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INVARIANT MEANS ON LOCALLY COMPACT GROUPS AND TRANSFORMATION GROUPS

by

ROY CAMERON SNELL

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ABSTRACT

This thesis deals with two separate questions in the area of invariant means on locally compact groups.

Granirer has shown that, for certain discrete semi-groups S, the range of a left invariant mean on the algebra m(S) is the entire [0,1] interval and further, that this range can be obtained on a nested family of left almost convergent subsets of S. We generalize the first part of his result to show that the range of every left invariant mean on $L^{\infty}(G)$ for a locally compact group G is [0,1]. If G is abelian we also show that this range is attained on a nested family of left almost convergent Borel subsets of G.

In the last chapter we deal with the problem of extending the concept of amenability for a locally compact group G to the situation where G acts on the space G/H of left cosets of G with respect to a closed subgroup H (a group acting in this way is called a transformation group). We introduce a definition of the amenable action of G on various closed subspaces of $L^{\infty}(G/H,\upsilon)$ (υ a quasi-invariant measure on G/H) which is equivalent to the one given by Greenleaf but is obtained by different methods. We also prove analogues of several well-known theorems concerning the amenability of locally compact groups.

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CHAPTER I

NOTATIONS AND TERMINOLOGY

1.1 NOTATION : In the material that follows Z, R and C will denote the integers, reals and complex numbers. The positive integers will be denoted by Z^+ and $T=\{\exp(2\pi ix| \ 0\leq x\leq 1\}$ contained in C is the circle group. Given two sets A and B, $A\sim B$ is their set theoretic difference and χ_A the characteristic function of the set A. If we are dealing with subsets of a set Y we will use 1_Y (or simply 1 if no confusion will result) to denote χ_Y - the constant function 1 on Y. The symbol $\emptyset$ will denote the empty set.

1.2 MEASURE THEORY : A measure space is a triple $(X,\mathcal{S},\mu)$ where X is a non-empty set; $\mathcal{S}$ a σ-field of subsets of X and μ a non-negative, countably additive set function defined on $\mathcal{S}$ with $\mu(\emptyset)= 0$. If $\mu(X)< \infty$ then μ is called finite or bounded and if $\mu(X)= 1$, μ is referred to as a probability measure. Given two measures μ_1 and μ_2 on the same space we write $\mu_1 << \mu_2$ to indicate that μ_1 is absolutely continuous with respect to μ_2 and $\mu_1 \equiv \mu_2$ to indicate that μ_1 and μ_2 are equivalent.

$L^1(X,\mu)$ is the space of absolutely μ-integrable real-valued functions on X and $L^\infty(X,\mu)$ the space of μ-essentially bounded real-valued functions on X. $L^1(X,\mu)$ is a Banach space under the norm $\| f\|_1 = \int_X |f(x)|d\mu(x)$ and $L^\infty(X,\mu)$ is a Banach space with

$$\| f\|_\infty = \inf\{\alpha| \ \mu(\{x\varepsilon X: \ |f(x)| > \alpha\})= 0\} \ .$$

SF($\mathcal{S}$) will denote the simple functions on $\mathcal{S}$ (i.e. all real-valued functions f on X of the form $f=\sum_{i=1}^{n} \alpha_i \cdot \chi_{A_i}$ where $\alpha_i \varepsilon R$ and $\{A_1,\ldots,A_n\}$ is a collection of pairwise disjoint members of $\mathcal{S}$). Note that the simple functions are norm dense in both $L^1(X,\mu)$ and $L^\infty(X,\mu)$.

For a topological space X we denote by B(X) the Borel subsets of X (the smallest σ-field of subsets containing the closed sets). If X is a locally compact space then a measure μ on X is a regular Borel measure if the Borel sets are μ-measurable and

i) $\mu(K)< \infty$ for every compact set $K\subset X$

ii) for every measurable set A

$$\mu(A) = \inf\{\mu(B) \mid B \text{ open, } A\subset B\}$$
$$= \sup\{\mu(K) \mid K \text{ compact, } K\subset A\} .$$

If K(X) denotes the continuous functions on X which vanish outside of a compact set then a regular Borel measure on X is completely determined if we specify the value of $\int_X f(x)d\mu(x)$ for every $f\varepsilon K(X)$. All the measures referred to in Chapter IV are regular Borel measures.

If G is a locally compact group then there exists a unique (up to a multiplicative constant) regular Borel measure λ on G, called the left Haar measure on G, with the property that for any measurable function f on G, $\int_G f(gx)d\lambda(x) = \int_G f(x)d\lambda(x)$ for all $g\varepsilon G$. There exists a continuous homomorphism Δ_G from G into the positive multiplicative reals called the modular function such that, for any measurable function f and any $g \in G$

$$\int_G f(xg^{-1})d\lambda(x) = \Delta_G(g)\cdot\int_G f(x)d\lambda(x) .$$

G is called unimodular if $\Delta_G(g)= 1$ for all $g\in G$. If <u>M(G)</u> denotes the space of finite regular Borel measures on G there exists a continuous bilinear operator $*$ called <u>convolution</u> mapping $M(G)\times L^1(G)\to L^1(G)$ and $M(G)\times L^\infty(G)\to L^\infty(G)$ defined by $\mu*f(x) = \int_G f(y^{-1}x)d\mu(y)$ for all $\mu\in M(G)$ and $f\in L^1(G)$ or $L^\infty(G)$. If we define a norm on M(G) by $\|\mu\|= \mu(G)$ then $\|\mu*f\|_1\leq \|\mu\|\cdot\|f\|_1$ and $\|\mu*g\|_\infty\leq \|\mu\|\cdot\|g\|_\infty$ for any $f\in L^1(G)$ and $g\in L^\infty(G)$. For $f\in L^1(G)$ we can define a finite regular Borel measure $\mu\in M(G)$ by setting $\int_G g(x)d\mu(x)= \int_G g(x)f(x)d\lambda(x)$ for each measurable function g on G so we can consider $L^1(G)$ isometrically embedded in M(G) since $\|\mu\|= \|f\|_1$.

1.3 <u>INVARIANT MEANS</u> : A <u>function algebra</u> A on a non-empty set X is a sup norm closed, point separating algebra of bounded real-valued functions on X which contains the constant functions. An example is <u>m(X)</u> - the algebra of all bounded real-valued functions on X. If X is a topological space we may consider <u>CB(X)</u> - the bounded continuous real-valued functions on X. If X is a topological semigroup (or group) <u>LUC(X)</u> is the algebra of bounded continuous functions f on X which have the property that whenever $x_\alpha\to x_o$ in X we have $\lim_\alpha \sup\{|f(x_\alpha x)-f(x_o x)|: x\in X\} = 0$. For a measure space $(X,\mathcal{S},\mu)$ the algebra <u>BM(X)</u> of bounded μ-measurable functions on X is a function algebra. We will often deal with $L^\infty(X)$ which is not strictly speaking a function algebra since functions are identified if they differ only on a null set.

If S is a semigroup then a function algebra A on S is called <u>left [right] invariant</u> if $L_x f \varepsilon A$ $[R_x f \varepsilon A]$ for all $x \varepsilon S$, $f \varepsilon A$ where $L_x f(t) = f(xt)$ $[R_x f(t) = f(tx)]$ for all $t \varepsilon S$. Note that each of the spaces m(S), CB(S), LUC(S), BM(S) and $L^\infty(S)$ is left invariant.

A <u>mean</u> ϕ on a function algebra A (or $L^\infty(X)$ for a measure space $(X,\mathcal{S},\mu)$) is a continuous linear functional on A (i.e. $\phi \varepsilon A^*$) with $\|\phi\| \leq 1$ and such that $\phi(f) \geq 0$ whenever $f \geq 0$. It is easily checked that the set of means on A is a w* compact subset of A*. A mean ϕ on a left invariant function algebra on a semigroup S is called a <u>left invariant mean</u> (LIM) if $\phi(L_x f) = \phi(f)$ for every $x \varepsilon S$ and $f \varepsilon A$. If $f \varepsilon A$ is such that $\phi(f) = c$ (constant) for every LIM ϕ on A then f is said to be <u>left almost convergent</u> to c. A set $B \subset S$ is called a left almost convergent set if χ_B is a left almost convergent function in A. A semigroup S is <u>left amenable</u> if there exists a LIM on m(S).

If G is a locally compact group then a mean υ on $L^\infty(G)$ is a <u>topological left invariant mean</u> (TLIM) if $\upsilon(\phi * g) = \upsilon(g)$ for every $\phi \varepsilon P(G) = \{f \varepsilon L^1(G) \mid f \geq 0, \int_G f(x) d\lambda(x) = 1\}$ and every $g \varepsilon L^\infty(G)$. It can be shown that the existence of a TLIM on $L^\infty(G)$ is equivalent to the existence of a LIM on $L^\infty(G)$. A locally compact group G is called <u>amenable</u> if there exists either a LIM or a TLIM on $L^\infty(G)$. It has been shown that all compact groups and all locally compact abelian groups are amenable. In later work we will use the fact that $P(G)$ is w* dense in the set of means on $L^{\infty}(G)$ (since $L^{1}(G)$ can be isometrically embedded in $L^{\infty}(G)^{*}$).

1.4 GROUP THEORY : If G is a group we will denote its identity by e. If H is a subgroup of G then G/H is the set of left cosets of G with respect to H. The set $\{x_i\}_{i \varepsilon I}$ of elements of G is a set of representatives for G/H if $x_i H \cap x_j H = \emptyset$ for $i \neq j$ and $G = \bigcup_{i \varepsilon I} x_i H$. In Chapter IV we will use $\dot{x}$ to denote the coset xH in G/H. The canonical projection Π_H: $G \to G/H$ is given by $\Pi_H(x) = xH$ for every $x \varepsilon G$. If G is a locally compact group and H a closed subgroup then G/H is a locally compact space and Π_H is continuous and open. A locally compact group G is compactly generated if there exists a compact set $K \subset G$ which generates G. A locally compact group G' is a direct factor of G if G/H is topologically isomorphic with G' for some closed, normal subgroup H of G.

CHAPTER II

THE RANGE OF INVARIANT MEANS ON LOCALLY COMPACT GROUPS AND SEMIGROUPS

2.0 INTRODUCTION : In this chapter we consider the range of left invariant means on the spaces of bounded and essentially bounded Borel measurable functions on locally compact groups and on subsemigroups of locally compact groups. The key lemma is Lemma 2.5.1 and the main results appear in Theorems 2.7.1 and 2.7.3.

2.1 BACKGROUND : Let $(X,\mathcal{B},\mu)$ be a measure space. A set $A\varepsilon\mathcal{B}$ is called an atom if $\mu(A)>0$ and for every $B\varepsilon\mathcal{B}$ with $B\subset A$ we have $\mu(B)=0$ or $\mu(A)$. The measure μ is called non-atomic if there are no atoms in $\mathcal{B}$.

A particular case of Liapounoff's Convexity Theorem for vector-valued measures (see [1]) yields the following

THEOREM 2.1.1 : Let $(X,\mathcal{B},\mu)$ be a finite measure space with μ a non-atomic measure. Then the set $\{\mu(A)|\ A\varepsilon\mathcal{B}\}$ is closed and convex.

As a consequence of this lemma we obtain

COROLLARY 2.1.2 : For any $B_o,B_1\varepsilon\mathcal{B}$ with $B_o\subset B_1$ we have

$$\{\mu(B')|\ B'\varepsilon\mathcal{B},B_o\subset B'\subset B_1\}=[\mu(B_o),\mu(B_1)].$$

PROOF : Define a measure υ on X by $\upsilon(A)=\mu(A\cap(B_1\sim B_o))$. Then υ is clearly finite and non-atomic with

$$0\leq \upsilon(A)\leq \mu(B_1\sim B_o) \text{ for every } A\varepsilon\mathcal{B}.$$

Since $\{\upsilon(A)|\ A\varepsilon\mathcal{B}\}$ is closed and convex by Theorem 2.1.1, it must constitute the entire interval

$$I=[0,\mu(B_1 \sim B_o)]=[0,\mu(B_1)-\mu(B_o)] .$$

If $\alpha\varepsilon[\mu(B_o),\mu(B_1)]$ then $\alpha-\mu(B_o)\varepsilon I$ so there exists $C\varepsilon\mathcal{B}$ with $\upsilon(C)=\alpha-\mu(B_o)$. Let $B'=B_o\cup(C\cap(B_1\sim B_o))$ so $B_o\subset B'\subset B_1$ and we have

$$\begin{aligned}\mu(B')&=\mu(B_o)+\mu(C\cap(B_1\sim B_o))\\ &=\mu(B_o)+\upsilon(C)=\alpha .\end{aligned}$$

From this result we see that for non-atomic probability measures μ, the range of μ is the entire [0,1] interval.

The next lemma indicates that the sets on which a measure μ attains its range can be assumed to be nested.

<u>LEMMA 2.1.3</u> : Let μ be a non-atomic probability measure on the measurable space $(X,\mathcal{B})$. Then there exists a family $\{A(t)|\ t\varepsilon[0,1]\}$ in $\mathcal{B}$ such that

i) $s\leq t$ implies $A(s)\subset A(t)$ and

ii) $\mu(A(t))=t$ for each $t\varepsilon[0,1]$.

<u>PROOF</u> : For $n\varepsilon Z^+$ let $Q_n=\{k/2^n|\ 0\leq k\leq 2^n, k\varepsilon Z\}$ and let $Q=\bigcup_{n\varepsilon Z^+}Q_n$. Then Q is countable and $Q_n\subset Q_{n+1}$ for each n. Assume that we have a family $\{B(t)|\ t\varepsilon Q_n\}$ of sets in $\mathcal{B}$ satisfying conditions i) and ii) for $s,t\varepsilon Q_n$. If $t=k/2^{n+1}$ is an element of Q_{n+1} let $s_1=k_1/2^n$ and $s_2=k_2/2^n$ be the largest and smallest elements of Q_n respectively for which

$$s_1\leq t\leq s_2 .$$

By Corollary 2.1.2 we have

$$\{\mu(B')|\ B'\varepsilon\mathcal{B}, B(s_1)\subset B'\subset B(s_2)\}=[s_1,s_2]$$ so there

exists $B'(t)\varepsilon\mathcal{S}$ with $B(s_1)\subset B'(t)\subset B(s_2)$ and $\mu(B'(t))=t$. In this manner we obtain a collection $\{B'(t) | t\varepsilon Q_{n+1}\}$ of sets in $\mathcal{S}$ which satisfies conditions i) and ii) and extends the family $\{B(t) | t\varepsilon Q_n\}$ (i.e. $B'(t)=B(t)$ for $t\varepsilon Q_{n+1}\cap Q_n$). By induction, there exists a family $\{A'(t) | t\varepsilon Q\}$ satisfying the required conditions and, for $t\varepsilon[0,1]$, if we let $A(t)=\bigcap\{A'(s) | s\varepsilon Q, t\leq s\}$ then $A(t)\varepsilon\mathcal{S}$ since Q is countable; $\{A(t) | t\varepsilon[0,1]\}$ is clearly nested and for $s_1\leq t\leq s_2$ with $s_1,s_2\varepsilon Q$ we have $A'(s_1)\subset A(t)\subset A'(s_2)$ consequently $s_1=\mu(A'(s_1))\leq \mu(A(t))\leq \mu(A'(s_2))=s_2$. Since Q is dense in [0,1] this implies that $\mu(A(t))=t$ as required.

A mean ϕ on the algebra of bounded measurable functions (or $L^\infty(X)$) of a measurable space $(X,\mathcal{S})$ can be regarded as a <u>finitely</u> additive measure μ_ϕ on X if we define $\mu_\phi(A)=\phi(\chi_A)$ for $A\varepsilon\mathcal{S}$.(For convenience of notation we will often write $\phi(A)$ rather than $\phi(\chi_A)$.)

In this setting it is possible to examine the range of ϕ in terms of the range of the corresponding measure. In particular we are interested in the case when ϕ is a left invariant mean (and hence μ_ϕ is a left invariant finitely additive measure). Liapounoff's Theorem does not apply here even if μ_ϕ is non-atomic due to the fact that the measure is not necessarily countably additive.

Granirer showed in [2] that for a certain class of discrete left amenable semigroups the range of each LIM

ϕ on m(S) is the entire interval [0,1] and further, that the nested family on which each ϕ attains its range can be chosen from the left almost convergent subsets of S.

THEOREM 2.1.4 :(Granirer [2]) If S is an infinite discrete right cancellation semigroup which is not an "AB group" (an "AB group" is an infinite amenable discrete group G in which each element has finite order and every infinite subgroup of G contains a finitely generated infinite subgroup) then there exists a nested family $\{A(t) \mid t \varepsilon [0,1]\}$ of (Borel) subsets of S such that $\phi(A(t))=t$ for any LIM ϕ on m(S).

Granirer conjectured that, since it is unlikely that "AB groups" exist, the result should hold for all infinite discrete right cancellation left amenable semigroups.

Chou was able to show in [3] that for every infinite amenable group G the range of each LIM ϕ on m(G) is the entire [0,1] interval. However the sets he obtained on which ϕ attains this range are not nested and depend heavily on ϕ so they are not left almost convergent.

2.2 PRELIMINARIES : In this chapter we generalize Chou's results to show that the range of a LIM on BM(S) or $L^{\infty}(S)$, where S is an infinite Borel subsemigroup of a locally compact group, is the interval [0,1]. The sets we obtain will be nested, however they will depend on the LIM ϕ being considered.

The method we use is based in part on Chou's technique of considering the relationship between left invariant means on m(G) and regular probability measures on the Stone-Cech compactification βG of G. In our case we consider the more general concept of the structure space of a Banach algebra.

We will require the following definition in order to state many of our results in a more concise manner.

DEFINITION 2.2.1 : Let ϕ be a mean on a function algebra A (or on $L^{\infty}(X)$ for a measure space $(X,\mathcal{B},\mu)$) and Q a subset of [0,1]. The family $\{A(t) \mid t\in Q\}$ is called a nested collection on which ϕ attains range Q if $\chi_{A(t)}\in A$ (or $A(t)\in\mathcal{B}$) for each t and

i) $s\leq t$ in Q implies that $A(s)\subset A(t)$

ii) $\phi(A(t))=t$ for each $t\in Q$.

