SOME SIZE AND STRUCTURE THEOREMS

FOR ULTRAPOWERS

by

MURRAY ALLAN JORGENSEN
BSc(Hons), University of Canterbury, 1967
MA, University of British Columbia, 1970

A THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

We accept this thesis as conforming to
the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
December 1971

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date 18 Feb. 1972

Supervisor: Dr. A. Adler

ABSTRACT

In this thesis we study the mapping $D \mapsto \mathbb{A}^I/D$, between ultrafilters and models, given by the ultrapower construction. Under this mapping homomorphisms of ultrapowers induce elementary embeddings of ultrapowers. Using these embeddings we investigate the dependence of the structure of an ultrapower $\mathbb{A}^I/D$ on the cardinality of the index set I.

With each ultrafilter D we associate a set of cardinals $\sigma(D)$ which we term the shadow of D. We investigate the form of the sets $\sigma(D)$. It is shown that if $\sigma(D)$ has "gaps" then isomorphisms arise between ultrapowers of different index sizes. In terms of $\sigma(D)$ we prove new results on the properties of the set of homomorphic images of an ultrafilter. Finally we introduce a new class of "quasicomplete" ultrafilters and prove several results about ultrapowers constructed using these.

Two results which can be mentioned here are the following:

Let α be a regular cardinal. We establish necessary and sufficient conditions on D

(i) for the cardinality of α to be raised in the passage to α^I/D.

(ii) for the confinality of α^I/D (regarded as an ordered set) to be greater than α^+.

Some of the results of this thesis depend on assumption of the Generalised Continuum Hypothesis. The result (i) above is a case in point.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

This thesis could not have reached its present form without many months of goading and guiding by my supervisor Dr. Andrew Adler nor without the efficient typing of Mrs. Edith Huang. Thanks also go to those who read my thesis in its early stages and made many helpful suggestions.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

A. Introduction

Starting from a relational structure $\mathbb{A}$ we can define, for each ultrafilter D on an index set I, a new relational structure $\mathbb{A}^I/D$ called an ultrapower of $\mathbb{A}$. $\mathbb{A}^I/D$ possesses the same first-order properties as $\mathbb{A}$ but is not in general isomorphic to $\mathbb{A}$. Most of the interest and attention given to ultrapowers has been directed to their application in model theory for constructing, from a given model of an axiom system, a new model of the same axioms with certain desired properties. Hence emphasis has been on building particular ultrapowers rather than on investigating general properties of ultrapowers. The acuteness of our ignorance in this latter area is emphasised by the many unresolved questions relating to the cardinality of ultrapowers. This present work represents a contribution to a "general theory" of ultrapowers whose preoccupation is with relating the form of the ultrapower $\mathbb{A}^I/D$ with the form of the ultrafilter D used to define it.

To be more precise let us suppose that $\mathbb{A}$ lists among its relations, functions, and constants all the finitary relations and operations on its underlying set A and all members of A so that $\mathbb{A}$ now carries as much structure as possible. Using this structure we define the natural notions of isomorphism between two ultrapowers $\mathbb{A}^I/D$ and $\mathbb{A}^J/E$ and isomorphic embedding of $\mathbb{A}^J/E$ into $\mathbb{A}^I/D$. Then we can ask

how closely the isomorphism class of the ultrapower $\mathbb{A}^I/D$ determines the ultrafilter D. For instance is it true that if D and E are ultrafilters on I and $\mathbb{A}^I/D \cong \mathbb{A}^I/E$ then D = E ? If not, what relation does D bear to E ? Also how many isomorphism classes is the set

$$\{ \mathbb{A}^I/D : D \text{ an ultrafilter on } I\}$$

broken into? Of course the answer to these and similar questions will depend on the size of $|A|$. For example if A is finite we have that $\mathbb{A}^I/D \cong \mathbb{A} \cong \mathbb{A}^I/E$ for <u>all</u> ultrafilters D and E on I.

Another kind of question is

(*) <u>Given A, I and D what cardinals can be written in the form $|J|$, where J is a set on which there resides a uniform ultrafilter E such that $\mathbb{A}^I/D \cong \mathbb{A}^J/E$</u> ?

In the remainder of this chapter we will fix on our notation and collect together some preliminary results known from the literature. In Chapter 2 we place a natural algebraic structure on the class of all ultrafilters. We show that if we restrict ourselves to ultrafilters on a given set I and if we take $|A|$ sufficiently large ($\geq |I|$) that the algebra of ultrapowers of $\mathbb{A}$ is completely determined by this algebra of ultrafilters up to an isomorphism of categories. We then collect and prove results about this algebra which find an immediate application in proving a size result (2.20) not previously derivable without the Generalized Continuum Hypothesis.

Chapter 3 applies some of the ideas of Chapter 2 to give partial answers to (*). Roughly speaking we investigate how "economically" an ultrapower can be obtained (up to isomorphism), it being "wasteful" to employ too large an index set. This idea leads to an

interesting classification of the substructures of an ultrapower.

Finally in Chapter 4 we present new results on the algebraic structure of an ultrafilter D and on the relationship of this structure to the cardinality of ultrapowers employing D. We define a notion of quasicompleteness for ultrafilters that helps to express some of these results.

B. Set-Theoretic Notation

We will use standard notational apparatus of set theory without special comment. Our informal set theory may be thought of as Zermelo-Fraenkel set theory including the Axiom of Choice, although we will refer to proper classes on occasion.

The letters GCH, respectively, LCH, attached to a result indicate that the result has been proved under the assumption of the Generalized Continuum Hypothesis, respectively, the Limit Cardinal Hypothesis. Ordinals are denoted by small Greek letters although we sometimes employ these letters to represent functions. Cardinals are taken to be initial ordinals and we specifically designated α, β, γ, δ and κ to range over cardinals. This will not inhibit us from declaring other letters to be ordinals or cardinals at various places in the thesis. We will allow the expression λ^{μ} to remain ambiguous, its interpretation to be suggested by the context. For example we can write both

$$f \in \lambda^{\mu} \quad \text{and} \quad \eta < \lambda^{\mu} = |\lambda^{\mu}|$$

In any event ordinal exponentiation is not intended. The symbol + is reserved for ordinal addition, but $\cdot$ always stands for cardinal multiplication. Σ and Π are used for infinite cardinal sums, resp., products.

For any set X

$$S(X) = \{Y : Y \subseteq X\}$$

$$S_\kappa(X) = \{Y : Y \subseteq X \text{ and } |Y| < \kappa\}$$

$$\bar{S}_\kappa(X) = \{Y : Y \subseteq X \text{ and } |X-Y| < \kappa\}$$

We will often think of functions as sequences and employ notation to suggest this. Thus if $f \in A^B$ and $\phi \in \lambda^\mu$ we may write

$$f = \langle f_b : b \in B\rangle \text{ and } \phi = \langle \phi_\eta : \eta < \mu\rangle$$

where $f_b = f(b)$ and $\phi_\eta = \phi(\eta)$. α^+ is the smallest cardinal greater than α. $\lambda^{(\mu)}$ is $\Sigma\langle\lambda^\nu : \nu < \mu\rangle$, the weak cardinal power.

C. Relational Structures

As implies in the Introduction we will be taking a <u>relational structure</u> to be an ordered 4-tuple

$$\mathbb{A} = \langle A, R, F, C\rangle$$

where A is a set, R is a sequence $\langle R_\eta : \eta < \lambda\rangle$ of finitary relations on A, F is a sequence $\langle F_\eta : \eta < \mu\rangle$ of finitary operations on A, and C is a sequence $\langle c_n : \eta < \nu\rangle$ of members of A. A is called the <u>underlying set</u> of $\mathbb{A}$. The (similarity) <u>type</u> of a relational structure A is a triple $\langle\sigma, \tau, \nu\rangle$ where $\sigma \in \omega^\lambda$, $\tau \in \omega^\mu$ such that

(i) for all $\eta < \lambda$, $R_\eta \subseteq A^{\sigma(\eta)}$

(ii) for all $\eta < \mu$, $F_\eta : A^{\tau(\eta)} \to A$

(iii) C has length ν.

$\mathbb{A}$ is said to be a <u>full structure on A</u> if the range of R includes all finitary relations on A, the range of F all finitary operations on A, and the range of C all members of A.

With every structure $\mathbb{A}$ (or strictly with every type $\langle\sigma,\tau,\nu\rangle$) there is associated a first-order language $L_{\mathbb{A}}$ (or $L_{\langle\sigma,\tau,\nu\rangle}$) defined as follows:

(1) <u>Individual Variables</u>

are elements of the countable set $\{v_n : n < \omega\}$.

(2) <u>Predicate Symbols</u>

for each $\eta > \lambda$, $L_{\mathbb{A}}$ has a $\sigma(\eta)$-ary predicate symbol P_η.

(3) <u>Function Symbols</u>

for each $\eta < \mu$, $L_{\mathbb{A}}$ has a $\tau(\eta)$-ary function symbol G_η.

(4) <u>Constant Symbols</u>

are elements of the set $\{d_\eta : \eta < \nu\}$.

(5) <u>Logical Symbols</u>

$=$, $\wedge$, $\neg$, and $\exists$, from which $\to$, $\vee$, and $\forall$ are defined in the usual way. (In defining $L_{\mathbb{A}}$ we have taken the type of $\mathbb{A}$ to be $\langle\sigma,\tau,\nu\rangle$ where range $(\sigma) = \lambda$ and range $(\tau) = \mu$. Unless another type is specified we will always use these letters to refer to the type of a relational system under discussion).

The notions of <u>formula</u> and <u>term</u> of $L_{\mathbb{A}}$ are defined by the standard induction on length of expression. <u>Free variable</u>, <u>bound variable</u>, and <u>sentence</u> also have their standard meaning. The <u>type</u> of $L_{\mathbb{A}}$ is just the type of $\mathbb{A}$. If $\mathbb{A}'$ is a structure of the same type as $L_{\mathbb{A}}$, ϕ is a formula of $L_{\mathbb{A}}$, and $x \in A'^{\omega}$ we shall say <u>the sequence x satisfies the formula ϕ in $\mathbb{A}'$</u> and write

$$\mathbb{A}' \models_x \phi$$

for the usual notion of satisfaction, in which = is to be interpreted as identity. (cf. [4], p.56). If ϕ is a sentence the particular sequence x in the preceding is immaterial and we can write $\mathbb{A}' \models \phi$ and say <u>$\mathbb{A}'$ is a model of ϕ</u> or <u>ϕ holds in $\mathbb{A}'$</u>. If $\sum$ is a set of sentences then <u>$\mathbb{A}'$ is a model of $\sum$</u> iff $\mathbb{A}' \models \phi$ for all $\phi \in \sum$

For each structure $\mathbb{A}$ the theory $T_{\mathbb{A}}$ is defined as the theory with language $L_{\mathbb{A}}$ and with axioms all sentences belonging to the set

$$\{\phi : \mathbb{A} \models \phi\} .$$

In general $T_{\mathbb{A}}$ is not recursively axiomatisable.

An <u>embedding</u> $f : \mathbb{A}_1 \to \mathbb{A}_2$ between two relational structures of the same type $\langle\sigma,\tau,\nu\rangle$ is a 1-1 map $f : \mathbb{A}_1 \to \mathbb{A}_2$ such that

(a) for all $\eta < \lambda$ and any elements $a_1, a_2, \ldots, a_{\sigma(\eta)} \in A_1$

$$\langle a_1,\ldots,a_{\sigma(\eta)}\rangle \in R_{1\eta} \text{ iff } \langle f(a_1),\ldots,f(a_{\sigma(\eta)})\rangle \in R_{2\eta}$$

(b) for all $\eta < \mu$ and elements $a_1,a_2,\ldots,a_{\tau(\eta)}$, $a \in A_1$

$$F_{1\eta}(a_1,\ldots,a_{\tau(\eta)}) = a \text{ iff } F_{2\eta}(f(a_1),\ldots,f(a_{\tau(\eta)})) = f(a)$$

(c) for all $\eta < \nu$

$$f(c_{1\eta}) = c_{2\eta}$$

Throughout the rest of this section $\mathbb{A}$ and $\mathbb{B}$ are structures of type $\langle\sigma,\tau,\nu\rangle$ and $L = L_{\mathbb{A}} = L_{\mathbb{B}}$.

If $\mathbb{A} \subseteq B$ we say that <u>$\mathbb{A}$ is a substructure of $\mathbb{B}$</u> and write $\mathbb{A} \subseteq \mathbb{B}$ iff the inclusion map $i : A \to B$ is an embedding of $\mathbb{A}$ into $\mathbb{B}$. It is easy to see that this is the case iff each relation and each function of $\mathbb{B}$ restricts to the corresponding relation or function of $\mathbb{A}$ and each constant of $\mathbb{B}$ is equal to the corresponding constant of $\mathbb{A}$. An embedding is said to be an <u>isomorphism</u> if it is onto. If there is an isomorphism between $\mathbb{A}$ and $\mathbb{B}$ we say that $\mathbb{A}$ and $\mathbb{B}$ are <u>isomorphic</u> and write $\mathbb{A} \cong \mathbb{B}$. If $\mathbb{A} \subseteq \mathbb{B}$ and for every every formula $\phi(v_0,\ldots,v_n)$ of L and any $a_0,\ldots,a_n \in A$ we have

$$\mathbb{A} \models \phi(a_0,\ldots,a_n) \quad \text{iff} \quad \mathbb{B} \models \phi(a_0,\ldots,a_n)$$

we say that $\mathbb{A}$ is an <u>elementary substructure</u> of $\mathbb{B}$ or that $\mathbb{B}$ is an <u>elementary extension</u> of $\mathbb{A}$ and write

$$\mathbb{A} \prec \mathbb{B}$$

If for every sentence ϕ of L we have that $\mathbb{A} \models \phi$ iff $\mathbb{B} \models \phi$ then we say that $\mathbb{A}$ and $\mathbb{B}$ are <u>elementarily equivalent</u> and write $\mathbb{A} \equiv \mathbb{B}$. Notice that $\equiv$ is an equivalence relation and that $\prec$ is transitive. An embedding $h : \mathbb{A} \to \mathbb{B}$ is called an <u>elementary embedding</u> if for any formula $\phi(v_0,\ldots,v_n)$ of L and any $a_0,\ldots,a_n \in A$

$$\mathbb{A} \models \phi(a_0,\ldots,a_n) \quad \text{iff} \quad \mathbb{B} \models \phi(h(a_0),\ldots,h(a_n)).$$

A first order theory T is said to be <u>model-complete</u> if whenever $\mathbb{B}$ and $\mathbb{C}$ are models of the set of axioms of T and $\mathbb{B} \subseteq \mathbb{C}$ then $\mathbb{B} \prec \mathbb{C}$.

1.1 Theorem

If $\mathbb{A}$ is a full structure then $T_{\mathbb{A}}$ is model-complete.

Proof:

(After Adler [2]). Let $\mathbb{B}$ and $\mathbb{C}$ be models of $T_{\mathbb{A}}$ such that $\mathbb{B} \subseteq \mathbb{C}$. We wish to show that for any formula $\phi(v_0,\ldots,v_n)$ of $L_{\mathbb{A}}$ and any members $b_0,\ldots,b_n \in B$ we have that $\mathbb{B} \models \phi(b_0,\ldots,b_n)$ iff $\mathbb{C} \models \phi(b_0,\ldots,b_n)$. We proceed by induction on the number of logical symbols in ϕ. It is easy to show that the result is true if ϕ is an atomic formula or if ϕ is ϕ_1 or $\phi_1 \wedge \phi_2$ where the result holds for ϕ_1 and ϕ_2. Suppose that $\phi = \exists x\psi(x,b_0,\ldots,b_n)$ where for all $b \in B$ $\mathbb{B} \models \psi(b,b_0,\ldots,b_n)$ iff $\mathbb{C} \models \psi(b,b_0,\ldots,b_n)$. Clearly if $\mathbb{B} \models \phi(b_0,\ldots,b_n)$ then $\mathbb{C} \models \phi(b_0,\ldots,b_n)$. The converse will fail only if $\mathbb{C} \models \phi(b_0,\ldots,b_n)$ but there is no $b \in B$ such that $\mathbb{B} \models \psi(b,b_0,\ldots,b_n)$. Assume that this is the case. Let $R(x,y)$ be a relation of $\mathbb{A}$ that well-orders A. Pick $a \in A$, and define $f \in A^{A^{n+1}}$ such that $f(a_0,\ldots,a_n)$ is the least element of A under R such that $\mathbb{A} \models \psi(f(a_0,\ldots,a_n), a_0,\ldots,a_n)$, or $f(a_0,\ldots,a_n) = a$ if no such least element exists. Then as $\mathbb{A}$ is full f is a function of $\mathbb{A}$, corresponding, say to a function symbol t.

Now by assumption $\mathbb{B} \models \psi(t(b_0,\ldots,b_n),b_0,\ldots,b_n)$ so that by the induction hypothesis $\mathbb{C} \models \psi(t(b_0,\ldots,b_n),b_0,\ldots,b_n)$. But for all $c_0,\ldots,c_n \in C$, if $\mathbb{C} \models \phi(c_0,\ldots,c_n)$ then $t(c_0,\ldots,c_n)$ is the R-least element of C such that $\mathbb{C} \models \psi(t(c_0,\ldots,c_n),c_0,\ldots,c_n)$. But this is a contradiction. So there is $b \in B$ such that $\mathbb{B} \models \psi(b,b_0,\ldots,b_n)$, $\mathbb{B} \models \exists(x)\psi(x,b_0,\ldots,b_n)$. //

In the following we denote the structure $\mathbb{A}$ by $\langle A, R_{\mathbb{A}}, F_{\mathbb{A}}, C_{\mathbb{A}}\rangle$. If $X, Y \subseteq A$ we say that X <u>generates</u> Y if Y is the smallest subset of A such that $X \cup \text{range}(C_{\mathbb{A}}) \subseteq Y$ and if $a_1, a_2, \ldots, a_n \in Y$ and F_η is an n-ary operation in range $(F_{\mathbb{A}})$ then $F_\eta(a_1, \ldots, a_n) \in Y$. If X is a subset of A that generates itself then we can define a substructure $\mathbb{H}$ of $\mathbb{A}$, the <u>restriction</u> of A to X as follows:

i) $H = X$

ii) $R_{\mathbb{H},\eta} = R_{\mathbb{A},\eta} \cap X^{\sigma(\eta)}$ for all $\eta < \lambda$

iii) $F_{\mathbb{H},\eta} = F_{\mathbb{A},\eta}|_{X^{\tau(n)}}$ for all $\eta < \mu$

iv) $C_{\mathbb{H}} = C_{\mathbb{A}}$

$\mathbb{H}$ is denoted by $\mathbb{A}|_X$.

D. Saturated Structures

Recall that the type of $\mathbb{A}$ is taken to be $\langle\sigma,\tau,\nu\rangle$. If $\chi = \langle e_\xi : \xi < \theta\rangle$ is a sequence of elements of A we define a structure $\mathbb{A}\circ\chi$ to be the structure formed from $\mathbb{A}$ by attaching the sequence χ after the sequence $C_{\mathbb{A}}$ to form a new sequence of constants. Thus $\mathbb{A}\circ\chi$ is a structure of type $\langle\sigma,\tau,\nu+\theta\rangle$. (The sequences of relations and functions and the underlying set of $\mathbb{A}$ are not changed.)

For any structure $\mathbb{B}$ of arbitrary type let $\mathcal{F}_{\mathbb{B}}$ be the set of formulas of $L_{\mathbb{B}}$ with at most v_0 free. Take $\Delta \subseteq \mathcal{F}_{\mathbb{B}}$. We shall say that Δ is <u>satisfiable in B</u> if there is a $b \in B$ such that

$$\mathbb{B} \models \phi(b) \quad \text{for all } \phi \in \Delta$$

and that Δ is <u>finitely satisfiable in B</u> if every finite subset of Δ

is satisfiable in $\mathbb{B}$.

Finally we define A to be <u>α-saturated</u> if, for all sequences χ of elements of A having length less than α, any subset of $\mathcal{F}_{\mathbb{A}\circ\chi}$ that is finitely satisfiable in $\mathbb{A}\circ\chi$ is also satisfiable in $\mathbb{A}\circ\chi$.

The theory of saturated structures forms a large body of results, but we need only the following:

<u>1.2 Theorem</u>

(a) If $\mathbb{A}$ and $\mathbb{B}$ are α-saturated, elementarily equivalent, structures of the same type, and if $|A| = |B| = \alpha$, then $\mathbb{A}$ and $\mathbb{B}$ are isomorphic.

(b) If $\mathbb{A}$ is an α-saturated structure of cardinality α and $\mathbb{B} \equiv \mathbb{A}$, where $|B| \leq \alpha$, then $\mathbb{B}$ can be elementarily embedded in $\mathbb{A}$.

<u>Proof</u>

(a) See [4], p.224, Th. 3.1.

(b) See [4], p.226, Th. 3.2. //

<u>E. Ultrafilters, Ultraproducts, Ultrapowers</u>

A <u>filter</u> D on a set I is a non-empty subset of S(I) such that (a) $X, Y \in D$ implies $X \cap Y \in D$; (b) $X \in D$ and $Y \supseteq X$ implies $Y \in D$. If $0 \notin D$, D is called a <u>proper</u> filter. A proper filter that is included in no other proper filter is called an <u>ultrafilter</u>. A necessary and sufficient condition for a proper filter to be an ultrafilter is (c) $X \subseteq I$ implies either $X \in D$ or $I - X \in D$. For

For any $i \in I$ the set

$$[i] = \{X \subseteq I : i \in X\}$$

is an ultrafilter on I. An ultrafilter of this form is called <u>principal</u>. A family $E \subseteq S(I)$ is said to have the <u>finite intersection property</u> (fip) if any finite subfamily of E has non-void intersection.

<u>1.3 Theorem</u>

Any family $E \subseteq S(I)$ having the fip can be extended to an ultrafilter on I.

<u>Proof</u>

A classical and easy application of Zorn's Lemma. //

<u>1.4 Theorem</u>

There are $2^{2^{|I|}}$ distinct ultrafilters on an infinite set I.

<u>Proof</u>

See [4], p.108, Theorem 1.5. //

As there are clearly only $|I|$ principal ultrafilters on I it follows that "most" ultrafilters on I are non-principal.

