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SOME PROBLEMS ON MOUNTAIN CLIMBING

by

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ABSTRACT

Let f and g be two continuous, real-valued functions defined on [0,1] with $f(0) = g(0)$ and $f(1) = g(1)$. The main result of this thesis is to characterize the property that (0,0) and (1,1) are in the same connected component of $G(f,g) = \{(x,y) | f(x)=g(y)\}$.

In Chapter I, we study conditions implying that (0,0) and (1,1) are in the same connected component of $G(f,g)$, where f and g are not necessarily real-valued functions. We obtain theorems to characterize [0,1].

In Chapter II, we give a simple proof of a theorem by Sikorski and Zarankiewicz.

In Chapter III, we obtain our main result.

In Chapter IV, we study pathwise connectedness in $G(f,g)$ and give some applications.

In Chapter V, we study the question of sliding a chord of given length along a path. An example is given to show that this is not always possible.

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INTRODUCTION

Let f and g be two real continuous functions defined on $[0,1]$ with $f(0) = g(0)$ and $f(1) = g(1)$. The question of whether there exist continuous functions h and j from $[0,1]$ to $[0,1]$ with $h(0) = j(0)$ and $h(1) = j(1)$ such that $fh(x) = gj(x)$ for all $x \varepsilon [0,1]$ has been studied by Sikorski and Zarankiewicz [1], Homma [7], Huneke [6] and Whittaker [2].

If we consider the graphs of f and g to be mountains, then the existence of h and j tells us that climbers can always climb the mountains maintaining a common elevation (h and j denote the horizontal progress of the climbers).

If we consider the set $G(f,g)$ defined by $\{(x,y) | f(x) = g(y)\}$, then the existence of h and j is equivalent to the existence of a path in $G(f,g)$ joining $(0,0)$ and $(1,1)$.

The main purpose of this thesis is to study the connectedness of $G(f,g)$ between $(0,0)$ and $(1,1)$. This is important when we want to know if the functions h and j exist.

The starting point will be a theorem of Sikorski and Zarankiewicz which states that if f and g are continuous functions from $[0,1]$ to $[0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

In Chapter I, we show that the above nice result is only true for functions from $[0,1] \to [0,1]$. It turns out we can characterize $[0,1]$ as well as the points 0 and 1 in terms of connectedness properties for $G(f,g)$. In Chapter II, we get a theorem which generalizes the above theorem proved by Sikorski and Zarankiewicz. Actually the result we get is known, but the original proof requires a lot of advanced topological means. The proof we have is a very simple direct proof. In Chapter III, we study the connectedness in $G(f,g)$ between $(0,0)$ and $(1,1)$ in more detail. We can give a necessary and sufficient condition that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$ in the most general case. In Chapter IV, we study some properties of pathwise connectedness in $G(f,g)$ and give some applications. In Chapter V, we discuss the question studied by Fenn [3] of sliding a chord of fixed length along a path from our point of view. An interesting counter-example is included. After I had the example, Professor R. O. Davies showed me a better example in terms of a simple curve.

CHAPTER I

CHARACTERIZATIONS OF [0,1]

Let a and b be two points of a topological space Y. Throughout this thesis, we shall denote by $F(a,b;Y)$ the class of all continuous functions from the closed unit interval I into Y with $f(0) = a$ and $f(1) = b$. For simplicity, we shall write $F(a,b)$ and F instead of $F(a,b;R)$, (where R is the set of all real numbers), and $F(0,1;I)$ respectively. If $f_1, f_2, \ldots, f_n$ belong to $F(a,b;Y)$, then $G(f_1,f_2,\ldots,f_n)$ is the set:

$$\{(x_1,x_2,\ldots,x_n) \varepsilon I^n \mid f_i(x_i) = f_j(x_j) \text{ for all } i,j\}.$$

Now we state two theorems which R. Sikorski and K. Zarankiewicz proved in [1].

<u>Theorem 1.1</u>:

Let f, g belong to F. Then the points $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

Theorem 1.2:

Let $f_1, f_2, \ldots, f_n$ belong to F with each f_i consisting of a finite number of monotone pieces. Then there exist $\psi_1, \psi_2, \ldots, \psi_n \varepsilon F$ such that

$$f_1\psi_1(x) = f_2\psi_2(x) = \ldots = f_n\psi_n(x)$$

for all x in [0,1].

In this chapter, we would like to consider Theorem 1.1 in the case where we do not assume f and g are real functions. Finally, we are able to give theorems to characterize [0,1].

Remark 1.3:

A space Y is called pathwise connected if for each pair of distinct points a, b in Y, there is a continuous function f from [0,1] to Y such that $f(0) = a$ and $f(1) = b$. If we can always choose f to be a homeomorphism, then we call Y arcwise connected.

Lemma 1.4:

Let a, b be two distinct points of a pathwise connected space Y. If for every f, g in F(a,b;Y), (0,0) and (1,1) are in the same connected component of $G(f,g)$, then every h in $F(a,b;Y)$ is surjective.

Proof:

If $h(I)$ is not equal to Y, then there exists a point $c \in Y - h(I)$. By pathwise connectedness, there exists $j \in F(a,b;Y)$ with $j(1/2) = c$. It is clear that $h(x)$ is not equal to $j(1/2)$ for any x in $[0,1]$. Therefore the set

$$\{(x,y) \in I^2 \mid x = 1/2\}$$

and

$$G(j,h)$$

are disjoint. Hence (0,0) and (1,1) are not in the same connected component of $G(j,h)$. This contradicts our assumption that (0,0) and (1,1) are in the same connected component of $G(j,h)$. Hence every $h \in F(a,b;Y)$ is surjective.

Lemma 1.5:

Let Y be a Hausdorff space and a, b be two distinct points in the same path-connected component of Y. Suppose that every f in $F(a,b;Y)$ is surjective. Then Y is homeomorphic to the closed unit interval I.

Proof:

Since a, b are in the same path-connected component of Y, there exist $g \in F(a,b;Y)$. Clearly, $g(I) = Y$. By the Hahn-Mazurkiewiez Theorem [5, p. 129], Y is a compact, locally connected, connected metric space, i.e., Y is a Peano space, and so Y is arcwise connected. Hence there exists a homeomorphism ¶ from I into Y with ¶$(0) = a$ and ¶$(1) = b$. Since ¶ $\in F(a,b;Y)$, so ¶$(I) = Y$. Hence Y is homeomorphic to the closed unit interval I.

Using the same argument as above, we immediately get a simple proof of the following known result.

Theorem 1.6:

If Y is a Hausdorff space, then Y is arcwise connected if and only if Y is pathwise connected.

Example 1.7:

The following example shows pathwise connectedness does not imply arcwise connectedness in general. Let Y consist of the two points $\{0,1\}$ with the topology

$T = \{\phi, \{0\}, Y\}$. We have only one pair of distinct points, i.e., 0, and 1 to consider. Define f from $[0,1]$ to Y with $f([0,1/2)) = 0$ and $f([1/2,1]) = 1$. Then f is continuous. Hence Y is pathwise connected space. Since Y has only two points, Y is not arcwise connected.