2.3 STRUCTURE SPACE OF AN ALGEBRA :

DEFINITION 2.3.1 : Let A be a function algebra (or $L^{\infty}(X)$ for a measure space $(X,\mathcal{B},\mu)$). The structure space $\Delta(A)$ of A is the set of all non-zero multiplicative linear functionals on A equipped with the weak * topology as a subset of A^{*}.

NOTE : As is well known, $\Delta(A)$ can be considered as the set of multiplicative means on A for $\phi\in\Delta(A)$ implies

$$\phi(1)=1 \text{ (since } 0\neq\phi(1)=\phi(1^2)=(\phi(1))^2)$$

and $\|\phi\|\leq 1$ (Choose $f\in A$ with $\|f\|\leq 1$. Then $f^n\in A$ with $\|f^n\|\leq 1$

for every integer n. Since

$$|\phi(f)|^n=|\phi(f^n)|\leq \|\phi\|\cdot\|f^n\|\leq \|\phi\|<\infty \text{ for each } n,$$

we must have $|\phi(f)|\leq 1$ so $\|\phi\|\leq 1$.).

The following well known result gives further properties of the structure space and its relationship to the algebra A.

THEOREM 2.3.2 :([4], Theorem C.25 p.479)

i) $\Delta(A)$ is compact and Hausdorff

ii) If we define $\hat{f}(\phi)=\phi(f)$ for $f\varepsilon A$ and $\phi\varepsilon\Delta(A)$ then the mapping $f\to\hat{f}$ is a linear isometry of A onto $C(\Delta(A))$- the continuous functions on the structure space.

In the cases we will consider, there is a convenient characterization of the topology of $\Delta(A)$ given by

LEMMA 2.3.3 : Let $\mathfrak{B}$ be a σ-field of subsets of a non-empty set X,(or $\mathfrak{B}=\mathcal{S}$ for the measure space $(X,\mathcal{S},\mu)$) and let A be the algebra of bounded $\mathfrak{B}$-measurable functions on X ($A=L^\infty(X)$). For $B\varepsilon\mathfrak{B}$ let

$$U_B=\{\phi\varepsilon\Delta(A)|\ \phi(B)=1\}$$

Then $\{U_B|\ B\varepsilon\mathfrak{B}\}$ is a base of open-closed (also compact) sets for the topology on $\Delta(A)$.

PROOF :(The proof is presented here for the sake of completeness) First note that $\chi_B^2=\chi_B$ for any set B so $\phi(B)=0$ or 1 for any $\phi\varepsilon\Delta(A)$, $B\varepsilon\mathfrak{B}$. Since $\hat{\chi}_B\varepsilon C(\Delta(A))$ and $U_B=\hat{\chi}_B^{-1}(1)=\hat{\chi}_B^{-1}\{(1/2,2)\}$ we have U_B open-closed (also compact as a closed subset of a compact space).

Let $g=\sum_{i=1}^{n}\alpha_i\chi_{B_i}\in SF(\mathcal{B})$ - the simple functions on $\mathcal{B}$ - and let $V=\{\phi\in\Delta(A): |\phi(g)-\phi_0(g)|<\varepsilon\}$ be a subbasic neighbourhood of $\phi_0\in\Delta(A)$. If $\phi_0(B_k)=1$ for some k then $\phi_0\in U_{B_k}$ and $\phi\in U_{B_k}$ implies $\phi(g)=\alpha_k=\phi_0(g)$ so $|\phi(g)-\phi_0(g)|<\varepsilon$ and $\phi\in V$. Therefore $\phi_0\in U_{B_k}\subset V$.

If $\phi_0(B_i)=0$ for all i, let $B=X\sim(\bigcup_{i=1}^{n}B_i)\in\mathcal{B}$. Then $\phi_0(B)=1$ and for $\phi\in U_B$ we have $\phi(g)=0=\phi_0(g)$ so $\phi_0\in U_B\subset V$.

For arbitrary $f\in A$ let $W=\{\phi\in\Delta(A): |\phi(f)-\phi_0(f)|<\varepsilon\}$ be a subbasic neighbourhood of ϕ_0. Choose $g\in SF(\mathcal{B})$ with $\|f-g\|_\infty<\varepsilon/3$ (this is possible since $SF(\mathcal{B})$ is dense in A) so we have $|\phi(f)-\phi(g)|<\varepsilon/3$ for every $\phi\in\Delta(A)$. If $U=\{\phi\in\Delta(A): |\phi(g)-\phi_0(g)|<\varepsilon/3\}$ then $\phi_0\in U\subset W$ and we can find $B\in\mathcal{B}$ with $\phi_0\in U_B\subset U\subset W$. Since $U_{B_1\cap B_2}=U_{B_1}\cap U_{B_2}$ for all $B_1,B_2\in\mathcal{B}$, the above implies that $\{U_B|\ B\in\mathcal{B}\}$ is a base for the topology.

2.4 PROBABILITY MEASURES ON $\Delta(A)$:

LEMMA 2.4.1 : A mean ϕ on A (or $L^\infty(X)$) gives rise to a regular probability measure μ_ϕ on $\Delta(A)$ ($\Delta(L^\infty(X))$) defined by $\int\hat{f}d\mu_\phi=\phi(f)$ for all $f\in A$ ($f\in L^\infty(X)$).

PROOF : This follows from the fact that for any compact Hausdorff space Y, $C(Y)^*$ is isometrically isomorphic to the space of bounded regular Borel measures on Y (see [8] p.265). Since $A\simeq C(\Delta(A))$ we have a homeomorphism between A^* and the bounded regular Borel measures on $\Delta(A)$ given by $\psi\to\mu_\psi$ where $\int\hat{f}d\mu_\psi=\psi(f)$ for $f\in A$. If ϕ is a mean then $\mu_\phi(\Delta(A))=\int\hat{1}d\mu_\phi=\phi(1)=1$ and μ_ϕ is a probability measure.

NOTE : For $B \in \mathcal{B}$, $\hat{\chi}_B = \chi_{U_B}$ in $\Delta(A)$ and the above formula shows that $\mu_\phi(U_B) = \phi(B)$.

DEFINITION 2.4.2 : A bounded regular Borel measure μ on a locally compact Hausdorff topological space X is called continuous if $\mu(\{x\})=0$ for every $x \in X$.

Using this definition we have

LEMMA 2.4.3 : If μ is a finite continuous measure on a locally compact Hausdorff space X then for any measurable sets B_o, B_1 with $B_o \subset B_1$ we have

$$\{\mu(B') \mid B' \in \mathcal{B}(X),\ B_o \subset B' \subset B_1\} = [\mu(B_o), \mu(B_1)] .$$

PROOF : In view of Corollary 2.1.2 it is sufficient to show that μ is non-atomic. If μ has atoms then, by regularity, there is a compact atom K. For each $x \in K$, since $\mu(\{x\})=0$, we can find an open set U_x containing x with $\mu(U_x) < \mu(K)$. By compactness of K there exists a finite collection $U_{x_1}, \ldots, U_{x_n}$ covering K. Since K is an atom, $\mu(K \cap U_{x_i})=0$ for each i which implies that $\mu(K)=0$ which is a contradiction.

2.5 MAIN LEMMA : The following lemma is the key tool used in obtaining the main results of this chapter. In its proof we will use, in part, an idea of Chou in [3] and one of Granirer in [2].

LEMMA 2.5.1 : Let X be a non-empty set, $\mathcal{B}$ a σ-field of subsets of X (or $\mathcal{B} = \mathcal{S}$ for the measure space $(X, \mathcal{S}, \mu)$) and

A the algebra of bounded $\mathfrak{B}$ measurable functions on X ($A=L^{\infty}(X)$). Let ϕ be a mean on A for which the corresponding probability measure μ_{ϕ} on $\Delta(A)$ is continuous. Then there exists a nested collection in A on which ϕ attains range $[0,1]$.

<u>PROOF</u> : Let $A_o, A_1 \varepsilon \mathfrak{B}$ with $A_o \subset A_1$ and let $\lambda \varepsilon [\phi(A_o), \phi(A_1)]$ and $\varepsilon > 0$ be given. By Lemma 2.4.3 there exists a measurable subset E of $\Delta(A)$ with $U_{A_o} \subset E \subset U_{A_1}$ and $\mu_{\phi}(E)=\lambda$ (since $\mu_{\phi}(U_{A_o}) = \phi(A_o)$, $\mu_{\phi}(U_{A_1}) = \phi(A_1)$ and $U_{A_o} \subset U_{A_1}$). Due to the fact that U_{A_o} is compact and U_{A_1} is open we can use the regularity of μ_{ϕ} to find a compact E_1 and an open E_2 with $U_{A_o} \subset E_1 \subset E \subset E_2 \subset U_{A_1}$ and $\mu_{\phi}(E_2 \sim E_1) < \varepsilon$. Since E_2 is open and $\{U_B | B \varepsilon \mathfrak{B}\}$ is a base for the topology on $\Delta(A)$, for each $x \varepsilon E_1$ there exists $B_x \varepsilon \mathfrak{B}$ with $x \varepsilon U_{B_x} \subset E_2$ and by the compactness of E_1 there exist $x_1, \ldots, x_n \varepsilon E_1$ with $E_1 \subset \bigcup_{i=1}^{n} U_{B_{x_i}} \subset E_2$. If $B = B_{x_1} \cup \ldots \cup B_{x_n} \varepsilon \mathfrak{B}$ then $U_B = \bigcup_{i=1}^{n} U_{B_{x_i}}$ so $E_1 \subset U_B \subset E_2$ and there- $|\phi(B)-\lambda| = |\mu_{\phi}(U_B) - \mu_{\phi}(E)| < \varepsilon$. Since $U_{A_o} \subset U_B \subset U_{A_1}$ implies $A_o \subset B \subset A_1$ this shows that $\{\phi(B) | B \varepsilon \mathfrak{B}, A_o \subset B \subset A_1\}$ is dense in $[\phi(A_o), \phi(A_1)]$.

Let $Q_n = \{k/2^n | 0 \leq k \leq 2^n, k \varepsilon Z^+\}$ for $n \varepsilon Z^+$ and let $Q = \bigcup_{n=1}^{\infty} Q_n$. We want to define a nested collection $\{A'(t) | t \varepsilon Q\}$ on which ϕ attains range Q. This is done inductively by defining $A(0) = \emptyset$, $A(1) = X$ and assuming that $\{A(t) | t \varepsilon Q_n\}$ is a nested collection on which ϕ attains range Q_n.

For $t\varepsilon Q_{n+1}\cap Q_n$ let $A'(t)=A(t)$. If $t\varepsilon Q_{n+1}\sim Q_n$ let $t_1,t_2\varepsilon Q_n$ be the maximum and minimum elements of Q_n for which $t_1< t< t_2$ and choose sequences $\alpha_n\uparrow t$ and $\beta_n\downarrow t$ in (t_1,t_2). Since $\{\phi(B)|\ B\varepsilon\mathfrak{B},\ A(t_1)\subset B\subset A(t_2)\}$ is dense in $[t_1,t_2]$, we can find $C_1\varepsilon\mathfrak{B}$ with $A(t_1)\subset C_1\subset A(t_2)$ and $\beta_2< \phi(C_1)< \beta_1$. Since $(\alpha_1,\alpha_2)\subset[t_1,\phi(C_1)]$ we can find $D_1\varepsilon\mathfrak{B}$ with $A(t_1)\subset D_1\subset C_1$ and $\alpha_1< \phi(D_1)< \alpha_2$. Since $(\beta_3,\beta_2)\subset[\phi(D_1),\phi(C_1)]$ there exists $C_2\varepsilon\mathfrak{B}$ with $D_1\subset C_2\subset C_1$ and $\beta_3< \phi(C_2)< \beta_2$. Continuing in this manner we obtain sequences $C_i\downarrow$ and $D_i\uparrow$ in $\mathfrak{B}$ with $A(t_1)\subset D_i\subset C_j\subset A(t_2)$ for all i,j and such that $\alpha_i< \phi(D_i)< \alpha_{i+1}$, $\beta_{i+1}< \phi(C_i)< \beta_i$ for all i. If we let $A'(t)=\bigcup_{i=1}^{\infty} D_i\varepsilon\mathfrak{B}$ we have $D_i\subset A'(t)\subset C_j$ for all i and j so $\alpha_i< \phi(D_i)\leq \phi(A'(t))\leq \phi(C_j)< \beta_j$ and since $\lim_{i\to\infty}\alpha_i = \lim_{j\to\infty}\beta_j = t$ this implies that $\phi(A'(t))= t$.

In this manner we obtain a nested collection $\{A'(t)|\ t\varepsilon Q_{n+1}\}$ which extends the collection $\{A(t)|\ t\varepsilon Q_n\}$ and on which ϕ attains range Q_{n+1}. By induction there exists a collection $\{A(t)|\ t\varepsilon Q\}$ on which ϕ attains range Q and the collection is nested for if $t_1,t_2\varepsilon Q$ with $t_1< t_2$ then $t_1,t_2\varepsilon Q_n$ for some n and thus $A(t_1)\subset A(t_2)$.

As in Lemma 2.1.3, for $t\varepsilon[0,1]$ we define

$$B(t)=\bigcap\{A(s)|\ s\varepsilon Q,\ t\leq s\} .$$

$B(t)\varepsilon\mathfrak{B}$ since Q is countable and the density of Q in [0,1] implies that $\{B(t) \mid t\varepsilon[0,1]\}$ is a nested collection in A on which ϕ attains range [0,1].

In view of this result the problem of determining the range of a mean on the algebras mentioned in the above lemma reduces to showing that the corresponding probability measure μ_ϕ is continuous.

2.6 ORBITS IN $\Delta(A)$:

DEFINITION 2.6.1 : Let A be a left invariant function algebra on a semigroup S. If $\psi\varepsilon\Delta(A)$ then the left orbit of ψ - $O(\psi)=\{L_t^*\psi \mid t\varepsilon S\}$
(where L_t^* denotes the adjoint of the left translation operator $L_t: A \to A$). Note that for each $\psi\varepsilon\Delta(A)$; $f,g\varepsilon A$ and $t\varepsilon S$

$$L_t^*\psi(f\cdot g)= \psi(L_t(f\cdot g))= \psi(L_tf\cdot L_tg)= \psi(L_tf)\cdot\psi(L_tg)$$
$$= (L_t^*\psi)(f)\cdot(L_t^*\psi)(g)$$

so $L_t^*\psi$ is also a multiplicative mean and we have $O(\psi)\subset\Delta(A)$.

The following lemma indicates the relationship between left orbits in $\Delta(A)$ and the measures μ_ϕ on $\Delta(A)$ corresponding to left invariant means ϕ on A.

LEMMA 2.6.2 : Let S be a semigroup, $\mathfrak{B}$ a left invariant σ-field of subsets of S and A the algebra of bounded $\mathfrak{B}$-measurable functions on S. If $O(\psi)$ is infinite for all $\psi\varepsilon\Delta(A)$ then, for any LIM ϕ on A, the corresponding regular probability measure μ_ϕ on $\Delta(A)$ is continuous.

PROOF : Choose $\psi\varepsilon\Delta(A)$ and $t\varepsilon S$. Note that if $L_t^*\psi\varepsilon U_B$ where

$B\varepsilon\mathfrak{B}$ then $\psi\varepsilon U_{t^{-1}B}$ since $\psi(\chi_{t^{-1}B}) = \psi(L_t\chi_B) = L_t^*\psi(B) = 1$.

Also, due to the fact that

$$\mu_\phi(U_B) = \phi(B) = \phi(L_t\chi_B) = \phi(\chi_{t^{-1}B}) = \mu_\phi(U_{t^{-1}B}),$$ we have $\mu_\phi(\{\psi\}) \leq \mu_\phi(U_{t^{-1}B}) = \mu_\phi(U_B)$ (since ϕ is left invariant) whenever $L_t^*\psi\varepsilon U_B$. The regularity of μ_ϕ and the fact that $\{U_B |\ B\varepsilon\mathfrak{B}\}$ is a base for the open sets in $\Delta(A)$ imply that

$$\mu_\phi(\{\psi\}) \leq \mu_\phi(\{L_t^*\psi\}) .$$

Since $O(\psi)$ is infinite, for any positive integer n we can find $t_1,\ldots,t_n\ \varepsilon S$ with $L_{t_i}^*\psi \neq L_{t_j}^*\psi$ for $i \neq j$. Therefore $1 = \mu_\phi(\Delta(A)) \geq \mu_\phi(\bigcup_{i=1}^{n}\{L_{t_i}^*\psi\}) = \sum_{i=1}^{n}\mu_\phi(\{L_{t_i}^*\psi\}) \geq n\cdot\mu_\phi(\{\psi\})$ and, since n is arbitrary, this implies that $\mu_\phi(\{\psi\}) = 0$ so μ_ϕ is continuous.

In the cases we are considering in this chapter we are able to call on the following theorem of Granirer and Lau

<u>THEOREM 2.6.3</u> :([5] Theorem 3) Let S be a subsemigroup of a locally compact group G and assume that LUC(S) (the bounded left uniformly continuous functions on S) admits a LIM ϕ of the type $\phi = \sum_{i=1}^{n}\alpha_i\phi_i$ where $\phi_i\varepsilon LUC(S)^*$ are multiplicative; $\phi_i \neq \phi_j$ for $i \neq j$; $\alpha_i > 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n}\alpha_i = 1$. Then S is a finite subgroup of G of order n.

Using this result we obtain

<u>THEOREM 2.6.4</u> : Let S be an infinite Borel subsemigroup

(of positive Haar measure) in a locally compact group G and let A be the algebra of bounded Borel measurable functions on S ($A= L^{\infty}(S)$). Then $O(\psi)$ is infinite for all $\psi \epsilon \Delta(A)$.

