilon \beta$ and for each $A' \varepsilon \beta$ there exists a finite set of disjoint $A_i \varepsilon \beta$ with $A_i \cap A' = \phi$ for each i and $\bigcup_i A_i \cup A' = A$. By a base mapping on σ we mean a mapping $\mathcal{B}$ on σ such that for each $A \varepsilon \sigma$, the image $\mathcal{B}(A)$ is a collection of bases in A satisfying the following axioms.

($\mathcal{B}$1) β_1 , $\beta_2 \in \mathcal{B}(A)$ implies $\beta_1 \cap \beta_2 \in \mathcal{B}(A)$.

($\mathcal{B}$2) $\beta \in \mathcal{B}(A)$ and $A' \in \beta$ imply $\beta_{A'} = \{A'' \mid A'' \in \beta$ and $A'' \subset A'\} \in \mathcal{B}(A')$.

($\mathcal{B}$3) $\beta_i \in \mathcal{B}(A_i)$, $A_i \in \sigma$ for $i = 1$, 2 and $A_1 \cap A_2 = \phi$, $A_1 \cup A_2 \in \sigma$ implies that $\beta_1 \oplus \beta_2 = \{A_1' \cup A_2' \mid A_i' \in \beta_i$ for $i = 1$, 2 , and $A_1' \cup A_2' \in \sigma\} \in \mathcal{B}(A_1 \cup A_2)$.

1.3. LEGITIMATE MAPPINGS.

Let F be an extended real-valued (set) function defined on γ , a collection of subsets of X . F is said to be superadditive on γ if $F(\bigcup_i A_i) \geq \sum_i F(A_i)$ for every finite collection $\{A_i\}$ of disjoint sets from γ for which $\bigcup_i A_i \in \gamma$ and the additions $\sum_i F(A_i)$ make sense. F is defined to be subadditive if and only if $-F$ is superadditive. If F is both superadditive and subadditive, we say that F is additive.

Given a base mapping $\mathcal{B}$ on σ let $\overline{M}$ be a mapping such that for each $A \in \sigma$, $\beta \in \mathcal{B}(A)$, the image $\overline{M}(A,\beta)$ is a semi-vector space of real-valued functions superadditive on β . If $\overline{M}$ satisfies further the following axioms, we say that $\overline{M}$ is a legitimate mapping on σ with base mapping $\mathcal{B}$.

($\overline{M}$1) For any β_1 , $\beta_2 \in \mathcal{B}(A)$ with $\beta_1 \subset \beta_2$, $\overline{M}(A,\beta_1) \supset \overline{M}(A,\beta_2)$.

($\overline{M}2$) For any $\beta \in \mathcal{B}(A)$ and any $A' \in \beta$,

$$\overline{M}(A , \beta) \mid A' = \{M \mid \beta_{A'} \mid M \in \overline{M}(A , \beta)\} \subset \overline{M}(A' ; \beta_{A'}).$$

($\overline{M}3$) For $A_1 , A_2 , \beta_1 , \beta_2$ as in ($\mathcal{B}3$), if $M_i \in \overline{M}(A_i , \beta_i)$ for $i = 1 , 2$, then $M_{12} \in \overline{M}(A_1 \cup A_2 , \beta_1 \oplus \beta_2)$, where $M_{12}(A') = M_1(A_1') + M_2(A_2')$ for any $A' = A_1' \cup A_2'$ in $\beta_1 \oplus \beta_2$ with $A_i' \in \beta_i$ for $i = 1 , 2$.

($\overline{M}4$) $M_1 = M_2$ on β and $M_1 \in \overline{M}(A , \beta)$ implies that $M_2 \in \overline{M}(A , \beta)$.

($\overline{M}5$) $\overline{M}(A , \beta)$ is closed under uniform sequence convergences in β. (i.e. if $F_n \in \overline{M}(A , \beta)$ for $n = 1 , 2 , 3 , \ldots ,$ and $F_n \to F$ uniformly in β, then $F \in \overline{M}(A , \beta)$.)

1.4. INEQUALITY PROPERTIES.

By an inequality property on set functions we mean a property $\overline{I}$ satisfying the following axioms.

($\overline{I}1$) If F_1 and F_2 are two set functions defined on a domain γ and if both F_1 and F_2 satisfy the property $\overline{I}$ on γ, then $\alpha_1F_1 + \alpha_2F_2$ satisfies $\overline{I}$ on γ whenever $\alpha_1F_1 + \alpha_2F_2$ makes sense, where α_1 and α_2 are non-negative real numbers.

($\overline{I}2$) If a set function satisfies $\overline{I}$ on domains γ_1 , γ_2 respectively, it does so on $\gamma_1 \cap \gamma_2$ and $\gamma_1 \cup \gamma_2$.

($\overline{I}3$) If F_1 and F_2 are two set functions on γ with $F_1 \geq F_2$ and F_2 satisfies $\overline{I}$ on γ, then F_1 satisfies $\overline{I}$ on γ.

If $\overline{I}$ is an inequality property, we denote its dual property by $\underline{I}$ and by this we mean that F satisfies $\underline{I}$ if and only if $-F$ satisfies $\overline{I}$. We will come across two kinds of inequality properties in the examples considered later; one is defined by means of inequalities containing the lower derivates of functions; the other is defined by means of inequalities containing the function values.

1.5. DERIVATE SYSTEMS.

Let N be a fixed collection of subsets of X closed under countable set unions, (i.e. $E_n \varepsilon N$ for $n = 1, 2, 3, \ldots$, imply that $\bigcup_n E_n \varepsilon N$). For convenience, we say that a property $P(x)$ is true almost everywhere (a.e.) in A if it is true for all x in A except at most for points of a set in N.

Given a legitimate mapping $\overline{M}$ on σ with a base mapping B, and an inequality properties $\overline{I}$, suppose that for each $A \varepsilon \sigma$, $\beta \varepsilon B(A)$, there exists a lower derivate operator $\underline{D}_{A\beta}$ on $\overline{M}(A,\beta)$. If the following axioms are satisfied, we say that $(\overline{M}, \underline{D}, B, N, \overline{I})$ is a derivate system on σ.

($\underline{D}\overline{M}$1) For $A_1 \varepsilon \sigma$, $A_2 \varepsilon \sigma_{A_1}$, $M_i \varepsilon \overline{M}(A_i, \beta_i)$, $i = 1, 2$, $M_1 = M_2$ on $\beta_1 \cap \beta_2$, one has $\underline{D}_{A_1\beta_1}(M_1, x) = \underline{D}_{A_2\beta_2}(M_2, x)$ for each $x \varepsilon A_2$.

($\underline{D}\overline{M}$2) If $M \varepsilon \overline{M}(A, \beta)$ with $\underline{D}_{A\beta}(M, x) \geq 0$ almost everywhere in A and M satisfies the inequality property $\overline{I}$, then $M \geq 0$ on β.

Note that by axiom $(\underline{\mathcal{D}}\overline{M}1)$, we can always (without any ambiguity) write $\underline{\mathcal{D}}(M , x)$ instead of $\underline{\mathcal{D}}_{A\beta}(M , x)$.

§2. THE INTEGRAL.

Given the set X , and σ a collection of subsets of X , we let $P \equiv (\overline{M} , \underline{\mathcal{D}} , \mathcal{B} , \mathcal{N} , \overline{I})$ be a derivate system on σ ; if we need other derivate systems on σ we will denote them by $P_1 \equiv (\overline{M}^1 , \underline{\mathcal{D}}_1 , \mathcal{B}_1 , \mathcal{N}_1 , \overline{I}_1)$ etc.

2.1. MAJOR AND MINOR FUNCTIONS.

Let $A \in \sigma$, $\beta \in \mathcal{B}(A)$, and f be an extended real-valued function defined and finite almost everywhere in A . A function M is a P - major function of f on A with base β , written $M \in \overline{M}_f(A , \beta)$, if

$(\overline{M}1)$ $M \in \overline{M}(A , \beta)$;

$(\overline{M}2)$ $\underline{\mathcal{D}}(M , x) \geq f(x)$ almost everywhere in A ;

$(\overline{M}3)$ M satisfies $\overline{I}$.

A function m is a P - minor function of f on A with base β , written $m \in \underline{M}_f(A , \beta)$, if $- m \in \overline{M}_{-f}(A , \beta)$. We will write $\underline{M}(A , \beta) = \{- M \mid M \in \overline{M}(A , \beta)\}$. It is easy to see that $m \in \underline{M}_f(A , \beta)$ if and only if

($\underline{m}1$) $m \in \underline{M}(A , \beta)$;

($\underline{m}2$) $\overline{D}(m,x) \leq f(x)$ almost everywhere in A ;

($\underline{m}3$) m satisfies $\underline{I}$.

The following lemma is fundamental for our theory.

LEMMA 1. For $M \in \overline{M}_f(A , \beta)$, $m \in \underline{M}_f(A , \beta)$, $M - m$ is superadditive and non-negative on β . In particular, $M(A) \geq m(A)$.

Proof. It is trivial that $M - m \in \overline{M}(A , \beta)$, so that $M - m$ is superadditive on β . As f is finite almost everywhere, it follows from ($\overline{M}2$) , ($\underline{m}2$) and ($\underline{D}2$) that $\underline{D}(M-m , x) \geq \underline{D}(M,x) - \overline{D}(m,x) \geq 0$ almost everywhere in A . Moreover, $M - m$ satisfies $\overline{I}$ by ($\overline{M}3$) , ($\underline{m}3$) and ($\overline{I}1$) . Hence, $M - m \geq 0$ on β by ($\underline{D}\overline{M}2$) , and the proof is completed.

2.2. THE DEFINITION OF THE INTEGRAL.

If both $\overline{M}_f(A , \beta)$ and $\underline{M}_f(A , \beta)$ are not empty and

$$\inf\{M(A) \mid M \in \overline{M}_f(A , \beta)\} = \sup \{m(A) \mid m \in \underline{M}_f(A , \beta)\} \neq \pm \infty ,$$

then we say that f is P - integrable on A with base β , and the common value, denoted by $(P) - \int_A^\beta f$, is called the P - integral of f on A with base β . The set of all P - integrable functions on A with base β will be denoted by $P(A , \beta)$.

The following lemma is an immediate consequence of lemma 1.

LEMMA 2. $f \in P(A , \beta)$ if and only if for each $\varepsilon > 0$ there exist $M \in \overline{M}_f(A , \beta), m \in \underline{M}_f(A , \beta)$ with $M(A) - m(A) < \varepsilon$.

LEMMA 3. Let β_1, $\beta_2 \in \mathcal{B}(A)$ with $\beta_1 \subset \beta_2$. If $f \in P(A, \beta_2)$, then $f \in P(A, \beta_1)$ and two P - integrals are equal. In particular, if β, $\beta' \in \mathcal{B}(A)$ and $f \in P(A, \beta)$, $f \in P(A, \beta')$, then $(P) - \int_A^{\beta} f = (P) - \int_A^{\beta'} f$.

Proof. This is immediate from $(\overline{M}1)$, $(\underline{D}\overline{M}1)$ and $(\mathcal{B}1)$.

Henceforward, we can often without any ambiguity leave the base unspecified.

2.3. ELEMENTARY PROPERTIES OF THE INTEGRAL.

THEOREM 1. $P(A, \beta)$ is a vector space and the P - integral is linear on $P(A, \beta)$.

Proof. First, we prove that if $f \in P(A, \beta)$ then $\alpha f \in P(A, \beta)$ for each real number α. For $\alpha = 0$, it is trivial from $(\underline{D}1)$. Suppose that $\alpha > 0$. By $(\underline{D}3)$ and $(\overline{D}3)$, it follows that $M \in \overline{M}_f$, $m \in \underline{M}_f \Rightarrow \alpha M \in \overline{M}_{\alpha f}$, $\alpha m \in \underline{M}_{\alpha f}$. Hence $\alpha f \in P(A, \beta)$ since $\alpha M(A) - \alpha m(A)$ can be made arbitrarily small with $M(A) - m(A)$. The equality $\int \alpha f = \alpha \int f$ follows from the inequalities $\alpha m(A) \leq \int \alpha f \leq \alpha M(A)$. For $\alpha < 0$, the proof is similar.

Secondly we prove that if $f_i \in P(A, \beta)$ for $i = 1, 2$ then $f_1 + f_2 \in P(A, \beta)$ and $\int (f_1 + f_2) = \int f_1 + \int f_2$. This follows from $(\underline{D}2)$ and $(\overline{D}2)$, and the proof of the theorem is completed.

THEOREM 2. If $f \in P(A, \beta)$ and $A' \in \beta$. Then $f \in P(A', \beta_{A'})$. Furthermore, if $A_1, A_2 \in \beta$ with $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$, then

$$\int_A f = \int_{A_1} f + \int_{A_2} f .$$

Proof. If $M \in \overline{M}_f(A, \beta)$, then $M \mid \beta_{A'} \in \overline{M}(A', \beta_{A'})$ by $(\overline{M}2)$ and $(\mathcal{D}\overline{M}1)$. Similar results hold for minor functions. By lemma 2, for each $\varepsilon > 0$, there are suitable major and minor functions M, m respectively with $M(A) - m(A) < \varepsilon$. By lemma 1, $M - m$ is superadditive and non-negative on β, so that $M(A') - m(A') \leq M(A) - m(A) < \varepsilon$. Thus, by lemma 2, $f \in P(A', \beta_{A'})$.

We now prove that $\int_A f = \int_{A_1} f + \int_{A_2} f$,

$$\int_A f = \inf\{M(A) \mid M \in \overline{M}_f(A, \beta)\} \geq \inf\{M(A_1) + M(A_2) \mid M \in \overline{M}_f(A, \beta)\}$$

$$\geq \inf\{M_1(A_1) + M_2(A_2) \mid M_i \in \overline{M}_f(A_i, \beta_{A_i}) \text{ for } i = 1, 2\}$$

$$\geq \inf\{M_1(A_1) \mid M_1 \in \overline{M}_f(A_1, \beta_{A_1})\} + \inf\{M_2(A_2) \mid M_2 \in \overline{M}_f(A_2, \beta_{A_2})\}$$

$= \int_{A_1} f + \int_{A_2} f$, where the first inequality follows from the superadditivity, while the second one follows from $(\overline{M}2)$. Similarly, using minor functions, it follows that $\int_A f \leq \int_{A_1} f + \int_{A_2} f$, completing the proof.

THEOREM 3. If $f \in P(A_i, \beta_i)$ for $i = 1, 2$, where $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 \in \sigma$, then $f \in P(A_1 \cup A_2, \beta_1 \oplus \beta_2)$ and $\int_{A_1 \cup A_2} f = \int_{A_1} f + \int_{A_2} f$.

Proof. This is immediate from $(\mathcal{B}3)$, $(\overline{M}3)$, $(\underline{\mathcal{D}}\overline{M}1)$ and Theorem 2.

THEOREM 4. Let $F \in \overline{M}(A, \beta) \cap \underline{M}(A, \beta)$ and satisfy both $\overline{I}$ and $\underline{I}$. If $\mathcal{D}F(x)$ exists and is finite almost everywhere in A , then $\mathcal{D}F \in P(A, \beta)$ and $\int_A \mathcal{D}F = F(A)$.

Proof. It is clear that $F \in \overline{M}_{\mathcal{D}F}(A, \beta) \cap \underline{M}_{\mathcal{D}F}(A, \beta)$, and the conclusion follows from lemma 2.

We close this subsection by remarking that if $f = g$ almost everywhere in A and $f \in P(A, \beta)$, then $g \in P(A, \beta)$ and the integrals of f is equal to the integral of g .

2.4. PRIMITIVES.

If $f \in P(A, \beta)$, then by Theorem 2, we see that $f \in P(A', \beta_{A'})$ for each $A' \in \beta$. Define $F(A') = \int_{A'} f$ for each $A' \in \beta$. F is called the primitive of f on A with base β . By Theorem 3, we know that F is additive on β , so that it is easy to obtain

THEOREM 5. Let $f \in P(A, \beta)$ with primitive F , and $M \in \overline{M}_f(A, \beta)$, $m \in \underline{M}_f(A, \beta)$. Then $M - F$, $F - m$ are both superadditive and non-negative on β .

LEMMA 4. If $f \in P(A, \beta)$ with primitive F , then there exists a sequence $\{M_k\} \subset \overline{M}_f(A, \beta)$ and a sequence $\{m_k\} \subset \underline{M}_f(A, \beta)$ such that $0 \leq M_k(A') - F(A') < \frac{1}{k}$ and $0 \leq F(A') - m_k(A') < \frac{1}{k}$ for each $A' \in \beta$.

Proof. This is immediate from Theorem 5.

THEOREM 6. If F is a primitive of $f \in P(A , \beta)$, then

$$F \in \overline{M}(A , \beta) \cap \underline{M}(A , \beta) .$$

Proof. This is immediate from lemma 4 and $(\overline{M}5)$.

The following general comparison theorem is a direct consequence of the definition of the integral.

THEOREM 7. Let $P_i = (\overline{M}^i , \underline{\mathcal{D}}_i , \mathcal{B}_i , \mathcal{N}_i , \overline{I}_i)$ be a derivate system on σ for $i = 1 , 2$. Suppose that $\overline{M}^1(A , \beta) \subset \overline{M}^2(A , \beta)$, $\mathcal{N}_1 \subset \mathcal{N}_2$, that each function satisfying $\overline{I}_1$ satisfies $\overline{I}_2$, and that $\underline{\mathcal{D}}_1(M,x) \leq \underline{\mathcal{D}}_2(M,x)$ for each $M \in \overline{M}^1(A , \beta)$, then $P_1(A , \beta) \subset P_2(A , \beta)$ and $(P_1) - \int f = (P_2) - \int f$ for each $f \in P_1(A , \beta)$.

§3. CONVERGENCE THEOREMS.

With some further reasonable restrictions on the derivate system $P = (\overline{M} , \underline{\mathcal{D}} , \mathcal{B} , \mathcal{N} , \overline{I})$, we will now obtain some convergence theorems for our integral similar to those for the Lebesgue integral. Throughout this section, we assume that $\overline{M}(A , \beta)$ satisfies the following additional axioms.