Theorem 1.8:

Let Y be a pathwise connected, Hausdorff space. Given two distinct points a, b of Y, the following conditions are equivalent:

(1) If f, g are in $F(a,b;Y)$, then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

(2) Y is homeomorphic to $[0,1]$ with $a = 0$ and $b = 1$ (or $a = 1$, $b = 0$).

Proof:

(2) implies (1). This is just Theorem 1.1.

Conversely if we assume (1), then by Lemma 1.4, every $h \in F(a,b;Y)$ is surjective. By Lemma 1.5 Y is homeomorphic to $[0,1]$. Now we claim $a = 0$ and $b = 1$ (or $a = 1$, $b = 0$). If this is not true, then a continuous function f can be constructed such that f in $F(a,b;Y)$ and f is not surjective. This is a contradiction, since we have proved that all such f are surjective. Hence $a = 0$ and $b = 1$ (or $a = 1$, $b = 0$).

Lemma 1.9:

Let f, g be two homeomorphisms from I to Y with $f(0) = g(0) = a$, $f(1) = g(1) = b$. If $f(I) \neq g(I)$, then $f(I) - g(I) \neq \phi$ and $g(I) - f(I) \neq \phi$.

Proof:

Assume $f(I) - g(I) = \phi$, i.e., $f(I)$ is a proper subset of $g(I)$. Then there exists a point t in $(0,1)$ such that $g(t) \notin f(I)$. It is clear $g(t)$ is a cut point of $g(I)$, i.e., $g(I) - g(t) = U \cup V$, where U and V are separated. Note we have $a \varepsilon U$, $b \varepsilon V$. Since $f(I)$ is contained in $g(I) - g(t)$ and is a connected set, so $f(I)$ is contained in one of U, V and not the other. But $f(1) = g(1) = b \varepsilon V$ and $F(0) = g(0) = a \varepsilon U$, which is impossible. Hence $f(I) - g(I) \neq \phi$. Similarly, $g(I) - f(I) \neq \phi$

Lemma 1.10:

Let Y be an arcwise connected space and p be a given point of Y. Suppose that for every $f \varepsilon F(p,p;Y)$, the points $(0,1)$ and $(1,0)$ are in the same connected component of $G(f,f) = \{(x,y) \mid f(x) = f(y)\}$. Then

(a) Y has a unique arc (Arc is a homeomorphic image of the closed unit interval I.) between any point x and p which we denote by $\overline{px}$.

(b) For any x_1, x_2 in Y, we have $\overline{px}_1 \subset \overline{px}_2$ or $\overline{px}_2 \subset \overline{px}_1$.

<u>Proof</u>:

(a) If we have two homeomorphisms f, g from I to Y with $f(0) = g(0) = p$, $f(1) = g(1) = x$ and $f(I) \neq g(I)$, then by Lemma 1.9, there exist t_1, t_2 in $[0,1]$ with

$$f^{-1}g(t_1) = \phi, \quad g^{-1}f(t_2) = \phi.$$

Let

$$j(t) = \begin{cases} f(2t) & \text{for } t \text{ in } [0,1/2] \\ g(2-2t) & \text{for } t \text{ in } [1/2,1] \end{cases}$$

Then $j \in F(p,p;Y)$. By the choice of t_2, $j(t) \neq j(t_2/2)$ for any t in $[1/2,1]$.

Furthermore, j is one to one on $[0,1/2]$. It follows that

$$j(t) \neq j(t_2/2) \quad \text{if} \quad t \neq t_2/2$$

Similarly,

$$j(t) \neq j((2-t_1)/2) \quad \text{if} \quad t \neq (2-t_1)/2$$

Clearly, $G(j,j)$ is disjoint from $S_1 \cup S_2$, where

$$S_1 = \{(x,y) \mid x = t_2/2, \quad t_2/2 < y \leq 1\}$$

and

$$S_2 = \{(x,y) \mid y = (2-t_1)/2, \quad 0 \leq x < (2-t_1)/2\}$$

Hence $(0,1)$ and $(1,0)$ are not in the same connected component of $G(j,j)$ and this is a contradiction.
Therefore for any two homeomorphisms f, g with $f(0) = g(0) = p$ and $f(1) = g(1) = x$, we have that $f(I) = g(I)$.

(b) Let f, g be two homeomorphisms from I to Y with $f(0) = g(0) = p$ and $f(1) = x_1$, $g(1) = x_2$.
Define

$$t_1 = \sup\{t \mid f(t) \in g(I)\}$$

Let $f(t_1) = x_3 = g(t_2)$. Then $f([0,t_1])$ is an arc from p to x_3 and $g([0,t_2])$ is an arc from p to x_3.
By part (a)

$$f([0,t_1]) = g([0,t_2]) = \overline{px}_3.$$

We claim that $x_3 = x_1$ or $x_3 = x_2$. If this is not true, then $t_1 \neq 1$ and $t_2 \neq 1$. Let

$$h(x) = \begin{cases} g(4x) & \text{for } x \text{ in } [0,1/4]. \\ g(4(t_2 - 1)x + 2 - t_2)) & \text{for } x \text{ in } [1/4,1/2]. \\ f(4(1 - t_1)x + 3t_1 - 2) & \text{for } x \text{ in } [1/2,3/4]. \\ f(4 - 4x) & \text{for } x \text{ in } [3/4,1]. \end{cases}$$

Clearly, h is in $F(p,p;Y)$. By the choice of t_1,

$$f(4(1 - t_1)x + 3t_1 - 2) \notin g(I)$$

for any x in $(1/2,3/4]$.
Therefore

$$h(y) \neq h(x)$$

for any $(x,y) \in S_1 = [0,1/2] \times (1/2,3/4]$.
Similarly,

$$h(x) \neq h(y)$$

for any $(x,y) \in S_2 = [1/4,1/2) \times [1/2,1]$

Hence $S_1 \cup S_2$ and $G(h,h)$ are disjoint. That means $(0,1)$ and $(1,0)$ are not in the same connected component of $G(h,h)$ and this contradicts our assumption. Therefore

$x_3 = x_1$ or $x_3 = x_2$, i.e., $\overline{px}_1 \subset \overline{px}_2$ or $\overline{px}_2 \subset \overline{px}_1$.

Theorem 1.11:

Let p be a given point of a Peano space Y. Suppose that for every $f \varepsilon F(p,p;Y)$, the points $(0,1)$ and $(1,0)$ are in the same connected component of $G(f,f)$. Then Y is homeomorphic to $[0,1]$.

Proof:

In order to prove that Y is homeomorphic to $[0,1]$, we would like to introduce an ordering on Y first. Define

$$x < y \text{ if } \overline{px} \subset \overline{py} \text{ for } x, y \text{ of } Y.$$

By part (b) of Lemma 1.10, $(Y,<)$ becomes a linearly ordered set.