<u>PROOF</u> : First note that for $\psi_1 \neq \psi_2$ in $\Delta(A)$ we can find $B \epsilon B(S)$ with $\psi_1(B) \neq \psi_2(B)$. Let $C= tB$ where $t \epsilon S$. Since S is a Borel set in G it is easily checked that

$$B(S)= \{B \epsilon B(G) \mid B \subset S\} .$$

Since $t \cdot B(G) \subset B(G)$ for all $t \epsilon G$, this implies that $C \epsilon B(S)$. Therefore $\chi_C \epsilon A$ with $L_t \chi_C = \chi_B$ so

$$L_t^* \psi_1(C)= \psi_1(L_t \chi_C)= \psi_1(B) \neq \psi_2(B)= \psi_2(L_t \chi_C)= L_t^* \psi_2(C)$$

and the mapping $L_t^*: \Delta(A) \to \Delta(A)$ is one-to one.

If $O(\psi)$ is finite for some $\psi \epsilon \Delta(A)$ we can find $t_1, \ldots, t_n \epsilon S$ such that i) $L_{t_i}^* \psi \neq L_{t_j}^* \psi$ for $i \neq j$ and

ii) for any $t \epsilon S$ we have $L_t^* \psi = L_{t_i}^* \psi$ for some i.

Let $\phi= (\sum_{i=1}^{n} L_{t_i}^* \psi)/n$. ϕ is clearly a mean on A and for $t \epsilon S$ we have $L_t^* \phi= (\sum_{i=1}^{n} L_t^* L_{t_i}^* \psi)/n = (\sum_{i=1}^{n} L_{t t_i}^* \psi)/n = (\sum_{i=1}^{n} L_{t_i}^* \psi)/n = \phi$ due to the properties i) and ii) above and the fact that L_t^* is one-to-one on $\Delta(A)$. Therefore ϕ is a LIM on A and so, by restriction, also a LIM on the closed subalgebra LUC(S) of A. However, since $L_{t_i}^* \psi$ is a multiplicative element of $LUC(S)^*$ (by restriction) for each i, Theorem 2.6.3 implies that S is finite which contradicts our assumption that S was infinite.

2.7 MAIN RESULTS :

THEOREM 2.7.1 : Let S be an infinite Borel subsemigroup of a locally compact group G. If A is the algebra of bounded Borel measurable functions on S and ϕ is a LIM on A then there exists a nested collection of Borel subsets of S on which ϕ attains range [0,1].

PROOF : By Lemma 2.6.4, $O(\psi)$ is infinite for each $\psi \varepsilon \Delta(A)$. Therefore Lemma 2.6.2 implies that μ_ϕ is continuous and the result follows from Lemma 2.5.1.

As a corollary to this theorem we obtain a stronger version of Chou's result in [3]

COROLLARY 2.7.2 : Let S be an infinite subsemigroup of a discrete group G. If ϕ is a LIM on m(S) there exists a nested collection of (Borel) subsets of S on which ϕ attains range [0,1].

The method of proof used in Lemma 2.6.2 will also work in the case where S is a Borel subsemigroup of positive Haar measure in a locally compact group G and $A=L^\infty(S)$ so, as before, combining Lemmas 2.5.1,2.6.2 and 2.6.4 we obtain

THEOREM 2.7.3 : Let S be a Borel subsemigroup of positive Haar measure in a locally compact group G. If ϕ is a LIM on $L^\infty(S)$ then there exists a nested collection of Borel subsets of S on which ϕ attains range [0,1].

CHAPTER III

THE RANGE OF INVARIANT MEANS ON LOCALLY COMPACT ABELIAN GROUPS

3.0 PRELIMINARIES : In this chapter we continue our examination of the range of left invariant means on $L^{\infty}(G)$ for a locally compact group G. Since the techniques used are also valid for the algebra of bounded Borel measurable functions on G, the results which follow for BM(G) as well as for $L^{\infty}(G)$.

In Chapter II we showed that the range of any LIM ϕ on $L^{\infty}(G)$ is the entire [0,1] interval and that this range is attained on a nested family of Borel subsets of G. However the sets obtained depend heavily on the LIM ϕ being considered and are thus not left almost convergent as was the case of the sets obtained by Granirer in [2].

The main result of this chapter obtained in Theorem 3.4.2 is the fact that for every locally compact abelian group there exists a nested family of left almost convergent Borel subsets of G on which every LIM ϕ attains the range [0,1]. This result is obtained using several structure theorems for locally compact, compactly generated abelian groups and, in the most difficult case of compact abelian groups, by employing in part a technique introduced by Granirer in [2].

In order to state many of the results in a more concise form we introduce the following definition.

DEFINITION 3.0.1 : If G is a locally compact group and Q is a subset of $[0,1]$ then a collection $\{A(t) \mid t \in Q\}$ of Borel subsets of G is called a nested collection with range Q if i) $s, t \in Q$ with $s \leq t$ implies $A(s) \subset A(t)$ and ii) $\phi(A(t)) = t$ for any LIM ϕ on $L^{\infty}(G)$ and each $t \in Q$.

3.1 NESTED COLLECTIONS IN QUOTIENT GROUPS :

In order that a locally compact amenable group G admit a nested collection with range $[0,1]$, it is sufficient that some continuous homomorphic image of G admit a nested collection with range Q where Q is a countable dense subset of $[0,1]$. This follows immediately from the following two lemmas.

LEMMA 3.1.1 : Let G be an amenable locally compact group and H a closed, normal subgroup of G. If there exists a nested collection with range Q in G/H then there exists a nested collection with range Q in G.

PROOF : First note that the amenability of G implies that the locally compact group G/H is also amenable (see [6] p.30).

Let us define a mapping $T: L^{\infty}(G/H) \to L^{\infty}(G)$ as follows - for $f \in L^{\infty}(G/H)$ let $Tf = f \circ \Pi_H$. Since Π_H is continuous and f is Borel measurable, Tf is a measurable function on G. If λ denotes left Haar measure on G and υ is a left Haar measure on G/H then a set A in G/H is υ-null if and only if $\Pi_H^{-1}(A)$ is λ-null in G (see Chapter IV Theorem 4.1.6). For any positive real number α,

$$\{x\varepsilon G: \ |Tf(x)| > \alpha\} = \Pi_H^{-1}\{xH\varepsilon G/H : \ |f(xH)| > \alpha\} .$$

Since $\|Tf\|_\infty = \inf\{\alpha| \ \lambda(\{x\varepsilon G: \ |Tf(x)| > \alpha\})=0\}$ the above implies that $\|Tf\|_\infty = \|f\|_\infty$ so $Tf\varepsilon L^\infty(G)$. Clearly $f\to Tf$ is linear therefore T is a linear isometry from $L^\infty(G/H)$ into $L^\infty(G)$. Consequently the adjoint T^* is a linear isometry from $L^\infty(G)^*$ into $L^\infty(G/H)^*$. If ϕ is a mean on $L^\infty(G)$ and $f\geq 0$ υ almost everywhere in G/H then $Tf\geq 0$ λ almost everywhere in G so $0\leq \phi(Tf)=T^*\phi(f)$. Also $T^*\phi(1_{G/H})= \phi(T(1_{G/H}))= \phi(1_G)= 1$ so $T^*\phi$ is a mean on $L^\infty(G/H)$. If, in addition, ϕ is left invariant then $T^*\phi(L_{xH}f)= \phi(T(L_{xH}f))= \phi(L_x Tf)= \phi(Tf)= T^*\phi(f)$ so $T^*\phi$ is a LIM on $L^\infty(G/H)$.

Let $\Omega=\{A(t)| \ t\varepsilon Q\}$ be a nested collection with range Q in G/H and define $B(t)= \Pi_H^{-1}(A(t))$ for each $t\varepsilon Q$. Since Π_H is continuous, B(t) is a Borel subset of G for each $t\varepsilon Q$ and $\Omega'=\{B(t)| \ t\varepsilon Q\}$ is clearly nested. If ϕ is a LIM on $L^\infty(G)$ then for each $t\varepsilon Q$ we have

$$t= T^*\phi(A(t))= \phi(T\chi_{A(t)})= \phi(B(t))$$

since $T^*\phi$ is a LIM on $L^\infty(G/H)$. Consequently Ω' is a nested collection with range Q in G.

<u>LEMMA 3.1.2</u> : Let G be an amenable locally compact group and Q a countable, dense subset of [0,1] such that there exists a nested collection in G with range Q. Then there exists a nested collection in G with range [0,1].

<u>PROOF</u> : Let $\{A(t)| \ t\varepsilon Q\}$ be nested with range Q. For

$t\varepsilon[0,1]$ let $A'(t)=\bigcap\{A(s)|\ s\varepsilon Q,\ t\leq s\}$. As in the proof of Lemma 2.1.3 it is easily checked that $\{A'(t)|\ t\varepsilon[0,1]\}$ is a nested collection in G with range [0,1].

3.2 NESTED COLLECTIONS IN T, R AND Z :

The structure theorems for certain locally compact groups which we employ later in the chapter will require the existence of nested collections with range [0,1] in the locally compact abelian groups T, R and Z (the circle group, reals and integers respectively), so we now prove

LEMMA 3.2.1 : There exists a nested collection with range [0,1] in each of the locally compact abelian groups T, R and Z.

PROOF : i) Let $Q=\{k/n|\ 0\leq k\leq n;\ k,n\varepsilon Z^+\}$ and for $k/n\varepsilon Q$ define $A(k/n)=\{\exp(2\pi ix)|\ 0\leq x<k/n\}$. Therefore $\Omega=\{A(k/n)|\ k/n\varepsilon Q\}$ is a nested family of Borel subsets of T. If we let $A_n=\{\exp(2\pi ix)|\ 0\leq x<1/n\}$ we have

$$T=\bigcup_{m=0}^{n-1} \exp(2\pi im/n)\cdot A_n \quad \text{(disjoint union) so for}$$

any LIM ϕ on $L^\infty(T)$, $1=\phi(T)=n\cdot\phi(A_n)$ and consequently $\phi(A_n)=1/n$. Since

$$A(k/n)=\bigcup_{m=0}^{k-1} \exp(2\pi im/n)\cdot A_n \quad \text{(disjoint union)}$$

this implies $\phi(A(k/n))=k\cdot\phi(A_n)=k/n$.

Therefore Ω is a nested collection in T with range Q and, since Q is a countable dense subset of [0,1], Lemma 3.1.2 implies the existence of a nested collection with range [0,1].

ii) Note that Z is a closed normal subgroup of R and T= R/Z. By part i) there exists a nested collection in R/Z with range [0,1] so by Lemma 3.1.1 such a collection also exists in R.

iii) Since Z is a discrete right cancellation, left amenable group which is abelian (hence not an " AB group") the result follows from Theorem 2.1.4 (due to Granirer). However we will give a different and constructive proof of this result in order to indicate how the desired family of left almost convergent sets can be obtained.

Our proof will depend on the fact that, for any non-negative integer n, the set $A=\{m\cdot n \mid m\in Z\}$ is left almost convergent to 1/n since

$$Z=\bigcup_{i=0}^{n-1}(i+A) \quad \text{(disjoint union) implies that}$$

$$1=\phi(Z)=\sum_{i=0}^{n-1}\phi(i+A)=n\cdot\phi(A) \quad \text{for any LIM } \phi \text{ on}$$

$L^{\infty}(Z)= m(Z)$. Thus given any number k/n where $1\leq k\leq n$ we can obtain a set B which is left almost convergent to k/n by setting $B=\bigcup_{i=1}^{k}(i+A)$.

In order to obtain a nested family of left almost convergent sets we proceed as follows :

- for each positive integer n let $Q_n=\{k/2^n \mid 0\leq k\leq 2^n\}$ and let $Q=\bigcup_{n=1}^{\infty}Q_n$. We obtain a nested collection in Z with range Q by induction. Assume we are given a permutation $\{x_1,\ldots,x_{2^n}\}$ of the integers $0,1,\ldots,2^n-1$ and define

$$A(k/2^n) = \bigcup_{i=1}^{k} (x_i + \{m \cdot 2^n \mid m \varepsilon Z\}) \quad \text{for } k/2^n \varepsilon Q_n$$

so, as indicated above, the family $\Omega_n = \{A(t) \mid t \varepsilon Q_n\}$ is a nested collection with range Q_n.

We want to extend Ω_n to a nested collection Ω_{n+1} with range Q_{n+1} and which is defined in the same manner as Ω_n. Define a permutation $\{y_1, \ldots, y_{2^{n+1}}\}$ by the formula

$$\begin{aligned} y_{2i-1} &= x_i \\ y_{2i} &= x_i + 2^n \end{aligned} \qquad \text{for } 1 \leq i \leq 2^n$$

It is clear that $y_i \neq y_j$ for $i \neq j$ so we have a permutation of the integers $0, 1, \ldots, 2^{n+1}-1$. As before define

$$B(k/2^{n+1}) = \bigcup_{i=1}^{k} (y_i + \{m \cdot 2^{n+1} \mid m \varepsilon Z\}) \ .$$

$\Omega_{n+1} = \{B(k/2^{n+1}) \mid k/2^{n+1} \varepsilon Q_{n+1}\}$ is clearly a nested collection with range Q_{n+1}. All that must be checked is that Ω_{n+1} extends Ω_n. Let $t = (k_1/2^n) = (k_2/2^{n+1}) \varepsilon Q_n \cap Q_{n+1}$. This implies that $k_2 = 2 \cdot k_1$ and we have

$$A(t) = \bigcup_{i=1}^{k_1} (x_i + \{m \cdot 2^n \mid m \varepsilon Z\}).$$

Since, for every i,

$$(x_i + \{m \cdot 2^n \mid m \varepsilon Z\}) = (y_{2i-1} + \{m \cdot 2^{n+1} \mid m \varepsilon Z\}) \cup (y_{2i} + \{m \cdot 2^{n+1} \mid m \varepsilon Z\})$$

this implies

$$A(t)= \bigcup_{i=1}^{k_1}((y_{2i-1}+\{m\cdot 2^{n+1}|\ m\varepsilon Z\})\cup(y_{2i}+\{m\cdot 2^{n+1}|\ m\varepsilon Z\}))$$

$$= \bigcup_{i=1}^{2k_1}(y_i+\{m\cdot 2^{n+1}|\ m\varepsilon Z\}) = B(t) .$$

If we let {0,1} be a permutation of the non-negative integers less than 2 and define $A(0)=\emptyset$, $A(1/2)=\{2\cdot m|\ m\varepsilon Z\}$ $A(1)= \{2\cdot m|\ m\varepsilon Z\}\cup\{2\cdot m+1|\ m\varepsilon Z\} = Z$, then by induction we can obtain a nested collection $\Omega=\{A(t)|\ t\varepsilon Q\}$ with range Q.

Since Q is countable and dense in [0,1], Lemma 3.1.2 implies the existence of a nested collection in Z with range [0,1].

<u>NOTE</u> : Combined with Lemma 3.1.1 this implies that any locally compact amenable group G with T, R or Z as a direct factor contains a nested collection with range [0,1] for we must have G/H topologically isomorphic with T, R or Z for some closed normal subgroup H of G.

3.3 <u>NESTED COLLECTIONS IN COMPACT ABELIAN GROUPS</u> :

In this section we show that any infinite compact abelian group has a nested collection with range [0,1]. The following structure theorem will be of prime importance in proving this result. The reader should note that $L^\infty(G)$ admits many invariant means if G is infinite.

<u>THEOREM 3.3 1</u> :([4] Theorem 9.5 p.89) Let G be a compact abelian group and U a neighbourhood of e in G. Then there exists a closed(normal) subgroup H of G with $H\subset U$ and G/H topologically isomorphic with $T^k\times F$ where k is a non-negative integer and F is a finite abelian group.

It is clear that if G contains a closed (normal) subgroup H with G/H topologically isomorphic with $T^k \times F$ as above with $k>0$ then G/H contains a nested collection with range [0,1] and consequently G contains such a collection by Lemma 3.1.1. It is therefore sufficient to consider the case where G has no closed subgroup H for which G/H has T as a direct factor. In this case we use in part a method introduced by Granirer in [2] to obtain the result. In order to apply the method we first require the following

LEMMA 3.3.2 : Let G be an infinite compact abelian group such that there is no closed subgroup H of G for which G/H has T as a direct factor. Then there exists a strictly decreasing sequence $\{H_n\}$ of open-closed subgroups of G such that G/H_n is a finite group for each n.

PROOF : Let $H_1=G$ and assume that H_k are defined for $k \leq n$ satisfying the required properties. Since H_n is open we can find a neighbourhood U of e with $U \subsetneq H_n$ and using Theorem 3.3.1 we obtain a closed subgroup H of G with $H \subset U \subsetneq H_n$ and such that G/H is topologically isomorphic with $T^k \times F$ for some non-negative integer k and finite abelian group F. The conditions on G imply that $k=0$ so G/H is finite. If we let $H_{n+1}=H$ then $H_{n+1} \subsetneq H_n$, G/H_{n+1} is finite and H_{n+1} is open-closed since, for any set $\{a_1=e, a_2, \ldots, a_q\}$ of representatives for G/H_{n+1} we have

$H_{n+1} = G \sim \bigcup_{i=2}^{q} a_i \cdot H_{n+1}$ and $\bigcup_{i=2}^{q} a_i \cdot H_{n+1}$ is closed because H_{n+1} is closed.

The result follows by induction.

This lemma allows us to take care of the remaining case by proving

LEMMA 3.3.3 : Let G be an amenable locally compact group for which there exists a strictly decreasing sequence $\{H_n\}$ of closed normal subgroups such that G/H_n is a finite group for each n. Then G contains a nested collection with range [0,1].

PROOF : Let p_n denote the order of G/H_n for each n. Then $\{p_n\}$ is a strictly increasing sequence of integers (since $H_{n+1} \subsetneq H_n$ implies that $G/H_n \subsetneq G/H_{n+1}$) and p_n divides p_{n+1} for each n (since G/H_n is isomorphic with a subgroup of G/H_{n+1}).