An ultrafilter D can be regarded as a measure on the set of subsets of I in the following way:

For each $X \subseteq I$ define

$$\mu_D(X) = \begin{cases} 1 & \text{if } X \in D \\ 0 & \text{if } X \notin D \end{cases}$$

then it is a consequence of (a), (b) and (c) that μ_D is a finitely

additive $\{0,1\}$-valued measure on I. This way of regarding D as a measure is a valuable aid to intuition as it makes available analogies with measure theory. In this spirit, if we have a proposition P concerning the members i of I we will say "P(i) holds for almost all i in I (mod D)" so long as

$$\{i \in I : P(i) \text{ holds}\} \in D .$$

If the context makes the meaning clear we will drop the qualification "(mod D)".

Consider, now, a sequence of relational structures $\langle \mathbb{A}_i : i \in I\rangle$ each of type $\langle\sigma,\tau,\nu\rangle$, and an ultrafilter D on I. We define the <u>ultraproduct</u> $B = \Pi\langle \mathbb{A}_i : i \in I\rangle / D$ of the sequence $\langle \mathbb{A}_i : i \in I\rangle$ modulo D as follows:

a) <u>Underlying set</u>

Let B' be the set $\Pi\langle A_i^* : i \in I\rangle$. Define an equivalence relation $\sim_D$ on B' as follows: if a, b $\in$ B' set

$$a \sim_D b \quad \text{iff} \quad \{i \in I : a_i = b_i\} \in D$$

i.e. a = b almost everywhere. Then we write a/D for the equivalence class of a under $\sim_D$ and take B to be the quotient set $\{b/D : b \in B'\}$ of B' under $\sim_D$.

b) <u>Relations</u>

Take $R_{i,\eta}$ to be the ηth relation of A_i where $\eta < \lambda$. Then if $\sigma(\eta) = n$ and $a_1/D, a_2/D, \ldots, a_n/D \in B$ we define $R_{\mathbb{B},\eta}$ so that

$$\langle a_1/D,\ldots,a_n/D\rangle \in R_{\mathbb{B},\eta} \text{ iff } \{i : \langle a_{1i},\ldots,a_{ni}\rangle \in R_{i,\eta}\} \in D.$$

c) Functions

Let $\tau(\eta) = m$, then we define $F_{\mathbb{B},\eta}$ by

$$F_{\mathbb{B},\eta}(a_1/D,\ldots,a_m/D) = a/D$$

iff

$$\{i : F_{i,\eta}(a_{1i},\ldots,a_{mi}) = a_i\} \in D, \quad (n < \mu).$$

d) Constants

$$c_{B,\eta} = g/D$$

where $g(i) = c_{i,\eta}$, $(n < \nu)$.

It remains to verify that $R_{\mathbb{B}}$, $F_{\mathbb{B}}$ and $C_{\mathbb{B}}$ are all well-defined by (b), (c) and (d). But this is an easy calculation.

If each factor $\mathbb{A}_i$ is the same, say $\mathbb{A}_i = \mathbb{A}$ for all $i \in I$, then we call the resulting structure $\mathbb{B}$ an ultrapower of $\mathbb{A}$ and write

$$\mathbb{B} = \mathbb{A}^I/D .$$

The importance of ultrapowers and ultraproducts arises from the following "Fundamental Theorem":

1.5 Theorem (Los' Theorem)

Let $\mathbb{A}_i$ $(i \in I)$ be as above and let ϕ be any formula of $L = L_{<\sigma,\tau,\nu>}$ and D any ultrafilter on I.

Then

$$\Pi < \mathbb{A}_i : i \in I>/D \models_x \phi$$

iff

$$\{i \in I : \mathbb{A}_i \models_{x_i} \phi\} \in D$$

where $x = <f_0/D, f_1/D, \ldots>$ is any countable sequence of members of

$\Pi\langle A_i : i \in I\rangle/D$ and $x_i = \langle f_0(i), f_1(i), \ldots\rangle$.

Proof

See [4], p.90, Theorem 2.1. //

1.6 Corollary

(i) if ϕ is a sentence of L

$$\Pi\langle A_i : i \in I\rangle/D \models \phi$$

iff $\mathbb{A}_i \models \phi$ for almost all $i \in I$.

(ii) $\mathbb{A} \equiv \mathbb{A}^I/D$.

Proof

Immediate from 1.5. //

In fact we can strengthen part (ii) of 1.6. For let $\iota : A \to A^I/D$ be defined by setting $\iota(a) = f_a/D$ where $f_a : I \to A$ has constant value a. Then we claim that ι is an elementary embedding of $\mathbb{A}$ into $\mathbb{A}^I/D$. For if $\phi(v_0,v_1,\ldots,v_n)$ is any formula of L and $a_0,\ldots,a_n$ any elements of A we have by 1.5 that

$$\mathbb{A}^I/D \models \phi(f_{a_0}/D, \ldots, f_{a_n}/D)$$

iff $\{i : \mathbb{A} \models \phi(f_{a_0}(i),\ldots,f_{a_n}(i))\} \in D$

iff $\{i : \mathbb{A} \models \phi(a_0,\ldots,a_n)\} \in D$

iff $\mathbb{A} \models \phi(a_0,\ldots,a_n)$.

ι is called the <u>canonical embedding</u> of $\mathbb{A}$ in $\mathbb{A}^I/D$.

We now discuss certain types of ultrafilters that are useful in the formation of ultrapowers. Let I be an arbitrary infinite set.

A principal ultrafilter on I is remarkable in that it contains an element of cardinality 1. By way of contrast we call an ultrafilter D on I <u>uniform</u> if for any $X \in D$ we have $|X| = |I|$. We have immediately:

<u>1.7 Theorem</u>

An ultrafilter on I is uniform iff it contains the filter $\bar{S}_\alpha(I)$, where $\alpha = |I|$. //

<u>1.8 Corollary</u>

Uniform ultrafilters exist on sets of arbitrary cardinality.

<u>Proof</u>

By 1.3 and 1.7. //

Principal ultrafilters are closed under arbitrary intersections of their elements: in fact this property characterises principal ultrafilters among all proper ultrafilters. For if D is a non-principal ultrafilter on a set I and we take $X_i = I - \{i\}$ then $X_i \in D$ for all $i \in I$ but

$$\cap\{X_i : i \in I\} = 0 \notin D.$$

We call an ultrafilter E <u>β-complete</u> if every intersection of β members of E belongs to E, otherwise E is <u>β-incomplete</u>. If E is β-complete for all $\beta < \gamma$ we say E is <u>$\widehat{\gamma}$-complete</u>, otherwise E is <u>$\widehat{\gamma}$-incomplete</u>. Thus if $|I| = \alpha$, every α-complete ultrafilter on I is principal. Every ultrafilter is $\widehat{\omega}$-complete by definition. If $|I| = \alpha > \omega$ it is natural to ask if there are any $\widehat{\alpha}$-complete non-principal ultrafilters on I. This question and the related question of whether there exist

any ω-complete non-principal ultrafilters on I are famous problems in the history of set theory. The first, and most seminal, work on these questions is the paper of Ulam [12]. It is known that the answer to both questions is negative for a wide class of cardinals α that includes all cardinals commonly encountered in most branches of mathematics. We have the following result.

1.9 Theorem

There exists an α and an ω-complete non-principal ultrafilter on α iff there exists a $\beta > \omega$ and a $\hat{\beta}$-complete non-principal ultrafilter on β.

Proof

[4], p.112, Theorem 1.11. //

Thus if we can find an ω-complete non-principal ultrafilter we can find an α-complete non-principal ultrafilter on some $\alpha > \omega$. The hypothesis that such ultrafilters exist is known as the axiom of measurable cardinals. This area of set theory is still under active study. The word "measurable" here results from thinking of an ultrafilter D as the $\{0,1\}$-valued measure μ_D, which turns out to be countably additive iff D is ω-complete. Measurable cardinals are those cardinals α which support ω-complete non-principal ultrafilters. Usually we work with ω-incomplete ultrafilters.

There is an equivalent way of stating α-completeness that we will sometimes use

1.10 Theorem

D is an α-complete ultrafilter on I iff for every collection $\{X_\eta : \eta < \alpha'\}$ of $\alpha' \leq \alpha$ subsets of I, $\cup\{X_\eta : \eta < \alpha'\} \in D$ implies $X_{\eta_0} \in D$ for some $\eta_0 < \alpha'$.

Proof

A routine application of de Morgan's laws. //

Ultrapowers taken using complete ultrafilters do not yield new structures: in fact we have the following result.

1.11 Theorem

If D is α-complete and $|A| \leq \alpha$ then $\mathbb{A}^I/D \cong \mathbb{A}$.

Proof

Under the hypotheses we will show that $\iota : \mathbb{A} \to \mathbb{A}^I/D$ is onto. For suppose $f/D \in A^I/D$, and let $X_a = \{i \in I : f(i) = f_a(i) = a\}$. Then $\cup\{X_a : a \in A\} = I \in D$ and so by 1.10 there is $a_0 \in A$ such that $X_{a_0} \in D$. Hence $f \sim_D f_{a_0}$ and $f/D \in \text{range}(\iota)$. //

We will now define another kind of property an ultrafilter can have which also stands in antithesis to the property of being principal.

Let κ be any cardinal. A family $\chi \subseteq S(I)$ forms a <u>κ-covering</u> of I if for each $i \in I$

$$|\{X \in \chi : i \in X\}| < \kappa .$$

An ultrafilter D on a set I is called <u>(κ,λ)-regular</u> if there exists a κ-covering χ of I such that $\chi \subseteq D$ and $|\chi| = \lambda$. An

ultrafilter is called simply <u>regular</u> if it is $(\omega,|I|)$-regular.

<u>1.12 Theorem</u>

(i) If $|I| \geq \omega$ there exists a regular ultrafilter on I.

(ii) If $|I| = \omega$ all non-principal ultrafilters on I are regular.

(iii) Any regular ultrafilter is uniform and ω-incomplete.

<u>Proof</u>

(i) Let $\phi : I \to S_\omega(I)$ be a 1-1 onto function, and set

$$X_i = \{j \in I : i \in \phi(j)\}.$$

Then we claim that $\chi = \{X_i : i \in I\}$ has the fip. For let $\{X_{i_1},\ldots,X_{i_n}\}$ be a finite subset of x. Then there exists $i_0 \in I$ such that $\phi(i_0) = \{i_1,\ldots,i_n\}$ and hence $i_0 \in X_{i_1} \cap \ldots \cap X_{i_n}$. So by 1.3 χ extends to an ultrafilter F on I. Finally x is an ω-covering as

$$|\{j : i \in X_j\}| = |\phi(i)| < \omega . \qquad /$$

(ii) As noted above all non-principal ultrafilters on a set of cardinal ω are ω-complete. Let $\{X_n : n<\omega\}$ be a collection of members of D such that $X = \cap\{X_n : n < \omega\} \notin D$. Then if $Y_n = X_0 \cap X_1 \cap \ldots \cap X_n - X$ it is clear that $Y_n \in D$ for all $n < \omega$ and that $\{Y_n : n < \omega\}$ is an ω-covering of I. /

(iii) If D is regular let $\{X_\eta : \eta < |I|\} = \chi$ be an ω-covering of I by members of D. Then $\cap\{X_\eta : \eta < \omega\} = 0 \notin D$

so that D is ω-incomplete. If D is not uniform let $X \in D$ be an element of the smallest cardinality. Then

$$|X| = |X \cap X_\eta| < |I|$$

for all $\eta < |I|$. But this implies there is $x \in X$ such that

$$|\{\eta : x \in X_\eta\}| = |I| ,$$

a contradiction as χ is an ω-covering. //

F. Operations on Ultrafilters

We will be employing various ways of constructing new ultrafilters from given ultrafilters in this thesis. We describe them here.

(1) Restriction

Given an ultrafilter D on I and a member X of D, the restriction of D to X is an ultrafilter $D|_X$ on X given by

$$D|_X = \{Y \cap X : Y \in D\}.$$

It is easy to check that $D|_X$ so defined is an ultrafilter. Note that any ultrafilter can be restricted to a uniform ultrafilter. (Take X to be a member of D having smallest cardinality.) Conversely if we have an ultrafilter E on a subset J of I we can expand E uniquely to an ultrafilter D on I by setting

$$D = \{X \subseteq I : X \cap J \in E\}$$

Thus D can be recovered from any of its restrictions.

(2) Product

Given ultrafilters D on I, E on J we define an ultrafilter $D \times E$ on $I \times J$ by

$$X \in D \times E \quad \text{iff} \quad \{j : X^{(j)} \in D\} \in E$$

where $X^{(j)} = \{i : \langle i,j\rangle \in X\}$.

(3) Infinite Sum

Suppose we have a family $\{I_\lambda : \lambda \in \Lambda\}$ of disjoint sets with ultrafilters D_λ on I_λ and E on Λ. Then we can define an ultrafilter D on $\cup\{I_\lambda : \lambda \in \Lambda\}$ by

$$X \in D \quad \text{iff} \quad \{\lambda : X \cap I_\lambda \in D_\lambda\} \in E$$

We write $D = \Sigma_E\{D_\lambda : \lambda \in \Lambda\}$.

(4) Image

If D is an ultrafilter on I and f is a map from I into J we can define an ultrafilter E on J by

$$X \in E \quad \text{iff} \quad f^{-1}(X) \in D.$$

We write $E = f(D)$ in this situation.

Now we prove the following theorem about (2).

1.13 Theorem

Let D be an ultrafilter on I, E an ultrafilter on J, and $\mathbb{A}$ any structure. Then

$$[\mathbb{A}^I/D]^J/E \cong \mathbb{A}^{I\times J} / D\times E .$$

Proof

Let $\phi : (A^I)^J \to A^{I\times J}$ be the standard identification, i.e. $\phi(\langle h_j : j \in J\rangle) = k$ where $k(i,j) = h_j(i)$. Then

$$\langle h_j/D : j\in J\rangle/E = \langle h'_j/D : j\in J\rangle/E$$

iff $\{j : \{i : h_j(i) = h'_j(i)\} \in D\} \in E$

iff $\{\langle i,j\rangle : h_j(i) = h'_j(i)\} \in D \times E$

iff $\{\langle i,j\rangle : k(i,j) = k'(i,j)\} \in D \times E$

iff $\phi(\langle h_j : j \in J\rangle)/D\times E = \phi(\langle h'_j : j \in J\rangle)/D\times E$

So if we define $\tilde{\phi} : [A^I/D]^J/E \to A^{I\times J}/D\times E$ by $\tilde{\phi}(\langle h_j/D : j\in J\rangle/E) = \phi(\langle h_j : j\in J\rangle)/D\times E$ then $\tilde{\phi}$ is well-defined, 1-1, and clearly onto.

We can easily check that $\tilde{\phi}$ preserves a binary relation R of $R_{\mathbb{A}}$ by repeating the above argument with "R" in place of "=". The argument for general relations and functions is no more difficult. Thus we see that $\tilde{\phi}$ is an isomorphism. //

G. Cardinality of Ultrapowers

We collect here some known results that will be used in the following chapters. In this section we will take D to be an ultrafilter on a set I of cardinal $\gamma \geq \omega$, and α and β arbitrary.

1.14 Theorem

(a) $\alpha \leq |\alpha^I/D| \leq \alpha^\gamma$

(b) If D is uniform $|\gamma^I/D| > \gamma$

(c) (i) $|\alpha^I/D|^\beta \leq |(\alpha^\beta)^I/D|$

(ii) $|(\alpha^\beta)^I/D| \leq |\alpha^I/D|^{|\beta^I/D|}$

(d) If α is finite $|\alpha^I/D| = \alpha$

(e) if D is regular and $\alpha \geq \omega$ then

$$|\alpha^I/D| = \alpha^\gamma$$

(f) If D is ω-incomplete then if $\alpha \geq \omega$

$$|\alpha^I/D|^\omega = |\alpha^I/D| \ .$$

<u>Proof</u>

(a) is trivial.

(b) is proved by a diagonal argument. For complete proof see [4], p.132, Theorem 3.19. /

(c) (i) Consider the map $\Phi : (\alpha^\beta)^I \to (\alpha^I)^\beta$ given by $\Phi(\langle h_i : i \in I\rangle) = k$ iff $k(\eta)(i) = h_i(\eta)$ for all $\eta < \beta$. Notice that if $\langle h_i : i\langle I\rangle \sim_D \langle h_i' : i \in I\rangle$ and $\Phi(\langle h_i' : i\in I\rangle) = k'$ then $k(\eta) \sim_D k'(\eta)$ for all $\eta < \beta$. So Φ induces a map

$$\breve{\Phi} : (\alpha^\beta)^I/D \to (\alpha^I/D)^\beta.$$

Further it is clear that Φ and $\breve{\Phi}$ are onto. (Φ is also 1-1, but $\breve{\Phi}$ need not be.) Consequently

$$|(\alpha^I/D)^\beta| \leq |(\alpha^\beta)^I/D|. \qquad /$$

(c) (ii) Consider the map

$$\psi : (\alpha^\beta)^I \to (\alpha^I)^{\beta^I}$$

given by $\psi(\langle h_i : i \in I\rangle) = k$ where k is given by $k(f) = g$

iff $g(i) = h_i(f(i)$ for all $i \in I$.

ψ induces a map

$$\breve{\psi} : (\alpha^\beta)^I/D \to (\alpha^I/D)^{\beta^I/D}$$

such that $\breve{\psi}(\langle h_i : i \in I\rangle/D) = \breve{k}$ where k is given by $\breve{k}(f/D) = g/D$ iff $g(i) = h_i(f(i))$ for almost all i in I (mod D).

It is readily checked that the definitions given for $\breve{k}$ and $\breve{\psi}$ do not depend on particular choices from equivalence classes. We now show that $\breve{\psi}$ is 1-1: suppose that $\langle h_i : i \in I\rangle/D \neq \langle h_i' : i \in I\rangle/D$ then

$$\{i : h_i \neq h_i'\} \in D$$

so there is an $f \in \beta^I$ such that

$$\{i : h_i(f(i)) \neq h_i'(f(i))\} \in D$$

hence if $\breve{k} = \breve{\psi}(\langle h_i : i \in I\rangle/D)$ and $\breve{k}' = \breve{\psi}(\langle h_i' : i \in I\rangle/D)$ then $\breve{k}(f/D) \neq \breve{k}'(f/D)$ and so $\breve{k} \neq \breve{k}'$. /

(d) follows from 1.11 as all ultrafilters are ω-complete. /

(e) We prove a little more: suppose D is (ω,λ)-regular and $\alpha \geq \omega$, then $|\alpha^I/D| \geq \alpha^\lambda$. Then (e) follows when $\lambda = \gamma$.

Let $\chi = \{X_\eta : \eta < \lambda\}$ be an ω-covering of I by members of D. Let $\Lambda_i = \{\eta : i \in X_\eta\}$, then Λ_i is finite for all $i \in I$. Let $\langle A_n : n < \omega\rangle$ be a sequence of disjoint subsets of α such that $|A_n| = \alpha$ and let $\psi_n : \alpha^n \to A_n$ be 1-1 maps.

If $f \in \alpha^\lambda$ and $\{\eta_1,\ldots,\eta_k\} = \Lambda \in S_\omega(\lambda)$ where $\eta_1 < \eta_2 < \ldots < \eta_k$ we set

$$f^*(\Lambda) = \langle f(\eta_1),\ldots,f(\eta_k)\rangle.$$

Now define $\theta : \alpha^\lambda \to \alpha^I$ by

$$(\theta f)(i) = \psi_{|\Lambda_i|}(f^*(\Lambda_i)) .$$

Let $\breve{\theta} : \alpha^{\lambda} \to \alpha^{I}/D$ be such that $\breve{\theta}f = \theta f/D$. If $f, g \in \alpha^{\lambda}$ and $f \neq g$ then there is $\eta < \lambda$ such that $f(\eta) \neq g(\eta)$ hence if $i \in X_{\eta}$, $f^*(\Lambda_i) \neq g^*(\Lambda_i)$ so that

$$\{i : \theta f(i) \neq \theta g(i)\} \in D$$

and $\breve{\theta}f \neq \breve{\theta}g$. Thus θ is 1-1 and the conclusion follows. /

(f) see [2], p.130, Theorem 3.16. This result is also a corollary of 1.16. /

Remark

The result (b) is proved in Frayne, Morel, Scott [7]. (c)(i) is found in Keisler [11]. (c)(ii) is a result of A. Adler and appears in [3]. (e) is proved in Keisler [8], where a discussion of the history of the result is given.

1.15 Corollary

(g) $2^{\beta} \leq |(2^{\beta})^{I}/D| \leq 2^{|\beta^{I}/D|}$.

(h) if $|(2^{\beta})^{I}/D| > 2^{\beta}$ then $|\beta^{I}/D| > \beta$.

Proof

(g) Set $\alpha = 2$ in (c) and apply (d).

(h) follows from (g). //

H. J. Keisler in [11] generalises the argument of 1.14(e) to prove the following result.

1.16 Theorem

Let D be (κ,λ)-regular and let $\{X_\eta : \eta < \lambda\} \subseteq D$ be a κ-covering of I, then

$$|\Pi\langle\alpha_i : i\in I\rangle/D|^\lambda \leq |\langle\alpha_i^{\kappa_i} : i\in I\rangle/D|$$

where $\kappa_i = |\{\eta : i \in X_\eta\}| < \kappa$. //

From this result can be derived (c)(i), (e) and (f). We shall refer to 1.16 as Keisler's Lemma. Later we will indicate other consequences of this result.

CHAPTER 2

IMAGES OF ULTRAFILTERS AND EMBEDDINGS OF ULTRAPOWERS

A. Isomorphism of Ultrapowers

In this chapter we shall explore a natural notion of homomorphism between ultrafilters. We will be using this notion of homomorphism to describe various embeddings and isomorphisms of ultrapowers. In the case of ultrafilters on countable sets this concept has been studied extensively in Booth [5], where it is attributed to H. J. Keisler and M. E. Rudin.

An ultrafilter pair is an ordered pair $\langle I,D\rangle$ in which the second component is an ultrafilter on the first. If $\langle I,D\rangle$ and $\langle J,E\rangle$ are ultrafilter pairs a map $h : I \to J$ is said to be a filter morphism if $E = h(D)$ i.e. $X \in E$ iff $h^{-1}(X) \in D$. We then write $h : \langle I,D\rangle \to \langle J,E\rangle$. If h is 1-1 onto in the above we shall say that h is a filter isomorphism and that the ultrafilters D and E are isomorphic (written $D \cong E$). An alternate definition of isomorphism of two ultrafilters would be to say that $D \cong E$ if there exists a map $\phi : D \to E$ with an inverse and preserving $\cup$ and $\cap$. It is proved in Sirota [15] that this is an equivalent definition.