Remark: $<$ is also a dense ordering in the following sense: Given any a and b of Y with $a < b$, there exist an element c such that $a < c < b$. Since $a < b$ means $\overline{pa} \subset \overline{pb}$, so there is a homeomorphism f with $f([0,1]) = \overline{pb}$, $f(0) = p$, $f(1) = b$. $f([0,t]) = \overline{pa}$ for some t in $(0,1)$. Take any $s \varepsilon (t,1)$ and let $f(s) = c$. It is clear that $\overline{pa} \subset \overline{pc} \subset \overline{pb}$, i.e., there is an element c with $a < c < b$. This proves our remark.

To prove the theorem, define $A(x) = \{y \varepsilon Y \mid x \leq y\}$, we want to prove that $A(x)$ is a closed set for any x in Y. Take any $z_1 \varepsilon Y - A(x)$. Note that $Y - A(x) = \overline{px} - \{x\}$. Hence $z_1 \varepsilon \overline{px}$ and there is an open set U of $\overline{px}$ such that $z_1 \varepsilon U$, $x \notin U$, $U = \overline{px} \cap V$, where V is an open set of Y. It is clear that x is not in V. Since Y is a Peano space, in particular Y is locally connected, so there is an open connected set K containing z_1 and contained in V. We know that a connected open subset of a Peano space is arcwise connected [5, p. 118]. Hence K is arcwise connected. If there is an element $z_2 \in K \cap A(x)$, then z_1, z_2 can be joined by an arc T in K. It is clear that $T \cup \overline{pz_1}$ is a pathwise connected, Hausdorff space. Therefore $T \cup \overline{pz_1}$ is arcwise connected. Hence $\overline{pz_2} \subset T \cup \overline{pz_1}$. Recall $x \notin T \cup \overline{pz_1}$. Since $z_2 \in A(x)$, we have $x < z_2$, i.e., $\overline{px} \subset \overline{pz_2}$. This is impossible because $x \notin \overline{pz_2}$. Hence $K \cap A(x) = \phi$. This proves that the complement of A(x) is open and so A(x) is closed.

Now the class of all A(x) has the finite intersection property. The compactness of Y implies that

$$\cap \{A(x) \mid x \in Y\} \neq \phi.$$

Now we claim that $\cap\{A(x) \mid x \in Y\}$ has exactly one element. If this is not true, say we have x_1, x_2 of $\cap\{A(x) \mid x \in Y\}$, then we have either $x_1 < x_2$ or $x_2 < x_1$. Let us say $x_1 < x_2$. By definition of $A(x_2)$, we know $x_1 \notin A(x_2)$ and so $x_1 \notin \cap\{A(x) \mid x \in Y\}$. This contradicts our choice of x_1. Hence $\cap\{A(x) \mid x \in Y\}$ has only one point q. Now we want to show $\overline{pq} = Y$. Take any $r \in Y$. If $q < r$, then $q \notin A(r)$ and that implies

$$q \notin \cap\{A(x) \mid x \in Y\}.$$

This contradicts our choice of q. Therefore $r \leq q$, i.e., $\overline{pr} \subset \overline{pq}$, and $r \in \overline{pq}$. Hence $\overline{pq} = Y$, so Y is homeomorphic to $[0,1]$.

Lemma 1.12:

Suppose f is a continuous function from $[0,1]$ to $[0,1]$ with $f(0) = f(1) = 0$. Then the points $(0,1)$ and $(1,0)$ are in the same connected component of $G(f,f) = \{(x,y) \mid f(x) = f(y)\}$.

Proof:

Let c be a point where f attains its maximum. Define:

$$h_1(x) = f(cx)/f(c) \qquad \text{for } x \text{ in } [0,1].$$

$$h_2(x) = f((c-1)x + 1)/f(c) \qquad \text{for } x \text{ in } [0,1].$$

Then h_1, h_2 belong to F. By Theorem 1.1, the points $(0,0)$, $(1,1)$ are in the same connected component of $G(h_1,h_2)$. Let

$$G_1 = \{(x,y) \in G(f,f) \mid 0 \leq x \leq c, \quad c \leq y \leq 1\}$$

The mapping:

$$(x,y) \longrightarrow (cx, \quad (c-1)y + 1)$$

defines a homeomorphism from $G(h_1,h_2)$ onto G_1. From the fact that the points $(0,0)$ and $(1,1)$ are in the same connected component of $G(h_1,h_2)$, we infer that the points $(0,1)$ and (c,c) are in the same connected component of $G_1 \subset G(f,f)$. Since $G(f,f)$ is symmetric with respect to to the diagonal set $\{(x,x) \mid x \in [0,1]\}$, we get that $(1,0)$ and (c,c) are in the same connected component of $G(f,f)$. Therefore $(1,0)$ and $(0,1)$ are in the same connected component of $G(f,f)$.

Theorem 1.13:

Let p be a given point of a Peano space Y. Then the following conditions are equivalent:

(1) $Y = [0,1]$ and $p = 0$ (or $p = 1$);

(2) For every f in $F(p,p;Y)$, the points $(0,1)$ and $(1,0)$ are in the same connected component of $G(f,f)$.

Proof:

(1) implies (2). This is just Lemma 1.12.

Let (2) hold. By Theorem 1.11, Y is homeomorphic to [0,1]. We claim that p is equal to either 1 or 0. If $p \neq 0$ and $p \neq 1$, then take $0 < x_2 < p < x_1 < 1$, so that $\overline{px_2}$ is not a subset of $\overline{px_1}$, and $\overline{px_1}$ is not a subset of $\overline{px_2}$. This contradicts part (b) of Lemma 1.10. Hence $p = 0$ or $p = 1$. This completes the proof of Theorem 1.13.

CHAPTER II

CONNECTEDNESS

In this chapter we shall prove the following theorem which is a generalization of Theorem 1.1.

Theorem 2.1:

If $f_1, f_2, \ldots, f_n$ belong to F, then the points $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$ are in the same connected component of $G(f_1,f_2,\ldots,f_n)$.

The proof of Theorem 2.1 is based on the following lemmas.

Lemma 2.2:

Suppose $f \in F$ and $\varepsilon > 0$. Then there exists $h \in F$, where h consists of a finite number of monotone pieces, such that $|f(x) - h(x)| < \varepsilon$ for any x in $[0,1]$.

Proof:

Because f is a continuous function defined on a compact set $[0,1]$, it is uniformly continuous on $[0,1]$.

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $|x - y| < \delta$. Choose $0 = a_0 < a_1 < \dots < a_n = 1$ such that $|a_i - a_{i-1}| < \delta$ for each i. Define $h(a_i) = f(a_i)$ and let h be linear in $[a_{i-1}, a_i]$. Then $h \in F$ and h consists of a finite number of monotone pieces. For any x in [0,1], there exists a j such that $x \in [a_{j-1}, a_j]$. Therefore

$$|f(x) - h(x)| \leq |f(x) - f(a_j)| + |f(a_j) - h(x)|$$

$$< \varepsilon/2 + |h(a_j) - h(x)|$$

$$< \varepsilon/2 + |h(a_j) - h(a_{j-1})|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This completes the proof of Lemma 2.2.

Lemma 2.3:

Suppose f, g belong to F and $\varepsilon > 0$. Then there exist ψ, ξ of F such that $|f\psi(x) - g\xi(x)| < \varepsilon$ for every x in [0,1].