Let $Q_n=\{k/p_n |\ k=1,2,\ldots,p_n\}$ and note that $Q_n \subset Q_{n+1}$ for all n. We will prove this lemma using a more general form of the method employed in Lemma 3.2.1 to show the existence of a nested collection in Z with range [0,1].

For any n, if $\{a_1,\ldots,a_{p_n}\}$ is a set of representatives for G/H_n and we define

$$A(k/p_n) = \bigcup_{i=1}^{k} a_i \cdot H_n \quad \text{for } k/p_n \in Q_n$$

then $\Omega=\{A(t)|\ t\in Q_n\}$ is a nested family of Borel subsets of G. Since $G = \bigcup_{i=1}^{p_n} a_i \cdot H_n$ (disjoint union), therefore

$$1 = \phi(G) = \sum_{i=1}^{p_n} \phi(a_i \cdot H_n) = p_n \cdot \phi(H_n)$$

for any LIM ϕ on $L^{\infty}(G)$ and consequently $\phi(H_n) = (1/p_n)$. So for any LIM ϕ we must have

$$\phi(A(k/p_n)) = \phi(\bigcup_{i=1}^{k} a_i \cdot H_n) = k \cdot \phi(H_n) = (k/p_n)$$

and Ω is a nested collection in G with range Q_n.

If Ω is such a collection we want to find a nested collection Ω' of the same type with range Q_{n+1} which extends Ω.

Let $\{c_1, \ldots, c_m\}$ be a set of representatives for the group H_n/H_{n+1} (which is a subgroup of G/H_{n+1} and therefore finite). Since $p_{n+1} = m \cdot p_n$, for every $1 \leq i \leq p_n$ we can write i uniquely as

$$i = \alpha(i) \cdot m + \beta(i) \text{ where } 0 \leq \alpha(i) < p_n, \ 1 \leq \beta(i) \leq m.$$

Defining $b_i = a_{\alpha(i)+1} \cdot c_{\beta(i)}$ we obtain a collection $\{b_1, \ldots, b_{p_{n+1}}\}$ of elements of G. Suppose $x \varepsilon b_i \cdot H_{n+1} \cap b_j \cdot H_{n+1}$ so $x = a_{\alpha(i)+1} \cdot c_{\beta(i)} \cdot h_1 = a_{\alpha(j)+1} \cdot c_{\beta(j)} \cdot h_2$ for some $h_1, h_2 \varepsilon H_{n+1}$. Since $c_{\beta(i)} \cdot h_1$, $c_{\beta(j)} \cdot h_2 \ \varepsilon H_n$ and the a's are representatives for G/H_n this implies that

$$a_{\alpha(i)+1} = a_{\alpha(j)+1} \quad \text{i.e. } \alpha(i) = \alpha(j).$$

Consequently $c_{\beta(i)} \cdot h_1 = c_{\beta(j)} \cdot h_2$ and since the c's are representatives for H_n/H_{n+1} this implies $c_{\beta(i)} = c_{\beta(j)}$

so $\beta(i)=\beta(j)$. Therefore $i=\alpha(i)\cdot m + \beta(i)$

$$=\alpha(j)\cdot m + \beta(j) = j$$

and we have shown that $b_i\cdot H_{n+1} \cap b_j\cdot H_{n+1} = \emptyset$ for $i\neq j$, hence the b's are a set of representatives for G/H_{n+1}. In the same manner as before we define a nested collection $\Omega'=\{B(t)|\ t\varepsilon Q_{n+1}\}$ with range Q_{n+1} by setting

$$B(k/p_{n+1})= \bigcup_{i=1}^{k} b_i\cdot H_{n+1} \ .$$

To show that Ω' extends Ω let $(k_1/p_n)=(k_2/p_{n+1})\varepsilon Q_{n+1}\cap Q_n$. This implies that $k_2= m\cdot k_1$ where m is the order of H_n/H_{n+1} and note that the manner in which the b_i's were defined gives us

$$\{a_i\cdot c_j|\ 1\leq i\leq k_1,\ 1\leq j\leq m\}=\{b_i|\ 1\leq i\leq k_1\cdot m= k_2\} \ .$$

Therefore $A(k_1/p_n)= \bigcup_{i=1}^{k_1} a_i\cdot H_n = \bigcup_{i=1}^{k_1} a_i\cdot(\bigcup_{j=1}^{m} c_j\cdot H_{n+1})$

$$= \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{m} a_i\cdot c_j\cdot H_{n+1} = \bigcup_{i=1}^{k_2} b_i\cdot H_{n+1}$$

$$= B(k_2/p_{n+1}) \ .$$

It is clear that by choosing any set of representatives for G/H_1 we can obtain a nested collection Ω_1 with range Q_1 by the method employed above. We have shown that any such collection Ω_n with range Q_n can be extended to a nested collection Ω_{n+1} of the same form with range Q_{n+1}. By induction we can obtain a nested collection $\Omega=\{A(t)|\ t\varepsilon Q\}$ with range Q where $Q=\bigcup_{n=1}^{\infty} Q_n$. Q is clearly countable and since H_n is strictly decreasing we have $\lim_{n\to\infty} p_n = \infty$ so Q is dense in [0,1]. The result now follows from Lemma 3.1.2.

We are now able to prove the main result of this section

THEOREM 3.3.4 : Let G be an infinite compact abelian group. Then there exists a nested collection in G with range [0,1].

PROOF : First note that G is amenable because it is abelian. If G contains a closed subgroup H for which $G_1=G/H$ has T as a direct factor then G_1 is topologically isomorphic with T×K for some closed subgroup K, consequently G_1/K is isomorphic with T and by Lemma 3.1.1 there exists a nested collection in G_1 with range [0,1]. Employing Lemma 3.1.1 again this implies that G also contains such a collection.

If, on the other hand, there is no closed subgroup H of G for which G/H has T as a direct factor then the result follows immediately from Lemmas 3.3.2 and 3.3.3 .

3.4 MAIN RESULTS : In order to prove the main result of this chapter we will require the following structure theorem for locally compact, compactly generated abelian groups.

THEOREM 3.4.1 :([4] p.90 Theorem 9.5) Every locally compact, compactly generated abelian group G is topologically isomorphic with $R^m \times Z^n \times F$ for some non-negative integers m and n and some compact abelian group F.

Using this we can now prove

THEOREM 3.4.2 : Let G be an infinite locally compact abelian group. Then there exists a nested collection in G with range [0,1].

PROOF : First consider the case where G is compactly generated. By Theorem 3.4.1 G is isomorphic with $R^m \times Z^n \times F$. If either m or n is non-zero then G has R or Z as a direct factor and G contains a nested collection with range [0,1] by Lemma 3.1.1. Otherwise G must be an infinite compact abelian group which we have shown in Theorem 3.3.4 contains the required family.

If G is not compactly generated, let U be a compact symmetric neighbourhood of e. If $H=\bigcup_{n=1}^{\infty} U^n$ then H is a subgroup of G, H is clearly open and therefore closed as well (see [4] p.34). Since G is not compactly generated, G/H is infinite and clearly abelian. Due to the fact that H is open, {xH} is open in G/H for every coset xH and consequently G/H is a discrete group. As an abelian group, G/H cannot be an "AB group" so by Theorem 2.1.4 there exists a nested collection in G/H with range [0,1] and thus the required collection exists in G by Lemma 3.1.1.

In view of Lemma 3.1.1 we can write this result in a stronger form as

COROLLARY 3.4.3 : Let G be an amenable locally compact group such that there exists a closed normal subgroup H of G for which G/H is infinite and abelian. Then G contains a nested collection with range [0,1].

3.5 REMARKS : We may apply Corollary 3.4.3 to show

LEMMA 3.5.1 : Let G be an infinite amenable non-unimodular locally compact group. Then there exists a nested collection in G with range [0,1].

PROOF : Let $H=\{x\varepsilon G \mid \Delta(x)=1\}$ (where Δ denotes the modular function on G). Then H is a normal subgroup of G since

$$\Delta(xhx^{-1})= \Delta(x)\cdot\Delta(h)\cdot\Delta(x^{-1})= \Delta(h)\cdot\Delta(x)\cdot\Delta(x^{-1})= \Delta(h)= 1$$

whenever $x\varepsilon G$ and $h\varepsilon H$. Also H is closed because Δ is continuous. The factor group G/H is abelian (as a subset of the positive multiplicative reals) and infinite since the range of Δ is infinite when G is non-unimodular. Consequently, by Corollary 3.4.3, there exists a nested collection in G with range [0,1].

CHAPTER IV

AMENABLE ACTIONS OF LOCALLY COMPACT GROUPS

4.0 INTRODUCTION : If G is a locally compact group and X a locally compact space then G is said to act continuously on X if there exists a jointly continuous mapping from $G \times X \to X$ denoted by $(g,x) \to gx$ which satisfies $g_1(g_2x)=(g_1g_2)x$ for any $g_1, g_2 \varepsilon G$ and $x \varepsilon X$. For example G acts continuously on the locally compact space G/H of left cosets of G with respect to a closed subgroup H if we define $g(xH)=gxH$ for $g \varepsilon G$ and $xH \varepsilon G/H$ (Note: for convenience of notatation, in the remainder of this chapter the coset xH will be denoted by $\dot{x}$).

In studying invariant means on a locally compact group G we are dealing with the case in which G acts on itself by left multiplication. There has been interest recently in studying the action of groups on certain locally compact spaces for the purpose of extending the concept of amenability to these more general situations. In [11] Greenleaf introduced the concept of the amenable action of a locally compact group on a locally compact space which supports a quasi-invariant measure and Eymard in [9] studied in detail the amenable action of a group on its space of left cosets with respect to a closed subgroup.

In Section 4.1 we examine the concept of a quasi-invariant measure on the space G/H of left cosets of a locally compact group G, particularly the relationship between such measures and the left Haar measure on G. The results presented in this section are well known and can be found, for example, in [7] and [10]. However the proofs given in these references are somewhat sketchy and the results themselves are presented in the midst of a great deal of material which is of little importance in the situation we are examining. We present proofs for some of these results since we feel that a thorough understanding of the relationships involved provides a much greater feeling for the material that is presented in the remainder of this chapter.

In section 4.2 we introduce a definition of the amenable action of a group on one of its coset spaces which is equivalent to the definition used by Greenleaf. However the technical details are very different and we feel that it is much easier to understand the relationship between our definition of amenable action and the concept of an amenable group in the usual sense.

We also present proofs of various results which have been obtained by Greenleaf and Eymard. In many cases the methods of proof used in this section are much more closely allied with those used to prove similar theorems for amenable locally compact groups than are those employed in [11].

4.1 QUASI-INVARIANT MEASURES : Let G be a locally compact group acting continuously on a locally compact space X. If μ is a measure on X define, for $g \varepsilon G$, the measure $\gamma(g)\mu$ on X

by $\gamma(g)\mu(A)=\mu(g^{-1}A)$ where $g^{-1}A=\{x\varepsilon X|\ gx\varepsilon A\}$. Note that this implies

$$\int_X f(gx)d\mu(x)=\int_X f(x)d(\gamma(g)\mu)(x)$$

for all $f\varepsilon K(X)$ (the continuous functions on X with compact support).

<u>DEFINITION 4.1.1</u> : i) If $\gamma(g)\mu=\mu$ for every $g\varepsilon G$ then μ is said to be <u>invariant</u> under the action of G

ii) If $\gamma(g)\mu$ is proportional to μ for every $g\varepsilon G$ then μ is said to be <u>relatively invariant</u> under the action of G.

iii) If $\gamma(g)\mu\equiv\mu$ (i.e. $\gamma(g)\mu$ is equivalent to μ) for every $g\varepsilon G$ then μ is said to be <u>quasi-invariant</u> under the action of G.

<u>REMARK</u> : For a relatively invariant measure μ, $\gamma(g)\mu$ is proportional to μ for each $g\varepsilon G$ so there exists a real constant $\alpha(g)$ with $\gamma(g)\mu=\alpha(g)\cdot\mu$. The mapping $\alpha:G\to R$ is called the <u>multiplier</u> of μ. It is clear that $\alpha(x)>0$ and $\alpha(xy)=\alpha(x)\alpha(y)$ for all $x,y\varepsilon G$. Also for any measure μ on X and $f\varepsilon K(X)$, $\int_X f(gx)d\mu(x)$ is a continuous function of g so α is a continuous function.

<u>EXAMPLES</u> : Let G=X=Z in the above context so we are considering the additive group of the integers acting on itself. If we let $\mu(A)=\text{card}(A)$ for $A\subset Z$ (where card(A) denotes the cardinality of A) then μ is a measure and $\mu(m+A)=\mu(A)$ for any $m\varepsilon Z$ consequently μ is invariant. In fact μ is a Haar measure on the locally compact discrete group Z.

If we define $\lambda(A)=\Sigma\{\exp(n)|\ n\varepsilon A\}$ then λ is a relatively invariant measure since $\lambda(m+A)=\exp(m)\cdot\lambda(A)$ for any $m\varepsilon Z$ and $A\subset Z$ but λ is clearly not invariant.

If $\upsilon(A)=\Sigma\{\exp(|n|)|\ n\varepsilon A\}$ then υ is a quasi-invariant measure since the only υ null set is Ø but υ is not relatively invariant (if $A=\{-1\}$ and $B=\{1\}$ then $\upsilon(A)=\upsilon(B)$ but $\upsilon(1+A)=1$ and $\upsilon(1+B)= e^2$so $\gamma(-1)\upsilon$ is not proportional to υ).

In the remainder of this chapter we will restrict ourselves to the case of a locally compact group G acting on the locally compact space G/H of left cosets of G with respect to a closed (not necessarily normal) subgroup H of G.(Note that such an action is necessarily continuous) We will denote by <u>λ a fixed left Haar measure on G</u> and by <u>β a fixed left Haar measure on H</u>. A thorough treatment of quasi-invariant measures on such coset spaces may be found in [7] and [10]. In this section we first present, without proofs, four key theorems which indicate the relationship between quasi-invariant measues on G/H and the measures λ and β mentioned above. Using these results we then develop further properties of quasi-invariant, relatively invariant and invariant measures on G/H which are needed in our treatment of the concept of amenability for G/H.

The key theorems we require are the following :

THEOREM 4.1.2 : Let υ be a measure on G. Then υ is quasi-invariant if and only if υ is equivalent to λ.

The proof of this theorem which completely characterizes the quasi-invariant measures on G can be found in [10] Chapter VII 1.9.

The next result provides a relationship between measures on G/H and those on G which allows us to examine the invariance properties of a measure on G/H by examining the corresponding measure on G.

THEOREM 4.1.3 : i) For $f \in K(G)$ define Tf on G/H by the formula $Tf(\dot{x}) = \int_H f(xh)d\beta(h)$ (Note that the value of the integral is independent of the choice of representative from $\dot{x}$ since β is a Haar integral on H). Then $Tf \in K(G/H)$ and T is a surjective linear mapping from K(G) onto K(G/H).

ii) Let υ be a measure on G/H. Then there exists a unique measure $\upsilon^{\#}$ on G such that

$$\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x}) = \int_G f(x)d\upsilon^{\#}(x) \text{ for every } f \in K(G)$$

The proof of this result can be found in [10] Chapter VII section 2 p.43.

Now that we have defined the measure $\upsilon^{\#}$ on G it is possible to extend the function T given in part i) above and obtain

THEOREM 4.1.4 : Let υ be a measure on G/H

i) If $f \in L^1(G,\upsilon^{\#})$ then the set of $\dot{x} \in G/H$ such that $h \to f(xh)$ is not β measurable is υ null; the function

Tf on G/H defined almost everywhere by the formula

$$Tf(\dot{x}) = \int_H f(xh)d\beta(h) \text{ is } \upsilon \text{ integrable and}$$
$$\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x}) = \int_G f(x)d\upsilon^{\#}(x)$$

ii) T is a continuous linear mapping of $L^1(G,\upsilon^{\#})$ onto $L^1(G/H,\upsilon)$ with $\|Tf\|_{1,\upsilon} \leq \|f\|_{1,\upsilon^{\#}}$ for all $f \in L^1(G,\upsilon^{\#})$.

A proof of this result can be found in [10] Chapter VII sections 3 and 4.

THEOREM 4.1.5 : Let υ be a measure on G/H. A υ-measurable set A in G/H is υ null if and only if $\Pi_H^{-1}(A)$ is $\upsilon^{\#}$ null in G.

See [10] Chapter VII Section 2.3 for a proof.

Since a measure is quasi-invariant if and only if its null sets are invariant under the action of G, Theorem 4.1.5 will be extremely useful as we attempt to characterize the quasi-invariant measures on G/H. One such characterization is given by

THEOREM 4.1.6 : Let υ be a measure on G/H. Then the following are equivalent

i) υ is quasi-invariant

ii) $\upsilon^{\#} \equiv \lambda$ (i.e. $\upsilon^{\#}$ is equivalent to λ)

iii) A is υ-null in G/H if and only if $\Pi_H^{-1}(A)$ is λ null in G.

PROOF : i) $\Rightarrow$ ii) First note that if $\upsilon_1 \ll \upsilon_2$ on G/H then $\upsilon_1^{\#} \ll \upsilon_2^{\#}$ on G. This can be seen as follows-

choose $A \subset G$ with $\upsilon_2^{\#}(A)=0$. By Theorem 4.1.4

$\int_{G/H} T\chi_A(\dot{x})d\upsilon_2(\dot{x}) = \upsilon_2^{\#}(A)=0$ and since $T\chi_A$ is non-negative this implies that $T\chi_A=0$ υ_2-almost everywhere in G/H. Since $\upsilon_1 << \upsilon_2$ implies that $T\chi_A=0$ υ_1-almost everywhere in G/H we have

$$\upsilon_1^{\#}(A) = \int_G \chi_A(x)d\upsilon_1^{\#}(x) = \int_G T\chi_A(\dot{x})d\upsilon_1(\dot{x})=0 \text{ and } \upsilon_1^{\#} << \upsilon_2^{\#}.$$

A consequence of this is the fact that $\upsilon_1^{\#}$ and $\upsilon_2^{\#}$ are equivalent if υ_1 and υ_2 are equivalent.

