CYLINDER MEASURES OVER VECTOR SPACES

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ABSTRACT

In this paper we present a theory of cylinder measures from the viewpoint of inverse systems of measure spaces. Specifically, we consider the problem of finding limits for the inverse system of measure spaces determined by a cylinder measure μ over a vector space X.

For any subspace Ω of the algebraic dual X^* such that (X,Ω) is a dual pair, we establish conditions on μ which ensure the existence of a limit measure on Ω.

For any regular topology G on Ω, finer than the topology of pointwise convergence, we give a necessary and sufficient condition on μ for it to have a limit measure on Ω Radon with respect to G.

We introduce the concept of a weighted system in a locally convex space. When X is a Hausdorff, locally convex space, and Ω is the topological dual of X, we use this concept in deriving further conditions under which μ will have a limit measure on Ω Radon with respect to G.

We apply our theory to the study of cylinder measures over Hilbertian spaces and ℓ^p-spaces, obtaining significant extensions and clarifications of many previously known results.

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INTRODUCTION

Cylinder measures were first introduced independently by I.M. Gelfand (Generalized random processes, [10]) and K. Ito (Stationary random distributions [15]) as a more general kind of stochastic process (J. Doob [7]), and arise naturally in probability theory when one defines a stochastic integral ([4] p. 137, [7] p. 426, [15] p. 211). On the other hand, the demands of theoretical physics (in particular, quantum field theory and statistical mechanics) have led to a considerable interest in the theory of integration over function spaces ([13], I.M. Gelfand and, A.M. Yaglom [12], I. Segal [43]), where the integrals considered are defined with respect to some cylinder measure (e.g. as in L. Gross [13] p. 53-54).

In the study of cylinder measures researchers have concentrated on two main approaches: in one, a cylinder measure is viewed as a linear map on a vector space into the space of measurable functions on some probability space (Gelfand [10], Ito [15], Gelfand and Vilenkin [11] Ch. IV , Fernique [9]); in the other, it is viewed as a set function on a family of cylinder sets of a vector space Ω (Minlos [25], Gelfand and Vilenkin [11] Ch. IV, Prohorov [33], Badrikian [1], L. Schwartz [39]). Inherent in both approaches is the notion of an inverse (or projective) system of measure spaces ([25] p. 293, [11] p. 309, [33] p. 409, [1] p. 2, [39] p. 832, [9] p. 34).

In this thesis we view a cylinder measure as an inverse system of measure spaces indexed by the finite dimensional subspaces of a vector space X. Moreover, we do so without any a priori choice of the "target" space Ω on which the limit measure is to live. The basic problem of finding a limit of the system on a space Ω of linear functionals is then analyzed with variable Ω in Chapter I. The key idea there is to examine the measure theoretic size of Ω in relation to the algebraic dual X^*. To this end the notion of "almost" sequential maximality is introduced.

Next, in Chapter II, we consider the more standard problem of finding a Radon limit measure on Ω when X is a topological vector space and Ω is its topological dual. When X is Hausdorff and locally convex, by introducing the concept of a weighted system in X, we establish a condition for the existence of such a Radon limit in terms of the notion of continuity with respect to a weighted system.

In Chapter III we apply the theory of Chapter II to the study of cylinder measures over Hilbertian, nuclear, and ℓ^p-spaces, thereby extending and clarifying several previously known results.

In the appendix we establish mainly technical results used in the proofs of Chapter III and present several counter-examples.

CHAPTER 0

PRELIMINARIES

1. Set-theoretic Notation.

In this work we shall use the following notation.

(1) $\emptyset$ is the empty set.

For any sets A and B ,

$A \sim B = \{x \in A : x \notin B\}$.

ω is the set of finite ordinals.

R is the field of real numbers.

$R^+ = \{t \in R : t \geq 0\}$.

$\mathbb{C}$ is the field of complex numbers.

In proofs we shall abbreviate "such that" to "s.t."

(2) For any set X and family $\mathcal{H}$ of subsets of X ,

$\bigsqcup \mathcal{H} = \bigcup_{H \in \mathcal{H}} H$, $\prod \mathcal{H} = \bigcap_{H \in \mathcal{H}} H$,

$\mathcal{P}(\mathcal{H}) = \{\mathcal{H}' \subset \mathcal{H} : \mathcal{H}'$ is countable, disjoint, and $\bigsqcup \mathcal{H}' = \bigsqcup \mathcal{H}\}$.

For any $A \subset X$,

$\mathcal{H} | A = \{H \cap A : H \in \mathcal{H}\}$.

$\mathcal{H}$ is a compact family iff for any $\mathcal{H}' \subset \mathcal{H}$,

if $\mathcal{A} \subset \mathcal{H}'$ is finite $\Rightarrow$ $\prod \mathcal{A} \neq \emptyset$, then $\prod \mathcal{H}' \neq \emptyset$.

For any topology $\mathcal{G}$ on X ,

$$\mathcal{K}(\mathcal{G}) = \{K \subset X : K \text{ is closed and compact in } \mathcal{G}\} .$$

(3) For any set X and $A \subset X$,

$$1_A : x \in X \to \begin{array}{ll} 1 \in \mathbb{C} & \text{if } x \in A , \\ 0 \in \mathbb{C} & \text{if } x \in X \sim A . \end{array}$$

For any $f : X \to Y$, $B \subset Y$,

$$f|A : x \in A \to f(x) \in Y ,$$

$$f[A] = \{f(x) : x \in A\} ,$$

$$f^{-1}[B] = \{x \in X : f(x) \in B\} .$$

For any sets X and Y , $I \subset X \times Y$, $x \in X$, $y \in Y$,

$$I_x = \{y \in Y : (x,y) \in I\} ,$$

$$I^y = \{x \in X : (x,y) \in I\} .$$

2. Outer Measures and Integrals.

Our measure-theoretic approach is essentially that of Caratheodory, as given by M. Sion in [44] and [45].

(1) For any set X and Caratheodory measure η on X ,

M_η is the family of η-measurable sets.

η is an $\mathcal{A}$-outer measure iff $\mathcal{A} \subset M_\eta$, and for any $A \subset X$,

$$\eta(A) = \inf\{\eta(A') : A \subset A' \in \mathcal{A}\} .$$

η is an outer measure iff η is an M_η-outer measure.

Throughout this work all measures considered will be outer measures.

η is the Caratheodory measure on X generated by τ and A

iff

A is a family of subsets of X with $\emptyset \in A$,

$\tau : A \to R^+$ with $\tau(\emptyset) = 0$, and for any $B \subset X$,

$\eta(B) = \inf\{ \sum_{H \in H} \tau(H) : H \subset A$ is countable, $B \subset \bigcup H\}$.

(X,η) is a measure space iff X is a set and η is an outer measure on X .

(2) Integration.

We observe that for any measure space (X,η) , $P(M_\eta)$ is directed by refinement.

In general, we shall be considering complex-valued functions on X , and therefore also complex-valued integrals. However, we point out that for any η-measurable $f : X \to R^+$,

$$\int f d_\eta = \lim_{P \in P(M_\eta)} \sum_{B \in P} (\inf f[B]) \cdot \eta(B)$$

$$= \lim_{P \in P(M_\eta)} \sum_{B \in P}{}' (\sup f[B]) \cdot \eta(B) \ .$$

Further, for any $f : X \to R^+$, the outer integral

$$\int^* f d_\eta = \lim_{P \in P(M_\eta)} \sum_{B \in P} (\sup f[B]) \cdot \eta(B)$$

is a well-defined point in $R^+ \cup \{\infty\}$.

(3) Radón Measures

In this paper, many of the measures we consider will in fact be Radon measures. We give the relevant definitions below.

For any set X and topology $\mathcal{G}$ on X ,

η is a $\mathcal{G}$-Radón measure on X iff

(i) η is a $\mathcal{G}$-outer measure on X ,

(ii) $K \in \mathcal{K}(\mathcal{G}) \Rightarrow \eta(K) < \infty$,

and for every $G \in \mathcal{G}$,

(iii) $\eta(G) = \sup\{\eta(K) : K \subset G , K \in \mathcal{K}(\mathcal{G})\}$.

For any $\mathcal{G}$-Radón measure η on X ,

supp η = support of η .

(4) Induced Radón Measures

Let Y be an abstract space. For any finite measure space (X,η) and $T : X \to Y$,

$T[\eta]$ is the Caratheodory measure on Y generated by $\eta \circ T^{-1}$ and $\{A \subset Y : T^{-1}[A] \in M_\eta\}$.

We shall use the following lemmas.

Lemmas

(1) For any $A \in M_{T[\eta]}$,

$$T^{-1}[A] \in M_\eta \text{ and therefore } T[\eta](A) = \eta(T^{-1}[A]) .$$

(2) For any space Z and $U : Y \to Z$,

$$U[T[\eta]] = (U \circ T)[\eta]$$

(3) If X and Y are topological spaces, T is continuous, and η is Radón, then $T[\eta]$ is Radón.

Proof of Lemma 4.1. Let

$$\mathcal{A} = \{A \subset Y : T^{-1}[A] \in M_\eta\} .$$

First we note that for all $A \in \mathcal{A}$,

$$T[\eta](A) = \eta(T^{-1}[A]) .$$

Let $A \in M_{T[\eta]}$. Since $\mathcal{A}$ is a σ-field and $T[\eta]$ is an $\mathcal{A}$-outer measure, there exists $A' \in \mathcal{A}$ s.t.

$$A \subset A' \quad \text{and} \quad T[\eta](A) = T[\eta](A') .$$

If $T[\eta](A) = 0$ then

(1) $0 \leq \eta(T^{-1}[A]) \leq \eta(T^{-1}[A']) = T[\eta](A') = 0$.

In general, since $T[\eta](Y) < \infty$,

$$T[\eta](A' \sim A) = 0$$

and therefore by (1),

$$\eta(T^{-1}[A' \sim A]) = 0 \quad \text{and} \quad T^{-1}[A' \sim A] \in M_\eta .$$

Hence, since $T^{-1}[A'] \in M_\eta$,

$$T^{-1}[A] = T^{-1}[A'] \sim T^{-1}[A' \sim A] \in M_\eta .$$

The second assertion now follows immediately from the fact that

$$T[\eta]|\mathcal{A} = \eta \circ T^{-1} . \qquad \blacksquare$$

Proof of Lemma 4.2. We need only observe that, by 4.1 above,

$$\{B \subset Z : (U \circ T)^{-1}[B] \in M_\eta\} = \{B \subset Z : U^{-1}[B] \in M_{T[\eta]}\} ,$$

and for any $B \subset Z$ s.t. $(U \circ T)^{-1}[B] \in M_\eta$,

$$\eta((U \circ T)^{-1}[B]) = T[\eta](U^{-1}[B]) . \qquad \blacksquare$$

Proof of Lemma 4.3. Let $A \in M_{T[\eta]}$ and $\varepsilon > 0$. By Lemma 4.1,

$$T^{-1}[A] \in M_\eta ,$$

and therefore, since η is a finite Radón measure, there exists a compact $K \subset T^{-1}[A]$ s.t.

$$\eta(T^{-1}[A]) - \eta(K) < \varepsilon .$$

Then $T[K] \subset A$ is compact and

$$T[\eta](A) - T[\eta](K) \leq \eta(T^{-1}[A]) - \eta(K) < \varepsilon$$

Hence, since $\varepsilon > 0$ was arbitrary,

(1) $T[\eta](A) = \sup\{T[\eta](C) : C \subset A \text{ is compact}\}$.

Since $T[\eta]$ is finite it then also follows that

$$T[\eta](A) = \inf\{T[\eta](G) : G \supset A \text{ is open}\} ,$$

and since $T[\eta]$ is an outer measure we therefore conclude that for all $B \subset Y$,

(2) $T[\eta](B) = \inf\{T[\eta](G) : G \supset B \text{ is open}\}$.

consequently $T[\eta]$ is Radón. $\blacksquare$

CHAPTER I

CYLINDER MEASURES OVER VECTOR SPACES

As indicated in the introduction, we shall treat cylinder measures as being special inverse systems of measure spaces (Choksi [6]). In the following section we introduce the basic notions and results that we shall require about such systems.

1. Inverse Systems of Measure Spaces.

Throughout this section,

$\mathcal{F}$ is an index set directed by a relation $<$.

For any $E \in \mathcal{F}$,

(X_E, μ_E) is a measure space,

$M_E = M_{\mu_E}$.

For any E and F in $\mathcal{F}$ with $E < F$,

$r_{E,F} : X_F \to X_E$ is surjective,

with

$r_{E,E}$ being the identity map.

1.1 Definitions

(1) $(X_F, \mu_F)_{F \in \mathcal{F}}$ is an inverse system of measure spaces relative to the maps $r_{E,F}$,

iff, for any E, F , and G in $\mathcal{F}$ with $E < F < G$,

$$r_{E,G} = r_{E,F} \circ r_{F,G} \quad ,$$

and, for all $A \in M_E$,

$$r_{E,F}^{-1}[A] \in M_F \quad ,$$

$$\mu_F(r_{E,F}^{-1}[A]) = \mu_E(A) \quad .$$

(2) Let $(X_F, \mu_F)_{F \in \mathcal{F}}$ be an inverse system of measure spaces relative to the maps $r_{E,F}$.

If Ω is a set, and for each $F \in \mathcal{F}$,

$p_F : \Omega \to X_F$ is surjective, then, we call (Ω, ξ) a limit relative to the maps p_F of the given inverse system of measure spaces, iff for each E and F in $\mathcal{F}$ with $E < F$

$$p_E = r_{E,F} \circ p_F \quad ,$$

and,

ξ is an outer measure on Ω such that for all $A \in M_E$,

$$p_E^{-1}[A] \in M \quad ,$$

$$\xi(p_E^{-1}[A]) = \mu_E(A) \quad .$$

For the rest of this section we assume that $(X_F, \mu_F)_{F \in \mathcal{F}}$ is an inverse system of measure spaces relative to the maps $r_{E,F}$.

1.2 Definitions

For any set Ω , and surjective maps $p_F : \Omega \to X_F$ such that for any E and F in $\mathcal{F}$ with $E < F$,

$$p_E = r_{E,F} \circ p_F \quad ,$$

(1) $\mathrm{Cyl}(\Omega,p) = \{p_F^{-1}[A] : F \in \mathcal{F},\ A \in M_F\}$,

(2) $\tau(\Omega,p) : p_F^{-1}[A] \in \mathrm{Cyl}(\Omega,p) \to \mu_F(A) \in R^+$,

(3) $\eta_{\Omega,p}$ is the Caratheodory measure on Ω generated by

$$\tau_{\Omega,p} \text{ and } \mathrm{Cyl}(\Omega,p) \quad .$$

When there can be no ambiguity we shall omit the subscripts Ω and p .

Remarks

The following assertions are readily established. (Choksi [6], Mallory and Sion [23]).

(1) $\mathrm{Cyl}(\Omega,p)$ is a field.

(2) τ is well-defined and is finitely additive on $\mathrm{Cyl}(\Omega,p)$.

(3) $\mathrm{Cyl}(\Omega,p) \subset M_\eta$.

(4) (Ω,η) is a limit relative to the maps p_F of the given inverse system of measure spaces iff

$$\eta|\mathrm{Cyl}(\Omega,p) = \tau \quad .$$

(5) There exists an outer measure ξ on Ω such that (Ω,ξ) is a limit relative to the maps p_F of the given inverse system of measure spaces $\Leftrightarrow$

τ is countably additive, in which case η is such a measure.

We now suppose that

$(\overline{\Omega}, \eta_{\overline{\Omega},\overline{p}})$ is a limit relative to the maps $\overline{p}_F$,

$\Omega \subset \overline{\Omega}$,

and for each $F \in \mathcal{F}$,

$p_F = \overline{p}_F | \Omega$ is surjective.

We shall be interested in determining when $(\Omega, \eta_{\Omega,p})$ itself is a limit relative to the maps p_F .

1.3 Lemmas.

With the above notation and hypotheses,

(1) $(\Omega, \eta_{\Omega,p})$ is a limit relative to the maps p_F of the given inverse system of measure spaces iff

$$\eta_{\overline{\Omega},\overline{p}}(\overline{A}) = \eta_{\overline{\Omega},\overline{p}}(\overline{A} \cap \Omega) \quad \text{for all} \quad \overline{A} \in Cyl(\overline{\Omega}, \overline{p}) \quad .$$

(2) For any $\mathcal{F}' \subset \mathcal{F}$, let

$\Lambda(\mathcal{F}')$ be the set of all $f \in \overline{\Omega}$ such that there does not exist $g \in \Omega$ with

$$\overline{p}_F(g) = \overline{p}_F(f) \quad \text{for every} \quad F \in \mathcal{F}' \quad .$$

If, for every $\{F_n\}_{n\varepsilon\omega} \subset \mathcal{F}$ with $F_n < F_{n+1}$ for each $n \in \omega$,

$$\eta_{\overline{\Omega},\overline{p}}(\Lambda(\{F_n\}_{n\varepsilon\omega})) = 0 \quad ,$$

then

$$\eta_{\overline{\Omega},\overline{p}}(\overline{A}) = \eta_{\overline{\Omega},\overline{p}}(\overline{A} \cap \Omega) \quad \text{for all} \quad \overline{A} \in Cyl(\overline{\Omega}, \overline{p}) \quad .$$

Proofs 1.

Let

$$\mathrm{Cyl}(\Omega,p) = \mathrm{Cyl} \ , \ \mathrm{Cyl}(\overline{\Omega},\overline{p}) = \overline{\mathrm{Cyl}} \ ,$$

$$\tau = \tau_{\Omega,p} \ , \ \overline{\tau} = \tau_{\overline{\Omega},\overline{p}} \ ,$$

$$\eta = \eta_{\Omega,p} \ , \ \overline{\eta} = \eta_{\overline{\Omega},\overline{p}} \ .$$

1.3.1 Since the maps p_E are surjective, for each A in Cyl, there exists a unique $\overline{A} \in \overline{\mathrm{Cyl}}$ s.t.

$$A = \overline{A} \cap \Omega \ .$$

Then,

$$A \in \mathrm{Cyl} \to \overline{A} \in \overline{\mathrm{Cyl}} \ \text{ is bijective}$$

and

$$\tau(A) = \overline{\tau}(\overline{A}) \ \text{ for all } A \in \mathrm{Cyl} \ .$$

Hence, for any $B \cap \overline{\Omega}$,

$$\overline{\eta}(B \cap \Omega) = \inf\{\underline{\Sigma}_{H\in \mathcal{H}} \ \overline{\tau}(\overline{H}) : \overline{\mathcal{H}} \subset \overline{\mathrm{Cyl}} \text{ is countable, } B \cap \Omega \subset \bigcup \overline{\mathcal{H}}\}$$

$$= \inf\{ \sum_{H\in\mathcal{H}} \tau(H) : \mathcal{H} \subset \mathrm{Cyl} \text{ is countable, } B \cap \Omega \subset \bigcup \mathcal{H}\}$$

$$= \eta(B \cap \Omega) \ .$$

Consequently, if (Ω,η) is a limit, then, by Remark 1.2.4., for any $\overline{A} \in \overline{\mathrm{Cyl}}$,

$$\overline{\eta}(\overline{A} \cap \Omega) = \eta(A) = \tau(A) = \overline{\tau}(\overline{A}) = \overline{\eta}(\overline{A}) \ .$$

On the other hand, if

$$\overline{\eta}(\overline{A}) = \overline{\eta}(\overline{A} \cap \Omega) \ \text{ for all } \overline{A} \in \overline{\mathrm{Cyl}} \ ,$$

then, again by Remark 1.2.4., for any $A \in \mathrm{Cyl}$,

$$\eta(A) = \overline{\eta}(\overline{A} \cap \Omega) = \overline{\eta}(\overline{A}) = \overline{\tau}(\overline{A}) = \tau(A) \ ,$$

and therefore (Ω,η) is a limit.

1.3.2. For any subfamily $\mathcal{F}'$ of $\mathcal{F}$; let

$$\overline{\mathrm{Cyl}}(\mathcal{F}') = \{\overline{p}_F^{-1}[B] : F \in \mathcal{F}' , B \in M_F\} .$$

We shall show that for any $A \in \overline{\mathrm{Cyl}}$ there exists $A' \subset \overline{\Omega}$ s.t.

$$\overline{\eta}(A') = 0 \quad \text{and} \quad \overline{\eta}(A \sim A') = \overline{\eta}(A \cap \Omega) .$$

In which case,

$$\overline{\eta}(A) = \overline{\eta}(A \sim A') + \overline{\eta}(A \cap A') = \overline{\eta}(A \sim A') = \overline{\eta}(A \cap \Omega) ,$$

and the lemma follows.

For each $n \in \omega$, let $\mathcal{H}_n \subset \overline{\mathrm{Cyl}}$ be countable with

$$A \cap \Omega \subset \bigsqcup \mathcal{H}_n \quad \text{and} \quad \sum_{H \in \mathcal{H}_n} \overline{\tau}(H) \leq \overline{\eta}(A \cap \Omega) + 1/n .$$

For each $n \in \omega$, choose countable $\mathcal{F}_n \subset \mathcal{F}$ with

(1) $\{A\} \cup \mathcal{H}_n \subset \overline{\mathrm{Cyl}}(\mathcal{F}_n)$.

Since $(X_F, \mu_F)_{F \in \mathcal{F}}$ is an inverse system of measure spaces relative to the maps $r_{E,F}$, we may further assume that

(2) $\mathcal{F}_n$ is a sequence $\{F_{n,j}\}_{j \in \omega}$ in $\mathcal{F}$ with $F_{n,j} < F_{n,j+1}$ for each $j \in \omega$.

Let

$$\Lambda_n = \Lambda(\mathcal{F}_n)$$

and

$$A' = \bigcup_{n \in \omega} \Lambda_n .$$

Since $\overline{\eta}(\Lambda_n) = 0$ for every $n \in \omega$, then

(3) $\overline{\eta}(A') = 0$.

Let $n \in \omega$. For each $f \in A \sim \Lambda_n$ there exists $g \in \Omega$ s.t.

$$\overline{p}_F(g) = \overline{p}_F(f) \quad \text{for all} \quad F \in \mathcal{F}_n .$$

In particular, by (1),

$$g \in A \quad .$$

Hence, for some $H \in \mathcal{H}_n$, with $H = \overline{p}_G^{-1}[\overline{p}_G^{-1}[H]]$ for some $G \in \mathcal{F}_n$,

$$g \in H \quad ,$$

and consequently,

$$f \in \overline{p}_G^{-1}[\overline{p}_G(f)] = \overline{p}_G^{-1}[\overline{p}_G(g)] \subset \overline{p}_G^{-1}[\overline{p}_G[H]] = H \quad .$$

It follows that

$$A \sim \Lambda_n \subset \bigsqcup \mathcal{H}_n \quad ,$$

and therefore

$$\text{(4)} \qquad \overline{\eta}(A \sim \Lambda_n) \leq \overline{\eta}(A \cap \Omega) + 1/n \quad .$$

Since $\Lambda_n \subset \overline{\Omega} \sim \Omega$ for each $n \in \omega$,

$$A' \subset \overline{\Omega} \sim \Omega \quad .$$

Hence

$$A \cap \Omega \subset A \sim A' \quad ,$$

and therefore, by (4), for each $n \in \omega$,

$$\overline{\eta}(A \cap \Omega) \leq \overline{\eta}(A \sim A') \leq \overline{\eta}(A \sim \Lambda_n) \leq \overline{\eta}(A \cap \Omega) + 1/n \quad .$$

Consequently,

$$\overline{\eta}(A \cap \Omega) = \overline{\eta}(A \sim A') \quad .$$

2. Cylinder Measures over Vector Spaces.

We shall view a cylinder measure over a vector space X as being an inverse system of measure spaces whose indexing set is the family of finite dimensional subspaces of X.

In this paper we shall consider only complex vector spaces, and we shall hereafter refer to them simply as vector spaces. By the term subspace we shall always mean vector subspace.

We note that if F is a finite-dimensional vector space, then there is a unique Hausdorff topology on F under which it is locally convex (the Euclidean topology). Since this is the only topology on F that we shall ever consider, explicit reference to it is hereafter omitted.

Throughout the remainder of this work, we shall use the following notation.

For any vector space X,

X^* is the set of linear functionals on X to $\mathbb{C}$,

w^* is the topology on X^* of pointwise convergence,

For any $A \subset X$,

$$A^{\circ} = \{f \in X^* : |f(x)| \leq 1 \text{ for all } x \in A\} .$$

$\mathcal{F}_X$ is the family of finite-dimensional subspaces of X directed by $\subset$. When there can be no ambiguity we shall omit the subscript X.

For any subspaces E and F of X with $E \subseteq F$,

$r_{E,F} : F^* \to E^*$ is the restriction map,

i.e. for all $f \in F^*$,

$$r_{E,F}(f) = f|E .$$

In what follows, E and F will always denote finite-dimensional vector spaces.

For any subspace Ω of X^*,

(X,Ω) is a dual pair iff $r_{F,X}|\Omega$ is surjective for every $F \in \mathcal{F}$.

Remark.

With the viewpoint of inverse systems discussed in the preceeding section, taking $\mathcal{F}_X$ as our index set and letting

$$X_F = F^* \quad \text{for each } F \in \mathcal{F}_X ,$$

we note that, for any E, F and G in $\mathcal{F}_X$ with $E \subset F \subset G$,

the restriction map $r_{E,F}$ is surjective and continuous,

and

$$r_{E,G} = r_{E,F} \circ r_{F,G} .$$

Thus, we shall make the following definition.

2.1 Definition.

(1) Let X be a vector space.

μ is a cylinder measure over X

iff

$\mu : F \in \mathcal{F} \to \mu_F$, a Radón measure on F^*,

is such that

$(F^*, \mu_F)_{F\in\mathcal{F}}$ is an inverse system of measure spaces relative to the restriction maps $r_{E,F}$.

(2) μ is a cylinder measure iff μ is a cylinder measure over some vector space X.

Remark.

Let X be a vector space. If

$$\mu : F \varepsilon \mathcal{F} \to \mu_F \quad , \text{ a finite Radon measure on } F^* \quad ,$$

then, by §0.4,

μ is a cylinder measure over X iff for any E and F in $\mathcal{F}$ with $E \subset F$,

$$\mu_E = r_{E,F}[\mu_F] \quad .$$

Let Ω be any subspace of X^*. For any E and F in $\mathcal{F}$ with $E \subset F$,

$$r_{E,X}|\Omega = r_{E,F} \circ (r_{F,X}|\Omega) \quad .$$

Hence, when (X,Ω) is a dual pair, the viewpoint of Definition 1.1.2 applies, with

$$p_E = r_{E,X}|\Omega \quad \text{for each } E \varepsilon \mathcal{F} \quad .$$

We shall therefore make the following definition.

2.2 Definition.

Let X be a vector space, μ a cylinder measure over X, and Ω be a subspace of X^* such that (X,Ω) is a dual pair.

For any outer measure ξ on Ω ,

ξ is a limit measure of $\dot{\mu}$ on Ω iff

(Ω,ξ) is a limit relative to the restriction maps $r_{F,X}|\dot{\Omega}$ of the inverse system of measure spaces $(F^*,\mu_F)_{F\varepsilon\mathcal{F}}$.

Remarks.

From the theory of inverse systems of measure spaces we know several conditions under which we can put a limit measure on the projective limit set L , where

$$L = \{\ell \;\varepsilon \prod_{E\varepsilon \mathcal{F}} E^* : \ell_E = r_{E,F}(\ell_F) \;,\; E \subset F\} \;.$$

Since there exists a set isomorphism

$$\Gamma : X^* \to L$$

such that

$$r_{F,X}(f) = (\Gamma(f))_F \quad \text{for all} \quad f \;\varepsilon\; X^* \quad \text{and} \quad F \;\varepsilon\; \mathcal{F} \;,$$

it follows that

L is sequentially maximal (Defn. 3.4).

Hence, by a theorem of Bochner ([4] p. 120), we deduce that μ always has a limit measure on X^* . However, little has been said about the properties such a limit measure can have. Therefore, in the next section, we shall construct one having special approximation properties.

Unfortunately, for most practical purposes X^* is far too unwieldy. We shall therefore be studying the problem of putting limit measures on subspaces of X^* .

3. Non-topological Limit Measures.

Given any cylinder measure μ over a vector space X, and subspace Ω of X^* such that (X,Ω) is a dual pair, we shall determine sufficient conditions on μ for it to have a limit measure on Ω.

Throughout this section we shall use the following notation.

X is a vector space.

For any cylinder measure μ over X, and subspace Ω of X^*,

$$\mathrm{Cyl}_\mu(\Omega) = \{\Omega \cap r^{-1}_{F,X}[A] : F \in \mathcal{F},\ A \in M_F\} ,$$

$$\tau_{\mu,\Omega} : \Omega \cap r^{-1}_{F,X}[A] \in \mathrm{Cyl}_\mu(\Omega) \to \mu_F(A) \in R^+ ,$$

μ_Ω is the Caratheodory measure on Ω generated by $\tau_{\mu,\Omega}$ and $\mathrm{Cyl}_\mu(\Omega)$.

$$\tau^*_\mu = \tau_{\mu,X^*}$$

and

$$\mu^* = \mu_{X^*} .$$

In what follows,

Ω will denote a subspace of X^* such that (X,Ω) is a dual pair.

