MATRICES WHICH, UNDER ROW PERMUTATIONS, GIVE SPECIFIED VALUES OF CERTAIN MATRIX FUNCTIONS.

by

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ABSTRACT

Let S_n denote the set of $n \times n$ permutation matrices; let T denote the set of transpositions in S_n ; let C denote the set of 3-cycles $\{(r, r+1, t) ; r = 1, \ldots, n-2; t = r+2, \ldots, n\}$ and let I denote the identity matrix in S_n . We shall denote the n-1st elementary symmetric function of the eigenvalues of A by $E_{n-1}(A)$.

In this thesis, we pose the following problems:

1. Let H be a subset of S_n and $a_1, \ldots, a_k$ be k-distinct real numbers. Determine the set of n-square matrices A such that $\{tr(PA):P \in H\} = \{a_1, \ldots, a_k\}$. We examine the cases when

(i) $H = S_n, k = 1$

(ii) $H = \{\text{2-cycles in } S_n\}, k = 1$

(iii) $H = S_n, k = 2$.

2. Determine the set of $n \times n$ matrices such that $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \in S_n$.

3. Examine those orthogonal matrices which can be

expressed as linear combinations of permutation matrices.

The main results are as follows:

If R' is the subspace of rank 1 matrices with all rows equal and if C' is the subspace of rank 1 matrices with all columns equal, then the $n \times n$ matrices A such that $tr(PA) = tr(A)$ for all $P \in S_n$ form a subspace $S = R' + C'$. This implies that the rank of A is ≤ 2.

If $tr(PA) = tr(A)$ for all $P \in T$, then such A's form a subspace which contains all $n \times n$ skew-symmetric matrices and is of dimension $n(\frac{n+1}{2})$.

Let A be an n-square matrix such that $\{tr(PA) : P \in S_n\} = \{a_1, a_2\}$, where $a_1 \neq a_2$. Then A is either of the form $C = A_1 + A_2$, where $A_1 \in (R' + C')$ and A_2 has entries $a_1 - a_2$ at (r_j, r_1), $j = 2, \ldots, k$ and zeros elsewhere, or of the form C^T.

The set $B_1 = \{P \in S_n : tr(PA) = a_1\}$ consists of all 2-cycles (r_1, r_j), $j = 2, \ldots, k$ and the products P of disjoint cycles $P_1, P_2, \ldots, P_m$, $m \geq 1$, for which one of the P_i has its graph with an edge $r_1 \rightarrow r_j$ (or $r_j \rightarrow r_1$) for $j = 2, \ldots, k$.

If A is rank $n-1$ n-square matrix with the property that $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \in S_n$, then A is of the form

$$A = \begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{n-1} \\ -\sum_{i=1}^{n-1} U_i \end{pmatrix}$$, where U_i are the row vectors.

Finally, if $\theta = \sum_{i=1}^{r} \alpha_i P_i$, where all P_i are from an independent set TUCUI of S_n , is an orthogonal matrix, then $\sum_{i=1}^{r} \alpha_i = \pm 1$.

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INTRODUCTION

Let S_n denote the set of $n \times n$ permutation matrices; let T denote the set of all transpositions of S_n; let C denote the set of 3-cycles $\{(r, r+1, t); r = 1, \ldots, n-2; t = r+2, \ldots, n\}$ and let I be the identity matrix of S_n.

One of the primary aims of this thesis is to characterise the following sets of $n \times n$ matrices:

(1) $\{A : tr(PA) = tr(A)$ for all $P \varepsilon H\}$ where $H = S_n$ or $H = T$.

(2) $\{A : tr(PA) \varepsilon \{a_1, a_2, \ldots, a_k\}$ for all $P \varepsilon S_n$; where $a_1, a_2, \ldots, a_k$ are distinct and reals$\}$ (we are only partially successful when $k > 2$).

(3) $\{A : E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n\}$.

We also consider the following problem:

(4) What linear combinations of independent permutation matrices give orthogonal matrices? In particular, what linear combinations of independent permutation matrices give symmetric orthogonal matrices? Problem (4) is included to give an application of the results obtained in Chapter I to the solution of certain matrix theory problems.

Problems (1), (2) and (4), listed above demand the construction of a suitable linearly independent set in S_n. Problem (3) requires a similar construction using members of the set $\{C_{n-1}(P) : P \in S_n\}$, where C_{n-1} is the (n-1)-compound. These constructions of linearly independent sets form the substance of Chapter I.

In Chapter I, we show first that $T \cup C \cup I$ is an independent set in S_n, and in fact generates the whole of the set S_n (Thm. 1.8). Furthermore, if the matrix P of S_n is a linear combination $\sum_{i=1}^{r} \alpha_i P_i$; $P_i \in T \cup C \cup I$, then $\sum_{i=1}^{r} \alpha_i = 1$ (Cor. 1.9). In the last section of this chapter, the set $\{C_{n-1}(P) : P \in S_n\}$ is characterised and an independent set which generates $\{C_{n-1}(P) : P \in S_n\}$ is constructed (Thm. 1.22). Some information on the coefficients of linear combinations of the generators of $\{C_{n-1}(P) : P \in S_n\}$ is obtained (Remark 1.28). The chapter concludes with an observation on the set $\{C_r(P) : P \in S_n\}$ where $r \neq 1$. (Note 1.29).

In Chapter II, we determine the structure of matrices A such that $tr(PA) = tr(A)$ for all $P \in S_n$ (Thm. 2.1). An immediate corollary of this structure theorem is that $rank(A) \leq 2$. Theorem (2.15), which actually generalises theorem (2.1), gives complete information about those matrices A such that $tr(PA) \in \{a_1, a_2\}$ for all $P \in S_n$ and where $a_1 \neq a_2$ are real. Some partial information about those matrices A for which $tr(PA) \in \{a_1, \ldots, a_k\}$ for all $P \in S_n$, where $a_i \neq a_j$ and a_i are reals for $i, j = 1, \ldots, k$ is given (Remark on Theorem (2.15)). Theorem (2.17) gives a necessary condition on A such that $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n$, and Theorem (2.27) characterises such A's completely.

Finally, Chapter III deals with those orthogonal matrices which can be expressed as linear combinations of permutation matrices. Three types of such linear combinations are considered.

First, we consider those orthogonal matrices which can be expressed as linear combinations of the elements of $T \cup I$. A necessary condition that such a linear combination be an orthogonal matrix is that the sum of the coefficients in the linear combination be ± 1 (Theorem (3.3) and Theorem (3.7)). Theorem (3.9) states that given a subset $\{(r, s)\}$ of 2-cycles for which the combined graph is strongly connected and complete (as an undirected graph), there exists an orthogonal matrix $\theta = \sum_r \sum_s \alpha_{rs}(r, s) + \lambda I$ such that every $\alpha_{rs} \neq 0$.

Secondly, we examine those orthogonal matrices which can be expressed as linear combinations of the elements in C (Theorem (3.10)).

Lastly, Theorem (3.14) gives a necessary condition for the linear combination to yield on orthogonal matrix when the independent permutations are chosen from the whole set $T \cup C \cup I$.

CHAPTER I

GENERATING SETS OF $n \times n$ PERMUTATION MATRICES AND THEIR $(n-1)$-COMPOUNDS

In order to facilitate computation in this chapter, we shall use a graphical representation of matrices. First, we need a few definitions concerning graphs [1] and matrices [2].

1.1 <u>Definitions</u>:

Denote the cartesian product of two sets P and Q by $P \times Q$. If $G \subseteq (V \times V) \times R$, where V is a non-empty set and R is the set of real numbers, then $G = \{(v, w; \alpha)\}$ is called a directed graph provided that for every pair (v, w) in $V \times V$, there is only one $\alpha \in R$. The elements of V are called vertices of the graph G and α is the weight on the edge joining v to w. Graphically,

$$v \xrightarrow{\alpha} w$$

.2 If V is a finite set, then G is called a finite graph, otherwise it is called an infinite graph. We shall consider finite graphs only.

.3 If a vertex v_1 in a directed graph $G = \{(v, w; \alpha)\}$ is such that in every triple (v, w, α), $\alpha = 0$ whenever $v = v_1$ or

$w = v_1$ except possibly for $v = w = v_1$, then the vertex v_1 is said to be an isolated vertex of G.

.4 By a graph of an $n \times n$ matrix (a_{ij}), we mean a directed graph $\{(v_i, v_j; a_{ij})\}$, where there are n vertices (v_i); $i = 1, \ldots, n$, and the weight on the edge joining v_i to v_j is a_{ij}

i.e. $$v_i \xrightarrow{a_{ij}} v_j$$

e.g. the graph of the identity matrix I

consists of $\{(v_i, v_j, \delta_{ij})\}$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. This graph consists of single loops of the form v_i at all v_i, $i = 1, \ldots, n$ and the weight on each loop is 1.

Similarly, a zero matrix corresponds to a zero graph i.e. a graph in which the weight on each edge is zero i.e. $G = \{(v, w; 0)\}$.

.5 If two directed graphs G_1 and G_2 are such that both have the same set of vertices (V) and if $G_1 = \{(v, w; \alpha_1)\}$, $G_2 = \{(v, w, \alpha_2)\}$, then $G_1 + G_2$ is also a directed graph with its set of vertices equal to V and it is given by $\{(v, w; \alpha_1 + \alpha_2)\}$.

.6 If $G = \{(v, w; \alpha)\}$ is a directed graph, then for $\beta \in R$, $\beta G = \{(v, w; \beta\alpha)\}$ is also a directed graph for any real β.

.7 Finally (cf. [2]), the r-th compound $C_r(A)$ of a $n\times n$ matrix A is the $\binom{n}{r}\times\binom{n}{r}$ matrix whose entries are $d(A[\alpha|\beta])$, $\alpha,\beta \in Q_{r,n}$ arranged lexicographically in α and β; where if $1 \leq k \leq n$, then $Q_{k,n}$ denotes the totality of strictly increasing sequences of k-integers chosen from $1, \ldots, n$; $d[\alpha|\beta]$ denotes the determinant of the submatrix of A lying in the rows indicated by integers in α and the columns by β.

Also, if A is an $r\times n$ matrix and the r-rows of A are denoted by $U_1, \ldots, U_r$ in succession $(1 \leq r \leq n)$, then $C_r(A)$ is an $\binom{n}{r}$ tuple and is sometimes called the Grassmann Product or Skew-symmetric Product of the vectors $U_1, \ldots, U_r$. The usual notation for this is $U_1 \wedge \ldots \wedge U_r$. From the properties of determinants, it follows, for a permutation σ in S_r, that

$$U_{\sigma(1)} \wedge \ldots \wedge U_{\sigma(r)} = \operatorname{sgn}\sigma\, U_1 \wedge \ldots \wedge U_r .$$

Furthermore, if B is an $n\times n$ matrix, then

$$C_r(B)\, U_1 \wedge \ldots \wedge U_r = B U_1 \wedge \ldots \wedge BU_r .$$

We denote the set of $n\times n$ permutation matrices by S_n. In S_n, we denote an m-cycle by $(r_1, \ldots, r_m)$. We shall use the terms permutation and permutation matrix interchangeably. Accordingly, by the graph of $(r_1, \ldots, r_m)$ we mean the graph of the corresponding permutation matrix. Its vertices are the integers $1, \ldots, n$. It should be noted that in the graph of $(r_1, \ldots, r_m)$, there are 1-cycles called loops at all the vertices j, $j \neq r_1, \ldots, r_m$. The following will give an independent set in S_n which generates S_n as linear combinations over the reals.

1.8 Theorem:

If T is the set of all 2-cycles, I the identity matrix and C is the set of 3-cycles $\{(r, r+1, k) ; r = 1, \ldots, n-2$ and $k = r+2, \ldots, n\}$ in S_n, then the set $T \cup C \cup I$ is an independent set in S_n, and it generates the whole set S_n as linear combinations over the reals. Furthermore, the cardinality of $T \cup C \cup I$ is $(n-1)^2 + 1$.

Proof: The number of elements in $T \cup C \cup I$ is $\frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} + 1 = (n-1)^2 + 1$. This is the dimension of a maximal independent set in S_n (see [2]; pp. 99-100). Thus we need only show that the set $T \cup C \cup I$ generates the set S_n.

First, we show that every 3-cycle in S_n is generated by the set $T \cup C \cup I$. Let us consider a cycle (r_1, r_2, r_3) not belonging to C.

Case (i) $r_1 < r_2 < r_3$

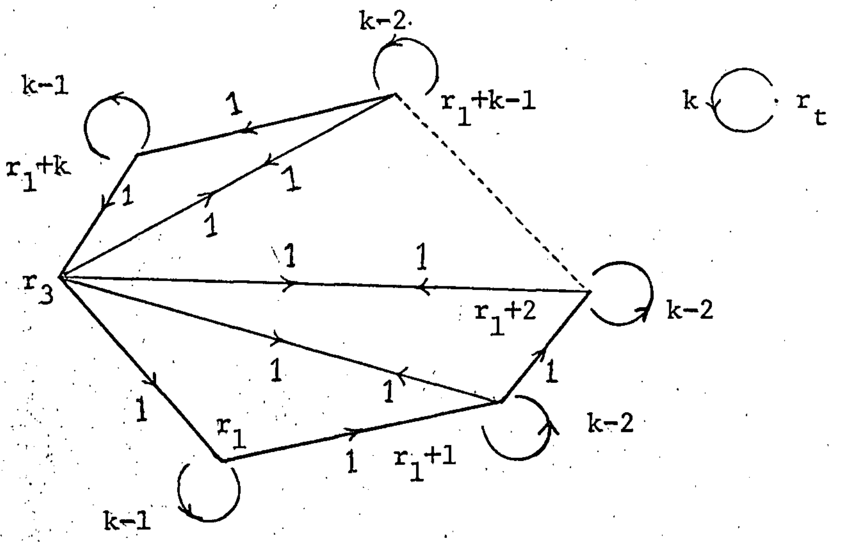
We write (r_1, r_2, r_3) as (r_1, r_1+k, r_3) and claim that

1.9 (r_1, r_1+k, r_3)

$$= \sum_{i=1}^{k} (r_1 + i-1, r_1+i, r_3) - \sum_{i=1}^{k-1} (r_1+i, r_3) + \sum_{i=1}^{k-1} (r_1+i-1, r_1+k)$$

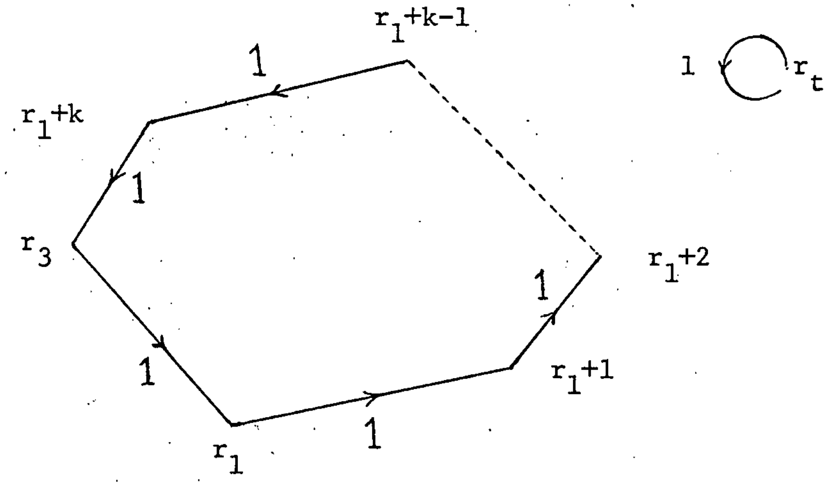
$$- \sum_{i=1}^{k-1} (r_1+i-1, r_1+i, r_1+k)$$

We show that the graph of the RHS is equal to that of the LHS. In the RHS of (1.9) the graph of the first sum is

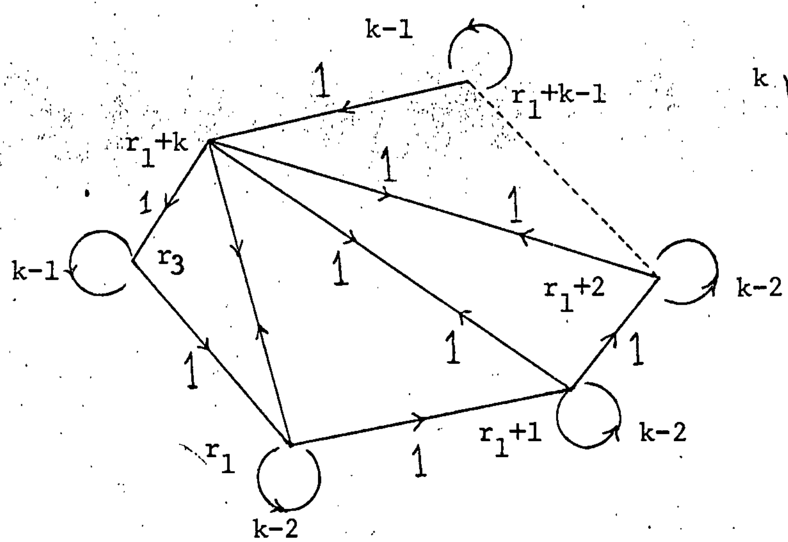


In this graph, the weight on the single loops at the vertices r_1 and r_1+k is $k-1$ whereas the weight on each of the single loops at the remaining vertices between r_1 and r_1+k is $k-2$

and the weight on the loop at r_3 is zero. Clearly, each of the isolated vertices (r_t) carry a weight k. The weights on the edges are shown in the graph. Now, subtracting from this the graph of the second sum in the RHS of (1.9), we get

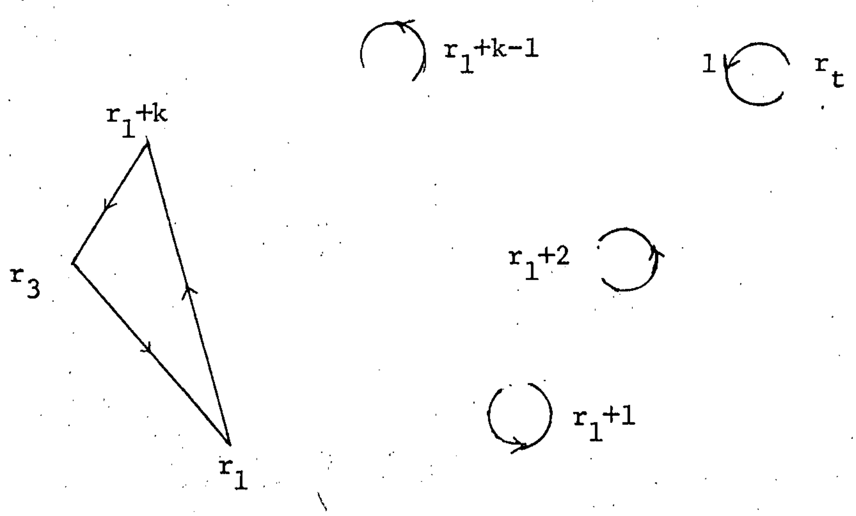


Adding the graph of the third sum of the RHS of (1.9) to the above graph, we get



Since we have added $(k-1)$ 2-cycles, therefore, the weight on single loops at the vertices $r_1, \ldots, r_1+k-2$ is $(k-2)$ and on the loops at the vertices r_1+k-1 and r_3 carry weight $(k-1)$ each. Also, on each of the isolated vertices, the weight on these loops is $(k-1) + 1 = k$.