If f_n is a sequence of functions defined on a domain E, by $f_n \uparrow f$ we mean that $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in E$ and $f_n(x) \leq f_{n+1}(x)$ for each n and for each $x \in E$.

($\overline{M}5'$) If $\{M_n\} \subset \overline{M}(A, \beta)$ and $M_n \uparrow M$, then $M \in \overline{M}(A, \beta)$.

($\underline{D}\overline{M}3$) For $M \in \overline{M}(A, \beta)$ with $M \geq 0$ on β, $\underline{D}(M,x) \geq 0$ for all $x \in A$.

REMARK. It is clear that ($\underline{D}\overline{M}3$) is a very natural axiom, however axiom ($\overline{M}5'$) seems to be too much of a restriction. However, in the particular examples in later chapters, the "interval" functions in $\overline{M}(A, \beta)$ are obtained from the "point" functions, so that the functions in $\overline{M}(A, \beta)$ will then be additive rather than only superadditive. If every function in $\overline{M}(A, \beta)$ is additive, then ($\overline{M}5'$) follows from axiom ($\overline{M}5$). To see this, let $M_n \in \overline{M}(A, \beta)$ for $n = 1, 2, 3, \ldots$, and $M_n \uparrow M$. We have to prove that $M \in \overline{M}(A, \beta)$. It is clear that $M(A') \geq M_n(A')$ for all $A' \in \beta$, and that M is additive on β. Thus, $M - M_n$ is non-negative and additive on β, so that $M(A) - M_n(A) \geq M(A') - M_n(A') \geq 0$ for each $A' \in \beta$. Now, as $M_n(A) \uparrow M(A)$, for each $\varepsilon > 0$ there exists a positive integer n_A such that $0 \leq M(A) - M_n(A) < \varepsilon$ for any $n \geq n_A$. Hence $0 \leq M(A') - M_n(A') < \varepsilon$ for each $n \geq n_A$ and for each $A' \in \beta$, i.e. M_n converges to M uniformly on β. That $M \in \overline{M}(A, \beta)$ then follows from axiom ($\overline{M}5$).

THEOREM 8. Suppose that f_n , f are functions defined and finite almost everywhere in A and $f_n \in P(A, \beta)$ for $n = 1, 2, 3, \ldots$, and $f_n(x) \uparrow f(x)$ almost everywhere in A . Then $f \in P(A, \beta)$ and $\lim_n \int f_n = \int f$.

Proof. First, note that if $g(x) \leq h(x)$ almost everywhere in A and $g, h \in P(A, \beta)$, then $\int g \leq \int h$. This follows directly from the definition of the integral.

Now, let F_n be the primitive of f_n for $n = 1, 2, 3, \ldots$. Then by Theorem 6, $F_n \in \overline{M}(A, \beta) \cap \underline{M}(A, \beta)$. For each $\varepsilon > 0$, choose $M_n \in \overline{M}_{f_n}(A, \beta)$ such that $0 \leq M_n - F_n < \varepsilon/2^n$ for $n = 1, 2, 3, \ldots$, which is possible by lemma 4. In β , let $N_1 = M_1$, and for $n \geq 2$, $N_n = M_n + \sum_{i=1}^{n-1} (M_i - F_i)$. Then $N_n \in \overline{M}(A, \beta)$ and $N_n \geq M_n$.

Furthermore,
$$\begin{aligned} N_{n+1} &= M_{n+1} + N_n - F_n \\ &\geq M_{n+1} + N_n - F_{n+1} \quad \text{since } F_n \leq F_{n+1} \\ &\geq N_n \quad \text{since } M_{n+1} \geq F_{n+1} . \end{aligned}$$

Thus, $N_n \uparrow N$ in β . By ($\overline{M}5'$) , $N \in \overline{M}(A, \beta)$. By ($\overline{I}3$) , N satisfies $\overline{I}$ since M_n does and $N \geq N_n \geq M_n$. Moreover, by ($\underline{D}2$) and ($\underline{D}\overline{M}3$) , it easily follows that $\underline{D}(N,x) \geq f(x)$ almost everywhere in A . Thus, we have proved that $N \in \overline{M}_f(A, \beta)$. Furthermore, since

$$N_n(A) \leq M_n(A) + \sum_{i=1}^{n-1} \varepsilon/2^i \leq F_n(A) + \varepsilon/2^n + \sum_{i=1}^{n-1} \varepsilon/2^i$$

$$= F_n(A) + \sum_{i=1}^{n} \varepsilon/2^i \quad \text{for} \quad n = 2, 3, 4, \ldots,$$

one has

$$\inf \{M(A) \mid M \in \overline{M}_f(A, \beta)\} \leq N(A) \leq \lim_n F_n(A) + \varepsilon .$$

As ε is arbitrary, we see that

$$\inf \{M(A) \mid M \in \overline{M}_f(A, \beta)\} \leq \lim_n F_n(A) .$$

Similarly, using minor functions, one can prove that $\underline{M}_f(A, \beta)$ is not empty and

$$\sup \{m(A) \mid m \in \underline{M}_f(A, \beta)\} \geq \lim_n F_n(A)$$

Thus, by lemma 1, $f \in P(A, \beta)$ and $\int f = \lim_n F_n = \lim \int f_n$, completing the proof.

<u>THEOREM 9</u>. Suppose that f_n, f are functions defined and finite almost everywhere in A, and $f_n \in P(A, \beta)$ for $n = 1, 2, 3, \ldots$. Further, suppose that $\liminf_n f_n(x) = f(x)$ almost everywhere in A. If $\inf \{M(A) \mid M \in \overline{M}_{\inf_n f_n}(A, \beta)\} > -\infty$, then

$\inf\{M(A)|M \in \overline{M}_f(A,\beta)\} \leq \liminf_n \int_A f_n$.

Proof. Let $g_n(x) = \inf_{k\geq n} f_k(x)$ almost everywhere in A . Then $f_k \geq g_n$ for each $k \geq n$ and $g_n(x) \uparrow f(x)$ almost everywhere in A . Hence, $\inf\{M(A)|M \in \overline{M}_{g_n}(A,\beta)\} \leq \inf\{M(A)|M \in \overline{M}_{f_k}(A,\beta)\}$ for $k \geq n$ since because $f_k \geq g_n$ we have that $\overline{M}_{f_k}(A,\beta) \subset \overline{M}_{g_n}(A,\beta)$. Thus, $\inf\{M(A)|M \in \overline{M}_{g_n}(A,\beta)\} \leq \inf_{k\geq n} \int_A f_k$ for $n = 1,2,3,\ldots$.

Hence

$$\lim_n [\inf\{M(A)|M \in \overline{M}_{g_n}(A,\beta)\}] \leq \liminf_n \int_A f_n .$$

Now, as $g_n \uparrow f$ and $\inf\{M(A)|M \in \overline{M}_{g_1}(A,\beta)\} > -\infty$, following an argument similar to that in the proof of theorem 8, one proves easily that

$$\lim_n [\inf\{M(A)|M \in \overline{M}_{g_n}(A,\beta)\}] \geq \inf\{M(A)|M \in \overline{M}_f(A,\beta)\} ,$$

and the proof is hence completed.

THEOREM 10. Suppose that $h, g, f_n \in P(A,\beta)$ and $g(x) \leq f_n(x) \leq h(x)$ almost everywhere in A for $n = 1,2,3,\ldots$. If f is a function defined and finite almost everywhere in A with $\lim_n f_n(x) = f(x)$ almost everywhere in A , then $f \in P(A,\beta)$ and $\int f = \lim_n \int f_n$.

Proof. Let $\phi_n = f_n - g$, $\phi = f - g$, $\psi = h - g$. Then $0 \leq \phi_n \leq \psi$ almost everywhere in A , so that $0 \leq \inf_n \phi_n \leq \sup_n \phi_n \leq \psi$ almost every in A . Hence $\inf \{M(A) | M \in \overline{M}_{\inf_n \phi_n}(A,\beta)\} > -\infty$ and

$$\sup \{m(A) | m \in \underline{M}_{\sup_n \phi_n}(A,\beta)\} < +\infty .$$

By Theorem 9 and its dual, we see that

$$a \equiv \inf \{M(A) | M \in \overline{M}_{\phi}(A,\beta)\} \leq \liminf_n \int_A \phi_n ,$$

$$b \equiv \sup \{m(A) | m \in \underline{M}_{\phi}(A,\beta)\} \geq \limsup_n \int_A \phi_n$$

as $\phi_n(x) \to \phi(x)$ almost everywhere in A . By lemma 1, we have $a \geq b$, and since in any case $\limsup \geq \liminf$, $\phi \in P(A,\beta)$ and $\int \phi = \lim_n \int \phi_n$.

Now, $f = \phi + g$, so that by theorem 1, we see that $f \in P(A,\beta)$ and $\int f = \lim_n \int f_n$, completing the proof.

§4. SOME GENERAL INTEGRALS AS PARTICULAR CASES.

We have mentioned in the introduction of this chapter that the integrals of Perron type defined by Romanovski [29] and by Pfeffer [26] can be obtained from our general theory. We now consider this point.

4.1 ROMANOVSKI'S INTEGRALS.

In [29], Romanovski defines an abstract space, which is called Romanovski space by Solomon in [37]. These spaces of Romanovski contain, as special cases, the Euclidean spaces of any dimension. We now give the definition of a Romanovski space and show how the P - and R - integrals, defined by Romanovski on this space, can be obtained from our general theory.

A triple (X,σ,μ) is a Romanovski space if X is a second countable, locally compact metric space, μ a non-negative countably additive set function, finite on Borel sets with compact closure in X and positive on non-empty open sets, σ a distinguished family of subsets of X satisfying ten axioms. For a precise description of these axioms, we refer to [29], [37].

Let $\mathcal{N}$ = the family of all subsets of X with zero μ-measure, $\mathcal{B}(A) = \{\sigma_A\}$ for each $A \in \sigma$. Then it is easy to see that $\mathcal{B}$ is a base mapping according to the definition in section 1.

Let F be a function defined on σ_A. Define

$$\underline{D}F(x) = \liminf_{\substack{A' \in \sigma_A \\ x \in \overline{A'}}} \frac{F(A')}{\mu(A')} ,$$

where $\overline{B}$ denote the closure of B in X. We say that F satisfies $\overline{I}$ on A if $\underline{D}F(x) > -\infty$ except perhaps for points of countably many boundaries of sets in σ. We define F to be AC on A if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i |F(A_i)| < \varepsilon$ whenever $\sum_i \mu(A_i) < \delta$ for any finite and disjoint $\{A_i\} \subset \sigma_A$. For any subset E of A,

$$\text{let } F_E(A') = F(A') \quad \text{if } \overline{A'} \cap E \neq \phi ,$$
$$= 0 \quad \text{if } \overline{A'} \cap E = \phi .$$

F is said AC on E if F_E is AC on A. Then F is ACG on A if A is a countable union of sets on each of which F is AC.

$$\text{Let } F^-(A') = F(A') \quad \text{if } F(A') < 0 ,$$
$$= 0 \quad \text{if } F(A') \geq 0 ;$$

then F is said to be $\underline{\text{ACG}}$ on A if F^- is ACG on A.

It is obvious that $\overline{I}$ and $\underline{\text{ACG}}$ defined above are both inequality properties as defined in section 1.

F is said to be continuous from interior on A if for each $A' \subset A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $A'' \subset A'$ and $\mu(A' \sim A'') < \delta$ imply $|F(A') - F(A'')| < \varepsilon$.

Let $\overline{M}(A,\sigma_A) = \overline{M}(A) = \{F | F$ is additive on σ_A and is continuous from interior on $A\}$, and $P = (\overline{M},\underline{D},\mathcal{B},\mathcal{N},\overline{I})$, $R = (\overline{M},\underline{D},\mathcal{B},\mathcal{N},\underline{\text{ACG}})$. Then it follows easily from lemmas on page 92 and page 95 in [29] that both P and R are derivate systems on σ. The P-integral and R-integral are just those defined by Romanovski in [29]. By the theorem on page 77 [29], we see easily that both P- and R-integral have differential properties as given in thereom 11 in next section. Whether there is a result similar to thereom 12 for the P-integral is an open question. The proof of theorem 12 depends on the Zahorski function on the real line, so if such a function could be constructed on an arbitrary Romanovski space, the question could be settled.

4.2 THE PFEFFER INTEGRAL.

We recall Pfeffer's setting [26] and show how his integral of Perron type is obtained.

Let X be a topological space and $X^{\sim} = X \cup \{\infty\}$ be the one-point compactification of X. For $A \subset X$, $\bar{A}$ denote the closure of A in X; for $A \subset X^{\sim}$, $\tilde{A}$ denote the closure of A in $X^{\sim}$. For each $x \in X^{\sim}$, choose once and for all a local base Γ_x of neighborhoods of x in $X^{\sim}$ such that the cardinality of Γ_x is the smallest cardinality of local bases at x. Further, assume that for each $x \in X$, $U \subset X$ for each $U \in \Gamma_x$.

Let σ be a pre-algebra of subsets of X such that $\Gamma_x \subset \sigma$ for each $x \in X$. Also, assume that there is a fixed integer $p \geq 1$ with the property that for each $U \in \Gamma_\infty$ there are disjoint sets $U_{1,\infty}, U_{2,\infty}, \ldots U_{p,\infty}$ from σ for which $\bigcup_{i=1}^{p} U_{i,\infty} = U \cap X$. By λ we shall denote the system of all sets $A \in \sigma$ such that $A \subset \bigcup_{i=1}^{n} U_i$, where $U_i \in \bigcup\{\Gamma_x | x \in X\}$ for $i = 1,2,3,\ldots,n$.

Let G be a function defined on σ, non-negative and additive on σ, and finite on λ.

For each $x \varepsilon X^{\sim}$, a certain family K_x of nets $\{Bu | u \varepsilon \Gamma, \subset\} \subset \sigma$ is associated, where Γ is a cofinal subset of Γ_x . This mapping is assumed to satisfy six axioms, see [26].

For a function defined on σ_A , and for $x \varepsilon X^{\sim}$, let $\#F(x,A) = \inf \{\liminf_{\alpha} F(B_\alpha) | \{B_\alpha\} \varepsilon K_A(\sigma_A)\}$, where $K_x(\sigma_A) = \{\{B_\alpha\} \varepsilon K_x | \{B_\alpha\} \subset \sigma_A\}$, and $*F(x,A) = \#(F/G)(x,A)$. If A is fixed, we write $\#F(x) = \#F(x,A)$, $*F(x) = *F(x,A)$.

Let $\overline{M}(A,\sigma_A) = \overline{M}(A) = \{F | F$ is superadditive on σ_A and there exists a countable set Z_F such that $\#F(x,A) \geq 0$ for each $x \varepsilon Z_F \cup \{\infty\}$ and $\#(-G)(x,A) \geq 0$ for each $x \varepsilon Z_F\}$.

Let N = the family of all countable sets in X , and $\mathcal{B}(A) = \{\sigma_A\}$. F is said to satisfy the inequality property $\overline{I}$ on A if $*F(x) > -\infty$ for all point in $\overline{A} \sim Z_F$. Let $P = (\overline{M}, *, \mathcal{B}, N, \overline{I})$. Then it follows from lemma 5.9 in [26] that P is a derivate system on σ . The P-integral is just that defined by Pfeffer in [26].

Whether this P-integral has a differential property requires further investigation.

§5. FURTHER PROPERTIES OF THE INTEGRAL ON THE REAL LINE.

Before studying some special cases, we are going to obtain a differential property of the integral and a characterization of integrability and also a very general integration by parts formula for an abstract derivate system on the real line. A different proof of the convergence theorem 10 is also given.

Throughout this section, let X be the real line, σ the family of all bounded half-open intervals like $[a,b[$, N the family of all subsets of Lebesgue measure zero. For each function defined on σ_A and for each $x \in \overline{A}$, let $\underline{D}(F,x) = \liminf\limits_{\substack{x \in [a,b] \\ [a,b[\in \sigma_A}} \frac{F([a,b[)}{b-a}$, the ordinary lower derivate of F at x .

Let $P = \{\overline{M}, \underline{\mathcal{D}}, \mathcal{B}, N, \overline{I}\}$ be a derivate system on σ satisfying $(\underline{\mathcal{D}}\overline{M}3)$ and the following additional axioms.

$(\underline{\mathcal{D}}5)$ Each $\underline{\mathcal{D}}M$ is Lebesgue measurable.

$(\underline{\mathcal{D}}6)$ $\underline{\mathcal{D}}(M,x) \geq \underline{D}(M,x)$.

We also assume that for each $\beta \in \mathcal{B}([a,b[)$, the set $B = \bigcup\limits_{A \in \beta} \dot{A}$ is of measure $b-a$, where $\dot{A}$ denote the end points of the interval A . Also, we assume that each $M \in \overline{M}([a,b[,\beta)$ is additive on β . Then the interval function M on β is in one-to-one correspondence to the point function M^* on B as follows

$$M^*(x) = M([a,x[) \quad \text{for each } x \in B \sim \{a\} ,$$

$$\qquad = 0 \qquad \text{for } x = a ,$$

$$M([x,y[) = M^*(y) - M^*(x) \quad \text{for each } [x,y[\in \beta .$$

Should no ambiguities arise, we will not distinguish the interval functions M on β and the point functions M^* on B. Furthermore, we may call B a base in $[a,b]$. Note that by the remark at the beginning of section 3 $(\overline{M}5')$ is also satisfied.

THEOREM 11. Suppose that $f \in P(A,\beta)$ with primitive F. Then $\mathcal{D}F(x)$ exists and is finite almost everywhere in A.