From Definition 1.1.2 and Remarks 1.2 we get the following assertions.

3.1 Propositions.

Let μ be a cylinder measure over X.

(1) For any outer measure ξ on Ω, ξ is a limit measure of μ iff

$$\mathrm{Cyl}_{\mu}(\Omega) \subset M_{\xi} \quad \text{and} \quad \xi | \mathrm{Cyl}_{\mu}(\Omega) = \tau_{\mu,\Omega} .$$

(2) If there exists any limit measure of μ on Ω, then μ_{Ω} is a limit measure of μ.

In view of Proposition 3.1.2, when looking for a limit measure of μ on Ω, we shall concentrate on μ_{Ω}.

When $\Omega = X^{*}$, we have the following result.

3.2 Theorem

For any cylinder measure μ over X,

μ^{*} is a limit measure of μ.

If

$$C = \{r^{-1}_{F,X}[K] : F \in F,\ K \subset F^{*} \text{ is compact}\}$$

then

C is a compact family,

and for any $A \in M_{\mu^*}$,

$$\mu^{*}(A) = \sup\{\mu^{*}(C) : C \subset A,\ C \in C_{\delta}\} .$$

(We note that C_{δ} is also a compact family.)

<u>Proof</u> For each $F \in \mathcal{F}$,

(1) μ_F is Radon and σ-finite.

Hence, for any $A \in \mathcal{C}yl_\mu(X^*)$,

(2) $\tau_\mu^*(A) = \sup\{\tau_\mu^*(C) : C \subseteq A , C \in \mathcal{C}\}$.

Since

$$\mu^*(A) \leq \tau_\mu^*(A) \quad \text{for all} \quad A \in \mathcal{C}yl_\mu(X^*) ,$$

we also deduce from (1) that

(3) μ^* is σ-finite.

Hence, by Thm II.2.5 of [23], the assertions of the theorem will follow once we show that $\mathcal{C}$ is a compact family. (Also see [24]).

For any $C_j = r^{-1}_{F_j,X}[K_j] \in \mathcal{C}$, $j = 1,2$, let F be the linear span of $F_1 \cup F_2$, and $K = \bigcap_{j=1,2} r^{-1}_{F_j,F}[K_j]$. Then K is compact and $C_1 \cap C_2 = r^{-1}_{F,X}[K]$.

Hence,

(4) $\mathcal{C}$ is closed under finite intersections.

For any $\mathcal{C}' \subseteq \mathcal{C}$ s.t. $\bigcap\alpha \neq \phi$ for every finite $\alpha \subset \mathcal{C}'$, let

$$\mathcal{A} = \{\textstyle\bigcap\alpha : \alpha \subset \mathcal{C}' \text{ is finite}\}$$

We note that $\mathcal{A}$ is a filterbase ([8] p. 211). In view of (4), for each finite $\alpha \subseteq \mathcal{C}'$, let

$$\textstyle\bigcap\alpha = r^{-1}_{F_\alpha,X}[K\ ] \quad \text{for some} \quad F_\alpha \in \mathcal{F} \quad \text{and compact}$$

$K_\alpha \subset F_\alpha^*$, and

$$Y = \cup\{F_\alpha : \alpha \subset \mathcal{C}' \text{ is finite}\} .$$

From the remarks preceding (4), we see that for any finite subfamilies α and β of C',

$$\alpha \subseteq \beta \Rightarrow F_\alpha \subset F_\beta ,$$

and therefore Y is a subspace of X.

Let U be a maximal filterbase in X^* ([8] p. 218) which is a subfilterbase of A ([8] p. 219, Thm. 7.3). Then, for each finite $\alpha \subset C'$,

$(r_{F_\alpha}[U])_{u \varepsilon U}$ is a maximal filterbase in F^*, and

there exists $u \varepsilon U$ s.t. $r_{F_\alpha}[u] \subset K_\alpha$.

Since K_α is compact and F_α^* is Hausdorff, this ultrafilter converges to a unique point

$$f_\alpha \varepsilon K_\alpha .$$

We note that if $F_\alpha = F_\beta$, then $f_\alpha = f_\beta$. Also, for any finite subfamilies α and β of C' with $\alpha \subset \beta$,

$$r_{F_\alpha, F_\beta}(f_\beta) = f_\alpha ,$$

since the restriction map is continuous and

$$r_{F_\alpha, X} = r_{F_\alpha, F_\beta} \circ r_{F_\beta, X} .$$

Consequently, there exists a unique $g \varepsilon Y^*$ s.t.

$$g|F_\alpha = f_\alpha \quad \text{for each finite } \alpha \subset C' .$$

If $f \varepsilon X^*$ is any linear extension of g, then, for each finite $\alpha \subset C'$.

$$f \varepsilon r^{-1}_{F_\alpha, X}[r_{F_\alpha, X}(f)] = r^{-1}_{F_\alpha, X}[f_\alpha] \subset r^{-1}_{F_\alpha, X}[K_\alpha] = \bigcap \alpha .$$

Hence,

$$\bigcap C' \neq \phi .$$

It follows that C is a compact family. ∎

Next, we consider the problem of finding a limit measure of μ on an arbitrary Ω .

Since μ^* is always a limit measure of μ , application of Lemma 1.3.1 yields the following basic result.

3.3 Lemma.

For any cylinder measure μ over X ,

μ has a limit measure on Ω iff

(1) $$\mu^*(A) = \mu^*(A \cap \Omega) \quad \text{for all} \quad A \in Cyl_\mu(X^*) \quad .$$

However, we are interested in finding intrinsic conditions on our systems which will guarantee the existence of a limit measure on Ω .

One such condition is the following, which is of considerable importance in the general theory of inverse systems of measure spaces (Bochner [4], p. 120, Choksi [6], Mallory and Sion [23]).

3.4 Definition

Ω is sequentially maximal iff

for any sequence $\{F_n\}_{n\varepsilon\omega}$ in $\mathcal{F}$ with $F_n \subset F_{n+1}$ for each $n \varepsilon \omega$, and $f_n \varepsilon F_n^*$ such that $r_{F_n,F_{n+1}}(f_{n+1}) = f_n$, there exists $g \varepsilon \Omega$ such that

$r_{F_n,X}(g) = f_n$ for each $n \varepsilon \omega$.

Remark.

We note that X^* is sequentially maximal. Consequently, the fact that μ^* is a limit measure of μ follows also from a theorem of Bochner ([4], p. 120).

Since μ^* is always a limit measure of μ, application of Lemmas 1.3.2 and 3.3 yields the following.

3.5 Proposition

If Ω is sequentially maximal, then every cylinder measure over X has a limit measure on Ω.

However we have the following.

3.6 Observation.

If X is a topological vector space containing a bounded, countable, linearly independent subset, and Ω is its continuous dual, then Ω is not sequentially maximal. (e.g. whenever X is an infinite-dimensional, metrizable, locally convex space).

Proof Let $\{a_n ; n \varepsilon \omega\}$ be a bounded, countable, linearly independent subset of X, and for each $n \varepsilon \omega$ let F_n be the linear span of $\{a_0,\ldots,a_n\}$. Then, for any $f \varepsilon X^*$ with $f(a_n) = n$ for every $n \varepsilon \omega$,

(1) $f[\{a_n ; n \in \omega\}] \subset \mathbb{C}$ is unbounded.

Hence, there cannot exist $g \in \Omega$ s.t. $g|F_n = f|F_n$ for every $n \in \omega$. For if so, then $g|\cup_{n\in\omega} F_n$ is continuous, and therefore $g[\{a_n : n \in \omega\}]$ is bounded, which contradicts (1). ∎

Since, in the theory of cylinder measure, Ω is often the continuous dual of metrizable l.c. space ([11], [39]), it follows that the condition of sequential maximality does not apply in many important situations.

In order that we might take fuller advantage of Lemma 3.3, we therefore weaken the notion of sequential maximality.

3.7 Definition

Let μ be a cylinder measure over X.

Ω is μ-sequentially maximal iff

for any sequence $\{F_n\}_{n\in\omega}$ in F with $F_n \subset F_{n+1}$ for every $n \in \omega$, and $\varepsilon > 0$

there exists $A_n \in M_{F_n}$ for each $n \in \omega$, such that

$$\sum_{n\in\omega} \mu_{F_n}(A_n) < \varepsilon ,$$

and for any sequence $\{f_n\}_{n\in\omega}$ with

$$f_n \in F_n^* \sim A_n , \; r_{F_n,F}(f_{n+1}) = f_n ,$$

there exists $g \in \Omega$ such that $r_{F_n,X}(g) = f_n$ for each $n \in \omega$.

The following key theorem of this section is now an immediate consequence of Lemma 1.3.2 and the above definition.

3.8 Theorem

Let μ be a cylinder measure over X .

If Ω is μ-sequentially maximal, then μ has a limit measure on Ω .

We now establish a condition on μ which ensures that Ω is μ-sequentially maximal.

3.9 Definition

Let μ be a cylinder measure over X .

For any family $\mathcal{H}$ of subsets of X^* ,

μ is $\mathcal{H}$-sequentially tight iff

for any sequence $\{F_n\}_{n\varepsilon\omega}$ in $\mathcal{F}$ with $F_n \subset F_{n+1}$ for each $n \varepsilon \omega$, $A \varepsilon M_{F_0}$ with $\mu_{F_0}(A) < \infty$, and $\varepsilon > 0$,

there exists $H \varepsilon \mathcal{H}$ such that

$$\mu_{F_n}(r^{-1}_{F_0,F_n}[A]) \sim r_{F_n,X}[H]) < \varepsilon \quad \text{for all } n \varepsilon \omega \ .$$

3.10 Theorem

Let μ be a cylinder measure over X .

If μ is $\mathcal{H}$-sequentially tight for some family $\mathcal{H}$ of w^*-compact subsets of Ω , then

Ω is μ-sequentially maximal,

and therefore μ has a limit measure on Ω .

We point out that under certain conditions μ-sequential maximality of Ω is also a necessary condition for μ to have a limit measure on Ω .

3.11 Proposition

Suppose that the Mackey topology on X induced by Ω ([47] p. 369) restricted to any subspace of countable dimension is metrizable. For any cylinder measure μ over X ,

if μ has a limit measure on Ω , then Ω is μ-sequentially maximal.

Proofs 3.

Lemma. Let

$\{F_n\}_{n\varepsilon\omega}$ be a sequence in $\mathcal{F}$ with $F_n \subset F_{n+1}$ for each $n \varepsilon \omega$,

K be a w^*-compact subset of X^* .

For any sequence $\{f_n\}_{n\varepsilon\omega}$ with

$$f_n \varepsilon r_{F_n,X}[K] \text{ and } r_{F_n,F_{n+1}}(f_{n+1}) = f_n .$$

there exists $g \varepsilon K$ s.t

$$r_{F_n,X}(g) = f_n \text{ for all } n \varepsilon \omega .$$

Proof For each $n \varepsilon \omega$,

(1) $$r^{-1}_{F_n,X}[f_n] \cap K \neq \phi .$$

Since $r_{F_n,F_{n+1}}[f_{n+1}] = f_n$,

(2) $$r^{-1}_{F_n,X}[f_n] \supset r^{-1}_{F_{n+1}}[f_{n+1}] \quad .$$

Also, since $r_{F_n,X}$ is w^*-continuous and K is w^*-compact,

(3) $$K \cap r^{-1}_{F_n,X}[f_n] \text{ is } w^*\text{-compact.}$$

Since w^* is a Hausdorff topology, it follows from

(1), (2) and (3), that

$$\bigcap_{n\varepsilon\omega} (K \cap r^{-1}_{F_n,X}[f_n]) \neq \phi \quad ,$$

and the lemma follows.

3.10 Let

$$\{F_n\}_{n\varepsilon\omega} \subset \mathcal{F} \text{ with } F_n \subset F_{n+1} \text{ for each } n \varepsilon \omega,$$

$$\{B\}_{j\varepsilon\omega} \subset M_{F_0}, \text{ with } \mu_{F_0}(B_j) < \infty \text{ for each } j \varepsilon \omega \quad ,$$

and $F_0^* = \bigcup_{j\varepsilon\omega} B_j$ . .

Since μ is $\mathcal{H}$-sequentially tight for some family $\mathcal{H}$ of w^*-compact subsets of Ω , given $\varepsilon > 0$, for each $j \varepsilon \omega$ choose a w^*-compact $K_j \subset \Omega$ s.t

$$\sup_{n\varepsilon\omega} \mu_{F_n}(r^{-1}_{F_0,F_n}[B_j] \sim r_{F_n,X}[K_j]) < \varepsilon/2^{j+1} \quad .$$

Let

$$C_n = \bigcup_{j\varepsilon\omega} (r^{-1}_{F_0,F_n}[B_j] \sim r_{F_n,X}[K_j])$$

$$A_0 = C_0$$

$$A_{n+1} = C_{n+1} \sim r^{-1}_{F_n,F_{n+1}}[C_n]$$

Then, for any $k \in \omega$,

$$\bigcup_{n=0}^{k} r^{-1}_{F_n,F_k}[A_n] = C_k$$

and

$$\sum_{n=0}^{k} \mu_{F_n}(A_n) = \mu_k(C_k) \leq \sum_{j=1}^{k} \varepsilon/2^{j+1} \quad .$$

Hence $\sum_{n\in\omega} \mu_{F_n}(A_n) < \varepsilon$.

If $\{f_n\}$ is a sequence s.t. for each $n \in \omega$,

$$f_n \in F_n^* \sim A_n \ , \ r_{F_n,F_{n+1}}(f_{n+1}) = f_n \ ,$$

then, for some $j \in \omega$,

$$f_0 \in r_{F_0,X}[K_j] \cap B_j$$

and hence for every $n \in \omega$,

$$f_{n+1} \in r_{F_{n+1},X}[K_j] \quad .$$

Consequently, by the Lemma, there exists $g \in \Omega$ s.t. $r_{F_n,X}(g) = f_n$ for all $n \in \omega$, and it follows that Ω is μ-sequentially maximal.

The last assertion is immediate from Thm. 3.8.

<u>Proof of 3.11</u>

For any $\{F_n\}_{n\in\omega} \subset \mathcal{F}$ with $F_n \subset F_{n+1}$ for each $n \in \omega$, let

$Y = \bigcup_{n\in\omega} F_n$ with the restricted topology,

$\Lambda = \{f \in X^*$: there does not exist $g \in \Omega$ s.t.

$$r_{F_n,X}(g) = r_{F_n,X}(f) \quad \text{for all} \quad n \in \omega\} \quad .$$

$\mathcal{B} = \{r^{-1}_{F_n,X}[A] : n \in \omega, A \in M_{F_n}\}$,

$\mathfrak{S}$ be the σ-field generated by $\mathcal{B}$, and

η the Caratheodory measure on X^* generated by $\tau^*_\mu|\mathcal{B}$ and $\mathcal{B}$. Since the topology of X restricted to Y is metrizable, choose a sequence $\{V_k\}_{k\varepsilon\omega}$ of absolutely convex neighbourhoods of the origin in X s.t. $\{V_k \cap Y\}_{k\varepsilon\omega}$ is a base for the neighbourhoods of the origin in Y . Using the Hahn-Banach extension theorem, one readily checks that

$$X^* \sim \Lambda = \bigcup_{k\varepsilon\omega} \bigcap_{n\varepsilon\omega} r^{-1}_{F_n,X}[r_{F_n,X}[V^0_k]] \ .$$

It then follows that

(1) $\qquad \Lambda \in \mathfrak{S}$,

and, since $\mathrm{Cyl}_\mu(X^*) \subset M_{\mu^*}$,

(2) $\qquad \Lambda \in M_{\mu^*}$.

We note that

(3) $\Lambda \subset X^* \sim \Omega$.

Since μ^* is σ-finite and $\mu^*(A) = \mu^*(A \cap \Omega)$ for all $A \in \mathrm{Cyl}_\mu(X^*)$, from (2) and (3) it follows that

(4) $\qquad \mu^*(\Lambda) = 0$

Since $\mathcal{B}$ is a field and τ^*_μ is countably additive on $\mathrm{Cyl}_\mu(X^*)$ (Thm. 3.2, Remark 1.2.5), we have that

$$\eta|\mathcal{B} = \tau^*_\mu|\mathcal{B} \quad \text{and} \quad \mu^*|\mathcal{B} = \tau^*_\mu|\mathcal{B} \ .$$

However, $\tau^*_\mu|\mathcal{B}$ has a unique countably additive extension to $\mathfrak{S}$. Hence, since $\mathfrak{S} \subset M_\eta \cap M^*_\mu$,

$$\eta|\mathfrak{S} = \mu^*|\mathfrak{S} ,$$

and therefore, by (1) and (4),

$$\eta(\Lambda) = 0 \ .$$

Consequently, given any $\varepsilon > 0$, there exists

$\{B_j\}_{j\varepsilon\omega} \subset \mathcal{B}$ s.t.

$$\Lambda \subset \bigcup_{j\varepsilon\omega} B_j \quad \text{and} \quad \sum_{j\varepsilon\omega} \tau_\mu^*(B_j) < \varepsilon \ .$$

For each $j \varepsilon \omega$, let

$$B_j = r^{-1}_{F_{n_j},X}[B'_j] \ , \ B'_j \ \varepsilon \ M_{F_{n_j}} \ ,$$

and

$$B_j \nsubseteq r^{-1}_{F_n,X}[B'] \quad \text{for any } n < n_j \text{ and } B' \ \varepsilon \ M_{F_n} \ .$$

For each $n \varepsilon \omega$, let

$$A_n = \bigsqcup \{B'_j : j \ \varepsilon \ \omega, \ n_j = n\}$$

Then,

(5) $$\sum_{n\varepsilon\omega} \mu_{F_n}(A_n) < \varepsilon \ .$$

Further, if $\{f_n\}_{n\varepsilon\omega}$ is any sequence s.t. for each $n \varepsilon \omega$,

$$f_n \ \varepsilon \ F_n \sim A_n \ , \ r_{F_n,F_{n+1}}(f_{n+1}) = f_n \ ,$$

then, there exists $f \ \varepsilon \ X^* \sim \bigcup_{n\varepsilon\omega} r^{-1}_{F_n,X}[A_n]$ s.t.

$$r_{F_n,X}(f) = f_n \quad \text{for each } n \ \varepsilon \ \omega \ .$$

Since $\Lambda \subset \bigcup_{j\varepsilon\omega} B_j = \bigcup_{n\varepsilon\omega} r^{-1}_{F_n,X}[A_n]$, from the definition of Λ it follows that

(6) there exists $g \ \varepsilon \ \Omega$ s.t. $r_{F_n,X}(g) = f_n$ for all $n \varepsilon \omega$.

Since the sequence $\{F_n\}_{n\varepsilon\omega}$ in $\mathcal{F}$ with $F_n \subsetneq F_{n+1}$ for each $n \varepsilon \omega$ was arbitrary, we conclude that Ω is μ-sequentially maximal.

4. Radón Limit Measures

In this section we shall consider the problem of finding Radón limit measures. The technique we use was communicated to us by C. Scheffer.

In this section we shall use the following notation.

X is a vector space,

Ω is a subspace of X^* such that (X,Ω) is a dual pair.

For any topology $\mathcal{G}$ on Ω and cylinder measure μ over X,

$g : A \subset X^* \to \inf \{\mu_F(r_{F,X}[A]) : F \in \mathcal{F}\}$,

$g_* : G \in \mathcal{G} \to \sup \{g(K) : K \in \mathcal{K}(\mathcal{G}), K \subset G\}$,

$\mu_{\mathcal{G}}$ is the Caratheodory measure on Ω generated by g_* and $\mathcal{G}$.

We shall hereafter assume that

μ is a fixed cylinder measure over X,

$\mathcal{G}$ is a regular Hausdorff topology on Ω which is finer than w^* restricted to Ω.

We have the following important assertions.

4.1 Propositions

(1) $\mu_{\mathcal{G}}$ is a $\mathcal{G}$-Radón measure on Ω, and $\mu_{\mathcal{G}}|\mathcal{G} = g_*$.

(2) $\mathrm{Cyl}_\mu(\Omega) \subset M_{\mu_{\mathcal{G}}}$.

(3) If there exists any $\mathcal{G}$-Radón limit measure of μ on Ω, then $\mu_{\mathcal{G}}$ is a limit measure of μ.

In view of the above propositions, when searching for a $\mathcal{G}$-Radón limit measure of μ, we shall restrict our attention to $\mu_{\mathcal{G}}$.

Following Scheffer [37] we make the following definition. Our terminology is slightly different.

4.2 <u>Definition</u> For any family $\mathcal{H}$ of subsets of X^*,

μ is $\mathcal{H}$-tight iff

for any $E \in \mathcal{F}$, $A \in M_E$ with $\mu_E(A) < \infty$, and $\varepsilon > 0$,

there exists $H \in \mathcal{H}$ such that

$\mu_F(r_{E,F}^{-1}[A] \sim r_{F,X}[H]) < \varepsilon$ for all $F \in \mathcal{F}$ with $E \subset F$.

We point out that the above definition is a "uniform" version of the definition of $\mathcal{H}$-sequential tightness (Defn. 3.9).

We now have the following key theorem concerning the existence of a $\mathcal{G}$-Radón limit measure of μ.

4.3 <u>Theorem</u>

μ has a $\mathcal{G}$-Radón limit measure on Ω

$\Leftrightarrow$ μ is $\mathcal{K}(\mathcal{G})$-tight.

<u>Remark</u> The above theorem extends a result due to Mourier [26], and Prohorov [33] (§5 Lemma 3). However, our approach is somewhat different from theirs.

Theorem 4.3 has a useful corollary.

Corollary If X is a metrizable, locally convex space, Ω is its continuous dual, and $\{V_n\}_{n\varepsilon\omega}$ is a base for the neighbourhoods of the origin in X, with $V_{n+1} \subset V_n$ for every $n \varepsilon \omega$, then,

μ has a w^*-Radon limit measure on Ω <=>

μ has a limit measure on Ω <=>

μ is $\{V_n^0\}_{n\varepsilon\omega}$-tight.

Proofs 4.

Notation Let

$$H = (Cyl_\mu(\Omega)) \cap (w^*|\Omega) \quad ,$$

$$\gamma = \mu_G \ ,$$

and $\tau = \tau_{\mu,\Omega}$.

We shall need the following lemmas.

L.1 For any $A \varepsilon Cyl_\mu(\Omega)$,

(1) $\tau(A) = \inf \{\tau(H) : A \subset H \varepsilon H\}$

(2) $\gamma(A) \leq \tau(A)$.

L.2 For any $K \varepsilon K(G)$,

(1) $r_{E,X}[K]$ is compact for every $E \varepsilon F$, therefore μ_E-measurable.

(2) $g(K) = \mu_\Omega(K) = \inf \{\tau(H) : K \subset H \varepsilon H\}$

(3) For any E and F in F with $E \subset F$, and $A \varepsilon M_E$,

$$\mu_E(r_{E,X}[K]) \geq \mu_F(r_{F,X}[K]) \quad .$$

Proof of L.1.1 For any $E \in F_X$ and $B \in M_E$, since $r_{E,X}$ is w^*-continuous and μ_E is Radón,

$$\tau(\Omega \cap r_{E,X}^{-1}[B]) = \mu_E(B) = \inf \{\mu_E(G) : B \subset G \subset E^*, G \text{ is open}\}$$

$$\geq \inf \{\tau(H) : \Omega \cap r_{E,X}^{-1}[B] \subset H \in \mathcal{H}\} \geq \tau(\Omega \cap r_{E,X}^{-1}[B]) .$$

Proof of L.1.2 We note that $\mathcal{H} \subset \mathcal{G}$, and for every $H \in \mathcal{H}$,

$$g_*(H) \leq \tau(H) .$$

Hence,

$$\gamma(A) = \inf \{ \sum_{H \in \mathcal{H}'} g_*(H) : \mathcal{H}' \subset \mathcal{H} \text{ is countable and } A \subset \bigcup \mathcal{H}'\}$$

$$\leq \inf \{ \sum_{H \in \mathcal{H}'} \tau(H) : \mathcal{H}' \subset \mathcal{H} \text{ is countable and } A \subset \bigcup \mathcal{H}'\}$$

$$\leq \inf \{\tau(H) : A \subset H \in \mathcal{H}\} = \tau(A) \quad \text{, by L.1.1.}$$

Proof of L.2.1 We only observe that for every $E \in F$, μ_E is Radon, $\mathcal{K}(\mathcal{G}) \subset \mathcal{K}(w^*)$, and $r_{E,X}$ is w^*-continuous.

Proof of L.2.2 For every $E \in F$, μ_E is Radón and $r_{E,X}$ is w^*-continuous. Therefore,

$$g(K) = \inf \{\mu_E(G) : E \in F, r_{E,X}[K] \subset G \subset E^*, G \text{ is open}\}$$

$$= \inf \{\tau(H) : K \subset H \in \mathcal{H}\} .$$

On the other hand, by L.1.1,

$$\mu_\Omega(K) = \inf \{ \sum_{H \in \mathcal{H}'} \tau(H) : \mathcal{H}' \subset \mathcal{H} \text{ is countable, } K \subset \bigcup \mathcal{H}'\}$$

$$= \inf \{\tau(H) : K \subset H \in \mathcal{H}\},$$

since,

K is $\mathcal{G}$-compact, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{H}$ is closed under finite unions, and τ is finitely subadditive on $Cyl_\mu(\Omega)$.

Proof of L.2.3 We have that

$$r_{F,X}[K] \subset r_{E,F}^{-1}[r_{E,X}[K]] \quad .$$

Hence,

$$\mu_E(r_{E,X}[K]) = \mu_F(r_{E,F}^{-1}[r_{E,X}[K]]) \geq \mu_F(r_{F,X}[K]) \quad .$$

4.1.1 To show that γ is a G-Radón measure, by Sion [44] Ch. V, Thm. 2.2, we need only show that

(1) $g(\phi) = 0$, g is positive, monotone, subadditive and additive on $\mathcal{K}(G)$,

(2) $\gamma(K) < \infty$ for all $K \in \mathcal{K}(G)$,

Except for additivity, the properties of g are immediate from L.2.2. We shall now establish the additivity of g on $\mathcal{K}(G)$.

Let K_1 and K_2 be in $\mathcal{K}(G)$ with $K_1 \cap K_2 = \phi$. Since $\mathcal{K}(G) \subset \mathcal{K}(w^*|\Omega)$, and $w^*|\Omega$ is regular and Hausdorff, there exists

$$G_j \in w^*|X,\ K_j \subset G_j,\ j = 1,2 \quad , \text{with}$$
$$G_1 \cap G_2 = \phi \quad .$$

However, $\mathcal{H}$ is a base for $w^*|\Omega$, and is closed under finite unions. Consequently, since K_1, K_2 are $w^*|\Omega$-compact, there exists

$$H_j \in \mathcal{H}\ ,\ K_j \subset H_j\ ,\ j = 1,2 \quad , \text{with}$$
$$H_1 \cap H_2 = \phi \quad .$$

Then, by L.2.2,

$$g(K_1) + g(K_2) = \mu_\Omega(K_1) + \mu_\Omega(K_2)$$

$$= \sum_{j=1,2} \inf \{\mu_\Omega(A^j) : K_j \subset A_j \in M_{\mu_\Omega}\}$$

$$= \sum_{j=1,2} \inf \{\mu_\Omega(A^j) : A \in M_{\mu_\Omega}, K_j \subset A^j \subset H_j\}$$

$$= \inf \{\mu_\Omega(A^1 \cup A^2) : A^j \in M_{\mu_\Omega}, K_j \subset A^j \subset H_j ; j = 1,2,\} ,$$

$$\leq \mu_\Omega(K_1 \cup K_2) = g(K_1 \cup K_2) .$$

Hence, by the subadditivity of g ,

$$g(K_1 \cup K_2) = g(K_1) + g(K_2) ,$$

Since K_1 and K_2 were arbitrary it follows that g is additive on $K(G)$.

It remains for us to prove (2).

Let $K \in H(G)$. For any $F \in F$, since μ_F is Radon and $r_{F,X}[K]$ is compact (L.2.1),

$$\Omega \cap r^{-1}_{F,X}[r_{F,X}[K]] \in Cyl_\mu(\Omega)$$

and by L.1.2,

$$\gamma(K) \leq \gamma(\Omega \cap r^{-1}_{F,X}[r_{F,X}[K]]) \leq \tau(\Omega \cap r^{-1}_{F,X}[r_{F,X}[K]])$$

$$= \mu_F(r_{F,X}[K]) < \infty .$$

Hence,

$$\gamma(K) < \infty \quad \text{for all} \quad K \in K(G) .$$

<u>4.1.2</u> Let $F \in F$ and $A \in M_F$.