We will say that D and E are equivalent if there are $X \in E$, $Y \in E$ such that $D|_X \cong E|_Y$. We write $E \approx D$. The relation is indeed an equivalence relation on the class of all ultrafilters. All principal ultrafilters form one class under equivalence. If D and E are ultrafilters on sets of the same cardinality then $D \cong E$

iff $D \approx E$. For if D and E are ultrafilters on sets I and J respectively such that $|I| = |J| \geq \omega$ and there are $X \in E$, $Y \in E$ such that $D|_X \cong E|_Y$ we can show $D \cong E$: We may assume $|I-X| = |J-Y| = |I|$. Let $f : X \to Y$ be a 1-1 onto map such that $V \in E|_Y$ iff $f^{-1}(V) \in D|_X$ and let $g : I-X \to J-Y$ be any 1-1 onto map. Then if we define $h : I \to J$ by $h|_X = f$, $h|_{I-X} = g$, it is easy to check that h is 1-1 onto and that $Z \in E$ iff $h^{-1}(Z) \in E$ and hence $D \cong E$.

Many properties of ultrafilters apply to all ultrafilters in a $\approx$-class if they apply to any in the class.

One example is (κ,λ)-regularity. In contrast the property of being uniform never applies to all ultrafilters in a $\approx$-class as each class contains ultrafilters of arbitrarily large cardinal, e.g. all expansions of a given ultrafilter. However we can define the <u>thickness</u> th(D) of an ultrafilter D to be the smallest cardinal of a set in D. Thus

$$\mathrm{th}(D) = \min\{|X| : X \in D\} .$$

It is clear that all ultrafilters in a given $\approx$-class have the same thickness. The following theorem is proved in Booth [5] for ultrafilters on a countable set. H. J. Keisler, M. E. Rudin, and K. Kunen are given credit for independent proofs of the result. The proof we give here is a slight modification of Booth's.

<u>2.1 Theorem</u>

Let $\langle I,D\rangle$ be an ultrafilter pair and $f : I \to I$ a function such that $f(D) = D$ then $\{i : f(i) = i\} \in D$.

Proof

Let $<$ be an arbitrary well-ordering of I and set $X = \{i : f(i) < i\}$ and $Y = \{i : f(i) > i\}$. We will show $X \notin D$ and $Y \notin D$, i.e. $X \cup Y \notin D$.

Let $f^{(n)}$ denote the n-fold iterate of f. Then set $X_n = \{i : n \text{ is the least integer such that } f^{(n)}(i) \notin X\}$ $(1 \le n < \omega)$. We have that $\cup\{X_n : 2 \le n < \omega\} = X$, for, as $<$ is a well-ordering, each $i \in I$ must belong to some X_n and $X_1 = I - X$. If $X_{odd} = \cup\{X_{2n+3} : n < \omega\}$ and $X_{even} = \cup\{X_{2n+2} : n < \omega\}$ then $X_{odd} \cap X_{even} = 0$, $X_{odd} \cup X_{even} = X$ and $X_{odd} = f^{-1}(X_{even})$. So by hypothesis $X_{odd} \in D$ iff $X_{even} \in D$, hence we must have $X_{odd}, X_{even} \notin D$ and $X \notin D$. If we define Y_{even} and Y_{odd} in the corresponding way we cannot prove $Y = Y_{even} \cup Y_{odd}$ but we can prove $Y_{even} \cup Y_{odd} \notin D$. If now remains to prove that $Y' = Y - (Y_{even} \cup Y_{odd})$ is not a member of D. For each $y \in Y'$ let $C_y = \{f^{(n)}(y) : 1 \le n < \omega\}$ be the orbit of y under f. Then it is easy to see that we can represent Y' as a disjoint union of orbits $\cup\{C_y : y \in Z\}$ for some $Z \subseteq Y'$. (Remember that $f(y) > y$ for all $y \in Y'$, the first member of Z chosen would be the least member of Y' under $<$). Now let

$$Y'_{odd} = \{y : y \text{ is an odd member of its orbit}\}$$

$$Y'_{even} = \{y : y \text{ is an even member of its orbit}\}$$

and argue as before to conclude $Y' \notin D$. //

Now we define $E \le D$ if there exist $Y \in E$, $X \in E$ and $f : X \to Y$ such that $E|_Y = f(D|_X)$. The next result shows that $\le$ is a partial ordering on the $\approx$ equivalence classes.

2.2 Theorem

(a) if $E \leq D$ and $D' \approx D$, $E' \approx E$ then $E' \leq D'$.

(b) $E \leq D$ and $D \leq E$ implies $E \approx D$.

(c) $E \leq D$ and $F \leq E$ implies $F \leq D$.

Proof

(a) and (c) are easy.

(b) suppose $E|_Y = f(D|_X)$, $D|_{X'} = g(E|_{Y'})$ where $X, X' \in D$ and $Y, Y' \in E$. By part (a) we can assume that $|X| = |X'| = th(D)$ and $|Y| = |Y'| = th(E)$. As $D|_X$ and $D|_{X'}$ are equivalent ultrafilters on sets of equal cardinality we have $D|_X \cong D|_{X'}$ and similarly $E|_Y \cong E|_{Y'}$ and there exist 1-1 onto maps $p : X' \to X$, $q : Y \to Y'$ such that $p(D|_{X'}) = D|_X$ and $q(E|_Y) = E|_{Y'}$. Then

$$p[g(q[f(D|_X)])] = D|_X .$$

So by 2.1 there is $X^* \in D$, $X^* \subseteq X$ such that if $i \in X^*$, $p(g(q(f(i))) = i$. But this implies $f|_{X^*}$ is 1-1 and as $Y^* = f(X^*) \in E$ we have $f : D|_{X^*} \cong E|_{Y^*}$, so that $D \approx E$. //

The following theorem shows the relation between homomorphisms of ultrafilters and embeddings of ultra powers.

2.3 Theorem

If $h : \langle I,D\rangle \to \langle J,E\rangle$ is a filter morphism then for any structure $\mathbb{A}$ if $h^{\#}_{\mathbb{A}} : \mathbb{A}^J/E \to \mathbb{A}^I/D$ is defined by

$h^{\#}_{\mathbb{A}}(g/E) = h \circ g/D$ [1] we have

(a) $h^{\#}_{\mathbb{A}}$ is an embedding

(b) if h is 1-1 then $h^{\#}_{\mathbb{A}}$ is an isomorphism.

<u>Proof</u>

(a) To see that $h^{\#}_{\mathbb{A}}$ is an embedding suppose that R_η is an n-ary relation of $\mathbb{A}$. Then if R'_η and R''_η are the corresponding relations of $\mathbb{A}^I/D$ and $\mathbb{A}^J/E$ respectively we have that

$$\langle g_1/E,\ldots,g_n/E\rangle \in R''_\eta$$

$$\text{iff} \quad \{j : \langle g_1(j),\ldots,g_n(j)\rangle \in R_\eta\} \in E$$

$$\text{iff} \quad h^{-1}(\{j : \langle g_1(j),\ldots,g_n(j)\rangle \in R_\eta\}) \in D$$

$$\text{iff} \quad \{i : \langle g_1(h(i)),\ldots,g_n(h(i))\rangle \in R_\eta\} \in D$$

$$\text{iff} \quad \langle h^{\#}(g_1/E),\ldots,h^{\#}(g_n/E)\rangle \in R'_\eta \ .$$

By similar arguments we can show that $h^{\#}_{\mathbb{A}}$ is 1-1 and that functions and constants are preserved.

(b) If h is 1-1 then every $f \in A^I$ can be written as $h \circ g$ for some $g \in A^J$. Hence $h^{\#}$ is onto, and hence an isomorphism. //

---

(1) We employ the diagram convention for writing compositions of functions:

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[rr, bend right, "h\circ g"'] & J \arrow[r, "g"] & A \end{tikzcd} \qquad \text{so} \quad h\circ g(i) = g(h(i))$$

2.4 Corollary

(a) If $E \leq D$ then $\mathbb{A}^J/E$ can be embedded in $\mathbb{A}^I/D$.

(b) If $E \approx D$ then $\mathbb{A}^J/E \cong \mathbb{A}^I/D$.

Proof

This corollary follows from 2.3 once we have proved that for any $\langle I,D\rangle$ if $X \in D$

$$\mathbb{A}^X/D|_X \cong \mathbb{A}^I/D \ .$$

In fact if we define a map

$$\phi : A^X/D|_X \to A^I/D$$

by $\phi(f/D|_X) = k/D$, where $k|_X = f$ and k is arbitrary on $I-X$, it is not hard to show that ϕ is an isomorphism. //

We frequently drop the subscript "$\mathbb{A}$" from "$h^{\#}_{\mathbb{A}}$" when there can be no confusion. We remark that the maps $h^{\#}_{\mathbb{A}}$ are always elementary embeddings. This is clear from 1.1 in the case that $\mathbb{A}$ is a full structure. For the general case note that any structure $\mathbb{A}$ can be expanded to a full structure $\mathbb{A}'$ by addition of new relations, functions, and constants. Then the argument of 2.3 shows that $h^{\#}_{\mathbb{A}} = h^{\#}_{\mathbb{A}'}$ is an embedding of $\mathbb{A}'^J/E$ into $\mathbb{A}'^I/D$, and hence is an elementary embedding with respect to the language $L_{\mathbb{A}'}$. But the language $L_{\mathbb{A}}$ is contained in $L_{\mathbb{A}'}$ so that $h^{\#}_{\mathbb{A}}$ is an elementary embedding with respect to $L_{\mathbb{A}}$. This result can also be proved directly.

To what extent are embeddings of the form $h^{\#}$ typical among embeddings $k : \mathbb{A}^J/E \to \mathbb{A}^I/D$? The following theorem shows

that if $\mathbb{A}$ is large enough and has enough relations and functions listed then, indeed, <u>all</u> embeddings are of this form.

<u>2.5 Theorem</u>

Suppose that $k : \mathbb{A}^J/E \to \mathbb{A}^I/D$ is an embedding. Then if $|A| \geq |J|$ and $\mathbb{A}$ lists all unary relations and unary functions of A then k can be written in the form $h^{\#}$ for some $h : I \to J$.

<u>Proof</u>

Let $f \in A^J$ be a 1-1 function and let $g \in A^I$ be any function such that $k(f/E) = g/D$. Pick a fixed $j_0 \in J$ and define $h : I \longrightarrow J$ by

$$h(i) = \begin{cases} f^{-1}(g(i)) & \text{if } g(i) \in f(J) \\ j_0 & \text{otherwise} \end{cases} .$$

We claim that $h(D) = E$ and that $k = h^{\#}$.

(a) <u>$h(D) = E$</u>

It suffices to prove that $f(E) = g(D)$. Let Y be any subset of A and let P_Y be a corresponding predicate symbol of $L_{\mathbb{A}}$.

Then $\quad \mathbb{A}^J/E \models P_Y(f/E)$ iff $\mathbb{A}^I/D \models P_Y(g/D)$

so $\quad \{j : \mathbb{A} \models P_Y(f(j))\} \in E$

iff $\{i : \mathbb{A} \models P_Y(g(i))\} \in D$

i.e. $\quad \{j : f(j) \in Y\} \in E$ iff $\{i : g(i) \in Y\} \in D$

i.e. $\quad f^{-1}(Y) \in E$ iff $g^{-1}(Y) \in D$

and hence $\quad f(E) = g(D)$.

(b) $\underline{h^{\#} = k}$

By part (a) h is a filter morphism $h : \langle I,D\rangle \to \langle J,E\rangle$ so by 2.3 $h^{\#} : \mathbb{A}^J/E \to \mathbb{A}^I/D$ is an embedding. Clearly we have $h^{\#}(f/E) = k(f/E) = g/D$. If $t \in A^A$ is any unary function of $\mathbb{A}$ and t_1, t_2 are the corresponding functions of $\mathbb{A}^J/E$, $\mathbb{A}^I/D$ respectively we have

$$h^{\#}(t_1(f/D)) = t_2(g/D)$$

$$k(t_1(f/D)) = t_2(g/D)$$

as k and $h^{\#}$ are homomorphisms. Thus $h^{\#}$ and k agree at all members of A^J/E of the form $t_1(f/E)$ for some unary function t_1 of $\mathbb{A}^J/E$. But, in fact, if ℓ/E is any element of A^J/E we can write $\ell = f\circ t$ for some $t \in A^A$ as f is 1-1. Hence $\ell/E = f\circ t/E = t_1(f/E)$. So $h^{\#}$ and k agree at all arguments in their common domain A^J/E. //

Remarks

(i) It is important to observe in connection with 2.5 that even if the embedding k is an isomorphism onto, h may still not be 1-1. (Or even "almost" 1-1.)

(ii) We may call a structure $\mathbb{A}$ that has unary relations for every subset of its underlying set A and unary operations for each function from A to A amongst its relations and operations a <u>full unary structure</u> on A. In particular 2.5 applies to all full structures $\mathbb{A}$.

2.6 Corollary

Let $\mathbb{A}$ be a full unary structure and let $\langle I,D\rangle$, $\langle J,E\rangle$ be such that $|I|$, $|J| \leq |A|$ then

$$\mathbb{A}^I/D \cong \mathbb{A}^J/E \quad \text{iff} \quad D \approx E.$$

Proof

2.4(b) gives us half the result. The converse direction follows from 2.5 and 2.2(b). //

We can also make the observation that if $\mathbb{A}$ is full unary and $|J| \leq |A|$ then any embedding of $\mathbb{A}^J/E$ into an ultrapower $\mathbb{A}^I/D$ is necessarily elementary, and is possible iff $E \leq D$. This is a consequence of 2.5, 2.4(a), and the fact that all the embeddings $h^{\#}_{\mathbb{A}}$ are elementary.

The following example shows that the cardinality restriction in 2.6 cannot be removed.

2.7 Theorem (GCH)

Let $\mathbb{A}$ be any structure on A. Then we can find $\langle I,D\rangle$, $\langle J,E\rangle$

(a) such that $|I| = |J| = |A|^+$

$$\mathbb{A}^I/D \cong \mathbb{A}^J/E$$

$$E \not\approx D$$

(b) such that $|I| = |J| = |A|^+$

$$\mathbb{A}^J/E \prec \mathbb{A}^I/D$$

$$E \not\leq D.$$

Proof (Informal)

Let $\kappa = |I|$. Keisler in [9] and [10] has investigated a class of ultrafilters on I which he terms the κ^+-good ultrafilters. The two results that we shall need about these ultrafilters are the following:

(1) There are 2^{2^κ} κ^+-good ultrafilters on I.

(2) If $\mathbb{A}$ is a structure with $|L_{\mathbb{A}}| < \kappa^+$ and D is a κ^+-good ultrafilter on I then $\mathbb{A}^I/D$ is a κ^+-saturated structure.

(a) As there are only 2^κ functions from I to I each ultrafilter on I is equivalent to at most 2^κ other ultrafilters on I. So by (1) there exist κ^+-good ultrafilters D and E on I such that $D \not\leq E$. If $\mathbb{A}$ is any structure on A then $|L_{\mathbb{A}}| \leq 2^{|A|} < 2^{|A|^+} = \kappa^+$ so that $\mathbb{A}^I/D$ and $\mathbb{A}^I/E$ are κ^+- saturated structures of cardinality κ^+. So $\mathbb{A}^I/D \cong \mathbb{A}^I/E$ by 1.2(a).

(b) Let <I,D> be as in (a). There are 2^κ ultrafilters F on I such that $F \leq D$, but 2^{2^κ} ultrafilters on I. Let E be any ultrafilter on I such that $E \not\leq D$. Then $|\mathbb{A}^I/E| \leq \kappa^+$ and hence $\mathbb{A}^I/E$ can be elementarily embedded in $\mathbb{A}^I/D$, by 1.2(b). //

We mention the following consequence of 2.6 and 2.7: For any full structure A we can find <I,D>, <J,E> such that

$$\mathbb{A}^I/D \cong \mathbb{A}^J/E \quad \text{but} \quad (\mathbb{A}^A)^I/D \not\cong (\mathbb{A}^A)^J/E$$

where $\mathbb{A}^A$ is a full structure on A^A. It is likely that this can be proved without the assumption of the GCH.

Theorem 2.7, in contrast with earlier results, shows up a strong distinction between ultrapowers with index set larger than the base A, and those with index size $\leq |A|$. We note next a result which shows that ultrapowers of this latter kind have a simple characterization.

2.8 Theorem (Adler)

Let $\mathbb{A}$ be a full unary structure on A. Then $\mathbb{A}^I/D$ is generated by a single element iff $\mathbb{A}^I/D \cong \mathbb{A}^J/E$ for some $\langle J,E\rangle$ with $|J| \leq |A|$.

Proof.

(a) <= We can assume $|I| \leq |A|$. Let $f : I \to A$ be any 1-1 function. We claim that f/D generates $\mathbb{A}^I/D$. For let g/D be any member of A^I/D. Then there is a function $t : A \to A$ such that $g = f \circ t$. But as $\mathbb{A}$ is full unary there is a function symbol $\underline{t}$ in $L_{\mathbb{A}}$ corresponding to t. Then if t_1 is the interpretation of $\underline{t}$ in $\mathbb{A}^I/D$ we have
$g/D = f \circ t/D = t_1(f/D)$.
Hence $\{f/D\}$ is a set that generates A^I/D.

(b) => Given $f/D \in A^I/D$, consider the set
$\{g/D : g/D = t_1(f/D) \text{ for some } t \in A^A\} = S$
(using the notation of part (a)). As $\mathbb{A}$ is full unary it is easy to see that S is closed under the functions of $F_{\mathbb{A}}$ and is hence the set generated by $\{f/D\}$. Now assume that f/D generates $\mathbb{A}^I/D$. Then

$S = A^I/D$ and each member of A^I/D can be written in the form

$$t_1(f/D) = f \circ t/D$$

for some $t \in A^A$.

Consider the embedding

$$f^{\#} : \mathbb{A}^A/E \to \mathbb{A}^I/D$$

where $E = f(D)$. By definition (c.f. 2.3) the range of $f^{\#}$ consists of all elements in A^I/D of form $f^{\#}(t/E) = f \circ t/D$ for some $t \in A^A$.

So $f^{\#}$ is in fact onto and hence $\mathbb{A}^A/E \cong \mathbb{A}^I/D$. //

<u>Remark</u>

In any ultrapower $\mathbb{A}^I/D$ if τ_X is a unary predicate symbol of $L_{\mathbb{A}}$ corresponding to $X \subseteq A$ we have

$$\mathbb{A}^I/D \models \tau_X(f/D) \quad \text{iff} \quad X \in f(D)$$

i.e. $f(D)$ is the "ultrafilter of properties" of f/D. Thus if $|I| \leq |A|$ we see by the above that if $\mathbb{A}$ is full unary, $\mathbb{A}^I/D$ is generated by f/D where f is 1-1. In this case $D \approx f(D)$ so that we can interpret D as being equivalent to the ultrafilter of properties of a generating element.

B. Images of Ultrafilters

In this section we collect and prove results about ultrafilters and filter morphisms. In particular we study the collection of images of a given ultrafilter.

2.9 Definitions

(a) $\sigma(D) \overset{\text{def}}{=} \{th(E) : E \leq D\}$

We call $\sigma(D)$ the <u>shadow</u> of D. It is the collection of cardinals α such that D has a uniform ultrafilter on α as an image.

(b) An ultrafilter D is <u>α-descendingly incomplete</u> (α-d.i.) if there exists a sequence $\langle X_\eta : \eta < \alpha\rangle$ of members of D such that $\eta' < \eta < \alpha$ implies $X_{\eta'} \supseteq X_\eta$ and such that $\cap\{X_\eta : \eta < \alpha\} = 0$. Such a sequence we will call an <u>α-chain</u> in D.

Descendingly incomplete ultrafilters were first studied by C. C. Chang in [6]. All ω-incomplete ultrafilters are ω-d.i., in fact an ultrafilter D is α-incomplete iff D is ω-d.i. for some β such that $\omega \leq \beta \leq \alpha$. Here the proof in the "if" direction is clear. To see "only if" suppose that D is α-incomplete for some $\alpha \geq \omega$. Then there is a collection $\{X_\eta : \eta < \alpha\} \subseteq D$ such that $\cap\{X_\eta : \eta < \alpha\} \notin D$. Define sets Y inductively so that $Y_0 = X_0$, $Y_{\eta+1} = Y_\eta \cap X_\eta$ and $Y_\lambda = \cap\{Y_\eta : \eta < \lambda\}$ if λ is a limit ordinal. Let μ be the least ordinal such that $Y_\mu \notin D$. Then it is clear that μ is a limit ordinal $\leq \alpha$. So for some cardinal β such that

$\omega \leq \beta \leq \alpha$ there is a sequence of ordinals $\langle \eta_\xi : \xi < \beta \rangle$ that is cofinal with μ. It is easy to see that $\langle (Y_{\eta_\xi} - Y_\mu) : \xi < \beta \rangle$ is a β-chain in D. Thus if an ultrafilter D is α-incomplete a knowledge of what chains exist tells us something about "in what way" D is incomplete. We also make the observation here that if D is α-d.i. and β is cofinal with α then D is α-d.i. Thus D is α-d.i. iff D is $cf(\alpha)$-d.i.

The next theorem shows the connection between the shadow $\alpha(D)$ of D and the set of cardinals α for which D is α-d.i.

2.10 Theorem

Let D be an ultrafilter on I, and α a regular infinite cardinal. Then $\alpha \in \sigma(D)$ iff D is α-d.i.

Proof

(i) Assume $\alpha \in \sigma(D)$. Then there is a pair $\langle J,E \rangle$ such that $E \leq D$ and $th(E) = \alpha$. We can assume that $|J| = \alpha$ and that $E = f(D)$ for some $f : I \to J$. Well-order J so that $J = \{j_\eta : \eta < \alpha\}$ and take X to be the set $\{j_\xi : \nu < \xi < \alpha\}$
Then $\langle X_\eta : \eta < \alpha \rangle$ is an α-chain in E. But then $\langle f^{-1}(X_\eta) : \eta < \alpha \rangle$ is an α-chain in D, so that D is α-d.i.