Subtracting the graph of the last sum in the RHS of (1.9) from the above graph, we get



This is also the graph of the LHS (1.9)

Case (ii) $\quad r_1 > r_2 > r_3$

In this case, we have

$$1.10 \quad (r_1, r_2, r_3) = (r_1, r_2) + (r_1, r_3) + (r_2, r_3) - (r_3, r_2, r_1) - I .$$

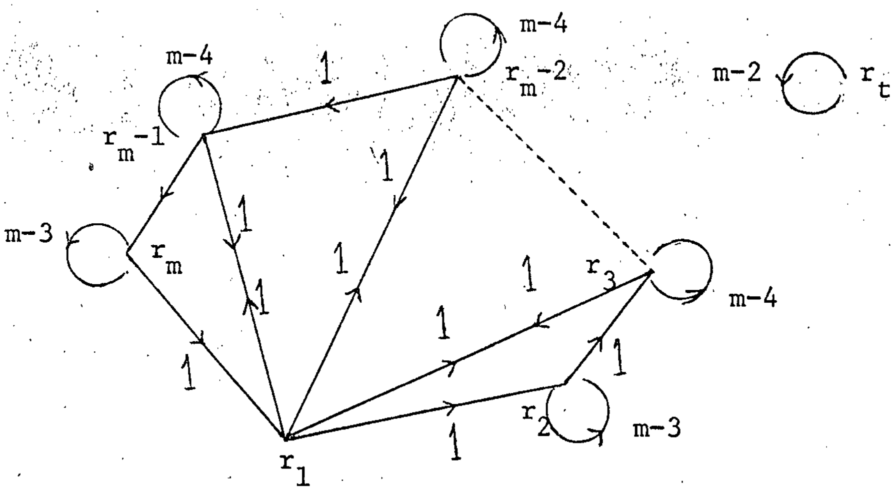
Clearly, the graphs of the RHS and the LHS of (1.10) are identical. Now the 3-cycle (r_3, r_2, r_1) is given by (1.9). Hence, the RHS of (1.10) consists of members of the set $T \cup C \cup I$.

From (1.9) and (1.10) it follows that every 3-cycle in S_n can be written as a linear combination of the members of $T \cup C \cup I$.

Now, consider any cycle $\sigma = (r_1, \ldots, r_m)$ we claim

1.11 $$(r_1, \ldots, r_m) = \sum_{i=1}^{m-2} (r_1, r_{i+1}, r_{i+2}) - \sum_{i=1}^{m-3} (r_1, r_{i+2}) .$$

In the RHS of (1.11), the graph of the first sum is



From this it follows immediately that the graph of the RHS of (1.11) is equal to the graph of the LHS of (1.11). Moreover, in the RHS of (1.11) every 3-cycle, not belonging to the set $T \cup C \cup I$

can be expressed in the form (1.9) or (1.10). Therefore, the RHS of (1.11) can be written as a linear combination of the members of $T \cup C \cup I$.

Finally, consider the case when a permutation is the product of two or more disjoint cycles. Let P_i ; $i = 1, \ldots, m$ be the disjoint cycles and $P = P_1 \ldots P_m$. We claim that

1.12 $$P_1 \ldots P_m = P_1 + \ldots + P_m - (m-1)I \; .$$

Clearly, the weight on the isolated vertices is 1, it therefore, follows that the graph of the RHS of (1.12) is equal to the graph of LHS (1.12). Also, (1.9), (1.10) and (1.11) express the RHS of (1.12) in terms of the members of the set $T \cup C \cup I$.

Hence, the set $T \cup C \cup I$ does generate the whole set S_n.

Q.E.D.

1.13 Notation:

We designate the set $T \cup C \cup I$ by M.

1.14 Corollary:

Every permutation matrix $P \varepsilon S_n$ can be written as

$$P = \sum_{i=1}^{r} \alpha_i P_i \; ; \quad P_i \varepsilon M \quad \text{such that} \quad \sum_{i=1}^{r} \alpha_i = 1 \; .$$

Proof: If $P \in M$, then there is nothing to prove, therefore suppose $P \notin M$. It is either a cycle or a product of disjoint cycles.

If P is a 3-cycle of the form (1.9), then from the RHS of (1.9) it is immediate that $\sum_{i=1}^{r} \alpha_i = 1$. If P is a 3-cycle of the form (1.10), then it is an easy consequence of the preceding statement that $\sum_{i=1}^{r} \alpha_i = 1$.

Similarly, if $P = (r_1, \ldots, r_m)$ or $P = P_1 \ \ldots \ P_m$ then from (1.11) and (1.12) respectively, it follows that $\sum_{i=1}^{r} \alpha_i = 1$.

The entries of $C_r(P)$; $P \in S_n$ are either 0, 1 or -1. In order to discuss such compounds, it is convenient to introduce the following notation:

1.15 Notation:

We have identified [pp. 6] the cycle $\sigma = (r_1, \ldots, r_m)$ with the permutation matrix P where $P_{r_i, r_{i+1}} = 1$ for $i = 1, \ldots, m$, $(r_{m+1} = r_1)$; $P_{kk} = 1$ for $k \in \{\{1, \ldots, n\} \setminus \{r_1, \ldots, r_m\}\}$ and $P_{ij} = 0$ otherwise. We now denote by $\tau = \overset{s_1, \ldots, s_m}{(r_1, \ldots, r_m)_t}$ the matrix Q such that $Q_{r_i, r_{i+1}} = s_i 1$ for $i = 1, \ldots, m$; $Q_{k,k} = t1$ for

$k \in \{\{1, \ldots, n\} \setminus \{r_1, \ldots, r_m\}\}$ and $Q_{ij} = 0$ otherwise, where each s_i and t are either the symbol $+$ or the symbol $-$.

For example: $\sigma = (1, 3, 4)$ is identified with

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

while $\tau = (1^+, 3^-, 4^+)_-$ is the matrix

$$\begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \end{bmatrix}$$

In fact $\sigma = (1^+, 3^+, 4^+)_+$ in the new notation.

1.16 Lemma:

$$C_{n-1}(r, s) = (n-r^{s_1}+1, n^{s_1}-s+1)_- \quad \text{where } n \geq 3 \text{ and}$$

$$\begin{aligned} s_1 &= + \quad \text{if } r+s \text{ is odd} \\ &= - \quad \text{if } r+s \text{ is even}. \end{aligned}$$

Proof: Suppose P is the matrix of a linear transformation of n-space R^n relative to the unitary basis $U_1, \ldots, U_n$. Then $C_{n-1}(P)$ is the matrix of a linear transformation of R^n relative to the basis $U_1 \wedge \dots \wedge U_{i-1} \wedge U_{i+1} \wedge \dots \wedge U_n$; $i = n, n-1, \dots, 1$. Now if $P = (r, s)$, then

$$C_{n-1}(P)U_1 \wedge \dots \wedge U_{i-1} \wedge U_{i+1} \wedge \dots \wedge U_n$$

$$= PU_1 \wedge \dots \wedge U_{i-1} \wedge PU_{i+1} \wedge \dots \wedge PU_n$$

$$= -U_1 \wedge \dots \wedge U_{i-1} \wedge U_{i+1} \wedge \dots \wedge U_n \qquad \text{if} \quad i \neq r, s ,$$

since $PU_1 \wedge \dots \wedge PU_{i-1} \wedge PU_{i+1} \wedge \dots \wedge PU_n$ is just $U_1 \wedge \dots \wedge U_{i-1} \wedge U_{i+1} \wedge \dots \wedge U_n$ with U_r and U_s interchanged. This shows that the (j, j) element of $C_{n-1}(P)$ is -1 , if $j \neq r, s$.

If $i = r$, then

$$C_{n-1}(P)U_1 \wedge \dots \wedge U_{i-1} \wedge U_{i+1} \wedge \dots \wedge U_n$$

$$= (-1)^{r+s-1} U_1 \wedge U_2 \wedge \dots \wedge U_{s-1} \wedge U_{s+1} \wedge \dots \wedge U_n ,$$

therefore the $(n-r+1, n-s+1)$ element of $C_{n-1}(P)$ is $(-1)^{r+s-1}$. Similarly, if $i = s$, the $(n-s+1, n-r+1)$ element is $(-1)^{r+s-1}$.

Q.E.D.

1.17 Lemma:

Let $\sigma = (r_1, \dots, r_m)$; $(r_{m+1} = r_1)$ be any cycle in S_n . If the r_i-th row and the r_{i+1}-th column are deleted from the matrix σ, then the determinant of the remaining $(n-1)\times(n-1)$ submatrix is $(-1)^{r_i+r_{i+1}}$ sgn σ.

<u>Proof:</u> Write

$$\sigma = (r_i, r_{i+1})(r_i, r_{i+2}) \ldots (r_i, r_{i-1}) \quad , \quad \text{then}$$

$$C_{n-1}(\sigma) = C_{n-1}(r_i, r_{i+1}) \ldots C_{n-1}(r_i, r_{i-1})$$

$$(C_{n-1}(AB) = C_{n-1}(A)C_{n-1}(B) \quad ; [2])$$

1.18 $$= (n-r_i+1^{s_1}, n-r_{i+1}^{s_1}+1)_- \ldots (n-r_i+1^{s_2}, n-r_{i-1}^{s_2}+1)_- \quad .$$

where $s_1, \ldots, s_2$ are given by (1.16).

The determinant of the submatrix of σ obtained by removing the r_i-th row and the r_{i+1}-th column is given by the entry in position $\{n-r_i+1, n-r_{i+1}+1\}$ in $C_{n-1}(\sigma)$. From (1.18) $C_{n-1}(\sigma) = (n-r_i+1^{s_1}, n-r_{i+1}^{s_1}+1)_-P$, where the sole nonzero element in the $(n-r_{i+1}+1)$-th row is $(-1)^{m+2}$ and this occurs on the diagonal of P. Thus the $\{n-r_i+1, n-r_{i+1}+1\}$ element of $C_{n-1}(\sigma)$ is $(-1)^{r_i+r_{i+1}+1}(-1)^{m+2} = (-1)^{r_i+r_{i+1}}$ sgn σ. Hence the lemma.

From the above Lemma, we get

1.19 Corollary:

$$C_{n-1}(\sigma) = (n-r_1+1^{s_1}, n-r_2+1^{s_2}, \ldots, n-r_m+1^{s_m})_{\mathrm{sgn}\,\sigma}$$

where $\sigma = (r_1, \ldots, r_m)$.

From (1.18), the sign of the diagonal elements in $C_{n-1}(\sigma)$ is $(-1) \ldots (-1)$, $(m-1)$ times i.e. $\mathrm{sgn}\,\sigma$. Moreover, $s_1, \ldots, s_m$ are determined by (1.17).

Thus given any permutation matrix, its $(n-1)$-compound can be computed by the above formulae.

e.g. if $\sigma = (2, 6, 5, 7, 8, 9)$, then for $n \geq 9$

$C_{n-1}(\sigma) = (n-1^-, n-5^+, n-4^-, n-6^+, n-7^+, n-8^+)_-$.

In this case $\mathrm{sgn}\,\sigma = -1$.

In the special case of a 3-cycle, where $\sigma = (r_1, r_2, r_3)$, then

1.20 $C_{n-1}(r_1, r_2, r_3) = (n-r_1+1^{s_1}, n-r_2+1^{s_2}, n-r_3+1^{s_3})_+$ where s_i is the sign of $(-1)^{r_i+r_{i+1}}$ and $r_4 = r_1$.

1.21 Notation:

Let $T_1 = \{C_{n-1}(P) : P \varepsilon T\}$, $C_1 = \{C_{n-1}(P) : P \varepsilon C\}$,

where T, C are defined in the Thm. (1.8).

1.22 Theorem:

The set $T_1 \cup C_1 \cup I$ is linearly independent and generates the set $\{C_{n-1}(P) : P \varepsilon S_n\}$ as linear combinations over the reals. The cardinality of $T_1 \cup C_1 \cup I$ is $(n-1)^2+1$.

Proof: First, we establish the independence of the set $T_1 \cup C_1 \cup I$. For all α's , β's and r real, we assume that

$$\alpha_{n-1\ n}\, C_{n-1}(1,\ 2) + \ldots + \alpha_{1\ n}\, C_{n-1}(1,\ n) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \alpha_{12}\, C_{n-1}(n-1,\ n) +$$

$$\beta_{n\ n-2}\, C_{n-1}(1,\ 2,\ 3) + \ldots + \beta_{n\ 1}\, C_{n-1}(1,\ 2,\ n) +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \beta_{31}\, C_{n-1}(n-2,\ n-1,\ n) +$$

$$rI = 0\ .$$

Using lemmas (1.16) and (1.17), we get

1.23 $$\alpha_{n-1\ n}(n-1^+,\ n^+)_- + \dots + \alpha_{1\ n}(1^{s_1},\ n^{s_1})_- +$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$+ \alpha_{12}(1^+,\ 2^+)_- +$$
$$\beta_{n\ n-2}(n^-,\ n-1^-,\ n-2^+)_+ + \dots + \beta_{n\ 1}(n^{s_2},\ n-1^{s_3},\ 1^{s_4})_+ +$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\beta_{31}(3^-,\ 2^-,\ 1^+)_+ + rI = 0$$

where the s_i are given by (1.16) and (1.20).

Consider the graph of the LHS of (1.23); the weight on the edge $2 \to 1$ is the sum of the coefficients of $(3^-,\ 2^-,\ 1^+)_+$ and $(1^+,\ 2^+)_-$ viz; $\alpha_{12} - \beta_{31}$. Similarly, the weight on the edge $1 \to 2$ is only α_{12}. In order to get a zero graph, the weight on each edge of the graph of the LHS of (1.23) must be zero, therefore, $\alpha_{12} = 0$, consequently $\beta_{31} = 0$.

We now use mathematical induction on the subscript s in β_{st} and $\alpha_{t\ s-1}$. Suppose that $\beta_{st} = 0$ and $\alpha_{t\ s-1} = 0$ for all $s = 3, \dots, k$ and for $t = s-2, \dots, 1$. We shall show that this implies $\beta_{st} = 0$ and $\alpha_{t\ s-1} = 0$; $s = k+1$ and $t = s-2, \dots, 1$.

We may visualize the α's and β's in a matrix

$$K = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1\ k-2} & \alpha_{1\ k-1} & \cdots & \alpha_{1n} \\ 0 & 0 & \alpha_{23} & \cdots & \alpha_{2\ k-2} & \alpha_{2\ k-1} & \cdots & \alpha_{2n} \\ \beta_{31} & 0 & 0 & \cdots & \alpha_{3\ k-2} & \alpha_{3\ k-1} & \cdots & \alpha_{3n} \\ \beta_{41} & \beta_{42} & 0 & \cdots & \alpha_{4\ k-2} & \alpha_{4\ k-1} & \cdots & \alpha_{4n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{k\ 1} & \beta_{k\ 2} & \beta_{k\ 3} & \cdots & \beta_{k\ k-2} & 0 & \cdots & \alpha_{kn} \\ \beta_{k+1\ 1} & \beta_{k+1\ 2} & \beta_{k+1\ 3} & \cdots & \beta_{k+1\ k-2} & \beta_{k+1\ k-1} & \cdots & \alpha_{k+1\ n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{n\ 1} & \beta_{n\ 2} & \beta_{n\ 3} & \cdots & \beta_{n\ k-2} & \beta_{n\ k-1} & \cdots & 0 \end{pmatrix}$$

In the induction process, we are assuming that all elements in the upper left hand $k \times (k-1)$ submatrix are zero, and we wish to show that the elements adjacent to this submatrix are also zero.

Case (i) $t \neq k-1$

The weight on the edge $k \to t$ is the sum of the coefficients of $(k^{s_1}, t^{s_1})_-$ and $(k+1^{s_2}, k^{s_3}, t^{s_4})_+$ viz; $s_1\alpha_{kt} + s_4\beta_{k+1\ t}$ and the weight on the edge $t \to k$ is the sum of the coefficients of $(k^{s_1}, t^{s_1})_-$ and $(k^{s'_2}, k-1^{s'_3}, t^{s'_4})_+$ viz; $s_1\alpha_{kt} + s'_4\beta_{kt}$. Since these weights vanish separately, therefore, we get

$$s_1\alpha_{kt} + s_4\beta_{k+1\ t} = 0 \qquad \text{and}$$

$$s_1\alpha_{kt} + s_4'\beta_{kt} \quad = 0 \qquad \text{by the induction}$$

hypothesis $\beta_{kt} = 0$. Therefore, $\alpha_{kt} = 0$ and $\beta_{k+1\ t} = 0$.