If $\mu_F(A) = 0$, then, by L.1.2,

$$\gamma(\Omega \cap r^{-1}_{F,X}[A]) = 0 .$$

and therefore

$$\Omega \cap r^{-1}_{F,X}[A] \in M_\gamma .$$

Otherwise, since μ_F is Radón, choose a Borel subset B of F^* with

$$A \subset B \quad \text{and} \quad \mu_F(B \sim A) = 0$$

By the preceding observation,

$$\Omega \cap r_{F,X}^{-1}[B \sim A] \in M_\gamma \quad .$$

However, $r_{F,X}|\Omega$ is $\mathcal{G}$-continuous since $w^*|\Omega \subset \mathcal{G}$, and by Prop. 4.1.1, γ is $\mathcal{G}$-Radón. Hence,

$$\Omega \cap r_{F,X}^{-1}[B] \in M_\gamma \quad ,$$

and therefore,

$$\Omega \cap r_{F,X}^{-1}[A] = \Omega \cap r_{F,X}^{-1}[B] \sim \Omega \cap r_{F,X}^{-1}[B \sim A] \in M_\gamma \quad .$$

We shall now establish another useful lemma.

<u>L.3</u> For every $K \in \mathcal{K}(\mathcal{G})$,

$$g(K) \leq \gamma(K) \quad .$$

If γ is a limit measure of μ , then, for every $K \in \mathcal{K}(\mathcal{G})$,

$$g(K) = \gamma(K) \quad .$$

<u>Proof of L.3</u> Let $K \in \mathcal{K}(\mathcal{G})$. By Prop. 4.1.1,

$$\gamma(K) = \inf\{g_*(G) : K \subset G \in \mathcal{G}\}$$
$$\geq g(K) \quad .$$

If γ is a limit measure of μ , then, by L.2.2 and Prop. 3.1,

$$g(K) = \inf\{\tau(H) : K \subset H \in \mathcal{H}\} \doteq \inf\{\gamma(H) : K \subset H \in \mathcal{H}\}$$
$$\geq \gamma(K) \quad , \text{since } \gamma \text{ is } \mathcal{G}\text{-Radón and } \mathcal{H} \subset \mathcal{G} \quad .$$

4.1.3 Let ξ be any G-Radón limit measure of μ on Ω.

For any $K \in K(G)$,

$\xi(K) = \inf\{\xi(G) : K \subset G \in G\}$

$\leq \inf\{\xi(H) : K \subset H \in H\}$ since $H \subset G$,

$= \inf\{\tau(H) : K \subset H \in H\}$ by Prop. 3.1.1

$= g(K)$ by L.2.2

$\leq \gamma(K)$ by L.3.

Hence, as ξ and γ are both G-Radón measures on Ω,

$\xi(A) \leq \gamma(A)$ for all $A \subset \Omega$.

In particular, by Prop. 3.1.1, for any $A \in Cyl_{\mu}(\Omega)$,

$\tau(A) = \xi(A) \leq \gamma(A)$,

and therefore, by L.1.2,

$\gamma(A) = \tau(A)$.

From Props. 3.1.1 and 4.1.2 it now follows that γ is a limit measure of μ.

4.3 By Prop. 4.1.3, if μ has any G-Radon limit measure on Ω, then γ is a G-Radon limit measure of μ. Hence, for any $E \in F$ and $A \in M_E$ with $\mu_E(A) < \infty$,

$\Omega \cap r_{E,X}^{-1}[A] \in M_{\gamma}$, $\gamma(\Omega \cap r_{E,X}^{-1}[A]) < \infty$,

and therefore

$\gamma(\Omega \cap r_{E,X}^{-1}[A]) = \sup\{\gamma(K) : K \in K(G), K \subset \Omega \cap r_{E,X}^{-1}[A]\}$.

Hence, for any $\varepsilon > 0$, there exists $K \in H(G)$ with $K \subset \Omega \cap r_{E,X}^{-1}[A]$ and

$\gamma(\Omega \cap r_{E,X}^{-1}[A] \sim K) < \varepsilon$.

In which case, for any $F \varepsilon F$ with $E \subset F$,

$$\mu_F(r^{-1}_{E,F}[A] \sim r_{F,X}[K])$$

$$= \gamma(\Omega \cap r^{-1}_{F,X}[r^{-1}_{E,F}[A]) \sim \Omega \cap r^{-1}_{F,X}[r_{F,X}[K]])$$

$$\leq \gamma(\Omega \cap r^{-1}_{E,X}[A] \sim K) < \varepsilon \quad .$$

It follows that μ is $K(G)$-tight.

We now show that $K(G)$-tightness of μ is a sufficient condition for μ to have a G-Radón limit measure on Ω . IN view of Props. 4.1 and 3.1.1, we need only show that

(1) μ is $K(G)$-tight $\Rightarrow \gamma | Cyl_\mu(\Omega) = \tau$.

If, for every $A \varepsilon Cyl_\mu(\Omega)$,

(2) $\tau(A) = \sup \{g(K) : K \varepsilon K(G) , K \subset A\}$,

then, for every $A \varepsilon Cyl_\mu(\Omega)$, since μ is σ-finite (L.1.2),

$$\gamma(A) = \sup \{\gamma(K) : K \varepsilon K(G) , K \subset A\} ,$$

$$\geq \sup \{g(K) : K \varepsilon K(G) , K \subset A\} \text{ by L.3,}$$

$$= \tau(A) \text{ by (1).}$$

Hence, by L.1.2,

$$\gamma(A) = \tau(A) \text{ for all } A \varepsilon Cyl_\mu(\Omega) ,$$

Consequently, (1) will have established when we show that,

(3) μ is $K(G)$-tight $\Rightarrow$ (2) holds for all $A \varepsilon Cyl_\mu(\Omega)$.

Suppose μ is $K(G)$-tight.

Let $E \varepsilon F$ and $B \varepsilon M_E$ with $\mu_E(B) < \infty$.

Given $\varepsilon > 0$, since μ_E is Radón, there exists a closed $C \subset B$ s.t.

$$\mu_E(B \sim C) < \varepsilon/2 \quad .$$

Since μ is $K(G)$-tight, there exists $K_1 \varepsilon K(G)$ s.t. for every $F \varepsilon F$ with $E \subset F$,

$$\mu_F(r^{-1}_{E,F}[B] \sim r_{F,X}[K_1]) < \varepsilon/2 \quad .$$

Let

$$K = K_1 \cap r_{E,X}^{-1}[C] \quad .$$

Since C is closed and $r_{E,X}|\Omega$ is $\mathcal{G}$-continuous,

(4) $K \varepsilon \mathcal{K}(\mathcal{G})$.

Further,

(5) $K \subset \Omega \cap r_{E,X}^{-1}[B]$.

Now, for any $F \varepsilon \mathcal{F}$ with $E \subset F$,

(6) $r_{F,X}[K] = r_{F,X}[K_1] \cap r_{E,F}^{-1}[C]$.

Hence,

$\mu_F(r_{E,F}^{-1}[B] \sim r_{F,X}[K])$

$= \mu_F((r_{E,F}^{-1}[B] \sim r_{F,X}[K_1]) \cup (r_{E,F}^{-1}[B] \sim r_{E,F}^{-1}[C]))$

$\leq \mu_F(r_{E,F}^{-1}[B] \sim r_{F,X}[K_1]) + \mu_E(B \sim C) < \varepsilon$.

Consequently, by L.2.3 and L.2.1,

$g(K) = \inf \{\mu_F(r_{F,X}[K]) : E \subset F \varepsilon \mathcal{F}\}$

$\geq \inf \{\mu_F(r_{E,F}^{-1}[B]) - \varepsilon : E \subset F \varepsilon \mathcal{F}\}$

$= \mu_E(B) - \varepsilon$.

Since ε was arbitrary it follows that (2) holds for all $A \varepsilon \mathrm{Cyl}_\mu(\Omega)$ with $\tau(A) < \infty$.

However, since μ_F is σ-finite for all $F \varepsilon \mathcal{F}$,

$\tau(A) = \sup \{\tau(A'): A' \subset A , A \varepsilon \mathrm{Cyl}_\mu(\Omega) , \tau(A') < \infty\}$,

Hence, (2) holds for all $A \varepsilon \mathrm{Cyl}_\mu(\Omega)$.

Proof of Cor. 4.3.

Let $K = \{V_n^0 : n \in \omega\}$.

We note that

(1) $\Omega = \bigsqcup K$, and

(2) $K \subset K(w^*|\Omega)$

By (2) and Thm. 4.3 we need only show that

(3) μ is K-tight whenever μ has a limit measure on Ω .

Suppose that μ has a limit measure on Ω . Let

$E \in F$ and $A \in M_E$ with $\mu_E(A) < \infty$.

Since μ_E is Radón, choose a closed $C \subset E^*$ s.t.

$C \subset A$ and $\mu_E(C) > \mu_E(A) - \varepsilon/2$,

and for each $n \in \omega$, let

$$K_n = V_n^0 \cap r_{E,X}^{-1}[C] \quad .$$

Since C is closed and $r_{E,X}$ is w^*-continuous, then, by (2),

(4) $K_n \in K(w^*|\Omega)$ for each $n \in \omega$.

Further, by (1)

(5) $\Omega \cap r_{E,X}^{-1}[C] = \bigcup_{n\in\omega} K_n$.

Since μ_Ω is an outer measure, and $K_n \subset K_{n+1}$ for each $n \in \omega$,

we deduce from (5) that

(6) $\mu_\Omega(\Omega \cap r_{E,X}^{-1}[C]) = \sup_{n\in\omega} \mu_\Omega(K_n)$.

By Prop. 3.1.2,

(7) μ_Ω is a limit measure of μ ,

Hence,

(8) $\mu_\Omega(\Omega \cap r_{E,X}^{-1}[C]) = \mu_E(C) < \mu_E(A) < \infty$.

Let $\varepsilon > 0$.

By (6) and (8), there exists $n \in \omega$ s.t.

(9) $\mu_\Omega(K_n) > \mu_\Omega(\Omega \cap r^{-1}_{E,X}[C]) - \varepsilon/2$.

Then, by (7),

$$\mu_\Omega(K_n) > \mu_E(C) - \varepsilon/2 > \mu_E(A) - \varepsilon \quad .$$

Hence, by (4) and L.2.2,

(10) $g(K_n) > \mu_E(A) - \varepsilon$.

We have that, for any $F \in \mathcal{F}$ with $E \in \mathcal{F}$,

$$r_{F,X}[K_n] \subset r^{-1}_{E,F}[C] \subset r^{-1}_{E,F}[A] \quad .$$

and therefore,

$\mu_F(r^{-1}_{E,F}[A] \sim r_{F,X}[K_n]) = \mu_F(r^{-1}_{E,F}[A]) - \mu_F(r_{F,X}[K_n])$

$= \mu_E(A) - \mu_F(r_{F,X}[K_n]) \leq \mu_E(A) - g(K_n)$

$< \varepsilon$, by (10).

However, $K_n \subset V^0_n$, and therefore, for every $F \in \mathcal{F}$ with $E \subset F$,

$$\mu_F(r^{-1}_{E,F}[A] \sim r_{F,X}[V^0_n]) < \varepsilon \ .$$

Since $\varepsilon > 0$, $E \in \mathcal{F}$, and $A \in M_E$ with $\mu_E(A) < \infty$, were all arbitrary, it follows that μ is $\mathcal{K}$-tight.

5. Finite Cylinder Measures

We shall specialize the results of the foregoing sections to the case of finite cylinder measures. By introducing the notion of a finite section of an arbitrary cylinder measure, we shall show that, with regard to the problem of finding limits, we can concentrate on finite cylinder measures.

5.1 Definition.

μ is a finite cylinder measure iff μ is a cylinder measure over a vector space X and for some $F \in \mathcal{F}_X$,

$$\mu_F(F^*) < \infty .$$

(We note that $\mu_F(F^*)$ is independent of $F \in \mathcal{F}_X$)

For the rest of this section we assume that

X is a vector space,

Ω is a subspace of X^* such that (X,Ω) is a dual pair,

$\mathcal{G}$ is a regular, Hausdorff topology on Ω which is finer than the w^*-topology restricted to Ω ,

μ is a cylinder measure over X .

The following lemmas indicate that the hypotheses of earlier results can be simplified when considering finite cylinder measures.

5.2 Lemmas.

If μ is a finite cylinder measure over X , then

(1) $\mu^*(A) = \mu^*(A \cap \Omega)$ for every $A \varepsilon Cyl_\mu(X^*)$ <=>

$$\mu^*(\Omega) = \mu^*(X^*) \quad .$$

For any family H of subsets of X^* ,

(2) μ is H-sequentially tight <=>

for any sequence $\{F_n\}_{n\varepsilon\omega}$ in F with $F_n \subset F_{n+1}$ for each $n \varepsilon \omega$, and $\varepsilon > 0$, there exists $H \varepsilon H$ such that

$$\mu_{F_n}(F_n^* \sim r_{F_n,X}[H]) < \varepsilon \text{ for all } n \varepsilon \omega .$$

(3) If $r_{F,X}[H] \varepsilon M_F$ for every $F \varepsilon F$ and $H \varepsilon H$, then,

μ is H-tight <=>

for any $\varepsilon > 0$ there exists $H \varepsilon H$ such that

$$\mu_F(F^* \sim r_{F,X}[H]) < \varepsilon \text{ for all } F \varepsilon F \quad .$$

Proof of 5.2.1

Certainly, if $\mu^*(\Omega \cap A) = \mu^*(A)$ for all $A \varepsilon Cyl_\mu(X^*)$, then

$$\mu^*(\Omega) = \mu^*(X^*) \quad .$$

On the other hand if $\mu^*(\Omega) = \mu^*(X^*)$, then, for any $A \varepsilon Cyl_\mu(X^*)$,

$$\mu^*(X^*) = \mu^*(\Omega) = \mu^*(\Omega \cap A) + \mu^*(\Omega \sim A) \le \mu^*(A) + \mu^*(X^* \sim A)$$
$$= \mu^*(X^*)$$

Hence $\mu^*(\Omega \cap A) = \mu^*(A)$.

Proof of 5.2.2

We observe only that for any $F \in \mathcal{F}$, $A \in M_F$ and $H \in \mathcal{H}$,

$$\mu_F(A \sim r_{F,X}[H]) \leq \mu_F(F^* \sim r_{F,X}[H]) .$$

Proof of 5.2.3

Together with the observation of Proof 5.2.2 above, we note that for any E and G in $\mathcal{F}$ with $E \subset G$, and $H \in \mathcal{H}$,

$$\mu_E(E^* \sim r_{E,X}[H]) \leq \mu_G(G^* \sim r_{G,X}[H])$$

The assertion is now immediate.

The following theorems are now immediate consequences of, respectively, Lemma 3.3, Theorem 3.10, and Theorem 4.3

5.3 Theorems

If μ is a finite cyliner measure over X, then

(1) μ has a limit measure on Ω <=>

$$\mu^*(\Omega) = \mu^*(X^*) . \qquad \text{(Silov [46])}.$$

(2) μ has a limit measure on Ω if,

for any sequence $\{F_n\}_{n \in \omega}$ in $\mathcal{F}$ with $F_n \subset F_{n+1}$ for each $n \in \omega$, and $\varepsilon > 0$, there exists a w^*-compact $K \subset \Omega$ such that

$$\mu_{F_n}(F_n^* \sim r_{F_n,X}[K]) < \varepsilon \quad \text{for all } n \in \omega .$$

(3) (Mourier-Prohorov, [3] §5 Lemma 3)

μ has a $\mathcal{G}$-Radón limit measure on Ω <=>

for any $\varepsilon > 0$ there exists $K \varepsilon \mathcal{K}(\mathcal{G})$ such that

$$\mu_F(F^* \sim r_{F,X}[K] < \varepsilon \quad \text{for all} \quad F \varepsilon \mathcal{F} \quad .$$

Remark. We point out that Theorem 5.3.2 does not seem to have been previously stated in the literature.

We shall now show that the problem of finding limits for arbitrary cylinder measures can be recuced to that for the finite case.

First, we make the following definition.

5.5 Definition ξ is a finite section of μ iff for some $E \varepsilon F$ and $A \varepsilon M_E$ with $\mu_E(A) < \infty$, ξ is the cylinder measure over X such that for every $F \varepsilon \mathcal{F}$,

$$\xi_F = r_{F,G}[\mu_G | r_{E,F}^{-1}[A]] \quad .$$

for some $G \varepsilon \mathcal{F}$ with $E \subset G$ and $F \subset G$.

Remark. If ξ is a finite section of μ then ξ is well-defined, ξ is in fact a cylinder measure over X , and

$$\xi^*(B) = \mu^*(r_{E,X}^{-1}[A] \cap B) \quad \text{for all} \quad B \subset X^* \quad .$$

(This remark is proved below.)

The following theorems are then readily established.

5.6 Theorems.

(1) μ has a limit measure on Ω iff every finite section of μ has a limit measure on Ω .

(2) μ is $K(G)$-tight iff every finite section of μ is $K(G)$-tight. Hence,

μ has a G-Radón limit measure on Ω iff every finite section of μ has a G-Radón limit measure on Ω .

Proof of Remark 5.5

For any $F \in F$ and $\alpha \subset F^*$ with $r_{F,G}^{-1}[\alpha] \in M_G$, in view of Remark 2.2.1 and Lemma 0.4.1,

$$\mu_G(r_{F,G}^{-1}[\alpha] \cap r_{E,G}^{-1}[A])$$

is independent of the choice of $G \in F$ with $E \subset G$ and $F \subset G$. Hence, so also is ξ_F .

We note that for any $F \in F$ with $E \subset F$,

(1) $\mu_F | r_{E,F}^{-1}[A]$ is Radón.

Consequently

(2) $\xi_F = \mu_F | r_{E,F}^{-1}[A]$ and is Radón.

Hence, by Lemma 0.4.3,

(3) ξ_F is Radón for every $F \in F$.

For any F and F_1 in F with $F \subset F_1$, if $G \in F$ with $E \cup F \cup F_1 \subset G$, then by Lemma 0.4.2,

$$\xi_F = r_{F,G}[\mu_G | r_{E,G}^{-1}[A]] = r_{F,F_1}[r_{F,F_1}[\mu_G | r_{E,G}^{-1}[A]]]$$

$$= r_{F,F_1}[\xi_{F_1}] .$$

Hence, by (3) and Lemma 0.4.1

ξ is a cylinder measure over X .

We shall now prove that

(4) $\xi^*(B) = \mu^*(r_{E,X}^{-1}[A] \cap B)$ for all $B \subset X^*$.

Let

$$\mathcal{H} = \{H \in \mathrm{Cyl}_\mu(X^*) : H \text{ is } w^*\text{-open}\} ,$$

$$\alpha = r_{E,X}^{-1}[A] .$$

We have that

$$\mathcal{H} \subset \mathrm{Cyl}_\xi(X^*) \cap \mathrm{Cyl}_\mu(X^*)$$

and by (2),

$$\tau_\xi^*(H) = \tau_\mu^*(H \cap \alpha) \quad \text{for every } H \in \mathcal{H} .$$

Hence, by Thm. 3.1,

(5) $\xi^*(H) = \tau_\mu^*(H \cap \alpha)$ for all $H \in \mathcal{H}_\sigma$.

Let $B \subset X^*$. If $\mu^*(B) < \infty$, then, since μ^* is an $\mathcal{H}_\sigma$-outer measure, for any $\varepsilon > 0$ there exists $H \in \mathcal{H}_\sigma$ s.t.

$$B \subset H \text{ and } \mu^*(H) < \mu^*(B) + \varepsilon .$$

Since $\alpha \in M_{\mu^*}$ we have that

$$\mu^*(B \cap \alpha) + \mu^*(B \sim \alpha) = \mu^*(B) > \mu^*(H) - \varepsilon$$

$$= \mu^*(H \cap \alpha) + \mu^*(H \sim \alpha) - \varepsilon$$

and therefore

$$\mu^*(H \cap \alpha) < \mu^*(B \cap \alpha) + \varepsilon .$$

Since μ^* is σ-finite it follows that for any $B \subset X^*$,

(6) $\mu^*(B \cap \alpha) = \inf \{\mu^*(H \cap \alpha) : B \subset H \in \mathcal{H}_\sigma\}$.

Since ξ^* and μ^* are both $\mathcal{H}_\sigma$-outer measures, (5) and (6) together imply that (4) holds.

Proof of 5.6.1.

By Lemma 3.3 we need only show that

(1) $\mu^*(A) = \mu^*(A \cap \Omega)$ for all $A \in Cyl_\mu(X^*)$

iff

(2) $\xi^*(X^*) = \xi^*(\Omega)$ for every finite section ξ of μ .

From Remark 5.5 it is immediate that (1) => (2).

However,

(2) => $\mu^*(A) = \mu^*(A \cap \Omega)$ for all $A \in Cyl_\mu(X^*)$ with $\mu^*(A) < \infty$.

For any $A \in Cyl_\mu(X^*)$, since μ^* is σ-finite, choose an increasing sequence $\{A_n\}_{n\in\omega}$ in $Cyl_\mu(X^*)$ s.t $\mu^*(A_n) < \infty$ for all $n \in \omega$ and

$$A = \bigcup_{n\in\omega} A_n \quad .$$

Since μ^* is an outer measure, we then have that

$$\mu^*(A \cap \Omega) = \lim_{n\in\omega} \mu^*(A_n \cap \Omega) = \lim_{n\in\omega} \mu^*(A_n) = \mu^*(A) \quad .$$

Hence (2) $\Rightarrow$ (1).

Proof of 5.6.2.

Let ξ be a finite section of μ determined by some $E \in \mathcal{F}$ and $A \in M_E$ with $\mu_E(A) < \infty$. By (2) in the proof of Remark 5.5, for any $F \in \mathcal{F}$ with $E \subset F$, and $K \in \mathcal{K}(G)$,

$$\mu_F(r^{-1}_{E,F}[A] \sim r_{F,X}[K]) = \xi_F(F^* \sim r_{F,X}[K]) \quad .$$

The assertion now follows from Lemma 5.1.2, and Thm. 4.3.

CHAPTER II

CYLINDER MEASURES OVER TOPOLOGICAL VECTOR SPACES

In this chapter, we are primarily interested in determining when a cylinder measure over a Hausdorff locally convex space X will have a limit measure on the topological dual X' which is Radón with respect to some given topology G on X'. Since (X,X') is a dual pair, the theory of the previous chapter applies with $\Omega = X'$. Hence, if G is regular and finer than the w^*-topology restricted to X', then, by Theorem I.4.3, μ will have a G-Radón limit measure on X' whenever μ is H-tight for some family $H \subset K(G)$. We shall take G to be one of three standard topologies, and these suggest that we take for H the particular family E defined below. Our main concern is then directed towards finding conditions under which μ is E-tight.

1. Notation

We point out that our topological vector spaces are not assumed to be necessarily Hausdorff.

In the rest of this paper we shall use the following notation. For any vector space X and $V \subset X$,

$$V^0 = \{f \in X^* : |f(x)| \leq 1 \text{ for all } x \in V\} .$$

For any topological vector space X ,

nbnd 0 in X is the family of neighbourhoods of the origin in X ,

$\mathcal{E}$ is the family of all sets $K \subset X^*$ such that K is w^*-closed and $K \subset V^0$ for some $V \in$ nbnd 0 in X .

$X' = \{f \in X^* : f$ is continuous$\}$

and for every $F \in \mathcal{F}_X$,

$$r_F = r_F^* | X' \quad .$$

In addition to the w^*-topology restricted to X' , we shall consider the following two topologies:

c^* is the topology on X' of uniform convergence on the compact subsets of X ,

s^* is the topology on X' of uniform convergence on the bounded subsets of X .

Remark We note that

$$\mathcal{E} \subset \mathcal{K}(w^*)$$

and

$$w^* | X' \subset c^* \subset s^* \quad .$$

2. $\mathcal{E}$-tight Cylinder Measures.

Throughout this section X is a topological vector space and μ is a cylinder measure over X .

When X is locally convex and Hausdorff we notice that $\mathcal{E}$ is nothing else but the family of w^*-closed equicontinuous subsets of X' . Hence, from Treves [47], Props. 32.5 and 32.8, we have that

$$\mathcal{E} \subset \mathcal{K}(c^*) \quad .$$

Consequently, application of Theorem I.4.3 yields the following assertion.

2.1 Theorem. Let X be Hausdorff and locally convex

μ is $\mathcal{E}$-tight => μ has a c^*-Radón limit measure on X' .

If $\mathcal{E} = \mathcal{K}(c^*)$, in particular, if X is barrelled ([47] Thm. 33.1) then,

μ is $\mathcal{E}$-tight <=> μ has a c^*-Radon limit measure on X' .

Sometimes $\mathcal{E}$-tightness of μ can also imply the existence of a limit measure on X' which is Radón with respect to the s^*-topology. For example, if X is a Montel space ([47] p. 356) then $\mathcal{E} = \mathcal{K}(s^*)$ ([47], Prop. 34.5); or, if X is a nuclear space ([47] p. 510) then $\mathcal{E} \subset \mathcal{K}(s^*)$ ([47] Prop. 50.2). Hence, on applying Theorem I.4.3, we obtain the following theorems.

2.2 Theorem

(1) If X is a Montel space, then,

μ is $\mathcal{E}$-tight <=> μ has an s^*-Radón limit measure on X' .

(2) If X is a nuclear space, then,

μ is $\mathcal{E}$-tight $\Rightarrow$ μ has an s^*-Radon limit measure on X' .

Even if $\mathcal{E} \not\subset \mathcal{K}(s^*)$, $\mathcal{E}$-tightness of μ can still imply that μ has an s^*-Radon limit measure on X' .

2.3 Theorem. Let X be Hausdorff and locally convex. For any $V \in$ nbnd 0 in X let

$$X'_V = \cup_{n\in\omega} nV^0 \text{ with the topology induced by the norm}$$
$$|\cdot|^*_V : f \in X'_V \to \sup_{x\in V}|f(x)| \in R^+ .$$

If there is a base $\mathcal{V}$ for nbnd 0 in X such that for each $V \in \mathcal{V}$,

X'_V is separable.

then,

(1) μ is $\mathcal{E}$-tight $\Rightarrow$ μ has an s^*-Radon limit measure on X' .

(2) If X' is a separable Banach space under s^* , then indeed every finite c^*-Radon measure on X' is s^*-Radon.

Proof of Theorem 2.3

We first establish the following lemma.

Lemma. For any $V \in$ nbnd 0 in X s.t. X'_V is separable, if $\mathcal{G}$ is the family of all open subsets of X'_V , then $\mathcal{G} \subset \sigma$-field generated by $(c^*|X'_V)$.

<u>Proof</u> Let $H = \{f + \varepsilon V^0 : \varepsilon > 0 , f \in X_V'\}$.

For any $\varepsilon > 0$ and $f \in X'$,

$g \in X' \to \varepsilon g \in X'$ and $g \in X' \to f + g \in X'$

are homeomorphisms with respect to c^* , since X' is a topological vector space under c^* .

Hence, since V^0 is w^*-closed and $w^* \subset c^*$,

H consists of c^*-closed subsets

and therefore

$H_\sigma \subset$ σ-field generated by $(c^*|X_V')$

However, X_V' is separable and metrizable.

Consequently,

$G \subset H_\sigma \subset$ σ-field generated by $(c^*|X_V')$

We now prove the theorem.

(1) By Thm. I.5.6.2, we may assume that μ is normalized.

In that case, by Lemma I.5.1.2, we need to prove that for any $\varepsilon > 0$, there exists $K \in K(s^*)$ s.t.

$\mu_F(F^* \sim r_F[K]) < \varepsilon$ for every $F \in F$.

With the notation of I.4, since μ is E-tight, then, by Thm. 2.1 and Props. I.4.1,

μ_{c^*} is c^*-Radon limit measure on X' of μ ,

and by L.3 of Proofs I.4,

(1) $\mu_{c^*}(K) = g(K)$ for all $K \in K(c^*)$.

Since μ is E-tight, for any $\varepsilon > 0$, there exists $V \in$ nbnd 0 in X s.t.

X_V' is separable,

and for all $F \in F$,

$\mu_F(F^* \sim r_F[V^0]) < \varepsilon/2$.

Hence, by L.2.1 of Proofs 1.4,

$$1 - g(V^0) < \varepsilon/2 \ ,$$

and therefore, by (1),

(2) $\mu_{c*}(X' - X'_V) < 1 - \mu_{c*}(V^0) < \varepsilon/2$.

If

$$\xi : A \subset X'_V \to \mu'_{c*}(A) \in R^+ \ ,$$

then ξ is a $c^*|X'_V$-outer measure on X'_V .

By the lemmas, and the fact that $c^*|X'_V \subset \mathcal{G}$,

it follows that ξ is a $\mathcal{G}$-outer measure on X'_V .

However, X'_V is complete ([47], Lemma 36.1, see also p. 477).

X'_V is also separable and metrizable. Hence, by Prohorov [32] Thm. 1.4, there exists $K \in \mathcal{K}(\mathcal{G})$ s.t.

(3) $\xi(X'_V \sim K) < \varepsilon/2$.