Let υ be quasi-invariant on G/H and note that for $f \varepsilon K(G)$ and $g \varepsilon G$ we have

$$\begin{aligned}
\int_G f(x)d(\gamma(g)\upsilon^{\#})(x) &= \int_G f(gx)d\upsilon^{\#}(x) \\
&= \int_{G/H}(\int_H f(gxh)d\beta(h))d\upsilon(\dot{x}) \\
&= \int_{G/H} Tf(g\dot{x})d\upsilon(\dot{x}) \\
&= \int_{G/H} Tf(\dot{x})d(\gamma(g)\upsilon)(\dot{x}) \\
&= \int_G f(x)d((\gamma(g)\upsilon)^{\#})(x)
\end{aligned}$$

so $\gamma(g)\upsilon^{\#}=(\gamma(g)\upsilon)^{\#}$ and, since $\gamma(g)\upsilon$ is equivalent to υ for every $g\varepsilon G$, this implies $\gamma(g)\upsilon^{\#}$ is equivalent to $\upsilon^{\#}$ i.e. $\upsilon^{\#}$ is quasi-invariant on G. The fact that $\upsilon^{\#}$ is equivalent to λ follows from Theorem 4.1.2.

iii) ⇒ i) $\quad \gamma(g)\upsilon(A)=\upsilon(g^{-1}A)=0$

iff $0=\lambda(\Pi_H^{-1}(g^{-1}A))=\lambda(g^{-1}\Pi_H^{-1}(A))=\lambda(\Pi_H^{-1}(A))$

(since λ is invariant)

iff $\upsilon(A)=0$ $\quad$ so $\gamma(g)\upsilon \equiv \upsilon$ and υ is quasi-invariant

ii) ⇒ iii) $\quad$ Assume $\upsilon^{\#} \equiv \lambda$. Then A is υ null in G/H

iff $\Pi_H^{-1}(A)$ is $\upsilon^{\#}$ null in G (by Theorem 4.1.5)
iff $\Pi_H^{-1}(A)$ is λ null in G .

The next theorem provides an extremely useful characterization of the quasi-invariant measures on G/H which we will use later to prove that G/H always admits such a measure.

THEOREM 4.1.7 : If υ is a quasi-invariant measure on G/H then there exists a strictly positive measurable function ρ on G satisfying

(*) for each $h \varepsilon H$, $\rho(xh)=\rho(x)\Delta_H(h)/\Delta_G(h)$ for λ almost all $x \varepsilon G$

and for which $\upsilon^{\#}=\rho\cdot\lambda$.

Conversely if ρ is a strictly positive measurable function on G satisfying (*) then $\rho\cdot\lambda=\upsilon^{\#}$ for some quasi-invariant measure υ on G/H.

PROOF : Assume that υ is quasi-invariant on G/H. By Theorem 4.1.6 $\upsilon^{\#}$ is equivalent to λ so applying the Radon Nikodym Theorem for locally compact spaces (see [4] p.144 Theorem 12.17) there exists a strictly positive measurable function ρ on G such that $\upsilon^{\#}=\rho\cdot\lambda$.

Choose $f \varepsilon K(G)$, $h_o \varepsilon H$ and let $g \varepsilon K(G)$ be defined by $g(x)=f(xh_o^{-1})$. Then

$$\int_G g(x)\rho(x)d\lambda(x)= \int_G f(xh_o^{-1})\rho(x)d\lambda(x)$$

$$= \int_G f(x)\Delta_G(h_o)\rho(xh_o)d\lambda(x) \quad \text{(replacing x by } xh_o)$$

Also $\mathbf{T}g(\dot{x})= \int_H g(xh)d\beta(h)= \int_H f(xhh_o^{-1})d\beta(h)$

$= \Delta_H(h_o) \int_H f(xh)d\beta(h)$ (replacing h by hh_o)

$= \Delta_H(h_o)\ Tf(\dot{x})$

Consequently $\int_G f(x)\Delta_G(h_o)\rho(xh_o)d\lambda(x) = \int_G g(x)\rho(x)d\lambda(x)$

$$= \int_G g(x)d\upsilon^{\#}(x)$$

$$= \int_{G/H} Tg(\dot{x})d\upsilon(\dot{x})$$

$$= \int_{G/H} \Delta_H(h_o)Tf(\dot{x})d\upsilon(\dot{x})$$

$$= \int_G \Delta_H(h_o)f(x)d\upsilon^{\#}(x)$$

$$= \int_G \Delta_H(h_o)f(x)\rho(x)d\lambda(x) \quad .$$

Since $f\varepsilon K(G)$ was arbitrary this implies

$$\Delta_G(h_o)\rho(xh_o) = \Delta_H(h_o)\rho(x)$$

for λ almost all $x\varepsilon G$ hence ρ satisfies (*).

Given a strictly positive measurable function ρ satisfying (*) we want to define a quasi-invariant measure υ on G/H with $\upsilon^{\#} = \rho\cdot\lambda$. The obvious way to do this is to set

$$\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x}) = \int_G f(x)\rho(x)d\lambda(x) \text{ for each } f\varepsilon K(G).$$

Since T: $K(G)\to K(G/H)$ is surjective and linear this formula will define a measure υ on G/H provided that the left hand side is well-defined for each $Tf\varepsilon K(G/H)$.

To show that υ is well-defined we choose $f\varepsilon K(G)$ with $Tf=0$ and show that $\int_G f(x)\rho(x)d\lambda(x)=0$. Let $g\varepsilon K(G)$ be such that $Tg(\dot{x}) = \int_H g(xh)d\beta(h)=1$ for all $x\varepsilon \mathrm{supp}(f)$ (this is possible due to Theorem 4.1.3 since $\Pi_H(\mathrm{supp}(f))$ is compact in G/H and T maps K(G) onto K(G/H)). We then have

$$\int_G f(x)\rho(x)d\lambda(x) = \int_G f(x)(\int_H g(xh)d\beta(h))\rho(x)d\lambda(x)$$

$$= \int_H(\int_G f(x)g(xh)\rho(x)d\lambda(x))d\beta(h) \quad \text{by Fubini's Thm.}$$

$$= \int_H(\int_G \Delta_G(h^{-1})f(xh^{-1})g(x)\rho(xh^{-1})d\lambda(x))d\beta(h)$$

$$\text{(replacing x by } xh^{-1})$$

$$= \int_H(\int_G \Delta_H(h^{-1})f(xh^{-1})g(x)\rho(x)d\lambda(x))d\beta(h) \quad \text{(using *)}$$

$$= \int_G g(x)\rho(x)(\int_H \Delta_H(h^{-1})f(xh^{-1})d\beta(h))d\lambda(x)$$

$$= \int_G g(x)\rho(x)(\int_H f(xh)d\beta(h))d\lambda(x) \quad \text{(replacing h by } h^{-1})$$

$$= 0 \quad \text{since } Tf=0 \text{ implies } \int_H f(xh)d\beta(h)=0 \text{ for } \lambda \text{ a.a. } x\in G.$$

Therefore υ is well-defined on G/H and

$$\int_G f(x)d\upsilon^{\#}(x) = \int_{G/H} Tf(\dot{x})d\upsilon(\dot{x}) = \int_G f(x)\rho(x)d\lambda(x)$$

for all $f\in K(G)$ hence $\upsilon^{\#}=\rho\cdot\lambda$. Since ρ is strictly positive we have $\rho\cdot\lambda$ equivalent to λ consequently $\upsilon^{\#}\equiv\lambda$ and υ is quasi-invariant by Theorem 4.1.6.

We can now use this result to obtain a similar characterization of the relatively invariant and invariant measures on G/H. The following appears in part as Theorems 15.22, 15.24 of [4].

<u>THEOREM 4.1.8</u> : If υ is a relatively invariant measure on G/H then there exists a strictly positive <u>continuous</u> function ρ on G satisfying

$$(**) \quad \rho(xy)=\rho(x)\rho(y)/a \text{ for all } x,y\in G$$

$$\rho(h) = a\cdot\Delta_H(h)/\Delta_G(h) \text{ for all } h\in H$$

where $a>0$ is some constant, and for which $\upsilon^{\#}=\rho\cdot\lambda$.

Conversely if ρ is a strictly positive continuous function on G satisfying (**) then $\rho\cdot\lambda=\upsilon^{\#}$ for some relatively invariant measure υ on G/H.

PROOF : Assume υ is a relatively invariant measure on G/H with multiplier α i.e. for every $y\varepsilon G$

$$\int_{G/H} Tf(y\dot{x})d\upsilon(\dot{x})= \alpha(y)\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x}) \text{ for all } Tf\varepsilon K(G/H)$$

For any $f\varepsilon K(G)$ we have

$$\int_G f(yx)d\upsilon^{\#}(x)= \int_{G/H} Tf(y\dot{x})d\upsilon(\dot{x})$$

$$= \alpha(y)\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x})= \alpha(y)\int_G f(x)d\upsilon^{\#}(x)$$

so $\upsilon^{\#}$ is relatively invariant on G with multiplier α. Let $\mu=\alpha\cdot\upsilon^{\#}$. For any $g\varepsilon G$ and $f\varepsilon K(G)$

$$\int_G f(gx)d\mu(x)= \int_G f(gx)\alpha(x)d\upsilon^{\#}(x)$$

$$=(1/\alpha(g))\int_G f(gx)\alpha(gx)d\upsilon^{\#}(x) \quad \text{since } \alpha(x)=\alpha(gx)/\alpha(g)$$

$$= \int_G f(x)\alpha(x)d\upsilon^{\#}(x) \quad \text{since } \upsilon^{\#} \text{ is relatively invariant}$$

$$= \int_G f(x)d\mu(x)$$

Therefore μ is an invariant measure on G so there exists a constant $a>0$ for which $\mu=a\cdot\lambda$ (due to the uniqueness of Haar measure).

By Theorem 4.1.7 there exists a strictly positive measurable function ρ satisfying (*) for which $\upsilon^{\#}=\rho\cdot\lambda$. Thus we have $\alpha\cdot\rho\cdot\lambda= \alpha\upsilon^{\#}=\mu=a\cdot\lambda$ which implies that $\alpha\cdot\rho=a$ λ almost everywhere in G i.e. $\rho=a/\alpha$ almost everywhere. Therefore we can assume that ρ is continuous (due to the continuity of α) and equal to a/α which implies that

$\rho(xy)= a/\alpha(xy)= a/(\alpha(x)\alpha(y))=\rho(x)\rho(y)/a$ for all $x,y\varepsilon G$ and since ρ satisfies (*), for any $h\varepsilon H, x\varepsilon G$ we have

$\rho(x)\Delta_H(h)/\Delta_G(h)=\rho(xh)=\rho(x)\rho(h)/a$ so $\rho(x)>0$ implies $\rho(h)= a\cdot\Delta_H(h)/\Delta_G(h)$.

Conversely assume that we are given a strictly positive continuous function ρ on G which satisfies (**). Then

$$\rho(xh)=\rho(x)\rho(h)/a=\rho(x)(1/a)\cdot a\cdot\Delta_H(h)/\Delta_G(h)=\rho(x)\Delta_H(h)/\Delta_G(h)$$

for all $x\varepsilon G, h\varepsilon H$. Since ρ also satisfies (*) of Theorem 4.1.7 there exists a measure υ on G/H with $\upsilon^{\#}=\rho\cdot\lambda$.

For any $f\varepsilon K(G)$ and $g\varepsilon G$ we have

$$\int_{G/H} Tf(g\dot{x})d\upsilon(\dot{x})= \int_G f(gx)d\upsilon^{\#}(x)= \int_G f(gx)\rho(x)d\lambda(x)$$

$$=(1/\rho(g))\int_G f(gx)\rho(gx)d\lambda(x) \quad \text{since } \rho(x)=\rho(gx)/\rho(g)$$

$$=(1/\rho(g))\int_G f(x)\rho(x)d\lambda(x) \quad \text{since } \lambda \text{ is invariant}$$

$$=(1/\rho(g))\int_G f(x)d\upsilon^{\#}(x)$$

$$=(1/\rho(g))\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x})$$

consequently υ is relatively invariant with multiplier $1/\rho$.

<u>COROLLARY 4.1.9</u> : There exists an invariant measure υ on G/H if and only if $\Delta_G(h)=\Delta_H(h)$ for all $h\varepsilon H$. Furthermore if υ is invariant then every invariant measure on G/H is of the form $b\cdot\upsilon$ for some constant $b>0$.

<u>PROOF</u> : Assume that υ is an invariant measure on G/H. Then υ is relatively invariant with multiplier $\alpha(y)=1$ for every $y\varepsilon G$. Since the strictly positive continuous ρ for which $\upsilon^{\#}=\rho\cdot\lambda$ equals a/α this implies that $\rho(x)=a$ for

all $x\varepsilon G$ hence $a=\rho(h)=a\cdot\Delta_H(h)/\Delta_G(h)$ for all $h\varepsilon H$ and therefore $\Delta_G(h)=\Delta_H(h)$.

If $\Delta_G(h)=\Delta_H(h)$ for all $h\varepsilon H$, let $a>0$ and define $\rho(x)=a$ for all $x\varepsilon G$. Then ρ satisfies (**) and there exists a relatively invariant measure υ on G/H with $\upsilon^{\#}=a\cdot\lambda$. For $Tf\varepsilon K(G/H)$ and $g\varepsilon G$ we have

$$\int_{G/H} Tf(g\dot{x})d\upsilon(\dot{x}) = \int_G f(gx)d\upsilon^{\#}(x) = \int_G f(gx)\cdot a\ d\lambda(x)$$
$$= \int_G f(x)\cdot a\ d\lambda(x) = \int_G f(x)d\upsilon^{\#}(x)$$
$$= \int_{G/H} Tf(\dot{x})d\upsilon(\dot{x})$$

and υ is invariant.

If υ_1 and υ_2 are invariant measures on G/H then $\upsilon_1^{\#}=a_1\cdot\lambda$ and $\upsilon_2^{\#}=a_2\cdot\lambda$ for some positive constants a_1 and a_2. This implies that $\upsilon_1^{\#}=(a_1/a_2)\cdot\upsilon_2^{\#}$ and it is easily checked that this gives $\upsilon_1=(a_1/a_2)\cdot\upsilon_2=b\cdot\upsilon_2$ where $b=a_1/a_2>0$.

We now apply Theorem 4.1.7 to prove

<u>THEOREM 4.1.10</u> : There exists a quasi-invariant measure on G/H.

<u>PROOF</u> : Choose any $f\neq 0$ in $K^+(G)$ and let $A=\{x\varepsilon G|\ f(x)>0\}$ so $\bar{A}$ is compact and A is open. By means of a purely topological argument (see [7] p.161 for details) there exists a subset Y of G such that the family $\{AyH\}_{y\varepsilon Y}$ covers G and is locally finite (i.e. every point of G has a neighbourhood which meets at most finitely many of the sets AyH). For each $y\varepsilon G$ define the function $f_y\varepsilon K(G)$ by

$$f_y(x)=f(xy^{-1}) \text{ for every } x\varepsilon G$$

and let $F(x)=\Sigma_{y\varepsilon Y}f_y(x)$. Since x is an element of at most finitely many of the sets $\{Ay\}_{y\varepsilon Y}$, F is well-defined as a finite sum and continuous since $\{Ay\}_{y\varepsilon Y}$ is locally finite. Define $\rho(x)= (\int_H F(xh)/\Delta_H(h)d\beta(h))/\Delta_G(x)$. Since $\{AyH\}_{y\varepsilon Y}$ covers G, for each $x\varepsilon G$ there is some $h\varepsilon H$ and $y\varepsilon Y$ with $xhy^{-1}\varepsilon A$ hence $F(xh)>0$. Since F is continuous this implies that $\rho(x)>0$ and ρ is clearly continuous as a quotient of continuous functions. If $h_o\varepsilon H$ then

$$\rho(xh_o)=(\int_H(F(xh_oh)/\Delta_H(h))d\beta(h))/\Delta_G(xh_o)$$

$$=(\int_H(F(xh)/\Delta_H(h_o^{-1}h))d\beta(h))/\Delta_G(xh_o) \quad \text{replacing } h \text{ by } h_o^{-1}h$$

$$=(\Delta_H(h_o)/\Delta_G(h_o)(\int_H(F(xh)/\Delta_H(h))d\beta(h))/\Delta_G(x)$$

$$=(\Delta_H(h_o)/\Delta_G(h_o))\cdot\rho(x)$$

Since ρ satisfies (*) of Theorem 4.1.7 there exists a quasi-invariant measure υ on G/H with $\upsilon^{\#}=\rho\cdot\lambda$.

<u>REMARK</u> : For the function F defined above, if we set

$$F_1(x)= \int_H F(xh)d\beta(h)$$

then F_1 is continuous and strictly positive and if we define $k_o(x)=F(x)/F_1(x)$ for each $x\varepsilon G$ we have a non-negative continuous function k_o on G such that for any compact set B in G, k_o coincides on BH with a function in $K^+(G)$ and $Tk_o(\dot{x})= \int_H k_o(xh)d\beta(h) = 1$ for all $x\varepsilon G$.
If υ is a quasi-invariant measure on G/H and $g\varepsilon L^1(G/H,\upsilon)$ it can be verified that the function $g'=k_o\cdot(g\circ\Pi_H)$ is in $L^1(G,\upsilon^{\#})$ and clearly $Tg'=g$. This is the method which is used in the proof of Thm. 4.1.4 to show that T is surjective.