Case (ii) $\quad t = k-1$

The weight on the edge $k \to t$ is the sum of the coefficients of $(k^s, t^s)_-$, $(1+k^{s_t}, k^{s_t'}, t^{s_t''})_+$, $\{(k^{s_{k-2}}, t^{s_{k-2}'}, k-2^{s_{k-2}''})_+ , \ldots, (k^{s_1}, t^{s_1'}, 1^{s_1''})_+\}$ viz;

$s\alpha_{kt} + s_t'\beta_{k+1\ t} + \sum_{r=1}^{k-2} s_r\beta_{kr}$, and the weight on the edge $t \to k$ is $s\alpha_{kt}$. Since $\beta_{kt} = 0$ for $t \in \{k-2, \ldots, 1\}$ (by induction), therefore, $\alpha_{kt} = 0$ and $\beta_{k+1\ t} = 0$. This completes the induction; all the $\beta_{st} = 0$ and all $\alpha_{t\ s-1} = 0$ for $s = 3, \ldots, n$, $t = s-2, \ldots, 1$. The α_{tn} where $t = n-2, \ldots, 1$ are however not yet accounted for. We have reduced equation (1.23) to

$$\alpha_{1n}(1^{t_1}, n^{t_1})_- + \ldots + \alpha_{n-1\ n}(n-1^+, n^+)_- + rI = 0 \quad .$$

Since, the graphs of (1, n), ..., (n-1, n) have no edge in common, their weights must be zero separately. Therefore, $r = 0$, $\alpha_{1n} = \ldots = \alpha_{n-1\ n} = 0$. This implies that $T_1 \cup C_1 \cup I$ is an independent set.

Now we prove that $T_1 \cup C_1 \cup I$ generates $\{C_{n-1}(P) ; P \varepsilon S_n\}$ as linear combinations over the reals. Since C_{n-1} is not a linear map, this does not follow from Theorem (1.8). The method, we use is somewhat similar to that used in Theorem (1.8).

To start with, we prove that every 3-cycle in $\{C_{n-1}(P) ; P \varepsilon S_n\}$ is thus generated. It follows from (1.20) that every 3-cycle of the form $C_{n-1}(P)$ is the compound of a 3-cycle. Let $C_{n-1}(r_1, r_2, r_3)$ be such a 3-cycle which does not belong to C_1.

Case (i) $r_1 < r_2 < r_3$

From (1.9), we have

$$(r_1, r_2, r_3) = \sum_{i=1}^{k} (r_1+i-1, r_1+i, r_3) - \sum_{i=1}^{k-1} (r_1+i, r_3)$$

$$+ \sum_{i=1}^{k-1} (r_1+i-1, r_1+k) - \sum_{i=1}^{k-1} (r_1+i-1, r_1+i, r_1+k) ,$$

where $r_2 = r_1 + k$.

We claim that

$$C_{n-1}(r_1, r_2, r_3) = \sum_{i=1}^{k} C_{n-1}(r_1+i-1, r_1+i, r_3) + \sum_{i=1}^{k-1} C_{n-1}(r_1+i, r_3)$$

$$- \sum_{i=1}^{k-1} C_{n-1}(r_1+i-1, r_1+k) - \sum_{i=1}^{k-1} C_{n-1}(r_1+i-1, r_1+i, r_1+k) .$$

To be specific we shall discuss the case where r_2+r_1 is even, r_2+r_3 odd and r_1+r_3 odd. The proof for the other possibilities is similar, and will not be included here. By virtue of (1.16) and (1.17), the above identity becomes:

1.24 $$(n-r_1+1^+,\ n-r_1-k^-+1,\ r^- - r_3+1)_+$$

$$= \sum_{i=1}^{k} (n-r_1-i^-+2,\ n-r_1^{\,s_i}-i+1,\ n-r_3^{\,-s_i}+1)_+$$

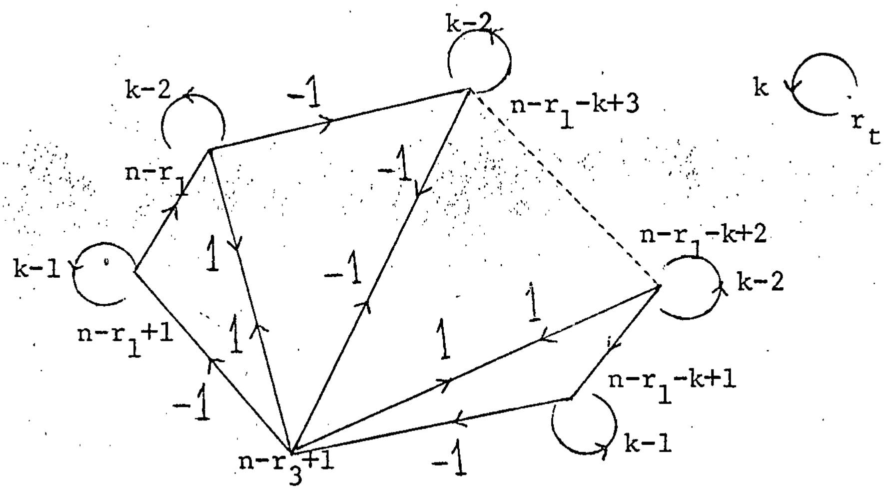
$$+ \sum_{i=1}^{k-1} (n-r_1-i^{-s_i}+1,\ n-r_3^{\,-s_i}+1)_-$$

$$- \sum_{i=1}^{k-1} (n-r_1-i^{s_i}+2,\ n-r_1^{\,s_i}-k+1)_-$$

$$- \sum_{i=1}^{k-1} (n-r_1-i^-+2,\ n-r_1^{\,s_i}-i+1,\ n-r_1^{\,-s_i}-k+1)_+$$

where $s_i = (-1)^{i+1}$.

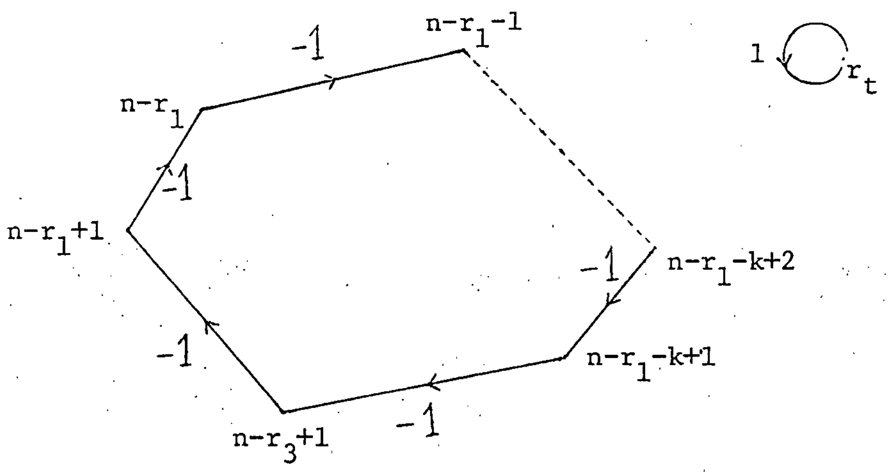
The graph of the first sum in the RHS of (1.24) is



The weights on the loops at the vertices are as follows:

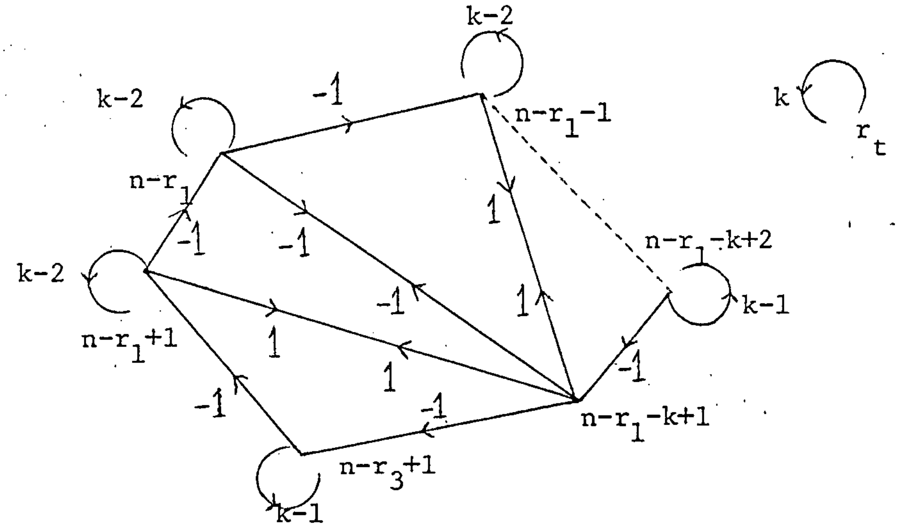
On the vertices $n-r_1-k+1$ and $n-r_1+1$ it is $k-1$, on each of the isolated vertices it is k and on each of the remaining vertices of the graph, the weight is $k-2$.

To this graph, we add the graph obtained from the second sum of the RHS of (1.24) and we get



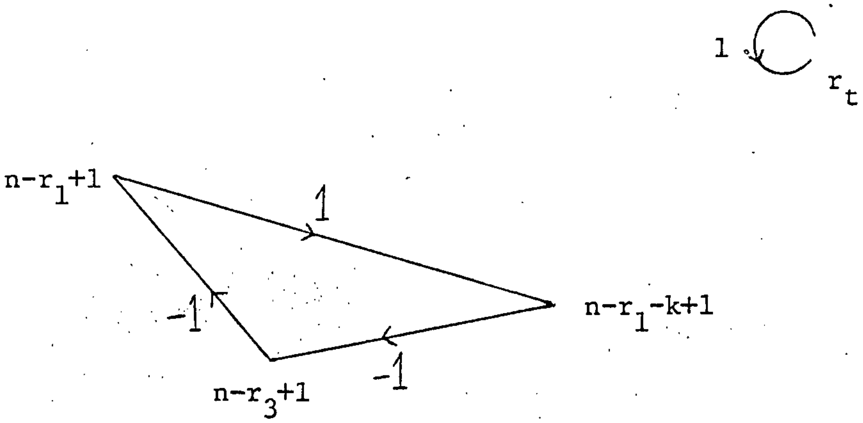
because the signs in the 2-cycles of the second sum in the RHS of (1.24) are opposite to the corresponding signs in the previous graph and therefore, these cancel each other.

Subtracting the graph of the third sum in the RHS of (1.24) from the above graph, we get



To see this, note that we have added $k-1$ 2-cycles (with opposite signs), hence the weights on the single loops at $n-r_3+1$ and $n-r_1-k+2$ are $k-1$. At each isolated vertex the weight is k, at the rest of the vertices of this graph the weight is $k-2$. The weights on the edges of the graph are as shown.

From the signs of the non-zero weights on the edges of the graph of the last sum in the RHS of (1.24), it is clear that if we subtract this graph from the above graph, we are left with:



Note that the weight on the single loop at $n-r_3+1$ becomes $(k-1)-(k-1) = 0$. Also, the weight on the single loop at $n-r_1-k+1$ remains zero, for every 3-cycle in the last sum of the RHS of (1.24) has $n-r_1-k+1$ as one of the vertices. At $n-r_1+1$, the loop has weight $(k-2) - (k-2) = 0$ and at each of the isolated vertices this weight is $k - (k-1) = 1$.

Since the above graph is also the graph of $(n-r_1+1^+, n-r_1-k^-+1, n^--r_3+1)_+$, which is the LHS of (1.24), therefore, the identity (1.24) holds.

Case (ii) $r_1 > r_2 > r_3$

From (1.10), we have

$$(r_1, r_2, r_3) = (r_1, r_2) + (r_1, r_3) + (r_2, r_3) - (r_3, r_2, r_1) - I .$$

We claim that

$$C_{n-1}(r_1, r_2, r_3) = -\{C_{n-1}(r_1, r_2) + C_{n-1}(r_1, r_3) + C_{n-1}(r_2, r_3)\}$$
$$- C_{n-1}(r_3, r_2, r_1) - I .$$

As in case (i), we consider only the case in which r_1+r_2 is even, r_1+r_3 odd and r_2+r_3 odd. The other cases can be varified similarly. Using lemmas (1.16) and (1.17) we get:

1.25 $\quad (n-r_1+1^+,\ n-r_2^-+1,\ n^- -r_3+1)_+$

$$= \{(n-r_1^+ +1,\ n^+ -r_2+1)_+ + (n-r_1^- +1,\ n^- -r_3+1)_+ + (n-r_2^- +1,\ n^- -r_3+1)_+\}$$

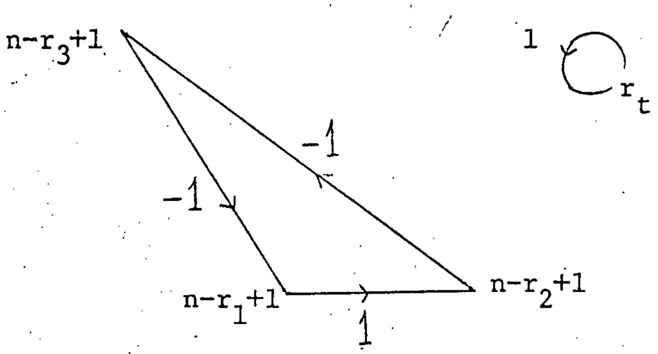
$$+ (n-r_3+1^+,\ n-r_2^- +1,\ n^+ -r_1+1)_- - I$$

where $(n-r_3^+ +1,\ n-r_2^- +1,\ n^+ -r_1+1)_-$ may be expanded as in (1.24).

Graphically, the weight in the RHS of (1.25) on the edge

$n-r_1+1 \rightarrow r-r_2+1$	is	1
$n-r_2+1 \rightarrow n-r_1+1$	is	$1 - 1 = 0$
$n-r_1+1 \rightarrow n-r_3+1$	is	$-1 + 1 = 0$
$n-r_3+1 \rightarrow n-r_1+1$	is	-1
$n-r_2+1 \rightarrow n-r_3+1$	is	-1
$n-r_3+1 \rightarrow n-r_2+1$	is	$-1 + 1 = 0$.

On each of the isolated loops the weight is $1 + 1 + 1 - 1 - 1 = 1$.
The resultant graph is, therefore,



which is the graph of the LHS of (1.25). Thus every 3-cycle in

$\{C_{n-1}(P) \; ; \; P \in S_n\}$ is generated by $T_1 \cup C_1 \cup I$.

As an example of the foregoing, consider the special case where $n = 6$ and $\sigma = (1, 4, 5)$. From (1.9) we have

$$(1, 4, 5) = (1, 2, 5) + (2, 3, 5) + (3, 4, 5) - (2, 5) - (3, 5) + (1, 4) + (2, 4) - (1, 2, 4) - (2, 3, 4) \; .$$

Hence

$$C_{n-1}(1, 4, 5) = C_{n-1}(1, 2, 5) + C_{n-1}(2, 3, 5) + C_{n-1}(3, 4, 5) + C_{n-1}(2, 5) + C_{n-1}(3, 5) - C_{n-1}(1, 4) - C_{n-1}(2, 4) - C_{n-1}(1, 2, 4) - C_{n-1}(2, 3, 4) \; .$$

That is

$$(6^-, 3^-, 2^+)_+ = (6^-, 5^-, 2^+)_+ + (5^-, 4^+, 2^-)_+ + (4^-, 3^-, 2^+)_+ + (5^+, 2^+)_- + (4^-, 2^-)_- + (6^-, 3^-)_+ + (5^+, 3^+)_+ + (6^+, 5^-, 3^+)_- + (5^+, 4^+, 3^-)_- \; .$$

Resuming the proof of the theorem, we now consider an arbitrary cycle $\sigma = (r_1, \ldots, r_m)$. From (1.11) we have:

$$\sigma = \sum_{i=1}^{m-2} (r_1, r_{i+1}, r_{i+2}) - \sum_{i=1}^{m-3} (r_1, r_{i+2}) \; .$$

We claim that $\operatorname{sgn} \sigma \; C_{n-1}(\sigma) = \sum_{i=1}^{m-2} C_{n-1}(r_1, r_{i+1}, r_{i+2}) + \sum_{i=1}^{m-3} C_{n-1}(r_1, r_{i+2})$.

By lemmas (1.16) and (1.17), this is equivalent to

1.26 $$\operatorname{sgn}\sigma\,(n-r_1^{s_1}+1,\ \ldots,\ n-r_m^{s_m}+1)_{\operatorname{sgn}\sigma}$$

$$= \sum_{i=1}^{m-2}(n-r_1+1^{s_i},\ n-r_i^{s_i'}+1^{+1},\ n^{s_i''}-r_{i+2}+1)_+$$

$$+ \sum_{i=1}^{m-3}(n-r_1^{t_i}+1,\ n^{t_i}-r_{i+2}+1)_-$$

with the help of Lemma (1.17), we infer that the weights on the common edges, of the graphs of the two sums in the RHS of (1.26) cancel each other and the resultant graph of the RHS of (1.26) is equal to the graph of the LHS of (1.26). It now follows from (1.24) and (1.25) that the RHS of (1.26) can be expressed as a linear combination of the elements in $T_1 \cup C_1 \cup I$.