However, $s^*|X'_V \subset \mathcal{G}$, hence

$$K \in \mathcal{K}(s^*) \ .$$

Then, certainly, $K \in \mathcal{K}(c^*)$, and by (1), (2) and (3),

$$g(K) = \mu'_{c*}(K) > 1 - \varepsilon \ .$$

From the definition of g , and L.2.1 of Proofs I.4,

$$\mu_F(F^* \sim r_F[K]) < \varepsilon \text{ for all } F \in \mathcal{F} \ .$$

(2) From the lemma we have any c^*-Radon measure on X' is an s^*-outer measure $(\mathcal{G} = s^*)$. The assertion now follows from Prohorov [32] Thm. 1.4

We are led by the above theorems to study conditions under which will be $\mathcal{E}$-tight. In view of Theorem I.5.6.2 we shall concentrate on the case when μ is finite.

Conditions for μ to be $\mathcal{E}$-tight will then be given in terms of the one-dimensional subspaces of X . We begin by indicating a necessary such condition.

2.4 Proposition Let X be a topological vector space, $r > 0$, and μ a finite cylinder measure over X.

If μ is $\mathcal{E}$-tight, then, for any $\varepsilon > 0$, there exists a w^*-Radón measure η on X^* with supp $\eta \in \mathcal{E}$, such that

$$x \in X \ , \ \int |f(x)|^r d\eta(f) \le 1 \Rightarrow \mu_{F_x}(\{f \in F_x^* : |f(x)| \ge 1\}) < \varepsilon \ ,$$

where F_x is the space spanned by x.

Proof. We assume that μ is normalized.

With the notation of I.4, by Th. I.4.3,

$\mu^*_{w^*}$ is a w^*-Radón limit measure on X^* for μ.

Since μ is $\mathcal{E}$-tight, by L.2.1. of Proofs I.4, for any $\varepsilon > 0$ there exists $V \in$ nbnd 0 in X s.t.

$$1 - g(V^0) < \varepsilon/2 \ ,$$

and therefore, by L.3 of Proofs I.4,

$$\mu^*_{w^*}(X^* \sim V^0) < \varepsilon/2 \ .$$

For any $x \in X$, let

$$I_x = \{f \in X^* : |f(x)| \ge 1\} \ .$$

If $\eta : A \subset X^* \to \frac{2}{\varepsilon} \mu^*_{w^*}(A \cap V^0) \in R^+$, then,

η is a w^*-Radon measure on X^* with supp $\eta \in \mathcal{E}$

and for any $x \in X$ s.t. $\int |f(x)|^r d\eta \le 1$,

$$\mu_{F_x}(\{f \in F_x^* : |f(x)| \ge 1\}) = \mu^*_{w^*}(I_x)$$

$$= \mu^*_{w^*}(I_x \cap V^0) + \mu^*_{w^*}(I_x \sim V^0)$$

$$\le \frac{\varepsilon}{2} \eta(I_x) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \int 1_{I_x} d\eta + \varepsilon/2$$

$$\le \frac{\varepsilon}{2} \int |f(x)|^r d\eta(f) + \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ . \qquad \blacksquare$$

The above proposition suggests the following definitions.

2.4 Notation. For any vector space X , $x \in X$ and cylinder measure μ over X ,

$$F_x = \text{space spanned by } x,$$

$$D_x = \{f \in F_x^* : |f(x)| \geq 1\}$$

and

$$\mu_x = \mu_{F_x} .$$

2.5 Definitions. Let X be a vector space, and $\mathcal{U}$ be a family of subsets U of X with $0 \in U$.

(1) For any finite cylinder measure μ over X ,

μ is $\mathcal{U}$-continuous iff for any $\varepsilon > 0$ there exists $U \in \mathcal{U}$ such that

$$x \in U \Rightarrow \mu_x(D_x) < \varepsilon .$$

(2) For any cylinder measure μ over X ,

μ is $\mathcal{U}$-continuous iff every finite section of $\mathcal{U}$ is $\mathcal{U}$-continuous.

(3) For any topological space Y and $T : X \to Y$,

T is $\mathcal{U}$-continuous iff for every neighbourhoods V of $T(0)$ there exists $U \in \mathcal{U}$ such that

$$T[U] \subset V .$$

(4) When X is a topological space, for any finite cylinder measure μ over X ,

μ is continuous iff μ is $\mathcal{V}$-continuous for some family $\mathcal{V}$ of neighbourhoods of the origin in X .

The discussion of limits in the rest of the chapter requires only the above concepts. However, to explain their relation to standard notions of continuity we introduce the following definitions.

Let X be a vector space.

For any $n \in \omega$ and $x = (x_0, \ldots, x_{n-1}) \in X^n$,

F_x = linear span of $\{x_0, \ldots, x_{n-1}\}$,

$\varphi_x : f \in F_x^* \to (f(x_0), \ldots, f(x_{n-1})) \in \mathbb{C}^n$,

and for any finite cylinder measure μ over X ,

$\tilde{\mu}_x = \varphi_x[\mu_{F_x}]$.

For any $n \in \omega$,

$M(\mathbb{C}^n)$ is the family of finite Radon measures on $\mathbb{C}^n$

endowed with the vague topology;

i.e. for any net $(\eta_j)_{j \in J}$ in $M(\mathbb{C}^n)$ and $\eta \in M(\mathbb{C}^n)$,

$\eta_j \to \eta$ in $M(\mathbb{C}^n)$ iff $\int f d\eta_j \to \int f d\eta$

for every bounded continuous $f : \mathbb{C}^n \to \mathbb{C}$.

We note that $\tilde{\mu}_x \in M(\mathbb{C}^n)$ for all $n \in \omega$ and $x \in X^n$.

We now have the following well-known proposition (Gelfand, Vilenkin [11] P. 310, Fernique [9] p. 37, which shows that one can naturally associate certain continuous maps with a continuous cylinder measure. (see also Appendix 1.7).

2.6 <u>Proposition</u> Let X be a topological vector space.

(1) μ is continuous

iff

(2) $\tilde{\mu} : x \in X \to \tilde{\mu}_x \in M(\mathbb{C})$ is continuous at 0

iff

(3) for each $n \in \omega$,

$\tilde{\mu}^n : x \in X^n \to \tilde{\mu}_x \in M(\mathbb{C}^n)$ is continuous with respect to the product topology on X^n .

<u>Remark</u> From the proof of the above proposition one readily checks that the following assertion also holds.

If $\mathcal{U}$ is a family of balanced, absorbent subsets of X, with $tU \in \mathcal{U}$ for every $U \in \mathcal{U}$ and $t > 0$, then,

μ is $\mathcal{U}$-continuous iff

$\tilde{\mu} : x \in X \to \tilde{\mu}_x \in M(\mathbb{C})$ is $\mathcal{U}$-continuous.

<u>Proof of Proposition 2.6.</u>

We show that (3) => (2) => (1) => (3)

(3) => (2) take $n = 1$.

(2) => (1) let $\chi : \mathbb{C} \to \mathbb{C}$ be bounded and continuous,

$$\chi(z) = \begin{cases} 1 & \text{if } |z| \geq 1 \\ 0 & \text{if } |z| \leq \frac{1}{2} \end{cases}$$

Let $\varepsilon > 0$. Choose $V \in$ nbnd 0 in X s.t.

$$x \in V \Rightarrow \left|\int \chi d\tilde{\mu}_x - \int \chi d\tilde{\mu}_0\right| < \varepsilon .$$

Since $\tilde{\mu}_0$ is concentrated at the origin in $\mathbb{C}$, and has finite mass, then

$$\int \chi d\tilde{\mu}_0 = 0 .$$

Hence, for any $x \in V$,

$$\mu_x(D_x) = \tilde{\mu}_x(\{z \in \mathbb{C} : |z| \geq 1\}) \leq \int \chi d\tilde{\mu}_x < \varepsilon .$$

(1) => (3) For any $F \in \mathcal{F}$, and $u > 0$, let

$$I(u) = \{(w,f) \in F \times F^* : |f(w)| \geq u\}$$

Let $n \in \omega$, $\chi : \mathbb{C}^n \to \mathbb{C}$ be bounded and continuous, and $x \in X^n$.

We shall show that for any $\varepsilon > 0$ there exists $V \in$ nbnd 0 in X s.t.

$$y \in x + V^n \Rightarrow \left| \int \chi d\tilde{\mu}_x - \int \chi d\tilde{\mu}_y \right| < \varepsilon \ .$$

(Note: $V^n \in$ nbnd 0 in X^n).

We assume that μ is normalized.

For any $z \in \mathbb{C}^n$, let $|z| = \sup \{|z_k| : k = 0,\dots,n-1\}$.

Let

(i) $M = \sup \{|\chi(z)| : z \in \mathbb{C}^n\}$,

(ii) W be an open nbnd of the origin in X s.t.

$w \in W \Rightarrow \mu_w(D_w) < \varepsilon/16nM$,

(iii) $t > 0$ be s.t. $x_k \in tW$ for each $k = 0,\dots,n-1$,

(iv) and let $\delta > 0$ be s.t.

$z^j \in \mathbb{C}^n$, $|z^j| \leq t$, $j = 0,1$, and $|z^0 - z^1| < \delta$

$\Rightarrow |\chi(z^0) - \chi(z^1)| < \varepsilon/4$.

Since W^n is an open nbnd of 0 in X^n ,

there exists $V \in$ nbnd 0 in X s.t.

(v) $\frac{1}{\delta} V \subset W$,

(vi) $x + V^n \subset tW^n$.

For any $y \in x + V^n$ and $F \in \mathcal{F}$ with $F_x \cup F_F \subset F$, let

$$\Psi_y = \varphi_y \circ r_{F_y} \ , \ \Psi_x = \varphi_x \circ r_{F_x} \ ,$$

$$A_y(\delta) = \{f \in F^* : |\Psi_y(f) - \Psi_x(f)| \geq \delta\}$$

and

$$B_y(t) = \bigcup_{k=0}^{k=n-1} I_{y_k}(t) \ .$$

Then,

$$\mu_F(A_y(\delta)) = \mu_F\left(\bigcup_{k=0}^{k=n-1} I_{x_k-y_k}(\delta)\right)$$

$$\leq \sum_{k=0}^{k=n-1} \mu_F(I_{(1/\delta)(x_k-y_k)}(1)) = \sum_{k=0}^{k=n-1} \mu_{(1/\delta)(x_k-y_k)}(D_{(1/\delta)(x_k-y_k)})$$

$< \varepsilon/16M$ by (ii) and (v).

From (vi), we have that

$$y \varepsilon tW^n \quad .$$

Hence, $\frac{1}{t} y_k \varepsilon W$ for each $k \varepsilon n$, and therefore, by reasoning as above,

$$\mu_F(B_y(t)) < \varepsilon/16M \quad .$$

In particular,

$$\mu_F(B_x(t)) < \varepsilon/16M \quad .$$

Consequently, if

$$B = B_x(t) \cup B_y(t) \cup A_y(\delta) \quad ,$$

then, $\mu_F(B) < 3\,\varepsilon/16M < \varepsilon/4M$,

and, by (iv),

$$f \varepsilon F^* \sim B \Rightarrow |\chi(\Psi_y(f)) - \chi(\Psi_x(f))| < \varepsilon/4$$

Hence,

$$|\int \chi d\tilde{\mu}_y - \int \chi d\tilde{\mu}_x| = |\int \chi \circ \Psi_y d\mu_F - \int \chi \circ \Psi_x d\mu_F\}$$

$$\leq \int_B | \chi \circ \Psi_y - \chi \circ \Psi_x \, d\mu_F + \int_{F^* \sim B} |\chi \circ \Psi_y - \chi \circ \Psi_x| d\mu_F$$

$$\leq 2M.\varepsilon/4M + \frac{\varepsilon}{4} \cdot \mu_F(F^* \sim B) < \varepsilon \quad .$$

i.e.

$$y \varepsilon x + V^n \Rightarrow |\int \chi d\tilde{\mu}_y - \int \chi d\tilde{\mu}_x| < \varepsilon \quad .$$

3. Limits of Continuous Cylinder Measures

Let X be a Hausdorff, locally convex space, and μ be a finite cylinder measure over X. In the previous section we have seen that conditions under which μ is E-tight are important for determining when μ has a c^*- or an s^*-Radón limit measure on X'. In terms of the one-dimensional subspaces of X Proposition 2.4 gives a necessary such condition. In seeking some kind of converse to that proposition, we are led to introduce the concept of a weighted system in X, which is defined below.

We shall use the following notation.

3.1 Notation

For any vector space X, absorbent absolutely convex $V \subset X$, and $F \in \mathcal{F}$,

$$\ker V = \{x \in X : x \in tV \text{ for every } t > 0\} ,$$

$$F_V = F \cap \ker V ,$$

$$(V \cap F)^0 = \{f \in F^* : |f(x)| \leq 1 \text{ for all } x \in V \cap F\} ,$$

$$F_V^a = \{f \in F^* : f(x) = 0 \text{ for all } x \in F_V\} .$$

For any $t > 0$,

$$I(t) = \{(x,f) \in F \times F^* : |f(x)| \geq t\} ,$$

$$I = I(1) .$$

So for any $x \in F$, $f \in F^*$,

$$I^f = \{x \in F : |f(x)| \geq 1\} ,$$

$$I_x = \{f \in F^* : |f(x)| \geq 1\} .$$

Remarks

We note that

(1) $F_V \varepsilon \mathcal{F}$;

and since $(x,f) \varepsilon F \times F^* \to f(x) \varepsilon \mathbb{C}$ is continuous with respect to the product topology on $F \times F^*$,

(2) for any $t > 0$,

$I(t)$ is closed.

3.2 Definitions

Let X be a locally convex space.

(1) $(\nu,\hat{\mathcal{F}},\mathcal{V})$ is a system of δ-weights in X if

$\delta > 0$;

$\mathcal{V}$ is a family of absolutely convex neighbourhoods of the origin in X ,

$\hat{\mathcal{F}} \subset \mathcal{F}$ is directed by $\subset$ and $\bigcup \hat{\mathcal{F}}$ is dense in X .

and

$\nu : V \varepsilon \mathcal{V},\ F \varepsilon \hat{\mathcal{F}} \to \nu_{V,F}$, a probability Radón measure measure on F for which

$f \varepsilon F_V^a \sim (V \cap F)^0 \Rightarrow \nu_{V,F}(I^f) \geq \delta$.

When $\mathcal{V}$ is a singleton $\{V\}$, we shall write $(\nu,\hat{\mathcal{F}},V)$ instead of $(\nu,\hat{\mathcal{F}},\mathcal{V})$.

(2) $\mathcal{W}$ is weighted by such a system $(\nu,\hat{\mathcal{F}},\mathcal{V})$ iff

$\mathcal{W}$ is a family of neighbourhoods of the origin in X ,

for each $W \varepsilon \mathcal{W}$ there exists $V \varepsilon \mathcal{V}$ such that $\ker V \subset W$,

and

$\nu_{V,F}(F \sim tW \cap F) \to 0$ as $t \to \infty$, uniformly for $F \varepsilon \hat{\mathcal{F}}$.

(3) $\mathcal{W}$ is a weighted system in X iff $\mathcal{W}$ is weighted by some system of δ-weights in X .

We shall now state and prove the fundamental results of this section.

3.3 Theorem

Let X be a Hausdorff, locally convex space, and μ be a finite cylinder measure over X .

If μ is $\mathcal{W}$-continuous for some weighted system in X , then μ is $\mathcal{E}$-tight.

Corollary

Let X be a Hausdorff, locally convex space, and μ be an arbitrary cylinder measure over X .

If μ is $\mathcal{W}$-continuous for some weighted system in X , then

μ is $\mathcal{E}$-tight,

and therefore,

μ has a c^*-Radón limit measure on X' .

Proof of Corollary

By Thms. 3.3, I. 5.6.2, and 2.1.

We shall need the following lemmas in the proof of Theorem 3.3. They are proved at the end of the section.

Lemmas

(1) Let F be a finite dimensional space, and $\varepsilon > 0$.

If ξ is a finite Radón measure on F^* such that

$$x \in F \Rightarrow \xi(I_x) < \varepsilon$$

then

$$\xi(F^* \sim \{0\}) \leq \varepsilon .$$

(2) Let X be a locally convex space and μ a continuous finite cylinder measure over X.

For any dense subspace Y of X, and $\varepsilon > 0$, if V is an absolutely convex neighbourhood of the origin in X such that

$$\mu_F(F^* \sim (V \cap F)^0) \leq \varepsilon \quad \text{for every} \quad F \in \mathcal{F}_Y ,$$

then

$$\mu_F(F^* \sim (V \cap F)^0) \leq \varepsilon \quad \text{for all} \quad F \in \mathcal{F}_X .$$

Proof of Theorem 3.3

By the Hahn-Banach extension theorem, we see that for any $F \in \mathcal{F}$, and absolutely convex $V \in \text{nbnd } 0$ in X,

(1) $$r_F[V^0] = (V \cap F)^0 .$$

Hence, by Lemma I.5.1.2, we need only prove that, for any $\varepsilon > 0$, there exists $V \in \text{nbnd } 0$ in X s.t.

(2) $$\mu_F(F^* \sim (V \wedge F)^0) \leq \varepsilon \quad \text{for all} \quad F \in \mathcal{F} .$$

We assume that μ is normalized.

Let $\mathcal{W}$ be a weighted system in X with respect to which μ is $\mathcal{W}$-continuous, and let $(\nu, \hat{\mathcal{F}}, \mathcal{V})$ be a system of δ-weights in X by which $\mathcal{W}$ is weighted.

For any $\varepsilon > 0$, let

$$0 < \varepsilon' < \min(\delta\varepsilon, \varepsilon) \quad ,$$

$$W \in \mathcal{W} \quad \text{s.t.} \quad x \in W \Rightarrow \mu_x(D_x) \leq \varepsilon'/4 \quad ,$$

$$V \in \mathcal{V} \quad \text{and} \quad t > 0 \quad \text{s.t.} \quad \ker V \subset W \quad \text{and}$$

$\nu_{V,F}(F \sim tW \cap F) < \varepsilon'/4$ for every $F \in \hat{F}$.

Let

$$U = V/t \quad .$$

Suppose that

(3) $$\mu_F(F^* \sim (U \cap F)^0) \leq \varepsilon \quad \text{for every} \quad F \in \hat{F} \quad .$$

Since $\hat{F}$ is directed by $\subset$, then $\bigsqcup \hat{F}$ is a subspace of X, and for any finite dimensional subspace E' of $\bigsqcup \hat{F}$ there exists $F \in \hat{F}$ with $E \subset F$. Hence, by (1), (3), and L.2.3 of Proofs 1.4,

$$\mu_E(E^* \sim (U \cap E)^0) \leq \varepsilon \quad .$$

Since $\mathcal{W}$ is a family of neighbourhoods of the origin in X, μ is necessarily continuous, and by hypothesis, $\bigsqcup \hat{F}$ is dense in X . Hence, from Lemma (2) and the foregoing remarks we conclude that (2) holds.

It remains for us to establish (3).

For any $F \in \hat{F}$,

(4) $$\mu_F(F^* \sim (U \cap F)^0) = \mu_F(F^* \sim t(V \cap F)^0)$$

$$= \mu_F(F_V^a \sim t(V \cap F)^0) + \mu_F(F^* \sim F_V^a)$$

since

$$(V \cap F)^0 \subset F_V^a \quad .$$

We show that each of the last two terms given in (4) is less than $\varepsilon/2$.

We estimate the first term.

Since $(\nu,\hat{F},V)$ is a system of δ-weights in X , then

$$f \in F_V^a \sim t\,.\,(V \cap F)^0 \Rightarrow \frac{1}{t} f \in F_V^a \sim (V \cap F)^0$$

$$\Rightarrow \nu_{V,F}(I^f(t)) = \nu_{V,F}(I^{f/t}) \geq \delta \quad .$$

Consequently,

$$\delta.\mu_F(F_V^a \sim t.(V \cap F)^0)$$

$$\leq \inf\{\nu_{V,F}(I^f(t)) : f \in F_V^a \sim t.(V \cap F)^0\}.\mu_F(F_V^a \sim t.(V \cap F)^0)$$

$\leq \nu_{V,F} \times \mu_F(I(t))$ by Sion [44] Ch. III, Thm. 1.2.6,

$= \int_F \int_{F^*} 1_{I(t)} d\mu_F d\nu_{V,F}$ by Fubini's theorem,

$$= \int \mu_F(I_x(t)) d\nu_{V,F}(x) = \int^* \mu_F(I_x(t)) d\nu_{V,F}$$

$$= \int^* \mu_F(I_{\frac{x}{t}}) d\nu_{V,F}(x) = \int^* \mu_{\frac{x}{t}}(D_{\frac{x}{t}}) d\nu_{V,F}(x)$$

$$\leq \int^* 1_{tW\ F} \cdot \mu_{\frac{x}{t}}(D_{\frac{x}{t}}) d\nu_{V,F}(x) + \int^* 1_{F \sim tW\ F} \cdot \mu_{\frac{x}{t}}(D_{\frac{x}{t}}) d\nu_{V,F}(x)$$

$$\leq \frac{\varepsilon'}{4} \cdot \nu_{V,F}(tW \cap F) + 1.\nu_{V,F}(F \sim tW \cap F)$$

$$\leq \frac{\varepsilon'}{4} \,.\, 1 + 1.\frac{\varepsilon'}{4} < \delta.\frac{\varepsilon}{2} \quad .$$

Hence,

(5) $$\mu_F(F_V^a \sim (U \cap F)^0) < \varepsilon/2 \quad .$$

We now estimate $\mu_F(F^* \sim F_V^a)$.

We have that

$$x \in F_V \Rightarrow x \in \ker V \Rightarrow x \in W \Rightarrow \mu_x(D_x) < \varepsilon'/4 \quad .$$

$\Rightarrow \mu_{F_V}(I_x) < \varepsilon'/4$ since $F_x \subset F_V \in \mathcal{F}$.

Hence, by Lemma (1),

$$\mu_{F_V}(F_V^* \sim \{0\}) \leq \varepsilon'/4 \quad .$$

Since

$$r_{F_V,F}[F^* \sim F_V^a] \subset F^* \sim \{0\} \quad ,$$

it therefore follows that

$$\mu_F(F^* \sim F_V^a) \leq \mu_F(r_{F_V,F}^{-1}[F_V^* \sim \{0\}])$$

$$= \mu_{F_V}(F^* \sim \{0\}) \leq \varepsilon'/4 < \varepsilon/4 \quad .$$

Then, certainly,

(6) $$\mu_F(F^* \sim F_V^a) < \varepsilon/2 \quad .$$

From (4), (5), and (6) we see that (3) holds.

Remark. We point out that the theorem still holds when we use a somewhat weaker notion of system of δ-weights, in which

$$\hat{F} : V \in \mathcal{V} \to \hat{F}_V \subset F \qquad \text{directed by } \subset \text{ and } \bigsqcup \hat{F}_V \text{ is}$$

dense in X .

The other definitions remain unchanged.

Proofs 3.

Proof of Lemma (1).

For any $x \in F$ and $n \in \omega$,

$$I_x(\tfrac{1}{n}) \subset I_x(\tfrac{1}{n+1}) \; , \; I_x(\tfrac{1}{n}) \in M_\xi \quad ,$$

and

$$\xi(I_x(\tfrac{1}{n})) = \xi(I_{nx}) < \varepsilon \; .$$

Consequently, for any $x \in F$, $x \neq 0$,

$$\xi(\{f \in F^* : f(x) \neq 0\}) = \xi(\bigcup_{n\in\omega} I_x(\tfrac{1}{n}))$$

$$= \lim_{n\in\omega} \xi(I_x(\tfrac{1}{n})) \leq \varepsilon \quad .$$

Hence if

(1) there exists $y \in F$ s.t.

$$\xi(\{f \in F^* : f(y) = 0\} \sim \{0\}) = 0 \quad ,$$

then

$$\xi(F^* \sim \{0\}) = \xi(\{f \in F^* : f(y) \neq 0\}) + \xi(\{f \in F^* : f(y) = 0\} \sim \{0\})$$
$$\leq \varepsilon \quad .$$

We shall establish (1) by induction.

For any subspace E of F, let

$$E^a = \{f \in F^* : f(x) = 0 \text{ for all } x \in E\} \quad .$$

Let $\dim F = n$. If $n = 1$ then (1) holds. We therefore assume that $n \geq 2$. For any k-dimensional subspace G of F with $2 \leq k \leq n$,

(2) $\{H^a \sim G^a : H$ is a $(k-1)$-dimensional subspace of $G\}$

is an uncountable, disjoint subfamily of M.

Let $G_0 = F$. Then, by (2) and the finiteness of ξ, there exists an $(n-1)$-dimensional subspace G_1 of F s.t.

$$\xi(G_1^a \sim \{0\}) = \xi(G_1^a \sim G_0^a) = 0 \quad .$$

For any $0 \leq k \leq n-2$, if there exists an $(n-k)$-dimensional subspace G_k of F s.t.

$$\xi(G_k^a \sim \{0\}) = 0 \quad ,$$

Then, by (2) and the finiteness of ξ, there exists an $(n-k-1)$-dimensional subspace G_{k+1} of G_k s.t.

$$\xi(G_{k+1}^a \sim G^a_k) = 0 \quad .$$

Consequently,

$$\xi(G_{k+1}^a \sim \{0\}) = \xi(G_{k+1}^a \sim G_k^a) + \xi(G_k^a \sim \{0\}) = 0 \quad .$$

Hence, there exists a one-dimensional subspace G_{n-1} of F s.t.

$$\xi(G_{n-1}^a \sim \{0\}) = 0 \quad .$$

i.e. (1) holds.

Proof of Lemma (2).

For any $F \varepsilon \mathcal{F}_X$, $n \varepsilon \omega$, $x \varepsilon F^n$, and $t > 0$, let

$$A_x^F(t) = \bigcap_{k=0}^{k=n-1} \{f \varepsilon F^* : |f(x_k)| < t\} .$$

We shall assume that μ is normalized. Since $(V \cap F)^0 = (V_0 \cap F)^0$, where V_0 is the interior of V, we shall further assume that V is open.

Let $E \varepsilon \mathcal{F}_X$.

Since E is separable there exists a countable, dense subset $\{x_n\}_{n\varepsilon\omega}$ of $V \cap E$. Then,

$$(V \cap E)^0 = \bigcap_{k\varepsilon\omega} \{f \varepsilon E^* : |f(x_k)| \leq 1\} .$$

Now, for any $n \varepsilon \omega$,

$$\bigcap_{k=0}^{k=n-1} \{f \varepsilon E^* : |f(x_k)| \leq 1\} = \bigcap_{m\varepsilon\omega} A^E_{(x_0,\ldots,x_{n-1})}(1 + \tfrac{1}{m}) .$$

Consequently, for any $\delta > 0$, there exists $n \varepsilon \omega$ and $m \varepsilon \omega$ s.t.

$$\mu_E((V \cap E)^0) \leq \mu_E(A^E_{(x_0,\ldots,x_{n-1})}(1 + \tfrac{1}{m})) < \mu_E((V \cap E)^0) + \frac{\delta}{2} .$$

Since μ is continuous, there exists $U \varepsilon$ nbnd 0 in X s.t.

(1) $u \varepsilon U \Rightarrow \mu_u(D_u) < \delta/2n$,

and since Y is dense in X and V is open there exists $\{y_0,\ldots,y_{n-1}\}$ in V s.t.

(2) $x_k - y_k \varepsilon \frac{1}{m} U$ for all $k = 0,\ldots,n-1$.

Let $F \varepsilon \mathcal{F}_X$ be such that $E \cup \{y_0,\ldots,y_{n-1}\} \subset F$ and let

$$x = (x_0,\ldots,x_{n-1}) , \; y = (y_0,\ldots,y_{n-1}) .$$

Then,

$$\mu_F(F^* \sim A^F_{x-y}(\tfrac{1}{m})) = \mu_F(\bigcup_{k=0}^{k=n-1} \{f \in F^* : |f(x_k - y_k)| \geq \tfrac{1}{m}\})$$

$$\leq \sum_{k=0}^{k=n-1} \mu_{x_k - y_k}(D_{m(x_k-y_k)}) < \delta/2 \quad \text{by (1) and (2).}$$

Further,

$$A^F_{x-y}(\tfrac{1}{m}) \cap A^F_Y(1) \subset A^F_X(1 + \tfrac{1}{m}) \quad .$$

Hence,

$$\mu_F(A^F_Y(1)) = \mu_F(A^F_{x-y}(\tfrac{1}{m}) \cap A^F_Y(1)) + \mu_F(A^F_Y(1) \sim A^F_{x-y}(\tfrac{1}{m}))$$

$$\leq \mu_F(A^F_x(1 + \tfrac{1}{m})) + \frac{\delta}{2} = \mu_E(A^E_x(1 + \tfrac{1}{m})) + \frac{\delta}{2} \quad .$$

$$< \mu_E((V \cap E)^0) + \delta \quad .$$

i.e.

$$(3) \quad \mu_F(A^F_Y(1)) \leq \mu_E((V \cap E)^0) + \delta \quad .$$

However, if F_y denotes the linear span of $\{y_0, \ldots, y_{n-1}\}$, we observe that

$$A^F_y(1) \supset r^{-1}_{F_y, F}[(V \cap F_y)^0] \quad .$$

Since $F_y \in \mathcal{F}_Y$, and $(V \cap F_y)^0$ is closed, we have that $(V \cap F_y)^0 \in M_{F_y}$ and

$$\mu_F(A^F_y(1)) \geq \mu_F(r^{-1}_{F_y, F}[(V \cap F_y)^0]) = \mu_{F_y}((V \cap F_y)^0) > 1 - \varepsilon \quad .$$

Hence, by (3),

$$\mu_E((V \cap E)^0) > 1 - \varepsilon - \delta \quad .$$

Since δ was arbitrary, it follows that

$$\mu_E((V \cap E)^0) \geq 1 - \varepsilon \quad .$$

Consequently, since $(V \cap E)^0 \in M_E$,

$$\mu_E(E^* \sim (V \cap E)^0) < \varepsilon \quad .$$

4. Induced Cylinder Measures.

It can happen that a finite cylinder measure over a Hausdorff, locally convex space X is given indirectly. For example, it may have been induced by a finite cylinder measure μ over a vector space Y and a linear map T on X to Y ([11] p. 311). In such a situation we shall be interested in obtaining conditions on μ and T which will ensure that the induced cylinder measure over X will have a limit measure on X', Radón with respect to some given topology on X'. This kind of problem seems to have been first mentioned in [11] Ch. IV. It has been studied extensively, by L. Schwartz, S. Kwapien, and others, in a series of papers ([19], [20], [39] - [42]).