(ii) Assume D is α-d.i., and let $\langle X_\eta : \eta < \alpha \rangle$ be an α-chain in D. Define $f : I \to \alpha$ by taking $f(i)$ to be the least ordinal ξ such that $i \notin X_\xi$. Let $E = f(D)$. We claim that E is uniform (i.e. $th(E) = \alpha$) if α is regular. Suppose that $Y \subseteq \alpha$ and $|Y| < \alpha$. Then by regularity of α there is an $\eta < \alpha$ such that $Y \subseteq \eta$. But then

$$f^{-1}(Y) \subseteq f^{-1}(\eta) \subseteq I - X_{\eta+1}$$

so that $f^{-1}(Y) \notin D$ and hence $Y \notin E$. Hence E is uniform, and thus $\alpha \in \sigma(D)$. //

2.11 Corollary

For all α, all D,

D is α-d.i. iff $cf(\alpha) \in \sigma(D)$. //

Notice that (i) does not use the regularity of α so that for every α, $\alpha \in \sigma(D)$ implies D is α-d.i. Also the proof of (i) shows that if E is a uniform ultrafilter on J then $|J| \in \sigma(E)$ and E is $|J|$-d.i.

For all $\langle I,D\rangle$ we have by definition that $\sigma(D) \subseteq \{\alpha : \alpha \leq |I|\}$. Of course if $1 < n < \omega$ then $n \notin \sigma(D)$ as there are no uniform ultrafilters on finite sets with more than one element. Apart from this restraint $\sigma(D)$ can be large.

2.12 Theorem

For any set I there is an ultrafilter D such that $\omega \leq \alpha \leq |I|$ implies $\alpha \in \sigma(D)$.

Proof

We shall define a sequence $\langle D_\eta : \eta \in On\rangle$ of ultrafilters with the following properties:

i) D_η is a uniform ultrafilter on ω_η

ii) $\eta' \leq \eta$ implies $D_{\eta'} \leq D_\eta$

The construction is by induction. Let D_0 be any uniform ultrafilter on ω and suppose that the ultrafilters D_η have been constructed for all $\eta < \lambda$.

Case 1. $\lambda = \beta+1$

Let F be any uniform ultrafilter on ω_λ and consider the ultrafilter $D_\beta \times F$ on $\omega_\beta \times \omega_\lambda$. It is easy to see that if $E \leq D_\beta$ then $E \leq D_\beta \times F$, so if we define D on ω_λ by $D_\lambda = \phi(D_\beta \times F)$ where $\phi : \omega_\beta \times \omega_\lambda \to \omega_\lambda$ is 1-1 onto then $D_\lambda \cong D_\beta \times F$ and D_λ satisfies (i) and (ii) (with $\eta = \lambda$).

Case 2. λ is a limit ordinal.

Let F be an ultrafilter on λ which extends the collection of sets $\{C_\xi : \xi < \lambda\}$ where $C_\xi = \{\eta : \xi \leq \eta < \lambda\}$. Let D'_η be an ultrafilter on $\omega_\eta \times \{\eta\} = I_\eta$ isomorphic to D_η and consider the ultrafilter $D' = \Sigma_F\{D'_\eta : \eta < \lambda\}$ on $\cup\{I_\eta : \eta < \lambda\} = I'$. We claim $D_\eta \leq D'$ for all $\eta < \lambda$. For suppose $\eta < \lambda$, then $D_\eta \leq D_\xi$ for all $\xi < \lambda$ such that $\xi \geq \eta$.

So by 2.2(a) we have $D'_\xi \leq D'_\xi$ for $\eta \leq \xi < \lambda$. For each such ξ let $f_\xi : I_\xi \to I_\eta$ be such that $f_\xi(D'_\xi) = D'_\eta$, and let $f : I' \to I_\eta$ be such that $f|_{I_\xi} = f_\xi$. (It is immaterial how f is defined on $\cup\{I'_\zeta : \zeta < \eta\}$.) Then $f(D') = D'_\eta$, for if $X \subseteq I_\eta$ then $f^{-1}(X) = \cup\{f^{-1}(X) : \xi < \lambda\}$. If $X \in D'_\eta$ then $f_\xi^{-1}(x) \in D'_\xi$ for all ξ such that $\eta \leq \xi < \lambda$ so that $f^{-1}(X) \in D'$, and if $X \notin D'_\eta$ it follows in the same way that $f^{-1}(X) \notin D'$. All this is by definition of D' and the fact that $C_\xi \in F$.

Now $|I'| = \omega_\lambda$ and as $\eta < \lambda$ implies $D_\eta \leq D'$ we have that $th(D') \geq \omega_\eta$ for all $\eta < \lambda$, and so $th(D') = \omega_\lambda$.

Finally we define D_λ to be an ultrafilter isomorphic to D' on ω_λ. It is clear that D_λ has the required properties. //

If an ultrafilter D is regular then D is α-d.i. for all α such that $\omega \leq \alpha \leq |I|$. This follows from lemma 2.2 of [12] which essentially observes that if λ is regular and $\langle X_\eta : \eta<\lambda\rangle$ is a κ-covering of I by elements of D where $\kappa \leq \lambda$, then $\langle Y_\eta : \eta < \lambda\rangle$ is a λ-chain in D where

$$Y_\eta = \cup\{X_\xi : \eta < \xi < \lambda\}.$$

This observation can be easily checked by the reader. Note that this does not render 2.12 trivial as it merely proves that if D is regular then $\alpha \in \sigma(D)$ for all <u>regular</u> α such that $\omega \leq \alpha \leq |I|$.

It is possible to relate the descending incompleteness of a product ultrafilter to the descending incompleteness of its factors. In fact we have the following result.

<u>2.13 Theorem</u>

Let D and E be any ultrafilters and β any cardinal. Then $D\times E$ is β-d.i. iff either D or E is β-d.i.

<u>Proof</u>

Suppose $\langle X_\eta : \eta < \beta\rangle$ is a β-chain in D: then $\langle X_\eta \times J : \eta < \beta\rangle$ is a β-chain in $D\times E$. Similarly if $\langle Y_\eta : \eta < \beta\rangle$ is a β-chain in E then $\langle I \times Y_\eta : \eta < \beta\rangle$ is a β-chain in $D\times E$.

Conversely, suppose that $\langle Z_\eta : \eta < \beta\rangle$ is a β-chain in $D\times E$, then for all $\eta < \beta$

$$U_\eta = \{j : \{i : \langle i,j\rangle \in Z_\eta\} \in D\} \in E.$$

If $U = \cap\{U_\eta : \eta < \beta\} = 0$ then we have that $\langle U_\eta : \eta < \beta\rangle$ is a β-chain in E. Otherwise we have that for all $j \in U$

$$V_\eta^j = \{i : \langle i,j\rangle \in Z_\eta\} \in D \qquad (\eta < \beta).$$

Pick any $j \in U$, then $\langle V_\eta^j : \eta < \beta\rangle$ is clearly decreasing and if $i \in \cap\{V_\eta^j : \eta < \beta\}$ then $\langle i,j\rangle \in \cap\{Z_\eta : \eta < \beta\} = 0$. This contradiction shows that $\cap\{V_\eta^j : \eta < \beta\} = 0$ and hence that $\langle V_\eta^j : \eta < \beta\rangle$ is a β-chain in D. //

By the same methods we can show that $\Sigma_E\langle D_j : j \in J\rangle$ is β-d.i. iff either E is β-d.i. or D_j is β-d.i. for almost all $j \in J$ (mod E).

We define the <u>spread</u> of an element f/D of an ultrapower A^I/D by

$$\text{spread } (f/D) = \min\{|\text{range}(f')| : f' \sim_D f\}$$

With this definition we can generalise 1.11. Note that the canonical embedding $\iota : A \rightarrow A^I/D$ is onto iff each element of A^I/D has spread 1.

<u>2.14 Theorem</u>

The ultrapower A^I/D contains an element of spread $\beta \leq |A|$ iff $\beta \in \sigma(D)$.

Proof

Suppose that $f/D \in A^I/D$ and spread $(f/D) = \beta$. We can assume that $|\text{range}(f)| = \beta$. Let $E = f(D)$. Clearly $\text{th}(E) \leq \beta$. If $\text{th}(E) < \beta$ there is an $X \in E$ such that $|X| < \beta$, $f^{-1}(X) \in D$. Pick $x_0 \in X$ and define $f' : I \to A$ by

$$f'(i) = \begin{cases} f(i) & \text{if } i \in f^{-1}(X) \\ x_0 & \text{otherwise} \end{cases}$$

Then $f' \sim_D f$ but $|\text{range}(f')| < \beta$ a contradiction, so that $\text{th}(E) = \beta$.

Conversely suppose $\beta \in \sigma(D)$. Then if $\beta \leq |A|$ we can find E on A and $g : I \to A$ such that $E = g(D)$ and $\text{th}(E) = \beta$. We claim that $\text{spread}(g/D) = \beta$:

(i) spread $(g/D) \geq \beta$ as if $g^* \sim_D g$ and $|\text{range}(g^*)| < \beta$ then $g^*(D) = g(D) = E$ and so $\text{th}(E) < \beta$.

(ii) spread $(g/D) \geq \beta$ for, as in the first part of the proof, we can find $g' \sim_D g$ with $|\text{range}(g')| \leq \beta$. //

It is natural to ask what sets of cardinals can be expressed as shadows of ultrafilters. Our result 2.12 shows that any set of the form $\{\beta : \beta = 1 \text{ or } \omega \leq \beta \leq \alpha\}$ can be so expressed. We also must require of any set S that is a shadow that $\alpha \in S$ implies $\text{cf}(\alpha) \in S$. Another condition on shadows is a consequence of the following result of Chang [6].

Chang's Theorem

Let D be any ultrafilter.

(a) If D is κ^+-d.i. and κ is regular, then D is κ-d.i.

(b) If D is κ^+-d.i. and $\mu = cf(\kappa) < \kappa$ then either D is μ-d.i. or for some $\beta < \kappa$ D is γ-d.i. for all regular γ such that $\beta < \gamma < \kappa$.

Chang's proof of this result uses the G.C.H. but subsequent proofs of the result by Kunen and Prikry in [12] are free of this hypothesis. We will now give the proof of Prikry that appeared in [12] of a result slightly stronger than Chang's Theorem.

First we need a combinatorial lemma from Ulam [17].

2.15 Lemma

Let λ be any infinite cardinal. Then there is an array

$$\langle A_\sigma^\eta : \sigma < \lambda, \quad \eta < \lambda^+ \rangle$$

of subsets of λ^+ with the following properties:

(i) $\sigma \neq \sigma'$ implies $A_\sigma^\eta \cap A_{\sigma'}^\eta = 0$

(ii) $\eta \neq \eta'$ implies $A_\sigma^\eta \cap A_\sigma^{\eta'} = 0$

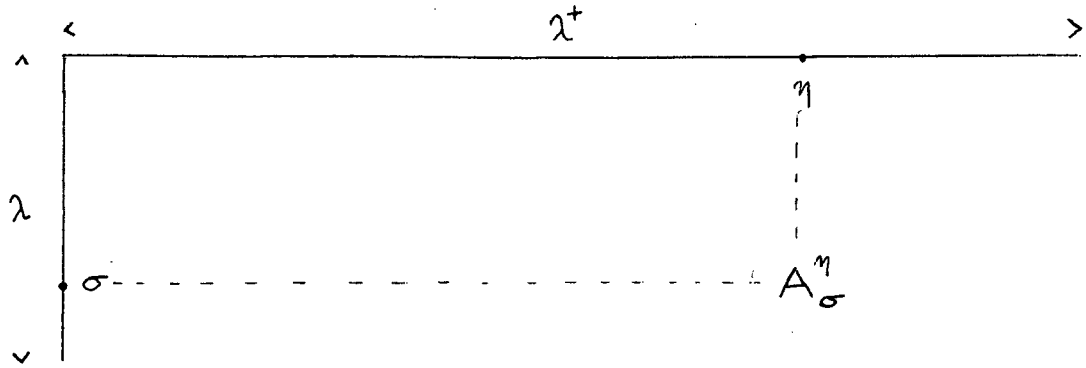
(iii) $|\lambda^+ - \{A_\rho^\eta : \rho < \lambda\}| \leq \lambda$

for every $\eta < \lambda^+$.

Proof

For every $\xi < \lambda^+$ let $\langle \alpha_\rho^\xi : \rho < \xi \rangle$ be a sequence of distinct elements of λ. Take $A_\sigma^\eta = \{\xi \in \lambda^+ : \alpha_\eta^\xi = \sigma\}$, then it is clear that (i) and (ii) are satisfied. To see that (iii) holds notice that if $\xi \in \lambda^+ - \cup\{A_\rho^\eta : \rho < \lambda\}$ then α_η^ξ is not defined, so that $\xi \leq \eta$. Hence $|\lambda^+ - \cup\{A_\rho^\eta : \rho < \lambda\}| \leq |\eta| \leq \lambda$. //

We can regard this array as a $\lambda\times\lambda^+$ matrix with set entries.



Then each row and each column forms a pairwise disjoint collection of subsets of λ^+, and the columns, in fact, partition λ^+ except for a set of cardinality at most λ. These and similar arrays are called <u>Ulam matrices</u>.

<u>2.16 Theorem</u> (Prikry)

Suppose D is λ^+-d.i., then either

(i) D is $cf(\lambda)$-d.i.

or (ii) there is a regular cardinal $\alpha \le \lambda$ such that D is (α,λ^+)-regular.

<u>Proof</u>

Let $\langle A^\eta_\sigma : \sigma < \lambda,\ \eta < \lambda^+\rangle$ be the Ulam matrix of 2.15. Set $B^\eta_\sigma = \cup\{A^\eta_\rho : \rho < \sigma\}$.

We claim that $S \subseteq \lambda^+$, $\sigma < \lambda$ and $|S| > |\sigma|$ implies

$$(*) \qquad \cap\{B^\xi_\sigma : \xi \in S\} = 0$$

For suppose that $\tau \in \cap\{B^\xi_\sigma : \xi \in S\}$. Then for each $\xi \in S$ there is a $\rho_\xi < \sigma$ such that $\tau \in A^\xi_{\rho_\xi}$.

Since $|S| > |\sigma|$ there are $\xi, \eta \in S$ such that $\xi \neq \eta$ and $\rho_\xi = \rho_\eta$ so that $\tau \in A^\xi_{\rho_\xi} \cap A^\eta_{\rho_\xi}$ in contradiction to 2.15(ii). Thus the claim (*) is verified.

Let D be λ^+-d.i., then since $\lambda^+ \in \sigma(D)$ D has an image $D' = f(D)$ on λ^+ that is uniform. Since any α-chain or α-covering we may find in D' can be lifted via f^{-1} to D we can start off by assuming that D is a uniform ultrafilter on λ^+. There are two cases.

Case I

For some $\eta < \lambda^+$ all $B^\eta_\sigma \notin D$ $(\sigma < \lambda)$. If $Y = \lambda^+ - \cup\{A^\eta_\rho : \rho < \lambda\} = \lambda^+ - \cup\{B^\eta_\sigma : \sigma < \lambda\}$ then $Y \notin D$ as $|Y| \leq \lambda$. If we set

$$X_\sigma = \lambda^+ - (Y \cup B^\eta_\sigma)$$

then $\langle X_\sigma : \sigma < \lambda\rangle$ is a λ-chain in D and hence D is λ-d.i. and consequently $cf(\lambda)$-d.i.

Case II

Case I fails. Then for each $\eta < \lambda^+$ there is a $\sigma_\eta < \lambda$ such that $B^\eta_{\sigma_\eta} \in D$. Then there is a $T \subseteq \lambda^+$ and a $\sigma < \lambda$ such that $|T| = \lambda^+$ and $\sigma_\eta = \sigma$ for all $\eta \in T$.

Then by the claim (*) proved above $\langle B^\eta_\sigma : \eta \in T\rangle$ is a $|\sigma|^+$-covering of λ^+ by members of D. So D is (α,λ^+)-regular where $\alpha = |\sigma|^+$. //

In view of the remarks following the result 2.12 on the construction of chains from coverings it is clear that 2.16 implies Chang's Theorem. We will be proving additional results on the form of shadows $\sigma(D)$ later in Chapter 4.

C. The Ultrapower as an Ordered Set

We consider now the situation in which the structure $\mathbb{A}$ has no operations or constants but merely a single binary relation R_0, where R_0 is a linear ordering relation. We shall write $<$ for R_0 and $\langle A, < \rangle$ for $\mathbb{A}$. The relation $<$ on A induces a relation $<'$ on A^I/D given by

$$f/D <' g/D \text{ iff } f(i) < g(i) \text{ for almost all } i \pmod D.$$

We can see either directly or from Los' Theorem that $<'$ is a linear ordering. If A is infinite and $<$ is a well-ordering of A it is not hard to see that $<'$ is a well-ordering iff D is ω-complete. This fact is proved in [4], p.134.

The following theorem was suggested by ideas of Keisler [8], p.405, and shows the connection between $<'$ and the descending in completeness properties of D. Chang quotes similar results of Keisler in his paper [6]. Recall that $\iota : A \to A^I/D$ is the canonical embedding.

2.17 Theorem

Let A be well-ordered by $<$ in type $|A| = \alpha$. Then $\iota(A)$ is confinal in A^I/D (w.r.t. $<'$) iff D is not α-d.i.

Proof

(a) $\leftarrow$. Assume that $\iota(A)$ is bounded in A^I/D by some element f/D. We will show that D is α-d.i. Let $\{a_\eta : \eta < \alpha\}$ be the elements of A as well-ordered by $<$. Then $\iota(A) = \{g_\eta/D : \eta<\alpha\}$ where $g_\eta(i) = a_\eta$ for all $i \in I$, $\eta < \alpha$. Now for all $\eta < \alpha$ we have $g_\eta/D <' f/D$.

So $X_\eta = \{i : a_\eta < f(i)\} \in D$ and it is clear that $\langle X_\eta : \eta < \alpha\rangle$ is an α-chain in D.

(b) $\rightarrow$. Suppose that $\langle Y_\eta : \eta < \alpha\rangle$ is a chain in D. Define $f : I \rightarrow A$ so that $f(i) = a_{\eta+1}$ whenever $i \in Y_\eta - Y_{\eta+1}$.

It is easy to see that f/D is an upper bound for $\iota(A)$ in A^I/D. //

We will often drop some formality and regard results like the above as pertaining to ultrapowers of cardinals. Thus we will say: "α is confinal in α^I/D iff D is not α-d.i." (The embedding and orderings being understood.)

Each member f/D of α^I/D determines an initial segment

$$[f/D] = \{g/D\epsilon\alpha^I/D : g/D<' f/D\}.$$

There is a natural map of the ultraproduct $\Pi\langle\eta_i : i\epsilon I\rangle/D$ into α^I/D induced by the inclusion map

$$\mathcal{J} : \Pi\langle\eta_i : i \in I\rangle \rightarrow \alpha^I$$

where $\eta_i = f(i)$.

It is easy to see that $\mathcal{J}$ is well-defined, 1-1, and that its range is [f/D]. Thus we can pass freely between ultraproducts and initial segments of ultrapowers. Notice, though, that one initial segment corresponds to many (closely related) ultraproducts.

We now prove a slightly less trivial result which connects the regularity properties of an ultrafilter to the order properties of an ultrapower.

2.18 Theorem

Let λ be a regular cardinal, D a uniform ultrafilter on λ^+.

Then every set of $\leq \lambda^+$ elements of λ^{λ^+}/D is bounded above in λ^{λ^+}/D iff D is (λ,λ^+)-regular.

Proof

We will be using the Ulam matrix $\langle A_\sigma^\eta : \sigma<\lambda, \eta<\lambda^+\rangle$ of 2.15 and the sets B_σ^η of 2.16. We employ a case division slightly different from 2.16.

Case (I) of 2.16 asserts that there exists a column, say the ηth column of the Ulam matrix for which $B_\sigma^\eta \notin D$ for all $\sigma < \lambda$. In fact we will assume for our Case I* that there exist λ^+ such columns, i.e. there is a subset $S \subseteq \lambda^+$ such that $|S| = \lambda^+$ and $B_\sigma^\eta \notin D$ for all $\eta \in S$, $\sigma < \lambda$.

We can still recover the conclusion of Case II from Case II* = not Case I*. For if (II*) applies there are at most λ columns with the peculiarity described above. These can be deleted from the matrix to form a new matrix to which the argument of (II) applies.

(i) $\rightarrow$. Assume that any set of λ^+ members of λ^{λ^+}/D is bounded above. If Case II* holds we can conclude as in 2.16 that D is (α^+,λ^+)-regular for some $\alpha < \lambda$ and so that D is (λ,λ^+)-regular. If Case I* holds there is $S \subseteq \lambda^+$ such that $|S| = \lambda^+$ and $B^\eta_\sigma \notin D$ for all $\eta \in S$ and all $\sigma < \lambda$. We define functions $\{f_\xi : \xi \in S\} \subseteq \lambda^{\lambda^+}$ by

$$f_\xi(i) = \begin{cases} \rho & \text{if } i \in A^\xi_\rho \\ 0 & \text{if } i \in \lambda^+ - \cup\{A^\xi_\rho \quad \rho < \lambda\} \end{cases}$$

Let g/D be an upper bound in λ^{λ^+}/D for the set $\{f\ /D : \xi \in S\}$. Then we have

$$X_\xi = \{i : 0 < f_\xi(i) < g(i)\} \in D.$$

But if $\rho = f_\xi(i) = f_{\xi'}(i) > 0$ then $i \in A^\xi_\rho \cap A^{\xi'}_\rho$ and so $\xi = \xi'$ by 2.15(ii). Hence no two f's take the same positive value at i, for any $i \in \lambda^+$.

Hence

$$|\{\xi : i \in X_\xi\}| \leq |g(i)| < \lambda$$

so that $\{X_\xi : \xi \in S\}$ is a λ-covering of λ^+ by members of D, and D is (λ,λ^+)-regular.

(ii) $\leftarrow$. Assume that D is (λ,λ^+)-regular and that $\{f_\eta/D : \eta<\lambda^+\}$ is any set of λ^+ elements of λ^{λ^+}/D. Let $\{X_\eta : \eta < \lambda^+\}$ be a λ-covering of λ^+ by members of D.

Define

$$g_\eta(i) = \begin{cases} f_\eta(i) & i \in X_\eta \\ 0 & \text{otherwise} \end{cases}$$

then $g_\eta \sim_D f_\eta$ and as $|\{\eta : i \in X_\eta\}| < \lambda$ and λ is regular we have that

$$\sup \{g_\eta(i) : \eta < \lambda^+\}$$

exists, and is less than λ. We denote this sup by $g(i)$.

Then we can show easily that g/D is an upper bound for $\{g\ /D : \eta<\lambda\ \} = \{f_\eta/D : \eta<\lambda^+\}$. //

<u>Remarks</u>

(A) We used the assumption that λ was regular only in section (ii) of the proof. An inspection of this section shows that for all λ we have the implications:

(1) D is $(cf(\lambda),\lambda^+)$-regular

implies

(2) Every subset of λ^{λ^+}/D of power $\leq \lambda^+$ is bounded above in λ^{λ^+}/D

implies

(3) D is (λ,λ^+)-regular.