Finally, we consider the general case:

$P = P_1 \ldots P_m$, where P_i for $i = 1, \ldots, m$ are disjoint cycles. From (1.12), we have $P = \sum_{i=1}^{m} P_i - (m-1)I$. We show that $\operatorname{sgn} P\, C_{n-1}(P) = \sum_{i=1}^{m} \operatorname{sgn} P_i\, C_{n-1}(P_i) - (m-1)I$. Since $\operatorname{sgn} P = \prod_{i=1}^{m} \operatorname{sgn} P_i$, therefore this is the same as

1.27 $$(\operatorname{sgn} P_1\, C_{n-1}(P_1)) \ldots (\operatorname{sgn} P_m\, C_{n-1}(P_m))$$

$$= \sum_{i=1}^{m} \operatorname{sgn} P_i\, C_{n-1}(P_i) - (m-1)I \quad .$$

It is evident that the weights on the corresponding edges of the graphs of the LHS and the RHS of (1.27) are the same; and the same is true for the weights on loops at the non-isolated vertices. There remains to show that the weight of each of the isolated loops has the same value for each side of (1.27). For this, note that in case of an odd cycle (i.e. $\operatorname{sgn}\sigma = 1$) each of the isolated loops carry a weight $+1$, while for an even cycle this weight -1. When we attach $\operatorname{sgn} P_i$ to each P_i, we change the weight on each of the isolated loops to $+1$. Hence, in the graph of $\sum_{i=1}^{m} \operatorname{sgn} P_i \, C_{n-1}(P_i)$, the isolated loops have weight m. When we add $-(m-1)I$ to this sum, we reduce the weight on each of the isolated loops to 1. Hence the identity (1.27) holds.

By (1.26) each $C_{n-1}(P_i)$ is contained in the linear combinations of the elements of $T_1 \cup C_1 \cup I$. Hence, by (1.27), $C_{n-1}(P)$ can also be expressed as a linear combination of the members of $T_1 \cup C_1 \cup I$.

Thus we have shown that the set $T_1 \cup C_1 \cup I$ generates the whole set $\{C_{n-1}(P) \; ; \; P \in S_n\}$.

Finally, the number of elements in $T_1 \cup C_1 \cup I$ is the same as the number of elements in the set $T \cup C \cup I$; viz; $(n-1)^2 + 1$.

Q.E.D.

1.28 Remark:

It is interesting to note that the parallel theorem to Corollary (1.14) involving $T_1 \cup C_1 \cup I$ and $\{C_{n-1}(P) \; ; \; P \; \varepsilon \; S_n\}$, is not true, as the following shows:

There exists $\sigma \; \varepsilon \; \{C_{n-1}(P) \; ; \; P \; \varepsilon \; S_n\}$ such that $\sigma = \sum_{i=1}^{r} \beta_i P_i$, where $P_i \; \varepsilon \; T_1 \cup C_1 \cup I$ and $\sum_{i=1}^{r} \beta_i \neq 1$. Let $\sigma = (r_3, r_1+1, r_1)$ where $r_2 = r_1+1$, then

$$\sigma = (r_3, r_1+1) + (r_3, r_1) + (r_1+1, r_1) - (r_1, r_1+1, r_3) - I \quad ,$$

and

$$C_{n-1}(\sigma) = -\{C_{n-1}(r_3, r_1+1) + C_{n-1}(r_3, r_1) + C_{n-1}(r_1+1, r_1)\}$$
$$- C_{n-1}(r_1, r_1+1, r_3) - I \qquad \text{(by (1.25))}.$$

Each compound appearing in the RHS of this expression is a member of $T_1 \cup C_1 \cup I$. However, $\sum_{i=1}^{5} \beta_i = -5 \neq 1$, where the β_i are the coefficients of these compounds.

We shall make use of this result in Chapter II.

1.29 Remark:

We close this chapter with a note about the set of r-compounds $\{C_r(P) \; ; \; P \; \varepsilon \; S_n\}$. We have so far been unable to establish theorems similar to (1.22) and (1.28) except in the special cases when

$r = n-1, 1$. For $r = 2$ and $n = 4$, we can say that this set contains an independent subset of cardinality 18 which is greater than $(n-1)^2+1$ viz; 10. Moreover, the above remark (1.28) remains true in this sepcial case; and we conjecture that it is true for the general set $\{C_r(P) ; P \in S_n\}$.

CHAPTER II

CHARACTERISATION OF $n\times n$ MATRICES A FOR WHICH TRACE(PA) $\varepsilon \{a_1, \ldots, a_k\}$, AND MATRICES B FOR WHICH $E_{n-1}(PB) = E_{n-1}(B) \forall P \varepsilon S_n$

In this chapter, we pose the following problem: Let H be a subset of S_n and let $a_1, \ldots, a_k$ be k distinct real numbers. Determine the set of square matrices A such that

$$\{tr(PA) \mid P \varepsilon H\} = \{a_1, \ldots, a_k\}$$

We provide solutions in the following cases:

(1) $H = S_n$; $k = 1$

(2) $H = \{$2-cycles in $S_n\}$, $k = 1$

(3) $H = S_n$; $k = 2$.

A second problem which we solve is the determination of the structure of $n\times n$ matrices A such that $E_{n-1}(PA) = E_{n-1}(A)$ $\forall$ $P \varepsilon S_n$ $(n \geq 3)$.

The r-th elementary symmetric function of the $n\times n$ matrix A (denoted by $E_r(A)$) is used to designate

$E_r(\lambda_1, \ldots, \lambda_n) = \sum_{1\leq i_1 \ldots < i_r \leq n} \prod_{j=1}^{r} \lambda_{ij}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. As is well known, $E_r(A)$ is equal to the sum of all the principal $r\times r$ subdeterminants of A. Again $E_r(A) = \text{trace } C_r(A) = tr(C_r(A))$, where $C_r(A)$ is the r-th compound of A. In particular, $E_n(A) = d(A)$ and $E_1(A) = \sum_{i=1}^{n} a_{ii} = tr(A)$ if $A = (a_{ij})$.

We first consider case (1) above; viz; the set of $n\times n$ matrices $\{A : tr(PA) = tr(A) \text{ ; for all } P \in S_n\}$.

By Corollary (1.14) we know that every $P \in S_n$ can be written as $P = \sum_{i=1}^{r} \alpha_i P_i$, where $P_i \in M$, $M = T \cup C \cup I$ (see (1.13)) and $P \in S_n$, $i = 1, \ldots, r$, and $\sum_{i=1}^{r} \alpha_i = 1$. Thus if $tr(QA) = tr(A)$ for all $Q \in M$, then for any $P \in S_n$ $(P = \sum_{i=1}^{r} \alpha_i P_i \text{ ; } P_i \in M$ and $\sum_{i=1}^{r} \alpha_i = 1)$ it follows that

$$tr(PA) = tr(\sum_{i=1}^{r} \alpha_i P_i A) = \sum_{i=1}^{r} \alpha_i tr(P_i A) \qquad ([2]; \text{pp-}18)$$

$$= \sum_{i=1}^{r} \alpha_i (tr(A))$$

$$= (\sum_{i=1}^{r} \alpha_i)(tr(A)) = tr(A) \quad .$$

Therefore, the set $\{A : tr(PA) = tr(A) \quad P \in S_n\}$ is just the set $\{A : tr(PA) = tr(A) \quad P \in M\}$.

Let $A = (a_{ij})$. Then, by assumption $tr(1, k)(k, r)A = tr(A) = tr(k, r)A$. Hence $a_{11} - a_{1k} = a_{r1} - a_{rk}$ for all k, r . To simplify the notation, let $\alpha_i = a_{1i}$ and $\delta_i = a_{i1} - a_{11}$ for $i = 1, \ldots, n$. Then $a_{rk} = \alpha_k + \delta_r$, and

$$A = \begin{pmatrix} \alpha_1+\delta_1 & \alpha_2+\delta_1 & \cdots & \alpha_n+\delta_1 \\ \alpha_1+\delta_2 & \alpha_2+\delta_2 & \cdots & \alpha_n+\delta_2 \\ \vdots & \vdots & & \vdots \\ \alpha_1+\delta_n & \alpha_2+\delta_n & \cdots & \alpha_n+\delta_n \end{pmatrix} .$$

This can be written

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} + \begin{pmatrix} \delta_1 & \delta_1 & \cdots & \delta_1 \\ \delta_2 & \delta_2 & \cdots & \delta_2 \\ \vdots & \vdots & & \vdots \\ \delta_n & \delta_n & \cdots & \delta_n \end{pmatrix} .$$

If R' = subspace of rank 1 matrices with one row repeated n-times and if C' = subspace of rank 1 matrices with one column repeated n-times, then clearly

$$A \in R' + C' .$$

On the other hand if $A \in R' + C'$, then clearly $tr(PA) = tr(A)$ for all $P \in S_n$. We have, therefore

2.1 Theorem:

The $n \times n$ matrices A such that $tr(PA) = tr(A)$ for all $P \in S_n$, form a subspace $S = R' + C'$, where R' = subspace of rank 1 matrices with all rows equal and C' = subspace of rank 1 matrices with all columns equal.

This is our result for the case (1) listed on page (33).

2.2 Corollary:

The rank of A such that $tr(PA) = tr(A)$ for all $P \in S_n$ is ≤ 2 .

Turning to case (2) on page (33), suppose we restrict the set H to interchanges only. It follows immediately that $S_1 = \{A : tr(PA) = tr(A)$ for all $P \in T\}$ is a subspace and contains all $n \times n$ skew-symmetric matrices.

It also contains the $n \times n$ matrices of the type

$$\begin{pmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & a \end{pmatrix} .$$

Since these two subspaces meet in the zero matrix only, we have

$$\dim S_1 \geq \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \quad .$$

Now the subspace S_1 is complementary to the subspace formed by $n \times n$ matrices of the type

$$\begin{pmatrix} 0 & * & * & \cdots & * & \cdots & * \\ 0 & 0 & * & \cdots & * & \cdots & * \\ \vdots & \vdots & & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & & * \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & \cdots & . & \cdots & 0 \end{pmatrix}$$

This subspace has dim $\frac{n(n-1)}{2}$ and therefore,

$$\dim S_1 \leq n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

$$\therefore \quad \dim S_1 = \frac{n(n+1)}{2} \quad .$$

For case (3) on page (33) we now consider the set of $n \times n$ matrices $\{A : \{tr(PA)\} = \{a_1, a_2\}$, for all $P \in S_n\}$ we assume $a_1 \neq a_2$.

We begin by studying the decomposition $B_1 \cup B_2$ of S_n given by

$$B_i = \{P \in S_n : tr(PA) = a_i\} \quad ; \quad i = 1, 2 \quad .$$

It is clear that equations (1.9), (1.10), (1.11) and (1.12) impose restrictions on the partitions $B_1 \cup B_2$ of S_n which are possible or "admissible". Our task now is to find the "admissible" partitions of S_n. In the first instance, we partition the set T of all 2-cycles in S_n. It is convenient to recall here the following:

2.3 Definition:

If for every pair (v, w) ; $v \neq w$, in a directed graph there exists a sequence $\{(v, v_1; \alpha_1), \ldots, (v_r, w; \alpha_{r+1})\}$ as well as $\{(w, w_1; \beta_1), \ldots, (w_r, v, \beta_{r+1})\}$, with all α_i and β_i nonzero, then the directed graph is said to be strongly connected. If the corresponding undirected graph [1] is complete (i.e. every two distinct vertices are joined by an edge), then we will call such a directed graph as strongly connected complete graph.

2.4 If H is a subgraph of G , the number of vertices in H is said to be the order of H .

Assume that $I \in B_2$ i.e. $tr(A) = a_2$, then we have

2.5 Lemma:

The graph of all the 2-cycles in $T \cap B_2$ contains a strongly connected complete subgraph of order $n-1$.

Proof: First, we observe that $T \cap B_1$ can not contain disjoint 2-cycles. For, if it does contain two disjoint 2-cycles (r_1, s_1) and (r_2, s_2), then by equation (1.12) we have

$$(r_1, s_1)(r_2, s_2) = (r_1, s_1) + (r_2, s_2) - I ,$$

$$(r_1, s_1)(r_2, s_2)A = (r_1, s_1)A + (r_2, s_2)A - A \quad \text{and}$$

$$tr(r_1, s_1)(r_2, s_2)A = a_1 + a_1 - a_2 = 2a_1 - a_2 .$$

Then $2a_1-a_2$ equals a_1 or a_2. In either case, $a_1 = a_2$, which is contrary to our assumption that $a_1 \neq a_2$. It follows that $T \cap B_1$ contains a strongly connected complete subgraph of order 3 or the 2-cycles of $T \cap B_1$ contain a vertex in common.

In case (r_1, s_1), (r_1, s_2) and (s_1, s_2) are in $T \cap B_1$, we have, from (1.10),

$$(r_1, s_2, s_1) = (r_1, s_1) + (r_1, s_2) + (s_1, s_2) - I - (r_1, s_1, s_2)$$

which gives

$$(r_1, s_2, s_1)A = (r_1, s_1)A + (r_1, s_2)A + (s_1, s_2)A - A - (r_1, s_1, s_2)A .$$

Taking the trace of both the sides, we obtain

$$\begin{aligned} tr(r_1, s_2, s_1)A &= a_1 + a_1 + a_1 - a_2 - tr(r_1, s_1, s_2)A \\ &= 3a_1 - a_2 - tr(r_1, s_1, s_2)A . \end{aligned}$$

In case $tr(r_1, s_1, s_2)A = a_1$, then $tr(r_1, s_2, s_1)A = 2a_1 - a_2$ which we have already found leads to a contradiction.

Now, if $tr(r_1, s_1, s_2)A = a_2$, we get $tr(r_1, s_2, s_1)A = 3a_1 - 2a_2$.

Then $3a_1 - 2a_2$ is equal to either a_1 or a_2 and in both these cases we get $a_1 = a_2$, contrary to our assumption that $a_1 \neq a_2$. We infer, therefore, that $T \cap B_1$ cannot contain a strongly connected (complete) subgraph of order greater than 2.

The only possibility remaining is that all the 2-cycles in $T \cap B_1$ have one vertex in common, i.e. these 2-cycles form rays from a vertex (r_1-say). Since $T \cap B_2 = T \sim T \cap B_1$, the graph of all the 2-cycles in $T \cap B_2$ contains a strongly connected complete subgraph of order $n-1$; viz; a graph with its set of vertices $\{V - (r_1)\}$, where V is the set of vertices of S_n; viz; $\{1, \ldots, n\}$.

We now look at the 3-cycles in B_1 and B_2. If the two cycles (r_1, s_1) and (r_1, s_2) are in $T \cap B_1$, we claim that the 3-cycles (r_1, s_1, s_2) and (r_1, s_2, s_1) are also in B_1. From the preceding paragraph, (s_1, s_2) must be in $T \cap B_2$. By (1.10), we get

$$(r_1, s_2, s_1) = (r_1, s_1) + (r_1, s_2) + (s_1, s_2) - I - (r_1, s_1, s_2) ,$$

2.6 $(r_1, s_2, s_1)A = (r_1, s_1)A + (r_1, s_2)A + (s_1, s_2)A - A - (r_1, s_1, s_2)A$

and $\quad tr(r_1, s_2, s_1)A = a_1 + a_1 + a_2 - a_2 - tr(r_1, s_1, s_2)A$

$$(\because \; I \varepsilon B_2)$$

$$= 2a_1 - tr(r_1, s_1, s_2)A \; .$$

If $tr(r_1, s_1, s_2)A = a_2$, then from the above equation $tr(r_1, s_2, s_1)A = 2a_1 - a_2$, which we know is impossible. Therefore, $tr(r_1, s_1, s_2)A = a_1$ and consequently, $tr(r_1, s_2, s_1)A = a_1$. Thus we have proved

2.7 Lemma:

If (r_1, s_1) and (r_1, s_2) are in $T \cap B_1$ then (r_1, s_1, s_2) and (r_1, s_2, s_1) are in B_1 .

2.8 Remark:

Suppose $T \cap B_1$ contains (r_1, s_1) but not (r_1, s_2) . In the equation (2.6), we take the trace of both the sides and obtain

$$tr(r_1, s_2, s_1)A = a_1 + a_2 + a_2 - a_2 - tr(r_1, s_1, s_2)A \; .$$

If $tr(r_1, s_1, s_2)A = a_1$, then $tr(r_1, s_2, s_1)A = a_2$ and vice versa. Therefore in this case the 3-cycles (r_1, s_1, s_2) and (r_1, s_2, s_1) are divided among B_1 and B_2.

Concerning the set B_2, we have the:

2.9 Lemma:

If (r_2, s_1), (r_2, s_2) and (s_1, s_2) are in $T \cap B_2$, then both (r_2, s_1, s_2) and (r_2, s_2, s_1) are in B_2.

Proof: By (1.10)

$$(r_2, s_2, s_1) = (r_2, s_1) + (r_2, s_2) + (s_1, s_2) - I - (r_2, s_1, s_2) ,$$

$$(r_2, s_2, s_1)A = (r_2, s_1)A + (r_2, s_2)A + (s_1, s_2)A - A - (r_2, s_1, s_2)A$$

and
$$\begin{aligned} tr(r_2, s_2, s_1)A &= a_2 + a_2 + a_2 - a_2 - tr(r_2, s_1, s_2)A \\ &= 2a_2 - tr(r_2, s_1, s_2)A . \end{aligned}$$

If $tr(r_2, s_1, s_2)A = a_1$, then $tr(r_2, s_2, s_1)A = 2a_2 - a_1$ which we know is impossible, therefore, $tr(r_2, s_1, s_2)A = a_2$ and consequently $tr(r_2, s_2, s_1)A = a_2$. Hence the lemma.