In view of the previous theory, our emphasis will be on determining conditions under which the induced cylinder measure will be $\mathcal{E}$-tight. Using the notions of continuity and weighted system we readily obtain such conditions.

4.1 Definition

For any vector spaces X and Y, linear map $T : X \to Y$, and finite cylinder measure μ over Y,

the cylinder measure ξ over X induced by μ and T is defined as follows: for each $F \in \mathcal{F}_X$,

$$\xi_F = T_F^*[\mu_{T[F]}] ,$$

where

$$T_F^* \text{ is the adjoint of } T|F ,$$

i.e.

$$T_F^* : f \in (T[F])^* \to f \circ (T|F) \in F^* .$$

We shall denote this induced cylinder measure ξ by

$\mu \sqsupset T$.

We prove below that ξ is indeed a cylinder measure over X .

Proof For each $E \in \mathcal{F}_X$, T_E^* is continuous.

Hence, by §0.4,

ξ_E is a finite Radón measure on E^* .

Since all the maps considered are continuous, then, by §0.4 and Remark I.2.1, for any E and F in $\mathcal{F}_X$ with $E \subset F$,

$$r_{E,F}[\xi_F] = r_{E,F}[T_F^*[\mu_{T[F]}]] = r_{E,F} \circ T_F^*[\mu_{T[F]}]$$

$$= T_E^* \circ r_{T[E],T[F]}[\mu_{T[F]}] = T_E^*[r_{T[E],T[F]}[\mu_{T[F]}]]$$

$$= T_E^*[\mu_{T[E]}] = \xi_E ,$$

and therefore, again by Remark I.2.1,

ξ is a cylinder measure over X . ∎

We now prove the following important lemma.

4.2 Lemma

For any vector space X , family $\mathcal{U}$ of subsets U of X with $0 \in U$, topological vector space Y , and linear $T : X \to Y$,

if T is $\mathcal{U}$-continuous, then $\mu \sqsupset T$ is $\mathcal{U}$-continuous for every continuous finite cylinder measure over Y .

<u>Proof</u>. Let μ be a continuous finite cylinder measure over Y .

For any $x \in X$, by Lemma 0.4.2,

$$(\mu \sqcap T)_x(D_x) = T^*_{F_x}[\mu_{Tx}](D_x) = \mu_{Tx}(T^{*-1}_{F_x}[D_x]) = \mu_{Tx}(D_{Tx}) \quad .$$

For any $\varepsilon > 0$, there exists $V \varepsilon$ nbnd 0 in Y s.t.

$$y \in V \Rightarrow \mu_y(D_y) < \varepsilon \quad ,$$

and there exists $U \in \mathcal{U}$ s.t.

$$T[U] \subset V \quad .$$

Then, by the first assertion,

$$x \in U \Rightarrow Tx \in V \Rightarrow (\mu \sqcap T)_x(D_x) = \mu_{Tx}(D_{Tx}) < \varepsilon \quad .$$

It follows that $\mu \sqcap T$ is $\mathcal{U}$-continuous. ∎

Our key theorem on induced cylinder measures is now an immediate consequence of Theorems 3.3, 2.1, and the above lemma.

4.3 <u>Theorem</u>.

Let X be a Hausdorff, locally convex space, Y be a topological vector space, and T be a linear map on X to Y .

If T is $\mathcal{W}$-continuous for some weighted system $\mathcal{W}$ in X , then for every continuous finite cylinder measure over Y ,

$\mu \sqcap T$ is $\mathcal{E}$-tight

are therefore

$\mu \sqcap T$ has a c^*-Radon limit measure over X .

<u>Remark</u>.

It is clear that this theorem reduces to the finite case of Corollary 3.3 when $X = Y$ and T is the identity map.

CHAPTER III

APPLICATIONS

We shall apply the theory of the previous chapter to a study of cylinder measures over Hilbertian and ℓ^p-spaces. Our results on cylinder measures over arbitrary Hausdorff, Hilbertian spaces generalize and clarify many known theorems (Minlos [25], Sazonov [35], Badrikian [1], Fernique [9]). In the case of ℓ^p-spaces we obtain significant extensions of formerly known results (L. Schwartz [39], Kwapien [19]).

Our main tool is Corollary II.3.3, which requires us to construct weighted systems in the above spaces. In view of Proposition II.2.4, it is the search for such systems which leads us to consider the families $\mathcal{S}^r$, for $r > 0$, defined below.

1. Preliminaries

For any vector spaces X and Y ,

$L[X,Y]$ is the set of linear maps on X to Y .

For any topological vector space X ,

$CM(X)$ is the family of continuous finite cylinder measures over X .

Remarks.

From Appendix 3.1.1 and 3.2.1, we have that for any family $\mathcal{C}$ of finite cylinder measures over a vector space X, there exists a coarsest topology on X under which it is a topological vector space, and such that

$$\mu \in \mathcal{C} \Rightarrow \mu \text{ is continuous.}$$

This topology is called the $\mathcal{C}$-topology.

For any topological vector space X, if the topology of X is the $CM(X)$-topology, then we call X a CM-space (Appendix 3.1.2).

For any topological vector space X, $0 < r < \infty$, and w^*-Radón measure η on X^* with supp $\eta \in E$,

$$S_{r,\eta} = \{x \in X : \int |f(x)|^r d\eta(f) \leq 1\} .$$

For each $r > 0$,

$\mathcal{S}^r$ is the family of all sets $S_{r,\eta} \subset X$.

1.1 Remarks

Let X be a topological vector space.

(1) For each $r > 0$, there is a unique topology on X under which X is a topological vector space having $\mathcal{S}^r$ as a base for its neighbourhoods of the origin. When $r \geq 1$, this topology is locally convex.

We shall call this topology the $\mathcal{S}^r$-topology.

(2) If $0 < r < t$, then $\mathcal{S}^t$ is finer than $\mathcal{S}^r$, i.e. for every $\alpha \in \mathcal{S}^r$ there exists $\beta \in \mathcal{S}^t$ with $\beta \subset \alpha$.

(3) If X is locally convex, then, for each $r > 0$

$\mathcal{S}^r$ is a family of neighbourhoods of the origin in X.

We prove only 1.1.2.

Proof of 1.1.2.

For any finite measure space (Ω,η) and integrable $f : \Omega \to \mathbb{C}$, if

$$p = t/r \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1 \quad ,$$

then, by Holder's inequality,

$$\int |f|^r d\eta \leq (\int |f|^{rp} d\eta)^{1/p} \ \eta(\Omega)^{1/q} \quad .$$

Hence,

(1) $$(\int |f|^r d\eta)^{1/r} \leq (\int |f|^t d\eta)^{1/t} \ \eta(\Omega)^{(t-r)/rt} \quad .$$

For any $S_{r,\eta} \in \mathcal{S}^r$,

$$\eta(X^*) < \infty \quad ,$$

since $\operatorname{supp} \eta \in E \subset K(w^*)$ and η is w^*-Radon.

Consequently, by (1), if

$$\xi = \eta(X^*)^{(t-r)/rt}.\eta$$

then

$$S_{t,\xi} \subset S_{r,\eta} \quad .$$

The assertion follows. ∎

To point the significance of the families $\mathcal{S}^r$ we note that Proposition II.2.4 can be restated as follows.

1.2 Proposition

Let X be a topological vector space. For any finite cylinder measure μ over X ,

μ is $\mathcal{E}$-tight => μ is $\mathcal{S}^r$-continuous for every $r > 0$. ▨

When X is Hausdorff and locally convex, the above proposition and Theorem II.3.3 yield the following assertion:

if μ is $\mathcal{W}$-continuous for some weighted system $\mathcal{W}$ in X , then μ is $\mathcal{S}^r$-continuous for each $r > 0$.

In view of this, when searching for weighted systems in X we shall look for suitable subfamilies of $\mathcal{S}^r$.

In general, $\mathcal{S}^r$-continuity for some $r > 0$ does not imply $\mathcal{E}$-tightness. (Example 1, Appendix 4).

We shall need the following result on induced cylinder measures.

1.3. Proposition.

Let X be a topological vector space, Y be a vector space, and $T \in L[X,Y]$. For any family $\mathcal{C}$ of finite cylinder measures over Y , if

$\mu \circ T$ is $\mathcal{E}$-tight for every $\mu \in \mathcal{C}$,

then, for each $r > 0$,

T is $\mathcal{S}^r$-continuous with respect to the $\mathcal{C}$-topology on Y .

<u>Proof</u>. Let $r > 0$. By Prop. 1.2,

$$\mu \in \mathcal{C} \Rightarrow \mu \sqcap T \text{ is } \mathcal{S}^r\text{-continuous.}$$

Hence, by Appendix 3.2.1,

the $\mathcal{S}^r$-topology is finer than the $(\mathcal{C} \sqcap T)$-topology on X .

By Appendix 3.2.2, this says exactly that

T is $\mathcal{S}^r$-continuous with respect to the $\mathcal{C}$-topology on Y . ▨

2. Hilbertian Spaces.

Throughout this section,

X is a Hausdorff, Hilbertian space ([1]).

i.e. X is a Hausdorff, locally convex space, for which there exists a family Γ of pseudo-inner products on X, such that nbnd 0 in X has as a base the family of all sets $\{x \in X : [x,x] \leq 1\}$, $[.,.] \in \Gamma$.

The fundamental theorem of this section is the following.

2.1 Theorem

For each $0 < r < \infty$,

$\mathcal{S}^r$ is a weighted system in X.

The proofs of this and other assertions will be given at the end of the section. Now, we concentrate on the consequences of the above theorem.

2.2 Theorems.

Let $\ddot{\mu}$ be a cylinder measure over X, and $0 < r < \infty$.

Then,

(1) $\ddot{\mu}$ is $\mathcal{E}$-tight $\iff \ddot{\mu}$ is $\mathcal{S}^r$-continuous.

(2) $\ddot{\mu}$ is $\mathcal{S}^r$-continuous $\Rightarrow \ddot{\mu}$ has a c^*-Radón limit measure on X'.

(3) If $K(c^*) = E$, in particular, if X is barrelled, then

μ is S^r-continuous <=>

μ is E-tight <=>

μ has a c^*-Radón limit measure on X' .

Using Theorems 2.2, we can now characterize certain positive-definite functions on X (Appendix 2).

2.3 Theorem.

Let ψ be a positive-definite function on X and $0 < r < \infty$.

Then,

ψ is S^r-continuous => there exists some finite c^*-Radón measure ξ on X' such that

$$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f) \quad \text{for every } x \in X .$$

If $K(c^*) = E$, in particular, if X is barrelled, then

ψ is S^r-continuous <=> there exists some finite c^*-Radón measure ξ on X' such that

$$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f) \quad \text{for every } x \in X .$$

Remarks

We note that Theorem 2.2.2 generalizes a result of Minlos ([25] p. 303 Thm. 1). Theorem 2.3 generalizes results due to Minlos ([25] P. 310), and Badrikian ([1] p. 16 Cor. 1). The special case when X is a Hilbert space will be discussed below (§2.7).

We point out that, with the viewpoint of §I.4, the assertions of Theorems 2.2.2 and 2.3 for the case $r = 2$ can be established by using the technique of characteristic functionals ([1], p. 9, Lemma 1, Prohorov [33]). Also, it can be shown that the S^2-topology is nothing else but the Gross-Sazonov topology on X ([35], [1], [13] p. 65).

By means of Proposition 1.2 and Remark 1.1.2 we can deduce the assertions above for $0 < r < 2$ from the case $r = 2$. We have been unable to give a similar deduction for the case $r > 2$. However, in this context, we draw attention to §2.6 below.

As consequences of Theorems 2.1, II 4.3, and Proposition 1.3, we have the following assertion concerning induced cylinder measures over X.

2.4 Theorem

Let Y be a vector space, $T \in L[X,Y]$, and $0 < r < \infty$. For any family C of finite cylinder measures over Y,

$\mu \,\square\, T$ is E-tight for every $\mu \in C$ <=>

T is S^r-continuous with respect to the C-topology on Y.

The above theorem yields immediately the corollaries given below. Corollary (2) significantly generalizes a result in [11] (p. 349).

Corollaries

Let Y be a topological vector space, $T \in L[X,Y]$, and $r > 0$.

(1) If Y is a CM-space, then

$\mu \circ T$ is E-tight for every $\mu \in CM(Y)$ <=>

T is S^r-continuous.

(2) If T is S^r-continuous, then, for every $\mu \in CM(Y)$,

$\mu \circ T$ is E-tight,

and therefore has a c^*-Radón limit measure on X' .

2.5 Remarks

Under certain circumstances one can readily strengthen the assertions of Theorems 2.2 - 2.4.

Let μ be a cylinder measure over X .

(1) (Theorem II. 2.3) If there exists a base $\mathcal{U}$ for nbnd 0 in X such that for each $U \in \mathcal{U}$,

the Banach space X'_U is separable,

then,

μ is E-tight => μ has an $\mathfrak{s}^*$-Radón limit measure on X' .

Hence, in those theorems involving the existence of a c^*-Radón limit measure on X' , we may replace c^* by $\mathfrak{s}^*$.

(2) Let G be a regular topology on X' with $w^*|X' \subset G$.

If

$E \subset K(G)$, or $E = K(G)$,

then the foregoing theorems may be modified as indicated by Theorem I.4.3.

In pariticular, we note that when X is a Montel space,

$$E = K(s^*) \quad . \quad \text{(cf. Thm. II.2.2.1)}$$

The theorems above allow us to make some interesting assertions about the S^r-topologies.

2.6 Theorems.

(1) For all $0 < r < \infty$, the families of S^r-continuous cylinder measures coincide.

(2) For all $0 < r \leq 2$, the S^r-topologies coincide.

(3) Let Y be a topological vector space, and for each $r > 0$,

$$T_r = \{T \in L[X,Y] : T \text{ is } S^r\text{-continuous}\} \quad .$$

If Y is a CM-space, then, for all $0 < r < \infty$, the families T_r coincide.

Remark.

In general, the S^r-topologies do not coincide for $r \geq 2$ (Example 3, Appendix 4). ∎

Clearly, we may interpret all of our results for the special case when X is a Hilbert space. In particular, we have the following theorems.

2.7 Theorems.

Let X be a Hilbert space,

(1) Let $0 < r < \infty$. For any cylinder measure μ over X,

μ is S^r-continuous <=>

μ is E-tight <=>

μ has a c^*-Radón limit measure on X'.

(2) Let $0 < r < \infty$, and ψ be a positive-definite function on X.

ψ is S^r-continuous <=>

for some finite c^*-Radón measure ξ on X',

$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f)$ for all $x \in X$.

(3) Let Y be a Hilbert space, and $T \in L[X,Y]$.

$\mu \square T$ has a c^*-Radón limit measure on X' for every $\mu \in CM(Y)$ <=>

T is a Hilbert-Schmidt map ([36] p. 177).

(4) ([42] VIII, Pietsch [31], Pełczyński [28]). Let Y be a Hilbert space.

For all $0 < r < \infty$,

$\{T \in L[X,Y] : T$ is r-summable$\}$

$= \{T \in L[X,Y] : T$ is Hilbert-Schmidt$\}$.

(For the definition of r-summability, see [31], and [42] p. VII. 3).

Remark

By Theorem II.2.3.2, when X is a separable Hilbert space, every c^*-Radón measure on X' is s^*-Radón. Hence, in Theorems 2.7, we can replace c^* by s^* when X is separable.

We point out that Theorems 2.7.1 and 2.7.2 are equivalent (Cor. I.4.3, Thm. I.5.6.2, Appendix 2.5 and 2.6).

We observe that even when X is a Hilbert space our work extends previously known results. Sazonov in [35] discusses the case when X is separable, obtaining Theorem 2.7.2 for the case $r = 2$. Waldenfels in [48] extends Sazonov's theorem to the non-separable case. Theorem 2.7.3 extends a result given in [11] (p. 349), where X is assumed to be separable and $r = 2$.

From Appendix 3.5 and Proof 2.7.4 we see that theorem 2.6.3 significantly generalizes the Pietsch-Pełczyński theorem given above (Theorem 2.7.4).

Proofs 2.

We shall need the following lemma.

Lemma

Let X be a locally convex space, $r > 0$, and $S = S_{r,\eta} \in \mathcal{S}^r$.

Let

$$\mathcal{P} = \mathcal{P}(M_\eta|\text{supp } \eta) \quad \text{directed by refinement,}$$

and for each $P \in \mathcal{P}$,

$$S'_P = \{x \in X : \sum_{B \in P} \inf_{f \in B} |f(x)|^r . \eta(B) > 1\} .$$

Then, for any $F \in \mathcal{F}_Y$, Radon measure ξ on F and $t > 0$,

$$\xi(F \sim tS) = \lim_{P \in \mathcal{P}} \xi(F \cap tS'_P) \quad .$$

<u>Proof of Lemma</u>

We first make the following observations.

(1) If $P \in \mathcal{P}$, $Q \in \mathcal{P}$, with Q finer than P, then

$$S'_P \subset S'_Q \quad .$$

(2) For any $u > 0$,

$$X \sim uS = \bigcup_{P \in \mathcal{P}} u\, S'_P \quad .$$

(3) For every $P \in \mathcal{P}$,

$$S'_P \text{ is open in } X \quad .$$

We prove only (3).

Let $P \in \mathcal{P}$. We have that

(4) $$S'_P = \bigcup_{\mathcal{B}} \{x \in X : \sum_{B \in \mathcal{B}} \inf_{f \in B} |f(x)|^r . \eta(B) > 1\} ,$$

where the union is taken over all finite $\mathcal{B} \subset P$. Hence, since $\operatorname{supp} \eta \in \mathcal{E}$ is equicontinuous, for every $B \in P$,

$$x \in X \to \inf_{f \in B} |f(x)| \in R \text{ is continuous.}$$

Hence, for any finite $\mathcal{B} \subsetneq P$,

(5) $$x \in X \to \sum_{B \in \mathcal{B}} (\inf_{f \in B} |f(x)|)^r . \eta(B) \text{ is continuous,}$$

and therefore, by (4), (3) holds.

From (3) we deduce that

$$tF \cap S'_P \text{ is open for every } P \in \mathcal{P} .$$

Consequently, as $\mathcal{P}$ is directed by refinement, from (2) it follows that for any compact C in F with $C \subset F \sim tS$, there exists $P \in \mathcal{P}$ s.t.

$C \subset tF \cap S'_P$.

Hence, since ξ is Radón and $F \sim tS$ is open in F ,

$$\xi(F \sim tS) = \sup \{\xi(C) : C \subset F \sim tS \text{ is compact}\}$$
$$= \sup \{\xi(tF \cap S'_P : P \in \mathcal{P}\}$$
$$= \lim_{P \in \mathcal{P}} \xi(tF \cap S'_P) \ .$$

2.1 Let $\mathcal{V}$ be a base for nbnd 0 in X s.t. for each $V \in \mathcal{V}$ there exists a pseudo-inner-product $[.,.]_V$ on X for which

$$V = \{x \in X : [x,x]_V \leq 1\} \ .$$

Let δ_2, $C_{2,r}$ be as given in Appendix 1.1.

For each $V \in \mathcal{V}$, let

$$\hat{\mathcal{F}} = \mathcal{F} \ .$$

For each $V \in \mathcal{V}$ and $F \in \mathcal{F}$, let

$\nu_{V,F}$ be a probability Radón measure on F related to $[.,.]_V|F \times F$ as in Appendix 1.3.

Then, from Appendix 1.3 we see that

$(\nu,\hat{\mathcal{F}},\mathcal{V})$ is a system of δ_2-weights in X .

By Remark 1.1.3,

(1) $\mathcal{S}^r \subset$ nbnd 0 in X .

Let $S = S_{r,\eta} \in \mathcal{S}^r$.

Since $\mathcal{V}$ is a base for nbnd 0 in X and supp $\eta \in \mathcal{E}$, there exists $V \in \mathcal{V}$ with

$$\text{supp } \eta \subset V^0 \ .$$

Then,

$$x \in \ker V \Rightarrow \sup_{f \in V^0} |f(x)| = 0 \Rightarrow \int |f(x)|^r d\eta = 0 \Rightarrow x \in S \quad .$$

i.e.

(2) $\quad \ker V \subset S \quad .$

Let $t > 0$. For any $B \subset X'$, let

$$f_B \in B \quad \text{and} \quad g_B = \frac{1}{t}.\eta(B)^{1/r}.f_B \quad .$$

Using the notation of the above Lemma, for any $P \in \mathcal{P}$,

$$F \cap tS'_P \subset \{x \in F : \sum_{B \in P} |f_B(x)|^r.\eta(B) > t^r\}$$

Hence,

$$\nu_{V,F}(F \cap tS'_P) \leq \nu_{V,F}(\{x \in F : \sum_{B \in P} |g_B(x)|^r > 1\})$$

$$\leq C_{2,r} \sum_{B \in P} \sup_{x \in V\ F} |g_B(x)|^r \text{ by Appendix 1.3.2,}$$

$$\leq C_{2,r} \sum_{B \in P} \sup_{x \in V} |g_B(x)|^r \leq \frac{1}{t^r} C_{2,r} \sum_{B \in P} \sup_{x \in V} |f_B(x)|^r.\eta(B)$$

$$\leq \frac{1}{t^r} C_{2,r}\eta(X') \text{ since } f_B \in V^0 \text{ for every } B \in P \quad .$$

From the above lemma it now follows that

$$\nu_{V,F}(F \sim tS) \leq \frac{1}{t^r} C_{2,r}\eta(X') \quad .$$

Since $C_{2,r}\eta(X') < \infty$, we conclude that

(3) $\quad \nu_{V,F}(F \sim tS) \to 0$ as $t \to \infty$ uniformly for $F \in \mathcal{F}$.

From (1), (2), and (3) we see that

$\quad \mathcal{S}^r$ is weighted by $(\nu, \hat{\mathcal{F}}, \mathcal{V})$.

2.2.1 $\quad$ By Cor. II.3.3, Thm. I.5.6.2 and Prop. 1.2 .

2.2.2 and 2.2.3 By Thms. II.2.1 and 2.2.1.

2.3 By Remark 1.1.3 and Appendix 2.2.5,

ψ is $\mathcal{S}^r$-continuous => ψ is continuous at 0

=> ψ is continuous => $\psi|F$ is continuous for all $F \in \mathcal{F}$.

Hence, by 2.4 and 2.5 of the Appendix,

there exists a finite $\mathcal{S}^r$-continuous cylinder measure

μ over X s.t.

$$\psi(x) = \int \exp i \operatorname{Re} f(x) d\mu_x(f) \quad \text{for all} \quad x \in X \quad .$$

By Thm. 2.2.2,

μ has a c^*-Radon limit measure ξ on X' .

Then, for every $x \in X$,

$$\psi(x) = \int_{F_x} \exp i \operatorname{Re} f(x) d\mu_x(f) = \int_{X'} \exp i \operatorname{Re} f(x) d\xi(f) \quad .$$

Suppose now that $E = K(c^*)$, and for some finite Radon measure ξ on X' ,

$$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f) \quad \text{for all} \quad x \in X \quad .$$

We note that for every $F \in \mathcal{F}$ and Borel subset H of F^* ,

$$r_F^{-1}[H] \in M_\xi \quad .$$

If, for each $F \in \mathcal{F}$,

$$\mu_F = r_F[\xi] \quad ,$$

then, by Lemmas 0.4 and Remark I.2.1,

μ is a finite cylinder measure over X .

Further, by Lemma 0.4.2,

ξ is a limit measure of μ ,

and therefore it follows that

ψ is the characteristic functional of μ .

Since ξ is c^*-Radón and $K(c^*) = E$, then, from Lemma I.5.1.2 and the definition of μ we see that

μ is E-tight.

Hence, by Prop. 1.2,

μ is S^r-continuous,

and therefore, by Appendix 2.5,

ψ is S^r-continuous.

2.6.1. By Thm. 2.2.1.

2.6.2. By Thm. 2.6.1, Cor. 2 of Appendix 3.5, and Appendix 3.2.1.

2.6.3. By Cor. 1 of Thm. 2.4.

2.7.1 and 2.7.2. are consequences respectively of Thms. 2.2.3 and 2.3, since Hilbert spaces are barrelled.

2.7.3. Since X and Y are Banach spaces, by [31] p. 339, Thm. 1,

T is S^2-continuous <=> T is Hilbert-Schmidt.

The assertion is now a consequence of Cor. 1 of 2.4, and Cor. 3 of Appendix 3.5.

2.7.4. Since X and Y are Banach spaces, by [42] p. VII. 3, § 2, for any $r > 0$,

T is S^r-continuous <=> T is r-absolutely summable.

The assertion now follows from Thm. 2.6.3 and Cor. 3 of Appendix 3.5.

3. Nuclear Spaces.

Nuclear spaces comprise one particularly important family of Hausdorff, Hilbertian spaces (Grothendieck [14], see also [36] and [47]). We shall therefore interpret the results of the previous section for the case when X is a nuclear space. As a consequence of the special structure of nuclear spaces, we shall be able to strengthen considerably the theorems concerning cylinder measures over arbitrary Hausdorff, Hilbertian spaces. We point out that many of the common spaces of distributions are in fact nuclear (Treves [47] Ch. 51).

For our definition of a nuclear space we shall use a characterization due to Pietsch ([29], [36] p. 178).

3.1 Definition.

X is a nuclear space iff X is a Hausdorff, locally convex space with the following property:

for any neighbourhood U of 0 in X, there exists another neighbourhood V of 0 in X, and a w^*-Radón measure η on X^* with supp $\eta \subset V^0$, such that

$$\{x \in X : \int |f(x)| d\eta(f) \leq 1\} \subset U \quad .$$

Remarks

If X is a Hausdorff, locally convex space, then, from Remark 1.1.3 and the above definition, we see that

(1) X is nuclear iff S^1 is a base for nbnd 0 in X .

For any nuclear space X , from (1) above, Remarks 1.1.2 and 1.1.3, it follows that

(2) the S^r-topologies on X coincide for $r \geq 1$.

In particular, taking S^2 as a base for nbnd 0 in X , we deduce that

(3) X is a Hilbertian space ([36] p. 102).

As in Treves [47], p. 519, we can prove that

(4) $E \subset K(s^*)$.

Hence, if X is barrelled, then

(5) $E = K(s^*)$.

We point out that coincidence of all the S^r-topologies for $r > 0$ is a consequence of (2), (3), and Theorem 2.6.2.

The theorems given below in 3.2 are direct consequences of the above remarks, and assertions from the previous section, specifically, Theorems 2.2, Theorem 2.3, and Remark 2.5.2.

3.2 Theorems.

Let X be a nuclear space, and μ be a cylinder measure over X .

(1) μ is continuous <=> μ is E-tight.

(2) μ is continuous => μ has an s^*-Radon limit measure on X' .

(3) If $\mathcal{K}(s^*) = \mathcal{E}$, in particular, if X is barrelled, then,

μ is continuous <=>

μ is $\mathcal{E}$-tight <=>

μ has an s^*-Radón limit measure on X' .

(4) Let ψ be a positive-definite function on X .

ψ is continuous =>

there exists an s^*-Radón measure ξ on X' such that

$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f)$ for all $x \varepsilon X$.

If $\mathcal{E} = \mathcal{K}(s^*)$, in particular, if X is barrelled, then

ψ is continuous <=>

there exists a finite s^*-Radón measure ξ on X' such that

$\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f)$ for all $x \varepsilon X$.

Theorem 3.2.2 extends a result of Minlos ([25], p. 303, Thm. 1), who considered finite cylinder measures over countably normed nuclear spaces ([11] p. 56). Vilenkin extended that result to the case of countable strict inductive limits of such spaces ([11] Ch. IV 2.4). Theorem 3.2.4 extends results due to Minlos ([25] p. 310) and Badrikian ([1] p. 17). We note that the theorems of 3.2 completely resolve a conjecture of I. Gelfand ([25] p. 310, [18], p. 222), that every finite continuous cylinder measure over a nuclear space X has a limit measure on the continuous dual X' .