Proof. Let k, ε be arbitrary given positive numbers. By lemma 2, there exist $M \in \overline{M}_f(A,\beta)$, $m \in \underline{M}_f(A,\beta)$ such that $M(A) - m(A) < k\varepsilon$, and also $M(A) - F(A) < k\varepsilon$. Let E_o be the set of points x for which at least one of the following inequalities $\underline{\mathcal{D}}M(x) \geq f(x)$, $\overline{\mathcal{D}}m(x) \leq f(x)$ fails to hold. Then E_o is of measure zero. Observe that by theorem 6, $F \in \overline{M}(A,\beta)$ so that $M - F \in \overline{M}(A,\beta)$. Hence by $(\underline{\mathcal{D}}5)$, $\underline{\mathcal{D}}(M-F)$ is measurable, so that the set E_k of points x in A on which $\underline{\mathcal{D}}(M-F,x) \geq k$ is measurable. We prove that $\mu(E_k) < \varepsilon$ as follows, where μ is the Lebesgue measure on the real line.

Let $R(A') = M(A') - F(A')$ for each $A' \in \beta$ and $R_1(A') = R(A')$ for $A' \in \beta$, $R_1(A') = \sup\limits_{A'' \varepsilon \sigma_{A'}} R(A'')$ for $A' \in \sigma_A \sim \beta$. Then

$R_1 \in \overline{M}(A,\beta)$ by $(\overline{M}4)$, and R_1 is non-negative on σ_A. Therefore, $D(R_1,x)$, and hence $\mathcal{D}(R_1,x)$ by $(\underline{\mathcal{D}}6)$, exists and is finite almost everywhere in A, and $(L) - \int_A \mathcal{D}(R_1,x)dx = (L) - \int_A D(R_1,x)dx \leq R_1(A) =$

$R(A) = M(A) - F(A) < k\varepsilon$, where (L) denote that the integral concerned is the Lebesgue integral. But

$$(L)-\int_A \mathcal{D}(R_1,x) = (L)-\int_A \underline{\mathcal{D}}(R_1,x)dx \geq (L)-\int_{E_k} \underline{\mathcal{D}}(R_1,x)dx \geq k\mu\ (E_k) \quad ,$$

so that $\varepsilon > \mu(E_k)$, which is what we want to prove.

Now, for $x \notin E_k \cup E_o$, $\underline{\mathcal{D}}(F,x) \geq \underline{\mathcal{D}}(M,x) - \overline{\mathcal{D}}(R,x) \geq f(x) - k$. As k and ε are arbitrary, it follows that $\underline{\mathcal{D}}(F,x) \geq f(x)$ almost everywhere in A .

In a like manner, using minor functions, we can prove that $\overline{\mathcal{D}}F(x) \leq f(x)$ almost everywhere in A . Then it follows that $\mathcal{D}F(x)$ exists and $\mathcal{D}F(x) = f(x)$ almost everywhere in A , completing the proof.

COROLLARY 1. If $f \varepsilon P(A,\beta)$, then f is measurable in A .

COROLLARY 2. If $f \varepsilon P(A,\beta)$, $M \varepsilon \overline{M}_f(A,\beta)$, $m \varepsilon \underline{M}_f(A,\beta)$, then $\mathcal{D}M(x)$ and $\mathcal{D}m(x)$ exist and are finite almost everywhere in A .

Suppose that the above derivate system P satisfies in addition the following two axioms. Then we can obtain a characterization of integrability similar to that of McGregor in [22].

($\overline{M}6$) Each function M continuous in $[a,b]$ belongs to $\overline{M}([a,b[,\beta)$ in the sense that the function $M^*([x,y[) = M(y) - M(x)$ for $[x,y[\,\varepsilon\,\sigma_{[a,b[}$ belongs to $\overline{M}([a,b[,\beta)$.

($\overline{I}4$) Let $\mathcal{C} \subset \mathcal{N}$ be closed under finite set unions. A function F satisfies the inequality property $\overline{I}$ if and only if $\underline{\mathcal{D}}F(x) > -\infty$ except perhaps for points of a set in $\mathcal{C}$.

For convenience, we say that a property P(x) is true nearly everywhere (n.e.) in A if P(x) is true for all x in A except at most for points of a set in $\mathcal{C}$. Note that the property $\overline{I}$ defined in ($\overline{I}4$) is a inequality property, but not every inequality property can be defined in this way.

THEOREM 12. Let f be a function finite almost everywhere in $A = [a,b[$. Then $f \in P(A,\beta)$ if and only if for each $\varepsilon > 0$, there exist functions T, t such that

(i) $T \in \overline{M}(A,\beta)$, $t \in \underline{M}(A,\beta)$;

(ii) $\mathcal{D}T(x)$, $\mathcal{D}t(x)$ exist n.e. in A and are finite a.e. in A ;

(iii) $+\infty \neq \mathcal{D}t(x) \leq f(x) \leq \mathcal{D}T(x) \neq -\infty$ n.e. in A ;

(iv) $T(A) - t(A) < \varepsilon$.

Proof. It is clear that the conditions are sufficient. To see that the conditions are necessary, let $f \in P(A,\beta)$. Then for each $\varepsilon > 0$, take $M \in \overline{M}(A,\beta)$, $m \in \underline{M}(A,\beta)$ with $M(A) - m(A) < \varepsilon/2$, which is possible by lemma 2. By corollary 2 to theorem 11, $\mathcal{D}M(x)$, and $\mathcal{D}m(x)$ exist and are finite a.e. in A . Let E be the subset of A where at least one of M , m fails to have a finite $\mathcal{D}$-derivative. The set E is of measure zero, so that there is a set E_1 of measure zero and of type G_δ such that $E \subset E_1 \subset A$.

Let ω be a point function defined on $[a,b]$ with the following properties:

(1) ω is AC on $[a,b]$;

(2) ω is differentiable in the ordinary sense;

(3) $\omega'(x) = +\infty$ for $x \varepsilon E_1$;

(4) $0 \leq \omega'(x) < +\infty$ for $x \varepsilon [a,b] \sim E_1$;

(5) $\omega(a) = 0$ and $\omega(b) < \varepsilon/4$.

That such a function exists is well-known; see Zahorski [40] or McGregor [22]. As ω is continuous in $[a,b]$ the corresponding interval function on σ_A , also denoted by ω , belongs to $\overline{M}(A,\beta)$ by $(\overline{M}6)$. Let $T = M + \omega$, $t = m - \omega$. Then $T \varepsilon \overline{M}(A,\beta)$ and $t \varepsilon \underline{M}(A,\beta)$.

Let C be the set of points x on which $\underline{\mathcal{D}}M(x) > -\infty$ fails to hold. For $x \varepsilon E_1 \sim C$. $\underline{\mathcal{D}}T(x) \geq \underline{\mathcal{D}}M(x) + D\omega(x) \geq +\infty$, so that $\mathcal{D}T(x) = +\infty \geq f(x)$. For $x \varepsilon A \sim [E_1 \cup C]$, $\mathcal{D}M(x)$ exists and is finite, so that $\mathcal{D}T(x)$ exists and is finite, and $\mathcal{D}T(x) = \mathcal{D}M(x) + D\omega(x) \geq \mathcal{D}M(x) \geq f(x)$.

Similarly, $\mathcal{D}t(x)$ exists n.e. in A and $+\infty \neq \mathcal{D}(t,x) \leq f(x)$ a.e. in A . Furthermore, $T(A) - t(A) = M(A) + \omega(A) - m(A) + \omega(A) < \varepsilon$. Thus, T,t satisfy all the required conditions, and hence the proof is completed.

COROLLARY. Let $P_1 = \{\overline{M},\underline{\mathcal{D}}_1,\mathcal{B},\mathcal{N},\overline{I}\}$ be another such derivate system on σ with $\mathcal{D}_1M(x) = \mathcal{D}M(x)$ n.e. in A whenever one of $\mathcal{D}_1M(x)$, $\mathcal{D}M(x)$ exists n.e. in A . Then $P(A,\beta) = P_1(A,\beta)$ and two integrals of the same function are equal.

By theorem 12, we see that

(A) we can use the $\mathcal{D}$-derivatives instead of $\mathcal{D}$-derivates in the definition of major functions and minor functions;

(B) the "almost everywhere" in $(\overline{M}2)$ and $(\underline{m}2)$ can be replaced by "nearly everywhere".

Statement (B) is well-known for most of the particular integrals of Perron type while statement (A) is due to McGregor [22] for the classical Perron integral. The proof here is essential that of [22]. For a similar result for the P^n-integral, see Bullen [3]. We will use theorem 12 and its corollary to prove the equivalence of the SCP-integral and the MZ-integral in chapter III.

If the derivate system does not satisfy some extra conditions, one can not get any reasonable integration by parts formula; but with some reasonable mild restrictions, which are unfortunately hard to check in particular cases, we obtain the following theorem.

THEOREM 13. Let $f \in P([a,b[,\beta)$ and U be a bounded non-negative point function on $[a,b]$, such that $U'(x)$ exists and is non-negative a.e. in $[a,b]$, and such that the following inequalities make sense for each $M \in \overline{\mathcal{M}}_f([a,b[,\beta)$, $m \in \underline{\mathcal{M}}_f([a,b[,\beta)$.

$$\underline{\mathcal{D}}(MU)(x) \geq M(x)U'(x) + U(x)\underline{\mathcal{D}}M(x) \qquad \text{a.e. in } [a,b] ,$$

$$\overline{\mathcal{D}}(mU)(x) \leq m(x)U'(x) + U(x)\overline{\mathcal{D}}m(x) \qquad \text{a.e. in } [a,b] ,$$

$$\underline{\mathcal{D}}(MU)(x) > -\infty \qquad \text{n.e. in } [a,b] ,$$

$$\overline{\mathcal{D}}(mU)(x) < +\infty \qquad \text{n.e. in } [a,b] .$$

Then if F is the primitive of f , $fU + FU' \varepsilon P([a,b[,\beta)$, and

$$\int_{[a,b[} (fU + FU') = F([a,b[)U([a,b[) \quad .$$

If, in addition, $FU' \varepsilon P([a,b[,\beta)$ then so is fU and

$$\int_{[a,b[} fU = F([a,b[)U([a,b[) - \int_{[a,b[} FU' \quad .$$

Proof. Under the hypotheses, one can easily see that if $M \varepsilon \overline{M}_f$, then $MU \varepsilon \overline{M}_{fU+FU'}$, and if $m \varepsilon \underline{M}_f$ then $mU \varepsilon \underline{M}_{fU+FU'}$. Also, $MU([a,b[) - mU([a,b[)$ is small when $M([a,b[) - m([a,b[)$ is small. Hence the required result follows easily.

Now, we are in a position to give another proof of theorem 10 for the derivate system P on σ satisfying the additional axioms $(\mathcal{D}\overline{M}3)$, $(\underline{\mathcal{D}}5)$, $(\underline{\mathcal{D}}6)$ and $(\overline{M}6)$, and also that a function F satisfies $\overline{I}$ whenever $\underline{D}F(x) > -\infty$ except perhaps for a countable set of points.

LEMMA 5. If a function is L-integrable, it is P-integrable and two integrals are equal.

Proof. By theorem 7, that if a function is Perron integrable (see section II.1), it is P-integrable and two integrals are equal. It is well-known that if a function is Lebesgue integrable, it is Perron integrable and two integrals are equal. The conclusion then follows.

LEMMA 6. Let $f(x) \geq 0$ almost everywhere in $[a,b]$. Then f is L-integrable on $[a,b]$ if and only if f is P-integrable on $[a,b[$ with a base $B \varepsilon \mathcal{B}([a,b[)$.

Proof. The one implication is given by lemma 5. To prove the other implication, let f be P-integrable on $[a,b[$ with a base B. Since $|f| = f$ almost everywhere in $[a,b]$, it follows that $|f|$ is also P-integrable on $[a,b]$ with base B. Clearly, the zero function $0 \in \underline{M}_{|f|}([a,b[,B)$. Let $M \in \overline{M}_{|f|}([a,b[,B)$. Then by lemma 1, $M(= M - 0)$ is monotone increasing in B. Define $M_1(x) = M(x)$ for $x \in B$, $M_1(x) = \sup\limits_{t \in B \cap [a,x]} M(t)$ for $x \in [a,b] \sim B$. Then M_1 is monotone increasing in $[a,b]$, so that $M_1'(x)$ is L-integrable on $[a,b]$ and hence so is $\underline{D}M$ since $M_1'(x) = \underline{D}M(x)$ almost everywhere in $[a,b]$ by $(\underline{D}\overline{M}1)$ and $(\underline{D}6)$. As f is measurable by corollary 1 to theorem 11, it follows that f is L-integrable on $[a,b]$ since $|f(x)| \leq \underline{D}M(x)$ almost everywhere in $[a,b]$. The proof is hence completed.

COROLLARY. Let f_1 be P-integrable on $[a,b[$ with base B, and f_2 be L-integrable on $[a,b]$ and $f_1 \geq f_2$ almost everywhere in $[a,b]$. Then f_1 is also L-integrable on $[a,b]$.

THEOREM 10'. Let $g,h,f_n (n = 1,2,3,\ldots)$ be P-integrable on $[a,b]$ with a base B, and $g(x) \leq f_n(x) \leq h(x)$ almost everywhere in $[a,b]$ for each n, and $\lim\limits_n f_n(x) = f(x)$ almost everywhere in $[a,b]$. Then f is P-integrable on $[a,b]$ with base B and $\int f = \lim\limits_n \int f_n$.

Proof. Since $0 \leq f_n(x) - g(x) \leq h(x) - g(x)$ almost everywhere in $[a,b]$, both $f_n - g$ and $h - g$ are L-integrable on $[a,b]$ by lemma 6. By Lebesgue dominated convergence theorem, we have $\lim_n (L)-\int(f_n-g) = (L)-\int(f-g)$. Hence, by lemma 5, $\lim_n (P)-\int(f_n-g) = (P)-\int(f-g)$. Now, g is P-integrable, so that $f = (f-g) + g$ is also P-integrable and $\lim_n (P)-\int f_n = (P)-\int f$, completing the proof.

We close this section by remarking that Kubota's abstract integral of Perron type [17], is a particular case of the integral in this section. In fact, taking $\mathcal{B}(A) = \{\sigma_A\}$, $\mathcal{N} = \{\phi\}$ and the inequality property $\overline{\text{I}}$ to mean $\underline{\mathcal{D}}F(x) > -\infty$, one gets Kubota's setting and his integral if axiom $(\underline{\mathcal{D}}4)$ is replaced by the equivalent axiom:

$(\underline{\mathcal{D}}4')$ $\underline{\mathcal{D}}(v_1+v_2)(x) = Dv_1(x) + \underline{\mathcal{D}}v_2(x)$ whenever the ordinary derivative $Dv_1(x)$ exists.

Axioms $(\underline{\mathcal{D}}4)$ and $(\underline{\mathcal{D}}4')$ are equivalent in the sense that one follows from the other by axioms $(\underline{\mathcal{D}}2)$, $(\underline{\mathcal{D}}3)$, $(\underline{\mathcal{D}}6)$ and the corresponding properties $(\overline{\mathcal{D}}2)$, $(\overline{\mathcal{D}}3)$ and $(\overline{\mathcal{D}}6)$. Incidently, note that Kubota did not assume axiom $(\underline{\mathcal{D}}5)$ explicitly. However, in proving a result corresponding to our lemma 6, he did use implicitly (see the second last sentence in his proof of theorem 3.8 [17]) our corollary 1 to theorem 11, and axiom $(\underline{\mathcal{D}}5)$ is essential in the proof of this corollary.

CHAPTER II. THE C_nP-INTEGRAL

The C_nP-integral was first defined by Burkill in [5], [6]. Since then, many authors have shown an interest in this integral; see for instance Bosanquet [1], James [13], Kubota [18], Sargent [31] - [34], and Skvorcov [35], [36]. We will show how to obtain the C_nP-integral from our general theory, and also state an integration by parts formula, which will be used extensively in Chapter III.

The theory of C_nP-integral based on theorem 2(below). There is a defect in Burkill's original proof in [6] (see line 9, page 546). This defect was noted recently and independently by Verblunsky in [38]. We give a new and correct proof of this theorem. For different proofs of stronger results, we refer to Sargent [31] and Verblunsky[39].

Sargent has defined a C_nD-integral [32] equivalent to the C_nP-integral. However, there is a defect in her proof for theorem 4 (below). This has also been given a correct proof recently and independently by Verblunsky in [38]. We supply another proof, which seems simpler and more direct, in the sense that we do not appeal to the deep de la Vallee Poussin decomposition theorem used by Verblunsky.

Throughout this chapter, as in section I.5, X is to be the real line, σ the family of all bounded half-open intervals like $[a,b[$, N the family of all subsets of Lebesgue measure zero. For each $A \varepsilon \sigma$, let $\mathcal{B}(A) = \{\sigma_A\}$. It is easy to see that $\mathcal{B}$ is a base mapping. Legitimate mappings and derivate operators will be defined later. Once a derivate operator $\underline{\mathcal{D}}$ is defined, a function F will be said to satisfy the property $\overline{I}$ if and only if $\underline{\mathcal{D}}F(x) > -\infty$ except perhaps for a countable set of points. Note that for all the derivate operators used, the property $\overline{I}$ defined above is an inequality property.

As we have noted in section I.5, corresponding point functions and the additive interval functions will not be distinguished if this causes no ambiguities.

§1 THE CLASSICAL PERRON INTEGRAL.

The classical Perron integral is the C_o-P-integral of the next section. We single it out in this separate section because by doing so, we can make the induction arguments in next section clearer.

For each $A = [a,b[$, let $\overline{\mathcal{M}^o}(A) = \{M | M$ is additive on σ_A and the corresponding point function is continuous in $[a,b]\}$. Let $\underline{C_o D}M(x)$ or $\underline{D}M(x)$ be the ordinary lower drivate of M at x . Then it is easy to show that $P_o = (\overline{\mathcal{M}^o}, \underline{D}, \mathcal{B}, \mathcal{N}, \overline{I})$ is a derivate system satisfying all the additional axioms $(\underline{\mathcal{D}}\overline{\mathcal{M}}3)$, $(\underline{\mathcal{D}}5)$, $(\underline{\mathcal{D}}6)$, $(\overline{\mathcal{M}}6)$, $(\overline{I}4)$ in section I.5, and also $(\overline{\mathcal{M}}5')$ in section I.3. The P_o-integral is just the classical Perron integral; see [23], [30].