We now use our representation of the permutation matrices (1.9), (1.10), (1.11) and (1.12) to complete the characterisation of the admissible partitions of S_n. The discussion falls naturally into the following cases:

Case (i) $T \cap B_1$ contains all the 2-cycles with common vertex (r_1) viz; $(r_1, r_2), \ldots, (r_1, r_n)$ where $r_i \varepsilon \{1, \ldots, n\}$. There are $(n-1)$ such 2-cycles. From Lemma (2.7), it follows that it contains all the 3-cycles with r_1 in them. Furthermore, we claim that B_1 contains all cycles with the integer r_1 in them. In order to show this, consider any such cycle $(r_1, s_1, \ldots, s_m)$. From (1.11) we have

$$\begin{aligned}&(r_1, s_1, \ldots, s_m)\\&= (r_1, s_1, s_2) + (r_1, s_2, s_3) + \ldots + (r_1, s_{m-1}, s_m) - (r_1, s_2)\\&- \ldots - (r_1, s_{m-1}) .\end{aligned}$$

Multiplying on the right by A and taking the trace of both the sides, we obtain

$$\operatorname{tr}(r_1, s_1, \ldots, s_m)A = (m-1)a_1 - (m-2)a_1 = a_1 .$$

We can now conclude in this case that B_1 consists of those products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i contains r_1. For, suppose r_1 is involved in P_1. Then, by (1.12)

$$P = \sum_{i=1}^{m} P_i - (m-1)I \qquad \text{and}$$

$$\operatorname{tr}(PA) = \sum_{i=1}^{m} \operatorname{tr}(P_iA) - (m-1)\operatorname{tr} A = a_1+(m-1)a_2-(m-1)a_2 = a_1 .$$

Case (ii) $T \cap B_1$ does not contain all the 2-cycles with the vertex r_1 in common. Let $(r_1, r_2), \ldots, (r_1, r_k)$ be the 2-cycles of B_1 and let $\{t_1, \ldots, t_{n-k}\} = \{1, \ldots, n\} \setminus \{r_1, \ldots, r_k\}$. Consider the cycles (r_1, r_i, t_j) and (r_1, t_j, r_i). By Remark (2.8) if $(r_1, r_i, t_j) \in B_1$, then $(r_1, t_j, r_i) \in B_2$, and vice versa. Our argument breaks into three subcases; viz,

(a) $\{(r_1, r_i, t_j) : i = 2, \ldots, k, j = 1, \ldots, n-k\} \subseteq B_1$;

(b) $\{(r_1, r_i, t_j) : i = 2, \ldots, k ; j = 1, \ldots, n-k\} \subseteq B_2$;

(c) $\{(r_1, r_i, t_j) : i = 2, \ldots, k, j = 1, \ldots, n-k\}$ intersects both B_1 and B_2 nontrivially.

Case (a) $\{(r_1,r_i,t_j) : i = 2, \ldots, k, j = 1, \ldots, n-k\} \subseteq B_1$

In this case, B_1 contains all 3-cycles of the type (r_1, r_i, r_j), where $i,j \in \{1, \ldots, k\}$ (by lemma (2.7)) and all 3-cycles of the type (r_1, r_i, t_j). Furthermore, B_1 contains all cycles σ such that the graph of σ contains an edge $r_1 \to r_i$. To show this, we note that

$$(r_1, r_i, s_1, \ldots, s_m)$$

$$= (r_1, r_i, s_1) + (r_1, s_1, s_2) + \ldots + (r_1, s_{m-1}, s_m) - (r_1, s_1)$$

$$- \ldots - (r_1, s_{m-1}) ,$$

by (1.11) . If $\{s_1, \ldots, s_m\} \subseteq \{t_1, \ldots, t_{n-k}\}$, then $tr(r_1, r_i, s_1, \ldots, s_m)A = a_1 + (m-1)a_2 - (m-1)a_2 = a_1$. If, however, $\{s_1, \ldots, s_t\} \subseteq \{r_1, \ldots, r_k\}$ and $\{s_{t+1}, \ldots, s_m\} \subseteq \{t_1, \ldots, t_{n-k}\}$, then

$$\begin{aligned} & tr(r_1, r_i, s_1, \ldots, s_m)A \\ &= a_1 + ta_1 + (m-1-t)a_2 - ta_1 - (m-1-t)a_2 \\ &= a_1 , \end{aligned}$$

because $(r_1, s_i, s_{i+1}) \in B_1$ iff (r_1, s_i) and (r_1, s_{i+1}) are in B_1 .

It can easily be shown that B_1 does not contain a cycle whose graph has no edge $r_1 \to r_i$. From this, it follows immediately that B_1 consists of those products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i has a graph with an edge $r_1 \to r_i$.

Case (b) In this case, $\{(r_1, r_i, t_j) : i = 2, \ldots, k ,$ $j = 1, \ldots, n-k\} \subseteq B_2$, and an argument similar to that of case (a) can be made leading to the conclusion that B_1 consists of those products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i has its graph with an edge $r_i \to r_1$.

Case (c) $\{(r_1, r_i, t_j) : i = 2, \ldots, k \ ; j = 1, \ldots, n-k\}$ intersects both B_1 and B_2 nontrivially.

First, we show that if for a fixed r_i, $(r_1, r_i, t_p) \varepsilon B_1$ for some $p \varepsilon \{1, \ldots, n-k\}$, then $(r_1, r_i, t_j) \varepsilon B_1$ for all $j \varepsilon \{1, \ldots, n-k\}$. Suppose $(r_1, r_i, t_q) \notin B_1$ for some $q \varepsilon \{1, \ldots, n-k\}$. We have

$$(r_1, r_i, t_q) = (r_1, r_i, t_p) + (t_p, r_i, t_q) - (r_i, t_p) + (r_1, t_q) - (r_1, t_q, t_p) .$$

$$\begin{aligned} tr(r_1, r_i, t_q)A &= tr(r_1, r_i, t_p)A + tr(t_p, r_i, t_q)A - tr(r_i,t_p)A + tr(r_1,t_q)A \\ &\qquad - tr(r_1, t_q, t_p)A . \\ &= a_1 + a_2 - a_2 + a_2 - a_2 = a_1 \end{aligned}$$

, which contradicts the assumption that $(r_1,r_i,t_q) \notin B_1$. Hence, for each $r_i \varepsilon \{r_2, \ldots, r_k\}$ if $(r_1, r_i, t_p) \varepsilon B_1$ for some p, then $(r_1, r_i, t_j) \varepsilon B_1$ for all $j \varepsilon \{1, \ldots, n-k\}$.

Similarly, if for a fixed r_i, $(r_1, t_p, r_i) \varepsilon B_1$ for some $p \varepsilon \{1, \ldots, n-k\}$, then $(r_1, t_j, r_i) \varepsilon B_1$ for all $j \varepsilon \{1, \ldots, n-k\}$.

Now, let $\{(r_1, s_i, t_j) : s_i \varepsilon \{s_1, \ldots, s_m\} \subseteq \{r_1, \ldots, r_k\} ;$ $j = 1, \ldots, n-k\} \subseteq B_1$ and $\{(r_1, r_i, t_j) : r_i \varepsilon \{r_1, \ldots, r_k\} \setminus \{s_1, \ldots, s_m\} ;$ $j = 1, \ldots, n-k\} \subseteq B_2$. Clearly, B_1 also contains (r_1, r_i, r_j) and (r_1, r_j, r_i) for all $r_i, r_j \varepsilon \{r_1, \ldots, r_k\}$ (lemma (2.7)). It is a matter of simple verification that B_1 contains all products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i has a graph with an edge $r_1 \to s_i$ or an edge $r_i \to r_1$ for $r_i \in \{r_1, \ldots, r_k\} \setminus \{s_1, \ldots, s_m\}$. We omit the details.

Thus we have characterized the possible partitions $B_1 \cup B_2$ of S_n . We now look for the possible structures of A which can occur in each of the possible partitions $B_1 \cup B_2$. It will be seen that the case (ii) (c) is not possible as long as our assumption that $a_1 \neq a_2$ stands.

We want A to be such that

$$\begin{aligned} tr(PA) &= a_1 \quad \text{for all } P \in B_1 \\ &= tr(A) = a_2 \quad \text{for all } P \in B_2 \text{ and} \end{aligned}$$

$S_n = B_1 \cup B_2$. First, we consider the partition of S_n given by Case (i).

Assume, for simplicity, that $r_1 = 1$ and $T \cap B_1$ contains $(1,2), \ldots, (1,n)$. Let A' be the $(n-1) \times (n-1)$ submatrix obtained by deleting the 1^{st} row and the 1^{st} column of A . Then,

$$tr(PA') = tr(A') \forall P \in B_2 = S_{n-1} \text{ where}$$

S_{n-1} is over $\{2, \ldots, n\}$. By Thm. (2.1)

$$2.10 \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \alpha_1+\delta_1 & \alpha_2+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ a_{31} & \alpha_1+\delta_2 & \alpha_2+\delta_2 & \cdots & \alpha_{n-1}+\delta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \alpha_1+\delta_{n-1} & \alpha_2+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix}$$

For $r,m \in \{2, \ldots, n\}$, $(1,r)$, $(1,m,r)$ and $(1,r,m)$ are in B_1 . Therefore,

$$tr(1, r)A = tr(1, m, r)A$$

and
$$a_{r1} - a_{m1} = \delta_{r-1} - \delta_{m-1} .$$

Set $\alpha_n = a_{21} - \delta_1$; then

$$a_{r1} - \delta_{r-1} = a_{21} - \delta_1 = \alpha_n$$

Hence
$$a_{r1} = \alpha_n + \delta_{r-1} , \ r \in \{2, \ldots, n\} .$$

Also, $tr(1, r)A = tr(1, r, m)A$.

which gives
$$a_{1r} - a_{1m} = \alpha_{r-1} - \alpha_{m-1} .$$

Set $\delta_n = a_{12} - \alpha_1$; then

$$a_{1r} - \alpha_{r-1} = a_{12} - \alpha_1 = \delta_n$$

Hence $$a_{1r} = \alpha_{r-1} + \delta_n \ .$$

We determine a_{11} as follows:

$$a_2 = \mathrm{tr}(A) = a_{11} + \sum_{i \neq 1}^{n} a_{ii} = a_{11} + \sum_{i=1}^{n-1} (\alpha_i + \delta_i) \ ,$$

and $$a_1 = a_{12} + a_{21} + \sum_{i \neq 1,2}^{n} a_{ii} = \alpha_n + \delta_n + \sum_{i=1}^{n-1} (\alpha_i + \delta_i)$$

$$a_2 - a_1 = a_{11} - (\alpha_n + \delta_n)$$

or $$a_{11} = (a_2 - a_1) + (\alpha_n + \delta_n)$$

Hence, the matrix A is completely determined, and is given by

$$A = \begin{pmatrix} (a_2-a_1)+\alpha_n+\delta_n & \cdots & \alpha_1+\delta_n & \cdots & \alpha_{n-1}+\delta_n \\ \alpha_n+\delta_1 & & \alpha_1+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ \vdots & & \vdots & & \vdots \\ \alpha_n+\delta_{n-1} & & \alpha_1+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix}$$

In general, if the i-th vertex is common to all the members of $T \cap B_1$, then

$$2.11 \qquad A = \begin{array}{c} \begin{bmatrix} \alpha_1+\delta_1 & \cdots & \alpha_n+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ \\ \alpha_1+\delta_n & \cdots & \alpha_n+\delta_n+(a_2-a_1) & \cdots & \alpha_{n-1}+\delta_n \\ \\ \alpha_1+\delta_{n-1} & \cdots & \alpha_n+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \cdots i\text{-th} \\ \\ \\ \end{array}$$

Clearly, $a_1 = a_2$ reduces (2.11) to the same form of A as was obtained in Thm. (2.1).

Now, consider the partition of S_n given by case (ii) (a).

Assume that $r_1 = 1$ and B_1 contains $(1, 2), \ldots, (1, k)$ $(k<n)$ and all cycles with an edge $1 \to r$ for $r \in \{2, \ldots, k\}$. Again, by lemma (2.5), A reduces to the form (2.10). Also, we know that $(1, r)$, $(1, r, m)$ and $(1, m, r)$ are in B_1 for $r, m \in \{2, \ldots, k\}$. Therefore, for all $m \neq 1, r$, we have

$$\operatorname{tr}(1, r)A = \operatorname{tr}(1, r, m)A$$

It follows that
$$a_{1r} - a_{1m} = a_{mr} - a_{mm} = \alpha_{r-1} - \alpha_{m-1}$$

$$a_{1m} = \alpha_{m-1} + (a_{1r} - \alpha_{r-1})$$

Set $a_{1r} = \alpha_{r-1} + \delta_n$; then

$a_{1m} = \alpha_{m-1} + \delta_n$. This gives all elements of the first row in (2.10) except a_{11} . Let us, now, find the first column of (2.10).

We know that $(1, r)$ and $(1, m, r)$ are in B_1 for $r, m \varepsilon \{2, \ldots, k\}$ (lemma (2.7)). Therefore,

$$tr(1, r)A = tr(1, m, r)A \quad ,$$

and hence $$a_{1r} - a_{m1} = a_{rm} - a_{mm}$$

Set $a_{21} - \delta_1 = \beta$; then

$a_{r1} = \delta_{r-1} + (a_{21} - \delta_1) = \beta + \delta_{r-1}$ for $r \varepsilon \{2, \ldots, k\}$.

We claim that this relation does not hold good for $r \varepsilon \{k+1, \ldots, n\}$, $m \varepsilon \{2, \ldots, k\}$, for if so, then $a_{r1} - a_{m1} = a_{rm} - a_{mm}$ implies that $tr(1, r)A = tr(1, m, r)A$. But in this case $(1, r) \varepsilon B_2$ and $(1, m, r) \varepsilon B_1$, which means that $tr(1, r)A \neq tr(1, m, r)A$, a contradiction. Therefore, the above chain of relations stops at a_{k1} : $a_{21} = \beta + \delta_1, a_{31} = \beta + \delta_2, \ldots, a_{k1} = \beta + \delta_{k-1}$.

However, if $m, r \notin \{2, \ldots, k\}$, then $(1, r)$ and $(1, m, r)$ belong to B_2 (lemma (2.9)). In particular, $(1, m)$, $(1, m+1, m)$ are in B_2 for $m \notin \{2, \ldots, k\}$, therefore,

$$tr(1, m)A = tr(1, m+1, m)A \ ,$$

and $$a_{m1} - a_{m+11} = a_{m\ m+1} - a_{m+1\ m+1} = \delta_{m-1} - \delta_m$$

Set $a_{k+1\ 1} - \delta_k = \alpha_n$; then $a_{k+2\ 1} = \alpha_n + \delta_{k+1}$, $\ldots$, $a_{n1} = \alpha_n + \delta_{n-1}$.

In order to determine a_{11} , we use

$$tr(1, m+1, m)A = tr(m, m+1)A, \ m \notin \{2, \ldots, k\} \ .$$

Here $$a_{m+1\ 1} - a_{11} = \delta_m - \delta_n \ ,$$

and hence $$a_{11} = \delta_n + (a_{m+1\ 1} - \delta_m)$$

$$= \delta_n + \alpha_n \ .$$ Consequently, the first column of A is $\{\alpha_n + \delta_n, \beta + \delta_1, \ldots, \beta + \delta_{k-1}, \alpha_n + \delta_k, \ldots, \alpha_n + \delta_{n-1}\}$, where β is obtained as follows:

Since $tr(1, 2)A = a_1$ and

$$tr(1, k+1)A = a_2$$

therefore, $$a_{21} + a_{12} + \sum_{i \neq 1,2}^{n} a_{ii} = a_1$$

and $$a_{1\ k+1} + a_{k+1\ 1} + \sum_{i \neq 1,k+1}^{n} a_{ii} = a_2$$

By subtracting the 2^{nd} equation from the first, we get

$$\alpha_1 + \delta_n + \beta + \delta_1 - \alpha_k - \delta_n - \alpha_n - \delta_k + \alpha_k + \delta_k - \alpha_1 - \delta_1 = a_1 - a_2$$

or
$$\beta - \alpha_n = a_1 - a_2$$

$$\beta = (a_1 - a_2) + \alpha_n$$

Thus the matrix A takes the form

$$2.12 \quad A = \begin{array}{cc} \begin{pmatrix} \alpha_n+\delta_n & \alpha_1+\delta_n & \cdots & \alpha_{n-1}+\delta_n \\ (a_1-a_2)+\alpha_n+\delta_1 & \alpha_1+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ (a_1-a_2)+\alpha_n+\delta_2 & \alpha_1+\delta_2 & \cdots & \alpha_{n-1}+\delta_2 \\ \vdots & & & \\ (a_1-a_2)+\alpha_n+\delta_{k-1} & \alpha_1+\delta_{k-1} & \cdots & \alpha_{n-1}+\delta_k \\ \alpha_n+\delta_k & \alpha_1+\delta_k & \cdots & \alpha_{n-1}+\delta_k \\ \vdots & \vdots & & \vdots \\ \alpha_n+\delta_{n-1} & \alpha_1+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix} & \begin{array}{l} \\ \\ \\ \\ \cdots \quad k\text{-th} \\ \cdots \\ \\ \\ \end{array} \end{array}$$

Here the rows 2 to k of the first column are "distinguished" since we assumed, for simplicity, that 2-cycles of B_1 had 1 as a common vertex, and $2, \ldots, k$ as the other vertices. In the general case, the "distinguished" elements of A are those in column r_1 and rows $r_2, r_3, \ldots, r_k$. Conversely, if A is of the above form, then $\{tr(PA) : P \in S_n\} = \{a_1, a_2\}$

In this case, also, $a_1 = a_2$ gives the same form of A as was obtained in Thm. (2.1).

Case (ii) (b) is analogous to case (ii) (a). In this case, we can assume that $T \cap B_1$ contains $(1, 2), \ldots, (1, k)$ $(k<n)$. Also, B_1 contains 3-cycles $(1, m, r)$ for $r \in \{2, \ldots, k\}$ and $m \in \{2, \ldots, n\}$, $m \neq r$. The equation $tr(1, m, r)A = tr(1, r)A$ determines the first column of A in (2.10) and the remaining computation is similar to the preceding analysis. We omit the details. We obtain

$$2.13 \quad A = \begin{pmatrix} \alpha_n+\delta_n & (a_1-a_2)+\alpha_1+\delta_n & \cdots & (a_1-a_2)+\alpha_{k-1}+\delta_n & \cdots & \alpha_{n-1}+\delta_n \\ \alpha_n+\delta_1 & \alpha_1+\delta_1 & \cdots & \alpha_{k-1}+\delta_1 & & \alpha_{n-1}+\delta_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_n+\delta_{n-1} & \alpha_1+\delta_{n-1} & \cdots & \alpha_{k-1}+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix}$$

Again, in the general case, the "distinguished" elements of A are those in row r_1 and columns $r_2, \ldots, r_k$. Conversely, if the matrix A is of the above form, then $\{tr(PA) : P \in S_n\} = \{a_1, a_2\}$.