Theorem 3.2.1 has a partial converse which extends a result of Minlos ([25]. Thm. 4).

3.3 Theorem

Let X be a Hausdorff, locally convex space.

If X is a CM-space and

$$\mu \in CM(X) \Rightarrow \mu \text{ is } \mathcal{E}\text{-tight},$$

then X is nuclear.

Proof By Prop. 1.2,

$$\mu \in CM(X) \Rightarrow \mu \text{ is } \mathcal{S}^1\text{-continuous.}$$

Hence, by Appendix 3.2.1,

the $\mathcal{S}^1$-topology is finer than the CM(X)-topology.

On the other hand, by Cor. 2 of Appendix 3.5, and Remark 1.1.3,

the CM(X)-topology is finer than the $\mathcal{S}^1$-topology.

Consequently,

the CM(X)-topology = the $\mathcal{S}^1$-topology.

Since X is a CM-space, it follows from Remark 3.1.1 that

X is nuclear. ∎

Remark.

We note that a Hausdorff, locally convex space is not necessarily a CM-space (Example 4.3, Appendix 4). When X is not a CM-space we see from the above proof that the best assertion possible is the following.

If $\mu \in CM(X) \Rightarrow \mu$ is $\mathcal{E}$-tight,

then,

the $\mathcal{S}^1$-topology and CM(X)-topology coincide.

Theorems 3.3 and 3.2.1 lead to the following new characterization of nuclear spaces (Remark 3.1.3, Cor. 3 of Appendix 3.5).

3.4 Theorem

Let X be a Hausdorff, locally convex space.

X is nuclear iff X is a $\mathcal{C}M$-space and

$\mu \in \mathcal{C}M(X) \Rightarrow \mu$ is $\mathcal{E}$-tight.

Concerning induced cylinder measures, Remark 3.1.4 enables us to strengthen Corollary (2) of Theorem 2.4. In view of Remark 2.5.2, the following assertion is immediate.

3.5 Theorem

Let X be a nuclear space, Y be a topological vector space, and $T \in L[X,Y]$.

If T is continuous, then, for every $\mu \in \mathcal{C}M(Y)$,

$\mu \circ T$ is $\mathcal{E}$-tight,

and therefore has an s^*-Radón limit measure on X' .

We observe that an infinite-dimensional normed space cannot be nuclear ([47], p. 520). As a consequence of this fact we can assert that certain cylinder measures over such a space X cannot have a limit measure on X' .

3.6 Proposition

Let X be an infinite-dimensional normed space

If μ is a finite cylinder measure over X such that the topology of X is the $\{\mu\}$-topology, then μ does not have a limit measure on X' .

Proof. By Cor. I.4.3, Prop. 1.2, and Appendix 3.2.1,

μ has a limit measure on X'

=> μ is $\mathcal{E}$-tight

=> μ is S^1-continuous

=> $\{\mu\}$-topology is coarser than the S^1-topology

=> X is nuclear, by Remarks 1.1.3 and 3.1.1.

Since X is an infinite-dimensional normed space the last assertion cannot hold, and therefore μ cannot have a limit measure on X' .

Corollary

Let A be an index set. For any $1 \leq p \leq 2$, if is the finite cylinder measure over $\ell^p(A)$ with characteristic functional (Remark, Appendix 2.4)

$$x \in \ell^p(A) \to \exp - (\|x\|_p)^p \in \mathbb{C} \quad ,$$

then μ does not have a limit measure on $(\ell^p(A))'$.

Proof See (i) in Proof of Example 4.2, Appendix 4, and Proof 3.1.1 of Appendix 3.

Remark

For $p = 2$ the above corollary is well known (Gross [13]). We have not seen a treatment of the case $1 \leq p < 2$ in the literature.

4. ℓ^p-spaces.

Applied to ℓ^p-spaces, $1 \leq p \leq \infty$, the theory of the previous chapter yields results analagous to those for Hilbertian spaces. Since ℓ^2-is a Hilbert space this case has already been discussed in §2.5. The results given there are stronger than those we shall obtain here for an arbitrary ℓ^p-space.

Notation

Let A be an index set. For any $0 < r \leq \infty$,

$$\ell^r(A) = \begin{cases} \{x \in \mathbb{C}^A : \sum_{\alpha \in A} |x(\alpha)|^r < \infty\} & \text{when } r \leq \infty, \\ \\ \{x \in \mathbb{C}^A : \sup_{\alpha \in A} |x(\alpha)| < \infty\} & \text{when } r = \infty. \end{cases}$$

We give $\ell^r(A)$ the usual topology. i.e.,

when $r < 1$, the topology generated by the quasi-norm (Appendix 3.3)

$$b_r : x \in \ell^r(A) \to \sum_{\alpha \in A} |x(\alpha)|^r \in R^+ ;$$

when $r \geq 1$, the topology generated by the norm

$$\|\cdot\|_r : x \in \ell^r(A) \to (\sum_{\alpha \in A} |x(\alpha)|^r)^{1/r} \in R^+ ,$$

where we take

$$(\sum_{\alpha \in A} |x(\alpha)|^r)^{1/r} = \sup_{\alpha \in A} |x(\alpha)| \quad \text{if } r = \infty .$$

For any $1 \leq p \leq 2$,

$$U_p = \{x \in \ell^p(A) : \sum_{\alpha \in A} |x(\alpha)|^2 \leq 1\} .$$

For any outer measure η on a space Ω,

$$\ell\nu(\eta) = \lim_{P\varepsilon P(M_\eta)} \sum_{B\varepsilon P} \eta(B)|\ln\eta(B)|$$

where we take

$$u \ln u = 0 \quad \text{when} \quad u = 0 \ .$$

The heart of this section is the following group of results, which assert that certain families of subsets of $\ell^p(A)$, $1 \leq p \leq \infty$, are weighted systems in $\ell^p(A)$.

4.1. Theorem.

Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$.

For any $r > 0$, let

$\overline{S}^r \subset S^r$ consist of those sets $S_{r,\eta} \varepsilon S^r$ for which η satisfies the added condition

$$\ell\nu(\eta) < \infty \ ,$$

and when $1 \leq p \leq 2$, let

$\tilde{S}^r \subset S^r$ consist of those sets $S_{r,\eta} \varepsilon S^r$ for which η satisfies the added condition

$$\int^* (\sup_{x\varepsilon U_p} |f(x)|^r d\eta(f) < \infty \ .$$

(1) If $2 < p \leq \infty$ and $0 < r < q$ then S^r is a weighted system in $\ell^p(A)$.

(2) If $2 < p \leq \infty$ and $r = q$ then $\overline{S}^r$ is a weighted system in $\ell^p(A)$.

(3) If $1 \leq p \leq 2$ and $0 < r < \infty$, then $\tilde{S}^r$ is a weighted system in $\ell^p(A)$.

(We note that $\tilde{S}^r = S^r$ when $p = 2$.)

The proof of the above theorem will be given at the end of the section. Now, we point out its immediate consequences when taken together with Corollary II.3.3.

4.2 Theorems

Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

(1) If $2 < p \leq \infty$ and $0 < r < q$, then, for any cylinder measure μ over $\ell^p(A)$,

μ is $\mathcal{S}^r$-continuous $\Leftrightarrow$

μ is $\mathcal{E}$-tight $\Leftrightarrow$

μ has a c^*-Radon limit measure on $(\ell^p(A))'$.

(Here, we also use Prop. 1.2 and Thm. II.2.1, noting that $\ell^p(A)$ is a Banach space and is therefore barrelled.)

(2) If $2 < p \leq \infty$ and $r = q$, then, for any cylinder measure μ over $\ell^p(A)$,

μ is $\overline{\mathcal{S}}^r$-continuous $\Rightarrow$

μ is $\mathcal{E}$-tight $\Rightarrow$

μ has a c^*-Radon limit measure on $(\ell^p(A))'$.

(3) If $1 \leq p \leq 2$ and $r > 0$, then, for any cylinder measure μ over $\ell^p(A)$,

μ is $\tilde{\mathcal{S}}^r$-continuous $\Rightarrow$

μ is $\mathcal{E}$-tight $\Rightarrow$

μ has a c^*-Radon limit measure on $(\ell^p(A))'$.

Using Theorems 4.2 we can represent certain positive-definite functions on $\ell^p(A)$ as Fourier transforms of measures on $(\ell^p(A))'$. The proofs of the assertions given below are similar to the proof of Theorem 2.3, and are therefore omitted.

4.3 Theorems

Let $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and ψ be a positive-definite function on $\ell^p(A)$.

(1) If $2 < p \leq \infty$ and $0 < r < q$, then,

ψ is S^r-continuous <=> for some finite c^*-Radón measure ξ on $(\ell^p(A))'$,

$$\psi(x) = \int \exp i \operatorname{Re} f(x)\, d\xi(f) \quad \text{for all} \quad x \in \ell^p(A) .$$

(2) If $2 < p \leq \infty$ and $r = q$, then,

ψ is $\overline{S}^r$-continuous => for some finite c^*-Radón measure ξ on $(\ell^p(A))'$,

$$\psi(x) = \int \exp i \operatorname{Re} f(x)\, d\xi(f) \quad \text{for all} \quad x \in \ell^p(A) .$$

(3) If $1 \leq p \leq 2$ and $r > 0$, then,

ψ is $\tilde{S}^r$-continuous => for some finite c^*-Radón measure ξ on $(\ell^p(A))'$,

$$\psi(x) = \int \exp i \operatorname{Re} f(x)\, d\xi(f) \quad \text{for all} \quad x \in \ell^p(A) .$$

Concerning induced cylinder measures, Theorem 4.1 yields the following results when taken together with Theorem II.4.3.

4.4 Theorems.

Let $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, Y be a vector space, $\mathcal{C}$ be a family of finite cylinder measures over Y , and $T \in [\ell^p(A), Y]$.

(1) If $2 < p \leq \infty$ and $0 < r < q$, then,

T is S^r-continuous with respect to the $\mathcal{C}$-topology on Y

$\Leftrightarrow$ for every $\mu \in \mathcal{C}$,

$\mu \,\square\, T$ has a c^*-Radoń limit measure on $(\ell^p(A))'$.

(Here, as for Thm. 4.2, we also use Prop. 1.3 and Thm. II.2.1.).

(2) If $2 < p \leq \infty$ and $r = q$, then,

T is $\overline{S}^r$-continuous with respect to the $\mathcal{C}$-topology on Y $\Rightarrow$

for every $\mu \in \mathcal{C}$,

$\mu \,\square\, T$ is $\mathcal{E}$-tight $\Rightarrow$

for every $\mu \in \mathcal{C}$,

$\mu \,\square\, T$ has a c^*-Radoń limit measure on $(\ell^p(A))'$.

(3) If $1 \leq p \leq 2$ and $r > 0$, then,

T is $\tilde{S}^r$-continuous with respect to the $\mathcal{C}$-topology on Y $\Rightarrow$

for every $\mu \in \mathcal{C}$,

$\mu \,\square\, T$ is $\mathcal{E}$-tight $\Rightarrow$

for every $\mu \in \mathcal{C}$,

$\mu \,\square\, T$ has a c^*-Radoń limit measure on $(\ell^p(A))'$.

As consequences of Theorems 4.4 we have the following extensions of results due L. Schwartz [39] and Kwapien [19]. They consider only the case when $r = q$ and A is countable.

Corollaries.

(1) If $2 < p \leq \infty$, $0 < r < q$, $y \in \ell^r(A)$ and

$$T : x \in \ell^p(A) \to (x(\alpha)y(\alpha))_{\alpha\in A} \in \ell^r(A) \quad ,$$

then, for every $\mu \in CM(\ell^r(A))$,

$\mu \boxdot T$ has a c^*-Radon limit measure on $(\ell^p(A))'$.

(2) If $2 < p \leq \infty$, $r = q$, $y \in \ell^r(A)$ with

$$\sum_{\alpha\in A} |y(\alpha)|^r |\ln|y(\alpha)|| < \infty \quad ,$$

and

$$T : x \in \ell^p(A) \to (x(\alpha)y(\alpha))_{\alpha\in A} \in \ell^r(A) \quad ,$$

then, for every $\mu \in CM(\ell^r(A))$,

$\mu \boxdot T$ has a c^*-Radon limit measure on $(\ell^p(A))'$.

(3) If $1 \leq p \leq 2$, $r > 0$, $y \in \ell^r(A)$, and

$$T : x \in \ell^p(A) \to (x(\alpha)y(\alpha))_{\alpha\in A} \in \ell^r(A) \quad ,$$

then, for every $\mu \in CM(\ell^r(A))$,

$\mu \boxdot T$ has a c^*-Radon limit measure on $(\ell^p(A))'$.

We give here the proof of only Corollary (1). The other proofs are similar.

Proof of Corollary (1).

For each $\alpha \in A$, let

$$e_\alpha \in (\ell^p(A))' : x \in \ell^p(A) \to x(\alpha) \in C \quad ,$$

and η be the discrete measure on $(\ell^p(A))'$ with

$$\eta(\{e_\alpha\}) = |y(\alpha)|^r \quad \text{for each } \alpha \in A \ .$$

Then,

$$\operatorname{supp} \eta \in \mathcal{E} \quad ,$$

and for any $x \in \ell^p(A)$,

$$\sum_{\alpha \in A} |(Tx)_\alpha|^r = \int |f(x)|^r d\eta(f) \quad .$$

It follows that T is $\mathcal{S}^r$-continuous, and the corollary is now an immediate consequence of Thm. 4.4.1. ∎

4.5 Remarks.

(1) If A is countable, then $(\ell^p(A))'$ is separable. Consequently, by Theorem II.2.3.2, every c^*-Radon measure on $(\ell^p(A))'$ is in fact s^*-Radon. The foregoing theorems may then be suitably modified.

(2) We point out that Corollary 4.4.2 is the best result possible when $2 < p \leq \infty$ and $r = q$. If

$$y \in \ell^q(A) \text{ with } \sum_{\alpha \in A} |y(\alpha)|^q |\ln|y(\alpha)|| = \infty \quad ,$$

and T is as given in the corollary, then by Example 4.2 of Appendix 4,

there exists $\mu \in CM(\ell^q(A))$ such that $\mu \,\square\, T$ fails to be $\mathcal{E}$-tight.

(3) With the notation of (2) above, as in the proof of Corollary 4.4.1, we see that

T is $\mathcal{S}^q$-continuous,

and therefore, by Lemma II.4.2,

$\mu \,\square\, T$ is $\mathcal{S}^q$-continuous.

From Remark (2) above, and Theorem II.3.3, it now follows that for any $2 < p \leq \infty$, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$\mathcal{S}^q$ is not a weighted system in $\ell^p(A)$.

However, Proposition 1.2 suggests that when searching for a weighted system in $\ell^p(A)$ we ought to look for a subfamily of S^q. Remark (2) then indicates that $\overline{S^q}$ is in fact an appropriate subfamily of S^q for us to consider.

(4) When $2 \leq p \leq \infty$, the construction which we use for producing a system of δ-weights in $\ell^p(A)$ depends on the fact that for any finite set K,

$$x \in \mathbb{C}^K \to \exp - \sum_{\alpha \in K} |x(\alpha)|^q \in \mathbb{C}$$

is a positive-definite function on $\mathbb{C}^K$, where $\frac{1}{p} + \frac{1}{q} = 1$. (Remark (1) of Proofs (4), and Proof 2.2 of Appendix 2.) If $1 \leq p < 2$, then $q > 2$ and the function given above is no longer positive-definite (Schoenberg [38] p. 532). The construction therefore breaks down when $1 \leq p < 2$. We can show that construction of a system of δ-weights in $\ell^p(A)$, $1 \leq p < 2$, would be possible if there were a $X : R^+ \to \mathbb{C}$ such that for any finite set K,

$$x \in \mathbb{C}^K \to X(\sum_{\alpha \in K} |x(\alpha)|^q) \in \mathbb{C}$$

was positive-definite on $\mathbb{C}^K$. If such a function X existed, then, by Appendix 2.2.4,

(i) $$x \in \ell^q(A) \to X(\sum_{\alpha \in A} |x(\alpha)|^q) \in \mathbb{C}$$

would be a positive-definite function on $\ell^q(A)$. However, when $q > 2$, one can show as in [5] that there does not exist $X : R^+ \to \mathbb{C}$ such that (i) holds.

Nonetheless, we can still obtain a system of δ-weights in $\ell^p(A)$, $1 \leq p < 2$, if we use the system of δ_2-weights induced by the canonical imbedding

$$x \in \ell^p(A) \to x \in \ell^2(A) \quad . \quad \text{(Remark (2) of Proofs 4.)}$$

(5) Remarks (4), Proposition 1.2, and the proof of Theorem 4.1.1, led us to believe that $\tilde{S}^r \subset S^r$, $r > 0$, would be a suitable family to study when searching for a weighted system in $\ell^p(A)$, $1 \leq p < 2$.

(6) We point out that for $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the appearance of q in the hypotheses arises at the finite-dimensional level (Remarks (4) and Appendix 1). Thus, although $(\ell^p(A))'$ may be identified with $\ell^q(A)$, we have avoided doing this, as carrying out such an identification might have suggested that the relationship between $\ell^p(A)$ and $\ell^q(A)$ was crucial to our argument.

Proofs 4.

Together with the notations of Appendix 1.1 and Proofs 2, for any $p \geq 1$, let

$$\frac{1}{p} + \frac{1}{q} = 1 \quad ,$$

$$X_p = \ell^p(A) \quad ,$$

$$V_p = \{x \in X_p : |x|_p \leq 1\} \quad .$$

$$|\cdot|_q : f \in X_p' \to \sup_{x \in V_p} |f(x)| \quad .$$

For any finite $K \subset A$, let

$$|\cdot|_{q,K} : f \in (\mathbb{C}^K)^* \to \sup \{|f(x)| : x \in V_p \cap \mathbb{C}^K\} \quad ,$$

$$r_K = r_{\mathbb{C}^K, X_p} \quad .$$

Let

$\hat{F} = \{\mathbb{C}^K : K \subset A$ is finite$\}$ directed by inclusion, For any $2 \leq p \leq \infty$ and finite $K \subset A$, let

γ_p^K be the product measure on $\mathbb{C}^K$ generated by γ_p on C,

and $\nu^p : \mathbb{C}^K \in \hat{F} \to \gamma_p^K$, which is Radon.

Remarks

(1) By Appendix 1.2, for each $2 < p \leq \infty$,

$(\nu^p, \hat{F}, V_p)$ is a system of δ_p-weights in X_p.

(2) For each $1 \leq p \leq 2$, since

$(V_2 \cap C^K)^0 \subset (V_p \cap C^K)^0$ for every finite $K \subset A$,

then, by Appendix 1.2, for every $1 \leq p \leq 2$,

(ν^2, F, V_p) is a system of δ_2-weights in X_p.

4.1. We observe that for any $p \geq 1$ and $r > 0$,

(1) $S^r \subset$ nbnd 0 in X_p (Remark 1.1.3).

and

(2) $\ker V_p = \{0\} \in S$ for every $S \in S^r$.

Now, for any $S = S_{r,\eta} \in S^r$, and each $B \subset X_p'$, let

$f_B \in B$, $g_B = \eta^{1/r}(B).f_B$

and

$s = \sup \{|f|_q : f \in \operatorname{supp} \eta\}$.

Then, as in Proof 2.2.1, for any $P \in P$, finite $K \subset A$, and $t > 0$, we have that

(5) $\nu_K^p(\mathbb{C}^K \cap tS_p') \leq \nu_K^p(\{x \in \mathbb{C}^K : \sum_{B \in P} |\frac{1}{t} g_B(x)|^r > 1\})$,

and, by the lemma of Proofs 2,

(6) $\nu_K^p(\mathbb{C}^K \sim tS) = \lim_{P \in P(S)} \nu_K^p(\mathbb{C}^K \cap tS_p')$.

Case 1. $(2 < p \leq \infty,\ 0 < r < q)$.

Since $\operatorname{supp} \eta \in E$,

$$0 < s < \infty \quad .$$

Then, for any $t > \frac{1}{s} \eta^{1/r}(X_p')$,

$$B \subset \operatorname{supp} \eta \Rightarrow \left|\frac{1}{t} g_B\right|_q < 1 \quad .$$

Hence, by Appendix 1.2.3, the right-hand-side of (5) is majorized by

$$C_{p,r} \sum_{B\varepsilon P} \left|r_K(\frac{1}{t} g_B)\right|_{q,K}^r + 2\pi C_p(\frac{q-r+1}{q-r}) \sum_{B\varepsilon P} \left|r_K(\frac{1}{t} g_B)\right|_{q,K}^q$$

$$\leq \frac{1}{t^r} C_{p,r} s^r \eta(X_p') + \frac{1}{t^q} 2\pi C_p(\frac{q-r+1}{q-r}) s^q \eta^{q/r}(X_p') \quad .$$

(Since $q/r > 1$, then

$$\sum_{B\varepsilon P} \eta^{q/r}(B) = \eta^{q/r}(X_p') \sum_{B\varepsilon P} [\eta(B)/\eta(X_p')]^{q/r}$$

$$\leq \eta^{q/r}(X_p') \sum_{B\varepsilon P} \eta(B)/\eta(X_p') = \eta^{q/r}(X_p').)$$

Whence, by (5) and (6),

$$(7) \quad \gamma_p^K(\mathbb{C}^K \sim tS) \leq \frac{1}{t^r} C_{p,r} s^r \eta(X_p') + \frac{1}{t^q} 2\pi C_p(\frac{q-r+1}{q-r}) s^q \eta^{q/r}(X_p') \quad .$$

Since the coefficients of $1/t^r$ and $1/t^q$ are finite and independent of K it follows that

$$(8) \quad \gamma_p^K(\mathbb{C}^K \sim tS) \to 0 \text{ as } t \to \infty \text{ uniformly for all finite } K \subset A \quad .$$

By (1), (2), and (8),

S^r is weighted by $(\nu^P, \hat{F}, V_p)$.

Case 2 $(2 < p \leq \infty,\ r = q,\ S \in \overline{S}^r)$.

If

$$c = \sup_{0 \leq u \leq s} u^q \ln u \quad .$$

then

$$0 < c < \infty \quad .$$

For any $t > \frac{1}{s}\,\eta^{1/r}(X_p')$, by Appendix 1.2.3, the right-hand-side of (5) is majorized by

$$(9)\quad C_{p,q} \sum_{B\in P} |r_K(\tfrac{1}{t}\, g_B)|_{q,K}^q + 2\pi C_p \sum_{B\in P} |r_K(\tfrac{1}{t}\, g_B)|_{q,K}^q \left|\ln|r_K(\tfrac{1}{t}\, g_B)|_{q,K}\right| \quad .$$

The first term of (9) is majorized by

$$\frac{1}{t^q}\, C_{p,q}\, s^q \eta(X_p') \quad .$$

The second term of (9) is majorized by

$$2\pi C_p \sum_{B\in P} \frac{1}{t^q} (|f_B|_q)^q \eta(B)[|\ln t| + \frac{1}{q}|\ln\eta(B)| + |\ln|f_B|_q|]$$

$$\leq \frac{|\ln t|}{t^q}\, 2\pi C_p s^q \eta(X_p') + \frac{1}{t^q}\, 2\pi C_p s^q\, \frac{1}{q} \sum_{B\in P} \eta(B)|\ln\eta(B)| \quad ,$$

$$+ \frac{1}{t^q}\, 2\pi C_p\, c\, \eta(X_p') \quad .$$

Hence, by (5), (6) and (9),

$$\gamma_p^K(\mathbb{C}^K \sim tS)$$

$$\leq \frac{1}{t^q}\,[C_{p,q} s^q \eta(X_p') + \frac{2\pi C_p}{q}\, q_{\ell\nu}(\eta) + 2\pi C_p\, c\, \eta(X_p')]$$

$$+ \frac{|\ln t|}{t^q}\,[2\pi C_p s^q \eta(X_p')] \quad .$$

By the hypotheses, the coefficients of $1/t^q$ and $|\ln t|/t^q$ are finite and independent of K . Hence,

(10) $\nu_K^p(C^K \sim tS) \to 0$ as $t \to \infty$ uniformly for all finite $K \subset A$.

By (1), (2) and (10)

$\overline{S}^r$ is weighted by $(\nu^p, \hat{F}, V_p)$.

<u>Case 3</u> $(1 \le p \le 2 \ , \ r > 0 \ , \ S \in \tilde{S}^r)$.

By Appendix 1.2.3, for any $t > 0$, the right-hand-side of (5) is majorized by

$$C_{2,r} \sum_{B \in P} |r_K(\tfrac{1}{t} g_B)|_{2,K}^r \ .$$

Hence, by (5) and (6),

$$\nu_K^2(\mathbb{C}^K \sim tS) = \lim_{P \in \mathcal{P}(S)} \gamma_2^K(C^K \cap tS_p')$$

$$\le \lim_{P \in \mathcal{P}} C_{2,r} \sum_{B \in P} |r_K(\tfrac{1}{t} g_B)|_{2,K}^r$$

$$= \frac{1}{t^r} C_{2,r} \cdot \lim_{P \in \mathcal{P}} \sum_{B \in P} \sup_{x \in U_p \cap C^K} |f_B(x)|)^r . \eta(B)$$

$$\le \frac{1}{t^r} C_{2,r} \lim_{P \in \mathcal{P}} \sum_{B \in P} (\sup_{f \in B} (\sup_{x \in U_p} |f(x)|)^r) \ \eta(B) \ .$$

$$= \frac{1}{t^r} C_{2,r} \int^* (\sup_{x \in U_p} |f(x)|)^r d\eta(f) \ .$$

By hypothesis, the coefficient of $1/t^r$ is finite and independent of K . Hence

(11) $\nu_K^2(\mathbb{C}^K \sim tS) \to 0$ as $t \to \infty$ uniformly for all finite $K \subset A$.

By (1), (2) and (11),

$\tilde{S}^r$ is weighted by $(\nu^2, \hat{F}, V_p)$.

APPENDIX

In this Appendix we establish a number of results and constructions which are necessary for the discussions of Chapter III. In the last section we give some counterexamples which complement the considerations of Chapter III.

1. Special Measures on Finite-Dimensional Spaces.

In this section we shall construct special measures on finite-dimensional spaces. The existence of these measures enables us to produce systems of δ-weights in Hilbertian spaces and in ℓ^p-spaces, $p \geq 1$.

Notation

K is a finite set.

For any $1 \leq p \leq \infty$,

$$\frac{1}{p} + \frac{1}{q} = 1 \quad ,$$

$$V_p = \begin{cases} \{x \in \mathbb{C}^K : \Sigma_{\alpha\varepsilon K} |x(\alpha)|^p \leq 1\} & \text{when } p < \infty, \\ \{x \in \mathbb{C}^K : \sup_{\alpha\varepsilon K} |x(\alpha)| \leq 1\} & \text{when } p = \infty \end{cases} .$$

For any $f \varepsilon (\mathbb{C}^K)^*$,

$$|f|_q = \sup_{x\varepsilon V_p} |f(x)|$$

(We note that $V_p^0 = \{f \varepsilon (\mathbb{C}^K)^* : |f|_q \leq 1\}$)

λ is the Lebesque measure on $\mathbb{C}$.

. For any finite dimensional space F ,

$$I = \{(x,f) \in F \times F^* : |f(x)| \geq 1\} .$$

The constructions of this section will be based on the assertions given below.

1.1 Lemmas

Let $2 \leq p \leq \infty$ and $r > 0$

(1) There exists a strictly-positive, continuous

$$\theta_p : \mathbb{C} \to R^+$$

such that

$$\exp - |w|^q = \int (\exp i \, Re\bar{w}z)\theta_p(z)dw(z) \quad \text{for all} \quad w \in \mathbb{C} .$$

(2) When $2 < p \leq \infty$, there exists

(i) $$0 < C_p < \infty$$

such that

(ii) $$\theta_p(z) < C_p/|z|^{2+q} \quad \text{for all} \quad z \in \mathbb{C} .$$

Hence, when $r \leq q$, for any $u > 0$,

(iii) $$\int_{1\leq|z|\leq u} |z|^r \theta_p(z)d\lambda(z) \leq \begin{cases} \dfrac{2\pi C_p}{q-r}[1 + 1/u^{q-r}] & \text{if } r < q , \\ 2\pi C_p\,|\ln u| & \text{if } r = q , \end{cases}$$

and

(iv) $$\int_{u\leq|z|} \theta_p(z)d\lambda(z) \leq 2\pi C_p|u|^{-q}$$

(3) When $p = 2$,

$$\theta_2(z) = \frac{1}{4\pi} \exp - \frac{|z|^2}{4} \quad \text{for every} \quad z \in \mathbb{C} .$$

Notation

For each $2 \leq p \leq \infty$, let

$$\gamma_p : B \subset \mathbb{C} \to \int^* 1_B \, \theta_p \, d\lambda \in R^+ ,$$

and

$$\delta_p = \gamma_p(\{z \in \mathbb{C} : |z| \geq 1\}) .$$

For any $r > 0$, let

$$C_{2,r} = \int |z|^r \, \theta_2(z) \, d\lambda(z) .$$

For each $2 < p \leq \infty$, and any $0 < r \leq q$, let

C_p be the constant of Lemma 1.1.2,

and

$$C_{p,r} = \begin{cases} \dfrac{2\pi C_p}{q-r} + \displaystyle\int_{|z|\leq 1} |z|^r \, \theta_p(z) d\lambda(z) & \text{if } r < q \\ 2\pi C_p + \displaystyle\int_{|z|\leq 1} |z|^r \, \theta_p(z) d\lambda(z) & \text{if } r = q . \end{cases}$$

Remarks

We note that in view of Lemmas 1.1.1 and 1.1.2,

γ_p is a probability Radón measure on $\mathbb{C}$,

$\delta_p > 0$

$0 < C_p < \infty,$

$0 < C_{p,r} < \infty$.