Proof:

By Lemma 2.2, there exist h, j of F with each consisting of a finite number of monotone pieces, such that

$$|f(x) - h(x)| < \varepsilon/2$$

and

$$|g(x) - j(x)| < \varepsilon/2$$

for any x in $[0,1]$.

By Theorem 1.2, there exist ψ, ξ of F such that

$$h\psi(x) = j\xi(x) \quad \text{for every } x \text{ in } [0,1].$$

Because

$$|f\psi(x) - h\psi(x)| < \varepsilon/2 \quad \text{and} \quad |g\xi(x) - j\xi(x)| < \varepsilon/2,$$

therefore

$$\begin{aligned} |f\psi(x) - g\xi(x)| &\leq |f\psi(x) - h\psi(x)| + |h\psi(x) - g\xi(x)| \\ &< \varepsilon/2 + |j\xi(x) - g\xi(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

<u>Lemma 2.4</u>:

Let $f_1, f_2, \ldots, f_n \in F$ and $\varepsilon > 0$. Then there exist $\psi_1, \psi_2, \ldots, \psi_n \in F$ such that

$$|f_i\psi_i(x) - f_j\psi_j(x)| < \varepsilon$$

for any x in $[0,1]$ and $i, j = 1, 2, \ldots, n$.

Proof:

We shall prove it by induction.

It is true for $n = 2$ (Lemma 2.3).

Assume it is true for $n = k$,

i.e., there exist $\xi_1, \xi_2, \ldots, \xi_k$ of F such that

$$|f_j\xi_j(x) - f_i\xi_i(x)| < \varepsilon/2$$

for any x in [0,1] and i, j = 1,2,..., k.

Since $f_1\xi_1$, $f_{k+1} \varepsilon F$, then by Lemma 2.3, there exist h_1, $h_2 \varepsilon F$ such that

$$|f_1\xi_1h_1(x) - f_{k+1}h_2(x)| < \varepsilon/2.$$

Let $\xi_ih_1 = \psi_i$ for $i = 1, 2, \ldots, k.$ and $h_2 = \psi_{k+1}$

Then

(1) $$|f_j\psi_j(x) - f_i\psi_i(x)| < \varepsilon/2$$

i, j = 1, 2,..., k.

In particular,

$$|f_1\psi_1(x) - f_j\psi_j(x)| < \varepsilon/2$$

j = 1, 2,..., k.

Since

$$|f_1\psi_1(x) - f_{k+1}\psi_{k+1}(x)| < \varepsilon/2,$$

we have

(2) $$|f_j\psi_j(x) - f_{k+1}\psi_{k+1}(x)| < \varepsilon$$

$j = 1, 2, \ldots, k$.

From (1) and (2),

$$|f_i\psi_i(x) - f_j\psi_j(x)| < \varepsilon$$

for any $x \in [0,1]$ and $i, j = 1, 2, \ldots, k+1$.

Hence the proof of Lemma 2.4 is completed.

Lemma 2.5:

Let X be a compact Hausdorff space and G be a compact subset of X. If a and b are not in the same component of G, then there exist disjoint open sets U, V of X such that $G \subset V \cup U$ and $a \in U$, $b \in V$.

Proof:

Since G is a compact Hausdorff space, the components and quasi-components are identical. That a and b are not in the same connected component of G implies that a and b are not in the same quasi-component of G. By definition of quasi-component, there exist disjoint closed sets H, K of G with $G = H \cup K$, $a \in H$ and $b \in K$. Since G is closed in X, so H and K are closed in X. Every compact Hausdorff space is normal. Therefore there exist disjoint open sets U and V of X such that $H \subset U$ and $K \subset V$. It is clear that $G \subset U \cup V$ and $a \varepsilon U$, $b \varepsilon V$. This completes the proof of Lemma 2.5.

Now we have established enough lemmas to prove Theorem 2.1.

Proof of Theorem 2.1:

First note that $G(f_1, f_2, \ldots, f_n)$ is a compact subset of the compact Hausdorff space I^n.

Assume $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ are not in the same connected component of $G = G(f_1, f_2, \ldots, f_n)$. Then by Lemma 2.5, there exist disjoint open sets U, V, of I^n with $G \subset U \cup V$ and $(0, 0, \ldots, 0) \varepsilon U$, $(1, 1, \ldots, 1) \varepsilon V$. For every positive integer k, there exist $\psi_{1k}, \psi_{2k}, \ldots, \psi_{nk} \varepsilon F$ such that

$$|f_i\psi_{ik}(x) - f_j\psi_{jk}(x)| < \frac{1}{k}$$

for $i,j = 1, 2, \ldots, n$, and the mapping

$$t \rightarrow (\psi_{1k}(t), \psi_{2k}(t), \ldots, \psi_{nk}(t))$$

defines a path joining $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Let

$$A_k = \{(\psi_{1k}(t), \psi_{2k}(t), \ldots, \psi_{nk}(t)) \mid t \in [0,1]\}.$$

Since A_k is connected and contains $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, by our choice of U, V, we have $A_k - (U \cup V) \neq \phi$. Let $(x_{1k}, x_{2k}, \ldots, x_{nk}) \in A_k - (U \cup V)$, where $x_{ik} = \psi_{ik}(t)$ for some t. The sequence $\{y_k\}$ where $y_k = (x_{1k}, x_{2k}, \ldots, x_{nk})$ contains a convergent subsequence. So without loss of generality, we may assume $\{y_k\}$ converges to $y = (x_1, x_2, \ldots, x_n)$. Given $\varepsilon > 0$, we can choose k large enough so that

$$\begin{aligned}
&|f_i(x_i) - f_j(x_j)| \\
< \; &|f_i(x_i) - f_i(x_{ik})| + |f_i(x_{ik}) - f_j(x_{jk})| + |f_j(x_{jk}) - f_j(x_j)| \\
< \; &\varepsilon/3 + |f_i\psi_{ik}(t) - f_j\psi_{jk}(t)| + \varepsilon/3 \\
< \; &\varepsilon/3 + 1/k + \varepsilon/3 \\
< \; &\varepsilon.
\end{aligned}$$

This means that

$$|f_i(x_i) - f_j(x_j)| = 0$$

for every $i,j = 1, 2, \ldots, n$. Hence

$$(1) \qquad y = (x_1, x_2, \ldots, x_n) \in G \subset U \cup V.$$

On the other hand, $y_k \in A_k - (U \cup V) \subset I^n - (U \cup V)$. Since U and V are open, $I^n - (U \cup V)$ is closed. Hence

$$(2) \qquad y \in I^n - (U \cup V).$$

From (1) (2), we get a contradiction. This means $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ are in the same connected component of $G(f_1, f_2, \ldots, f_n)$. This completes the proof of Theorem 2.1.

Corollary 2.6:

Suppose f is a real-valued continuous function defined on $[0,1]$, $f(x) \geq 0$ for every x in $[0,1]$ and $f(a_1) = f(a_2) = \ldots = f(a_n) = 0$ for $0 = a_1 < a_2 < \ldots < a_n = 1$. Then the n^n points of the form $(b_1, b_2, \ldots, b_n)$, where every $b_i = a_j$ for some j (b_s may equal b_k if $s \neq k$), are in the same connected component of $G(f, f, \ldots, f)$.