Clearly, $a_1 = a_2$ gives the same form of A as was obtained in theorem (2.1).

Finally, we consider the structure of A when the partition of S_n is given by case (ii) (c).

Let $(1, 2), \ldots, (1, k)$ be in $T \cap B_1$, $k<n$. For simplicity, assume that $\{(1, r, m) : r \varepsilon \{k_1+1, \ldots, k\} ; m = k+1, \ldots, n\} \subseteq B_1$ and $\{(1, r, m) : r \varepsilon \{2, \ldots, k_1\} ; m = k+1, \ldots, n\} \subseteq B_2$.

Lemma (2.5) reduces A to the form (2.10) and by Lemma (2.7), $(1, r)$, $(1, m, r)$ and $(1, r, m)$ are in B_1 for $r, m \varepsilon \{2, \ldots, k\} ; m \neq r$. Therefore, as before,

$$tr(1, r)A = tr(1, m, r)A \text{ gives}$$

$$a_{r1} - a_{m1} = a_{rm} - a_{mm}$$

and $$a_{r1} = \delta_{r-1} + (a_{21} - \delta_1) \quad r \varepsilon \{2, \ldots, k\} .$$

Also, for $r \varepsilon \{k+1, \ldots, n\}$, $(1, r)$ and $(1, 2, r)$ are in B_2. Hence $tr(1, r)A = tr(1, 2, r)A$

implies $a_{r1} = \delta_{r-1} + (a_{21} - \delta_1)$, for $r \varepsilon \{2, \ldots, n\}$.

Set $a_{21} - \delta_1 = \alpha_n$.

Then $$a_{r1} = \alpha_n + \delta_{r-1} ,$$

and $$a_{21} = \alpha_n + \delta_1, \ldots, a_{n1} = \alpha_n + \delta_{n-1} .$$

To find the first row of (2.10), note that $(1, r)$, $(1, m, r)$ and $(1, r, m)$ are in B_1 for $m, r \varepsilon \{2, \ldots, k\}$. Hence, 2.14 $\quad a_{1r} = \alpha_{r-1} + (a_{12} - \alpha_1) \quad \text{for} \quad r \in \{2, \ldots, k\}$.

Since $r \in \{k+1, \ldots, n\}$, $(1, r)$ and $(1, r, 2)$ are in B_2 ,

$$tr(1, r)A = tr(1, r, 2)A$$

and

$$a_{1r} = \alpha_{r-1} + (a_{12} - \alpha_1) .$$

By (2.14),

$$a_{1r} = \alpha_{r-1} + (a_{12} - \alpha_1) \quad \text{for} \quad r \in \{3, \ldots, n\}$$

Setting $a_{12} - \alpha_1 = \delta_n$,

we have

$$a_{1r} = \alpha_{r-1} + \delta_n \quad \text{for} \quad r \in \{3, \ldots, n\} .$$

In order to determine the element a_{11} , we use the fact that $(m, m+1)$ and $(1, m+1, m)$ are in B_2 for $m \in \{k+1, \ldots, n\}$. We have

$$tr(m, m+1)A = tr(1, m+1, m)A$$

and

$$\begin{aligned} a_{11} &= a_{m+1\,1} - a_{m+1\,m} + a_{1m} \\ &= \alpha_n + \delta_m - \alpha_{m-1} - \delta_m + \alpha_{m-1} + \delta_n \\ &= \alpha_n + \delta_n \end{aligned}$$

Thus (2.10) becomes

$$A = \begin{pmatrix} \alpha_n+\delta_n & \alpha_1+\delta_n & \cdots & \alpha_{n-1}+\delta_n \\ \alpha_n+\delta_1 & \alpha_1+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ \vdots & \vdots & & \vdots \\ \alpha_n+\delta_{n-1} & \alpha_1+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix}$$

We know from Theorem (2.1), that for such matrices $tr(PA) = tr(A) \forall P \in S_n$, which means that a_1 cannot be different from a_2. Hence case (ii) (c) cannot arise if $a_1 \neq a_2$.

We bring our results together in the

2.15 Theorem: Let A be an n-square matrix such that $\{tr(PA) : P \in S_n\} = \{a_1, a_2\}$, where $a_1 \neq a_2$. Then A is either of the form:

$$C = \begin{pmatrix} \alpha_1+\delta_1 & \cdots & \alpha_{r_1-1}+\delta_1 & \alpha_{r_1}+\delta_1 & \alpha_{r_1+1}+\delta_1 & \cdots & \alpha_n+\delta_1 \\ \alpha_1+\delta_2 & & \alpha_{r_1-1}+\delta_2 & \alpha_{r_1}+\delta_2 & \alpha_{r_1+1}+\delta_2 & \cdots & \alpha_n+\delta_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_1+\delta_{r_2} & \cdots & \alpha_{r_1-1}+\delta_{r_2} & (a_1-a_2)+\alpha_{r_1}+\delta_{r_2} & \alpha_{r_1+1}+\delta_{r_2} & \cdots & \alpha_n+\delta_{r_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_1+\delta_{r_k} & \cdots & \alpha_{r_1-1}+\delta_{r_k} & (a_1-a_2)+\alpha_{r_1}+\delta_{r_k} & \alpha_{r_1+1}+\delta_{r_k} & \cdots & \alpha_n+\delta_{r_k} \\ \alpha_1+\delta_{r_{k+1}} & \cdots & \alpha_{r_1-1}+\delta_{r_{k+1}} & \alpha_{r_1}+\delta_{r_{k-1}} & \alpha_{r_1+1}+\delta_{r_{k+1}} & \cdots & \alpha_n+\delta_{r_{k+1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_1+\delta_n & & \alpha_{r_1-1}+\delta_n & \alpha_{r_1}+\delta_n & \alpha_{r_1+1}+\delta_n & \cdots & \alpha_n+\delta_n \end{pmatrix}$$

or of the form C^T .

The set $B_1 = \{P \varepsilon S_n : tr(PA) = a_1\}$ consists of all 2-cycles (r_1, r_j) , $j = 2, \ldots, k$ and the products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i has its graph with an edge $r_1 \to r_j$ (or $r_j \to r_1$) for $j = 2, \ldots, k$.

Conversely, for every matrix of the form C or C^T $\{tr(PA) : P \varepsilon S_n\} = \{a_1, a_2\}$.

Proof: The foregoing discussion proves all but the last statement, which is trivial to verify

Corollary: If A is such that $\{tr(PA) : P \varepsilon S_n\} = \{a_1, a_2\}$, where $a_1 \neq a_2$, there exist permutation matrices θ_1 and θ_2 in S_n and an integer k , $1 \leq k \leq n$, such that either $\theta_1 A \theta_2$ or $(\theta_1 A \theta_2)^T$ is equal to:

$$C = \begin{pmatrix} \alpha_n+\delta_n & \alpha_1+\delta_n & \cdots & \alpha_{n-1}+\delta_n \\ (a_1-a_2)+\alpha_n+\delta_1 & \alpha_1+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\ (a_1-a_2)+\alpha_n+\delta_2 & \alpha_1+\delta_2 & \cdots & \alpha_{n-1}+\delta_2 \\ \vdots & \vdots & & \vdots \\ (a_1-a_2)+\alpha_n+\delta_{k-1} & \alpha_1+\delta_{k-1} & \cdots & \alpha_{n-1}+\delta_{k-1} \\ \alpha_n+\delta_k & \alpha_1+\delta_k & \cdots & \alpha_{n-1}+\delta_k \\ \vdots & \vdots & & \vdots \\ \alpha_n+\delta_{n-1} & \alpha_1+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1} \end{pmatrix}$$

The set $B_1 = \{P \varepsilon S_n : tr(PA) = a_1\}$ consists of all 2-cycles $(\theta_2(1), \theta_1(j))$, $j = 2, \ldots, k$ and the products P of disjoint cycles $P_1, \ldots, P_m$, $m \geq 1$, for which one of the P_i has its graph with an edge $\theta_2(1) \to \theta_1(j)$ (or $\theta_1(j) \to \theta_2(1)$) for $j = 2, \ldots, k$.

Remark: The general case of a matrix A for which $tr(PA)$ takes on k distinct values $a_1, \ldots, a_k$ as P ranges over S_n presents formidable combinatorial difficulties. We can, however, indicate one or two very special results which are possible.

If $k = n$ and $tr(1, r)A = a_{r-1}$ for $r = 2, \ldots, k$ and $tr(PA) = a_n$ for all other 2-cycles P in T, we can show that A has the form:

$$\left(\begin{array}{c|cccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
\hline
a_{21} & \alpha_1+\delta_1 & \alpha_2+\delta_1 & \cdots & \alpha_{n-1}+\delta_1 \\
a_{31} & \alpha_1+\delta_2 & \alpha_2+\delta_2 & \cdots & \alpha_{n-1}+\delta_2 \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n1} & \alpha_1+\delta_{n-1} & \alpha_2+\delta_{n-1} & \cdots & \alpha_{n-1}+\delta_{n-1}
\end{array}\right)$$

We do not give the details of proof, but the admissibility of such a partition of T follows from (1.9), (1.10), (1.11) and (1.12) and the structure is obtained by lemma (2.5).

Similarly, if $k > n$, an admissible partition of T is given by: $tr(1, 2)A = a_1, \ldots, tr(1, n)A = a_{n-1}, tr(2, 3)A = a_n, \ldots, tr(r, s) = a_{k-1}$ and $tr(PA) = a_k$ for all other 2-cycles in T. In this case A has the form:

$$\left[\begin{array}{c|ccc} * & & * & \\ \hline & \alpha_1+\delta_1 & \cdots & \alpha_{n-r}+\delta_1 \\ * & \vdots & & \vdots \\ & \alpha_1+\delta_{n-r} & \cdots & \alpha_{n-1}+\delta_{n-r} \end{array}\right]$$

To conclude this chapter, we consider the set of $n \times n$ matrices $\{A : E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n\}$, where $E_{n-1}(A)$ denotes the $(n-1)$st elementary symmetric function of A. We claim that for such matrices $E_{n-1}(A) = 0$.

By Remark (1.28), for a $P \varepsilon S_n$,

$$C_{n-1}(P) = \sum_{i=1}^{r} \beta_i C_{n-1}(P_i) \quad , \text{where } P_i \varepsilon M \text{ and}$$

$\sum_{i=1}^{r} \beta_i \neq 1$, therefore,

$$C_{n-1}(P)C_{n-1}(A) = \sum_{i=1}^{r} \beta_i C_{n-1}(P_i)C_{n-1}(A) \quad .$$

By taking the trace of both the sides, we get

$$2.16 \quad tr(C_{n-1}(PA)) = tr(C_{n-1}(P)C_{n-1}(A)) = \sum_{i=1}^{r} \beta_i tr(C_{n-1}(P_i)C_{n-1}(A))$$

By our assumption that $E_{n-1}(PA) = E_{n-1}(A) \forall P \varepsilon S_n$ and the fact that $trC_{n-1}(A) = E_{n-1}(A)$, we have

$$tr(C_{n-1}(P)C_{n-1}(A)) = tr(C_{n-1}(A)) = \lambda \quad , \text{say.}$$

Substituting λ in (2.16), we get

$\lambda = \lambda\sum_{i=1}^{r} \beta_i$, which implies $\lambda = 0$, since $\sum_{i=1}^{r} \beta_i \neq 1$.

$$E_{n-1}(A) = tr(C_{n-1}(A) = 0$$

Thus we have

2.17 Theorem: A necessary condition that the $n \times n$ $(n \geq 3)$ matrix A have the property, $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n$, is that $E_{n-1}(A) = 0$.

NOTE: The corresponding theorem for an $n \times n$ matrix A ; namely that $tr(PA) = tr(A)$ for all $P \varepsilon S_n$ implies $tr(A) = 0$ is not true. For, in this case corollary (1.14) asserts that if

$P \varepsilon S_n$, then $P = \sum_{i=1}^{r} \alpha_i P_i$, where $P_i \varepsilon M$ and $\sum_{i=1}^{r} \alpha_i = 1$. This fact saves tr(A) from becoming zero.

2.18 Notation: Write $C_{n-1}(A) = B$ and $C_{n-1}(P) = P'$.

In order to gain further insight into matrices with the property $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n$, we establish:

2.19 Theorem: For an $n\times n$ $(n \geq 3)$ matrix A satisfying $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \varepsilon S_n$, $\det A = 0$.

Proof: By Theorem (2.17), $E_{n-1}(PA) = tr(C_{n-1}(PA)) = tr(C_{n-1}(P)C_{n-1}(A)) = tr(P'B)$ implies $tr(P'B) = 0$ for all $P \varepsilon S_n$. In fact, $tr(QB) = 0$ for all linear combinations Q of such P''s

Since $\det B = \det C_{n-1}(A) = (\det A)^{n-1}$ ([2]; pp. 17), it is sufficient to establish that $\det B = 0$.

Now $B = (b_{ij})$ is an $n\times n$ matrix; let

$$B_o = \begin{pmatrix} b_{11}+b_{12} & b_{12}+b_{13} & b_{13} & \cdots & b_{1n} \\ b_{21}+b_{22} & b_{22}+b_{23} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \\ b_{k1}+b_{k2} & b_{k2}+b_{k3} & b_{k3} & \cdots & b_{kn} \\ \vdots & \vdots & \vdots & & \\ b_{n1}+b_{n2} & b_{n2}+b_{n3} & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

Then $\det B = \det B_o$. We claim that the first two columns of B_o are of the form $\{\alpha_r, -\alpha_r, \ldots, (-1)^{n-1}\alpha_r\}^T$, $r = 1, 2$ for certain real numbers α_1, α_2 . In the following, we shall want to derive conclusions from the fact that $tr(QB) = 0$ for any linear combination Q of P''s and the fact that $E_{n-1}(A) = \sum_{i=1}^{n} b_{ii} = 0$. We shall simply say " Q gives ....", where the dots indicate the simplified form of the equation $tr(QB) = 0$ obtained by using $\sum_{i=1}^{n} b_{ii} = 0$. Thus, for example, for $P' = (\overset{+}{1}, \overset{+}{2})_-$, $tr(P'B) = b_{12} + b_{21} - \sum_{i \neq 1,2}^{n} b_{ii} = 0$. Using $\sum_{i=1}^{n} b_{ii} = 0$, we find that "$(\overset{+}{1}, \overset{+}{2})_-$ gives $b_{12} + b_{21} = -(b_{11} + b_{22})$" ; and $b_{11} + b_{12} = -(b_{21} + b_{22})$. In the same way $(r, s)_-$ gives $b_{sr} + b_{rs} = \pm(b_{rr} + b_{ss})$, where the +ve sign is used if $r+s$ is even and the −ve sign is used otherwise.

If we let $b_{11} + b_{12} = \alpha_1$, then $b_{21} + b_{22} = -\alpha_1$, which says that the first two elements of the first column of B_o are α_1 and $-\alpha_1$, respectively. By judiciously picking sums of P''s , we can show that the remaining elements in the first column of B_o are also $\pm\alpha_1$ with correct signs. For example, $Q_1 = (\overset{+}{1}, \overset{+}{2}, \overset{+}{3}, \overset{+}{4})_- + (\overset{+}{1}, \overset{-}{3}, \overset{-}{4})_+$ gives $b_{31} + b_{32} = -(b_{21} + b_{22}) = \alpha_1$, since

$$0 = tr(Q_1B) = tr\left(\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & 0 & \ldots & 0 \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}\right) B$$

$$= b_{21} + b_{22} + b_{31} + b_{32} \quad , \text{ for } \sum_{i=1}^{n} b_{ii} = 0 \quad .$$

In general, for k even, $(\overset{+}{1}, \overset{+}{2}, \overset{+}{k-1}, \overset{+}{k})_- + (\overset{+}{1}, \overset{-}{k-1}, \overset{-}{k})_+$ gives $b_{21} + b_{k-12} + b_{kk-1} + b_{1k} + b_{k-11} - b_{kk-1} + b_{1k} + b_{22} = 0$ and

$$b_{k-11} + b_{k-12} = - (b_{21} + b_{22}) = \alpha_1 \quad .$$

If k is odd, then $(\overset{+}{1}, \overset{-}{2}, \overset{+}{k-1}, \overset{-}{k})_- + (\overset{-}{1}, \overset{-}{k-1}, \overset{+}{k})_+$ gives $b_{k-11} + b_{k-12} = (b_{21} + b_{22}) = - \alpha_1$.

These results give us all but the last element in the first column of B_o . Now, we show that the n-th element of the first column is obtained as follows:

For n odd, the matrix $(\overset{+}{1}, \overset{-}{2}, \overset{+}{n-1}, \overset{-}{n})_- + (\overset{+}{2}, \overset{-}{n-1}, \overset{-}{n})_+$ gives $b_{21} - b_{n-12} + b_{nn-1} - b_{1n} + b_{n-12} - b_{nn-1} - b_{2n} + b_{11} = 0$; and hence $- (b_{1n} + b_{2n}) + (b_{11} + b_{21}) = 0$. This can be rewritten as: $(b_{21} + b_{12}) - b_{12} - (b_{1n} + b_{n1}) + b_{n1} - (b_{2n} + b_{n2}) + b_{n2} + b_{11} = 0$.