1.2 Lemmas

Let $2 \leq p \leq \infty$ and

γ_p^K be the product measure on $\mathbb{C}^K$ generated by the measure γ_p on $\mathbb{C}$.

(1) γ_p^K is a probability Radon measure on $\mathbb{C}^K$.

(2) $f \in (\mathbb{C}^K)^* \sim V_p^0 \Rightarrow \gamma_p^K(I^f) \geq \delta_p$.

(3) For any sequence $\{f_n\}_{n\in\omega} \subset (\mathbb{C}^K)^*$, and $r > 0$, let

$$B = \{x \in C^K : \sum_{n\in\omega} |f_n(x)|^r > 1\}$$

(i) If $p = 2$, then $q = 2$ and

$$\gamma_2^K(B) \leq C_{2,r} \sum_{n\in\omega} |f_n|_2^r \ .$$

(ii) If $p > 2$, $r \leq q$, and $|f_n|_q \leq 1$ for every $n \in \omega$, then,

$$\gamma_p^K(B) \leq \begin{cases} C_{p,r} \sum_{n\in\omega} |f_n|_q^r + 2\pi C_p \left(\frac{q-r+1}{q-r}\right) \sum_{n\in\omega} |f_n|_q^2 & \text{if } r < q , \\ C_{p,q} \sum_{n\in\omega} |f_n|_q^q + 2\pi C_p \sum_{n\in\omega} |f_n|_q^q |\ln|f_n|_q| & \text{if } r = q . \end{cases}$$

1.3. Lemma.

Let F be a finite-dimensional vector space.

If $[.,.]$ is a pseudo-inner product on F and

$$V = \{x \in F : [x,x] \leq 1\} ,$$

then, there exists a probability Radón measure ξ on F such that

(1) $f \in (\ker V)^a \sim V^0 \Rightarrow \xi(I^f) \geq \delta_2$.

(2) For any sequence $\{f_n\}_{n\epsilon\omega}$ in F^* ,

$$\xi(\{x \in F : \sum_{n\epsilon\omega} |f_n(x)|^r > 1\}) \leq C_{2,r} \sum_{n\epsilon\omega} (\sup_{x\epsilon V} |f_n(x)|)^r .$$

Proofs 1.

1.1 Let $\tilde{\lambda}$ be the Lebesque measure on R^2 .

From Blumenthal and Getoor [3] p. 263, we have the following facts. (See also Levy [21] Ch. VII.)

For any $0 < q \leq 2$, there exists a strictly positive continuous

$$\tilde{\theta}_p : R^2 \to R^+$$

s.t.

(i) $\exp - |t| = \int [\exp i(t.u)]\, \tilde{\theta}_p(u) d\tilde{\lambda}(u)$ for all $t \in R^2$,

where, for any $t \in R^2$, $u \in R^2$,

$$t.u = t_0u_0 + t_1u_1$$

and

$$|t| = \sqrt{(t_0^2 + t_1^2)} .$$

If $q < 2$, there exists $0 < c_q < \infty$ s.t.

$$\lim_{|t|\to\infty} |t|^{2+q}\, \tilde{\theta}_p(t) = c_q .$$

Hence, if

(ii) $C_p = \sup_{\tau\epsilon R^2} |t|^{2+q} \tilde{\theta}_p(t)$,

then,

(iii) $0 < C_p < \infty$

and

(iv) $\tilde{\theta}_p(t) \leq C_p/|t|^{2+q}$ for all $t \in R^2$.

Consequently, for any $0 < r \leq q$, and $u > 0$,

$$\int |t|^r \, \tilde{\theta}_p(t)\, d\tilde{\lambda}(t) \leq C_p \int_{1 \leq |t| \leq u} |t|^r \cdot \frac{1}{|t|^{2+q}}\, d\tilde{\lambda}(t)$$

$$= \quad 2\pi C_p \int_{1 \leq \rho \leq u} (\rho^{1+q-r})^{-1} d\rho \quad \text{using polar coordinates.}$$

By integrating the last term it follows that

(v)

$$\int_{1 \leq |t| \leq u} |\tau| \tilde{\theta}_p(t)\, d\tilde{\lambda}(t) \leq \begin{cases} \dfrac{2\pi C_p}{q-r}[1-1/u^{q-r}] \leq \dfrac{2\pi C_p}{q-r}[1+1/u^{q-r}] & , \text{ if } r < q , \\[2ex] 2\pi C_p |\ln u| \quad \text{if} \quad r = q . \end{cases}$$

For $q = 2$, using the fact that

$$(1/\sqrt{2\pi}) \int_R (\exp i\, xy) \exp - \frac{x^2}{2}\, dx = \exp - \frac{y^2}{2} ,$$

a direct computation shows that

(vi) $$\theta_2(t) = \frac{1}{4\pi} \exp - \frac{|t|^2}{4} .$$

Hence, for any $r > 0$,

(vii) $$\int |t|^r \tilde{\theta}_2(t)\, d\tilde{\lambda}(t) < \infty .$$

Let

$$T : z \in \mathbb{C} \to (\text{Re } z, \text{Im } z) \in R^2 ,$$

and for each $0 < q \leq 2$,

$$\theta_p = \tilde{\theta}_p \circ T .$$

The assertions of Lemmas 1.1 now follow from (i) - (vii) above, and the properties of the map T , namely,

for any z and w in $\mathbb{C}$,

$$\text{Re } \overline{w}z = (Tw).(Tz) ,$$

and T is an isometric, measure preserving, homeomorphism.

1.2.

Notation For any $f \in (\mathbb{C}^K)^*$, let

$$f' = f/|f|_q \quad ,$$

$$\varphi^f : x \in \mathbb{C}^K \to f(x) \in \mathbb{C} \quad ,$$

$$\tilde{\gamma}_p^{K,f} = \varphi^f[\gamma_p^K]$$

and let

$$\tilde{\psi}^f : w \in \mathbb{C} \to \int (\exp i \operatorname{Re} \overline{w}z)\, d\tilde{\gamma}_p^{K,f}(z) \in \mathbb{C} \quad .$$

1.2.1 This follows from the fact that γ_p is a probability Radon measure on $\mathbb{C}$.

1.2.2 For each $\alpha \in K$, let

$$e_\alpha = 1_{\{\alpha\}} \in \mathbb{C}^K \quad .$$

Then, for any $f \in (\mathbb{C}^K)^*$, and $w \in \mathbb{C}$,

$$(1) \quad \tilde{\psi}^f(w) = \int \exp i \operatorname{Re} \overline{w}z \; d\tilde{\gamma}_p^{K,f}(z)$$

$$= \int \exp i \operatorname{Re} f(\overline{w}x)\, d\tilde{\gamma}_p^{K}(x)$$

$$= \prod_{\alpha \in K} \left[\int (\exp i \operatorname{Re} \overline{w} f(e_\alpha) x(\alpha))\, d\gamma_p(x(\alpha))\right]$$

$$= \exp - \sum_{\alpha \in K} |\overline{w\, f(e_\alpha)}|^q \quad \text{by Lemma 1.1.1,}$$

$$= \exp - |w|^q |f|_q^q \quad .$$

Since a Radon measure on a finite-dimensional space is uniquely determined by its Fourier transform (Bochner's Theorem, Appendix 2.3), it follows that

$$(2) \quad |f|_q = 1 \Rightarrow \tilde{\gamma}_p^{K,f} = \gamma_p \quad .$$

However,

$$|f'|_q = 1 \quad .$$

and

$$f \in (\mathbb{C}^K)^* \sim V_p^0 \Rightarrow |f|_q \geq 1 \quad .$$

Hence,

$$\gamma_p^K(I^f) \geq \gamma_p^K(I^{f'}) = \tilde{\gamma}_p^{K,f'}(\{z \in \mathbb{C} : |z| \geq 1\})$$

$$= \delta_p \quad \text{, by (2) above.}$$

1.2.3.

(i) Since $\sum_{n\in\omega} |f_n'(x)|^r$ is a series of positive γ_2^K-measurable functions on C^K ,

$$\gamma_2^K(B) = \int 1_B \cdot d\gamma_2^K \leq \int (\sum_{n\in\omega} |f_n(x)^r) d\gamma_2^K(x)$$

$$= \sum_{n\in\omega} |f_n|_2^r \int |f_n'(x)|^r d\gamma_2^K(x)$$

$$= \sum_{n\in\omega} |f_n|_2^r \int |z|^r \, d\tilde{\gamma}_2^{K,f_n'}(z)$$

$$= C_{2,r} \sum_{n\in\omega} |f_n|_2^r \quad \text{by (2) of Proof 1.2.2.}$$

(ii) Let

$$H_n = \{x \in C^K : |f_n(x)| \leq 1\} \quad \text{for each } n \in \omega \quad ,$$

$$H = \bigcap_{n\in\omega} H_n$$

and

$$h : x \in \mathbb{C}^K \to \begin{cases} 1 & \text{if } x \in \mathbb{C}^K \sim H \\ \sum_{n\in\omega} |f_n(x)|^r & \text{if } x \in H \quad . \end{cases}$$

We have that

$$(1)\quad \gamma_p^K(B) \leq \int h\,d\gamma_p^K = \int 1_{\mathbb{C}^K \sim H}\,d\gamma_p^K + \int 1_H h\,d\gamma_p^K \ .$$

Now,

$$\int 1_{\mathbb{C}^K \sim H}\,d\gamma_p^K \leq \sum_{n\varepsilon\omega} \gamma_p^K(\mathbb{C}^K \sim H_n)$$

$$= \sum_{n\varepsilon\omega} \gamma_p^K(\{x \varepsilon \mathbb{C}^K : |f_n'(x)| > 1/|f_n|_q\})$$

$$= \sum_{n\varepsilon\omega} \tilde{\gamma}_p^{K,f_n'}(\{z \varepsilon \mathbb{C} : |z| > 1/|f_n|_q\})$$

$$(2)\qquad \leq 2\pi C_p \sum_{n\varepsilon\omega} |f_n|_q^q \quad \text{by (2) of Proof 1.2.2,}$$

and Lemma 1.1.2.(iv).

Since $\sum_{n\varepsilon\omega} |f_n'(x)|^r$ is a series of positive γ_p^K-measurable functions on $\mathbb{C}^K$, we also have that

$$\int 1_H h\,d\gamma_p^K = \int 1_H[\sum_{n\varepsilon\omega} |f_n|_q^r|f_n'(x)|^r]\,d\gamma_p^K(x)$$

$$= \sum_{n\varepsilon\omega} |f_n|_q^r \int 1_H|f_n'(x)|^r d\gamma_p^K(x)$$

$$\leq \sum_{n\varepsilon\omega} |f_n|_q^r \int 1_{H_n}|f_n'(x)|^r d\gamma_p^K(x)$$

$$= \sum_{n\varepsilon\omega} |f_n|_q^r \int_{|z|\leq 1/|f_n|_q} |z|^r\,d\tilde{\gamma}_p^{K,f_n'}(z)$$

$$= \sum_{n\varepsilon\omega} |f_n|_q^r \int_{|z|\leq 1/|f_n|_q} |z|^r\,\theta_p(z)d\lambda(z) \qquad \text{by (2) of Proof 1.2.2.}$$

Hence

$$\int 1_H h\,d\gamma_p^K \leq \sum_{n\varepsilon\omega} |f_n|_q^r[\int_{|z|\leq 1} |z|^r\,\theta_p(z)d\lambda(z) + \int_{1\leq|z|\leq 1/|f_n|_q} |z|^r\,\theta_p(z)d\lambda(z)]$$

Letting $a_{p,r} = \int_{|z|\leq 1} |z|^r\,\theta_p(z)d\lambda(z)$, from Lemma 1.1.2 (iii), we have that

$$(3) \int 1_H h d\gamma_p^K \le \begin{cases} \sum_{n\varepsilon\omega} |f_n|_q^r (a_{p,r} + \frac{2\pi C_p}{q-r}[1 + |f_n|_q^{q-r}]) & \text{if } r < q , \\ \sum_{n\varepsilon\omega} |f_n|_q^r (a_{p,r} + 2\pi C_p |\ln|f_n|_q|) & \text{if } r = q . \end{cases}$$

Consequently, if $r < q$, then, from (1), (2) and (3),

$$\gamma_p^K(B) \le 2\pi C_p \sum_{n\varepsilon\omega} |f_n|_q^q + \sum_{n\varepsilon\omega} |f_n|_q^r (a_{p,r} + \frac{2\pi C_p}{q-r}[1 + |f_n|_q^{q-r}])$$

$$= C_{p,r} \sum_{n\varepsilon\omega} |f_n|_q^r + 2\pi C_p (\frac{q-r+1}{q-r}) \sum_{n\varepsilon\omega} |f_n|_q^r ,$$

since all terms are positive.

The case $r = q$ is established from (1), (2) and (3) similarly.

<u>1.3</u> We first suppose that $[.,.]$ is non-degenerate. If so, choose a $[.,.]$-orthonormal basis K for F. Let

$$T : z \varepsilon \mathbb{C}^K \to \sum_{\alpha\varepsilon K} z_\alpha . \alpha \varepsilon F .$$

and

$$f \varepsilon F^* \to \hat{f} = f \circ T \varepsilon (\mathbb{C}^K)^* .$$

Then,

T is a homeomorphism.

Further, for any $x' \varepsilon \mathbb{C}^K$, $y' \varepsilon \mathbb{C}^K$, and $f \varepsilon F^*$,

(1) $[Tx',Ty'] = \langle x',y' \rangle$, where $\langle .,. \rangle$ denotes the inner product in $\mathbb{C}^K$,

and

(2) $\sup_{x\varepsilon V} |f(x)| = |\hat{f}|_2$

Whence, if

$$\xi_1 = \gamma_2^K \circ T^{-1}$$

then

$$\xi_1(I^f) = \gamma_2^K(I^{\hat{f}})$$

and

$$\xi(B) = \gamma_2^K(\{x' \in \mathbb{C}^K : \sum_{n\in\omega} |\hat{f}_n(x')|^r > 1\})$$

The assertions now follow from (3) and Lemmas 1.2.

When $[.,.]$ is degenerate, let F_1 be any subspace of F s.t. F is the direct sum of F_1 and $\ker V$. (Possibly, $F_1 = \{0\}$). We have that

$$[.,.]|F_1 \times F_1 \text{ is non-degenerate,}$$

since $F_1 \cap \ker V = \{0\}$.

Let ξ_1 be the probability Radon measure on F_1 determined as above, and

$$\xi : H \subset F \to \xi(H \cap F_1) \in R^+ .$$

Then,

(3) ξ is a probability Radon measure on F.

Since F is the direct sum of F_1 and $\ker V$, for any $x \in X$ there exists a unique representation

$$x = x_1 + x_2 \text{, with } x_1 \in F_1 \text{ and } x_2 \in \ker V .$$

Consequently,

(4) $f \in (\ker V)^a \Rightarrow \sup_{x\in V} |f(x)| = \sup_{x\in V\cap F_1} |f(x)|$

and therefore, from the non-degenerate case above,

(5) $f \in (\ker V)^a \sim V^0 \Rightarrow r_{F_1,F}(f) \in F_1^* \sim (V \cap F_1)^0$

$$\Rightarrow \xi(I^f) = \xi_1(I^{r_{F_1,F}(f)}) > \delta_2 .$$

If $f_n \in F^* \sim (\ker V)^a$ for any $n \in \omega$, then

$$\sup_{x \in V} |f_n(x)| = \infty$$

and therefore (2) of the lemma holds.

If $f_n \in (\ker V)^a$ for every $n \in \omega$, then from the non-degenerate case above,

$$\begin{aligned}
\xi(B) &= \xi_1(\{x \in F_1 : \sum_{n \in \omega} |f_n(x)|^r > 1\}) \\
&\leq C_{2,r} \sum_{n \in \omega} \sup_{x \in V \cap F_1} |f_n(x)|^r \\
&= C_{2,r} \sum_{n \in \omega} \sup_{x \in V} |f_n(x)|^r \quad \text{by (4).}
\end{aligned}$$

2. Positive-definite Functions on Vector Spaces.

In this section we give a number of useful results concerning positive-definite functions on vector spaces.

2.1 Definition

Let X be a commutative group.

ψ is a positive-definite function on X iff

$\psi : X \to \mathbb{C}$,

and for any $n \in \omega$, $\{x_0,\ldots,x_{n-1}\} \subset X$, $\{z_0,\ldots,z_{n-1}\} \subset \mathbb{C}$,

$$\sum_{k,\ell=0}^{n-1} z_k \bar{z}_\ell \, \psi(x_k - x_\ell) \geq 0 \quad .$$

We shall need the following elementary assertions about positive-definite functions on groups.

2.2 Propositions

Let X be a commutative group.

(1) If ψ is a positive-definite function on X , then

$0 \leq \psi(0) < \infty$.

(2) Let ψ be a positive-definite function on X , and Y be a commutative group.

If $T : Y \to X$ is a homomorphism, then $\psi \circ T$ is a positive-definite function on X .

(3) If φ and ψ are positive-definite functions on X , then

$\varphi\psi$ is a positive-definite function on X .

(4) If $(\psi_j)_{j\varepsilon J}$ is a net of positive definite functions on X, and $\psi : X \to C$ is such that

$$\psi(x) = \lim_{j\varepsilon J} \psi_j(x) \quad \text{for all} \quad x \varepsilon X \ ,$$

then,

ψ is a positive-definite function on X.

(5) If X is a topological group, and ψ is positive-definite function on X, then

ψ is continuous on $X \Leftrightarrow \psi$ is continuous at $0 \varepsilon X$.

For finite-dimensional spaces we have the following version of a well known representation theorem (Rudin [34] p. 19 1.4.3, Bochner [4] p. 58).

2.3 Theorem

Let F be a finite-dimensional vector space.

ψ is a continuous positive-definite function on F

iff

there exists a unique finite Radón measure ξ on F^* such that $\psi(x) = \int \exp i \operatorname{Re} f(x) d\xi(f)$ for all $x \varepsilon F$.

Using the above theorem, as in [11] (p. 349) one readily establishes its following infinite-dimensional analogue. We omit the proof. (The theorem given in [11] is formulated only for real vector spaces. See also [48]).

2.4 Theorem

Let X be a vector space.

ψ is a positive-definite function on X with $\psi|F$ continuous for every $F \varepsilon \mathcal{F}$, iff

there exists a unique finite cylinder measure μ over X such that

$$\psi(x) = \int \exp i \operatorname{Re} f(x) d\mu_X(f) \quad \text{for all} \quad x \varepsilon X .$$

Remark

When μ and ψ are related as in the foregoing theorem, we call ψ the characteristic functional of μ (Prohorov [33]).

The final theorem of this section is useful for determining continuity properties of cylinder measures. As an adjunct to Proposition II.2.6, it further motivates the terminology "continuous cylinder measure", introduced in Definitions II.2.5.1.

2.5 Theorem

Let X be a vector space, μ be a finite cylinder measure over X , and $\mathcal{V}$ be a family of balanced, absorbent subsets V of X with $uV \varepsilon \mathcal{V}$ for every $u > 0$.

If ψ is the characteristic functional of μ , then

$$\mu \text{ is } \mathcal{V}\text{-continuous} \iff \psi \text{ is } \mathcal{V}\text{-continuous.}$$

Corollary

Let X be a topological space, and μ be a finite cylinder measure over X .

If ψ is the characteristic functional of μ , then

μ is continuous $\Leftrightarrow$ ψ is continuous.

Proofs 2

2.2.1 Taking $n = 1$, $x_0 = 0$, and $z_0 = 1$, the assertion follows immediately from Defn. 2.1.

2.2.2 For any $n \in \omega$, $\{x_0,\ldots,x_{n-1}\} \subset X$, and $\{z_0,\ldots,z_{n-1}\} \subset \mathbb{C}$,

$$\sum_{k,\ell=0}^{n-1} z_k \bar{z}_\ell \, \psi \circ T(x_k - x_\ell) = \sum_{k,\ell=0}^{n-1} z_k \bar{z}_\ell \, \psi(T_{x_k} - T_{x_\ell}) \geq 0 \ .$$

2.2.3 From [48] we have that

$$\varphi(x) = \overline{\varphi(-x)} \quad \text{for all } x \in X \ .$$

Hence, by Defn.2.1, for any $n \in \omega$, $\{x_0,\ldots,x_{n-1}\} \subset X$, and $\{z_0,\ldots,z_{n-1}\} \subset \mathbb{C}$,

$$M = (\varphi(x_k - x_\ell))_{k,\ell=0}^{n-1} \quad \text{is a positive-definite}$$

Hermitian matrix.

Hence, there exists an $n \times n$-matrix T s.t.

$$M = TT^* \ ,$$

where $T = (t_{k,s})$, $T^* = (t^*_{k,s})$, and $t^*_{k,s} = \bar{t}_{s,k}$.

Consequently,

$$\sum_{k,\ell=0}^{n-1} z_k\bar{z}_\ell \varphi(x_k - x_\ell)\psi(x_k - x_\ell)$$

$$\sum_{k,\ell=0}^{n-1} z_k\bar{z}_\ell \left(\sum_{s=0}^{n-1} t_{k,s}t^*_{s,\ell}\right)\psi(x_k - x_\ell)$$

$$\sum_{s=0}^{n-1}\sum_{k,\ell=0}^{n-1} (z_k t_{k,s})\overline{(z_k t_{\ell,s})}\psi(x_k - x_\ell) \geq 0 \quad ,$$

and therefore ψ is positive-definite.

2.2.4 For any $n \in \omega$, $\{x_0,\ldots,x_{n-1}\} \subset X$, and $\{z_0,\ldots,z_{n-1}\} \subset \mathbb{C}$,

$$\sum_{k,\ell=0}^{n-1} z_k\bar{z}_\ell\, \psi(x_k - x_\ell) = \lim_{j\in J} \sum_{k,\ell=0}^{n-1} z_k\bar{z}_\ell\, \psi_j(x_k - x_\ell) \geq 0 \quad .$$

2.2.5 From Rudin [34], p. 18, 1.4.1 (4), we have that for any x and y in X,

$$|\psi(x) - \psi(y)| \leq 2\psi(0) \operatorname{Re} (\psi(0) - \psi(x - y)) \quad .$$

The assertion follows.

2.3 This theorem is a special case of a general theorem in Harmonic Analysis ([34], p. 19 1.4.3). However, it is readily derived from the real case treated by Bochner ([4] p. 58).

2.5 Together with the notations of II.2.4 and II.2.6, for each $x \in X$, let

$$\tilde{\psi}_x(w) = \int \exp i \operatorname{Re} \bar{w}z d\tilde{\mu}_x(z) \quad \text{for every} \quad z \in \mathbb{C} \quad .$$

We note that

(1) $$\tilde{\psi}_x(w) = \psi(\bar{w}x) \quad \text{for every} \quad z \in \mathbb{C}$$

Suppose that μ is $\mathcal{V}$-continuous. Since

$$z \in \mathbb{C} \to \exp i \operatorname{Re} z \in \mathbb{C}$$

is bounded and continuous, by (1) and Prop. II.2.6 we have that

(2) ψ is $\mathcal{V}$-continuous

Suppose that ψ is $\mathcal{V}$-continuous. Since ψ/c is positive-definite for every $c > 0$, by Prop. 2.2.1 we may assume that

$$\psi(0) = 1 \quad .$$

Given any $\varepsilon > 0$, choose

$$0 < \varepsilon' < \frac{\varepsilon}{2}\left(\frac{\sqrt{e}-1}{\sqrt{e}}\right) \quad ,$$

$$t > 2/\sqrt{\varepsilon'} \quad ,$$

and $V \in \mathcal{V}$ s.t.

$$x \in V \Rightarrow 1 - \operatorname{Re} \psi(x) < \varepsilon' \quad .$$

By (1) and the fact that V is balanced,

(3) $$x \in V \text{ , } z \in \mathbb{C} \text{ , } |z| \leq 1 \Rightarrow 1 - \tilde{\psi}_x(z) < \varepsilon' \quad .$$

If $z = u_1 + iu_2 \in \mathbb{C}$, then $|z|^2 = u_1^2 + u_2^2$, and therefore, by (3), for any $x \in V$,

(4) $$u_1^2 + u_2^2 \leq 1 \Rightarrow 1 - \tilde{\psi}_x(V) < \varepsilon' \quad .$$

Hence, by the lemma given by Kolmogorov in [17], for any $x \in V$,

(5) $$\mu_x(D_x) = \tilde{\mu}_x(\{z \in \mathbb{C} : |z| \geq 1\}) \leq \frac{\sqrt{e}}{\sqrt{e}-1}(\varepsilon' + \frac{4}{t^2})$$

$$< \varepsilon \quad .$$

Since ε was arbitrary, it follows that

μ is $\mathcal{V}$-continuous

Proof of Corollary 2.5

We note that nbnd 0 in X has a base $\mathcal{V}$ consisting of balanced, absorbent sets V , with $\varepsilon V \in \mathcal{V}$ for every $\varepsilon > 0$. Hence, by the above theorem and Prop. 2.2.5,

$$\begin{aligned} \mu \text{ is continuous } &\Leftrightarrow \mu \text{ is } \mathcal{V}\text{-continuous} \\ &\Leftrightarrow \psi \text{ is } \mathcal{V}\text{-continuous} \\ &\Leftrightarrow \psi \text{ is continuous.} \end{aligned}$$

3. CM-spaces.

For any family C of finite cylinder measures over a vector space X, we shall define the C-topology on X and give some of its properties. We shall establish examples of topological vector spaces whose topologies are exactly those determined by the families of continuous finite cylinder measures.

We note that for sets X and Y, topology G on Y, and $T : X \to Y$,

$$\{T^{-1}[G] : G \in G\}$$

is a topology on X. We shall refer to it as the topology on X induced by G and T.

3.1 Definition

Let X be a vector space.

(1) For any family C of finite cylinder measures over X, the C-topology is the topology on X having for a base all subsets V of X with

$$V = x + \varepsilon \bigcap_{\mu \varepsilon H} \{y \varepsilon X : \mu_y(D_y) < \varepsilon\}$$

for some $x \varepsilon X$, finite $H \subset C$, and $\varepsilon > 0$.

(2) X is a CM-space iff X is a topological vector space whose topology is the $CM(X)$-topology.

Concerning C-topologies we have the following assertions.

3.2 Propositions

Let Y be a vector space, and $\mathcal{C}$ be a family of finite cylinder measures over Y .

(1) Y is a topological vector space under the $\mathcal{C}$-topology, which is the coarsest such topology with respect to which $\mathcal{C}$ is a family of continuous cylinder measures.

In particular, when Y is a topological vector space, and $\mathcal{C} = CM(Y)$,

the $\mathcal{C}$-topology is coarser than the original topology of Y .

(2) For any vector space X and $T \in L[X,Y]$ if

$$\mathcal{C} \circ T = \{\mu \circ T : \mu \in \mathcal{C}\} ,$$

then,

the $\mathcal{C} \circ T$-topology is the topology on X induced by the $\mathcal{C}$-topology and T .

We shall now show that the class of CM-spaces contains many interesting topological vector spaces. However, not all topological vector spaces are CM-spaces. In Appendix 4 we give an example of a Banach space which is not a CM-space (Example 4.2).

3.3 Definitions

(1) Let X be a vector space

$b : X \to R^+$ is a pseudo-quasi-norm on X iff

$$b(0) = 0 \quad ,$$

for any x and y in X ,

$$b(-x) = b(x) \quad ,$$

$$b(x + y) \leq b(x) + b(y) \quad ,$$

and

$$z \in \mathbb{C} \to b(zx) \in R^+ \text{ is continuous at } 0 \in \mathbb{C} \quad .$$

(For any family $\{b_j\}_{j\in J}$ of pseudo-quasi-norm on X , as in Yosida [49] p. 31, one can show that X is a topological vector space under the coarsest topology on X making b_j continuous for every $j \in J$).

(2) For any measure space (A,η) , and $r > 0$,

$$L^r(A,\eta) = \{f \in \mathbb{C}^A : f \text{ is } \eta\text{-measurable}, \int |f|^r d\eta < \infty\} \ ,$$

$$b_r : f \in L^r(A,\eta) \to \int |f|^r d\eta \in R^+ \quad ,$$

and when $r \geq 1$,

$$|\cdot|_r : f \in L^r(A,\eta) \to (\int |f|^r d\eta)^{1/r} \in R^+ \quad .$$

Remarks

When $0 < r < 1$, b_r is a pseudo-quasi-norm on $L^r(A,\eta)$, which is therefore a topological vector space under the coarsest topology making b_r continuous.