Proof:

If $f = 0$, then the result is true. Let $f \neq 0$. Assume that $f(x)$ attains its maximum at point c. Note $c \neq a_1, a_2, \ldots, a_n$. Let $f(c) = M$. For every positive integer k, define $g_k(x) = f((c - b_k)x + b_k)/M$, where

$(b_1, b_2, \ldots, b_n)$ is a point of the required form. The g_k's satisfy the conditions of Theorem 2.1. Hence $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ are in the same connected component of $G(g_1, g_2, \ldots, g_n)$. Let G' be the set:

$$\{(x_1,x_2,\ldots,x_n) \in G(f,f,\ldots,f) \mid x_i \text{ is between } b_i \text{ and } c\}.$$

The mapping $(x_1, x_2, \ldots, x_n) \to$

$$\big((c - b_1)x_1 + b_1, (c - b_2)x_2 + b_2, \ldots, (c - b_n)x_n + b_n\big)$$

defines a homeomorphism from $G(g_1,g_2,\ldots,g_n)$ onto G'. Because $(0, 0, \ldots, 0) \to (b_1, b_2, \ldots, b_n)$ and $(1, 1, \ldots, 1) \to (c, c, \ldots, c)$, we have $(b_1, b_2, \ldots, b_n)$ and $(c, c, \ldots, c)$ in the same connected component of $G' \subset G(f,f,\ldots,f)$. This is true for any $(b_1, b_2, \ldots, b_n)$, where every $b_i = a_j$ for some j. Hence the n^n points of this form are in the same connected component of $G(f,f,\ldots,f)$.

We can use the same method as in Corollary 2.6 and use Theorem 1.2 to prove the following:

Corollary 2.7:

Suppose f is a continuous function defined on $[0,1]$, $f(x) \geq 0$ for every x in $[0,1]$, f consists of a finite number of monotone pieces and

$$f(a_1) = f(a_2) = \cdots = f(a_n) = 0$$

for $0 = a_1 < a_2 < \ldots < a_n = 1$, Then the n^n points of the form $(b_1, b_2, \ldots, b_n)$, where every $b_i = a_j$ for some j (b_s may equal b_k if $s \neq k$), are in the same pathwise connected component of $G(f,f,\ldots,f)$.

CHAPTER III

COMPATIBILITIES OF FUNCTIONS

From Theorem 1.1, we know that if we have two continuous functions f, g, from $[0,1]$ to $[0,1]$ with $f(0) = g(0) = a = 0$ and $f(1) = g(1) = b = 1$, then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. If we do not assume $a = 0$ or $b = 1$, then the points $(0,0)$ and $(1,1)$ need not be in the same connected component of $G(f,g)$, as the following example shows. Let

$$f(x) = \begin{cases} 2x & \text{for } x \in [0,\frac{1}{2}] \\ \frac{3}{2} - x & \text{for } x \in [\frac{1}{2},1] \end{cases}$$

$$g(x) = \begin{cases} 4x & \text{for } x \in [0,\frac{1}{4}] \\ -4x + 2 & \text{for } x \in [\frac{1}{4},\frac{1}{2}] \\ x - \frac{1}{2} & \text{for } x \in [\frac{1}{2},1]. \end{cases}$$

But in this case we are still able to give a necessary and sufficient condition that $(0,0)$ and $(1,1)$ will lie in the same connected component of $G(f,g)$.

Definition 3.1:

We say that a closed interval $[p,q]$ is an α-interval of f if

$$f[p,q] = \begin{cases} [f(p),f(q)] & \text{if } f(p) < f(q) \\ [f(q),f(p)] & \text{if } f(q) < f(p). \end{cases}$$

Definition 3.2:

$f_{|[s,t]}$ is called right compatible with $g_{|[c,d]}$ if condition (A) or (B) holds.

Condition (A): there exist points $s = x_o < x_1 < \dots < x_m = t$, $c = y_o < y_1 < \dots < y_m = d$ such that

1. $f(x_i) = g(y_i)$ $i = 0, 1, \dots, m$.
2. Every closed interval $[x_i, x_{i+1}]$ is an α-interval of f.
3. Every closed interval $[y_i, y_{i+1}]$ is an α-interval of g.

Condition (B): there exist strictly monotone increasing sequences $\{x_i\}$, $\{y_i\}$ such that

1. $x_o = s$, $y_o = c$ and $x_i < t$, $y_i < d$ for all i.
2. If $\lim x_i = x_\infty$ and $\lim y_n = y_\infty$, then
   $f([x_\infty,t]) = g([y_\infty,d]) = f(t)$.
3. $f(x_i) = g(x_i)$ for all i.
4. Every closed interval $[x_i,x_{i+1}]$ is an α-interval of f.
5. Every closed interval $[y_i,y_{i+1}]$ is an α-interval of g.

Definition 3.3:

$f_{|[s,t]}$ is compatible with $g_{|[c,d]}$ if condition (A) holds.

Definition 3.4:

$f_{|[s,t]}$ is left compatible with $g_{|[c,d]}$ if condition (A') or (B') holds:

Condition (A'): there exist points $s = x_{-n} < x_{-n+1} < \cdots$ $x_o = t$, $c = y_{-n} < y_{-n+1} < \ldots < y_o = d$ such that

1. $f(x_{-i}) = g(y_{-i})$ for $i = 0, 1, \ldots, n$.
2. Every closed interval $[x_{-i},x_{-i+1}]$ is an α-interval of f.
3. Every closed interval $[y_{-i},y_{-i+1}]$ is an α-interval of g.

Condition (B'): there exist strictly monotone decreasing sequences $\{x_{-i}\}$ and $\{y_{-i}\}$ satisfying the following relations:

1. $x_o = t$, $y_o = d$ and $x_{-i} > s$, $y_{-i} > c$ for all i.
2. If $\lim x_{-i} = x_{-\infty}$ and $\lim y_{-i} = y_{-\infty}$, then $f([s,x_{-\infty}]) = g([c,y_{-\infty}]) = f(s)$.
3. $f(x_{-i}) = g(y_{-i})$ for all i.
4. Every closed interval $[x_{-i},x_{-i+1}]$ is an α-interval of f.
5. Every closed interval $[y_{-i},y_{-i+1}])$ is an α-interval of g.

Theorem 3.5:

Suppose that f, g, belong to $F(0,b;[0,\infty))$. Then the points (0,0) and (1,1) are in the same connected component of G(f,g) if and only if $f_{|[0,1]}$ is right compatible with $g_{|[0,1]}$.

Proof:

First let us suppose that $f_{|[0,1]}$ is right compatible with $g_{|[0,1]}$.