Since $(r, s)_-$ gives $b_{rs} + b_{sr} = \pm (b_{rr} + b_{ss})$,

we get $$b_{n1} + b_{n2} = b_{11} + b_{12} = \alpha_1$$

Similarly, when n is even the matrix $(\overset{+}{1}, \overset{+}{2}, \overset{+}{n-1}, \overset{+}{n})_- + (\overset{-}{2}, \overset{-}{n-1}, \overset{+}{n})_+$ gives $b_{n1} + b_{n2} = - (b_{11} + b_{12}) = - \alpha_1$.

Hence, the first column of B_o is $\{\alpha_1, -\alpha_1, \ldots, (-1)^{n-1}\alpha_1\}^T$.

In order to get the second column of B_o , we observe that the matrix $(\overset{+}{1}, \overset{+}{2}, \overset{+}{3}, \overset{+}{4})_- + (\overset{+}{1}, \overset{-}{3}, \overset{-}{4})_+$ gives $b_{21} + b_{31} + b_{32} + b_{22} = 0$. This can be re-written as $(b_{12} + b_{21}) - b_{12} + (b_{31} + b_{13}) - b_{13} + (b_{32} + b_{23}) - b_{23} + b_{22} = 0$. By using the fact that $b_{12} + b_{21} = -(b_{11} + b_{22})$, $b_{31} + b_{13} = b_{11} + b_{33}$ and $b_{32} + b_{23} = -(b_{22} + b_{33})$, we get

$b_{22} + b_{23} = -(b_{12} + b_{13})$. Set $b_{12} + b_{13} = \alpha_2$;

then $b_{22} + b_{23} = -\alpha_2$. Also, $(\overset{+}{2}, \overset{+}{3})_-$ gives

$b_{33} + b_{32} = -(b_{22} + b_{23}) = \alpha_2$. Thus, the first three elements of the second column of B_o are α_2, $-\alpha_2$ and α_2 , respectively. The remaining elements of this column are obtained by examining the matrices: $\{(2, 3, k-1, k)_- + (2, k-1, k)_+, k = 5, \ldots, n\}$ and $(2, 3, n-1, n)_- + (3, n-1, n)_+$. We omit the details. Hence, we have

$$B_o = \begin{pmatrix} \alpha_1 & \alpha_2 & b_{13} & \cdots & b_{1n} \\ -\alpha_1 & -\alpha_2 & b_{23} & \cdots & b_{2n} \\ \alpha_1 & \alpha_2 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-1}\alpha_1 & (-1)^{n-1}\alpha_2 & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

and $\det B_o = 0 = \det B$. Therefore, $\det A = 0$. QED.

2.20 Remark. Let matrix B' be obtained from the matrix $B = C_{n-1}(A) = (b_{ij})$, $i, j = 1, \ldots, n$, by replacing the first $n-1$ columns by $(b_{ij} + b_{i,j+1})$, $i = 1, \ldots, n$; $j = 1, \ldots, n-1$ and by keeping the last column as it is. We claim that the first $n-1$ columns of B' are of the form $\{\alpha_r, -\alpha_r, \ldots, (-1)^{n-1}\alpha_r\}^T$, $r = 1, \ldots, n-1$.

In the preceding theorem, we have seen that it is true for the first two columns of B' . We assert that it is true for any k-th column of B' .

Suppose, the k-th element of this column is $(-1)^{k-1}\alpha_k$. In this column, the matrices $\{(k^{s_1}, k+1^{s_2}, k+r^{s_3}, k+r+1^{s_4})_- + (k^{s_5}, k+r^{-s_3}, k+r+1^{-s_4})_+$, $r = 2, \ldots, n-k-1$ and $(k^{t_1}, k+1^{t_2}, n^{t_3}-1, n^{t_4})_- + (k+1^{-t_2}, n^{-t_3}-1, n^{t_5})_+\}$ give the last $n-k-1$ elements, the matrix $(k^+, k^+ +1)_-$ gives the $(k+1)$st element, the matrix $(k^+ -1, k^+, k^+ +1, k^+ +2)_- + (k-1^+, k+1^-, k+2^-)_+$ gives the $(k-1)$st element and the first $k-2$ elements are given by $\{(r, k-1, k)_+ + (r, k)_-$, $r = 1, \ldots, k-2\}$. Thus, without the details, we infer that the k-th column is of the form $\{\alpha_k, -\alpha_k, \ldots, (-1)^{n-1}\alpha_k\}$, $k \le n-1$.

If A is an $m \times n$ matrix with $\rho(A) = k$, then

$\rho(C_r(A)) = \binom{k}{r}$ ([2]; pp. 28). Clearly, for $k < r$, $\rho(C_r(A)) = 0$, i.e. every entry of $C_r(A)$ is zero.

In the present case, when $r = n-1$, $\rho(B) = \binom{k}{n-1}$. Since $\det A = 0$, therefore $\rho(A) < n$. But for $\rho(A) < n-1$, we have the trivial case in which $\rho(B) = 0$. In this case $B = 0$ and it is trivially true that for any A (of rank $< n-1$) $tr(C_{n-1}(A)) = tr(C_{n-1}(PA))$ for all $P \varepsilon S_n$.

We consider now the structure of B and A when $\rho(A) = n-1$ and $\rho(B)$ is (consequently) 1.

Assume that B has the form:

$$B = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ k_1a_1 & k_1a_2 & \cdots & k_1a_n \\ \vdots & \vdots & & \vdots \\ k_{n-1}a_1 & k_{n-1}a_2 & \cdots & k_{n-1}a_n \end{pmatrix}$$

If we examine the first $n-1$ columns of B in the light of Remark (2.20), we find that

2.21 $(a_r + a_{r+1})(1 + k_1) = (a_r + a_{r+1})(1 - k_2) = \ldots = (a_r + a_{r+1})(1 + (-1)^{n-1}k_{n-1}) = 0$ for $r = 1, \ldots, n-1$.

We now consider the form of B in the two cases which arise when $a_r + a_{r+1} \neq 0$ for some $r \in \{1, \ldots, n-1\}$ and when $a_r + a_{r+1} = 0$ for all $r \in \{1, \ldots, n-1\}$.

Case (i) $\quad a_r + a_{r+1} \neq 0$ for $r = k$.

From (2.21), we have $k_r = (-1)^r$ and the matrix B becomes

$$2.22 \quad B = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ -a_1 & -a_2 & \cdots & -a_n \\ & & & \\ & & & \\ (-1)^{n-1}a_1 & (-1)^{n-1}a_2 & \cdots & (-1)^{n-1}a_n \end{pmatrix}$$

Case (ii) $\quad a_r + a_{r+1} = 0$ for all $r \in \{1, \ldots, n-1\}$.

In this case the matrix B takes the form

$$2.23 \quad B = \begin{pmatrix} a_1 & -a_1 & a_1 & \cdots & (-1)^{n-1}a_1 \\ k_1a_1 & -k_1a_1 & k_1a_1 & \cdots & k_1(-1)^{n-1}a_1 \\ \vdots & \vdots & \vdots & & \vdots \\ k_{n-1}a_1 & -k_{n-1}a_1 & k_{n-1}a_1 & \cdots & k_{n-1}(-1)^{n-1}a_1 \end{pmatrix}$$

Note that as far as form is concerned, (2.23) is essentially the transpose of (2.22). This implies that the structure of a matrix A whose compound is a matrix of the form (2.23) is the transpose of a matrix whose compound has the form (2.22). It is therefore sufficient to determine the form of A when B is given by (2.22).

If U_i, $i = 1, \ldots, n$ are the rows of A, then by definition (1.7)

2.24 $$U_1 \wedge U_2 \wedge \ldots \wedge \hat{U}_n = (-1)^{n-i} U_1 \wedge \ldots \wedge \hat{U}_i \wedge \ldots \wedge U_n ,$$

where $\hat{U}_i$ implies the absence of U_i in the Grassmann product.

Now A is of rank n-1. Let us suppose that the row vectors $U_1, \ldots, U_{n-1}$ are independent, and

2.25 $$U_n = \beta_1 U_1 + \ldots + \beta_{n-1} U_{n-1}$$

Taking the left Grassmann product of (2.25) with $U_1 \wedge \ldots \wedge \hat{U}_i \wedge \ldots \wedge \hat{U}_n$, we get

$$\begin{aligned} U_1 \wedge U_2 \wedge \ldots \wedge \hat{U}_i \wedge \ldots \wedge U_n &= \beta_i U_1 \wedge \ldots \wedge \hat{U}_i \wedge \ldots \wedge \hat{U}_n \wedge U_i \\ &= (-1)^{n-i-1} \beta_i U_1 \wedge \ldots \wedge \hat{U}_n \end{aligned}$$

Using (2.24), we obtain

$$(-1)^{n-i} U_1 \wedge \ldots \wedge \hat{U}_n = (-1)^{n-i-1} \beta_i U_1 \wedge \ldots \wedge \hat{U}_n ,$$ which implies that $$\beta_i = -1 \text{ for all } i \in \{1, \ldots, n-1\}$$

Thus $$U_n = - \sum_{i=1}^{n-1} U_i \quad , \text{ and}$$

2.26 $$A = \begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{n-1} \\ -\sum_{i=1}^{n-1} U_i \end{pmatrix}$$

Note that we are led to this form, no matter which rows of A we assume to be linearly independent.

Similarly, the form of A, when B is given by (2.23), is the transpose of (2.26).

We summarize these results in

2.27 Theorem: If A is rank $n-1$ n-square matrix with the property that $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \in S_n$ $(n \geq 3)$, then A is of the form (2.26) or its transpose.

However, for an n-square matrix A of rank less than $n-1$, $E_{n-1}(PA) = E_{n-1}(A)$ for all $P \in S_n$ is trivially true.

CHAPTER III

ORTHOGONAL MATRICES AS LINEAR COMBINATIONS OF PERMUTATION MATRICES.

3.2 It is an interesting fact that some orthogonal matrices, such as any permutation matrix, or $\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$ can be expressed as linear combinations of permutation matrices; while others, such as $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$ cannot. In this Chapter we shall take a look at orthogonal matrices of the former type. Our main result will be to show that, if the orthogonal matrix θ can be written as a sum $\sum_{i=1}^{k} \alpha_i P_i$, then $\sum_{i=1}^{k} \alpha_i = \pm 1$.

In view of theorem (1.8) such linear combinations can always be expressed in terms of the members of TUCUI. It turns out to be convenient to treat this question under three possibilities which can arise. First, we examine those orthogonal matrices which can be expressed as linear combinations of the elements of $T \cup I$. Obviously such matrices are always symmetric. Parenthatically, we note that not every symmetric orthogonal matrix can be expressed as a linear combination of $T \cup I$, as the example $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ shows. Secondly, we consider those orthogonal matrices which can be expressed as linear combinations of permutation matrices from the set C. Finally, we look at those orthogonal matrices which lie outside the two previous categories and require permutations from both C and $T \cup I$ in their representation.

Suppose the orthogonal matrix θ is a linear combination of elements of $T \cup I$. Let all $(r, s) \in T$ be arranged in lexicographic order $(<)$ and let the coefficient of (r, s) be denoted by α_{rs}. In any product $\alpha_{rs}\alpha_{tu}$, as a matter of convenience it will be taken for granted that $(r, s) < (t, u)$. Furthermore, we denote $\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs}$ by $\Sigma\alpha_{rs}$, the sum of the products of all α_{rs} taken two at a time by $\Sigma\, \alpha_{rs}\alpha_{tu}$, and the sum of the squares of α_{rs} by $\Sigma\, \alpha_{rs}^2$. Let $\Sigma_{r_1} = \sum_{r,s \neq r_1} \alpha_{rs}$ be the sum of all α's which are the coefficients of those permutations (r, s) which keep r_1 fixed, let $\Sigma_{r_1,s_1} = \sum_{r,s \neq r_1,s_1} \alpha_{rs}$ = sum of all α's which are the coefficients of those (r, s) which leave r_1 and s_1 unaltered, let σ_{r_1} be the set of α's in the summation Σ_{r_1} and let $\sum_{\sigma_{r_1}} \alpha_{rs}\alpha_{tu}$ be the sum of the products of all α_{rs} in σ_{r_1} taken two at a time.

3.3 Theorem: If $\theta = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs}(r, s)$, where the α_{rs} are real, is an orthogonal matrix, then $\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs} = \Sigma\alpha_{rs} = \pm 1$.

Proof: We note first that, in the special case in which $\alpha_{r_1 s} = 0$, $s = 1, \ldots, n$; $s \neq r_1$, for a fixed $r_1 \in \{1, \ldots, n\}$;

$$\theta = \begin{pmatrix} & & & 0 & & \\ & & & 0 & & \\ & * & & \vdots & * & \\ 0 & 0 & \cdots & \Sigma\alpha_{rs} & \cdots & 0 \\ & * & & \vdots & * & \\ & & & 0 & & \\ & & & \vdots\, r_1\text{-th} & & \end{pmatrix} \begin{matrix} \\ \\ \\ \ldots r_1\text{-th} \\ \\ \\ \\ \end{matrix}$$

From this it is evident that $\Sigma\alpha_{rs} = \pm 1$.

For the general case, there are technically two approaches that we could take. Since θ is symmetric and orthogonal, $\theta^2 = I = (\Sigma\, \alpha_{rs}(r, s))(\Sigma\, \alpha_{rs}(r, s))$. We could consider both sides of this equation as representations of I as linear combinations of elements of M (Theorem 1.8), and equate coefficients. We prefer, however, a second approach which just uses the fact that the row vectors of θ form an orthonormal set of vectors. Written out, θ has the form

$$\theta = \begin{pmatrix} \Sigma_1 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n-1} & \alpha_{1n} \\ \alpha_{12} & \Sigma_2 & \alpha_{23} & \cdots & \alpha_{2n-1} & \alpha_{2n} \\ \alpha_{13} & \alpha_{23} & \Sigma_3 & \cdots & \alpha_{3n-1} & \alpha_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{1n-1} & \alpha_{2n-1} & \alpha_{3n-1} & \cdots & \Sigma_{n-1} & \alpha_{n-1n} \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \cdots & \alpha_{n-1n} & \Sigma_n \end{pmatrix}$$

Using the fact that each row of the matrix θ is of norm 1, we obtain the n equations:

$$\Sigma\, \alpha_{rs}^2 + 2k_1 = 1 \quad , \text{ where } \quad k_1 = \sum_{\sigma_1} \alpha_{rs}\alpha_{tu}$$

$$\Sigma\, \alpha_{rs}^2 + 2k_2 = 1 \quad , \text{ where } \quad k_2 = \sum_{\sigma_1} \alpha_{rs}\alpha_{tu}$$

....

....

$$\Sigma\, \alpha_{rs}^2 + 2k_n = 1 \quad , \text{ where } \quad k_n = \sum_{\sigma_n} \alpha_{rs}\alpha_{tu}$$

It follows that $k_1 = k_2 = \ldots = k_n = k$, say. Clearly, each k_i contains $\binom{m_1}{2}$ terms, where $m_1 = \binom{n-1}{2}$. Also,

$$\sum_{i=1}^{n} k_i = nk = \sum_{\sigma_1} \alpha_{rs}\alpha_{tu} + \ldots + \sum_{\sigma_n} \alpha_{rs}\alpha_{tu} \quad \text{, and}$$

3.4 $$nk = (n-3)\Sigma\, \alpha_{rs}\alpha_{tu} - R \quad \text{, where}$$

$$R = \alpha_{12}[\Sigma_{1,2}] + \ldots + \alpha_{rs}[\Sigma_{r,s}] + \ldots + \alpha_{n-1n}[\Sigma_{n-1,n}]$$

Here $[\Sigma_{r,s}]$ denotes the sum of those $\alpha_{kp} \varepsilon\, \sigma_{r,s}$ such that $(k, p) > (r, s)$.

Now, using the fact that the inner product of any two rows of θ is zero, we get the following $\binom{n}{2}$ equations:

3.5 $$\begin{cases} \alpha_{12}(\Sigma_{1,2} + \Sigma\alpha_{rs} - \alpha_{12}) + \alpha_{13}\alpha_{23} + \ldots + \alpha_{1n-1}\alpha_{2n-1} + \alpha_{1n}\alpha_{2n} = 0 \\ \alpha_{13}(\Sigma_{1,3} + \Sigma\alpha_{rs} - \alpha_{13}) + \alpha_{12}\alpha_{23} + \ldots + \alpha_{1n-1}\alpha_{3n-1} + \alpha_{1n}\alpha_{3n} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_{n-1n}(\Sigma_{n-1,n} + \Sigma\alpha_{rs} - \alpha_{n-1n}) + \alpha_{1n-1}\alpha_{1n} + \ldots + \alpha_{n-1n-2}\alpha_{nn-2} = 0 \end{cases}$$

Each equation in this set has $n(n-2)$ terms. Adding these equations, we get

$$3\, \Sigma\alpha_{rs}\alpha_{tu} + R = 0$$. Adding this equation to (3.4), we obtain

$$nk = n\, \Sigma\alpha_{rs}\alpha_{tu} \quad \text{, which implies}$$

3.6 $$k = \Sigma\alpha_{rs}\alpha_{tu}$$

Since $\Sigma\alpha_{rs}^2 + 2k_i = 1$ and $k_i = k$, therefore

$$(\Sigma\alpha_{rs})^2 = 1 \qquad \text{Q.E.D.}$$

3.7 Theorem: If $\theta = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs}(r, s) + \lambda I$ is an orthogonal matrix, then $\Sigma\alpha_{rs} + \lambda = \pm 1$.

Proof: The argument is similar to the one in Theorem (3.3). In this case equation (3.4) becomes

$$nk = (n-3)\Sigma\alpha_{rs}\alpha_{tu} - R + (n-3)\lambda\Sigma\alpha_{rs} + \lambda\Sigma\alpha_{rs} \quad , \text{ and}$$

the set of equations (3.5) when added, gives

$$R + 3\,\Sigma\alpha_{rs}\alpha_{tu} + 3\,\lambda\Sigma\alpha_{rs} - \lambda\Sigma\alpha_{rs} = 0$$

Adding these two equations, we obtain

$n\lambda\Sigma\alpha_{rs} + n\Sigma\alpha_{rs}\alpha_{tu} = nk$. Using this in

$\Sigma\alpha_{rs}^2 + \lambda^2 + 2k = 1$, we get

$$\Sigma\alpha_{rs} + \lambda = \pm 1 \quad .$$

Q.E.D.