4.2 <u>AMENABLE ACTIONS OF G ON G/H</u> : In defining the concepts of invariant and topological invariant means on $L^\infty(G)$ for a locally compact group G, strong use is made of the convolution operations which provide continuous bilinear mappings from $M(G)\times L^1(G)\to L^1(G)$ and $M(G)\times L^\infty(G)\to L^\infty(G)$ (where M(G) denotes the space of finite Radon measures on G). In order to extend the definitions to include the case where G acts on the space G/H of left cosets of G with respect to a closed subgroup H, we must define similar operations from $M(G)\times L^1(G/H)\to L^1(G/H)$ and $M(G)\times L^\infty(G/H)\to L^\infty(G/H)$ which coincide with convolution in the case that $H=\{e\}$ and $G/H=G$. The operations we define are equivalent to those introduced by Greenleaf in [11] however our approach is different due to the fact that we rely heavily on the material developed in Section 4.1 which allows for a more explicit definition of the required operations.

In view of Theorem 4.1.10 we know that there exist quasi-invariant measures on G/H and any two such measures are equivalent, since by Theorem 4.1.6 the null sets of a quasi-invariant measure on G/H are precisely the subsets A of G/H for which $\Pi_H^{-1}(A)$ is λ null in G. For the remainder of this chapter <u>υ will denote a fixed quasi-invariant measure on G/H</u> and, since we will be examining the space $L^\infty(G/H,\upsilon)$ of υ essentially bounded Borel measurable functions on G/H, our results will hold for any other quasi-invariant measure because $L^\infty(G/H)$ does not depend on υ.

For the quasi-invariant measure υ we have chosen, we know, by Theorem 4.1.7, that there exists a strictly positive measurable function ρ on G with $\upsilon^{\#}=\rho\cdot\lambda$. In view of Theorem 4.1.10 establishing the existence of a quasi-invariant measure on G/H, we will assume that υ has been chosen so that ρ is also continuous with

$$\rho(xh)= (\rho(x)\cdot\Delta_H(h))/\Delta_G(h) \quad \text{for all } x\varepsilon G,\ h\varepsilon H .$$

For $f\varepsilon L^1(G,\lambda)$ let $Tf=T(f/\rho)$ (with T defined as in Theorem 4.1.4). Since

$$\int_G (f(x)/\rho(x))d\upsilon^{\#}(x)= \int_G f(x)d\lambda(x)$$ the function

(f/ρ) is in $L^1(G,\upsilon^{\#})$ consequently $Tf\varepsilon L^1(G/H,\upsilon)$ with

$$Tf(\dot{x})= \int_H (f(xh)/\rho(xh))d\beta(h) \quad \text{almost everywhere}$$

in G/H. Since T is continuous, linear and surjective, T is also a continuous linear surjection from $L^1(G,\lambda)$ onto $L^1(G/H,\upsilon)$ with $\|Tf\|_{1,\upsilon}=\|T(f/\rho)\|_{1,\upsilon}\leq \|f/\rho\|_{1,\upsilon^{\#}} = \|f\|_{1,\lambda}$. Therefore $L^1(G/H,\upsilon)$ (which will be referred to henceforth as $L^1(G/H)$) is isometrically isomorphic with $L^1(G)/J(T)$ - where $J(T)=\{f\varepsilon L^1(G)|\ Tf=0\}$ denotes the kernel of T. It is shown in [7] Chapter 8 Section 2.5 that J(T) can be characterized as the closed linear span of

$$\{A_h f-f|\ h\varepsilon H,\ f\varepsilon K(G)\} \text{ where } A_h f(x)= f(xh)\cdot\Delta_G(h)$$

for all $x\varepsilon G$.

In working with topological invariant means on $L^{\infty}(G)$ for a locally compact group G reference is often made to the set $P(G)=\{f\varepsilon L^1(G)|\ f\geq 0,\ \int_G f(x)d\lambda(x)= 1\}$. In the material that follows we will refer to

$$P(G/H)=\{f\varepsilon L^1(G/H) \mid f\geq 0, \int_{G/H} f(\dot{x})d\upsilon(\dot{x})= 1\}.$$

If $f\varepsilon P(G)$ then $Tf\geq 0$ and $\int_{G/H} Tf(\dot{x})d\upsilon(\dot{x})= \int_G f(x)d\lambda(x)= 1$ so $Tf\varepsilon P(G/H)$. Let $g\varepsilon P(G/H)$ and let k_0 be the non-negative continuous function on G defined in the remark following Theorem 4.1.10. If $g'= k_0\cdot(f\circ\Pi_H)$ then $g'\varepsilon L^1(G,\upsilon^{\#})$ and $Tg'=g$. If $\bar{g}$ is defined on G by $\bar{g}= g'\cdot\rho$ then $\bar{g}\geq 0$ since k_0, f, ρ and Π_H are non-negative. Also

$$\int_G \bar{g}(x)d\lambda(x)= \int_G g'(x)\rho(x)d\lambda(x)= \int_G g'(x)d\upsilon^{\#}(x)$$

$$= \int_{G/H} Tg'(\dot{x})d\upsilon(\dot{x})= \int_{G/H} g(\dot{x})d\upsilon(\dot{x})= 1$$

(since $g\varepsilon P(G/H)$), consequently $\bar{g}\varepsilon P(G)$ with

$T\bar{g}= T(\bar{g}/\rho)= T(g')= g$ and T maps P(G) onto P(G/H).

The operator T provides a convenient method for working with functions in $L^1(G/H)$ by dealing with their pre-images in $L^1(G)$. The following lemma provides a similar relationship between functions in $L^\infty(G/H)$ and $L^\infty(G)$.

<u>LEMMA 4.2.1</u> : Let $f\varepsilon L^\infty(G/H)$ and define $\hat{f}$ on G by $\hat{f} = f\circ\Pi_H$. Then $\hat{f}\varepsilon L^\infty(G)$ with $\|\hat{f}\|_\infty= \|f\|_\infty$.

Furthermore if $g\varepsilon L^\infty(G)$ is such that for every $h\varepsilon H$

$$g(xh)= g(x) \text{ for almost all } x\varepsilon G$$

there exists a function $\bar{g}\varepsilon L^\infty(G/H)$ such that $\hat{\bar{g}} = g$.

<u>PROOF</u> : Since Π_H is continuous and f is measurable, we have f measurable on G. By Theorem 4.1.6 a set A in G/H is υ-null if and only if $\Pi_H^{-1}(A)$ is λ-null in G.

For a given real number α let

$$A_\alpha = \{\dot{x} \in G/H : |f(\dot{x})| > \alpha\} \text{ so we have}$$

$$\Pi_H^{-1}(A_\alpha) = \{x \in G : |\hat{f}(x)| > \alpha\} .$$

Since $\upsilon(A_\alpha) = 0$ if and only if $\lambda(\Pi_H^{-1}(A_\alpha)) = 0$ this implies that $\|f\|_\infty = \|\hat{f}\|_\infty$ consequently the mapping $f \to \hat{f}$ is an isometry from $L^\infty(G/H)$ into $L^\infty(G)$.

Let $g \in L^\infty(G)$ be such that, for every $h \in H$, $g(xh) = g(x)$ for almost all $x \in G$. First note that this property is independent of the choice of representative from the equivalence class denoted by g. This can be seen as follows : let $g_1, g_2 \in BM(G)$ with $g_1 = g_2$ almost everywhere and such that, for every $h \in H$, $g_1(xh) = g_1(x)$ for almost all $x \in G$. Let $h \in H$ be fixed and let

$$N = \{x \in G \mid g_1(x) \neq g_2(x)\}, \quad A = \{x \in G \mid g_1(xh) \neq g_1(x)\}$$

so $\lambda(N) = \lambda(A) = \lambda(Nh^{-1}) = 0$. If $B = A \cup N \cup Nh^{-1}$ then $\lambda(B) = 0$ and for $x \notin B$ we have

$$g_2(xh) = g_1(xh) = g_1(x) = g_2(x) \text{ since } xh \notin N,\ x \notin A \text{ and } x \notin N$$

consequently $g_2(xh) = g_2(x)$ almost everywhere and the property is independent on the choice of representative.

Choosing $h_0 \in H$ and $f \in K(G)$ we have

$$\int_G f(xh_0)\Delta_G(h_0)g(x)\,d\lambda(x)$$

$$= \int_G f(x)g(xh_0^{-1})\,d\lambda(x) \qquad \text{replacing } x \text{ by } xh_0^{-1}$$

$$= \int_G f(x)g(x)\,d\lambda(x) \qquad \text{since } g(x) = g(xh_0^{-1}) \text{ a.e.}$$

Therefore $\int_G (A_{h_0}f - f)(x)g(x)\,d\lambda(x) =$

$$= \int_G (f(xh_0)\Delta_G(h_0) - f(x))g(x)\,d\lambda(x) = 0$$

for all $h_0 \in H$, $f \in K(G)$. Since $J(T)$ is the closed linear span of $\{A_h f - f \mid f \in K(G),\ h \in H\}$ and $f \to \int_G f(x)g(x)d\lambda(x)$ is continuous, we have $\int_G f(x)g(x)d\lambda(x) = 0$ for all $f \in J(T)$.

If we define $\Phi(f) = \int_G f(x)g(x)d\lambda(x)$ for $f \in L^1(G)$ then Φ is a continuous linear functional on $L^1(G)$ which vanishes on $J(T)$. Since $L^1(G/H) \simeq L^1(G)/J(T)$ we can define the continuous linear functional Φ' on $L^1(G/H)$ by setting

$$\Phi'(Tf) = \Phi(f) \text{ for } Tf \in L^1(G/H) .$$

Since $L^1(G/H)^* = L^\infty(G/H)$ (see [4] p.148 Theorem 12.18) there exists a function $\bar{g} \in L^\infty(G/H)$ such that

$$\Phi'(Tf) = \int_{G/H} Tf(\dot{x})\bar{g}(\dot{x})d\upsilon(\dot{x}) .$$

Since
$$T(f\cdot\hat{\bar{g}})(\dot{x}) = \int_H (f(xh)\hat{\bar{g}}(xh)/\rho(xh))d\beta(h)$$
$$= \int_H (f(xh)\bar{g}(\dot{x})/\rho(xh))d\beta(h) = \bar{g}(\dot{x})Tf(\dot{x})$$

this implies that

$$\int_G f(x)\hat{\bar{g}}(x)d\lambda(x) = \int_{G/H} \bar{g}(\dot{x})Tf(\dot{x})d\upsilon(\dot{x})$$
$$= \Phi'(Tf) = \Phi(f)$$
$$= \int_G f(x)g(x)d\lambda(x) \quad \text{for all } f \in L^1(G)$$

so $\hat{\bar{g}} = g$.

With the operators $T: L^1(G) \to L^1(G/H)$ and $\hat{}: L^\infty(G/H) \to L^\infty(G)$ defined we now introduce the convolution type operators which will be used in the material that follows.

<u>DEFINITION 4.2.2</u> : a) If $\mu \in M(G)$ and $f \in L^\infty(G/H)$ let $\mu \circledast f$ denote the unique function in $L^\infty(G/H)$ for which

$$\widehat{\mu \circledast f} = \mu * \hat{f} \text{ on } G.$$

b) If $\mu\epsilon M(G)$ and $f\epsilon L^1(G/H)$ then $\mu\circledast f\epsilon L^1(G/H)$ is defined by $\mu\circledast f = T(\mu * T^{-1}(f))$.

<u>REMARKS</u> : 1. Since $\hat{f}\epsilon L^\infty(G)$ for all $f\epsilon L^\infty(G/H)$ we have $\mu*\hat{f}\epsilon L^\infty(G)$ with, for every $h\epsilon H$,

$$\mu*\hat{f}(xh) = \int_G \hat{f}(y^{-1}xh)d\mu(y) = \int_G f\circ\Pi_H(y^{-1}xh)d\mu(y)$$
$$= \int_G f\circ\Pi_H(y^{-1}x)d\mu(y) = \int_G \hat{f}(y^{-1}x)d\mu(y)$$
$$= \mu*\hat{f}(x)$$

for almost all $x\epsilon G$ so by Lemma 4.2.1 there exists a function $\mu\circledast f\epsilon L^\infty(G/H)$ with $\widehat{\mu\circledast f} = \mu*\hat{f}$ and $\mu\circledast f$ is well-defined.

2. If $h_0\epsilon H$ and $f\epsilon K(G)$ then, for $\mu\epsilon M(G)$,

$$T(\mu*A_{h_0}f)(\dot{x}) = \int_H (\mu*A_{h_0}f(xh)/\rho(xh)\ d\beta(h)$$
$$= \int_H\int_G (f(y^{-1}xhh_0)\Delta_G(h_0)/\rho(xh))d\mu(y)d\beta(h)$$
$$= \int_G\int_H (f(y^{-1}xhh_0)\Delta_G(h_0)/\rho(xh))d\beta(h)d\mu(y)$$
$$= \int_G\int_H (f(y^{-1}xh)\Delta_G(h_0)\Delta_H(h_0^{-1})/\rho(xhh_0^{-1}))d\beta(h)d\mu(y)$$
(replacing h by hh_0^{-1})
$$= \int_G\int_H (f(y^{-1}xh)/\rho(xh))d\beta(h)d\mu(y)$$
(since $\rho(xhh_0^{-1}) = \rho(xh)\Delta_H(h_0^{-1})/\Delta_G(h_0^{-1})$)
$$= \int_H\int_G (f(y^{-1}xh)/\rho(xh))d\mu(y)d\beta(h)$$
$$= \int_H ((\mu*f)(xh)/\rho(xh))d\beta(h)$$
$$= T(\mu*f)(\dot{x})$$

so $T(\mu*(A_{h_0}f-f)) = 0$ for all $f\epsilon K(G)$ and $h_0\epsilon H$. Since the mapping $g\to T(\mu*g)$ is continuous this implies that

$T(\mu * g) = 0$ whenever $g \in J(T)$ consequently the function $\mu \circledast f \in L^1(G/H)$ is well-defined for all $\mu \in M(G)$ and $f \in L^1(G/H)$.

3. It should be noted that the function $\mu \odot f$ defined above is independent of the quasi-invariant measure we have chosen on G/H.

If υ_1 and υ_2 are different quasi-invariant measures on G/H then they are equivalent by Theorem 4.1.6 so there exists a strictly positive measurable function k on G/H such that, for every $f \in L^1(G/H, \upsilon_1)$ we have $f' = f \cdot k \in L^1(G/H, \upsilon_2)$ with $\int_{G/H} f(\dot{x}) d\upsilon_1(\dot{x}) = \int_{G/H} f'(\dot{x}) d\upsilon_2(\dot{x})$. Thus $\Lambda: f \to f'$ is an isometric isomorphism of $L^1(G/H, \upsilon_1)$ onto $L^1(G/H, \upsilon_2)$. It is easily checked that

$$\Lambda(\mu \circledast_1 f) = \mu \circledast_2 (\Lambda f) \quad \text{for all } \mu \in M(G),\ f \in L^1(G/H, \upsilon_1)$$

(where $\circledast_1$, $\circledast_2$ denote the convolution type operations corresponding to υ_1 and υ_2 respectively).

Some properties of the operations $\circledast$ and $\odot$ which we will need in later work are the following.

<u>LEMMA 4.2.3</u> : Let $\mu, \mu_1 \in M(G)$; $f \in L^\infty(G/H)$ and $g \in L^1(G/H)$ Then

i) $\odot$: $M(G) \times L^\infty(G/H) \to L^\infty(G/H)$ is a jointly continuous bilinear operator with $\|\mu \odot f\|_\infty \leq \|\mu\| \cdot \|f\|_\infty$.

ii) $\circledast$: $M(G) \times L^1(G/H) \to L^1(G/H)$ is a jointly continuous bilinear operator with $\|\mu \circledast g\|_1 \leq \|\mu\| \cdot \|g\|_1$.

iii) $(\mu * \mu_1) \odot f = \mu \odot (\mu_1 \odot f)$

PROOF : i) Since $\widehat{\mu\odot f} = \mu*\hat{f}$, we have by Lemma 4.2.1, $\|\mu\odot f\|_\infty = \|\widehat{\mu\odot f}\|_\infty = \|\mu*\hat{f}\|_\infty \leq \|\mu\|\cdot\|\hat{f}\|_\infty = \|\mu\|\cdot\|f\|_\infty$ which implies that $\odot$ is jointly continuous.

It is easily checked that the mapping $f\to\hat{f}$ is linear and since $(\mu,\bar{f})\to\mu*\bar{f}$,where $\mu\in M(G)$, $\bar{f}\in L^\infty(G)$,is bilinear, the fact that $\odot$ is bilinear follows immediately.

ii) In Remark 2 above we showed that $T(\mu*k)= 0$ whenever $k\in J(T)$. Let $\bar{g}\in L^1(G)$ with $T\bar{g}= g$ and let $k\in J(T)$. Then

$$\begin{aligned}\|\mu\circledast g\|_1 &= \|T(\mu*\bar{g})\|_1 = \|T(\mu*(\bar{g}+k))\|_1 \\ &\leq \|\mu*(\bar{g}+k)\|_1 \qquad \text{since } \|T\| \leq 1 \\ &\leq \|\mu\|\cdot\|\bar{g}+k\|_1 \quad .\end{aligned}$$

Since $L^1(G/H)\simeq L^1(G)/J(T)$ we have $\|g\|_1 = \inf_{k\in J(T)} \|\bar{g}+k\|_1$ and the fact that $k\in J(T)$ was arbitrary in the calculation above implies that $\|\mu\circledast g\|_1 \leq \|\mu\|\cdot\|g\|_1$ and $\circledast$ is jointly continuous.

Since T is linear and $(\mu,\bar{g})\to\mu*\bar{g}$, where $\mu\in M(G)$ and $\bar{g}\in L^1(G)$, is bilinear, it is clear that $\circledast$ is bilinear.

iii) Note that

$$\begin{aligned}\widehat{(\mu*\mu_1)\odot f} &= (\mu*\mu_1)*\hat{f} = \mu*(\mu_1*\hat{f}) \\ &= \mu*(\widehat{\mu_1\odot f}) = \widehat{\mu\odot(\mu_1\odot f)}\end{aligned}$$

and the fact that ^ is an isometry from $L^\infty(G/H)$ into $L^\infty(G)$ implies that $(\mu*\mu_1)\odot f = \mu\odot(\mu_1\odot f)$.