From theorem I.6, for P_o-integral, we see that the P_o-primitives are continuous. Moreover, it is well-known that a P_o-primitive is ACG^* (see Saks [30]), and conversely an ACG^* function is a P_o-primitive of its D-derivative. It is also well-known that in the definition of major functions, "$\underline{D}M(x) > -\infty$ n.e." can be replaced by "$\underline{D}M(x) > -\infty$ everywhere" without affecting the generality of the resulting integral. An integration by parts formula for the P_o-integral reads as follows.

THEOREM 1. Let $f \in P_o([a,b[)$ and g be of bounded variation on $[a,b]$. Then $fg \in P_o([a,b[)$ and

$$\int_A fg = F(A)g(A) - \int_A F(t)dg(t) \quad ,$$

where $A = [a,b[$, and F is the P_o-primitive of f and the integral in the right hand side is the Stieltjes integral.

This theorem will be used later. For the proof, we refer to Saks [30], McShane [23[, or Gordon and Lasher [10], who provided a more direct proof from the definition of P_o-integrals.

The following notions will be used and extended later. For an additive interval function F on $\sigma_{[a,b[}$, suppose that the corresponding point function F is P_o-integrable in a neighborhood of $x \in [a,b]$. For $h \neq 0$, $x + h$ in the neighborhood, write $C_1(F;x,x+h) = \frac{1}{h} \int_{(x,h)} F$, where $(x,h) = [x,x+h[$ if $h > 0$, $= [x+h, x[$ if $h < 0$. Then F

is said to be C_1-continuous at x if $\lim_{h\to 0} C_1(F;x,x+h) = F(x)$; and SC_1-continuous at $x \varepsilon\ ]a,b[$ if $\lim_{h\to 0+} \{C_1(F;x,x+h) - C_1(F;x,x-h)\} = 0$, and SC_1-continuous at a or b if it is C_1-continuous there.

We end this section by remarking that $SP_o = (\overline{M}, \underline{SD}, \mathcal{B}, \mathcal{N}, \overline{I})$ is also a derivate system on σ , where $\underline{SD}M(x) = \liminf_{h\to 0+} \frac{M(x+h)-M(x-h)}{2h}$, the symmetric lower derivate of M at x . This can be checked easily noting the recent result due to Mukhopadhyay [21],

PROPOSITION. If $\underline{SD}M(x) \geq 0$ a.e. in $[a,b]$ and $\underline{SD}M(x) > -\infty$ n.e. in $[a,b]$, then M is monotone increasing in $[a,b]$, where M is a continuous function on $[a,b]$.

The SP_o-integral is more general than the P_o-integral, and might be more suitable for application to the trigonometric series (cf. [7] or section III.6). We may consider this SP_o-integral as the first of the SCP-scale of integrals defined below in chapter III.

§2. THE C_nP-INTEGRAL.

We define a scale of derivate systems on σ by induction as follows. For each $[a,b[$, let $\overline{M}^1([a,b[;\sigma_{[a,b[}) = \overline{M}^1([a,b[) = \{M | M$ is C_1-continuous in $[a,b]\}$, and for each $M \varepsilon \overline{M}^1([a,b[)$ and for each $x \varepsilon [a,b]$, let $\underline{C_1D}M(x) = \liminf_{h\to 0} \frac{C_1(M,x,x+h) - M(x)}{h/2}$. That the $P_1 = (\overline{M^1}, \underline{C_1}D, \mathcal{B}, \mathcal{N}, \overline{I})$ is a derivate system on σ follows easily from lemma on page 316 and lemma on page 319 [5].

Suppose that for $n \geq 2$, the derivate system $P_{n-1} = (\overline{M^{n-1}}, \underline{C_{n-1}}D, \mathcal{B}, \mathcal{N}, \overline{I})$ has been defined. For each $M \varepsilon P_{n-1}([a,b[)$ and for each $x \varepsilon [a,b]$, $h \neq 0$ with $x + h \varepsilon [a,b]$, let

$$C_n(M;x,x+h) = \frac{n}{h^n} (P_{n-1}) - \int_x^{x+h} (x+h-t)^{n-1} M(t)dt \quad .$$

Then M is said to be C_n-continuous at x if $\lim_{h\to 0} C_n(M;x,x+h) = M(x)$.
Let $\overline{M^n}([a,b[) = \{M | M$ is C_n-continuous in $[a,b]\}$, and for each M

define
$$\underline{C_n}DM(x) = \liminf_{h\to 0} \frac{C_n(M;x,x+h) - M(x)}{h/n+1} \quad .$$

Then it can be shown that $P_n = \{\overline{M^n}, \underline{C_n}D, \mathcal{B}, \mathcal{N}, \overline{I}\}$ is a derivate system on σ. The P_n-integral is in fact equivalent (see Bosanquet [1]) to the C_nP-integral of Burkill in [5], [6].

That the P_n defined above is in fact a derivate system is easy to check. We only prove the following theorem, of which the significance has been mentioned in the introduction.

THEOREM 2. Let M be C_n-continuous in $[a,b]$ and $\underline{C_n}DM(x) \geq 0$ a.e. in $[a,b]$ and $\underline{C_n}\,DM(x) > -\infty$ n.e. in $[a,b]$. Then M as a point function is monotone increasing in $[a,b]$.

To prove this, we recall some notions introduced by Sargent in [32]. For $n \geq 1$, a function F is said to be $\underline{AC_n}^*$ on a set E if it is $C_{n-1}P$-integrable on an interval containing E, and if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_r \inf_{a_r<x<b_r} \{C_n(F;a_r,x) - F(a_r)\} > -\varepsilon$$

$$\sum_r \inf_{a_r<x<b_r} \{F(b_r) - C_n(F;b_r,x)\} > -\varepsilon$$

for all finite sets of non-overlapping intervals $\{[a_r,b_r]\}$ with end points in E and such that $\sum_r (b_r-a_r) \leq \delta$. The concept $\overline{AC}_n^*$ is defined in a similar way. If F is both $\underline{AC}_n^*$ and $\overline{AC}_n^*$, then F is said to be AC_n^*. Applying the method in the proof of theorem I in [32], lemma IV in [32] reads as follows.

LEMMA 1. Let F be C_n-continuous in $[a,b]$, and $\underline{C_nD}F(x) > -\infty$ n.e. in $[a,b]$. Then $[a,b]$ is the union of a countable closed sets over each of which F is $\underline{AC}_n^*$.

Generalizing the concept of AC functions (see Saks [30]), we say that a function F is $\underline{AC}$ on a set E if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_r \{F(b_r) - F(a_r)\} > -\varepsilon$

for all finite sets of non-overlapping intervals $\{[a_r,b_r]\}$ with end points in E and such that $\sum(b_r-a_r) \leq \delta$. $\overline{AC}$ is defined in an obvious way. These notions $\underline{AC}$ and $\overline{AC}$ were first introduced by J. Ridder in [28].

Following parts of the argument used in the proof of lemma III in [32], we have

LEMMA 2. Let F be $C_{n-1}P$-integrable on $[c,d]$, and

$$\underline{W} = \min\{\inf[C_n(F;c,x) - F(c)] , \inf[F(d) - C_n(F;d,x)]\} .$$

Then there exists a constant α independent of c,d such that

$$F(d) - F(c) \geq -\alpha\underline{W} .$$

Proof of THEOREM 2. By lemma 1 there exists a sequence $\{E_k\}$ of closed sets with union $[a,b]$ and on each of which M is $\underline{AC^*_n}$. By lemma 2, M is $\underline{AC}$ on each E_k , $k = 1,2,3,...$.

Let A be the set of points in $[a,b]$ such that if $x \in A$, then there is no interval containing x on which M is monotone increasing. Then A is closed and hence by the Baire category theorem, if A is not empty there is an interval $[\ell,m]$ and an integer k such that $A \cap]\ell,m[$ is not empty and $A \cap [\ell,m] = E_k \cap [\ell,m]$. As M is $\underline{AC}$

on E_k , M is $\underline{AC}$ on $A \cap [\ell,m]$. As M is monotone increasing on each of the intervals contiguous to $A \cap [\ell,m]$ w.r.t. $[\ell,m]$, by the C_n-continuity of M , it follows that M is $\underline{AC}$ on $[\ell,m]$.

Now, letting $\varepsilon > 0$ be given, we prove that for each $x \in [\ell,m]$ with $\underline{C_n D}M(x) \geq 0$, there exists a sequence of points x_i with $x_i \to x$ for which $\frac{M(x_i)-M(x)}{x_i-x} > -\varepsilon$. Suppose to the contrary that $\frac{M(t)-M(x)}{t-x} \leq -\varepsilon$ for all t with $t - x < \delta$ for any $\delta > 0$. Then for $0 < h < \delta$, one has

$$\frac{(n+1)n}{h^{n+1}} \int_x^{x+h} (x+h-t)^{n-1}(M(t) - M(x))dt$$

$$\leq \frac{(n+1)n}{h^{n+1}} \int_x^{x+h} (x+h-t)^{n-1}(-\varepsilon(t-x))dt = -\varepsilon ,$$

so that $\underline{C_n D}M(x) \leq -\varepsilon$, contradicting to $\underline{C_n D}M(x) \geq 0$,

Let G be the set of points in $[\ell,m]$ such that for $x \in G$, $\underline{C_n D}M(x) \geq 0$. Then the measure of G is $m-\ell$. By the above assertion, for each $x \in G$ we can take a sequence of intervals $]x,x_i[$ with $x_i - x \to 0$ for which $\frac{M(x_i) - M(x)}{x_i-x} > - \varepsilon$. This associates a Vitali family of intervals with each point in G . Hence by the Vitali covering theorem, there is a finite mutually exclusive set $\{]x_i,x_i'[\}$ of the family with $\sum(x_i' - x_i) > (m-\ell) - \eta$, η arbitrary, for which

$$\sum_i \{M(x_i') - M(x_i)\} > -\varepsilon \sum (x_i' - x_i) \quad .$$

Let $\{]t_j, t_j'[\}$ be the subintervals of $[\ell, m]$ complementary to the set $\{[x_i, x_i']\}$. Then $\sum_j (t_j' - t_j) < \eta$. Hence, as M is $\underline{AC}$ in $[\ell, m]$, one has, for sufficiently small η ,

$$M(m) - M(\ell) \geq \sum_i \{M(x_i') - M(x_i)\} + \sum_i \{M(t_j') - M(t_j)\}$$

$$\geq -\varepsilon \sum (x_i' - x_i) - \varepsilon \geq -\varepsilon\,[(m-\ell) - \eta - 1] \quad .$$

As ε is arbitrary, one concludes that $M(m) \geq M(\ell)$.

If $\ell \leq c < d \leq m$, it can be shown in the same way that $M(d) \geq M(c)$, so that M is monotone increasing in $[\ell, m]$. This is a contradiction since $A \cap]\ell, m[$ is not empty. Thus, we conclude that A is empty. Therefore, for each $x \in [a,b]$, there is an interval containing x such that M is monotone increasing in the interval. By Heine-Borel theorem, there is then a finite set of such intervals covering $[a,b]$ and it then easily follows that M is monotone increasing in $[a,b]$, completing the proof.

We remark that it is well-known that $\overline{M^n}(A) \subset \overline{M^{n+1}}(A)$ and $\underline{C_n D}M(x) \leq \underline{C_{n+1} D}M(x)$ for each $M \in \overline{M^n}(A)$, $x \in \overline{A}$, for each $n = 0, 1, 2, 3, \ldots$, where $\underline{C_o D} = \underline{D}$. Hence by theorem I.7, we have the consistency of the P_n-scale (or $C_n P$-scale) starting from the classical Perron integral (i.e. P_o-integral in §1).

The following integration by parts formula will be needed in the next chapter. For a proof, see Burkill [6].

THEOREM 3. For $a \le x \le b$, let $F(x) = (P_n) - \int_a^x f(t)dt$, and

$$G_n(x) = \int_a^x \int_a^{\xi_1} \int_a^{\xi_2} \cdots \int_a^{\xi_{n-1}} g(\xi_n) d\xi_n \, d\xi_{n-1} \cdots d\xi_2 \, d\xi_1 ,$$

where g is of bounded variation in $[a,b]$. Then $fG_n \in P_n([a,b[)$ and

$$(P_n) - \int_\alpha^\beta (fG_n)(t)dt = [FG_n]_\alpha^\beta - (P_{n-1}) - \int_\alpha^\beta (FG_{n-1})(t)dt ,$$

where $a \le \alpha < \beta \le b$.

§3. THE C_nD-INTEGRAL AND THE C_nP-INTEGRAL.

In [32], Sargent has defined the C_nD-integral by induction. The C_oD-integral is just the special Denjoy integral, which is equivalent to the P_o-integral in §1. For $n \ge 1$, assuming that $C_{n-1}D$-integral has been defined and is equivalent to the P_{n-1}-integral (i.e. $C_{n-1}P$-integral in Burkill's notation), the C_nD-integral is then defined as follows. A function f on $[a,b]$ is C_nD-integrable on $[a,b]$ if there is a function F C_n-continuous in $[a,b]$ and AC_nG^* on $[a,b]$ such that $C_nDF(x) = f(x)$ a.e. in $[a,b]$. That F is AC_nG^* on $[a,b]$ means that there is a sequence of sets with union $[a,b]$ such that

F is AC_n^* (see §2) on each of the sets.

Sargent proved that the P_n-integral is more general than the C_nD-integral; see theorem XI in [32]. In the proof of the converse, that the C_nD-integral is more general than (and hence of course equivalent to) the P_n-integral, theorem VIII [32], we noticed that there is a defect since the set E_n (defined there) depends on the choice of ε, so that the argument breaks down. The purpose of this section is to supply a correct proof.

THEOREM 4. Let f be P_n-integrable on $[a,b]$ with primitive F. Then F is AC_nG^* on $[a,b]$.

Proof. Given $\varepsilon_o > 0$, by lemma I.2., there exist a P_n-major function M_o and a P_n-minor function m_o such that $M_o(b) - m_o(b) < \varepsilon_o$, and also $M_o(b) - F(b) < \varepsilon_o$, $F(b) - m_o(b) < \varepsilon_o$.

By lemma 1, there exists a sequence $\{E_1^o\}$ of closed sets such that M_o is $\underline{AC}_n^*$ and m_o is $\overline{AC}_n^*$ on each E_k^o, $k = 1,2,3,\ldots$, where $\bigcup_k E_k^o = [a,b]$.

For fixed $k = 1,2,3,\ldots$, let $]c_r,d_r[$ be the contiguous intervals of E_k^o in $[a,b]$. As M_o is $\underline{AC}_n^*$ on E_k^o, we have

(1) $$\sum_r \inf_{c_r<x<d_r} \{C_n(M_o;c_r x) - M_o(c_r)\} > -\infty \quad ,$$

(2) $$\sum_r \inf_{c_r<x<d_r} \{M_o(d_r) - C_n(M_o;d_r,x)\} > -\infty \ ,$$

and similarly,

(3) $$\sum_r \sup \{C_n(m_o;c_r,x) - m_o(c_r)\} < +\infty \quad ,$$

(4) $$\sum_r \sup \{m_o(d_r) - C_n(m_o;d_r,x)\} < +\infty \quad ,$$

Suppose that $c_r < x < d_r$. Then

$$C_n(F;c_r,x) - F(c_r) = C_n(M_o;c_r,x) - M_o(c_r)$$
$$- \frac{n}{(x-c_r)^n} \int_{c_r}^{x} (x-t)^{n-1}\{M_o(t) - F(t)\}dt + M_o(c_r) - F(c_r)$$
$$\geq C_n(M_o;c_r,x) - M_o(c_r) - \{M_o(d_r) - F(d_r)\} + \{M_o(c_r) - F(c_r)\}$$

since $M-F$ is monotone increasing in $[a,b]$ by theorem I.5. It follows that

$$\sum_r \inf_{c_r<x<d_r} \{C_n(F;c_r,x) - F(c_r)\}$$
$$\geq \sum_r \inf \{C_n(M_o;c_r,x) - M_o(c_r)\} - \{M_o(b) - F(b)\} > -\infty$$

by (1) and the fact $M_o(b) - F(b) < \varepsilon_o$.

Similarly, using (2), (3), (4), we have

$$\sum_r \inf \{F(d_r) - C_n(F;d_r,x)\} > -\infty \ , \ \sum_r \sup \{C_n(F;c_r,x) - F(c_r)\} < +\infty \quad ,$$

and $\sum_r \sup \{F(d_r) - C_n(F;d_r,x)\} < +\infty$. Hence we have

(5) $$\sum_r \sup |C_n(F;c_r,x) - F(c_r)| < +\infty$$

(6) $$\sum_r \sup |F(d_r) - C_n(F;d_r,x)| < +\infty \quad .$$

Now, we show that F is AC on each E_k^o. First, note that by lemma 2, we have that M_o is $\underline{AC}$ and m_o is $\overline{AC}$ on E_k^o, so that there exists a constant A such that $\sum_i \{M_o(x_i') - M_o(x_i)\} > -A$ and $\sum_i \{m_o(x_i') - m_o(x_i)\} < A$ for any finite set $\{[x_i,x_i']\}$ of non-overlapping intervals with end points in E_k^o. For such finite set $\{[x_i,x_i']\}$ we have

$$0 \leq \sum\{M_o(x_i') - M_o(x_i)\} - \sum\{m_o(x_i') - m_o(x_i)\} \leq M_o(b) - m_o(b) < \varepsilon_o$$

since $M_o - m_o$ is monotone increasing and non-negative. Combining the above inequalities, we have for any relevant set $\{[x_i,x_i']\}$,

$$- A < \sum\{M_o(x_i') - M_o(x_i)\} < A + \varepsilon_o,$$

$$- A -\varepsilon_o < \sum\{m_o(x_i') - m_o(x_i)\} < A \quad ,$$ so that we have

(7) Both M_o and m_o are BV on E_k^o.

We prove further that

(8) if $M \in \overline{M_f^n}([a,b])$, $m \in \underline{M_f^n}([a,b])$, then both M and m are BV on E_k^o.