The following example shows that there exist orthogonal matrices θ of the form $\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs}(r, s) + \lambda I$, in which none of the α_{rs} is zero.

For the matrix

$$3.8 \quad \theta = \begin{pmatrix} \frac{2}{n} & \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} & \frac{2}{n} & \frac{-(n-2)}{n} \\ \frac{2}{n} & \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} & \frac{-(n-2)}{n} & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n} & \frac{2}{n} & \cdots & \frac{-(n-2)}{n} & \frac{2}{n} & \frac{2}{n} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \frac{-(n-2)}{n} & \cdots & \frac{2}{n} & \frac{2}{n} & \frac{2}{n} \\ \frac{2}{n} & \frac{-(n-2)}{n} & \frac{2}{n} & \cdots & \frac{2}{n} & \frac{2}{n} & \frac{2}{n} \\ \frac{-(n-2)}{n} & \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} & \frac{2}{n} & \frac{2}{n} \end{pmatrix}$$: it is

easy to show that when n is odd, then $\lambda = \frac{3-n}{2}$; and when n is even, $\lambda = \frac{4-n}{2}$. In both cases, $\Sigma\alpha_{rs} + \lambda = 1$.

More generally we have

3.9 Theorem: Given a subset $\{(r, s)\}$ of 2-cycles for which the combined graph is strongly connected and complete, there exists an orthogonal matrix $\theta = \sum_r \sum_s \alpha_{rs}(r, s) + \lambda I$ such that every $\alpha_{rs} \neq 0$.

Proof: The fact that $\{(r, s)\}$ form a strongly connected complete subgroup implies that $\{(r, s)\}$ is the whole set of 2-cycles in S_k for some $k \leq n$. The preceding example then gives appropriate nonzero values of α_{rs} .

Now we consider those orthogonal matrices which can be expressed as linear combinations of permutation matrices from the set $C = \{(r, r+1, t);\ r = 1, \ldots, n-2;\ t = r+2, \ldots, n\}$.

Suppose the orthogonal matrix θ is a linear combination of elements of C . Let β_{rt} be the coefficient of $(r, r+1, t)$ and let $b_r = \beta_{rr+2} + \beta_{rr+3} + \cdots + \beta_{rn}$.

3.10 Theorem: If $\theta = \sum_{r=1}^{n-2} \sum_{t=r+2}^{n} \beta_{rt}(r, r+1, t)$, $b_r \geq 0$ for all r , is an orthogonal matrix, then all but one of the β_{rt} are zero.

Proof: The matrix θ has the form

$$\theta = \begin{pmatrix} \sum\limits_{r\neq 1} b_r & b_1 & 0 & 0 & \cdots & 0 \\ 0 & \sum\limits_{r\neq 1,2} b_r & \beta_{13}+b_2 & \beta_{14} & \cdots & \beta_{1n} \\ \beta_{13} & 0 & -\beta_{13}+\sum\limits_{r\neq 2,3} b_r & \beta_{24}+b_3 & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{1n} & \beta_{2n} & \beta_{3n} & \beta_{4n} & \cdots & \sum\limits_{r=1}^{n-2} b_r - \sum\limits_{i=1}^{n-2} \beta_{in} \end{pmatrix}$$

We shall obtain our result by using induction on n of S_n. Since θ is an orthogonal matrix, the inner products of the first row with every other row of θ are zero, we get

3.11 $$b_1 \sum_{r \neq 1,2} b_r = 0$$

3.12 $$\beta_{13} \sum_{r \neq 1} b_r = 0$$

3.13 $$\begin{cases} \beta_{14} \sum_{r \neq 1} b_r + \beta_{24} b_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_{1n} \sum_{r \neq 1} b_r + \beta_{2n} b_1 = 0 \end{cases}$$

By (3.11) and (3.12),

either $b_1 = 0$ or $\sum_{r \neq 1,2} b_r = 0$ and

either $\beta_{13} = 0$ or $\sum_{r \neq 1} b_r = 0$. Thus we have the following three cases:

(i) $b_1 = 0$

(ii) $\sum_{r \neq 1,2} b_r = 0$ and $\sum_{r \neq 1} b_r = 0$

(iii) $\sum_{r \neq 1,2} b_r = 0$ and $\beta_{13} = 0$.

Let us examine these cases.

Case (i) $\qquad b_1 = 0$

This implies that $\sum_{r \neq 1} b_r = 1$ since θ is an orthogonal, $\beta_{13} = \beta_{14} = \ldots = \beta_{1n} = 0$, and θ reduces to the form where induction is applicable to its $(n-1) \times (n-1)$ principal submatrix.

Case (ii) $\qquad \sum_{r \neq 1,2} b_r = 0$ and $\sum_{r \neq 1} b_r = 0$

Let $\theta = (a_{ij})$. $\sum_{r \neq 1} b_r = 0$ implies that $b_1 = 1$ and $\beta_{24} = \beta_{25} = \ldots = \beta_{2n} = 0$. It also implies that $b_r = 0$ for all $r = 2, \ldots, n-2$ $(b_r \geq 0)$. The fact that the third row of θ has norm 1 gives

$$\beta_{13}^2 + (-\beta_{13} + \sum_{r \neq 2,3} b_r)^2 + b_3^2 = 1 \quad ,$$

and hence $\qquad \beta_{13}^2 + (1-\beta_{13})^2 = 1$.

Thus $\qquad \beta_{13} = 0$ or $\beta_{13} = 1$

If $\beta_{13} = 1$, then $a_{33} = 1-\beta_{13} = 0$. In this case we can show that θ reduces to a 3-cycle $(1, 2, 3)$. We achieve this by showing that $a_{tt} = 1$ for all $t > 3$. Note that we now have $\beta_{14} = \ldots = \beta_{1n} = 0$ and $a_{23} = \beta_{13} = 1$. This implies that $\beta_{35} = \ldots = \beta_{3n} = 0$, and hence the only nonzero entry in the fourth row of the matrix θ is a_{44} . This means that $a_{44} = 1$. Assume that $a_{rr} = 1$ for $r = 4, \ldots, t-1$. We shall show that $a_{tt} = 1$. From the form of θ it is clear that $a_{tr} = 0$ for $r \neq t$ and $r = 1, \ldots, n$, for $a_{rr} = 1, r = 4, \ldots, t-1$. Thus a_{tt} is the only nonzero entry in the t-th row of the matrix θ and it is 1 . Hence, in this case $\theta = (1, 2, 3)$.

Consider the case when $\beta_{13} = 0$. This implies $a_{23} = 0$ and $a_{33} = 1$. Since the fourth row of θ is of norm 1 , we get

$\beta_{14}(1-\beta_{14}) = 0$. Therefore,

$\beta_{14} = 0$ or $\beta_{14} = 1$. Suppose that

$\beta_{1,r-1} = \beta_{1,r-2} = \ldots = \beta_{13} = 0$ and $\beta_{1r} \neq 0 \neq a_{2r}$.

It is clear that $a_{t-1t-1} = 1$ for $t \leq r$. Since $a_{r-1r-1} = 1$, therefore, $a_{r-1r+1} = a_{r-1r+2} = \ldots = a_{r-1n} = 0$. Consider the elements of the r-th row of θ ; $\beta_{ir} \neq 0$ and $a_{rr-1} = \ldots = a_{r2} = 0$. Also, $a_{rr+1} = a_{r-1r+1} = 0, \ldots, a_{rn} = a_{r-1n} = 0$. Thus the r-th row has only two nonzero elements; viz, a_{r1} and a_{rr} . This implies that $\beta_{1r}(1-\beta_{1r}) = 0$, since the norm of r-th row is 1 . Thus $\beta_{1r} = 1$ since $\beta_{1r} \neq 0$ for $a_{2r} \neq 0$. The fact that $a_{tt} = 1$ for $t > r$ follows from an argument similar to that used when $\beta_{13} = 1$. Hence, in this instance, the matrix θ reduces to the cycle $(1, 2, r)$.

Case (iii) $\sum_{r\neq 1,2} b_r = 0$ and $\beta_{13} = 0$

Adding the equations in (3.12) and (3.13), we obtain

$$b_1 \sum_{r\neq 1} b_r + b_1 b_2 = 0 \quad \text{and}$$

$$b_1 \left(\sum_{r\neq 1} b_r + b_2 \right) = 0$$

Suppose $b_1 = 0$. This case reduces to case (i).

If, however, $\sum_{r\neq 1} b_r + b_2 = 0$, then

$\sum_{r\neq 1,2} b_r + 2b_2 = 0$. Since $\sum_{r\neq 1,2} b_r = 0$,

therefore $b_2 = 0$ and $\sum_{r\neq 1} b_r = 0$. This implies that $b_1 = 1$ and this case reduces to case (ii). Q.E.D.

We conclude this chapter with

3.14 Theorem: If $\theta = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \alpha_{rs}(r, s) + \sum_{r=1}^{n-2} \sum_{k=r+2}^{n} \beta_{rk}(r, r+1, k)$ $+ \lambda I$, where the α's, β's and λ are real, is an orthogonal matrix, then $\Sigma\alpha_{rs} + \Sigma\beta_{rk} + \lambda = \pm 1$.

Proof: The matrix θ is of the form:

$$\theta = \begin{bmatrix} \sum\limits_{r\neq 1} b_r+\Sigma_i+\lambda & \alpha_{12}+b_1 & \cdots & \alpha_{1i} & \cdots\alpha_{1n} \\ \alpha_{12} & \sum\limits_{r\neq 1,2} b_r+\lambda+\Sigma_2 & \cdots & \alpha_{2i}+\beta_{1i} & \cdots\alpha_{2n}+\beta_{1n} \\ & & & & \\ \alpha_{1i}+\beta_{1i} & \alpha_{2i}+\beta_{2i} & \cdots & \sum\limits_{r\neq i,i-1} b_r+\Sigma_i-\sum\limits_{r=1}^{i-2}\beta_{ri}+\lambda & \cdots\alpha_{in}+\beta_{i-1n} \\ & & & & \\ \alpha_{1n}+\beta_{1n} & \alpha_{2n}+\beta_{2n} & \cdots & \alpha_{in}+\beta_{in} & \cdots\Sigma_n+\lambda+\Sigma b_r-\sum\limits_{r=1}^{n-2}\beta_{rn} \end{bmatrix}$$

In addition to the notation in theorem (3.3), we let d_i denote the set of β's occurring in the expression for the element in the $\{i, i\}$ position of θ, let $\overline{d}_i$ denote the set of β's occurring in the non-diagonal elements of the i-th column of θ, and let $\overset{m}{\Sigma}\beta_{rs}\beta_{tu}$ denote the sum of those products $\beta_{rs}\beta_{tu}$ for which $\{r, r+1, s\}$ and $\{t, t+1, u\}$ have m $(m<3)$ integers in common; $\overset{m}{\Sigma}\alpha_{rs}\beta_{tu}$ is defined in the same way. As usual, the sum of β's and the sum of the products of β's, taken two at a time, in any set S are denoted by $\underset{s}{\Sigma}\beta_{rs}$ and $\underset{s}{\Sigma}\beta_{rs}\beta_{tu}$, respectively.

Now by using the fact that each row of the matrix is of norm 1, we obtain n equations

3.15 $$\lambda^2 + \Sigma\alpha_{rs}^2 + \Sigma\beta_{pq}^2 + 2k_i = 1,\ i = 1, \ldots, n \quad \text{, where}$$

3.16 $k_i = \lambda(\Sigma_i + \sum_{d_i} \beta_{pq}) + \sum_{\sigma_i} \alpha_{rs}\alpha_{tu} + (\sum_{d_i} \beta_{pq}\beta_{rs} + \sum_{\{i,i+1\}} \beta_{rs}\beta_{tu}) +$

$$\{(\Sigma_i)(\sum_{d_i} \beta_{pq}) + \sum_{r=1}^{i-2} \alpha_{ri}\beta_{ri} + \alpha_{ii+1}(b_i + \beta_{i-1i+1})\}$$

By (3.15), it follows that $k_1 = \ldots = k_n = k$, say,

3.17 and $$nk = \sum_{i=1}^{n} k_i$$

We now proceed to simplify the RHS of (3.17).

By theorem (3.7) we can replace $\sum_{i=1}^{n} \lambda\Sigma_i$ by $(n-2)\lambda\Sigma\alpha_{rs}$; by theorem (3.3), we can replace $\sum_{i=1}^{n} \sum_{\sigma_i} \alpha_{rs}\alpha_{tu}$ by $(n-3)\ \Sigma\alpha_{rs}\alpha_{tu} - R$. We shall now show that $\sum_{i=1}^{n} (\sum_{d_i} \beta_{pq}\beta_{rs} + \sum_{\{i,i+1\}} \beta_{pq}\beta_{rs})$ can be replaced by $(n-3)\overset{2}{\Sigma}\beta_{rs}\beta_{tu} + (n-5)\overset{1}{\Sigma}\beta_{rs}\beta_{tu} + (n-6)\overset{o}{\Sigma}\beta_{rs}\beta_{tu}$. Let $\{r, r+1, s\}$ and $\{t, t+1, u\}$ have two integers in common. This implies that β_{rs} and β_{tu} are together, as a sum, in $(n-4)$ diagonal and one non-diagonal positions; the non-diagonal position corresponds to the pair of integers common to $\{r, r+1, s\}$ and $\{t, t+1, u\}$. Thus the term $\beta_{rs}\beta_{tu}$ will occur in the k_i's for $(n-3)$ times. Similarly, the term $\beta_{rs}\beta_{tu}$ is repeated $(n-5)$ or $(n-6)$ times in k_i's when $\{r, r+1, s\}$, $\{t, t+1, u\}$ have one or no integer in common, respectively.

Finally, we show that $\sum_{i=1}^{n} \{(\Sigma_i)(\sum_{d_i} \beta_{pq}) + \sum_{k=1}^{i-2} \alpha_{ri}\beta_{ri} + \alpha_{ii+1}(b_i + \beta_{i-1i+1})\}$ can be replaced by $(n-2)\overset{2}{\Sigma}\alpha_{rs}\beta_{tu} + (n-4)\overset{1}{\Sigma}\alpha_{rs}\beta_{tu} +$

$(n-5)\overset{o}{\Sigma}\alpha_{rs}\beta_{tu}$. Let $\{r, s\}$ and $\{t, t+1, u\}$ have two integers in common. This implies that α_{rs} and β_{tu} are together, as a sum, in $(n-3)$ diagonal and one non-diagonal positions; the non-diagonal position corresponds to the pair of integers common to $\{r, s\}$, $\{t, t+1, u\}$. Therefore, in the k_i's , the term $\alpha_{rs}\beta_{tu}$ occurs $(n-2)$ times. The arguments for the cases when $\{r, s\}$, $\{t, t+1, u\}$ have one or no integer in common, are similar to the preceding one.

Thus, the equation (3.17) takes the form

$$3.18 \quad nk = (n-2)\lambda\Sigma\alpha_{rs} + \lambda\sum_{i=1}^{n}\sum_{d_i}\beta_{pq} + (n-3)\Sigma\alpha_{rs}\alpha_{tu} - R + (n-3)\overset{2}{\Sigma}\beta_{rs}\beta_{tu} + (n-5)\overset{1}{\Sigma}\beta_{rs}\beta_{tu} + (n-6)\overset{o}{\Sigma}\beta_{rs}\beta_{tu} + (n-2)\overset{2}{\Sigma}\alpha_{rs}\beta_{tu} + (n-4)\overset{1}{\Sigma}\alpha_{rs}\beta_{tu} + (n-5)\overset{o}{\Sigma}\alpha_{rs}\beta_{tu} .$$

Now, by using the fact that the inner product of any two rows of θ is zero, we obtain $\binom{n}{2}$ equations. The sum of these equations is put in a simplified form by using the fact that the sum of α's or β's in each row (column) is the same and by using the information given by the equivalence of equations (3.17) and (3.18). This simplified form is

$$3.19 \quad 0 = 2\lambda\Sigma\alpha_{rs} + \lambda\sum_{i=1}^{n}\sum_{d_i}\beta_{pq} + 3\Sigma\alpha_{rs}\alpha_{tu} + R + 3\overset{2}{\Sigma}\beta_{rs}\beta_{tu} + 5\overset{1}{\Sigma}\beta_{rs}\beta_{tu} + 6\overset{o}{\Sigma}\beta_{rs}\beta_{tu} + 2\overset{2}{\Sigma}\alpha_{rs}\beta_{tu} + 4\overset{1}{\Sigma}\alpha_{rs}\beta_{tu} + 5\overset{o}{\Sigma}\alpha_{rs}\beta_{tu} .$$

Adding (3.18) and (3.19), we get

$$nk = n\{\lambda(\Sigma\alpha_{rs} + \Sigma\beta_{pq}) + \Sigma\alpha_{rs}\alpha_{tu} + \Sigma\beta_{rs}\beta_{tu} + \Sigma\alpha_{rs}\Sigma\beta_{tu}\}$$

Using this in (3.15), we get

$$(\Sigma\alpha_{rs} + \Sigma\beta_{rs} + \lambda)^2 = 1$$

Hence $\quad \Sigma\alpha_{rs} + \Sigma\beta_{rs} + \lambda = \pm 1$. Q.E.D.

3.20 Corollary: If $P = \sum_{i=1}^{r} \alpha_i P_i$, where $P \in S_n$ and $P_i \in$ TUCUI (theorem (1.8)), then $\sum_{i=1}^{r} \alpha_i = 1$.

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