We are now able to define the concepts of the amenable action of a group on its coset spaces as follows

DEFINITION 4.2.4 : Let G be a locally compact group and H a closed subgroup of G. If X is any of the spaces $L^{\infty}(G/H)$, CB(G/H), LUC(G/H) and μ is a mean on X then μ is said to be

i) a left invariant mean (LIM) if $\mu(\delta_g \circledast f) = \mu(f)$ for every $g\epsilon G$, $f\epsilon X$ (where $\delta_g \epsilon M(G)$ denotes point mass at g).

ii) a topological left invariant mean (TLIM) if $\mu(\phi \circledast f) = \mu(f)$ for every $\phi \epsilon P(G)=\{g\epsilon L^1(G) \mid f \geq 0, \int_G f(x)d\lambda(x)=1\}$ (note that $\phi \circledast f$ is defined since $L^1(G)$ can be embedded in M(G)).

We say that X is G (topologically) amenable if such a (TLIM) LIM exists.

REMARKS :

1. Note that $\delta_g \circledast f(\dot{x}) = f(g^{-1} \cdot \dot{x}) = L_{g^{-1}} f(\dot{x})$ for $f\epsilon L^{\infty}(G/H)$.

2. In the above LUC(G/H) denotes the space of bounded continuous functions f on G/H such that whenever $g_\alpha \to g_o$ G we have $\| \delta_{g_\alpha} \circledast f - \delta_{g_o} \circledast f \|_\infty \to 0$.

3. If $f\epsilon L^{\infty}(G/H)$ and $\phi \epsilon L^1(G)$ then $\phi \circledast f \epsilon LUC(G/H)$ so $P(G) \circledast X \subset X$ for any of the spaces mentioned above and $\mu(\phi \circledast f)$ is defined when μ is a mean on X.(This follows from the fact that the mapping $t \to L_t \phi$ is left uniformly continuous so whenever $x_\alpha \to x_o$ in G we can find α_o such that $\alpha \geq \alpha_o$ implies $\| L_{x_\alpha} \phi - L_{x_o} \phi \|_1 < (\varepsilon / \| f \|_\infty)$. Hence for any $\dot{x} \epsilon G/H$

$$|\phi \circledast f(x_\alpha \dot{x}) - \phi \circledast f(x_o \dot{x})| = |\int_G \phi(y) f(y^{-1} x_\alpha \dot{x}) d\lambda(y) - \int_G \phi(y) f(y^{-1} x_o \dot{x}) d\lambda(y)|$$

$$= |\int_G \phi(x_\alpha y) f(y^{-1} \dot{x}) d\lambda(y) - \int_G \phi(x_o y) f(y^{-1} \dot{x}) d\lambda(y)|$$

$$= \left|\int_G (L_{x_\alpha}\phi - L_{x_o}\phi)(y) f(y^{-1}\dot{x}) d\lambda(y)\right|$$

$$\leq \|f\|_\infty \cdot \int_G |L_{x_\alpha}\phi - L_{x_o}\phi|(y) d\lambda(y)$$

$$\leq \|f\|_\infty \, \|L_{x_\alpha}\phi - L_{x_o}\phi\|_1 \leq \varepsilon$$

when $\alpha \geq \alpha_o$ so $\phi \circledcirc f \varepsilon LUC(G/H)$.)

This definition of left invariant and topological left invariant means coincides with the usual one in the case where H is normal in G and G/H is a locally compact group. In this situation let υ_1 denote a left Haar measure on G/H. By Theorem 4.1.2 υ is equivalent to υ_1 (since υ is quasi-invariant) so, as in Remark 3 following Definition 4.2.2, there exists a strictly positive measurable function k on G/H such that for every $f \varepsilon L^1(G/H,\upsilon)$ we have $f' = f \cdot k \varepsilon L^1(G/H,\upsilon_1)$ with $\int_{G/H} f(\dot{x}) d\upsilon(\dot{x}) = \int_{G/H} f'(\dot{x}) d\upsilon_1(\dot{x})$. If $\phi \varepsilon P(G)$ we have shown that $T\phi \varepsilon P(G/H,\upsilon)$ consequently $T\phi \cdot k \varepsilon P(G/H,\upsilon_1)$ and conversely every element in $P(G/H,\upsilon_1)$ is of the form $T\phi \cdot k$ for some $\phi \varepsilon P(G)$. For $\phi \varepsilon P(G)$ and $f \varepsilon L^\infty(G/H)$ if we let $q(y) = \phi(y) f(y^{-1}\dot{x})$ then

$$Tq(\dot{y}) = \int_H (\phi(yh) f(h^{-1}y^{-1}\dot{x})/\rho(yh)) d\beta(h)$$

$$= \int_H (\phi(yh) f((\dot{y}^{-1})\dot{x})/\rho(yh)) d\beta(h) \quad \text{since H is normal}$$

$$= f((\dot{y}^{-1})\dot{x}) \cdot T\phi(\dot{y}) \quad \text{therefore}$$

$$\phi \circledcirc f(\dot{x}) = \int_G \phi(y) f(y^{-1}\dot{x}) d\lambda(y) = \int_{G/H} T\phi(\dot{y}) f((\dot{y}^{-1})\dot{x}) d\upsilon(\dot{y})$$

$$= \int_{G/H} T\phi(\dot{y}) k(\dot{y}) f((\dot{y}^{-1})\dot{x}) d\upsilon_1(\dot{y}) = (T\phi \cdot k) * f(\dot{x}) .$$

In view of the remarks above this implies that

$\{\phi \circledcirc f \mid \phi \varepsilon P(G)\} = \{(T\phi \cdot k) * f \mid (T\phi \cdot k) \varepsilon P(G/H, \upsilon_1)\}$

$= \{\phi' * f \mid \phi' \varepsilon P(G/H, \upsilon_1)\}$

so μ is a TLIM on G/H in the usual sense

iff $\mu(\phi' * f) = \mu(f)$ for all $\phi' \varepsilon P(G/H, \upsilon_1)$

iff $\mu(\phi \circledcirc f) = \mu(f)$ for all $\phi \varepsilon P(G)$

iff μ is a TLIM on G/H in the sense of Definition 4.2.4

As mentioned in Remark 1 above, $\delta_g \circledcirc f = L_g -1(f)$ for $g \varepsilon G$ and $f \varepsilon L^\infty(G/H)$ so it is clear that μ is a LIM on G/H in the usual sense if and only if μ is a LIM on G/H in the sense of Definition 4.2.4.

As in the case of the usual definition of the amenability of a group acting on itself, the following theorem shows that the notions of X being G amenable or G topologically amenable are equivalent when X is any of the spaces $L^\infty(G/H)$, CB(G/H) or LUC(G/H).

<u>THEOREM 4.2.5</u> : Let G be a locally compact group and H a closed subgroup of G. Consider the following statements

0 - G is amenable (in the usual sense)

1 [1A] - there exists a [topological] LIM on $L^\infty(G/H)$

2 [2A] - there exists a [topological] LIM on CB(G/H)

3 [3A] - there exists a [topological] LIM on LUC(G/H) .

Then statement (0) implies each of the others and the rest are equivalent .

<u>PROOF</u> : We will show that $0 \to 1 \to 2 \to 3 \to 3A \to 2A \to 1A \to 1$.

0→1) Let μ be a LIM on $L^\infty(G)$ and for $f\varepsilon L^\infty(G/H)$ define $\bar{\mu}(f) = \mu(\hat{f})$. If $f \geq 0$ on G/H then $\hat{f} \geq 0$ on G so $\bar{\mu}(f)=\mu(\hat{f}) \geq 0$. Also $\hat{1}_{G/H} = 1_G$ hence $\bar{\mu}(1_{G/H}) = 1$ and $\bar{\mu}$ is a mean on $L^\infty(G/H)$. If $g\varepsilon G$ then $\widehat{(\delta_g \circledast f)} = (\delta_g * \hat{f}) = L_{g^{-1}}(\hat{f})$ so

$$\bar{\mu}(\delta_g \circledast f) = \mu(L_{g^{-1}}(\hat{f})) = \mu(\hat{f}) = \bar{\mu}(f) \quad \text{and } \bar{\mu} \text{ is a LIM}$$

on $L^\infty(G/H)$.

1→2→3) Since CB(G/H) is a closed invariant subspace of $L^\infty(G/H)$ and LUC(G/H) is a closed invariant subspace of CB(G/H) the required left invariant means can be obtained by restriction.

3→3A) Let $\{e_\alpha\}_{\alpha\varepsilon\mathcal{A}}$ be an approximate identity in $L^1(G)$. This means that $e_\alpha \varepsilon P(G)$ for each α in the directed set $\mathcal{A}$ and, for any neighbourhood V of the identity in G, there exists $\alpha_o \varepsilon \mathcal{A}$ such that $\alpha \geq \alpha_o$ implies e_α vanishes outside of V. Also for any $g\varepsilon L^1(G)$ we have $\lim_\alpha \|g * e_\alpha - g\|_1 = 0$ and $\lim_\alpha \|e_\alpha * g - g\|_1 = 0$. (The existence of such an approximate identity for $L^1(G)$ is well known. A proof can be found in [12] p.124.)

If $f\varepsilon LUC(G/H)$ we can show that $\lim_\alpha \|e_\alpha \circledast f - f\|_\infty = 0$ as follows : given $\varepsilon > 0$ we use the fact that f is left uniformly continuous to choose a neighbourhood V of the identity in G with $\|\delta_y \circledast f - f\|_\infty < \varepsilon$ for all $y\varepsilon V$. If $\alpha_o \varepsilon \mathcal{A}$ is chosen so that e_α vanishes outside of V for $\alpha \geq \alpha_o$ then we have

$$|e_\alpha \circledast f(\dot{x}) - f(\dot{x})| = |\int_G e_\alpha(y) f(y^{-1}\dot{x})\,d\lambda(y) \;-\; \int_G e_\alpha(y) f(\dot{x})\,d\lambda(y)|$$

(since $\int_G e_\alpha(y)\,d\lambda(y) = 1$)

$$= |\int_G e_\alpha(y)(\delta_y \circledast f(\dot{x}) - f(\dot{x}))\,d\lambda(y)|$$

$$\leq \|\delta_y \circledast f - f\|_\infty \cdot \int_G e_\alpha(y)\,d\lambda(y) \quad \text{for } \alpha \geq \alpha_o$$ (since e_α vanishes outside of V)

$$< \varepsilon$$

consequently $\| e_\alpha \circledast f - f\|_\infty < \varepsilon$ whenever $\alpha \geq \alpha_o$ so $\lim_\alpha \|e_\alpha \circledast f - f\|_\infty = 0$.

With this fact established we can employ a method of Namioka to obtain the desired result. Let μ be a LIM on LUC(G/H), $f \varepsilon$ LUC(G/H) and define a functional Q_f on $L^1(G)$ by the formula $Q_f(\phi) = \mu(\phi \circledast f)$. Q_f is clearly a bounded linear functional on $L^1(G)$ and since, for $g \varepsilon G$

$$(L_g\phi) \circledast f(\dot{x}) = \int_G \phi(gy) f(y^{-1}\dot{x})\,d\lambda(y) = \int_G \phi(y) f(y^{-1}g\dot{x})\,d\lambda(y)$$

(replacing y by $g^{-1}y$)

$$= (\phi \circledast f)(g\dot{x}) = L_g(\phi \circledast f)(\dot{x})$$

for all $\dot{x} \varepsilon G/H$ we have

$$Q_f(L_g\phi) = \mu((L_g\phi) \circledast f) = \mu(L_g(\phi \circledast f)) = \mu(\phi \circledast f) = Q_f(\phi)$$

therefore Q_f is left invariant. By the uniqueness of Haar measure on G, there must exist a constant k(f) with

$$\mu(\phi \circledast f) = k(f) \cdot \int_G \phi(y)\,d\lambda(y) \text{ for all } \phi \varepsilon L^1(G).$$

Therefore for any $\phi \varepsilon P(G)$, $Q_f(\phi) = \mu(\phi \circledast f) = k(f)$ and

$$|\mu(f) - \mu(\phi \circledast f)| \leq |\mu(f) - \mu(e_\alpha \circledast f)| + |\mu(e_\alpha \circledast f) - \mu((\phi * e_\alpha) \circledast f)|$$
$$+ |\mu((\phi * e_\alpha) \circledast f) - \mu(\phi \circledast f)|$$

$\leq \|\mu\| \cdot \|f-e_\alpha \circledcirc f\|_\infty + |Q_f(e_\alpha)-Q_f(\phi * e_\alpha)| + |\mu((\phi * e_\alpha - \phi)\circledcirc f)|$

$\leq \|f-e_\alpha \circledcirc f\|_\infty + |k(f)-k(f)| + \|\mu\| \cdot \|\phi * e_\alpha - \phi\|_1 \cdot \|f\|_\infty$ (since $\phi * e_\alpha \varepsilon P(G)$)

and since $\|f-e_\alpha \circledcirc f\|_\infty$ and $\|\phi * e_\alpha - \phi\|_1$ tend to 0 we have

$\mu(f) = \mu(\phi \circledcirc f)$ so μ is also a TLIM on LUC(G/H).

3A→2A→1A) First note that 1A→2A→3A as in the proof of 1→2→3 so it suffices to show that 3A→1A. Let μ be a TLIM on LUC(G/H) and let E be a compact neighbourhood of the identity in G with ϕ_E its normalized characteristic function (i.e. $\phi_E = (1/\lambda(E)) \cdot \chi_E$). As mentioned in Remark 3 following Definition 4.2.4 $\phi_E \circledcirc f \varepsilon LUC(G/H)$ for any $f \varepsilon L^\infty(G/H)$ (since $\phi_E \varepsilon P(G)$) so we can define a mean $\bar{\mu}$ on $L^\infty(G/H)$ by $\bar{\mu}(f) = \mu(\phi_E \circledcirc f)$.

Again let $\{e_\alpha\}_{\alpha \varepsilon \mathfrak{a}}$ be an approximate identity in $L^1(G)$ and choose $\phi \varepsilon P(G)$, $f \varepsilon L^\infty(G/H)$. We have

$\bar{\mu}(\phi \circledcirc f) = \mu(\phi_E \circledcirc (\phi \circledcirc f))$

$= \mu((\phi_E * \phi)\circledcirc f)$ by Lemma 4.2.3 (c)

$= \lim_\alpha \mu((\phi_E * \phi)\circledcirc(e_\alpha \circledcirc f))$ since μ is continuous and $\|f-e_\alpha \circledcirc f\|_\infty \to 0$

$= \lim_\alpha \mu(e_\alpha \circledcirc f)$ since $\phi_E * \phi \varepsilon P(G)$, $e_\alpha \circledcirc f \varepsilon LUC(G/H)$ and μ is a TLIM

$= \lim_\alpha \mu(\phi_E \circledcirc (e_\alpha \circledcirc f))$ since $\phi_E \varepsilon P(G)$ and μ is a TLIM

$= \lim_\alpha \mu((\phi_E * e_\alpha)\circledcirc f)$ by Lemma 4.2.3 (c)

$= \mu(\phi_E \circledcirc f)$ since μ is continuous and $\|\phi_E * e_\alpha - \phi_E\|_1 \to 0$

$= \bar{\mu}(f)$

Thus $\bar{\mu}$ is a TLIM on $L^{\infty}(G/H)$ and we have shown that 3A→1A.

1A→1) Let μ be a TLIM on $L^{\infty}(G/H)$ and choose $\phi \varepsilon P(G)$ $x \varepsilon G$ and $f \varepsilon L^{\infty}(G/H)$. Then

$$\mu(\delta_x \circledast f) = \mu(\phi \circledast (\delta_x \circledast f)) = \mu((\phi * \delta_x) \circledast f) = \mu(f)$$

since $\phi * \delta_x = \Delta(x^{-1}) \cdot R_{x^{-1}}(\phi) \varepsilon P(G)$ for all $x \varepsilon G$, $\phi \varepsilon P(G)$ so μ is a LIM on $L^{\infty}(G/H)$.

In view of this Theorem we may adopt the terminology that <u>G acts amenably on G/H</u> to indicate the existence of a LIM or a TLIM on any of the spaces $L^{\infty}(G/H)$, CB(G/H) and LUC(G/H).

<u>NOTE</u> : The methods used in the proof of this theorem are very close to the ones employed in [6] Theorem 2.2.1 to establish the same type of result concerning the existence of left invariant and topological left invariant means on the spaces $L^{\infty}(G)$, CB(G), LUC(G) and UCB(G) for a locally compact group G. Our Theorem 4.2.5 is exactly Theorem 3.3 of Greenleaf's paper [11] however, due to the approach he uses in defining the convolution type operations, his proofs of many of the implications are quite different.

A valuable tool for studying amenable groups has been Day's concept of nets of finite means (i.e. elements of P(G)) converging to left invariance. We are able to introduce a similar concept which is useful in examining the amenable action of G on G/H. The definition which follows is equivalent to the one used by Greenleaf in [11].

DEFINITION 4.2.6 : A net $\{\phi_\alpha\} \subset P(G/H)$ is said to converge

i) weakly [strongly] to left invariance if $(\delta_g \circledast \phi_\alpha - \phi_\alpha) \to 0$ in the weak* topology on $L^1(G/H)$ for all $g \in G$ (i.e. $(\delta_g \circledast \phi_\alpha - \phi_\alpha, f) = \int_{G/H} (\delta_g \circledast \phi_\alpha - \phi_\alpha)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) \to 0$ for every $g \in G$ and $f \in L^\infty(G/H)$)
[if $\| \delta_g \circledast \phi_\alpha - \phi_\alpha \|_1 \to 0$ for all $g \in G$].

ii) weakly [strongly] to topological left invariance if $(\phi \circledast \phi_\alpha - \phi_\alpha) \to 0$ in the weak* topology on $L^1(G/H)$ for all $\phi \in P(G)$. [if $\| \phi \circledast \phi_\alpha - \phi_\alpha \|_1 \to 0$ for all $\phi \in P(G)$.]