When $r \geq 1$, $|\cdot|_r$ is a pseudo-norm on $L^r(A,\eta)$, which is therefore a locally convex space under the coarsest topology making $|\cdot|_r$ continuous.

We shall hereafter assume that $L^r(A,\eta)$, $r > 0$, carries the appropriate topology indicated by the foregoing observations.

We shall need the following lemmas, which are of independent interest.

3.4 <u>Lemmas</u>

(1) Let X be a vector space, and b be a pseudo-quasi-norm on X . If ψ is a positive-definite function on X such that the coarsest topology on X making ψ continuous coincides with the coarsest topology making b continuous, then, there exists a finite cylinder measure μ over X whose characteristic function is ψ , and,

the $\{\mu\}$-topology on X is the coarsest topology on X with respect to which b is continuous.

(2) Let X be a vector space. If $\{b_V\}_{V\varepsilon \mathcal{V}}$ is a family of pseudo-quasi-norms on X such that for each $V \varepsilon \mathcal{V}$, there exists a positive-definite function ψ_V on X satisfying the hypothesis given in (1) above, then,

X is a CM-space under the coarsest topology making b_V continuous for each $V \varepsilon \mathcal{V}$.

(3) Let (A,η) be a measure space. For any $0 < r \leq 2$,

$$f \varepsilon L^r(A,\eta) \to \exp - b_r(f) \varepsilon \mathbb{C}$$

is a positive-definite function on $L^r(A,\eta)$.

The following theorem and its corollaries indicate that many of the topological vector spaces considered in this paper are in fact CM-spaces.

3.5 Theorem

X is a CM-space whenever X is a topological vector space having a family $\mathcal{V}$ of neighbourhoods of 0 which satisfies the following conditions:

(i) $\{\varepsilon V : V \in \mathcal{V}, \varepsilon > 0\}$ is a base for nbnd 0 in X.

(ii) For each $V \in \mathcal{V}$, there exists a measure space (A_V, η_V), $0 < r_V \leq 2$, and $T_V \in L[X, \mathcal{L}^{r_V}(A_V, \eta_V)]$, such that

$$V = \{x \in X : \int |T_V(x)|^{r_V} d\eta_V \leq 1\} .$$

Corollaries.

(1) Let (A, η) be a measure space. For any $0 < r \leq 2$, $\mathcal{L}^r(A, \eta)$ is a CM-space.

In particular,

$\ell^r(A)$ is a CM-space.

(2) Let X be a topological vector space. For any $0 < r \leq 2$, X with the $\mathcal{S}^r$-topology is a CM-space.

(3) Every Hilbertian space is a CM-space.

Proofs 3

Notation

For any vector space X , $\varepsilon > 0$, $y \in X$,

$$I(\varepsilon) = \{(x,f) \in X \times X^* : |f(x)| \geq \varepsilon\} ,$$

$$D_{y,\varepsilon} = \{f \in F_y^* : |f(y)| \geq \varepsilon\} ,$$

and for any family $\mathcal{C}$ of finite cylinder measures over X ,

$$V(\mathcal{C},\varepsilon) = \bigcap_{\mu\in\mathcal{C}} \{x \in X : \mu_x(D_{x,\varepsilon}) < \varepsilon\}$$

Remark Since

$$D_{x,\varepsilon} = D_{\frac{x}{\varepsilon}} \text{ for all } x \in X \text{ and } \varepsilon > 0 ,$$

it follows that

$$V(\mathcal{C},\varepsilon) = \varepsilon \bigcap_{\mu\in\mathcal{C}} \{x \in X : \mu_x(D_x) < \varepsilon\}$$

3.2.1 We shall only prove the first assertion. The second then follows immediately from the definition. Let

$$\mathcal{V} = \{V(\mathcal{H},\varepsilon) : \mathcal{H} \subset \mathcal{C} \text{ is finite, } \varepsilon > 0\} .$$

In view of the remark above, it will be sufficient if we show that $\mathcal{V}$ has the following properties.

(i) $0 \in V$ for every $V \in \mathcal{V}$.

(ii) $\mathcal{V}$ is a filterbase.

For each $V \in \mathcal{V}$,

(iii) there exists $U \in \mathcal{V}$ s.t. $U + U \subset V$.

(iv) V is absorbent,

(v) V is balanced. (Treves [47] p. 21)

Proofs of (i) - (v).

(i) For any $\varepsilon > 0$,

$$\mu_0(D_{0,\varepsilon}) = \mu_0(\emptyset) = 0 \quad .$$

and therefore $0 \in V$ for every $V \in \mathcal{V}$.

(ii) For any $0 < \delta < \varepsilon$, $\mu \in \mathcal{C}$, and $y \in X$,

$$\mu_y(D_{y,\delta}) \geq \mu_y(D_{y,\varepsilon}) \quad .$$

Hence, if $V(\varepsilon_j,\mathcal{C}_j) \in \mathcal{V}$, $j = 0,1$, and $\varepsilon = \min\{\varepsilon_0,\varepsilon_1\}$, then,

$$V(\varepsilon,\mathcal{C}_0 \cup \mathcal{C}_1) \subset \bigcap_{j=0,1} V(\varepsilon_j,\mathcal{C}_j) \quad .$$

(iii) Let $V = V(\varepsilon,\mathcal{H})$, and $U = V(\varepsilon/2,\mathcal{H})$.

For any $x \in X$, $y \in Y$, and $f \in F^*_{(x,y)}$ (II.2.6),

$$|f(x) + f(y)| \leq |f(x)| + |f(y)| \quad ,$$

and therefore,

$$I_{x+y}(\varepsilon) \subset I_x(\varepsilon/2) \cup I_t(\varepsilon/2) \quad .$$

Consequently, for any $x \in U$, $y \in U$, and $\mu \in \mathcal{H}$,

$$\mu_{x+y}(D_{x+y,\varepsilon}) = \mu_{F_{(x,y)}}(I_{x+y}(\varepsilon))$$

$$\leq \mu_{F_{(x,y)}}(I_x(\varepsilon/2)) + \mu_{F_{(x,y)}}(I_y(\varepsilon/2))$$

$$= \mu_x(D_{x,\varepsilon/2}) + \mu_y(D_{y,\varepsilon/2}) < \varepsilon \quad .$$

i.e.

$$U + U \subset V \quad .$$

(iv) For any $x \in X$, $\varepsilon > 0$, $t > 0$, and $\mu \in \mathcal{C}$,

$$\mu_{tx}(D_{tx,\varepsilon}) = \mu_x(D_{x,\varepsilon/t})$$

and

$$0 < u < t \Rightarrow D_{x,\varepsilon/u} \subset D_{x,\varepsilon/t} \quad .$$

Consequently, since μ_x is finite,

$$\lim_{n\to\infty} \mu_{x/n}(D_{x/n,\varepsilon}) = \lim_{n\to\infty} \mu_x(D_{x,n\varepsilon})$$

$$= \mu\left(\bigcap_{n\varepsilon\omega} D_{x,n\varepsilon}\right) = \mu_x(\emptyset) = 0 \quad .$$

Hence, for any $V \varepsilon \mathcal{V}$, there exists $n \varepsilon \omega$ s.t.

$$x/n \ \varepsilon\ V \quad .$$

i.e. V is absorbent.

(v) For any $\mu \varepsilon C$, $x \varepsilon X$, $\varepsilon > 0$ and $z \varepsilon C$ with $|z| \leq 1$,

$$\mu_{zx}(D_{zx,\varepsilon}) = \mu_x(D_{|z|x,\varepsilon}) = \mu_x(D_{x,\varepsilon/|z|})$$

$$\leq \mu_x(D_{x,\varepsilon}) \quad \text{since} \quad D_{x,\varepsilon/|z|} \subset D_{x,\varepsilon} \quad .$$

Hence, for any $V \varepsilon \mathcal{V}$,

$$zV \subset V \quad .$$

<u>3.2.2.</u> As in Lemma II.4.2, for any $x \varepsilon X$, $\varepsilon > 0$, and $\mu \varepsilon C$,

$$(\mu \sqsupset T)_x(D_{x,\varepsilon}) = \mu_{T_x}(D_{Tx,\varepsilon}) \quad .$$

Hence, for any $\varepsilon > 0$ and finite subfamily H of C ,

$$T^{-1}\left[\bigcap_{\mu\varepsilon H} \{y \ \varepsilon\ Y : \mu_y(D_{y,\varepsilon}) < \varepsilon\}\right]$$

$$= \bigcap_{\mu\varepsilon H} \{x \ \varepsilon\ X : \mu_{T_x}(D_{Tx,\varepsilon}) < \varepsilon\}$$

$$= \bigcap_{\mu\varepsilon H} \{x \ \varepsilon\ X : (\mu \sqsupset T)_x(D_{x,\varepsilon}) < \varepsilon\} \quad .$$

It follows that $\{T^{-1}[V] : V \varepsilon \mathcal{V}\}$ is a base for the $C \sqsupset T$-topology neighbourhoods of 0 in X , where $\mathcal{V}$ is as defined in Proof 3.2.1. However, from Proof 3.2.1 we see that $\mathcal{V}$ is a base for the C-topology neighbourhoods of 0 in Y . The assertion now follows from Prop. 3.2.1 and the linearity of T .

Lemma

Let F be a finite-dimensional space. If b is a pseudo-quasi-norm on F, then b is continuous on F.

Proof of Lemma. Let K be a basis of F.

Every $x \in F$ has a unique representation

$$\sum_{\alpha \in K} z_\alpha(x)\alpha \quad ,$$

and the norm

$$x \in F \to \sum_{\alpha \in K} |z_\alpha(x)| \in R^+$$

generates the topology of F.

For any net $(x_j)_{j \in J}$ in F,

$x_j \to 0 \Rightarrow \sum_{\alpha \in K} |z_\alpha(x_j)| \to 0 \Rightarrow$

$z_\alpha(x_j) \to 0$ for each $\alpha \in K \Rightarrow$

$b(z_\alpha(x_j)\alpha) \to 0$ for each $\alpha \in K \Rightarrow$

$\sum_{\alpha \in K} b(z_\alpha(x_j)\alpha) \to 0 \Rightarrow b(x_j) \to 0$, since

$$b(x_j) \leq \sum_{\alpha \in K} b(z_\alpha(x_j)\alpha) \quad .$$

Hence b is continuous at $0 \in F$. However, for any x and y in F,

$$|b(x) - b(y)| \leq b(x - y) \quad ,$$

and therefore continuity of b at $0 \in F$ implies continuity of b on F.

3.4.1 By the above Lemma,

$\psi|F$ is continuous for every $F \in \mathcal{F}$,

and therefore, by Thm. 2.4, there is a cylinder measure μ over X whose characteristic functional is ψ . From the hypothesis, X is a topological vector space under the coarsest topology making ψ continuous. Hence, by Cor. 2.5 and Prop. 3.1.1,

the $\{\mu\}$-topology

= coarsest topology making μ continuous

= coarsest topology making ψ continuous

= coarsest topology making b continuous.

3.4.2 By Prop. 3.1.1 and Lemma 3.4.1.

3.4.3 Let $\alpha : B \in M_\eta \to \alpha_B \in B$.

For any $P \in \mathcal{P}(M_\eta)$, let

$d(P)$ be the family of finite subsets of P directed by inclusion.

Then, for any $f \in L^r(A,\eta)$,

$$b_r(f) = \lim_{P\in\mathcal{P}(M_\eta)} \lim_{K\in d(P)} \sum_{B\in K} |f(\alpha_B)|^r \cdot \eta(B) .$$

Consequently, since

$t \in R \to \exp - t \in R$ is continuous,

we have that

$$\exp - b_r(f) = \lim_{P\in\mathcal{P}(M_\eta)} \lim_{Q\in d(P)} \prod_{B\in Q} \exp - |\eta(B)^{1/r} f(\alpha_B)|^r .$$

Since $f \in L^r(A,\eta) \to \eta(B)^{1/r} f(\alpha_B)$ is linear for every $B \in M_\eta$, we deduce from Lemma 1.1.1 and Props. 2.2.2 - 2.2.4 that $f \in L^r(A,\eta) \to \exp - b_r(f) \in \mathbb{C}$ is positive-definite.

3.5 For each $V \in \mathcal{V}$, let

$$\varepsilon_V = \begin{cases} 1 & \text{when } r_V < 1 , \\ 1/r_V & \text{when } r_V \geq 1 , \end{cases}$$

$$b_V : x \in X \to \left(\int |T_V(x)|^{r_V} d\eta_V\right)^{\varepsilon_V} ,$$

and

$$\psi_V : x \in X \to \exp - b_V(x) \in \mathbb{C} .$$

For each $V \in \mathcal{V}$, we have that

b_V is a pseudo-quasi-norm on X,

and for every $t > 0$,

$$tV = \{x \in X : b_V(x) \leq t^{r_V \varepsilon_V}\}$$

Since the topology of X is completely determined by its neighbourhoods of 0, it follows that the topology of X is the coarsest topology making b_V continuous for every $V \in \mathcal{V}$. However, by Lemma 3.4.3, ψ_V is positive-definite, and since $t \in R^+ \to \exp - t \in (0,1]$ is a homeomorphism, it follows that

the coarsest topology on X making ψ_V continuous

= the coarsest topology on X making b_V continuous.

The theorem is now a consequence of Lemma 3.4.2.

Corollaries (1) and (2) are immediate consequences of the theorem.

Proof of Corollary (3) Recalling the definition of a Hilbertian space (§ III.2), we need only make the following observation.

Let X be a vector space. For any pseudo-inner-product $[.,.]$ on X, there exists a measure space (A,η) (A is an index set and η is counting measure on A), and $T \in L[X,L^2(A,\eta)]$, such that

$$[x,x] = \int |T(x)|^2 d\eta \quad \text{for all } x \in X .$$

(Treves [47], p. 115 - 116.)

4. Examples

4.1 Example

There exists a Banach space X and finite cylinder measure μ over X such that

μ is $\mathcal{S}^1$-continuous but is not $\mathcal{E}$-tight.

Proof Let A be a set.

Together with Notation 1.1, let

$X = \ell^1(A)$ with the usual topology (Notation III.4),

$$[.,.] : (x,y) \in X \times X \to \sum_{\alpha \in A} x(\alpha)\overline{y(\alpha)} \in \mathbb{C} \quad ,$$

$$\psi : x \in X \to \exp - [x,x] \in \mathbb{C} \quad .$$

For any finite $K \subset A$,

$$T_K : w \in \mathbb{C}^K \to f_w \in (\mathbb{C}^K)^* \quad , \text{ where}$$

$$f_w(x) = \sum_{\alpha \in K} x(\alpha)\overline{w(\alpha)} \quad \text{for all } x \in \mathbb{C}^K \quad .$$

Since

$$[x,x] = \sum_{\alpha \in A} |x(\alpha)|^2 \quad \text{for all } x \in \ell^1(A) \quad ,$$

as in the proof of Lemma 3.4.3, we deduce that ψ is a positive-definite function on $\ell^1(A)$. Since $x \in X \to \sqrt{[x,x]}$ is a norm on $\ell^1(A)$, we further deduce that $\psi|F$ is continuous for every $F \in \mathcal{F}$.

By Thm. 2.4, there exists a cylinder measure μ over $\ell^1(A)$ whose characteristic function is ψ . Then, for any finite $K \subset A$, and $x \in \mathbb{C}^K$, as in Proof 1.2.2,

$$\int \exp i \operatorname{Re} f(x) d(\gamma_2^K \circ T_K^{-1})(f) = \psi(x)$$

$$= \int \exp i \operatorname{Re} f(x) d\mu_x(f) = \int \exp i \operatorname{Re} f(x) d\mu_{\mathbb{C}^K}(f) \quad .$$

(Note that T_K is a homeomorphism, and therefore $\gamma_2^K \circ \Gamma_K$ is Radon.)

Hence, by Thm. 2.3,

(1) $\mu_{\mathbb{C}^K} = \gamma_2^K \circ T_K^{-1}$.

Consequently, for any $t > 0$, with the notation of Proofs. III.4,

$$\mu_{\mathbb{C}^K}(r_K[tV_1^0]) = \mu_{\mathbb{C}^K}(t(V_1 \cap \mathbb{C}^K)^0)'$$

$$= \gamma_2^K(\{w \in \mathbb{C}^K : \sup_{\alpha\in K} |w(\alpha)| \leq t\})$$

$$= \prod_{\alpha\in K} \int_{|w(\alpha)|\leq t} \theta_2 \, (w(\alpha)) d\lambda(w(\alpha))$$

However,

$$\int_{|z|\leq t} \theta_2(z) d\lambda(z) < 1 \quad ,$$

since $\int \theta_2(z) d\lambda(z) = 1$ and θ_2 is strictly positive (Lemmas 1.1). It follows that

$$\inf_{\text{finite } K \subset A} \mu_{\mathbb{C}^K}(r_K[tV_1^0]) = 0 \quad ,$$

and therefore, by Lemma I.5.1.2, μ cannot be E-tight.

On the other hand, by Pietsch [30] p. 82, Prop. 4, there exists $S_{1,\eta} \in S^1$ s.t.

$$[x,x] \leq \int | f(x)| d\eta(f) \quad \text{for all } x \in X \quad ,$$

and therefore

$$x \in X \to [x,x] \in R^+ \text{ is } \mathcal{S}^1\text{-continuous.}$$

Hence ψ is S^1-continuous. Consequently, by Thm. 2.5, μ is S^1-continuous.

4.2 Example

Let A be a set, $2 < p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$.
If $y \varepsilon \ell^q(A)$ is such that

$$\sum_{\alpha \varepsilon A} |y(\alpha)|^q |\ln|y(\alpha)|| = \infty$$

and

$$T : x \varepsilon \ell^p(A) \to (x(\alpha)y(\alpha))_{\alpha\varepsilon A} \varepsilon \ell^q(A) ,$$

then there exists $\mu \varepsilon CM(\ell^q(A))$ such that $\mu \,\square\, T$ is not E-tight.

Notation. Together with the notations of §2.1 and Proofs III.4, for any $t \geq 0$, , let

$$b(t) = t^q|\ln t|, \; t > 0 \text{ , and } b(0) = 0 \quad .$$

We shall need the following lemma ([41] Lemma 2].)

Lemma. Let

$$w : A \to \mathbb{C} \text{ with } |w(\alpha)| \leq 1 \text{ for all } \alpha \varepsilon A \quad .$$

There exists a constant $0 < C < \infty$ such that

for every finite K A ,

$$\gamma_p^K(\{z \varepsilon \mathbb{C}^K : \sum_{\alpha\varepsilon K} |z(\alpha)w(\alpha)|^q > 1\}) \geq e^{-1} - \exp - \sum_{\alpha\varepsilon K} b(|w(\alpha)|) \quad .$$

Proof of Lemma Let θ_p be the function of Lemma 1.1.1.

From [3] p. 263,

$$0 < \lim_{|v|\to\infty} |v|^{q+2}\, \theta_p(v) < \infty \quad .$$

Hence, there exists $0 < C' < \infty$ s.t.

(1) $\theta_p(v) \geq C'/|v|^{q+2}$ for all $v \in \mathbb{C}$ with $|v| \geq 1$.

By Taylor's theorem, for any $0 \leq t \leq 1$,

$$1 - \exp - t = t \exp - t' \quad \text{for some} \quad 0 \leq t' \leq 1 ,$$

and therefore

(2) $1 - \exp - t \geq t\, e^{-1}$ for all $0 \leq t \leq 1$.

Let

(3) $C = 2\pi e^{-1}C'$.

Then, for each $\alpha \in A$,

(4) $\int (1 - \exp - |vw(\alpha)|^q)\theta_p(v)\,d\lambda(v)$

$$\geq e^{-1} \int_{0 \leq |vw(\alpha)| \leq 1} |vw(\alpha)|^q \theta_p(v)\,d\lambda(v) \quad \text{by (2)},$$

$$\geq e^{-1}C' \int_{1 \leq |v| \leq 1/|w(\alpha)|} |vw(\alpha)|^q \cdot \frac{1}{|v|^{q+2}}\,d\lambda(v) \quad \text{by (1)}$$

$$= 2\pi e^{-1}C'|w(\alpha)|^q \int_{1 \leq \rho \leq 1/|w(\alpha)|} 1/\rho \quad d\rho$$

$$= C\, b(|w(\alpha)|) \quad \text{by (3)}.$$

For any finite $K \subset A$, if

$$B_K = \{z \in \mathbb{C}^K : \sum_{\alpha \in K} |z(\alpha)w(\alpha)|^q > 1\} ,$$

then,

$$\gamma_p^K(B_K) \geq \int 1_{B_K}(z)(1 - \exp - \sum_{\alpha \in K} |z(\alpha)w(\alpha)|^q)\,d\gamma_p^K(z)$$

$$= \int (1 - \exp - \sum_{\alpha \in K} |z(\alpha)w(\alpha)|^q)\,d\gamma_p^K(z)$$

$$- \int 1_{\mathbb{C}^K \sim B_K}(z)(1 - \exp - \sum_{\alpha \in K} |z(\alpha)w(\alpha)|^q)\,d\gamma_p^K(z)$$

$$\geq \int (1 - \exp - \sum_{\alpha \in K} |z(\alpha)w(\alpha)|^q)\,d\gamma_p^K(z) - (1 - e^{-1})$$

$$= e^{-1} - \prod_{\alpha \in K} \int \exp - |z(\alpha)w(\alpha)|^q d\gamma_p(z(\alpha)) \quad .$$

However, for each $\alpha \varepsilon K$,

$$\int \exp - |z(\alpha)w(\alpha)|^q d\gamma_p(z(\alpha)) = 1 - \int(1 - \exp - |z(\alpha)w(\alpha)|^q) d\gamma_p(z(\alpha))$$

$\leq 1 - Cb(w(\alpha))$ by (4) above.

Hence,

$$\gamma_p^K(B_K) \geq e^{-1} - \prod_{\alpha\varepsilon K} [1 - Cb(w(\alpha))]$$

$$\geq e^{-1} \prod_{\alpha\varepsilon K} \exp - Cb(w(\alpha)) \quad \text{since } 1 - u \leq e^{-u} \text{ for all } u > 0 \ ,$$

$$= e^{-1} - \exp - C \sum_{\alpha\varepsilon K} b(w(\alpha)) \ .$$

Proof of Example 4.2

If

$$h : x \ \varepsilon \ \ell^q(A) \to \exp - \sum_{\alpha\varepsilon A} |x(\alpha)|^q \ \varepsilon \ \mathbb{C} \ ,$$

then h is continuous. By Lemma 3.4.3 and the lemma of Proofs 3., h is positive definite and $h|F$ is continuous for every finite dimensional subspace F of $\ell^q(A)$.

Hence, by Thms. 2.4 and 2.5,

(1) there exists a continuous finite cylinder measure μ over $\ell^q(A)$ with characteristic functional h .

Clearly,

(2) $\mu \ \varepsilon \ CM(\ell^q(A))$.

Choose

(3) $t > 0$ s.t. $|y(\alpha)|/t \leq 1$ for all $\alpha \varepsilon A$.

Let $0 < \delta < e^{-1}$.

For any finite subfamily K of A , let

$h_K : z \ \varepsilon \ \mathbb{C}^K \to h_K(z) \ \varepsilon \ (\mathbb{C}^K)^*$, with

$$h_K(z)(x) = \sum_{\alpha\varepsilon K} x(\alpha)\overline{z(\alpha)} \quad \text{for all } x \ \varepsilon \ C^K \ ,$$

then, as in earlier proofs (Proof 1.2.2 (1), Proof 4.1(1),

(4) $\mu_{\mathbb{C}^K} = \gamma_p^K \circ h_K^{-1}$.

Hence, for any finite $K \subset A$ s.t. $\alpha \in K \Rightarrow y(\alpha) \neq 0$,

$$(\mu \,\square\, T)_{\mathbb{C}^K}((\mathbb{C}^K)^* \sim t(V_p \cap \mathbb{C}^K)^0)$$

$= \mu_{\mathbb{C}^K}((\mathbb{C}^K)^* \sim tT_{\mathbb{C}^K}^{*-1}(V_p \cap C^K)^0)$ by Lemma 0.4.2, and the fact that $\mathbb{C}^K = T[\mathbb{C}^K]$;

$$= \gamma_p^K(\{z \in \mathbb{C}^K : \sum_{\alpha\in K} |\frac{y(\alpha)}{t} z(\alpha)|^q > 1\})$$

$\geq e^{-1} - \exp - C \sum_{\alpha\in K} b(|y(\alpha)|/t)$ by the Lemma.

Now,

$$\sum_{\alpha\in A} b(|y(\alpha)|/t) = \infty \quad \text{for any} \quad t > 0 \quad ,$$

Therefore there exists finite $J \subset A$ s.t. $\alpha \in J \Rightarrow y(\alpha) \neq 0$, and

$$e^{-1} - \exp - C \sum_{\alpha\in J} b(|y(\alpha)|/t) > \delta \quad .$$

Hence,

$$(\mu \,\square\, T)_{\mathbb{C}^J}((\mathbb{C}^J)^* \sim t(V_p \cap \mathbb{C}^J)^0) > \delta \quad .$$

Since $t > \sup_{\alpha\in A} |y(\alpha)|$, and $0 < \delta < e^{-1}$ were arbitrary, and with the notation of Proofs III.4,

$$r_J(tV_p^0) = t(V_p \cap \mathbb{C}^J)^0 \quad ,$$

it follows from Lemma I.5.1.2 that

$\mu \,\square\, T$ is not E-tight.

4.3 Example

There exists a Banach space which is not a C M-space.

Proof. Let

$$c_0 = \{x \in \mathbb{C}^{\omega} : \lim_{n\to\infty} x(n) = 0\} ,$$

$$|\cdot|_\infty : x \in c_0 \to \sup_{n\in\omega} |x(n)| \in R^+ ,$$

$$\ell^2 = \ell^2(\omega)$$

and

$T : \ell^2 \to c_0$ be the canonical imbedding.

As is well known, c_0 is a Banach space under the topology generated by the norm $|\cdot|_\infty$.

From Pietsch [30] p. 83, Remark 2.2,

(1) T is not S^1-continuous.

From Kwapien [19] we have that

$$\mu \in CM(c_0) \Rightarrow \mu \sqsupset T \text{ has a limit measure on } (\ell^2)' ,$$

and therefore, by Cor. I.4.3,

$$\mu \in CM(c_0) \Rightarrow \mu \sqsupset T \text{ is } E\text{-tight.}$$

Hence, by Prop. III.1.3,

(2) T is S^1-continuous with respect to the $CM(c_0)$-topology on c_0 .

From (1) and (2) it follows that the $CM(c_0)$-topology does not coincide with the norm topology, i.e.

c_0 is not a CM-space.

4.4 Example

For any $r \geq 4$, the S^r-topology on $\ell^2(\omega)$ does not coincide with the S^2-topology.

Proof We shall construct a $T : \ell^2(\omega) \to \ell^2(\omega)$ which will be S^r-continuous but not S^2-continuous, from which it follows that the S^r-and S^2-topologies do not coincide.

For each $n \in \omega$, let

$$e_n = 1_{\{n\}}$$

$$a_n = n^{-2/r} ,$$

$$T : x \in \ell^2(\omega) \to (a_n x_n)_{n\in\omega} \in \ell^r(\omega) .$$

As in the proof of Cor. III.4.4.1, we conclude that

(1) T is S^r-continuous.

If T were also S^2-continuous, then, there would exist a w^*-Radón measure η on $(\ell^2(\omega))^*$ with $\operatorname{supp} \eta \in E$, s.t.

$$\left(\int |f(x)|^2 d\eta(x)\right)^{1/2} \leq 1 \Rightarrow |Tx|_r \leq 1 .$$

Hence,

$$|Tx|_r^2 \leq \int |f(x)|^2 d\eta \quad \text{for all} \quad x \in \ell^2(\omega) .$$

Consequently, for any $k \in \omega$

$$\sum_{n\leq k} a_n^2 = \sum_{n\leq k} |Te_n|_r^2 \leq \int \sum_{n\leq k} |f(e_n)|^2 d\eta(f)$$

$$\leq \int \left(\sup_{|x|_2\leq 1} |f(x)|\right)^2 d\eta(f) ,$$

since $\{e_n\}_{n\in\omega}$ is a orthonormal basis of the Hilbert space $\ell^2(w)$.

Since $\operatorname{supp} \eta \in E$, and $k \in \omega$ was arbitrary, it follows that

$$\sum_{n\in\omega} a_n^2 < \infty .$$

However, this is impossible, since

$$a_n^2 = n^{-4/r} \geq n^{-1} \quad , \text{ and } \quad \sum_{n \varepsilon \omega} 1/n = \infty \quad .$$

Hence

T is not S^2-continuous.

Remark.

In view of Theorem III. 2.6.3, from the above example we see that for every $r \geq 4$,

$\ell^r(\omega)$ is not a CM-space.

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