This in fact follows from $M = m_o + (M - m_o)$ and $m = M_o - (M_o - m)$ since M_o and m_o are BV on E_k^o by (7), and as $M_o - m$ and $M - m_o$ as both are monotone in $[a,b]$, they are also BV on E_k^o .

We have noticed that M_o is $\underline{AC}$ on E_k^o and m_o is $\overline{AC}$ on E_k^o . With the result (8), we prove further that

(9) if $M \varepsilon \overline{M}_f^n([a,b])$, $m \varepsilon \underline{M}_f^n([a,b])$, then M is $\underline{AC}$ and m is $\overline{AC}$ on E_k^o .

To see this, let $\{]c_r,d_r[\}$ be the intervals contiguous to E_k^o in $[a,b]$. Define $M_*(x) = M(x)$ for x on E_k^o , and on each $]c_r,d_r[$ M_* is defined such that the graph of M_* is the linear segment joining the points $(c_r,M(c_r))$ and $(d_r,M(d_r))$. Then it is easy to see that M_* is C_n-continuous in $[a,b]$ and $\underline{C_n}DM_*(x) > -\infty$ n.e. in $[a,b]$. Hence by lemma 1 and lemma 2. M_* is $(\underline{ACG})$ in $[a,b]$, that is $[a,b] = \bigcup_k E_k$, and M_* is $\underline{AC}$ on each E_k , where E_k is closed. Also, M_* is BV in $[a,b]$ since M is BV on E_k^o by (8). Let $G(x) = M_*(x) - (L)\int_a^x M_*'(t)dt$.

Then G as a difference of an $(\underline{ACG})$ function and an AC function is itself $(\underline{ACG})$. Furthermore, $G'(x) = 0$ a.e. in $[a,b]$. Hence, using the Baire's category theorem and the Vitali covering theorem, it can be shown that G is monotone increasing in $[a,b]$ and hence is also non-negative in $[a,b]$. Therefore, $M_*(x) - (L)\int_a^x M_*'(t)dt \geq 0$ for each x in $[a,b]$, and

$$M_*(x_i') - M_*(x_i) \geq \int_{x_i}^{x_i'} M_*'(t)dt \quad \text{for any } [x_i,x_i'] \subset [a,b] .$$

As $\int_a^x M_*'(t)dt$ is AC on $[a,b]$, it follows that M_* is $\underline{AC}$ on $[a,b]$.

As $M = M_*$ on E_k^o , it follows that M is $\underline{AC}$ on E_k^o . Similar arguments hold for P_n-minor functions, and (9) is hence proved.

Now, we are in a position to prove that

$$(10) \qquad F \text{ is AC on } E_k^o .$$

To do this, let $\varepsilon > 0$ be given. Choose $M \varepsilon \overline{M}_f$, $m \varepsilon \underline{M}_f$ with $M(b) - m(b) < \varepsilon/2$. Then for each finite set $\{[x_i,x_i']\}$ of nonoverlapping intervals in $[a,b]$, we have

$$0 \leq \sum\{M(x_i') - M(x_i)\} - \sum\{m(x_i') - m(x_i)\} \leq M(b) - m(b) < \varepsilon/2.$$

$$\sum \{m(x_i') - m(x_i)\} \leq \sum\{F(x_i') - F(x_i)\} \leq \sum\{M(x_i') - M(x_i)\}$$

since $M - m$, $M - F$, $F - m$ are all non-negative and monotone increasing in $[a,b]$. For such relevant $\{[x_i,x_i']\}$, if $x_i,x_i' \varepsilon E_k^o$, and if $\sum(x_i' - x_i)$ is sufficiently small, by (9), we have

$\sum\{M(x_i') - M(x_i)\} > -\varepsilon/2$ and $\sum\{m(x_i') - m(x_i)\} < \varepsilon/2$.

Combining all the above inequalities gives

$$-\varepsilon < \sum \{F(x_i') - F(x_i)\} < \varepsilon ,$$

so that F is AC on E_k^o , proving (10).

As the $C_{n-1}D$-integral and the P_{n-1}-integral are equivalent by induction hypotheses, it follows from (5), (6) and (10) that F is AC_n^* on E_k^o by theorem II in [32]. As k is arbitrary and $\cup E_k^o = [a,b]$, it follows that F is AC_nG^* on $[a,b]$, completing the proof of theorem 4.

We remark that the technique used in this chapter is motivated by studying the paper [16], where the C_1P-integral was investigaged in great detail.

CHAPTER III

A SCALE OF SYMMETRIC CP-INTEGRALS AND THE MZ-INTEGRAL

Burkill has defined a SCP-integral in [7], which is more suitable for application to the trigonometric series than the CP-integral. Although this SCP-integral has been investigated by many people, no scale corresponding to the CP-scale has appeared in the literature. One of our purposes in this chapter is to use the general theory developed in chapter I to give an SCP-scale of integrals.

As a preliminary, we prove some lemmas concerning the de la Vallée Poussin derivatives in section 1 and state two well-known theorems concerning n-convex functions in section 2. The results essential to the definition of our scale of integrals are proved in section 3. After developing the SC_nP-integral in section 4, section 5 is devoted to its connection to the James symmetric P^{n+1}-integral scale [13].

By the MZ-integral, we mean the integral defined by Marcinkiewicz and Zygmund in [21]. This MZ-integral solves the coefficient problem of the convergent trigonometric series. Burkill also used the SCP-integral to solve the same problem. However, in his proof, he used an integration by parts formula, which remains unproved up to now. We prove in the last section that the MZ-integral and the SCP-integral are in fact equivalent. This implies that the SCP-integral does solve the coefficient problem.

§1. THE SYMMETRIC de la VALLÉE POUSSIN DERIVATIVES.

Let F be a function defined on a bounded closed interval $[a,b]$, and $x \in]a,b[$. If there are constants $\beta_0, \beta_2, \ldots, \beta_{2r}, (r \geq 0)$, depending on x but not on h such that

$$\frac{1}{2}\{F(x+h) + F(x-h)\} - \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} = o(h^{2r})$$

as $h \to 0$, then β_{2r} is called the symmetric de la Vallée Poussin (s.d.l.V.P.) derivative of order $2r$ of F at x, and we write $\beta_{2r} = D_{2r}F(x)$. It is clear that if $D_{2r}F(x)$ exists, so does $D_{2k}F(x)$ for $k = 0,1,2,\ldots,r-1$, and $D_{2k}F(x) = \beta_{2k}$.

If $D_{2k}F(x)$ exists for $0 \leq k \leq m-1$, $(m \geq 1)$, define $\theta_{2m}(x,h) = \theta_{2m}(F;x,h)$ by

$$\frac{h^{2m}}{(2m)!}\theta_{2m}(x,h) = \frac{1}{2}\{F(x+h) + F(x-h)\} - \sum_{k=0}^{m-1} \frac{h^{2k}}{(2k)!} D_{2k}F(x) \quad , \tag{2}$$

and let

$$\begin{aligned} \overline{D}_{2m}F(x) &= \limsup_{h\to 0} \theta_{2m}(x,h) \quad , \\ \underline{D}_{2m}F(x) &= \liminf_{h\to 0} \theta_{2m}(x,h) \quad . \end{aligned} \tag{3}$$

Then a finite common value for $\overline{D}_{2m}F(x)$ and $\underline{D}_{2m}F(x)$ implies that $D_{2m}F(x)$ exists and equals this common value.

In a similar way, the odd-ordered s.d.l.V.P. derivative is defined by replacing (1) by

$$(1') \qquad \tfrac{1}{2}\{F(x+h) - F(x-h)\} - \sum_{k=1}^{r} \beta_{2k+1} \frac{h^{2k+1}}{(2k+1)!} = o(h^{2r+1})$$

as $h \to 0$. Similar changes can be made in (2), (3).

The following lemma is an extension and generalization of lemma 4, (i) in [33]. For a partial converse in the non-symmetric case, see lemma 10 in [21].

LEMMA 1. Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_o . If for some n , $D_nG(x_o)$ exists, then $D_{n+1}H(x_o)$ exists and is equal to $D_nG(x_o)$.

Proof. The proof is by induction on n . To see that it is true for $n = 1$, consider for sufficiently small $h > 0$,

$$\Theta_2(H;x_o,h) = \frac{2!}{h^2}\{\tfrac{1}{2}[H(x_o+h) + H_o(x-h)] - H(x_o)\} .$$

In order to apply l'Hôpital's rule, let

$$f(h) = \tfrac{1}{2}[H(x_o+h) + H(x_o-h)] - H(x_o) , \ g(h) = \frac{h^2}{2!} .$$

Then $f(h) \to 0$ as $h \to 0$ since H is clearly continuous in a neighborhood of x_o . Also, $g(h) \to 0$ as $h \to 0$. Furthermore, $g'(h) = h \neq 0$, and $\frac{f'(h)}{g'(h)} = \frac{H'(x_o+h) - H'(x_o-h)}{2h} = \frac{G(x_o+h) - G(x_o-h)}{2h}$,

which approaches to $D_1G(x_o)$ as $h \to 0$. Hence

$\lim \Theta_2(H;x_o,h) = D_1G(x_o)$, which is what we want to prove.

Now, suppose that the conclusion of the lemma is true for $n < r$, where $r \geq 2$. Then we prove that it is also true for $n = r$ as follows. Suppose r is even, $r = 2m$ say. As $D_{2m}G(x_o)$ exists, so does $D_{2k}G(x_o)$ for $0 \leq k \leq m - 1$, and hence by the induction hypothese, $D_{2k+1}H(x_o)$ exists and equals $D_{2k}G(x_o)$ for $0 \leq k \leq m - 1$. Consider

$$\Theta_{2m+1}(H;x_o,h) = \frac{(2m+1)!}{h^{2m+1}} \{\tfrac{1}{2}[H(x_o+h) - H(x_o-h)] - \sum_{k=0}^{m-1} \frac{h^{2k+1}}{(2k+1)!} D_{2k+1}H(x_o)\} .$$

Applying l'Hôpital's rule, one gets

$\lim_{h\to 0} \Theta_{2m+1}(H;x_o,h) = D_{2m}G(x_o)$, which complete the proof for even r .

A similar argument will give the case for r odd.

Note that, in particular, we can apply lemma 1 to the case that H is the Lebesgue integral of a continuous function G in some interval.

Following James [13], we say that a function F is n-smooth at x if $D_{n-2}F(x)$ exists and $\lim_{h\to 0} h\Theta_n(F;x,h) = 0$. By a similar argument in the proof of lemma 1, one has

LEMMA 2. Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_o . Then H is $(n+1)$-smooth at x_o if G is n-smooth at x_o .

LEMMA 3. Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_o. Then for $n \geq 1$,

$$\overline{D}_n G(x_o) \geq \overline{D}_{n+1} H(x_o) \geq \underline{D}_{n+1} H(x_o) \geq \underline{D}_n G(x_o) \tag{4}$$

whenever $\Theta_n(G;x_o,h)$ makes sense.

Proof. By lemma 1, if $\Theta_n(G;x_o,h)$ makes sense, so does $\Theta_{n+1}(H;x_o,h)$. The inequalities (4) then follow from the inequalities (cf [12], p. 359).

$$\limsup_{h\to 0} \frac{f'(h)}{g'(h)} \geq \limsup_{h\to 0} \frac{f(h)}{g(h)} \geq \liminf_{h\to 0} \frac{f(h)}{g(h)} \geq \liminf_{h\to 0} \frac{f'(h)}{g'(h)}$$

for suitable choices of f and g.

§2. SOME PROPERTIES OF n-CONVEX FUNCTIONS.

For the definition of n-convex functions, we refer to the papers mentioned below. The first result we want, due to James [13], [15] but is proved in a more complete form by Bullen [2], gives a set of conditions which are sufficient for a function to be n-convex. The second result gives some important properties of an n-convex function. Before stating these, we recall some concepts.

A function F defined on $[a,b]$ is said to satisfy the condition (C_{2r}) in $[a,b]$ if

(a) F is continuous in $[a,b]$;

(b) $D_{2k}F$ exists, is finite and has no simple discontinuities in $]a,b[$ for $0 \leq k \leq r-1$;

(c) F is $2r$-smooth at all points in $]a,b[$ except perhaps for points of a countable set.

Similarly, the condition (C_{2r+1}) is defined, so that the condition (C_n) makes sense for all integer $n \geq 2$.

A linear set is called a scattered set if it contains no subset that is dense-in-itself. Note that the union of two scattered sets is also scattered [20].

If it is true that

$$F(x+h) - F(x) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r) \quad \text{as} \quad h \to 0 \quad ,$$

then α_k $(1 \leq k \leq r)$ is called the Peano derivative of order k of F at x , written $\alpha_k = F_{(k)}(x)$, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are constants depending on x only, not on h . It is clear that if $F_{(k)}(x)$ exists, so does $D_k F(x)$ and two are equal. But the converse is not true in general.

If F posseses Peano derivative $F_{(k)}(x)$, $1 \leq k \leq r-1$, write

$$\frac{h^r}{r!} \gamma_r(F;x,h) = F(x+h) - F(x) - \sum_{k=1}^{r-1} F_{(k)}(x) \quad .$$

Then define $\overline{F}_{(r),+}(x) = \limsup_{h \to 0+} \gamma(F;x,h)$.

$\underline{F}_{(r),+}$, $\overline{F}_{(r),-}$, $\underline{F}_{(r),-}$ are similarly defined, and then

$F_{(r),+}$, $F_{(r),-}$ are defined in a usual way.

THEOREM 1. (cf.[2], theorem 16). Let F satisfy the condition (C_n) in $[a,b]$ and

(i) $\overline{D}_n F(x) \geq 0$ almost everywhere in $]a,b[$;

(ii) $\overline{D}_n F(x) > -\infty$ for $x \in\,]a,b[\sim S$, S a scattered set;

(iii) $\limsup_{x \to 0} h\, \Theta_n(F;x,h) \geq 0 \geq \liminf_{h \to 0} h\, \Theta_n(F;x,h)$ for $x \in S$.

Then F is n-convex in $[a,b]$.

THEOREM 2. ([2], theorem 7). Let F be n-convex in $[a,b]$. Then

(i) $F^{(r)}$ exists and is continuous in $[a,b]$ for $1 \leq r \leq n-2$, where $F^{(r)}(x)$ denote the ordinary r^{th} derivative of F at x ;

(ii) both $F_{(k-1),-}$, $F_{(n-1),+}$ are monotone increasing in $[a,b]$;

(iii) $F_{(n-1),+} = (F^{(n-2)})'_+$, and $F_{(n-1),-} = (F^{(n-2)})'_-$;

(iv) $F^{(n-1)}(x)$ exists at all except a countable set of points.

§3. THE SC_r-DERIVATIVE AND THE SC_r-CONTINUITY.

Let $r \geq 1$, F be $C_{r-1}P$ (i.e. P_{r-1} of chapter II)-integrable on $[a,b]$, $x \in]a,b[$, $C_r(F;x,x+h)$ as defined in chapter II, and

$$\Delta_r(F;x,h) = \frac{r+1}{2h}\{C_r(F;x,x+h) - C_r(F;x,x-h)\} \quad ,$$

$$\underline{SC_r D}\, F(x) = \liminf_{h\to 0} \Delta_r(F;x,h) \quad .$$

The notations $\overline{SC_r D}$, $SC_r D$ then have the obvious meanings. We call $SC_r DF(x)$, if exists, the symmetric Cesaro derivative of order r of F at x , or simply SC_r-derivative of F at x .

If $\lim_{h\to 0+} \{C_r(F;x,x+h) - C_r(F;x,x-h)\} = 0$, F is said to be SC_r-continuous at x . It is clear that F is SC_r-continuous at x whenever it is C_r-continuous at x , and $SC_r DF$ exists and equals $C_r DF(x)$ whenever $C_r DF(x)$ exists. But, neither of the converses is true. It is also easy to check that $\underline{SC_r DF}$ is measurable (cf. theorem 8 below).

LEMMA 4. For $r \geq 0$, let F be C_r-continuous in $[a,b]$. Then F has no simple discontinuities in $[a,b]$. In particular, every $C_r P$-primitive of a function on $[a,b]$ has no simple discontinuities in $[a,b]$.

Proof. For $r = 0$, the result is immediate since the C_o-continuity is just the ordinary continuity. For $r \geq 1$, suppose that $x_o \in]a,b]$, and $\lim_{x\to x_o^-} F(x) = B$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$B - \varepsilon < F(x) < B + \varepsilon \qquad \text{for } x_o - \delta < x < x_o ,$$

or

$$B - \varepsilon < F(x) < B + \varepsilon \qquad \text{for } x_o - h \leq x < x_o ,$$

where h is such that $0 < h < \delta$. Hence

$$(B-\varepsilon)(x - x_o + h)^{r-1} \leq F(x)(x-x_o+h)^{r-1} \leq (B+\varepsilon)(x-x_o+h)^{r-1}$$

for $x_o - h \leq x < x_o$, which implies that

$$B - \varepsilon \leq \frac{r}{h^r} (C_{r-1}P) - \int_{x_o-h}^{x_o} (x-x_o+h)^{r-1} F(x)dx \leq B + \varepsilon$$

for $0 < h < \delta$, so that $\lim_{h\to 0+} C_r(F;x_o,x_o-h) = B$.

But $F(x_o) = \lim_{h\to 0} C_r(F;x_o,x_o-h) = \lim_{h\to 0+} C_r(F;x_o,x-h)$. Hence $F(x_o) = B$.

Similarly, if $x_o \ \varepsilon\ [a,b[$, and $\lim_{x\to x_o+} F(x) = B'$, then $F(x_o) = B'$. Hence F has no simple discontinuities in $[a,b]$.

The last statement of the lemma is now immediate since by theorem I.6, every $C_r P$-primitive is C_r-continuous.