Suppose condition (A) holds. Then by Theorem 1.1,

(x_{i+1}, y_{i+1}) and (x_i, y_i) are in the same connected component of $G_i = G(f_i, g_i)$, where $f_i = f|_{[x_i, x_{i+1}]}$ and $g_i = g|_{[y_i, y_{i+1}]}$. Since $(0,0) = (x_o, y_o)$ and (x_1, y_1) are in the same connected component of G_o and (x_1, y_1), (x_2, y_2) are in the same connected component of G_1, hence $(0,0)$, (x_2, y_2) are in the same connected component of $G_o \cup G_1$. Using the same argument, we can show that $(0,0)$ and $(1,1) = (x_m, y_m)$ are in the same connected component of $\bigcup_{i=o}^{m-1} G_i$. Because $G_i \subset G(f,g)$ for each i, we have $\bigcup_{i=o}^{m-1} G_i \subset G(f,g)$. Hence $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

Suppose condition (B) holds. Then it is clear that (x_∞, y_∞) and $(1,1)$ are in the same connected component of $G(f,g)$ because $f([x_\infty,1]) = g([y_\infty,1])$. For each i, (x_i, y_i) and (x_{i+1}, y_{i+1}) are in the same connected component of $G_i = G(f_i, g_i)$, where $f_i = f|_{[x_i, x_{i+1}]}$ and $g_i = g|_{[y_i, y_{i+1}]}$ If we define $H_j = \bigcup_{i=o}^{j} G_i$, then we have that $(0,0)$ and (x_{j+1}, y_{j+1}) are in the same connected component of H_j. Let $H = \bigcup_{j=0}^{\infty} H_j$, then H has a component C containing (x_i, y_i) for each i. By definition of x_∞, y_∞, we have that (x_∞, y_∞) belongs to the closure of C relative to the usual topology of R^2. Since C is contained in $G(f,g)$ and

$G(f,g)$ is a closed set of R^2, the closure of C is a subset of $G(f,g)$. Therefore the closure of C is a connected set containing both $(0,0)$ and (x_∞,y_∞). We have proved that (x_∞,y_∞) and $(1,1)$ are in the same connected component of $G(f,g)$. Hence the points $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

Now assume that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. It is clear that $f(I) = g(I)$, for otherwise there exists q such that $f^{-1}g(q) = \phi$. Then $(0,0)$ and $(1,1)$ are not in the same connected component of $G(f,g)$. Now we want to show that condition (A) or (B) holds. Define $x_o = 0$, $y_o = 0$, and

$$x_1 = \sup \{x \in [0,1] \mid f(x) \text{ attains its absolute maximum}\}$$

$$y_1 = \sup \{y \in [0,1] \mid g(y) \text{ attains its absolute maximum}\}.$$

Now f and g are continuous functions defined on a compact set $[0,1]$, so x_1 and y_1 are well-defined. Since $f(I) = g(I)$, we have $f(x_1) = g(y_1)$. Define

$$x_2 = \sup \{x \in [x_1,1] \mid f(x) \text{ attains its minimum in } [x_1,1]\}$$

$$y_2 = \sup \{x \in [y_1,1] \mid g(y) \text{ attains its minimum in } [y_1,1]\}.$$

It is clear that $y_1 < y_2$ and $x_1 < x_2$. We claim that

$f(x_2) = g(y_2)$. Suppose $f(x_2) > g(y_2)$. Then $f(x) \neq g(y_2)$ for $x \geq x_1$ and $f(x_1) \neq g(y)$ for $y > y_1$. Therefore the set $G(f,g)$ is disjoint from $L_1 \cup L_2$, where

$$L_1 = \{(x,y) \mid x = x_1, \; y > y_1\}$$

$$L_2 = \{(x,y) \mid x \geq x_1, \; y = y_2\}.$$

This contradicts our assumption that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. Similarly, we cannot assume that $f(x_2) < g(y_2)$. Hence $f(x_2) = g(y_2)$.

Suppose that x_j and y_j have been constructed. Assume $x_j \neq 1$ and j is even, i.e., $f(x_j) < f(1) = b$. Define:

$$x_{j+1} = \sup \{x \in [x_j,1] \mid f(x) \text{ attains its maximum in } [x_j,1]\}$$

$$y_{j+1} = \sup \{y \in [y_j,1] \mid g(y) \text{ attains its maximum in } [y_j,1]\}$$

We claim that $f(x_{j+1}) = g(y_{j+1})$. Assume $f(x_{j+1}) < g(y_{j+1})$. Then $f(x) \neq g(y_{j+1})$ for x in $[x_j,1]$ and $f(x_j) \neq g(y)$ for $y > y_j$. Therefore the set $G(f,g)$ is disjoint from $L' \cup L^*$, where

$$L' = \{(x,y) \mid y = y_{j+1}, \; x \geq x_j\}$$

$$L^* = \{(x,y) \mid x = x_j, \ y > y_j\}.$$

This contradicts our assumption that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. Similarly, we cannot assume that $f(x_{j+1}) > g(y_{j+1})$. Hence $f(x_{j+1}) = g(y_{j+1})$. Suppose that $x_j \neq 1$ and j is odd, i.e., $f(x_j) > f(1) = b$. Define:

$$x_{j+1} = \sup \{x \in [x_j,1] \mid f(x) \text{ attains its minimum in } [x_j,1]\}$$

$$y_{j+1} = \sup \{y \in [y_j,1] \mid g(y) \text{ attains its minimum in } [y_j,1]\}.$$

Using the same method as when we proved that $f(x_2) = g(y_2)$, we can prove that $f(x_{j+1}) = g(y_{j+1})$. This completes the induction step to construct $\{x_i\}$, $\{y_i\}$.

If for some m, $x_m = 1$, then from $f(x_m) = g(y_m)$ and $f(x_m) = f(1) = b$, we have $g(y_m) = b$. By definition of y_m, we get $y_m = 1$. In this case, we get condition (A).

If for any i, $x_i \neq 1$, then $y_i \neq 1$ for any i. Because $\{x_i\}$ and $\{y_i\}$ are bounded monotone increasing sequences, we have $\lim x_i = x_\infty$ and $\lim y_i = y_\infty$. By our construction of $\{x_i\}$ and $\{y_i\}$, we have $f(x_i) > b$ if i is odd and $f(x_i) < b$ if i is even. Hence $f(x\ ) =$ $f(x_\infty) = g(x_\infty) = b$. In order to get condition (B), the only thing to prove is $f([x_\infty,1]) = g([y_\infty,1]) = b$. Assume that $x_\infty \neq 1$ and $f(x) \neq b$ for some x in $[x_\infty,1]$; say $f(x) < b$. Since $f(x_{2i})$ is monotone increasing to b, there is an x_{2j} such that $f(x) < f(x_{2j}) < b$. By continuity of f, there is a point c such that $x_\infty < c < x$ and $f(c) = f(x_{2j})$. The facts $x_{2j} < c$ and $f(c) = f(x_{2j})$ contradict our choice of x_{2j}. That means $f(x)$ cannot be less than b whenever x is in $[x_\infty,1]$. For the same reason, $f(x)$ cannot be greater than b whenever x is in $[x_\infty,1]$. Hence $f([x_\infty,1]) = b$. Similarly, $g([y_\infty,1]) = b$. This completes the proof of Theorem 3.5.

It is clear that the following theorems are true.