(B) We have shown in 2.7 that it may be true that

$$\mathbb{A}^{\lambda^+}/D \cong \mathbb{A}^{\lambda^+}/E \quad ,$$

where $\mathbb{A} = \langle A, < \rangle$ and $|A| = \lambda$, but that $D \not\cong E$. Notwithstanding this negative result we see by 2.18 that D and E have this much in common: D is (λ,λ^+)-regular iff E is.

D. Shadow and Cardinality, I

Many cardinality questions can be settled when the shadow $\sigma(D)$ of an ultrafilter D is known. For example if D is an ultrafilter on I and $cf(\gamma) \in \sigma(D)$ we have

$$(*) \qquad |\alpha^I/D|^\gamma \leq |(\alpha^{(\gamma)})^I/D| \quad .$$

This follows from Keisler's Lemma (1.16) as follows:

Let $\chi = \{X_\eta : \eta < \gamma\}$ be a γ-chain in D. Then

$$\gamma_i = |\{\eta : i \in X_\eta\}| < \gamma$$

for all $i \in I$, so that χ is a γ-covering and we may apply 1.16 with $\kappa = \lambda = \gamma$.

Thus

$$\begin{aligned} |\alpha^I/D|^\gamma &\leq |\Pi\langle\alpha^{\gamma_i} : i \in I\rangle / D| \\ &\leq |\Pi\langle\alpha^{(\gamma)} : i \in I\rangle / D| \\ &= |(\alpha^{(\gamma)})^I/D| \end{aligned}$$

as $\alpha^{(\gamma)} = \sup\{\alpha^\mu : \mu < \gamma\} \geq \alpha^{\gamma_i}$ for all $i \in I$. So (*) holds. Putting $\gamma = \omega$ we see that (*) generalises 1.14 (f). If G.C.H. holds we have the following consequence of (*):

2.19 Theorem (GCH)

Let D be an ultrafilter on I such that $cf(\alpha) \in \sigma(D)$. Then if $|A| = \alpha$ we have

(i) $|A^I/D| > |A|$

(ii) $cf(|A^I/D|) > |A|$.

Proof

The GCH implies that $\alpha^{(\alpha)} = \alpha$ so that if we take $\gamma = \alpha$ in (*) we have that

$$|\alpha^I/D|^\alpha = |\alpha^I/D|.$$

But it is well-known from set theory that under GCH

$$\beta^\alpha = \beta \quad \text{iff} \quad \alpha < cf(\beta) .$$

So that we can conclude

ii) $cf(|A^I/D|) > |A|$, from which i) follows. //

However we can prove a fragment of this result without the use of GCH by the following argument:

2.20 Theorem

Let D be an ultrafilter on I. Then $|A| \in \sigma(D)$ implies $|A^I/D| > |A|$.

Proof

If $|A| \in \sigma(D)$ then D has a uniform image E on A. Suppose $E = h(D)$ for some $h : I \to A$. But then

$$h^\# : A^A/E \to A^I/D$$

is 1-1, and we know by 1.14(b) that $|A^A/E| > |A|$. Consequently $|A^I/D| > |A|$. //

Later we shall prove a converse to this theorem in the case where $|A|$ is regular. The proof of the converse given employs GCH.

E. Categorical Formulation

Here we indicate briefly an interpretation of the main ideas of this chapter in terms of category theory.

(1) The category PAIRS is the category with objects all ultrafilter pairs $\langle I,D\rangle$ and morphisms the filter morphisms $f : \langle I,D\rangle \to \langle J,E\rangle$, i.e. maps $f : I \to J$ such that $f(D) = E$.

(2) For any relational structure $\mathbb{A}$ we have the category MODELS OF $\mathbb{A}$ whose objects are all relational structures $\mathbb{B}$ of the same type as $\mathbb{A}$ and such that $\mathbb{B} \equiv \mathbb{A}$, and whose morphisms are the elementary embeddings $\phi : \mathbb{B}_1 \to \mathbb{B}_2$.

(3) The category ULTRAPOWERS of $\mathbb{A}$ is the full subcategory of MODELS OF $\mathbb{A}$ determined by the ultrapowers of $\mathbb{A}$. For each $\mathbb{A}$ we have a contravarient functor

$$F_{\mathbb{A}} : \text{PAIRS} \to \text{ULTRAPOWERS OF } \mathbb{A}$$

such that if $\langle I,D\rangle$ is an object of PAIRS then

$$F_{\mathbb{A}}(\langle I,D\rangle) = \mathbb{A}^I/D$$

and if $h \in \text{Hom}(\langle I,D\rangle, \langle J,E\rangle)$ then

$$F_{\mathbb{A}}(h) = h^{\#}_{\mathbb{A}} \in \text{Hom}(\mathbb{A}^J/E, \mathbb{A}^I/D).$$

(4) Let PAIRS* be the category with the same objects as PAIRS but where the morphisms are given by

$$h/D \in \text{Hom}^*(\langle I,D\rangle,\langle J,E\rangle) \text{ iff } h \in \text{Hom}(\langle I,D\rangle,\langle J,E\rangle)$$

i.e. the morphisms are equivalence classes of the morphisms of PAIRS. The functor

$$F^*_{\mathbb{A}} : \text{PAIRS}^* \to \text{ULTRAPOWERS OF } \mathbb{A}$$

is defined in the obvious manner.

(5) For each set I, we define I-PAIRS* and I-ULTRAPOWERS OF A to be the full subcategories of PAIRS* and ULTRAFILTERS OF $\mathbb{A}$ formed by looking only at ultrafilters on the set I. Then these are small categories and it can be shown that $F^*_{\mathbb{A}}$ restricts to a functor from I-PAIRS* to I-ULTRAPOWERS OF $\mathbb{A}$ that is an isomorphism of categories when $|I| \leq |A|$ and $\mathbb{A}$ is a full unary structure.

Note on Authorship in Chapter 2

Much of Section A is well-known, in particular 2.1 which appears in Booth [5]. Theorem 2.2 is also in [5], but here the generalisation to uncountable index sets is slightly less trivial. Theorem 2.5 is a joint result of A. Adler and the writer. Theorem 2.7 is a result of the writer and disproves an unpublished conjecture of A. Adler. The remaining results of Section A are largely "housekeeping".

As far as the writer can ascertain the notions of "shadow" and "spread" are new, although similar ideas have certainly been touched upon by some authors in the course of a proof.

Apart from the results of Chang, Ulam, and Prikry, the results of Section B are the writer's, as are the results 2.18 and 2.20 of Sections C and D.

All results in Chapters 3 and 4 where no explicit ascription is given, are due to the writer.

CHAPTER 3

REDUCTION OF INDEX SIZE

In Chapter 2 we have already noted a connection between the structural properties of ultrapowers $\mathbb{A}^I/D$ and the cardinality of the index set I. For example we have the theorem that if $\mathbb{A}$ is a full structure $\mathbb{A}^I/D$ is generated by a single element iff $\mathbb{A}^I/D \cong \mathbb{A}^J/E$ for some $\langle J,E\rangle$ such that $|J| \leq |A|$. In fact, if we examine the proof of this result, we see that this isomorphism arises in a particular way: We have a morphism $h : \langle I,D\rangle \to \langle J,E\rangle$ such that the induced embedding $h^{\#} : \mathbb{A}^J/E \to \mathbb{A}^I/D$ turns out to be onto. In this chapter we will prove various results involving the construction of isomorphisms of this kind. We will show, roughly, that if an ultrapower $\mathbb{A}^I/D$ is "small" compared with I then an isomorphic ultrapower $\mathbb{A}^J/E$ exists with $|J| < |I|$. If the ultrafilter D is uniform on I one might expect that this situation cannot arise - however we will show in Chapter 4 that this can happen if $|I|$ is measurable.

A. When $h^{\#}$ is onto

If f is a function with domain I we let Π_f be the equivalence relation on I given by $\langle i,i'\rangle \in \Pi_f$ iff $i\Pi_f i'$ iff $f(i) = f(i')$. We identify the equivalence relation Π_f with the

partition of I if induces. Thus we speak of the partition $\Pi_f = \{f^{-1}(j) : j \in \text{range}(f)\}$.

3.1 Lemma (A. Adler)

Let $h : \langle I,D\rangle \to \langle J,E\rangle$ be a filter morphism. The following are equivalent conditions on $\mathbb{A}$:

(a) $h^{\#}_{\mathbb{A}}$ is onto.

(b) for each $f \in A^I$ there is an $f' \in A^I$ such that $f \sim_D f'$ and f' is constant on each cell* of Π_h.

(c) if $I_j = h^{-1}(j)$ for each $j \in J$ and $\{I_j^\eta : \eta < \alpha\}$ is any partition of I_j $(j \in J)$, where $\alpha \leq |A|$, then there is a sequence $\langle \eta_j : j \in J\rangle$ such that

$$\cup\{I_j^{\eta_j} : j \in J\} \in D.$$

Proof

As $\text{range}(h^{\#}) = \{h\circ g/D : g\in A^J\}$ and a function from I to A is constant on each cell of Π_h iff it can be written as $h\circ g$ for some $g\in A^I$ it is clear that (a) and (b) are equivalent.

Now well-order A as $\{a_\eta : \eta < |A|\}$. If (b) holds and

$$\{I_j^\eta : \eta < \alpha\} \qquad (j \in J)$$

is a system of partitions as in (c), let $f : I \to A$ be defined by

$$f(i) = a_\eta \qquad \text{if} \quad i \in I_j^\eta$$

---

*We will speak of the members of partitions as cells.

for some $j \in J$.

Let f' be a function constant on each cell of Π_h and such that $f' \sim_D f$. For each $j \in J$ let $K_j = \{i : f(i) = f'(i)\} \cap I_j$. Then it is easy to see that either $K_j = 0$ or $K_j = I_j^{\xi}$ for some $\xi < \alpha$. If $K_j = I_j^{\eta_j}$, otherwise set $\eta_j = 0$. Then

$$\cup\{I_j^{\eta_j} : j \in J\} \supseteq \{i : f(i) = f'(i)\} \in D$$

and so (c) holds. This argument reverses without great difficulty to show that (c) implies (b). //

3.2 Theorem (A. Adler)

Let $\mathbb{A}^I/D = \mathbb{B}$ be any ultrapower of A. Then there is an image $E = h(D)$ of D on the set $J = A^B$ such that $h^{\#} : \mathbb{A}^J/E \to \mathbb{A}^I/D$ is an isomorphism.

Proof

Select a representative f_b from each $\sim_D$-class $b \in B$. Define a partition Π of I by setting $i\Pi i'$ iff $f_b(i) = f_b(i')$ for all $b \in B$. Then Π has at most A^B cells. In fact $\Pi = \Pi_h$ where $h : I \to A^B$ is given by

$$h(i)(b) = f_b(i) \qquad (i \in I, \ b \in B).$$

If $f \in A^I$ then for some $b \in B$ (in fact for $b = f/D$) we have $f \sim_D f_b$. But f_b is clearly constant on the cells of Π_h so the result follows by 3.1. //

The result 3.2 tells us that, up to isomorphism, an ultrapower $\mathbb{B}$ of $\mathbb{A}$ can always be taken to have index size less than or equal to $|A^B|$. If $|B|$ is regular we can do better (with a little help from the GCH).

3.3 Theorem (GCH)

Let $\mathbb{A}^I/D = \mathbb{B}$ be an ultrapower of $\mathbb{A}$. If $|B|$ is regular then there is an image $E = h(E)$ of D on a set J such that $|J| \leq |B|$ and $h^{\#} : \mathbb{A}^J/E \to \mathbb{A}^I/D$ is an isomorphism.

Proof

By 3.2 there is an image $E_1 = h_1(D)$ of D on $J_1 = A^B$ such that $h_1^{\#}$ is an isomorphism.

(1) If E_1 is not uniform we can find $J \in E_1$ such that $|J| \leq |B|$ because the GCH and the fact that $|B| \geq |A|$ imply that $|J_1| = |A^B| = |B|^+$. Then if $h_2 : J_1 \to J$ is any map that extends the identity on J we have $h_2(E_1) = E_1|_J$.

(2) Consider, now, the case that E_1 is uniform on J_1. Then, by the observation following 2.11, we have that E_1 is $|B|^+$-d.i., and hence by Chang's Theorem, that E_1 is $|B|$-d.i.

Let $\langle X_\eta : \eta < |B| \rangle$ be a $|B|$-chain in E_1 and let $\langle f_\eta/D : \eta < |B| \rangle$ be a well-ordering of A^{J_1}/E_1. Select $a \in A$.

For each $\eta < |B|$ define $g_\eta \sim_D f_\eta$ by

$$g_\eta(i) = \begin{cases} f_\eta(i) & \text{if} \quad i \in X_\eta \\ a & \text{if} \quad i \in J_1 - X_\eta \end{cases}$$

The partition Π is defined by $i\Pi i'$ iff $g_\eta(i) = g_\eta(i')$ for all $\eta < |B|$. The subset $X_\eta - X_{\eta+1}$ of J_1 is broken up by Π into at most $|A^\eta|$ pieces. So $|\Pi|$, the number of cells of Π, is at most

$$\Sigma\{|A^\eta| : \eta < |B|\}$$

but as $|A| \leq |B|$ and by the GCH we have that $|A^\eta| \leq |B|$ for all $\eta < |B|$. Hence $|\Pi| \leq |B|$.

So we can write $\Pi = \Pi_{h_2}$ where $h_2 : J_1 \to J$ is a function with range J such that $|J| \leq |B|$. As each g_η is constant on each cell of Π_{h_2} we have by 3.1 that $h_2^\# : \mathbb{A}^J/E \to \mathbb{A}^{J_1}/E_1$ is an isomorphism where $E = h_2(E_1)$.

In case (1) also it is clear that $h_2^\#$ is an isomorphism. (c.f. proof of 2.4.) In either case if we take $h = h_1 \circ h_2$ we have $E = h(D)$ and that

$$h^\# = h_2^\# \circ h_1^\# : \mathbb{A}^J/E \to \mathbb{A}^I/D$$

is an isomorphism. //

B. Transcendence Degree of an Ultrapower

We have three correlated scales for measuring the "size" of an ultrapower $\mathbb{B} = \mathbb{A}^I/D$: (a) the cardinality of B (b) the smallest number $|J|$ such that there is an ultrafilter E on J with $\mathbb{A}^J/E \cong \mathbb{B}$ (c) the smallest cardinal number $|X|$ where X generates $\mathbb{B}$. The results of section A concerned the relationship of (b) to (a). Hence we consider the relationship of (c) to (a).

3.4 Definition

The <u>degree</u> of a relational structure $\mathbb{B}$ is the smallest cardinal β such that there exists a set $\{b_\eta : \eta < \beta\}$ which generates $\mathbb{B}$.

If $\mathbb{B} \equiv \mathbb{A}$, where $\mathbb{A}$ is a full structure, then there is a natural embedding of $\mathbb{A}$ into $\mathbb{B}$ that sends an element $a \in A$ to the element of B with the same name. Thus we can regard $\mathbb{B}$ as an extension of $\mathbb{A}$. If the degree of $\mathbb{B}$ is zero then it isn't hard to see that the embedding is onto and hence $\mathbb{B}$ is isomorphic to $\mathbb{A}$. If the degree of $\mathbb{B}$ is one then $\mathbb{B}$ can be generated from $\mathbb{A}$ with the adjunction of a single element.

If a subset X of B generates X' in $\mathbb{B}$, where $\mathbb{B} = \langle B,R,F,C\rangle$ and $\mathbb{B} \equiv \mathbb{A}$, $\mathbb{A}$ being a full structure, then X' consists of all elements of the form $F_\eta(b_1,b_2,\ldots,b_n)$ where $F_\eta \in$ range (F) and $b_1,b_2,\ldots,b_n \in X \cup$ range(C).

To see this notice that all such elements clearly belong to X' and that the set of all such elements is closed under the application of functions from range(F). This is because range($F_{\mathbb{A}}$)

is closed under composition by the full-ness of $\mathbb{A}$ and hence as $\mathbb{B} \equiv \mathbb{A}$ we have that range $(F_{\mathbb{B}})$ = range(F) is closed under composition. More generally, with no assumptions on $\mathbb{B}$ we can write X' as the set of all elements $\psi(b_1,\ldots,b_n)$, where $b_1,\ldots,b_n \in \text{range}(C) \cup X$ and ψ is a composition of members of range(F).

It follows from these observations that if X is a generating set for $\mathbb{B}$ and $|X| = \beta$ then

$$\beta \leq |B| \leq \max(\kappa,\omega,\beta,\gamma)$$

where $\kappa = |\text{range}(F)|$, $\gamma = |\text{range}(C)|$.

For the set of compositions of members of range(F) has cardinality $\max(\kappa,\omega)$, and the set of finite sequences from $X \cup \text{range}(C)$ has cardinality $\max(\beta,\gamma)$. If $\mathbb{B} \equiv \mathbb{A}$ and $\mathbb{A}$ is a full structure then $\kappa = |A^A|$, $\gamma = |A|$, so that we have

3.5 Theorem

If $\mathbb{B}$ is of degree β and $\mathbb{B} \equiv \mathbb{A}$ where $\mathbb{A}$ is a full structure* then $\beta \leq |B| \leq \max(\beta,|A^A|,\omega)$. Consequently if $|B| \geq |A^A|, \omega$ then $\beta = |B|$. //

So, with large models of $T_{\mathbb{A}}$, degree coincides with cardinality. Generalising the proof of 3.2 we prove:

---

* or any structure with $|L_A| = |A^A|$.

3.6 Theorem

Let $\mathbb{A}$ be any structure and let $\mathbb{B} = A^I/D$ have degree β. Then there is an image $E = h(D)$ of D on A^β such that $h^\# : \mathbb{A}^{A^\beta}/E \to \mathbb{A}^I/D$ is an isomorphism.

Proof (Outline)

Let $\{f_\eta/D : \eta < \beta\}$ generate B. A partition Π of I is defined by

$$i\Pi i' \quad \text{iff} \quad f_\eta(i) = f_\eta(i') \quad \text{for all} \quad \eta < \beta.$$

$|\Pi| = |A^\beta|$, so Π determines an image $E = h(D)$ of D on A^β. Each member of B can be expressed as k/D where for all $i \in I$

$$k(i) = \psi(f_{\eta_1}(i), f_{\eta_2}(i), \ldots, f_{\eta_n}(i))$$

for some $\eta_1, \eta_2, \ldots, \eta_n < \beta$ and $\psi \in A^{A^n}$. These functions k are all constant on the cells of Π, so the natural embedding $h^\# : \mathbb{A}^{A^\beta}/E \to \mathbb{A}^I/D$ is onto. //

For the remainder of this section we will assume that $\mathbb{A}$ is a full unary structure, with $|A| \geq \omega$.

3.7 Corollary (GCH)

An ultrapower $B = \mathbb{A}^I/D$ either has degree ≤ 1 or degree $\geq cf(|A|)$.

Proof

If $\mathbb{B}$ has degree $\beta < cf(|A|)$ then $|A^\beta| = |A|$ and so by 3.6 $\mathbb{B} \cong \mathbb{A}^A/E$ for some ultrafilter E on A. But then by 2.8 we have $\beta \leq 1$. //

Note that GCH is not needed if we replace $cf(|A|)$ by $cf^*(|A|)$ where

$$cf^*(\kappa) = \min \{\alpha : \kappa^\alpha > \kappa\} .$$

3.8 Corollary (GCH)

If $|A|$ is regular then the ultrapower $\mathbb{B} = \mathbb{A}^I/D$ has degree either 0, 1 or $|B|$.

Proof (Outline)

Let B have degree β, where $\beta < |B|$. Then by 3.5 $\beta < |A^A|$ and so by GCH $\beta \leq |A|$. If $\beta < |A|$ then $\beta \leq 1$ by 3.7, so we can assume $\beta = |A|$. By 3.6 we can take $|I|$ to be $|A^A| = |A|^+$. An argument which combines the techniques of 3.6 and 3.3 shows that $\mathbb{B} \cong \mathbb{A}^A/E$ for some ultrafilter E on A.

But, by 2.8, this implies that $\beta \leq 1$, contradicting the possibility that $\beta = |A|$. //

Thus, under the GCH, an ultrapower B of A can have degree strictly between 1 and $|B|$ only if $|A|$ is singular and $cf(|A|) \leq \beta \leq |A|$. Whether this state of affairs can occur is an open question.

C. Ultrapowers that are Direct Limits

Since so many cases are known of ultrapowers $\mathbb{A}^I/D$ where $|A^I/D| = |A^I|$, it might be conjectured that if the GCH holds and $|A| < |A^J/E| = \gamma^+$ then $\mathbb{A}^J/E \cong \mathbb{A}^I/D$ for some $\langle I,D\rangle$ such that $|I| = \gamma$. That is "most" ultrapowers are isomorphic to an ultrapower of "proper" index size. We have come close to such a result in section A of this chapter, except that an application of 3.3 will yield only a pair $\langle K,F\rangle$ such that $\mathbb{A}^J/E = \mathbb{A}^K/F$ and $|K| = \gamma^+$.

In this section we will show that if γ is regular then $|A^J/E| \leq \gamma^+$ iff $\mathbb{A}^J/E$ can be expressed as a direct limit of ultrapowers of index size γ, over a well-ordered set of length γ^+.

A partition Π of the index set I of an ultrapower $\mathbb{A}^I/D$ determines an elementary substructure $\mathbb{C}$ of $\mathbb{A}^I/D$ where $f/D \in C$ iff $f \sim_D g$, g being constant on each cell of Π. We have used this fact to construct isomerphisms and embeddings in Chapter 2 and in sections A and B of this chapter. More generally if we have a set $\mathcal{P}$ of partitions of I we can define a subset $Q_{\mathcal{P}}$ of A^I/D by $f/D \in Q_{\mathcal{P}}$ iff $f \sim_D g$ where g is constant on the cells of some $\Pi \in \mathcal{P}$.