LEMMA 5. For $n \geq 0$, let F be C_nP-integrable on $[a,b]$, and for $x \in [a,b]$, let

$$G_n(x) = (C_nP) - \int_a^x F(t)dt \quad ,$$

$$G_k(x) = (C_kP) - \int_a^x G_{k+1}(t)dt \quad , \quad 0 \leq k \leq n-1 \quad ,$$

$$G(x) = G_o(x) \quad .$$

Then (i) G is continuous in $[a,b]$;

(ii) if F is SC_{n+1}-continuous at x, then $D_nG(x)$ exists and $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leq k \leq [\frac{n}{2}]$, and G is $(n+2)$-smooth at x, and $\Theta_{n+2}(G;x,x+h) = \Delta_{n+1}(F;x,h)$;

(iii) if F is C_{n+1}-continuous at x, then $G_{(n+1)}(x)$ exists and $G_{(k)}(x) = G_k(x)$ for $0 \leq k \leq n+1$, where $G_{n+1} = F$.

Proof. (i) is immediate since G is just a C_oP-primitive.

For (ii) and (iii), note that by integration by parts,

(5)
$$C_{n+1}(F;x,x+h) = \frac{(n+1)!}{h^{n+1}} \{G(x+h) - G(x) - \sum_{k=1}^{n} \frac{h^k}{k!} G_k(x)\} \quad ,$$

$$C_{n+1}(F;x,x-h) = \frac{(n+1)!}{(-h)^{n+1}} \{G(x-h) - G(x) - \sum_{k=1}^{n} \frac{(-h)^k}{k!} G_k(x)\}$$

for $h \neq 0$ with $x + h \in [a,b]$. Hence for n even, say $n = 2m$,

(5e) $$C_{n+1}(F;x,x+h) - C_{n+1}(F;x,x-h)$$
$$= \frac{(2m+1)!}{h^{2m+1}} \{G(x+h) + G(x-h) - 2 \sum_{k=1}^{m} \frac{h^{2k}}{(2k)!} G_{2k}(x)\} \quad ;$$

and for n odd, say $n = 2m + 1$,

(5o) $$C_{n+1}(F;x,x+h) - C_{n+1}(F;x,x-h)$$
$$= \frac{(2m+2)!}{h^{2m+2}} \{G(x+h) - G(x-h) - 2 \sum_{k=0}^{m} \frac{h^{2k+1}}{(2k+1)!} G_{2k+1}(x)\} \quad .$$

For both cases, if F is SC_{n+1}-continuous at x , then $D_nG(x)$ exists and $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leq k \leq [\frac{n}{2}]$, and G is $(n+2)$-smooth at x , where $[\frac{n}{2}]$ = the greatest integer less that $\frac{n}{2} + 1$. Furthermore, $\Theta_{n+2}(G;x,h) = \Delta_{n+1}(F;x,h)$, proving (ii). (iii) follows from the equality (5).

REMARK. If $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leq k \leq [\frac{n}{2}]$, and G is $(n+2)$-smooth at x , then F is SC_{n+1}-continuous at x . This is clear since replacing $G_{n-2k}(x)$ by $D_{n-2k}G(x)$ in (5e) and (5o) one has that

$$C_{n+1}(F;x,x+h) - C_{n+1}(F;x,x-h) = \frac{2}{n+2}\, h\, \Theta_{n+2}(G;x,h) \quad .$$

LEMMA 6. For $n \geq 0$, let F be C_nP-integrable on $[a,b]$, and SC_{n+1}-continuous in $]a,b[$, and G_n be defined as in lemma 5. If

If (a) $\overline{SC_{n+1}D}F(x) \geq 0$ a.e. in $[a,b]$,

(b) $\overline{SC_{n+1}D}F(x) > -\infty$ for $x \in \,]a,b[\sim S$, S, a scattered set,

then G is $(n+2)$-convex in $[a,b]$.

Proof. This is immediate since by lemma 5, (ii), and lemma 4, G satisfies all the conditions in theorem 1 with $n + 2$ replacing n.

THEOREM 3. For $n \geq 0$, let F be C_nP-integrable on $[a,b]$ and SC_{n+1}-continuous in $]a,b[$. If

(a) $\overline{SC_{n+1}D}F(x) \geq 0$ a.e. in $[a,b]$,

(b) $\overline{SC_{n+1}D}F(x) > -\infty$ for $x \in \,]a,b[\sim S$, S scattered,

(c) F is C_{n+1}-continuous in B $[a,b]$,

then F is monotone increasing in B.

Proof. Let G be defined as in lemma 5. Then by lemma 6, G is $(n+2)$-convex in $[a,b]$, so that by theorem 2, (iv), $G^{(n+1)}$ and hence $G_{(n+1)}$ exists at all except a countable set of points. By theorem 2, (ii), $G_{(n+1)}$ is monotone increasing where it exists. Thus the condition (c) and lemma 5, (iii) imply that F is monotone increasing in B.

THEOREM 4. For $n \geq 0$, let F be C_nP-integrable on $[a,b]$, and $x_o \in \,]a,b[$. If F is SC_{n+1}-continuous at x_o, then F is SC_{n+2}-continuous at x_o, and

$$* \qquad \overline{SC_{n+1}}DF(x_o) \geq \overline{SC_{n+2}}DF(x_o) \geq \underline{SC_{n+2}}DF(x_o) \geq \underline{SC_{n+1}}DF(x_o) \ .$$

Proof. Note first that F is $C_{n+1}P$-integrable on $[a,b]$ by the consistency of the CP-scale. Let, for $x \in [a,b]$,

$$G_n(x) = (C_nP) - \int_a^x F(t)dt \ ,$$

$$G_k(x) = (C_kP) - \int_a^x G_{k+1}(t)dt \quad \text{for } 0 \leq k \leq n-1 \ ,$$

$$H_{n+1}(x) = (C_{n+1}P) - \int_a^x F(t)dt \ ,$$

$$H_k(x) = (C_kP) - \int_a^x H_{k+1}(t)dt \quad \text{for } 0 \leq k \leq n \ .$$

Then $H_{k+1}(x) = G_k(x)$ for $0 \leq k \leq n$ and $H_o(x) = (L) - \int_a^x G_o(t)dt$. By lemma (5), (ii), G_o is $(n+2)$-smooth at x_o, so that H_o is $(n+3)$-smooth at x_o by lemma 2. Hence by the remark following lemma 5, F is SC_{n+2}-continuous at x_o . The inequalities * follow from lemma 5 and lemma 3, completing the proof.

THEOREM 5. Let $\{M_k\}$ be a sequence of SC_n-continuous functions in $]a,b[$, and each M_k is C_n-continuous in a set $B \subset [a,b]$ with $a,b \in B$ and the measure of B is $b - a$. Suppose that $M_k(x) \to M(x)$ as $k \to \infty$

uniformly in B . Then M is SC_n-continuous in $]a,b[$ and C_n-continuous in B .

Proof. Given $\varepsilon > 0$, choose k such that for all $x \varepsilon B$, $|M(x) - M_k(x)| < \frac{1}{3}\varepsilon$. For each $c \varepsilon B$, choose $\delta > 0$ such that $|C_n(M_k;c,c+h) - M_k(c)| < \frac{1}{3}\varepsilon$ whenever $|h| < \delta$ with $x + h \varepsilon [a,b]$. Then $|C_n(M;c,c+h) - C_n(M_k;c,c+h)| < \frac{1}{3}\varepsilon$, so that $|C_n(M;c,c+h) - M(c)| < \varepsilon$ whenever $|h| < \delta$ with $x + h \varepsilon [a,b]$, proving that M is C_n-continuous at c .

That M is SC_k-continuous at each point $c \varepsilon]a,b[$ is proved in a similar way, only replacing $M_k(c)$, $M(c)$ in the above argument by $C_n(M_k;c-h,c)$ and $C_n(M;c-h,c)$, h now being restricted to $c \pm h \varepsilon [a,b]$.

§4. THE SC_nP-INTEGRAL.

Let X be the real line σ the family of all half-open intervals, N the family of all subsets of measure zero. For each positive integer n and each lower derivate operator $\underline{SC_nD}$, $\overline{I}_n$ is defined by $\underline{SC_n}DF(x) > -\infty$ except perhaps for a scattered set of points. We are going to consider "point functions" instead of "interval functions", so that by a base B in $[a,b[$, we mean that $B \subset [a,b]$ and $a,b \varepsilon B$ and the measure of B is $b - a$. Throughout this section, we will consider the base mapping to be the another extreme case $\mathcal{B}(A)$ = the family of all bases in A .

For each interval $[a,b[$ and each base $\mathcal{B}$ in $[a,b[$, let

$$\overline{SM}^n([a,b[,\mathcal{B}) = \{M | M \text{ is } C_n\text{-continuous in } \mathcal{B} \text{ and } SC_n\text{-continuous in }]a,b[\} .$$

Define

$$SC_nP = (\overline{SM}^n, \underline{SC_n}D, \mathcal{B}, N, \overline{I}_n) .$$

Then by theorem 3, and theorem 5, it is easy to check that SC_nP is a derivate system on σ , which furthermore satisfies the additional axioms $(\underline{\mathcal{D}}5)$, $(\underline{\mathcal{D}}6)$, $(\overline{M}6)$, $(\overline{I}4)$ in section I.5, and also $(\overline{M}5')$. Therefore, we obtain a SC_nP-integral for $n = 1,2,3,...$, a scale of symmetric CP-integrals. It follows from theorem I.7 that this scale is more general than the scale of Burkill's CP-scale in chapter II since $\overline{SM}^n \supset \overline{M}^n$ and $\underline{SC_n}DF(x) \geq \underline{C_n}DF(x)$.

As for the CP-scale, we have the consistency theorem for our scale.

THEOREM 6. If f is SC_nP-integrable on $[a,b[$ with base $\mathcal{B}$, then f is also $SC_{n+1}P$-integrable on $[a,b[$ with base $\mathcal{B}$.

Proof. This is immediate from theorem 4 and theorem I, 7.

REMARKS.

(1) Note that the definitions of SC_1P-integral and Burkill's SCP-integral (see [7] or section 6 below) have different families C in $(\overline{I}4)$-scattered sets and countable sets respectively. However, the two integrals are equivalent. For, letting M_1 be a P_o-primitive of M , one has $\underline{SCD}M(x) = \underline{D_2}M_1(x)$. Hence from the remark by James at the end of [15], the set of points x

where $\underline{SCD}M(x) = -\infty$ is a G_σ set and, if at most countable, it must be scattered.

(2) Burkill in [7] listed an integration by parts formula for the SCP-integral and stated that the proof followed from that given for the CP-integral in [5]. This is not true since the proof in [5] used essentially the following inequality

$$\underline{CD}(MG)(x) \geq M(x)G'(x) + \underline{CD}M(x)G(x) \quad ,$$

but we do not have a similar inequality for the SCD-derivate. For example, let

$$M(x) = \begin{cases} x^{-\frac{1}{2}} & \text{for } x > 0 \ , \\ (-x)^{-\frac{1}{2}} & \text{for } x < 0 \ , \\ k & \text{for } x = 0 \ , \text{ where } k \text{ is any constant,} \end{cases}$$

and let $G(x) = -x$. Then

$$SC_1D(MG)(0) = -\infty \not\geq -k = M(0)G'(0) + (SC_1DM(0))G(0) \quad .$$

Thus, whether the formula for SCP-integral in [7] is true remains an open question. Burkill in a recent letter to me agreed with this and said that the same point had been made to him by a young Russian mathematician some years ago.

If such an integration by parts formula exists for the SC_1P-integral, then one can use this to define the SC_2P-integral instead of using the C_1P-integral. Then a more general scale would be obtained by induction. Such a scale would be useful in application to the Cesaro summable trigonometric series.

§5. THE SC_nP-INTEGRAL AND THE P^{n+1}-INTEGRAL.

As we mentioned in the introduction of this chapter, in this section we are going to investigate the relation of the P^{n+1}-integral and the SC_nP-integral.

By P^{n+1}-integral, we mean the modified symmetric one as in [15]. For convenience, we give the definition of its major functions here.

Let f be a function defined almost everywhere in $[a,b]$, and let a_i, $i = 1,2,3,\ldots, n+1$, be fixed points such that $a = a_1 < a_2 < \ldots < a_n < a_{n+1} = b$. A function Q is called a J_{n+1}-major function of f over (a_i) if

(a) Q satisfies the condition (C_{n+1}) in $[a,b]$ (cf §2);

(b) $\underline{D}_{n+1}\, Q(x) \geq f(x)$ almost everywhere in $[a,b]$;

(c) $\underline{D}_{n+1} Q(x) > -\infty$, $x \in \,]a,b[\, \sim S$, S a scattered set;

(d) $Q(a_i) = 0$ for $i = 1,2,3,\ldots,n+1$.

THEOREM 7. Let f be SC_nP-integrable on $[a,b[$ with base B. Then f is P^{n+1}-integrable over $(a_i;c)$, where $a = a_1 < a_2 < \ldots < a_n < a_{n+1} = b$, $c \in [a,b]$. Moreover, letting

$$F_n(x) = (SC_nP) - \int_a^x f(t) \ , \ x \in B \ ,$$

$$F_k(x) = (C_kP) - \int_a^x F_{k+1}(t)dt \ , \ x \in [a,b] \ , \ 0 \leq k \leq n-1 \ ,$$

$$F = F_o \ ,$$

one has for $a_s \leq c < a_{s+1}$,

$$* \qquad (-1)^s \int_{(a_i)}^{c} f(t)d_{n+1}t = F(c) - \sum_{i=1}^{n+1} \lambda(c;a_i)F(a_i) \ ,$$

where $\lambda(c;a_i) = \prod_{j\neq i} (c-a_j)/(a_i-a_j)$ is a polynomial in c of degree at most n .

Proof. Let M be a SC_nP-major function of f on $[a,b[$ with base B , and let

$$G(x) = (C_oP) - \int_a^x (C_1P) - \int_a^{t_1} (C_2P) - \int_a^{t_2} \ldots (C_{n-1}P) - \int_a^{t_{n-1}} M(t_n)dt_n dt_{n-1} \ldots dt_2 dt_1 \ .$$

Then by lemma 4 and lemma 5, G satisfies conditions (a), (b), (c) in the above definition. Hence if we set

$$Q(x) = G(x) - \sum_{i=1}^{n+1} \lambda(x;a_i)G(a_i) \ ,$$

then Q is a J_{n+1}-major function of f over (a_i) . Similarly, a SC_nP-minor function m yields a J_{n+1}-minor function

$$q(x) = g(x) - \sum_{i=1}^{n+1} \lambda(x;a_i)g(a_i) \ ,$$

where g is defined similar to G .

For $\varepsilon > 0$, if we choose M , m such that $M(b) - m(b) < \varepsilon/[1 + \sum_{i=1}^{n+1} (c;a_i)](b-a)^n$, then the corresponding Q , q have

$$|Q(c) - q(c)| \leq |G(c) - g(c)| + \sum |\lambda(c;a_i)| \; |G(a_i) - g(a_i)| \leq \varepsilon \; .$$

Hence, the P^{n+1}-integrability of f follows.

The equality * follows as above by using the property that F_n can be uniformly approximated in B by a sequence of SC_nP-major or minor functions.

COROLLARY 1. $F_{(n)}(x)$ exists for each x in B and $D_{n-1}F(x)$ exists for each $x \in \,]a,b[$. Furthermore, $F_{(n)} = F_n$ on B , and $D_kF = F_k$ on $]a,b[$ for $k = 0,1,2,\dots,n-1$, where F , F_k are those in theorem 6.

Proof. By theorem I.6, F_n is C_n-continuous in B and SC_n-continuous in $]a,b[$, so that the required results follow from lemma 5.

COROLLARY 2. There exists a function which is P^{n+1}-integrable on $[a,b]$ but not SC_nP-integrable on $[a,b]$.

Proof. This is similar to that of Cross in [8*] for $n = 1$. In fact, if n is odd, let

$$F(x) = \begin{cases} x \cos \frac{1}{x} & \text{for } x \neq 0 , \\ 0 & \text{for } x = 0 ; \end{cases}$$

if n is even, let

$$F(x) = \begin{cases} x \sin \frac{1}{x} & \text{for } x \neq 0 , \\ 0 & \text{for } x = 0 . \end{cases}$$

In either case, let $f(x) = F^{(n+1)}(x)$ for $x \neq 0$,

$\qquad\qquad 0$ for $x \neq 0$.

Then $D_{n+1}F(x) = f(x)$ for all x , including $x = 0$, and as shown by James in [13], f is P^{n+1}-integrable over any interval containing 0 . However f is not SC_nP-integrable over $[o,b[$ for any $b > 0$. For otherwise, it would follow from corollary 1 that $F_{(n)}(0)$ exists. But not even $F_{(1)}(0)$ exists.

COROLLARY 3. Let f be periodic with period $2b$, $b > 0$. For $n \geq 1$, let $m = [\frac{n-1}{2}]$. Then if f is SC_nP-integrable on $[-2(m+1)b , 2(n-m)b[$ with base B , one has

$$\frac{1}{(2b)^n} \binom{n+1}{m+1} \int_{(a_i)}^{o} f(t)d_{n+1}t = (SC_nP) - \int_{[-b,b[} f(t)dt \quad ,$$

where $(a_i) = (-2(m+1)b , -2mb , -2(m-1)b , \ldots, -2b, 2b, 4b, \ldots, 2(n-m)b)$. The proof, exactly similar to that of Cross in [8] for the unsymmetric case, is omitted.

REMARKS. (i) Skvorcov [36] has pointed out that a function P^2-integrable over two abutting intervals is not necessarily P^2-integrable over their union. We give an example to show that P^{n+1}-integral has the same property for $n \geq 2$. Let F be as defined in corollary 2. Consider the function f defined by $f(x) = F^{(n+1)}(x)$ for $x \in]0,\frac{i}{\pi}]$,

$\qquad\qquad 0$ for $x \in [-\frac{i}{\pi},0]$, where $i = 2$ if n is odd and $i = 1$ if n is even. Then (cf [13]) f is P^{n+1}-integrable over $[-\frac{i}{\pi},0]$ with P^{n+1}-primitive $G = 0$ on $[-\frac{i}{\pi},0]$, and P^{n+1}-integrable over $[0,\frac{i}{\pi}]$ with P^{n+1}-primitive F on $[0,\frac{i}{\pi}]$. For $n = 1$, it is well-known (cf [8**]) that f is not P^2-integrable over $[-\frac{i}{\pi},\frac{i}{\pi}]$. We show that it is also the case for $n \geq 2$.