Theorem 3.6:

Let $f, g \in F(a,b)$, where $f(0) = g(0) = a$ is the maximum value of $f(x)$ and $g(x)$. Then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$ if and only if $f_{|[0,1]}$ is right compatible with $g_{|[0,1]}$.

Theorem 3.7:

Let $f, g \in F(a,b)$, where $f(1) = g(1) = b$ is the maximum (minimum) value of $f(x)$ and $g(x)$. Then $(0,0)$ and

(1,1) are in the same connected component of $G(f,g)$ if and only if $f_{|[0,1]}$ is left compatible with $g_{|[0,1]}$.

Now we want to discuss more general case, i.e., without assuming anything about a or b.

<u>Definition 3.8</u>:

Define:

$$a_1 = \inf \{x \in [0,1] |\ f(x) \text{ attains its maximum}\}$$
$$a_2 = \sup \{x \in [0,1] |\ f(x) \text{ attains its maximum}\}$$
$$a_3 = \inf \{x \in [0,1] |\ f(x) \text{ attains its minimum}\}$$
$$a_4 = \sup \{x \in [0,1] |\ f(x) \text{ attains its minimum}\}$$
$$b_1 = \inf \{x \in [0,1] |\ g(x) \text{ attains its maximum}\}$$
$$b_2 = \sup \{x \in [0,1] |\ g(x) \text{ attains its maximum}\}$$
$$b_3 = \inf \{x \in [0,1] |\ g(x) \text{ attains its minimum}\}$$
$$b_4 = \sup \{x \in [0,1] |\ g(x) \text{ attains its minimum}\}.$$

If f and g are not constant functions, then there are six possible ways to order a_1, a_2, a_3, a_4 as well as b_1, b_2, b_3, b_4:

1. $a_1 \le a_2 < a_3 \le a_4$
2. $a_1 < a_3 < a_2 < a_4$
3. $a_1 < a_3 \le a_4 < a_2$
4. $a_3 \le a_4 < a_1 \le a_2$
5. $a_3 < a_1 < a_4 < a_2$
6. $a_3 < a_1 \le a_2 < a_4$.

Lemma 3.9:

Suppose that $f, g \in F(a,b)$ and $(0,0)$, $(1,1)$ are in the same connected component of $G(f,g)$. Then the orderings of the a_i's and b_i's have the same initial and terminal subscripts.

Remark:

As before, that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$ implies that $f(I) = g(I)$. Hence $f(a_1) = f(a_2) = g(b_1) = g(b_2)$ and

$f(a_3) = f(a_4) = g(b_3) = g(b_4)$.

Proof of Lemma 3.9:

Assume for the given f, we have $a_1 \leqslant a_2 < a_3 \leqslant a_4$. We want to show that for g we have $b_1 \leqslant b_2 < b_3 \leqslant b_4$ or $b_1 < b_3 < b_2 < b_4$. Since g cannot be a constant function, we have $b_3 < b_1$ or $b_1 < b_3$. Assume $b_3 < b_1$. By definition of b_1 and the fact that $f(a_1) = g(b_1)$, we have:

$$f(a_1) \neq g(y) \qquad \text{for any } y \in [0,b_1).$$

By definition of b_3 and the fact that $g(b_3) = f(a_3)$, we have:

$$f(x) \neq g(b_3) \qquad \text{for any } x \in [0,a_3).$$

This implies that $(0,0)$ and $(1,1)$ are not in the same connected component of $G(f,g)$. Hence $b_1 < b_3$. The set $\{(x,y) \mid x = a_4,\ y \in (b_4,1]\} \cup \{(x,y) \mid x \in (a_2,1],\ y = b_2\}$ is always disjoint from $G(f,g)$. If we assume $b_4 < b_2$, then the above set separates $(0,0)$ and $(1,1)$. This is impossible. Hence $b_2 < b_4$. Therefore if we assume $a_1 \leq a_2 < a_3 \leq a_4$, we have $b_1 \leq b_2 < b_3 \leq b_4$ or $b_1 < b_3 < b_2 < b_4$. Using the same kind of argument, we can check the other five cases. This completes the proof of Lemma 3.9.

<u>Theorem 3.10</u>:

Suppose that $f, g \in F(a,b)$. Then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$ if and only if there exist x_0, y_0 such that $f_{|[0,x_o]}$ is left compatible with $g_{|[0,y_o]}$ and $f_{|[x_o,1]}$ is right compatible with $g_{|[y_o,1]}$.

<u>Proof</u>:

Assume that $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$.

If we have (1) $a_1 \leq a_2 < a_3 \leq a_4$, or (2) $a_1 < a_3 < a_2 < a_4$, or (3) $a_1 < a_3 \leq a_4 < a_2$, then we choose:

$$x_o = a_1 \quad \text{and} \quad y_o = b_1.$$

If we have (4) $a_3 \leq a_4 < a_1 \leq a_2$, or (5) $a_3 < a_1 < a_4 < a_2$, or (6) $a_3 < a_1 \leq a_2 < a_4$, then we choose:

$$x_o = a_3 \quad \text{and} \quad y_o = b_3.$$

Now we shall discuss case (1). The other five cases can be treated similarly.

By definition of a_1 and b_1, we have $f(a_1) = g(b_1) \neq f(x)$ for any x in $[0,a_1)$ and $g(y) \neq f(a_1)$ for any y in $[0,b_1)$. Suppose that (a_1,b_1) and $(0,0)$ are not in the same connected component of $G(f,g)$. Then it is clear that $(0,0)$ and $(1,1)$ are not in the same connected component of $G(f,g)$. Therefore (a_1,b_1) and $(0,0)$ are in the same connected component of $G(f,g)$. By the above observation, we know that $(0,0)$ and (a_1,b_1) are in the same connected component of:

$$G^* = \{(x,y) \in G(f,g) \mid 0 \leq x \leq a_1, \ 0 \leq y \leq b_1\}.$$

By Theorem 3.7, $f_{|[0,a_1]}$ is left compatible with $g_{|[0,b_1]}$. Similarly, we can show that $f_{|[a_4,1]}$ is right compatible with $g_{|[b_4,1]}$. Since we assume $a_1 \leq a_2 < a_3 \leq a_4$, we have $b_1 < b_4$ by Lemma 3.9. Because $[a_1,a_4]$ and $[b_1,b_4]$ are α-intervals of f and g respectively, $f_{|[a_1,1]}$ is right compatible with $g_{|[b_1,1]}$.

Now suppose for some x_o, y_o, $f_{|[0,x_o]}$ is left compatible with $g_{|[0,y_o]}$ and $f_{|[x_o,1]}$ is right compatible with $g_{|[y_o,1]}$. Then we can show that (0,0) and (x_o,y_o) are in the same connected component of G(f,g) and that (x_o,y_o) and (1,1) are in the same connected component of G(f,g) by using the same method as we used in Theorem 3.5. Hence (0,0) and (1,1) are in the same connected component of G(f,g).

<u>Definition 3.11</u>:

For f, g ε F(a,b), define $f \simeq g$ if (0,0) and (1,1) are in the same connected component of G(f,g).