3.8 Lemma

Let A be any structure, then the structure $\mathbb{A}^I/D|_Q = \mathbb{Q}$ is a substructure of $\mathbb{A}^I/D$ if for every finite collection $\{\Pi_1,\Pi_2,\ldots,\Pi_n\}$ of partitions from $\mathcal{P}$ there is a partition $\Pi \in \mathcal{P}$ and a set $X \in D$ such that if $i, i' \in X$ then

$$i\Pi i' \text{ implies } i\Pi_k i' \text{ for all } k, \quad k = 1,2,\ldots,n.$$

Proof

Assume that $\mathcal{P}$ has the stated property. We will show that $Q_{\mathcal{P}}$ is closed under the application of functions and constants of $\mathbb{A}^I/D = \langle A^I/D, R^*, F^*, C^* \rangle$. It is easy to see that range $(C^*) \subseteq Q$. Now suppose that $f_1/D, \ldots, f_n/D \in Q_{\mathcal{P}}$, where f_k is constant on each cell of $\Pi_k \in \mathcal{P}$, and that $F^*_\eta \in$ range (F^*) is an n-ary function of A^I/D. We have

$$F^*_\eta(f_1/D,\ldots,f_n/D) = g/D$$

where g is given by $g(i) = F_\eta(f_1(i),\ldots,f_n(i))$ $(i \in I)$, F_η being the function of $\mathbb{A}$ corresponding to F^*_η. Let Π and X have the properties of the statement of the lemma and define $g' \in A^I$ by

$$g'(i) = \begin{cases} g(i) & i \in X \\ a_0 & i \notin X \end{cases}$$

where a_0 is a fixed element of A. Then g' is constant on each cell of Π and $g' \sim g$ so $g/D = g'/D \in Q_{\mathcal{P}}$. //

If D is an ultrafilter on I we shall call a set $\mathcal{P}$ of partitions of I <u>D-directed</u> if for every finite subset $\{\Pi_1,\Pi_2,\ldots,\Pi_n\}$ of $\mathcal{P}$ there is $\Pi \in \mathcal{P}$ and $X \in D$ such that if $i, i' \in X$, then $i\Pi i'$ implies $i\Pi_k i'$ for all $k = 1,2,\ldots,n$.

Less formally we shall say in this situation that Π refines each Π_k almost everywhere.

The substructures of ultrapowers given by D-directed sets of partitions are precisely the structures that Keisler [8] calls limit ultrapowers. Keisler works with filters of equivalence relations

rather than with D-directed sets of partitions.

We feel that presentation of limit ultrapowers by means of sets of partitions has the advantage of throwing attention on the natural embeddings of ultrapowers into limit ultrapowers. This is made explicit by the following lemma which relates to the situation we shall be interested in, namely, ultrapowers that are direct limits over a well-ordered set. The lemma has a functorial character which makes possible easy generalisations.

3.9 Lemma

Let $\mathbb{A}^I/D$ be an ultrapower of the structure $\mathbb{A}$. Suppose we have a sequence $\langle \Pi_\eta : \eta < \gamma \rangle$ of partitions of I such that

(a) each function $f \in A^I$ is $\sim_D$-equivalent to a function g constant on each cell of some Π_η.

(b) if $\eta' < \eta$ then Π_η refines $\Pi_{\eta'}$ almost everywhere.

Then there is a sequence of ultrapowers $\langle \mathbb{A}^{I_\eta}/D_\eta : \eta < \gamma \rangle$ and families of maps

$$h_\eta : I \to I_\eta \qquad (\eta < \gamma)$$

$$h_{\eta\eta'} : I_\eta \to I_{\eta'} \quad (\eta' < \eta < \gamma)$$

such that $\mathbb{A}^I/D$ is the direct limit of the sequence $\langle \mathbb{A}^{I_\eta}/D_\eta : \eta < \gamma \rangle$ via the embeddings

$$h_\eta^\# : \mathbb{A}^{I_\eta}/D_\eta \to \mathbb{A}^I/D$$

and

$$h_{\eta\eta'}^\# : \mathbb{A}^{I_{\eta'}}/D_{\eta'} \to \mathbb{A}^{I_\eta}/D \qquad (\eta' < \eta < \gamma).$$

<u>Proof</u>

Let the sets $\{I_\eta\}$ be chosen to index the partitions, thus let $\Pi_\eta = \{\sigma_j^\eta : j \in I_\eta\}$ for all $\eta < \gamma$. The functions $h_\eta : I \to I_\eta$ are defined by

$$h_\eta(i) = j \qquad \text{iff} \qquad i \in \sigma_j^\eta$$

and we take $D_\eta = h_\eta(D)$. We now define the functions $h^{\#}_{\eta\eta'} : I_\eta \to I_{\eta'}$. By hypothesis Π_η refines $\Pi_{\eta'}$ on a set $X = X(\eta,\eta')$ belonging to D. We will take $h_{\eta\eta'}(i) = i'$ if

$$\sigma^\eta \cap X \subseteq \sigma_{i'}^{\eta'} \cap X \quad .$$

This gives i' uniquely from i so long as $\sigma_i^\eta \cap X \neq 0$.

But

$$\{i \in I_\eta : \sigma_i^\eta \cap X = 0\} \notin D$$

as

$$h_\eta^{-1}(\{i \in I_\eta : \sigma_i^\eta \cap X = 0\}) \subseteq I - X \notin D.$$

So that $h_{\eta\eta'}$ has been defined for almost all $i \in I$ (mod D_η). Define the remaining values of $h_{\eta\eta'}$ arbitrarily.

It follows that the diagram

$$\begin{tikzcd}
 & I_\eta \arrow[dl, "h_{\eta\eta'}"'] \arrow[dr, "h_{\eta\eta''}"] & \\
I_{\eta'} \arrow[rr, "h_{\eta'\eta''}"'] & & I_{\eta''}
\end{tikzcd}$$

$\eta'' < \eta' < \eta < \gamma$ commutes on a set in D_η

and that

$$\begin{tikzcd} & I \arrow[dl, "h_\eta"'] \arrow[dr, "h_{\eta'}"] & \\ I_\eta \arrow[rr, "h_{\eta\eta'}"'] & & I_{\eta'} \end{tikzcd} \qquad \eta' < \eta < \gamma$$

commutes on a set in D. From this it is easy to see that $h^{\#}_{\eta'} = h^{\#}_{\eta\eta'} \circ h^{\#}_{\eta}$ and that $h^{\#}_{\eta\eta''} = h^{\#}_{\eta'\eta''} \circ h^{\#}_{\eta\eta'}$.

Finally let $\mathbb{B}$ be any structure for which there exist embeddings

$$k_\eta : \mathbb{A}^{I_\eta}/D_\eta \to \mathbb{B} \qquad \eta < \gamma$$

such that $k_{\eta'} = h^{\#}_{\eta\eta'} \circ k_\eta$ for all $\eta' < \eta < \gamma$. Then we define an embedding

$$k : \mathbb{A}^I/D \to \mathbb{B}$$

as follows:

if $f/D \in \mathbb{A}^I/D$ then there is $\eta < \gamma$ so that f is $\sim_D$-equivalent to a function constant on the classes of Π_η. Thus $f/D \in$ range $(h^{\#}_\eta)$ and there exists a unique $f_\eta/D \in \mathbb{A}_\eta^{I_\eta}/D_\eta$ such that $f/D = h^{\#}_\eta(f_\eta/D_\eta)$. We set $k(f/D) = k_\eta(f_\eta/D_\eta)$.

This definition of $k(f/D)$ is independent of which is chosen as if $f/D = h^{\#}_{\eta'}(f_{\eta'}/D_{\eta'})$ where $\eta' < \eta$ then it is easy to see that $f_\eta/D_\eta = h^{\#}_{\eta\eta'}(f_{\eta'}/D_{\eta'})$ and so

$$k_{\eta'}(f_{\eta'}/D_{\eta'}) = h^{\#}_{\eta\eta'} \circ k_{\eta}(f_{\eta'}/D_{\eta'})$$
$$= k_{\eta}(h^{\#}_{\eta\eta'}(f_{\eta'}/D_{\eta'}))$$
$$= k_{\eta}(f_{\eta}/D_{\eta}) .$$

Also k is 1-1 as if $f^1/D \neq f^2/D$ then for some $\eta > \gamma$ there exist f^1_η/D_η, f^2_η/D_η such that $f^1_\eta/D_\eta \neq f^2_\eta/D_\eta$ and $f^i/D = h^{\#}_\eta(f^i_\eta/D_\eta)$ $i = 1,2$. Then

$$k(f^1/D) = k_\eta(f^1_\eta/D_\eta) \neq k_\eta(f^2_\eta/D_\eta) = k(f^2/D).$$

A similar argument shows that k is an isomorphism. //

If condition (a) of 3.9 is dropped the direct limit is obtained as a substructure of $\mathbb{A}^I/D$. In the result which now follows it is important that the limit is taken over a well-ordered chain because it is immediate from Adler [2] and can be deduced from Keisler [8] that <u>any</u> model of $T_{\mathbb{A}}$, where $\mathbb{A}$ is full, is a direct limit of ultrapowers with index size $\leq |A|$. We shall say in the situation of 3.9 that $\mathbb{A}^I/D$ is a <u>chain-limit</u> of the sequence of ultrapowers $< \mathbb{A}^{I_\eta}/D_\eta : \eta > \gamma >$. We speak of chain-limits of sets as well as chain-limits of structures.

We can now state

<u>3.10 Theorem</u> (GCH)

An ultrapower $\mathbb{B} = \mathbb{A}^I/D$ is a chain-limit of ultrapowers of index size $< |B|$ if either

(1) $|B| = \gamma^+$, where γ is regular

or (2) $|B|$ is a limit cardinal $> |A|$.

Proof

(1) $|B| = \gamma^+$, γ regular.

By 3.3 we can take $|I| = \gamma^+$. Let B be $\{f_\eta/D : \eta<\gamma^+\}$ and define Π_η for each $\eta < \gamma^+$ by $i\,\Pi_\eta\,i'$ iff $f(i) = f(i')$. We will construct a sequence $\langle\Pi^*_\eta : \eta < \gamma^+\rangle$ of partitions of I such that if $\xi < \gamma^+$ then Π^*_ξ refines every partition in the set

$$\{\Pi_\eta : \eta < \xi\} \cup \{\Pi^*_\eta : \eta < \xi\}$$

on some member of D, and such that for all $\eta < \gamma^+$, $|\Pi^*_\eta| \le \gamma$.

Suppose the partitions $\{\Pi^*_\eta : \eta < \xi\}$ are already appropriately defined, we will show how to define Π^*_ξ.

As $\xi < \gamma^+$ we have $|\xi| \le \gamma$ and we can re-order the set $\{\Pi_\eta : \eta < \xi\} \cup \{\Pi^*_\eta : \eta < \xi\}$ as $\{\Pi^{**}_\eta : \eta < \kappa\}$ where $\kappa = |\xi| \le \gamma$. Without loss of generality D is uniform on I and hence D is γ^+-d.i. So by Chang's Theorem D is γ-d.i. Let $\langle X_\eta : \eta < \gamma\rangle$ be a γ-chain in D.

Define an equivalence relation $\approx$ on I as follows:

(a) if i and i' belong to $X_\eta - X_{\eta+1}$ for some $\eta > \gamma$ then

$$i \approx i' \quad \text{iff} \quad i\Pi^{**}_{\eta'}i' \quad \text{for all } \eta'$$

such that $\eta' < \eta$, $\eta' < \kappa$.

(b) if i and i' belong to $I-X_0$ then $i \approx i'$.

(c) in all other cases $i \not\approx i'$.

Now $I = (I-X_0) \cup \cup\{X_\eta - X_{\eta+1} : \eta<\gamma\}$ is a disjoint decomposition of I and each $X_\eta - X_{\eta+1}$ is broken up by $\approx$ into $\le \gamma^{|\eta|}$

cells. As γ is regular and $|\eta| < \gamma$ we have by GCH that $\gamma^{|\eta|} = \gamma$. Hence $\approx$ breaks up I into $\leq \gamma \cdot \gamma = \gamma$ cells and we can take Π^*_ξ to be the partition of I determined by $\approx$.

It is clear that the sequence $\langle \Pi^*_\eta : \eta < \gamma^+ \rangle$ thus defined satisfies conditions (a) and (b) of 3.9. Hence by 3.9 $\mathbb{B}$ is a direct limit of a sequence $\langle \mathbb{A}^{I_\eta}/D : \eta < \gamma^+ \rangle$ where $|I_\eta| = |\Pi^*_\eta| \leq \gamma$. /

(2) $|B|$ a limit cardinal $> |A|$.

Define the partitions $\langle \Pi_\eta : \eta < |B| \rangle$ as in part (1). Let Π^*_ξ be the common refinement of $\{\Pi_\eta : \eta < \xi\}$ for each $\xi < |B|$.

For any $\xi < |B|$ we have

$$|\Pi^*_\xi| \leq |A^\xi|$$

So as $|A| < |B|$ we have by the GCH that $|A^\xi| < |B|$, so that $|\Pi^*_\xi| < |B|$ for all $\xi < |B|$. It is easy to see that $\langle \Pi^*_\xi : \xi < |B| \rangle$ then satisfies (a) and (b) of 3.9. //

Notice that if $|A| \leq \gamma^+$ and $\mathbb{B}$ is the chain-limit of a sequence of ultrapowers $\langle \mathbb{A}^{I_\eta}/D_\eta : \eta < \gamma^+ \rangle$ then we have that $|B| \leq 2^\gamma \cdot \gamma^+ = 2^\gamma$, so that we have proved

<u>3.11 Theorem</u> (GCH)

Let $\mathbb{B} = \mathbb{A}^I/D$ and let γ be a regular cardinal such that $|A| \leq \gamma^+$. Then it is necessary and sufficient for $|B| \leq \gamma^+$ that $\mathbb{B}$ is a chain-limit of a sequence of ultrapowers $\langle \mathbb{A}^{I_\eta}/D_\eta : \eta < \gamma^+ \rangle$, where $|I_\eta| \leq \gamma$ for all $\eta < \gamma^+$. //

Notice that in all of the above we have not excluded the possibility that the direct limits are trivial in the sense that

$$\mathbb{A}^{I_\eta}/D_\eta = \mathbb{A}^{I_{\eta_0}}/D_{\eta_0} \text{ for some } \eta_0 < \gamma^+ \text{ and all } \eta > \eta_0.$$

If we regard the least size of index $|I|$ such that an ultrapower $\mathbb{B}$ of a structure $\mathbb{A}$ is isomorphic to $\mathbb{A}^I/D$ for some ultrafilter D on I as a measure of the "complexity" of $\mathbb{B}$ then 3.10 represents certain ultrapowers as chain-limits of "simpler" ultrapowers.

D. A Hierarchy of Substructures

In order to generalise the results of the last section we define a hierarchy of substructures of $\mathbb{A}^I/D$. In fact we will work with arbitrary subsets of A^I/D.

3.12 Definition

Let $\mathbb{A}$ be any structure and let X be a subset of A^I/D. We say that X is <u>of level α</u> if α is the least cardinal such that there is a partition Π of I into α cells such that each element f/D of X has a representative $f' \sim_D f$ that is constant on each cell of Π. We write "level (X) = α". The level of a substructure of $\mathbb{A}^I/D$ is taken to be the level of its underlying set. //

Note that

(i) level $(\{f/D\})$ = spread(f/D).

(ii) if X is finite, level (X) $\leq |A|$.

(iii) if $\mathbb{A}(X)$ is the substructure of $\mathbb{A}^I/D$ generated by X then the level of $\mathbb{A}(X)$ is the same as the level of X.

(iv) the level of $\mathbb{A}^I/D$ is the least cardinal α such that D has an image $E = h(D)$ on α for which $h^{\#} : \mathbb{A}^{\alpha}/E \to \mathbb{A}^I/D$ is an isomorphism. In the rest of this section we assume that $\mathbb{A}^I/D$ is presented with $|I|$ already reduced in this way, i.e. that the level of $\mathbb{A}^I/D$ is $|I|$.

3.13 Definition

For all cardinals $\beta \leq |I|$ we define

$$L_\beta = \{X \subseteq A^I/D : \text{level } (X) < \beta\} \qquad //$$

3.14 Theorem

Let $\mathbb{B} = \mathbb{A}^I/D$.

(a) For all infinite $\beta \leq |I|$, L_β is an ideal in $S(B)$.

(b) (GCH) if D is γ-d.i., where γ is regular, then $L_{\gamma+}$ is γ-complete.

Proof

(a) It is clear that $X \in L_\beta$ and $Y \subseteq X$ implies $Y \in L_\beta$. Suppose that $X_1, X_2 \in L_\beta$ and that Π_1 and Π_2 are the corresponding partitions of I. Then $|\Pi_1|, |\Pi_2| < \beta$ and if $\Pi_1 \cdot \Pi_2$ is the common refinement of Π_1 and Π_2 it is clear that $|\Pi_1 \cdot \Pi_2| = |\Pi_1| \cdot |\Pi_2| < \beta$. Each element f/D of X_k $(k = 1,2)$ has a representative constant on each cell of Π_k and hence on each cell of $\Pi_1 \cdot \Pi_2$.

So each member of $X_1 \cup X_2$ has a representative constant on each cell of $\Pi_1 \cdot \Pi_2$. Thus $X_1 \cup X_2 \in L_\beta$ and hence L_β is an ideal. /

(b) Let

(i) $\{X_\eta : \eta < \gamma\}$ be a set of γ members of $L_{\gamma+}$.

(ii) $\langle Y_\eta : \eta < \gamma \rangle$ be a γ-chain in D.

(iii) $\{\Pi_\eta : \eta < \gamma\}$ be a set of γ partitions of I into $\leq \gamma$ cells, corresponding to the X_η's.

We define a partition Π of I as in the proof of 3.10, i.e. if $i, i' \in Y_\eta - Y_{\eta+1}$ then

$$i \Pi i' \text{ iff } i \Pi_{\eta'} i' \text{ for all } \eta' < \eta .$$

We have that

$$|\Pi| \leq \Sigma\{\gamma^{|\eta|} : \eta < \gamma\} = \gamma$$

(employing the GCH).

The partition Π refines each Π_η on $Y_{\eta+1} \in D$, so that each member of X_η has a representative constant on every cell of Π. Thus each member of $\cup\{X_\eta : \eta < \alpha\}$ has a representative constant on the cells of Π, so that $\cup\{X_\eta : \eta < \gamma\} \in L_{\gamma+}$. //

A theorem of Tarski [16] states that a $\hat{\beta}$-complete proper ideal L on a non-measurable set of cardinality β is either principal or admits a pairwise disjoint family $\{X_\eta : \eta < \beta\}$ of non-members of L.

If $|I| = \gamma^+$ we have by assumption that $L_{\gamma+}$ is a proper ideal. If γ is regular we have that $L_{\gamma+}$ is γ-complete, from the above. Also from 3.3 and GCH we infer that

$$|A^I/D| = |B| \geq \text{level } (B) = |I| \ .$$

If $|B| = |I|$, then $L_{\gamma+}$ is a $\overset{\frown}{\gamma^+}$-complete proper ideal on a set of cardinality γ^+. So by the result of Tarski memtioned above $L_{\gamma+}$ admits a pariwise disjoint family of non-members $\{X_\eta : \eta < \gamma^+\}$. In this situation any embedding of the form

$$h^\# : A^J/E \to A^I/D,$$

where $|J| \leq \gamma$, fails to be onto in a very strong way: in fact the range of $h^\#$ can include none of the sets X_η, $\eta < \gamma^+$, for to do so would contradict level $(X_\eta) = \gamma^+$ i.e. $X_\eta \notin L_{\gamma+}$.

CHAPTER 4

INDUCED ISOMORPHISMS AND THE SHADOW

In Chapter 3 we presented theorems where we constructed images $\langle J,E\rangle$ of an ultrafilter pair $\langle I,D\rangle$ such that the embedding

$$h^{\#}_{\mathbb{A}} : \mathbb{A}^J/E \to \mathbb{A}^I/D$$

was an isomorphism and where $|J|$ depended on $|A^I/D|$. We prove now results connecting the nature of the shadow $\sigma(D)$ of D with the problem of when embeddings like $h^{\#}_A$ are onto. These results will allow us to pass from results on $\sigma(D)$ to results on $|A^I/D|$ and vice-versa. In particular we shall obtain a partial converse to 2.20 and some results concerning ultrapowers $\mathbb{A}^I/D$ with $|A| < |A^I/D| < |I|$. We will also investigate a related notion of quasicompleteness for ultrafilters.

A. Three Algebraic Theorems

Throughout this section $h : \langle I,D\rangle \to \langle J,E\rangle$ is a fixed filter morphism. Recall that $h^{\#}_A : A^J/E \to A^I/D$ is the injection defined by $h^{\#}_A(g/E) = h\circ g/D$. As we have said above we are interested in when these maps $h^{\#}_A$ are onto. Some easy first results are given in the following lemma.

4.1 Lemma

(i) $h_A^{\#}$ is onto iff $h_{|A|}^{\#}$ is.

(ii) If $|A| \geq |I|$ then $h_A^{\#}$ is onto iff $E \approx D$.

(iii) If $h_\alpha^{\#}$ is onto and $\beta < \alpha$ then $h_\beta^{\#}$ is onto.

(iv) If $D \not\approx E$ there is a cardinal $\alpha_h \geq \omega$ such that $h_\beta^{\#}$ is onto iff $\beta < \alpha_h$.

Proof

Parts (i) and (iii) are simple consequences of 3.1. Part (iv) is a consequence of parts (ii) and (iii). We outline the proof of part (ii).

Suppose $E \approx D$, then we that $h(D) \approx D$. Employing 2.1 we have by an argument similar to that of the proof of 2.2(b) that there is a set $X \in D$ such that $h|_X$ is 1-1 and $h(X) \in E$. Then by 2.3(b) we see that $h_A^{\#}$ is onto.

Conversely if $h_A^{\#}$ is onto, choose (for each j) partitions $\{I_j^\eta : \eta < |A|\}$ of $I_j = h^{-1}(j)$ such that each cell I_j^η is a singleton. (We can do this as $|I_j| \leq |I| \leq |A|$.) Now we employ 3.1 to conclude the existence of a sequence $\langle \eta_j : j \in J \rangle$ such that

$$\cup \{I_j^{\eta_j} : j \in J\} \in D \quad .$$

The map which establishes $D \approx E$ is the map which sends j to the member of $I_j^{\eta_j}$. //

4.2 Theorem

If $\beta \in \sigma(D)$ is such that $|J| \leq \beta \leq |I|$ then there is a

uniform ultrafilter F on β and maps $f : I \to \beta$, $g : \beta \to J$ such that $f(D) = F$, $g(F) = E$ and $h_A^{\#}$ factors

$$\begin{tikzcd} A^J/E \arrow[rr, "h_A^{\#}"] \arrow[dr, "g_A^{\#}"'] & & A^I/D \\ & A^\beta/F \arrow[ur, "f_A^{\#}"'] & \end{tikzcd}$$

whence $h_A^{\#}$ is onto iff $g_A^{\#}$ and $f_A^{\#}$ are onto.