Suppose, to the contrary, that f is P^{n+1}-integrable over $[-\frac{i}{\pi},\frac{i}{\pi}]$ with P^{n+1}-primitive H. We show that first $H_{(1),-}(0)$ and then $H_{(1),+}(0)$ exists. Note that on $[-\frac{i}{\pi},0]$, $H-G$ is a polynomial of degree n at most ([13]), and so is $H-F$ on $[0,\frac{i}{\pi}]$. Hence both $(H-G)_{(1),-}(0)$ and $(H-F)_{(1),+}(0)$ exist. As $G_{(1),-}(0)$ exists, we see that $H_{(1),-}(0)$ exists. To see that $H_{(1),+}(0)$ exists, note first that $M-H$ is $(n+1)$-convex on $[-\frac{i}{\pi},\frac{i}{\pi}]$ (cf. [13]) for any J_{n+1}-major function M of f on $[-\frac{i}{\pi},\frac{i}{\pi}]$, so that $D_{n-1}H(0)$ exists since $H = M - (M-H)$. In particular, $D_iH(0)$ exists, where $i = 2$ if n is odd and $i = 1$ if n is even. If it is $i = 1$, then $H_{(1),+}(0)$ exists since $H_{(1),-}(0)$ exists. If it is $i = 2$, then H is smooth at 0, so that $H_{(1),+}(0)$ exists since $H_{(1),-}(0)$ exists. Thus, we have proved that both $(H-F)_{(1),+}(0)$ and $H_{(1),+}(0)$ exist. Then it follows that $F_{(1),+}(0)$ exists, a contradiction, and our proof is hence completed.

(ii) Unlike that for P^{n+1}-integral, note that our SC_nP-integral has the "additive" property by theorem I.3.

(iii) Necessary and sufficient conditions for a function P^{n+1}-integrable over two abutting intervals to be P^{n+1}-integrable over their union are under consideration. Note also that the comparison to Taylor's AP-integral might be interesting (cf. [8**]).

§6. THE MZ-INTEGRAL AND THE SCP-INTEGRAL.

Throughout this section, X , σ, $\mathcal{B}$, $\mathcal{N}$, will be the same as in section 4, and once a derivate operator $\underline{\mathcal{D}}$ is chosen, $\overline{I}$ will be defined by $\underline{\mathcal{D}}F(x) > -\infty$ except for a countable set of points. We show how to obtain using our general theory the MZ-integral of Marchinkiewicz and Zygmund [21] and the SCP-integral of Burkill [7], and then prove that they are in fact equivalent.

For each P_o-integrable function M (see section II.1) on $[a,b]$, and for each $x \in \,]a,b[$, let

$$\underline{Bs}M(x) = \liminf_{h\to 0+} \frac{1}{h} \liminf_{\varepsilon\to 0+} \int_{\varepsilon}^{h} \frac{M(x+u) - M(x-u)}{2u}\, du ,$$

and also

$$\underline{Bs}M(a) = \liminf_{h\to 0+} \frac{1}{h} \liminf_{\varepsilon\to 0+} \int_{\varepsilon}^{h} \frac{M(x+u) - M(a)}{u}\, du$$

$$\underline{Bs}M(b) = \liminf_{h\to 0+} \frac{1}{h} \liminf_{\varepsilon\to 0+} \int_{\varepsilon}^{h} \frac{M(b) - M(x-u)}{u}\, du .$$

These are called the lower Borel derivates. We have

THEOREM 8. $\underline{Bs}M$ is measurable.

Proof. First, note that the function $\phi(M;x,h,\varepsilon) = \int_{\varepsilon}^{h} \frac{M(x+u) - M(x-u)}{2u}\, du$ is continuous in x . For, by the second mean value theorem (see [29], [16]), there exists T with $\varepsilon \le T \le h$ such that

$\phi(M;x+\Delta x,h,\varepsilon)-\phi(M;x,h,\varepsilon)$

$$= \frac{1}{2\varepsilon}\{\int_T^{T+\Delta x} M(x+u)\,du - \int_\varepsilon^{\varepsilon+\Delta x} M(x+u)\,du - \int_{T-\Delta x}^{T} M(x-u)\,du + \int_\varepsilon^{\varepsilon+\Delta x} M(x-u)\,du\}$$

$$+ \frac{1}{2h}\{\int_h^{h+\Delta x} M(x+u)\,du - \int_T^{T+\Delta x} M(x+u)\,du - \int_{h-\Delta x}^{h} M(x-u)\,du + \int_T^{T-\Delta x} M(x-u)\,du\} \quad .$$

Note that T depends on Δx . However, as the P_o-primitive as a point function is continuous in the closed interval concerned, it is uniformly continuous there. Hence each integral in the right hand side of the above equality tends to zero with Δx . Hence $\phi(M;x+\Delta x,h,\varepsilon) \to \phi(M;x,h,\varepsilon)$ as $\Delta x \to 0$, proving the continuity of ϕ in x .

Now, let $\Phi(M;x,h) = \liminf_{\varepsilon\to 0+} \phi(M;x,h,\varepsilon)$. Then Φ is measurable in x since ϕ is continuous in ε . Furthermore, $\Phi(M;x,h)$ is continuous in h since by simple calculations,

$$\Phi(M;x,h+\Delta h) = \int_h^{h+\Delta h} \frac{M(x+u) - M(x-u)}{2u}\,du \to 0 \quad \text{as} \quad \Delta h \to 0 \quad .$$

Hence $\underline{B}s\ M(x) = \liminf_{h\to 0+} \frac{1}{h}\,\Phi(M;x,h)$ is measurable in x , completing the proof.

THEOREM 9. Let B be a base in $[a,b[$ and M be a function defined on $[a,b]$ such that M is C_1-continuous in B and SC_1-continuous in $]a,b[$, and furthermore $\lim_{\varepsilon\to 0+} \int_\varepsilon^h \frac{M(x+u) - M(x-u)}{2u}\,du$ exists (finite or infinite) for all x except perhaps for a countable set of points,

$h \neq 0$ with $x + h \in [a,b]$. If $\underline{B}sM(x) \geq 0$ almost everywhere in $]a,b[$ and $\underline{B}sM(x) > -\infty$ except for a countable set of points, then M is monotone increasing in B .

Proof. Let M_1 be the P_o-primitive of M . Then by lemma 30 in [21], $\overline{D}_2 M_1(x) \geq \overline{B}sM(x) \geq \underline{B}sM(x)$ except for a countable set of points, where $\overline{D}_2 M_1(x) = \limsup_{h\to 0+} \dfrac{M_1(x+h) + M_1(x-h) - 2M(x)}{h^2}$. Hence, as a point function, M_1 is convex in $[a,b]$ by theorem 2, and so M is monotone in B since $M_1{}'(x) = M(x)$ for x in B by the C_1-continuity of M in B , completing the proof.

Now, we are in a position to define the MZ-integral as well as the SCP-integral. For each base B in $[a,b[$ let $\overline{SM}([a,b[,B) = \{M|M$ is C_1-continuous in B and SC_1-continuous in $]a,b[\}$, (i.e. the $\overline{SM}^1$ of section 4) and

$$\overline{SMR}([a,b[,B) = \{M|M \in \overline{SM}([a,b[,B) \text{ and } \lim_{\varepsilon\to 0+} \int_\varepsilon^h \frac{M(x+u) - M(x-u)}{2u}\, du$$

exists (finite or infinite) except perhaps for a countable set of points$\}$.

Define $SCP = (\overline{SM}, \underline{SC_1}D, \mathcal{B}, \mathcal{N}, \overline{I})$, $SCP_R = (\overline{SMR}, \underline{SC_1}D, \mathcal{B}, \mathcal{N}, \overline{I})$, $MZ = (\overline{SMR}, \underline{B}_s, \mathcal{B}, \mathcal{N}, \overline{I})$.

It is easy to see that both SCP and SCP_R are derivate systems on σ . That MZ is also a derivate system on σ follows easily from theorem 8 and theorem 9. Thus, we can define the SCP-, SCP_R- and MZ-integrals. The SCP-integral is just that of Burkill's in [7], while the MZ-integral is just that of Marcinkiewicz and Zygmund in [21] except that the latter was defined by using Lebesgue integrals instead of the P_o-integral.

REMARK. It is easy to see that all the derivate systems SCP , SCP_R and MZ satisfy the extra axioms in section I.4, except that $\overline{SMR}([a,b[,B)$ may not contain all the functions continuous in $[a,b]$. However, the function ω used in the proof of theorem I.12 belongs to $\overline{SMR}([a,b[,B)$ so that all the results in section I.4 are applicable to the SCP-, SCP_R-, and MZ-integral. To see that $\omega \in \overline{SMR}([a,b[,B)$, we need only show that $\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{h} \frac{\omega(x+u) - \omega(x-u)}{2u}\,du$ exists (finite or infinite) except perhaps for a countable set of points. In fact, for $x \in [a,b] - E_1$, $\omega'(x)$ is finite, so that $\frac{\omega(x+u) - \omega(x-u)}{2u}$ is bounded for small u ; for $x \in E_1$, $\omega'(x) = +\infty$, so that $\frac{\omega(x+u) - \omega(x-u)}{2u}$ is positive for small u ; in both cases, we see that $\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{h} \frac{\omega(x+u) - \omega(x-u)}{2u}\,du$ exists.

Now, we establish two lemmas, which will be used to prove the main result of this section (i.e. theorem 10 below).

LEMMA 7. Let $M \in \overline{SMR}([a,b[,B)$. Then BsM(x) exists if and only if $SC_1DM(x)$ exists.

Proof. Let M_1 be the P_o-primitive of M . Then it is easy to see that $\underline{SC_1}DM(x) = \underline{D}_2M_1(x)$ and $\overline{SC_1}DM(x) = \overline{D}_2M_1(x)$ and so the conclusion follows from lemma 28 in [21].

LEMMA 8. Let $M \in \overline{SM}([a,b[,B)$ and $SC_1DM(x)$ exist n.e. in $[a,b]$. Then $M \in \overline{SMR}([a,b[,B)$.

Proof. Let $SC_1DM(x_o)$ exist and let $\psi(t) = \int_o^t \{M(x_o+u) - M(x_o-u)\}du$. For $0 < k < h$,

$$\int_k^h \frac{M(x_o+u) - M(x_o-u)}{2u} du = \int_k^h \frac{\psi'(u)}{2u} du$$

$$= \frac{1}{2}\{\frac{\psi(h)}{h} - \frac{\psi(k)}{k} + \int_k^h \frac{\psi(u)}{u^2} du\}$$

by integration by parts. By the SC_1-continuity of M , $\frac{\psi(k)}{k} \to 0$ as $k \to 0+$. For $SC_1DM(x_o)$ finite, $\frac{\psi(u)}{u^2}$ is bounded for small u ; for $SC_1DM(x) = +\infty$ or $-\infty$, $\frac{\psi(u)}{u^2}$ is of constant sign for small u . In all cases, one sees that

$$\lim_{k\to 0+} \int_k^h \frac{\psi(u)}{u^2} du \text{ exists, so that } \lim_{k\to 0+} \int_k^h \frac{M(x_o+u) - M(x_o-u)}{2u} du$$

exists (finite or infinite), completing the proof.

THEOREM 10. The SCP- , SCP_R- and MZ-integral are all equivalent.

Proof. By lemma 7, one sees that the corollary to theorem I.12 applies to the derivate systems $SCP_R(=P)$ and $MZ(=P_1)$, so that the SCP_R-integral and the MZ-integral are equivalent. To see that they are also equivalent to the SCP-integral, note that by theorem I.7, the SCP-integral is more general than the SCP_R-integral. It remains to show that the MZ-integral is more general than the SCP-integral. To do this, let f be a SCP-integrable function. Applying theorem I.12 to the derivate system SCP , one obtains for f an appropriate

SCP-major function T and an appropriate SCP-minor function t. Then by lemma 8 and lemma 7, one sees that T, t are respectively relevant MZ-major and minor functions for f, so that it is MZ-integrable, completing the proof.

We have remarked that the integration by parts formula for SCP-integral stated by Burkill in [7] remains unproved. Hence his proof of theorem 5.2 in [7] (- the SCP-integral solves the coefficient problem for the convergent trigonometric series) breaks down. However, this theorem remains true by our theorem 10 since it has been proved in [21] that the MZ-integral solves the coefficient problem. We remark that the proof in [21], without using integration by parts but using formal multiplication of series (also see James' P^2-integral), applies to the SCP-integral too.

CHAPTER IV

AN ACP-INTEGRAL AND A SCALE OF APPROXIMATELY MEAN-CONTINUOUS INTEGRALS.

Many authors have generalized the continuous classical Perron integral to integrals that are approximately continuous; see, for example, Burkill [4], Kubota [19]. It would be nice if one can generalize the Burkill's C_nP-integral in the same way. We are only able to do so for $n = 1$. One of our purpose in this chapter is to obtain such an ACP-integral, and then using a method due to Bullen in [3] to obtain an AP^2-integral, and prove that they are equivalent in some suitable sense.

Ellis [9] has defined a scale of mean-continuous integrals, of which the definition is simpler in the sense that the approximate derivative is used for all orders of this scale. With the same idea, we will obtain a scale of approximately mean-continuous integrals, which is more general than and seems more natural than (§1 below) the scale of Ellis.

§1. ON THE MEAN-CONTINUOUS FUNCTIONS.

We prove that the mean continuity scale of Ellis is just Burkill's scale of Cesaro continuity (theorem 1 below), which gives a motivation for a more natural approximately mean-continuous integral (section 2).

The GM-integral scale [9] starts from a function integrable in the general Denjoy sense (see Saks [30]). Ellis called such a function

F M_1-continuous if $\frac{1}{h}\int_x^{x+h} F(t)dt \to F(x)$ as $h \to 0$ for each x . By theorem 1 below, this is just a C_1-continuous function, and hence is special Denjoy integrable. This is why we say that the Ellis integral seems somewhat unnatural in the sense that it starts from the general Denjoy integrable functions.

We recall that the M_n-continuity in [9] was defined in the same way as the C_n-continuity (cf section II.2) except that the GM_{n-1}-integral was used instead of the C_nP-integral.

THEOREM 1. A function is M_n-continuous in an interval if and only if it is C_n-continuous in the interval.

This has been proved by Sargent in [33], page 120. However, we give another proof here.

Proof. Note that the GM_{n-1}-integral is more general than the $C_{n-1}P$-integral, so that a C_n-continuous function is M_n-continuous. To prove the converse, let F be M_n-continuous in $[a,b]$. Then F is GM_{n-1}-integrable on $[a,b]$, and

$$\frac{n}{h^n}(GM_{n-1}) - \int_x^{x+h} (x+h-t)^{n-1}F(t)dt = F(x) + o(1)$$

as $h \to 0$. Let

$$F_{n-1}(x) = (GM_{n-1}) - \int_a^x F(t)dt \quad ,$$

$$F_k(x) = (GM_k) - \int_a^x F_{k+1}(t)dt \quad \text{for} \quad 0 \le k \le n-2 \quad .$$

Then, using the integration by parts formula for the GM_k-integral, one gets that

$$\frac{1}{(n-1)!}\,(GM_{n-1})\int_x^{x+h} (x+h-t)^{n-1}F(t)\,dt = F_o(x+h) - F_o(x) - \sum_{k=1}^{n-1} \frac{h^k}{k!} F_k(x) \quad .$$

Hence as $h \to 0$,

$$F_o(x+h) = F_o(x) + \sum_{k=1}^{n} F_k(x) + o(h^n) \text{ , where } F_n = F \quad .$$

It then follows that F is the n^{th} Peano derivative of F_o . As F_o is continuous, it follows from lemma 11.1 of James [13], that F is C_n-continuous in $[a,b]$, completing the proof.

§2. A SCALE OF APPROXIMATELY CONTINUOUS INTEGRALS.

This scale will be defined inductively. In a manner analogous to the definition of the C_n-mean, the M_n-mean of a function F is defined to be

$$M_n(F;a,b) = \frac{n}{(b-a)^n} \int_a^b (b-t)^{n-1}F(t)\,dt$$

for any positive integer n , where the integral in the definition of M_1-mean is the general Denjoy integral, and the integral involved in the definition of M_n-mean for $n \geq 2$ is the $AM_{n-1}P$-integral defined below. The function F is said to be AM_n-continuous at x_o if

$$\operatorname*{applim}_{h\to 0} M_n(F;x,x+h) = F(x_o) \quad ,$$

where "applim" means "approximate limit" (cf Saks [30]).

A function F is $\underline{(ACG)}$ on a set E if E can be covered by a countable sequence of closed sets on each of which F is $\underline{AC}$ (see section II.2). Note that $\underline{(ACG)}$ is an inequality property as defined in section I.1.

Let X, σ, N be as in section I.4, and for each $A \varepsilon \sigma$, let $\mathcal{B}(A) = \{\sigma_A\}$. For each positive integer n, and for each $A \varepsilon \sigma$ let

$$\overline{AM^n}(A) = \overline{AM^n}(A,\sigma_A) = \{M | M \text{ is } AM_n\text{-continuous in } \overline{A}\} \quad ,$$

and for each M, and each x, let

$$\underline{AD}M(x) = \operatorname*{applim\,inf}_{\substack{h\to 0 \\ x+h\varepsilon \overline{A}}} \frac{M(x+h) - M(x)}{h} \quad .$$

Let $AM_nP = (\overline{AM^n}, \underline{AD}, \mathcal{B}, N, \underline{(ACG)})$. Then we are going to show that AM_nP is a derivate system on σ, so correspondingly we have an AM_nP-integral for $n = 1,2,3,\ldots$, thus obtaining a scale of approximately mean continuous integrals. The integral in the definition of M_n-mean for $n \geq 2$ is in the sense of $AM_{n-1}P$-integral. Thus, in defining the AM_nP-integral for $n \geq 1$, we assume that $AM_{n-1}P$-integral has been defined with some properties, where the AM_oP-integral is taken to be the general Denjoy integral (see remark at the end of this section).