The significance of these definitions and their usefulness in examining the concept of the amenable action of G on G/H is indicated by the following theorem. In the proof we use essentially the same methods as are employed to prove a similar result for amenable groups ([6] Theorems 2.4.2, 2.4.3]

THEOREM 4.2.7 : If G is a locally compact group and H is a closed subgroup of G then the following are equivalent

1) G acts amenably on G/H

2 [2A]) there exists a net $\{\phi_\alpha\} \subset P(G/H)$ converging weakly [strongly] to left invariance

3 [3A]) there exists a net $\{\phi_\alpha\} \subset P(G/H)$ converging weakly [strongly] to topological left invariance .

PROOF : First note that for $\phi \varepsilon P(G)$, $\phi_1 \varepsilon P(G/H)$, $f \varepsilon L^{\infty}(G/H)$ and $g \varepsilon G$ we have the following equalities –

$$(\dagger) \int_{G/H} (\delta_g \circledast \phi_1)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) = \int_{G/H} \phi_1(\dot{x})((\delta_{g^{-1}}) \circledcirc f)(\dot{x}) d\upsilon(\dot{x})$$

$$(\dagger\dagger) \int_{G/H} (\phi \circledast \phi_1)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) = \int_{G/H} \phi_1(\dot{x})(\overline{\overline{\phi}} \circledcirc f)(\dot{x}) d\upsilon(\dot{x})$$

where $\overline{\overline{\phi}}(y) = (\Delta_G(y^{-1}) \cdot \phi(y^{-1})$ for $y \varepsilon G$ (hence $\overline{\overline{\phi}} \varepsilon P(G)$).

To establish these equalities let $\psi \varepsilon P(G)$ be such that $T\psi = \phi_1$. Then

$$\int_{G/H} (\delta_g \circledast \phi_1)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) = \int_{G/H} T(\delta_g * \psi)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x})$$

$$= \int_G (\delta_g * \psi)(x) \hat{f}(x) d\lambda(x) \qquad \text{(since } T((\delta_g * \psi) \cdot \hat{f}) = T(\delta_g * \psi) \cdot f \text{)}$$

$$= \int_G \psi(g^{-1}x) \hat{f}(x) d\lambda(x)$$

$$= \int_G \psi(x) \hat{f}(gx) d\lambda(x) \qquad \text{replacing } x \text{ by } gx$$

$$= \int_G \psi(x)((\delta_{g^{-1}}) * \hat{f})(x) d\lambda(x)$$

$$= \int_{G/H} T\psi(\dot{x})((\delta_{g^{-1}}) \circledcirc f)(\dot{x}) d\upsilon(\dot{x})$$

$$\text{(since } T(\psi \cdot ((\delta_{g^{-1}}) * \hat{f})) = T\psi \cdot ((\delta_{g^{-1}}) \circledcirc f) \text{)}$$

$$= \int_{G/H} \phi_1(\dot{x})((\delta_{g^{-1}}) \circledcirc f)(\dot{x}) d\upsilon(\dot{x}) \quad .$$

In a similar fashion

$$\int_{G/H} (\phi \circledast \phi_1)(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) = \int_G (\phi * \psi)(x) \hat{f}(x) d\lambda(x)$$

$$= \int_G \int_G \phi(y) \psi(y^{-1}x) \hat{f}(x) d\lambda(y) d\lambda(x)$$

$$= \int_G \int_G (\Delta_G(y^{-1}) \phi(y^{-1})) \psi(yx) \hat{f}(x) d\lambda(y) d\lambda(x) \qquad \text{(replacing } y \text{ by } y^{-1}\text{)}$$

$$= \int_G \int_G \overline{\overline{\phi}}(y) \psi(yx) \hat{f}(x) d\lambda(x) d\lambda(y)$$

$= \int_G \int_G \overline{\overline{\phi}}(y)\psi(x)f(y^{-1}x)d\lambda(x)d\lambda(y)$ (replacing x by $y^{-1}x$)

$= \int_G \psi(x)(\int_G \overline{\overline{\phi}}(y)f(y^{-1}x)d\lambda(y))d\lambda(x)$

$= \int_G \psi(x)(\overline{\overline{\phi}} * f)(x)d\lambda(x)$

$= \int_{G/H} T\psi(\dot{x})(\overline{\overline{\phi}} \circledcirc f)(\dot{x})d\upsilon(\dot{x})$

$= \int_{G/H} \phi_1(\dot{x})(\overline{\overline{\phi}} \circledcirc f)(\dot{x})d\upsilon(\dot{x})$

2→1) Let $\{\phi_\alpha\} \subset P(G/H)$ converge weakly to left invariance. Since P(G/H) is contained in the weak* compact sets of means on $L^\infty(G/H)$, there exists a subnet $\{\phi_{\alpha_i}\}$ converging in the weak* topology to a mean μ. For any $f \varepsilon L^\infty(G/H)$ and $g \varepsilon G$ we have

$$\mu(\delta_g \circledcirc f) = \lim_{\alpha_i} \phi_{\alpha_i}(\delta_g \circledcirc f) = \lim \int_{G/H} \phi_{\alpha_i}(\dot{x})(\delta_g \circledcirc f)(\dot{x})d\upsilon(\dot{x})$$

$$= \lim \int_{G/H}((\delta_g - 1) \circledast \phi_{\alpha_i})(\dot{x})f(\dot{x})d\upsilon(\dot{x}) \quad \text{(using †)}$$

$$= \lim ((\delta_g - 1) \circledast \phi_{\alpha_i})(f)$$

$$= \lim_{\alpha_i} \phi_{\alpha_i}(f) = \mu(f)$$

so μ is a left invariant mean on $L^\infty(G/H)$ and G acts amenably on G/H.

3→1) If $\{\phi_\alpha\} \subset P(G/H)$ converges weakly to topological left invariance then, as above, there is a subnet $\{\phi_{\alpha_i}\}$ converging in the weak* topology to a mean μ. If $f \varepsilon L^\infty(G/H)$ and $\phi \varepsilon P(G)$ then

$$\mu(\phi \circledcirc f) = \lim_{\alpha_i} \int_{G/H} \phi_{\alpha_i}(\dot{x})(\phi \circledcirc f)(\dot{x})d\upsilon(\dot{x})$$

$$= \lim \int_{G/H} (\overline{\overline{\phi}} \circledast \phi_{\alpha_i})(\dot{x}) f(\dot{x}) d\upsilon(\dot{x}) \qquad \text{(using ††)}$$

$$= \lim (\overline{\overline{\phi}} \circledast \phi_{\alpha_i})(f)$$

$$= \lim_{\alpha_i} \phi_{\alpha_i}(f) \qquad (\text{since } \overline{\overline{\phi}} \in P(G))$$

$$= \mu(f)$$

so μ is a topological left invariant mean on $L^\infty(G/H)$ and consequently G acts amenably on G/H.

1→2) Let μ be a LIM on $L^\infty(G/H)$. Since $P(G/H)$ is dense in the set of means with the weak* topology, there exists a net $\{\phi_\alpha\} \subset P(G/H)$ with $w^*\text{-}\lim_\alpha \phi_\alpha = \mu$. Consequently for $f \in L^\infty(G/H)$, $g \in G$ we have

$$\lim_\alpha (\delta_g \circledast \phi_\alpha)(f) = \lim \phi_\alpha((\delta_g - 1) \circledcirc f) \qquad \text{(using †)}$$

$$= \mu((\delta_g - 1) \circledast f)$$

$$= \mu(f) = \lim_\alpha \phi_\alpha(f)$$

so $\{\phi_\alpha\}$ converges weakly to left invariance.

1→3) If μ is a TLIM on $L^\infty(G/H)$ and $\{\phi_\alpha\} \subset P(G/H)$ is such that $w^*\text{-}\lim_\alpha \phi_\alpha = \mu$ then for $\phi \in P(G)$ and $f \in L^\infty(G/H)$,

$$\lim_\alpha (\phi \circledast \phi_\alpha)(f) = \lim \phi_\alpha(\overline{\overline{\phi}} \circledcirc f) \qquad \text{(using ††)}$$

$$= \mu(\overline{\overline{\phi}} \circledcirc f) = \mu(f) \qquad (\text{since } \overline{\overline{\phi}} \in P(G))$$

$$= \lim_\alpha \phi_\alpha(f)$$

so $\{\phi_\alpha\}$ converges weakly to topological left invariance.

We have established the equivalence of 1,2 and 3. Since the norm topology on $L^1(G/H)$ is stronger than the weak* topology, it is clear that 2A→2 and 3A→3. The remaining implications can be established using a method due to Namioka.

$3 \to 3A$) For each $\phi \varepsilon P(G)$ take a copy of $L^1(G/H)$ and form the locally convex product space

$$E = \Pi\{L^1(G/H) \mid \phi \varepsilon P(G)\}$$

with the product of the norm topologies. Define the linear mapping $S: L^1(G/H) \to E$ by $Sf(\phi) = \phi \circledast f - f$ for $\phi \varepsilon P(G)$, $f \varepsilon L^1(G/H)$.

Let $\{\phi_\alpha\} \subset P(G/H)$ converge weakly to topological left invariance so we have $S\phi_\alpha(\phi) = \phi \circledast \phi_\alpha - \phi_\alpha \to 0$ in the weak* topology on $L^1(G/H)$ for every $\phi \varepsilon P(G)$. Since the weak topology on E coincides with the product of the weak* topologies (see [13] p.160) this implies that $w^*\text{-}\lim_\alpha S\phi_\alpha = 0$ and therefore 0 lies in the weak closure of $S(P(G/H))$ E. Since E is locally convex and $S(P(G/H))$ is a convex set, the weak and strong closures coincide (see [8] Chapter V Theorem 3.13 p.422) so there is some net $\{\psi_\alpha\} \subset P(G/H)$ such that $S(\psi_\alpha) \to 0$ in the norm topology on E i.e. for every $\phi \varepsilon P(G)$

$$\|\phi \circledast \psi_\alpha - \psi_\alpha\|_1 \to 0$$

Consequently $\{\psi_\alpha\} \subset P(G/H)$ converges strongly to topological left invariance.

$2 \to 2A$) In the same manner as above, for every $g \varepsilon G$ take a copy of $L^1(G/H)$ and form the locally convex product space $E' = \Pi\{L^1(G/H) \mid g \varepsilon G\}$ with the product of the norm topologies. Define the linear mapping $S': L^1(G/H) \to E'$ by $S'f(g) = \delta_g \odot f - f$ for $g \varepsilon G$, $f \varepsilon L^1(G/H)$. If $\{\phi_\alpha\} \subset P(G/H)$ converges weakly to left invariance we have $w^*\text{-}\lim S'(\phi_\alpha) = 0$ in E' and, as before, the weak and strong closures of S'(P(G/H)) coincide so there exists a net $\{\psi_\alpha\}$ P(G/H) with $\|\delta_g \circledast \psi_\alpha - \psi_\alpha\|_1 \to 0$ for all $g \in G$. Therefore $\{\psi_\alpha\}$ converges strongly to topological left invariance.

REMARK : As we have mentioned before, given any two quasi-invariant measures υ_1 and υ_2 on G/H the fact that $\upsilon_1 \equiv \upsilon_2$ implies that $L^1(G/H,\upsilon_1)$ is isometrically isomorphic with $L^1(G/H,\upsilon_2)$ under an isomorphism Λ which satisfies

$\Lambda(\mu \circledast_1 f) = \mu \circledast_2 (\Lambda f)$ for all $\mu \in M(G)$, $f \in L^1(G/H,\upsilon_1)$. If $\{\phi_\alpha\} \subset P(G/H,\upsilon_1)$ converges weakly (strongly) to left invariance then $\{\Lambda\phi_\alpha\}$ is a net in $P(G/H,\upsilon_2)$ with

$\delta_g \circledast_2 (\Lambda\phi_\alpha) - \Lambda\phi_\alpha = \Lambda(\delta_g \circledast_1 \phi_\alpha - \phi_\alpha)$ so $\{\Lambda\phi_\alpha\}$ converges weakly (strongly) to left invariance as well. A similar argument shows that the notion of nets of finite means converging to topological invariance is also independent of the quasi-invariant measure being considered.

One important application of this theorem occurs in the proof of the following result.

THEOREM 4.2.8 : G acts amenably on G/H if and only if given $\varepsilon > 0$ and a compact subset K of G there exists $\phi \in P(G/H)$ such that $\|\delta_x \circledast \phi - \phi\|_1 < \varepsilon$ for all $x \in K$.

PROOF : $\leftarrow$) Let $\Gamma = \{(\varepsilon,K) \mid \varepsilon > 0$, K compact subset of G$\}$ directed by the partial ordering $(\varepsilon_1,K_1) < (\varepsilon_2,K_2)$ if $\varepsilon_2 < \varepsilon_1$ and $K_1 \subset K_2$. For each $\alpha = (\varepsilon,K) \in \Gamma$ there exists $\phi_\alpha \in P(G/H)$ with $\|\delta_x \circledast \phi_\alpha - \phi_\alpha\|_1 < \varepsilon$ for all $x \in K$.

Given $\varepsilon > 0$ and $x \varepsilon G$ choose $\alpha_o = (\varepsilon, K_o) \varepsilon \Gamma$ where $x \varepsilon K_o$. If $\alpha \geq \alpha_o$ then $\| \delta_x \circledast \phi_\alpha - \phi_\alpha \|_1 < \varepsilon$ so $\{\phi_\alpha\}$ converges strongly to left invariance and, by Theorem 4.2.7, G acts amenably on G/H.

$\rightarrow$) Let $\{\phi_\alpha\} \subset P(G/H)$ converge strongly to topological left invariance (the existence of such a net is due to Theorem 4.2.7 and the fact that we are assuming that G acts amenably on G/H). Fix $\beta \varepsilon P(G)$ and choose a small compact neighbourhood E of the identity in G such that

$$\| \phi_E * \beta - \beta \|_1 < \varepsilon \quad \text{and} \quad \| \delta_x * \beta - \beta \|_1 < \varepsilon \text{ for all } x \varepsilon E$$

(where $\phi_E = (1/\lambda(E)) \cdot \chi_E$ denotes the normalized characteristic function of the set E). Choose $x_1 = e, x_2, \ldots, x_n \varepsilon G$ for which $K \subset \bigcup_{i=1}^{n} x_i \cdot E$ and let $\psi_i = \phi_{x_i \cdot E} = \delta_{x_i} * \phi_E$ for $i = 1, 2, \ldots, n$.

Since $\{\phi_\alpha\}$ converges strongly to topological left invariance there exists ϕ_{α_o} (which we shall denote by ϕ_o) with $\| \beta \circledast \phi_o - \phi_o \|_1 < \varepsilon$ and $\| \psi_i \circledast \phi_o - \phi_o \|_1 < \varepsilon$ for $i = 1, 2, \ldots, n$. Let $\phi = \beta \circledast \phi_o \ \varepsilon\ P(G/H)$ and note that, for every $x \varepsilon E$,

$$\| \phi_E \circledast \phi - \delta_x \circledast \phi \|_1 \leq \| \phi_E \circledast \phi - \phi \|_1 + \| \phi - \delta_x \circledast \phi \|_1$$

$$\leq \| \phi_E \circledast (\beta \circledast \phi_o) - \beta \circledast \phi_o \|_1 + \| \beta \circledast \phi_o - \delta_x \circledast (\beta \circledast \phi_o) \|_1$$

$$\leq \| (\phi_E * \beta - \beta) \circledast \phi_o \|_1 + \| (\beta - \delta_x * \beta) \circledast \phi_o \|_1$$

$$\leq \| \phi_E * \beta - \beta \|_1 \cdot \| \phi_o \|_1 + \| \beta - \delta_x * \beta \|_1 \cdot \| \phi_o \|_1$$

(using Lemma 4.2.3 ii))

$< 2 \cdot \varepsilon$ (due to the manner in which E was defined and the fact that $\| \phi_o \|_1 = 1$)

Thus if $x\varepsilon E$ we have

$$\|(\delta_{x_i}*\phi_E)\circledast\phi-\delta_{x_i x}\circledast\phi\|_1=\|\delta_{x_i}\circledast(\phi_E\circledast\phi)-\delta_{x_i}\circledast(\delta_x\circledast\phi)\|_1$$

$$\leq \|\delta_{x_i}\|\cdot\|\phi_E\circledast\phi-\delta_x\circledast\phi\|_1 < 2\cdot\varepsilon.$$

Therefore

$$\|\delta_{x_i x}\circledast\phi-\phi\|_1 \leq \|\delta_{x_i x}\circledast\phi-(\delta_{x_i}*\phi_E)\circledast\phi\|_1+\|(\delta_{x_i}*\phi_E)\circledast\phi-\phi\|_1$$

$$\leq 2\cdot\varepsilon + \|(\delta_{x_i}*\phi_E)\circledast(\beta\circledast\phi_o)-\beta\circledast\phi_o\|_1$$

$$\leq 2\cdot\varepsilon + \|(\delta_{x_i}*\phi_E)\circledast(\beta\circledast\phi_o)-(\delta_{x_i}*\phi_E)\circledast\phi_o\|_1+$$

$$\|(\delta_{x_i}*\phi_E)\circledast\phi_o-\phi_o\|_1 + \|\phi_o-\beta\circledast\phi_o\|_1$$

$$\leq 2\cdot\varepsilon + \|(\delta_{x_i}*\phi_E)\circledast(\beta\circledast\phi_o-\phi_o)\|_1$$

$$+\|\psi_i\circledast\phi_o-\phi_o\|_1 + \|\phi_o-\beta\circledast\phi_o\|_1$$

$$\leq 5\cdot\varepsilon \quad \text{for all } x\varepsilon E.$$

Since, for any $t\varepsilon K$, we have $t= x_i x$ for some i and $x\varepsilon E$ therefore $\|\delta_t\circledast\phi-\phi\|_1 < 6\cdot\varepsilon$ for all $t\varepsilon K$ and the result follows.

This theorem is an analogue of Theorem 3.2.1 of [6] for locally compact groups. It appears without proof as Theorem 3.5 in [11].

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