Proof

As $\beta \in \sigma(D)$ there is a partition of I into β cells that determines a uniform image F' of D on β. There is also a partition of I into $|J|$ cells that determine E as an image of D (viz: the partition Π_h determined by h). Taking the common refinement of these two partitions we obtain a partition of I into β cells that determines an ultrafilter F on β such that $E \leq F \leq D$, and $F' \leq F$ so that F is uniform. Then there are obvious maps $f : I \to \beta$, $g : \beta \to J$ such that $f(D) = F$, $G(F) = E$ and $h = f \circ g$. The rest of the theorem follows. //

4.3 Theorem

Suppose $h_\beta^{\#}$ is onto, and $\beta < \gamma < \alpha$ implies that $\gamma \notin \sigma(D)$. Then $h_\mu^{\#}$ is onto for all $\mu < \alpha$.

Proof

Let $\Pi_h = \{I_j : j \in J\}$, where $I_j = h^{-1}(j)$, be the partition of I determined by h. Suppose $f/D \in \mu^I/D$ where $\mu < \alpha$.

Then by 2.14 and hypothesis we have that spread $(f/D) \leq \beta$. So there is $f^* \sim_D f$ such that $|\text{range } (f^*)| \leq \beta$. Let range $(F^*) = V$. Then by hypothesis and 4.1 we have that $h_V^{\#}$ is onto. So by 3.1 there is $f \sim_D f^*$ such that f' is constant on each I_j $(j \in J)$. But then $f' \sim_D f$ and so by 3.1 again we have that $f/D \in \text{range } (h_\mu^{\#})$. //

4.4 Theorem

Let D be an ultrafilter on I such that $\beta < \gamma < \alpha$ implies $\gamma \notin \sigma(D)$. Then if $2^{2^{2^\beta}} < \alpha$ there is an ultrafilter E on β and a map $k : I \to \beta$ such that $E = k(D)$ and $k_\gamma^{\#}$ is onto for all $\gamma < \alpha$.

Proof

First we claim

(1) there is an ultrafilter E on β such that $E \leq D$ and if G is any ultrafilter on β such that $G \leq D$ then $G \leq E$.

For consider all ultrafilters on β that are images of D. There are at most 2^{2^β} of them and we can select $\leq 2^{2^\beta}$ partitions of I into β pieces corresponding to these. The common refinement of these 2^{2^β} partitions has at most $\beta^{(2^{2^\beta})} = 2^{2^{2^\beta}}$ cells, and hence determines an image E' of D on $2^{2^{2^\beta}}$ such that each image of D on β is an image of E'.

However $th(E') \leq \beta$, for otherwise there would be a uniform image of D on γ where $\beta < \gamma \leq 2^{2^{2^\beta}}$, contradicting $\gamma \notin \sigma(D)$. So $E' \approx E$ where E is an ultrafilter on β. Clearly E satisfies (1). Notice that (1) implies that E is unique up to $\approx$ (and hence by earlier remarks, up to isomorphism).

Select a map $k : I \to \beta$ such that $E = k(D)$. We claim

(2) $k^{\#}_{\beta}$ is onto.

Let k determine a partition $\Pi = \{I_j : j \in \beta\}$ of I. By 3.1 our claim (2) is verified if we can show that for any refinement $\Pi^* = \{I_j^{\eta} : j \in \beta, \eta < \beta\}$ of Π, where $I_j = \cup\{I_j^{\eta} : \eta < \beta\}$, there exists a sequence $\langle\eta_j : j \in \beta\rangle \in \beta^{\beta}$ such that

$$\cup \{I_j^{\eta_j} : j \in \beta\} \in D.$$

Now Π^* determines an image E^* of I on $\beta\times\beta$ such that $\pi_1(E^*) = E$ where $\pi_1 : \beta\times\beta \to \beta$ is the projection on the first factor. But $|\beta\times\beta| = \beta$ so E^* is isomorphic to an ultrafilter on β and hence by the universality property (1) of E there is a map $\psi : \beta \to \beta\times\beta$ such that $\psi(E) = E^*$. But then we have $(\pi_1\circ\psi)(E^*) = E^*$ so by 2.1 there is $X \in E^*$ such that $\pi_1\circ\psi|_X$ is the identity on X. Then if $Y = \pi_1(X)$, $Y \in E$ and $X = \psi(Y) = \{\psi(j) : j \in Y\}$. As $(\psi\circ\pi_1)(j) = j$ for $j \in Y$ we have $\psi(j) = \langle j,\eta_j\rangle$ for some $\beta_j \in \beta$ $(j \in Y)$.

Hence

$$X \subseteq \{\langle j,\eta_j\rangle : j \in \beta\}$$

where the η_j are chosen arbitrarily if $j \in \beta - Y$. So $\{\langle j,\eta_j\rangle : j \in \beta\} \in E^*$ and so

$$\cup \{I_j^{\eta_j} : j \in \beta\} \in D$$

by definition of E^*. Thus (2) is verified.

(3) $k^{\#}_{\gamma}$ is onto for all $\gamma < \alpha$.

This follows from (2) and 4.3. //

B. Quasicomplete Ultrafilters

We add some remarks on when the hypotheses of 4.4 are satisfied. If there is a regular cardinal $\gamma_0 < |I|$ such that $\gamma_0 \notin \sigma(D)$ then it follows from Chang's Theorem that γ_0^+, γ_0^{++}, γ_0^{+++}, etc. $\notin \sigma(D)$. Then the hypotheses of 4.4 will be satisfied if we take $\beta = \gamma_0$ and α the least cardinal $> \gamma_0$ such that $\alpha \in \sigma(D)$, provided we accept the limit cardinal hypothesis. For then $2^{2^{2^\beta}} = \beta^{(n+)}$ for some n $(3 \leq n < \omega)$. We have already exhibited ultrafilters D on I such that $\gamma \in \sigma(D)$ for all $\gamma \leq |I|$. The question has been posed (in Chang [6] for the case that $|I| = \omega_{\omega+1}$) as to whether there exist uniform ultrafilters D on I such that there is a regular $\gamma < |I|$ such that $\gamma \notin \sigma(D)$. We will construct an example of such an ultrafilter in the case where $|I| = \mu$, the least measurable cardinal: the situation in general and in particular for $|I| = \omega_{\omega+1}$ remains unresolved.

We will call a uniform ultrafilter D on a set I <u>(κ,λ)-quasicomplete</u> if $\kappa^+ < \lambda$ and $\gamma \notin \sigma(D)$ for all γ such that $\kappa < \gamma < \lambda$, and simply <u>$\kappa$-quasicomplete</u> if D is $(\kappa,|I|)$-quasicomplete. We will motivate these names a little later, but for the meantime note that these quasicomplete ultrafilters lack certain kinds of incompleteness. From our earlier remarks we see that if D is a uniform ultrafilter on I and γ is a regular cardinal $\notin \sigma(D)$ then D is (γ,λ)-quasicomplete, where λ is the least cardinal $> \gamma$ such that $\lambda \in \sigma(D)$.

Independently of this work Silver et al. [14] have studied ω-quasicomplete ultrafilters, which they call <u>indecomposible</u>

ultrafilters. Silver proves that if D is an ω-incomplete indecomposible ultrafilter on a strong limit cardinal $\lambda > \omega$ then there is a partition $\{I_n : n < \omega\}$ of λ such that $I_n \notin D$ for all $n < \omega$ and if each I_n is decomposed:

$$I_n = \cup \{I_n^m : m < \omega\} \qquad (n < \omega)$$

then there is a sequence $\langle m_n : n < \omega\rangle$ such that

$$\cup \{I_n^{m_n} : n < \omega\} \in D .$$

This result is a special case of 4.4.

Another result mentioned by Silver in [14] is ascribed to Chang and Prikry. It states that if GCH holds, and D is an indecomposible ultrafilter on λ then either λ is inaccessible or $cf(\lambda) = \omega$. We do not have available the proof of this result but in section E we will prove a similar theorem in the more general setting of κ-quasicomplete ultrafilters.

Silver goes on in [14] to show that if λ is a strongly inaccessible cardinal which supports an indecomposible ultrafilter then λ has several "large cardinal" properties. In the other direction we will show that if μ is the least measurable cardinal there exist κ-quasicomplete ultrafilters on μ for any κ such that $\omega \leq \kappa < \mu$. First we note the following result:

<u>4.5 Lemma</u>

If $\mathbb{A}$ is a full unary structure and $\mathbb{A}^I/D \cong \mathbb{A}^J/E$ then for all cardinals $\beta \leq |A|$ we have

$$\beta \in \sigma(D) \quad \text{iff} \quad \beta \in \sigma(E) .$$

Proof

Given the hypotheses, it is easy to show that for any structure $\mathbb{K}$ with $|K| \leq |A|$ we have

$$\mathbb{K}^I/D \cong \mathbb{K}^J/E ,$$

(at least with respect to the unary relations). Let K be a set of cardinality β and suppose $F = f(D)$ is a uniform image of D on K. Take K to be the structure which lists precisely the unary relations of K, and let $\phi : \mathbb{K}^I/D \to \mathbb{K}^J/E$ be an isomorphism. Then if $\phi(f/D) = g/E$ we claim $F = g(E)$. For if X is any subset of K and τ_X is the corresponding predicate of $L_{\mathbb{K}}$ then

$$K^I/D \models \tau_X(f/D) \quad \text{iff} \quad K^J/E \models \tau_X(g/E)$$

i.e. $\{i : f(i) \in X\} \in D$ iff $\{j : g(j) \in X\} \in E$

i.e. $f^{-1}(X) \in D$ iff $g^{-1}(X) \in E$

i.e. $X \in f(D)$ iff $X \in g(E)$

So $F = f(D) = g(E)$.

Thus $\beta \in \sigma(D)$ implies $\beta \in \sigma(E)$, and the converse holds by symmetry. //

As a corollary we have a converse to 4.4.

4.6 Corollary

If $E = h(D)$ is an image of D on β and $h^{\#}_{\gamma}$ is onto for all γ such that $\gamma < \alpha$ then $\gamma \notin \sigma(D)$ for all γ such that $\beta < \gamma < \alpha$.

Proof

As the embeddings $h^{\#}_{\gamma}$ induce isomorphisms of full structures on γ we have from 4.5 that if $\gamma < \alpha$ then $\gamma \in \sigma(D)$ iff $\gamma \in \sigma(E)$.

But $\gamma > \beta$ implies $\gamma \notin \sigma(E)$. Hence if γ satisfies $\beta < \gamma < \alpha$ then $\gamma \notin \sigma(D)$. //

To construct a κ-quasicomplete ultrafilter on μ we will make use of the construction of Adler [1] for an ω-incomplete non-regular ultrafilter on μ.

Let E be any uniform ultrafilter on κ and let F be a μ-complete uniform ultrafilter on μ. Take $I = \mu\times\kappa$, $D = F\times E$. If $\pi : I \to \kappa$ is the projection we have $\pi(D) = E$.

Suppose A is a full structure on a set of cardinality $\alpha < \mu$. We claim that

$$\pi^{\#}_{A} : \mathbb{A}^{\kappa}/E \to \mathbb{A}^{I}/D$$

is onto.

Let $I_j = \{\langle\eta,j\rangle : \eta < \mu\}$ for each $j \in \kappa$. By 3.1 we can show that $\pi^{\#}$ is onto if we can prove that whenever we have decompositions

$$I_j = \{I_j^{\eta} : \eta < \alpha\}$$

of each I_j into α parts, then there is a sequence $\langle\eta_j : j \in \kappa\rangle$ such that

$$\cup\{I_j^{\eta_j} : j \in \kappa\} \in D.$$

Assume that we are given such a sequence of decompositions and let $X_j^{\eta} = \{\xi : \langle\xi,j\rangle \in I_j^{\eta}\}$ then for each $j \in \kappa$ we have

$$\mu = \cup\{X_j^{\eta} : \eta < \alpha\} \in F .$$

As F is α-complete if follows that for each $j \in \kappa$ there is an $\eta_j < \alpha$ such that $X_j^{\eta_j} \in F$. But

$$\cup\{I_j^{\eta_j} : j \in \kappa\} \in D = F \times E \quad \text{iff} \quad \{j : X_j^{\eta_j} \in F\} \in E .$$

Hence if the sequence $\langle\eta_j : j \in \kappa\rangle$ is chosen so that $X_j^{\eta_j} \in F$ for all $j \in \kappa$ we have that

$$\cup \{I_j^{\eta_j} : j \in \kappa\} \in D.$$

So $\pi^{\#}_{\mathbb{A}}$ is onto, and hence an isomorphism. As this argument holds for all $\alpha < \mu$ we have by 4.6 that $\alpha \notin \sigma(D)$ for all α such that $\kappa < \alpha < \mu$, so that D is κ-quasicomplete.

C. Shadow and Cardinality II

We will prove here a converse to 2.20 under the assumption of the GCH and also some general results about $|A^I/D|$ where D is a uniform ultrafilter on I.

4.7 Theorem (LCH)

Let D be an ultrafilter on I and let $\alpha \leq |I|$ be such that $\alpha = \Sigma\{\alpha^\beta : \beta < \alpha\}$. Then

$$\alpha \in \sigma(D) \quad \text{iff} \quad |\alpha^I/D| > \alpha .$$

Proof

Note that any such α must be regular. The forward direction of the result follows from 2.20.

Assume, now, that $\alpha \in \sigma(D)$. Then by 2.10 and Chang's Theorem we have that $\alpha^{+}, \alpha^{++}, \alpha^{+++}, \ldots \notin \sigma(D)$, and hence by LCH that

$$\alpha \leq \gamma \leq 2^{2^{2^{\alpha}}} \text{ implies } \gamma \notin \sigma(D).$$

(1) If $2^{2^{2^{\alpha}}} < |I|$ then by 4.4 there is an image $E = h(D)$ of D on α such that

$$h^{\#} : \alpha^{\alpha}/E \to \alpha^{I}/D$$

is a bijection.

But $\beta = th(E) < \alpha$ as $\alpha \notin \sigma(D)$ implies $\alpha \notin \sigma(E)$, thus

$$|\alpha^{I}/D| = |\alpha^{\alpha}/E| = |\alpha^{\beta}/E'|$$

for some $E' \approx E$ on β.

So

$$|\alpha^{I}/D| \leq |\alpha^{\beta}| = \alpha$$

by hypothesis and 1.14(a).

(2) If $2^{2^{2^{\alpha}}} \geq |I|$ then we must have $th(D) = \beta' < \alpha$ and by a similar argument

$$|\alpha^{I}/D| \leq |\alpha^{\beta'}| = \alpha . \qquad //$$

<u>4.8 Corollary</u> (GCH)

Let D be an ultrafilter on I, and let α be a regular cardinal $\leq |I|$.

Then

$$\alpha \in \sigma(D) \quad \text{iff} \quad |\alpha^{I}/D| > \alpha .$$

Proof

If the GCH holds then for any regular cardinal α we have $\alpha = \Sigma\{\alpha^{\beta} : \beta < \alpha\}$. Certainly the LCH will then hold so we may apply 4.7. //

If α is singular the situation is more complex and we summarize it in the following diagram of implications

$$\begin{tikzcd}
 & \text{D is } cf(\alpha)\text{-d.i.} \arrow[dd, "2\ (\text{GCH})"] \\
\alpha \in \sigma(D) \arrow[ur, "1"] \arrow[dr, "3"'] & \\
 & |\alpha^{I}/D| > \alpha
\end{tikzcd}$$

The implication (1) cannot be reversed, for we may have $cf(\alpha) \leq |I| < \alpha$. Less trivially look at an ω-quasicomplete ultrafilter D on I as constructed in section B. ($|I| = \mu$). Then D is ω-d.i. but, e.g., $\omega_{\omega} \notin \sigma(D)$. The same examples show that (3) cannot be reversed.

Implication (2) presents some problems: Is GCH needed? Can (2) be reversed?

These remain open questions. Of related interest to these problems and results is the following result which makes no use of the GCH.

4.9 Theorem

If $|\alpha^I/D| > \alpha$ then

either (1) D is cf(α)-d.i.

or (2) there is a $\beta < \alpha$ such that $|\beta^I/D| > \alpha$.

Proof

Suppose that D is not cf(α)-d.i. Then by 2.17 $\iota(\alpha)$ is confinal in α^I/D where $\iota : \alpha \to \alpha^I/D$ is the canonical embedding and α^I/D is regarded as an ordered set. We take $<'$ to be the order on α^I/D.

Define C_η to be $\{f/D : f/D <' \iota(\eta)\}$ for each $\eta < \alpha$.

Then by our remarks following 2.17 we have that

$$|C_\eta| = |\eta^I/D|$$

for all $\eta < \alpha$.

But

$$\alpha^I/D = \cup\{C_\eta : \eta < \alpha\}$$

and so

$$(\dagger) \qquad |\alpha^I/D| = \sup\{|\beta^I/D| : \beta < \alpha\}$$

Hence if $|\alpha^I/D| > \alpha$ there is a $\beta < \alpha$ such that $|\beta^I/D| > \alpha$. //

This theorem rather nicely fills the gap in our understanding left by 4.8: it tells us that $|\alpha^I/D|$ is greater than α iff there is "good cause" for it to be.

A bonus from the proof of the above result is the principle (†) which holds whenever D is not cf(α)-d.i.

It is an open question in the theory of ultrapowers whether there exists a non-trivial ultrapower of singular cardinality, i.e. an ultrapower α^I/D where $|\alpha^I/D|$ is singular and greater than α. Using (†) we can envisage the following "scenario": that there exists an ultrafilter D that is not cf(α)-d.i. and such that for some γ, $\gamma < \beta < \alpha$ implies $\alpha < |\beta^I/D| < |\alpha^I/D|$. Then by (†) $|\alpha^I/D|$ would have to be singular.

By combining 4.9 with 1.15(h) we obtain

<u>4.10 Theorem</u>

If $|(2^\alpha)^I/D| > 2^\alpha$ then either D is cf(α)-d.i. or there is a $\beta < \alpha$ such that $|\beta^I/D| > \alpha$.

<u>Proof</u>

By 1.15(h) $|(2^\alpha)^I/D| > 2^\alpha$ implies $|\alpha^I/D| > \alpha$. The result then follows by 4.9. //

The result 4.10 bears an interesting relationship to Chang's Theorem: both results can be proved without the GCH, but if the GCH is assumed then 4.10 implies Chang's Theorem. For then we have by 4.8,

$$|(2^\alpha)^I/D| > 2^\alpha \quad \text{iff} \quad |(\alpha^+)^I/D| > \alpha^+ \quad \text{iff} \quad D \text{ is } \alpha^+\text{-d.i.}$$

Also if $\beta < \alpha$ is such that $|\beta^I/D| > \alpha$ then we have for all regular

γ such that $\beta \leq \gamma \leq \alpha$

$$|\gamma^I/D| \geq |\beta^I/D| > \alpha \geq \gamma$$

so that $|\gamma^I/D| > \gamma$ and by 4.8 D is γ-d.i. Chang's Theorem readily follows. //

The above results relate only to the question of when $|A^I/D| > |A|$ for ultrapowers A^I/D. If we wish for more precise information on where $|A^I/D|$ falls in the range of cardinals between $|A|$ and $|A^I|$ we find that we have to study much more difficult questions about the structure of the ultrafilter D.

The following theorem shows us that even the "simplest" further question "When is $|A^I/D| > |A|^+$?" leads us to a new level of difficulty.

<u>4.11 Theorem</u> (GCH)

Suppose $|A|$ is regular and that $|I| > |A|$.

Then $|A^I/D| \leq |A|^+$ iff A^I/D is a chain-limit of a sequence $\langle A^{I_\eta}/D_\eta : \eta < |A|^+ \rangle$ of ultrapowers with index size $|I_\eta| \leq |A|$.

<u>Proof</u>

Apply 3.11 with $\gamma = |A|$. //

In terms of the ultrafilter D we may express this theorem in the following way:

$$|A^I/D| \leq |A|^+$$

iff

there exist system of maps

$$\{h_\eta : \eta > |A|^+\} \subseteq A^I$$

$$\{h_{\eta\eta'} : \eta < \eta < |A|^+\} \subseteq A^A$$

such that the diagrams

$$\begin{tikzcd} & I \arrow[dl, "h_\eta"'] \arrow[dr, "h_{\eta'}"] & \\ A \arrow[rr, "h_{\eta\eta'}"'] & & A \end{tikzcd} \qquad (\eta' < \eta < |A|^+)$$

commute on a set in D, and if $D_\eta = h_\eta(D)$ for all $\eta < |A|^+$ then the diagrams

$$\begin{tikzcd} & A \arrow[dl, "h_{\eta\eta'}"'] \arrow[dr, "h_{\eta\eta''}"] & \\ A \arrow[rr, "h_{\eta'\eta''}"'] & & A \end{tikzcd}$$

commute on a set in D_η and further if g is any member of A^I there exists an $\eta > |A|^+$ and a $t \in A^A$ such that the diagram

$$\begin{tikzcd} & I \arrow[dl, "h_\eta"'] \arrow[dr, "g"] & \\ A \arrow[rr, "t"'] & & A \end{tikzcd}$$

commutes on a set in D.

4-31

The question of whether such ultrafilters exist for various $|A|$ and $|I|$ seems to be quite difficult. The result of Prikry [13] shows that it is consistent to assume that all uniform ultrafilters on ω_1 are regular. This assumption would imply that no ultrapower ω^{ω_1}/D, where D is uniform on ω_1, is a chain-limit of a sequence $\langle\omega^{\omega}/D_\eta : \eta < \omega_1\rangle$.

We shall close this section by proving a result which shows that if D is a uniform ultrafilter on I where $|I| > |A|$ then either $|A^I/D|$ is "reasonably large" or we have some interesting pathological situations.

<u>4.12 Theorem</u>

Let $B = A^I/D$ be any ultrapower of A, where D is uniform on I. Let $\beta = |A^\beta|$. Then either

(1) $\beta \geq |I|$

or (2) $\beta^+ \notin \sigma(D)$

or (3) D has images D_1 on β, D_2 uniform on β^+ and there exists $h : \beta^+ \to \beta$ such that $h(D_2) = D_1$ and the injection $h^\# : A^\beta/D_1 \to A^{\beta^+}/D_2$ is onto.

<u>Proof</u>

By 3.2 there is a map $k : I \to \beta$ such that

$$k^\# : A^\beta/E \to A^I/D$$

is onto, where $E = k(D)$.