We remark that we might define in a similar way another scale of integrals starting from the AP-integral of Burkill [4]. However, doing this, we are unable to prove the consistency of the scale.

That $AM_n P$ is in fact a derivate system on σ follows easily from theorem 2 and theorem 3 below.

THEOREM 2. For $n = 1,2,3,\ldots$, if F is AM_n-continuous in $[a,b]$ and $(\underline{ACG})$ in $[a,b]$ with $\underline{AD}F(x) \geq 0$ almost everywhere in $[a,b]$, then F is monotone increasing in $[a,b]$.

Proof. This follows from the usual proof of monotonicity (cf. the proof of theorem II.2) by applying the Baire category theorem, the Vitali covering theorem and the following lemma.

LEMMA 1. For $n = 1,2,3,\ldots$, let F be AM_n-continuous in $[a,b]$ and monotone increasing in $]a,b[$, then F is monotone increasing and continuous in $[a,b]$.

Proof. First, we prove that $F(a) \leq F(x)$ for each $x \in\,]a,b]$. Suppose to the contrary that $F(a) > F(x_o)$ for some $x_o \in\,]a,b]$. Let $\varepsilon = F(a) - F(x_o)$. Then $\varepsilon > 0$ and $F(a) - F(t) \geq F(a) - F(x_o) > \varepsilon/2$ for all $t \in\,]a,x_o]$ since F is monotone increasing in $]a,b[$. Hence for each $x \in\,]a,x_o]$,

$$M_n(F;a,x) = \frac{n}{(x-a)^n}\,(AM_{n-1}P)\int_a^x (x-t)^{n-1}F(t)\,dt$$

$$\leq \frac{n}{(x-a)^n}\,(AM_{n-1}P) - \int_a^x (x-t)^{n-1}(F(a) - \varepsilon/2)\,dt = F(a) - \varepsilon/2 ,$$

so that $\operatorname{applim}_{x \to a+} M_n(F;a,x) \leq F(a) - \varepsilon/2 < F(a)$, a contradiction of the fact that F is AM_n-continuous at a .

Similarly, one can prove that $F(b) \geq F(x)$ for each $x \in [a,b[$, and hence F is monotone increasing in $[a,b]$.

To show that F is continuous in $[a,b]$, suppose to the contrary that F is not continuous at x_o for some $x_o \in [a,b]$. Note that $F(x_o+)$ and $F(x_o-)$ exist (only one of them exists if $x_o = a$ or b) since F is monotone in $[a,b]$. Again, by the monotonicity, either $F(x_o-) < F(x_o)$ or $F(x_o) < F(x_o+)$. Suppose that $F(x_o) < F(x_o+)$, and let $T = F(x_o+) - F(x_o) > 0$. Then by a similar calculation to the one above, we have

$$\operatorname{applim}_{x \to x_o+} M_n(F;x_o,x) \geq F(x_o) + T/2 > F(x_o) \quad ,$$

a contradiction. If $F(x_o) > F(x_o-)$, a similar argument can be given. Thus F is continuous at each point of $[a,b]$.

Note that in the above arguments, we use the property that if f_1, f_2 are both $AM_{n-1}P$-integrable with $f_1 \leq f_2$, then $\int f_1 \leq \int f_2$. Thus, the proof of this lemma is completed by the following theorem, which we prove by induction.

THEOREM ($\leq$,n) Let f_1, f_2 be AM_nP-integrable on $[a,b[$ and $f_1 \leq f_2$ almost everywhere in $[a,b]$. Then

$$(AM_nP) - \int_a^b f(t)dt \leq (AM_nP) - \int_a^b f(t)dt \quad . \quad n = 0,1,2,3,\ldots \quad .$$

Proof. The assertion is true for $n = 0$, since the AM_oP-integral is just the general Denjoy integral. Suppose that the assertion is true for $n = k - 1$, $k \geq 1$. Then the assertion is also true for $n = k$ by the definition of AM_nP-integral, and the proof is then completed by induction.

THEOREM 3. For $n = 1,2,3,\ldots$, let $\{F_k\}$ be a sequence of AM_n-continuous functions such that $F_k \to F$ uniformly in $[a,b]$. Then F is AM_n-continuous.

Proof. We only prove it for $n = 1$. For $n > 1$, using Theorem ($\geq$,n-1), a similar proof can be given.

Let $c \in [a,b]$, and given $\varepsilon > 0$, choose k so that $|F_k(x) - F(x)| < \frac{1}{3}\varepsilon$ for all x in $[a,b]$. Then by theorem ($\leq$,0) , we have $|M_1(F_k;c,c+h) - M_1(F;c,c+h)| < \frac{1}{3}\varepsilon$ if $h > 0$ with $c + h \in [a,b]$. As F_k is AM_1-continuous at c , the set E_1 of points x for which $|M_1(F_k;c,x) - F_k(c)| > \frac{1}{3}\varepsilon$ has zero density at c . For each $x \notin E_1$ and x near c , we have

$$|M_1(F;c,x) - F(c)|$$

$$\leq |M_1(F;c,x) - M_1(F_k;c,x)| + |M_1(F_k;c,x) - F_k(c)| < \varepsilon \ .$$

As ε is arbitrary, we see that F is AM_1-continuous at c, completing the proof.

The general properties of our AM_nP-integral follow from the general theory in Chapter I. In addition we prove the consistency of this scale.

THEOREM 4. If $n \geq 1$ then an $AM_{n-1}P$-integrable function is also AM_nP-integrable and two integrals are equal.

Proof. For $n = 1$, let f be AM_oP-integrable with primitive F. Then F being continuous is AM_1-continuous. It is then easy to see that F is both an AM_1P-major and -minor function of f and the proof is then completed.

Now, suppose that it is true for $n = k$, $k \geq 1$. We prove that it is true for $n = k + 1$. To do this, by theorem I.7, it suffices to show that if F is AM_k-continuous, then F is AM_{k+1}-continuous. As F is AM_k-continuous, it is $AM_{k-1}P$-integrable and hence it is AM_kP-integrable and two integrals are equal by induction hypotheses. The AM_{k+1}-continuity of F then follows by applying the integration by parts formula, which we will prove below, (Theorem 5).

THEOREM 5. Let $F(x) = (AM_nP) - \int_a^x f(t)dt$, and

$$G_n(x) = \int_a^x \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_{n-1}} g(t_n)dt_n dt_{n-1} \cdots dt_2 dt_1 \text{ , for } x \in [a,b] \text{ ,}$$

where g is continuous and of bounded variation. Then fG_n is AM_nP-integrable over $[a,b]$ and

$$\int_\alpha^\beta (fG_n)(t)dt = [FG]_\alpha^\beta - \int_\alpha^\beta FG_{n-1}(t)dt$$

for $a \leq \alpha < \beta \leq b$.

Proof. We only prove it for $n = 1$. The general case can then be proved by induction.

Without loss of generality, we suppose that g (and hence G_1) is non-negative in $[a,b]$. Let M be an AM_1P-major function of f on $[a,b[$, we are going to show that MG_1 is an AM_1P-major function of $Fg + fG_1$. To do this, we have to show that MG_1 is AM_1-continuous, $\underline{(ACG)}$ in $[a,b]$ and $\underline{AD}(MG_1) \geq Fg + fG_1$ almost everywhere in $[a,b]$. That MG_1 has the last two properties is trivial. We prove that MG_1 is AM_1-continuous as follows.

It is clear that MG_1 is AM_oP-integrable. Using the integration by parts formula for the AM_oP-integral, we have

$$M_1(MG_1;x,x+h) = \frac{1}{h}\int_x^{x+h} MG_1 = \frac{1}{h}\ [(F_xG_1(t)\Big|_x^{x+h} - \int_x^{x+h} F_xg] \quad ,$$

where $F_x(t) = \int_x^t M(u)du$. Hence

$$M_1(MG_1;x,x+h) = \frac{1}{h}F_x(x+h)G_1(x+h) - \frac{1}{h}\int_x^{x+h} F_xg \quad .$$

As G_1 is continuous and M is AM_1-continuous, one has

$$\operatorname*{applim}_{h\to 0} \frac{1}{h}F_x(x+h)G_1(x+h) = M(x)G_1(x) \quad .$$

F_xg is continuous so that $\frac{1}{h}\int_x^{x+h} F_xg \to F_x(x)g(x) = 0$ as $h \to 0$. Hence $\operatorname*{applim}_{h\to 0} M_1(MG_1;x,x+h) = M(x)G_1(x) + 0 = M(x)G_1(x)$, proving that MG_1 is AM_1-continous at x .

A similar argument for an AM_1P-minor function proves that $Fg + fG_1$ is AM_1P-integrable and $\int_\alpha^\beta (Fg+fG_1) = FG_1\Big|_\alpha^\beta$.

Now, by theorem I.6, F is AM_1-continuous in $[a,b]$, so that F is general Denjoy integrable, and hence so is Fg . Hence Fg is AM_1P-integrable, so that $fG_1 = (Fg + fG_1) - Fg$ is AM_1P-integrable, by theorem I.1. Furthermore,

$$\int_\alpha^\beta fG_1 = FG_1\Big|_\alpha^\beta - \int_\alpha^\beta Fg \quad ,$$ completing the proof.

THEOREM 6. The AM_nP-integral is more general than the GM_n-integral of Ellis in [9].

Proof. It is true for $n = 0$, from the definition. Suppose that it is true for $n = k$, $k \geq 0$. Then we prove that it is true for $n = k + 1$. To this end, let f be GM_{k+1}-integrable. Then there exists a M_{k+1}-continuous (ACG) function F such that $ADF = f$ almost everywhere. Thus by the induction hypotheses, F is AM_kP-integrable and hence AM_{k+1}-continuous. Hence, it is easy to see that F serves as both $AM_{k+1}P$-major and -minor function of f, and hence f is $AM_{k+1}P$-integrable and $(AM_{k+1}P) - \int f = (GM_{k+1}) - \int f$, completing the proof.

We end this section by the following remarks.

REMARKS. (i) Instead of starting from the general Denjoy integral, we can start from the AP-integral, where AP is a derivate system defined by $AP = (\overline{M}^o, \underline{AD}, \mathcal{B}, N, \underline{(ACG)})$, where $\overline{M}^o$ is the legitimate mapping defined in section II.1, i.e. $\overline{M}^o(A) = \{M | M$ is continuous in $\overline{A}\}$. However, one can prove that in fact this AP-integral is equivalent to the general Denjoy integral.

(ii) For $n = 1,2,3,\ldots$, let $\overline{M}^n(A) = \{M | M$ is M_n-continuous in $\overline{A}\}$, and let $M_nP = \{\overline{M}^n, \underline{AD}, \mathcal{B}, N, \underline{(ACG)}\}$. Then M_nP is a derivate system. The M_nP-integral can be proved to be equivalent to Ellis GM_n-integral, which was defined by a descriptive method of Denjoy's.

§3. AN ACP-INTEGRAL AND AN AP^2-INTEGRAL.

For each general Denjoy integrable function F, let

$$\underline{ACD}\, F(x) = \operatorname*{applim\,inf}_{h\to 0} \frac{M_1(F;x,x+h) - M(x)}{h/2},$$

and let $\overline{I}$ be the inequality property defined by $\underline{ACD}\, F(x) > -\infty$ for all x. Let $ACP = (\overline{AM^1}, \underline{ACD}, \mathcal{B}, \mathcal{N}, \overline{I})$, where $\overline{AM^1}$, $\mathcal{B}$, $\mathcal{N}$ are defined as in section 2. Then, it can be checked that ACP is a derivate system on σ. This ACP-integral is just a special case of Ridder's CP_{app}-integral in [27]. We will prove that the ACP-integral is equivalent to an AP^2-integral defined below in the sense of theorem 6 below.

Before defining the AP^2-integral, we prove a lemma. Let F be function such that $ADF(x)$ $\left(= \operatorname*{applim}_{h\to 0} \frac{F(x+h) - F(x)}{h}\right)$ exists, and define $\underline{AD}_2\, F(x) = \operatorname*{applim\,inf}_{h\to 0} \frac{F(x+h) - F(x) - hADF(x)}{h^2}$, and similarly for $\overline{AD}_2\, F(x)$.

LEMMA 2. Let F be continuous such that $ADF(x)$ exists for each x in $[a,b]$ and $\overline{AD}_2 F(x) \geq 0$ in $[a,b]$. Then F is convex in $[a,b]$.

Proof. Let $G_n(x) = F(x) + \frac{1}{2n} x^2$ for x in $[a,b]$, $n = 1,2,3,\ldots$. Then $\overline{AD}_2 G_n(x) = \overline{AD}_2 F(x) + \frac{1}{n} > 0$. We prove that G_n is convex in $[a,b]$, so that F, the limit of G_n, is also convex in $[a,b]$, and the proof will then complete.

To show that G_n is convex in $[a,b]$, suppose to the contrary that G_n is not convex in $[a,b]$. Then there exists an interval $[\alpha,\beta] \subset [a,b]$ such that the function

$$H(x) = G_n(x) - \frac{(\beta-x)G_n(\alpha) - (x-\alpha)G_n(\beta)}{\beta-\alpha}$$

is sometimes positive in $[\alpha,\beta]$. As $H(\alpha) = H(\beta) = 0$, the continuous function H assumes a positive maximum in $]\alpha,\beta[$ at x_o say. Then we have $H(x_o) \geq H(x)$ for each $x \in [\alpha,\beta]$ and $A\,DH(x_o) = 0$, so that

$$\overline{AD}_2 H(x_o) \leq 0 ,$$

which contradicts the fact that $\overline{AD}_2 H(x_o) = \overline{AD}_2 G_n(x_o) > 0$.

Now, using the modified approach to the P^n-integrals used in [3], we define an AP^2-integral. Let f be a function defined on $[a,b]$. Then a function M continuous in $[a,b]$ is called an AP^2-major function of f on $[a,b]$ if

(a) $ADM(x)$ exists and is finite for each x in $[a,b]$;

(b) $\underline{AD}_2 M(x) \geq f(x)$ almost everywhere in $[a,b]$;

(c) $\underline{AD}_2 M(x) > -\infty$ for each x in $[a,b]$;

(d) $ADM(a) = 0 = M(a)$.

If $-m$ is an AP^2-major function of $-f$ on $[a,b]$, then m is called an AP^2-minor function of f on $[a,b]$.

LEMMA 3. Let M be an AP^2-major function and m an AP^2-minor function of f on $[a,b]$. Then $M-m$ is non-negative and convex on $[a,b]$.

Proof. Let $G = M-m$. Then G is continuous in $[a,b]$, $ADG(x)$ exists and is finite for each x in $[a,b]$, $\underline{AD}_2G(x) \geq 0$ for $x \in [a,b] \sim E$, where E is of measure zero, and $\underline{AD}_2G(x) > -\infty$ for each $x \in [a,b]$.

Let E_1 be a G_δ set of measure zero with $E \subset E_1 \subset [a,b]$, and let ω be the function used in the proof of theorem I.12 with $\varepsilon/4$ replacing $\varepsilon/b-a$, and write

$$\psi_\varepsilon(x) = (L) \int_a^x \omega(t)dt \quad ;$$

then $\psi'_\varepsilon(x) = \omega(x)$, ψ_ε is continuous, $AD\psi_\varepsilon(x) = \psi'_\varepsilon(x)$

exists and is finite for each x, $\underline{AD}_2\psi_\varepsilon(x) = \omega'(x) \geq 0$, $\underline{AD}_2\psi_\varepsilon(x) = +\infty$ for each $x \in E$, and $0 \leq \psi_\varepsilon(x) \leq \varepsilon$.

For each $\varepsilon = \frac{1}{h}$, write $\psi_n = \psi_\varepsilon$, and define $G_n = G + \psi_n$. Then by lemma 2, G_n is convex in $[a,b]$, so that the limit function G is convex in $[a,b]$. That $M-m$ is non-negative follows from the convexity and the conditions $M(a) - m(a) = 0 = ADM(a) - ADm(a)$, completing the proof.

In case that f has both AP^2-major functions M, and AP^2-minor functions m on $[a,b]$, and $\sup_m m(b) = \inf_M M(b) \neq \pm\infty$, we say that f is AP^2-integrable on $[a,b]$, and the common value, denoted by $(AP^2) - \int_a^b f(t)dt$, is called the AP^2-integral of f on $[a,b]$. It follows from lemma 3, that if f is AP^2-integrable on $[a,b]$, so is on each $[c,d] \subset [a,b]$.

THEOREM 6. f is ACP-integrable on $[a,b]$ if and only if f is AP^2-integrable on $[a,b]$. Furthermore, if $F(x) = (AP^2) - \int_a^x f(t)dt$, then $ADF(x)$ exists and $ADF(x) = (ACP) - \int_a^x f(t)dt$,

$$F(x) = (\mathcal{D}) - \int_a^x (ACP) - \int_a^u f(t)dt\, du \quad .$$

Proof. We will only prove the first assertion, since the proof of the last one being similar to that in section III.5.

(i) Suppose that f is ACP-integrable on $[a,b]$. Let M be an ACP-major function of f on $[a,b]$, and

$$G(x) = (\mathcal{D}) - \int_a^x M(t)dt \quad .$$

Then G is continuous on $[a,b]$ with $G(a) = 0$ and $ADG(x) = M(x)$, $ADG(a) = M(a) = 0$, $\underline{AD}_2G(x) = \underline{ACD}M(x)$, so that G is an AP^2-major function of f on $[a,b]$. A similar result holds for minor functions, and the AP^2-integrability of f follows.

(ii) Suppose that f is AP^2-integrable on $[a,b]$. Let G be an AP^2-major function of f on $[a,b]$. Then $ADG(x)$ exists and is finite on $[a,b]$, so that $ADG(x)$ is Denjoy integrable with G as a primitive. Furthermore, $ADG(a) = 0$, ADG is AC_1-continuous in $[a,b]$, and $\underline{ACD}(ADG)(x) = AD_2G(x)$, so that ADG is an ACP-major function of f on $[a,b]$. A similar argument for the minor functions completes the proof.

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