Assume that $\beta < |I|$ and $\beta^+ \in \sigma(D)$. Then there is a partition Π' of I into β^+ pieces that determine a uniform image

of D or β^+. Let Π be the common refinement of Π' and Π_k. Then $|\pi| = \beta^+ \cdot \beta = \beta^+$ and so $\Pi = \Pi_g$ for a map $g : I \to \beta^+$. By the construction of g there is a map $h : \beta^+ \to \beta$ we have

$$k = g \circ h, \qquad h(g(D)) = E.$$

Set $g(D) = F$, then because Π_g refines Π' and Π' determines a uniform image on β^+ we have that F is uniform on β^+.

We have embeddings

$$\begin{tikzcd} A^{\beta}/E \arrow[r, "h^{\#}"] \arrow[rrr, bend right, "k^{\#}"'] & A^{\beta^+}/F \arrow[r, "g^{\#}"] & A^{I}/D \end{tikzcd}$$

and as $k^{\#}$ is onto so are $h^{\#}$ and $g^{\#}$. The result follows if we set $D_1 = D$, $D_2 = F$. //

If we assume the GCH and $|B|$ is regular we can improve this result by using 3.3 instead of 3.2. We can then prove the above result with $\beta = |B|$ instead of $|A^B|$.

Looking at 4.12 again we see that Conclusion (1) gives us a cardinality result, Conclusion (2) some completeness properties of the ultrafilter, and Conclusion (3) isomorphism of ultrapowers of different index sizes. We venture a conjecture at this point.

4.13 Conjecture

Let A be a full structure on a set $\mathbb{A}$ and let $\langle I_k, D_k \rangle$, $k = 1,2$ be such that

(1) $|I_1| < |I_2|$

(2) D_2 is uniform on I_2

(3) $\alpha \leq |I_k|$ implies $\alpha \in \sigma(D_k)$ for $k = 1,2$.

Then
$$\mathbb{A}^{I_1}/D_1 \not\cong \mathbb{A}^{I_2}/D_2 \quad . \qquad //$$

We may weaken this a little by replacing (1) with (1'): $|I_1| < \mathrm{cf}|I_2|)$. Note that by 2.6 we have that 4.13 is a theorem if $|A| \geq |I_2|$ and also if $|I_1| < |A| < |I_2|$ we can show that 4.13 holds. For if $\mathbb{A}^{I_1}/D_1 \cong \mathbb{A}^{I_2}/D_2$ where $|I_1| < |A| < |I_2|$ then by 4.5 we can conclude $|A| \notin \sigma(D_2)$.

This conjecture, if verified, would settle some interesting cardinality questions. For example consider the ultrapower

$$\omega_p^{\omega_q} / D \qquad \text{where} \quad p < q < \omega$$

and D is uniform on ω_q.

By Chang's Theorem we know that $\sigma(D) \supseteq \{\alpha : \omega \leq \alpha \leq \omega_q\}$ so that by 2.20 we have

$$|\omega_p^{\omega_q}/D| \geq \omega_{p+1} \quad .$$

But if we assume GCH and 4.13 we can apply 4.12 with $\Omega = B$ to conclude

$$|\omega_p^{\omega_q}/D| \geq \omega_q$$

because 4.12(2) contradicts Chang's Theorem and 4.12(3) is ruled out

by 4.13. Conjecture 4.13 would be even more useful in conjunction with further results on the shadow $\sigma(D)$ of an ultrafilter D. We prove more about $\sigma(D)$ in the following section, but much still remains unknown in this area.

D. Further Results on the Shadow

We prove here two additional results about the form of sets of cardinals $\sigma(D)$, adding to the information that Chang's Theorem gives us. For simplicity of proof we shall employ the GCH in the proof of the next result. However, it is sufficient to assume the LCH, or in part (2), that α is a strong limit cardinal.

4.14 Theorem (GCH)

Let D be an ultrafilter on I and let α be a limit cardinal $\leq |I|$ such that $\alpha = \sup\{\beta < \alpha : \beta \in \sigma(D)\}$, (i.e., α is a limit point of $\sigma(D)$) then

(1) either $\alpha \in \sigma(D)$ or $\alpha^+ \in \sigma(D)$

(2) if $cf(\alpha) \in \sigma(D)$ then $\alpha \in \sigma(D)$.

Proof

Let $\{\alpha_\eta : \eta < \beta\} \subseteq \sigma(D)$ be such that $\alpha = \sup\{\alpha_\eta : \eta < \beta\}$. For each $\eta < \beta$ let π_η be a partition of I into α_η pieces that determines a uniform image D_η of D on α_η .

(1) The common refinement $\Lambda\{\Pi_\eta : \eta < \beta\}$ of all these partitions determines an image D^* of D on a set

of cardinal $\Pi\langle\alpha_\eta : \eta < \beta\rangle$. But

$$\Pi\langle\alpha_\eta : \eta < \beta\rangle \leq \alpha^\beta = \alpha^+ \quad \text{as } cf(\alpha) \leq \beta \leq \alpha .$$

Hence, by expansion if necessary, we can take D^* to be on α^+.

As it is clear that $D_\eta \leq D^*$ for all $\eta < \beta$ we must have

$$th(D^*) \geq \alpha_\eta \qquad \text{for all } \eta < \beta .$$

So $th(D^*) \geq \alpha$, hence $th(D^*) = \alpha$ or α^+. /

(2) The result is trivial unless $cf(\alpha) < \alpha$ and it is easily seen that we can take $\beta = cf(\alpha)$. Let E be a uniform image of D on β determined, say, by a partition $\{I_\eta : \eta < \beta\}$ of I. We construct a new partition Π of I as follows:

$i\Pi i'$ iff there exists $\eta < \beta$ such that $i, i' \in I_\eta$ and $i\Pi_\xi i'$ for all $\xi < \eta$.

It is easy to see that Π refines Π_ξ on the set

$$\cup\{I_\eta : \xi < \nu < \beta\} = Y_\xi .$$

But as E is uniform the sets Y_ξ $(\xi < \beta)$ all belong to D, so Π refines each Π_ξ almost everywhere.

Now

$$|\Pi| \leq \Sigma\langle\Pi\langle\alpha_\xi : \xi < \eta\rangle : \eta < \beta\rangle$$

as each I_η is broken up by Π into at most $\Pi\langle\alpha_\xi : \xi < \eta\rangle$ pieces.

So

$$|\Pi| \leq \Sigma\langle\alpha_\eta^{|\eta|} : \eta < \beta\rangle$$

$$\leq \Sigma\langle\alpha_\eta^{+} : \eta < \beta\rangle = \alpha ,$$

and we can take Π to determine an image $\hat{D}$ of D on α. By construction of Π, $D_\eta \leq \hat{D}$ for all $\eta < \beta$. Hence $th(\hat{D}) = \alpha$ and $\alpha \in \sigma(D)$. //

Our next theorem on $\sigma(D)$ depends on a cardinality result of Andrew Adler, which we state as a lemma:

4.15 Lemma (GCH) (A. Adler)

Let D be a uniform ultrafilter on α^+ and let E be any ultrafilter on β. Then if

$$|\alpha^{\alpha^+}/D| = |\alpha^\beta/E|$$

and $$|cf(\alpha)^{\alpha^+}/D| = |cf(\alpha)^\beta/E|$$

then we must have $\beta \geq \alpha$.

Proof

By 1.14(b) we have $|\alpha^{+\alpha^+}/D| > \alpha$ and so by 1.15(h), GCH we have $|\alpha^{\alpha^+}/D| > \alpha$. Thus

$$|\alpha^\beta| \geq |\alpha^\beta/E| > \alpha$$

which, with GCH, implies that $\beta \geq cf(\alpha)$. By 1.14(c)(ii) we have for any $\langle I,F\rangle$, γ, δ that

$$|(\alpha^\delta)^I/F| \leq |\gamma^I/F|^{|\delta^I/F|} .$$

In this inequality set $\gamma = \alpha$, $\delta = cf(\alpha)$, $I = \alpha^+$ and $F = D$ to obtain

$$|(\alpha^+)^{\alpha^+}/D| \leq |\alpha^{\alpha^+}/D|\,|cf(\alpha)^{\alpha^+}/D|$$

so

$$\alpha^+ < |\alpha^{\alpha^+}/D|\,|cf(\alpha)^{\alpha+}/D| \qquad (*)$$

If $\beta < \alpha$ then $|\alpha^\beta/E| = |\alpha^{\alpha^+}/D| = \alpha^+$. By hypothesis $|cf(\alpha)^\beta/E| = |cf(\alpha)^{\alpha^+}/D|$. Now $\beta \geq cf(\alpha)$ so

$$|cf(\alpha)^\beta/E| \leq \beta^\beta = \beta^+ \leq \alpha .$$

Thus

$$|\alpha^{\alpha^+}/D|\,|cf(\alpha)^{\alpha^+}/D| \leq (\alpha^+)^\alpha = \alpha^+$$

which contradicts (*). Hence $\beta \geq \alpha$. //

<u>4.16 Theorem</u> (GCH)

Let D be an ultrafilter on I, then if $\alpha^+ \in \sigma(D)$ either (1) $\alpha \in \sigma(D)$ or (2) α is a limit point of $\sigma(D)$.

<u>Proof</u>

Suppose to the contrary that there exists $\beta < \alpha$ such that $\beta < \gamma \leq \alpha$ implies $\gamma \notin \sigma(D)$. By Chang's Theorem we can take α to be a limit cardinal so that $2^{2^{2^\beta}} = \beta^{+++} < \alpha < \alpha^+$ and 4.4 applies to show that

$$h^\# : \alpha^\beta/E \to \alpha^{\alpha^+}/D$$

is an isomorphism (w.r.t. any structure on α) for some $h : \alpha^+ \to \beta$

and where $E = h(D)$.

But if A is a subset of α of power $cf(\alpha)$ then $h^{\#}$ preserves the corresponding unary predicate and so restricts to a bijection

$$h^* : A^{\beta}/E \to A^{\alpha^+}/D .$$

Thus

$$|\alpha^{\beta}/E| = |\alpha^{\alpha^+}/D|$$

and
$$|cf(\alpha)^{\beta}/E| = |cf(\alpha)^{\alpha^+}/D|$$

and so by 4.5 $\beta \geq \alpha$, a contradiction. //

We may prefer to express this theorem in the contrapositive:

If $\alpha \notin \sigma(D)$ then $\alpha^+ \in \sigma(D)$ iff α is a limit point of $\sigma(D)$.

E. Further Results on Quasicomplete Ultrafilters

An important and so yet not fully resolved question in the theory of ultrafilters is the question of whether there exist ultrafilters D on sets I of "modest" cardinality such that $\gamma \notin \sigma(D)$ for some regular cardinal $\gamma < |I|$. As we observed in Section B the existence of such a γ implies that D is (γ,λ)-quasicomplete for some $\lambda > \gamma$, in fact D will have a γ-quasicomplete image on λ. Equivalently we can ask for each $\kappa \geq \omega$ which cardinals $\lambda > \kappa^+$ support κ-quasicomplete ultrafilters. We already know from Section B that we can take $\lambda = \mu$, the first measurable cardinal. We also have the following theorem, which generalises

the result of Chang and Prikry for indecomposable ultrafilters that we mentioned in Section B.

4.17 Theorem (GCH)

If D is a κ-quasicomplete ultrafilter on a set of cardinal $\lambda > \kappa^+$ then either λ is inaccessible or $cf(\lambda) \leq \kappa$.

Proof

By 4.16 λ is a limit cardinal as if $\lambda = \delta^+$ for some then either $\delta \in \sigma(D)$ or δ is a limit point of $\sigma(D)$. But by hypothesis $\kappa < \gamma < \lambda$ implies $\gamma \notin \sigma(D)$.

Also as D is uniform on a set of cardinality λ we have $\lambda \in \sigma(D)$ and hence $cf(\lambda) \in \sigma(D)$. But then we must have either $cf(\lambda) = \lambda$ or $cf(\lambda) \leq \kappa$. //

Throughout the rest of this section D is a κ-quasicomplete ultrafilter on a set I of cardinality λ where $2^{2^{2^\kappa}} < \lambda$. By 4.4 this means that there is a map $h : I \to \kappa$ such that the injection

$$h_\gamma^{\#} : \gamma^\kappa/E \to \gamma^I/D \qquad (E = h(D))$$

is onto for all $\gamma < \lambda$ and hence induces isomorphisms $\mathbb{A}^\kappa/E \cong \mathbb{A}^I/D$ for all structures $\mathbb{A}$ such that $|A| = \gamma < \lambda$.

For uniformity of notation we will take $\kappa = J$ and set $h^{-1}(j) = I_j$ for all $j \in J$. We define a preordering on the subsets of I as follows:

If $X, Y \subseteq I$ set

$$X \preccurlyeq Y \qquad \text{iff} \quad \{j : X \cap I_j \subseteq Y \cap I_j\} \in E .$$

We are interested in the following statement:

$C(\beta)$: if $\{X_\eta : \eta < \beta\}$ is a collection of β elements of D then there is an $X \in D$ such that $X \leqslant X_\eta$ for all $\eta < \beta$.

We shall show that if λ is a strong limit cardinal (and, if LCH holds, it always is) that $C(\beta)$ holds for all $\beta < \lambda$. This is the "completeness" property that motivates the name "quasicomplete". Suppose, then, that λ is a strong limit cardinal and that $\{X_\eta : \eta < \beta\}$ is a collection of $\beta < \lambda$members of D. Define a partition Π of I by

$$i\Pi i' \longleftrightarrow \text{for all } \eta < \beta \quad i \in X_\eta \text{ iff } i' \in X_\eta$$

Then Π partitions I into at most 2^β cells. Suppose that Π breaks up each I_j so that

$$I_j = \cup\{I_j^\xi : \xi < 2^\beta\} \quad .$$

As $2^\beta < \lambda$ we have that $h^{\#}_{2^\beta}$ is onto and so by 3.1 there is a sequence $\langle\xi_j : j \in J\rangle \in (2^\beta)^J$ such that

$$\cup\{I_j^{\xi_j} : j \in J\} \in D.$$

Now take $X = \cup\{I_j^{\xi_j} : j \in J\}$ then it is clear that

$$\{j : X \cap I_j \text{ meets } X_\eta \cap I_j\} \in E$$

but $X \cap I_j = I_j^{\xi_j}$ so that if $X \cap I_j$ meets $X_\eta \cap I_j$ then $X \cap I_j \subseteq X_\eta \cap I_j$ so that

$$\{j : X \cap I_j \subseteq X_\eta \cap I_j\} \in E$$

and hence $X \preccurlyeq X_\eta$ for all $\eta < \beta$.

Notice that we have, in fact, shown that $C(\beta)$ holds for all β such that $2^\beta < \lambda$.

Returning to the assumption that λ is a strong limit cardinal we note that if $cf(\lambda) < \lambda$ (and hence $\leq \kappa$) we also have $C(\lambda)$. For let $\lambda = \cup\{\alpha_\xi : \xi < \beta\}$ where $\beta < \lambda$ and $\alpha_\xi < \lambda$ for all $\xi < \beta$ and let $\{X_\eta : \eta < \lambda\}$ be a collection of members of D. Then as $C(\alpha_\xi)$ holds there are sets $X^\xi \in D$ such that

$$X^\xi \preccurlyeq X_\eta \qquad \text{for all} \quad \eta < \alpha_\xi \quad (\xi < \beta)$$

and as $C(\beta)$ holds there is a set $X \in D$ such that

$$X \preccurlyeq X^\xi \qquad \text{for all} \quad \xi < \beta$$

and it follows easily that

$$X \preccurlyeq X_\eta \qquad \text{for all} \quad \eta < \lambda \ .$$

We have shown that if $\mathbb{A}$ is any structure such that $|A| = \alpha < \lambda$ then the embedding

$$h^{\#}_A : \mathbb{A}^J/E \to \mathbb{A}^I/D$$

is onto, and hence an isomorphism. On the other hand if $|A| = \lambda$ then by 2.6 $h^{\#}_{\mathbb{A}}$ is not onto as $E \not\approx D$ and hence $\mathbb{A}^J/E \not\cong \mathbb{A}^I/D$. However we have the following theorem which shows that the range of $h^{\#}_{\mathbb{A}}$ still froms a very comprehensive substructure of $\mathbb{A}^I/D$.

4.18 Theorem

Let $\mathbb{A}$ be any structure of cardinality λ. Then if Σ is any set of formulas from $\mathcal{F}_{\mathbb{A}}$ (c.f. Chapter 1, Section D) such that $|\Sigma| < \lambda$ then Σ is satisfiable in $\mathbb{A}^I/D$ iff Σ is satisfiable already in $h^{\#}_{\mathbb{A}}(\mathbb{A}^J/E)$.

Proof

Suppose $\Sigma = \{\phi_\eta : \eta < \beta\}$ where $\beta < \lambda$ and that $f/D \in \mathbb{A}^I/D$ is such that

$$\mathbb{A}^I/D \models \phi_\eta(f/D) \quad \text{for all } \eta < \beta .$$

Let $X_\eta = \{i \in I : \mathbb{A} \models \phi_\eta(f(i))\}$, then $X_\eta \in D$ for each $\eta < \beta$ by Los' Theorem. As $C(\beta)$ holds there is $X \in D$ such that $X \preccurlyeq X_\eta$ for all $\eta < \beta$.

We define $g \in A^I$ as follows:

if $X \cap I_j \neq \phi$ pick $i_0 \in X \cap I_j$ and define $g(i) = f(i_0)$ for all $i \in I_j$.

if $X \cap I_j = \phi$ pick $a_0 \in A$ and set $g(i) = a_0$ for all $i \in I_j$. $(j \in J)$.

Then g is constant on each I_j so that by 3.1, $g/D \in \text{range}(h^{\#}_{\mathbb{A}})$. But for each $\eta < \beta$ we have that

$$\{i : \mathbb{A} \models \phi_\eta(g(i))\} \supseteq Y_\eta$$

where $Y_\eta = \cup\{I_j : X \cap I_j \subseteq X \cap I_j\} \cap X$. For if $i \in Y_\eta$ then $g(i) = f(i_0)$ for some $i_0 \in X \cap I_j$ where $i \in I_j$

but then as $\qquad i_0 \in X_\eta \cap I_j$

we have $\qquad \mathbb{A} \models \phi_\eta(f(i_0))$

so that $\qquad \mathbb{A} \models \phi_\eta(g(i))$.

But $\qquad Y_\eta \in D \quad$ as

add remove

$$Y_\eta = h^{-1}(\{j : X \cap I_j \subseteq X_\eta \cap I_j\}) \cap X\}$$

$$= h^{-1}(Z_\eta) \cap X \qquad \text{where } Z_\eta \in E .$$

Hence $\{i : \mathbb{A} \models \phi_\eta(g(i))\} \in D$ so that $\mathbb{A}^I/D = \phi_\eta(g/D)$ for all $\eta < \beta$, and g/D satisfies Σ. As $g/D \in \text{range}\,(h^\#_{\mathbb{A}}) = h^\#_{\mathbb{A}}(\mathbb{A}^J/E)$ the theorem is proved. //

In the case of singular strong limit cardinals λ we have shown that $C(\lambda)$ holds and we can strengthen 4.18 to hold for sets of formulas Σ with $|\Sigma| = \lambda$.

As an illustration of 4.18 we will note a special case. Let $\underline{\alpha}$ be a structure on the cardinal α that has the well-order $<$ as one of its relations, and let B be an elementary extension of $\underline{\alpha}$. Then if β is any element of α we will say that <u>β is realised in B</u> if there is a $b \in B$ such that

$$B = \eta < b < \beta \qquad \text{for each } \eta < \beta.$$

Now regard $\underline{\lambda}$ and $\underline{\lambda}^J/E$ as substructures of $\underline{\lambda}^I/D$ by virtue of the embeddings ι and $h^\#$ respectively. If $\Sigma = \{(\eta<v_0<\beta) : \eta<\beta\}$ it is easy to see that realising β and satisfying Σ are equivalent conditions on extensions of $\underline{\lambda}$. Then we have immediately

4.19 Corollary

If $\beta < \lambda$ then β is realised in $\underline{\lambda}^I/D$ iff β is already realised in $\underline{\lambda}^J/E$. //

Informally, β is realised in $\underline{\alpha}^K/F$ iff β is not confinal in $\underline{\beta}^K/F(\subseteq \underline{\alpha}^K/F)$. We can show after the manner of 2.17 that β is realised in $\underline{\alpha}^K/F$ iff F is β-d.i. The result of 4.19 then would follow directly from the observation that $\sigma(D) = \sigma(E) \cup \{\lambda\}$.

BIBLIOGRAPHY

[1] A. Adler, Cardinalities of Ultrapowers: An Example, Proc. Amer. Math. Soc. 28 (1971) pp. 311-312.

[2] A. Adler, Representations of Models of Full Theories, to appear.

[3] A. Adler and M. A. Jorgensen, Inequalities for the cardinality of ultraproducts, to appear.

[4] J. L. Bell and A. B. Slomson, Models and Ultraproducts: An Introduction, North-Holland, Amsterdam (1969).

[5] D. Booth, Ultrafilters on a countable set, Annals of Math. Logic 2 (1970), pp. 1-24.

[6] C. C. Chang, Descendingly incomplete ultrafilters, Trans. Amer. Math. Soc. 126 (1967), pp. 108-118.

[7] T. Frayne, A. Morel, and D. Scott, Reduced direct products, Fund. Math. 51 (1962), pp. 195-228.

[8] H. J. Keisler, Limit ultrapowers, Trans. Amer. Math. Soc. 107 (1963), pp. 382-408.

[9] H. J. Keisler, Good ideals in fields of sets, Annals of Math. 79 (1964), pp. 338-459.

[10] H. J. Keisler, Ultrapowers and Saturated Models, Indeg. Math., 26 pp. 178-186.

[11] H. J. Keisler, On cardinalities of ultraproducts, Bull. Amer. Math. Soc. 70 (1964), pp. 644-647.

[12] K. Kunen and K. Prikry, On descendingly incomplete ultrafilters, to appear.

[13] K. Prikry, On a problem of Gillman and Keisler, Annals of math. logic 2 (1970), pp. 179-187.

[14] J. Silver, Indecomposable ultrafilters and $0^{\#}$, talk at Tarski Symposium, Berkeley, Calif., June 23-30, 1971.

[15] S. Sirota, On the problem of the classification of filters, Sov. Math. Doklady 11 (1970) pp. 408-411.

[16] A. Tarski, Ideale in vollständigen Mengenkörpen I, Fund. Math. 32 (1939) pp. 45-63.

[17] S. Ulam, Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math. 16 (1930), pp. 140-150.