<u>Theorem 3.12</u>:

$\simeq$ is an equivalence relation.

<u>Proof</u>:

$f \simeq f$ because the set $\{(x,x) | x \in [0,1]\}$ belongs to $G(f,g)$ and connects $(0,0)$ and $(1,1)$.

If $f \approx g$, then $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. Also $G(f,g)$ is homeomorphic to $G(g,f)$ under the mapping $(x,y) \to (y,x)$, and $(0,0)$, $(1,1)$ are mapped to $(0,0)$, $(1,1)$. Hence $g \approx f$.

Now we prove that $\approx$ is a transitive relation. From Theorem 3.5 and Theorem 3.10, we know that each f has a unique sequence associated with it. Assume that $f \approx g$. By Theorem 3.10, $f|_{[0,x_o]}$ is left compatible with $g|_{[0,y_o]}$ and $f|_{[x_o,1]}$ is right compatible with $g|_{[y_o,1]}$. And if $g \approx h$, then $g|_{[0,y'_o]}$ is left compatible with $h|_{[0,z_o]}$ and and $g|_{[y'_o,1]}$ is right compatible with $h|_{[z_o,1]}$. Because the sequences for f, g, h are unique as chosen in Theorem 3.10, we must have $y_i = y'_i$ and $f(x_j) = g(y_i) = h(z_i)$ for all i. Therefore $f|_{[0,x_o]}$ is left compatible with $h|_{[0,z_o]}$ and $f|_{[x_o,1]}$ is right compatible with $h|_{[z_o,1]}$. Hence $f \approx h$. Therefore $\approx$ is an equivalence relation in $F(a,b)$.

<u>Definition 3.13</u>:

Let f, $g \in F(p,p)$. Then define:

$$(f \circ g)(x) = \begin{cases} f(2x) & \text{for } x \text{ in } [0,\frac{1}{2}] \\ g(2x-1) & \text{for } x \text{ in } [\frac{1}{2},1]. \end{cases}$$

Lemma 3.14:

If $f_0 \approx f_1$ and $g_0 \approx g_1$, then $f_0 \circ g_0 \approx f_1 \circ g_1$.

Proof:

For every $(x,y) \in G(f_0,f_1)$, we have $f_0 \circ g_0(\frac{x}{2}) = f_0(x) = f_1(y) = f_1 \circ g_1(\frac{y}{2})$. Since $f_0 \approx f_1$, we have that $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ are in the same connected component of $G(f_0 \circ g_0, f_1 \circ g_1)$. For every $(x,y) \in G(g_0,g_1)$, we have $(\frac{x+1}{2},\frac{y+1}{2}) \in G(f_0 \circ g_0, f_1 \circ g_1)$. Since $g_0 \approx g_1$, we have that $(\frac{1}{2},\frac{1}{2})$ and $(1,1)$ are in the same connected component of $G(f_0 \circ g_0, f_1 \circ g_1)$. Hence $(0,0)$ and $(1,1)$ are in the same connected component of $G(f_0 \circ g_0, f_1 \circ g_1)$, i.e., $f_0 \circ g_0 \approx f_1 \circ g_1$.

Lemma 3.15:

The equivalence classes of $F(p,p)$ form a semi-group.

Remark 3.16:

Let f, $g \in F(a,b)$. Define $f \sim g$ if $(0,0)$ and $(1,1)$ are in the same path-connected component of $G(f,g)$. Then the following example shows that $\sim$ is not an equivalence relation.

Example 3.17:

Let f, $g \in F$. Then $f \sim g$ if and only if there exist h, $j \in F$ such that $fh(x) = gj(x)$ for all x in $[0,1]$. There exist f, $g \in F$ such that $f \not\sim g$ (cf.[1]). It is clear that if $i(x) = x$ for all $x \in [0,1]$, then $f \sim i$ and $i \sim g$ for any f, g. If we choose f, g such that $f \not\sim g$, then we cannot have the transitive relation.

We know from [1] that if f, $g \in F(a,b)$, and f, g consist of a finite number of monotone pieces, then $(0,0)$ and $(1,1)$ are in the same path-connected component of $G(f,g)$ if and only if $(0,0)$ and $(1,1)$ are in the same connected component of $G(f,g)$. So we immediately have the following theorem as a corollary of Theorem 3.10.

Theorem 3.18:

Suppose that f, $g \in F(a,b)$, and f, g consist of a finite number of monotone pieces. Then $(0,0)$ and $(1,1)$ lie in the same path-connected component of $G(f,g)$. if and only if $f_{|[0,1]}$ is compatible with $g_{|[0,1]}$.

Theorem 3.19:

Suppose that $f \varepsilon F(p,p)$ and f consists of a finite number of monotone pieces. Then $(0,1)$ and $(1,0)$ are in the same path-connected component of $G(f,f)$ if and only if there exist

$$0 = x_{-n} < x_{-n+1} < \cdots\cdots < x_o < x_1 < \cdots\cdots < x_n = 1$$

such that every interval $[x_i, x_{i+1}]$ is an α-interval of f and $f(x_i) = f(x_{-i})$ for all i.

Proof:

First we assume that the conditions hold. We want to prove that $(0,1)$ and $(1,0)$ are pathwise connected in $G(f,f)$. Define $f_1(x) = f(x_o x)$ and $f_2(x) = f\left((x_o - 1)x + 1\right)$. By Theorem 3.18, $(0,0)$ and $(1,1)$ are pathwise connected in $G(f_1,f_2)$. Let G' be the set:

$$\{(x,y) \varepsilon G(f,f) |\ x \varepsilon [0, x_o],\ \ y \varepsilon [x_o, 1]\}.$$

The mapping $(x,y) \to (x_0x, (x_0-1)y + 1)$ defines a homeomorphism from $G(f_1,f_2)$ onto G'. Hence $(0,1)$ and (x_0,x_0) are pathwise connected in $G' \subset G(f,f)$. Now $G(f,f)$ is symmetric with respect to the diagonal. Hence $(1,0)$ and (x_0,x_0) are also pathwise connected in $G(f,f)$. Therefore $(1,0)$ and $(0,1)$ are pathwise connected in $G(f,f)$.

Usually we have six possible ways to order a_1, a_2, a_3 and a_4, but if we assume $(0,1)$ and $(1,0)$ are pathwise connected in $G(f,f)$, then we claim that either $a_1 < a_3 \leq a_4 < a_2$ or $a_3 < a_1 \leq a_2 < a_4$.

By definition of a_1, a_2 and a_3, $f(x) \neq f(a_1)$ for $x \in (a_2,1]$ and $f(x) \neq f(a_3)$ for $x \in [0,a_3)$. If $a_1 \leq a_2 < a_3$, then it is clear that $(0,1)$ and $(1,0)$ are not in the same connected component of $G(f,f)$. Hence we cannot have $a_1 \leq a_2 < a_3 \leq a_4$. Similarly we cannot have the other three cases.

Now suppose $(0,1)$ and $(1,0)$ are in the same path-connected component of $G(f,f)$. If we have $a_1 < a_3 \leq a_4 < a_2$, then we want to show that $(0,1)$ and (a_1,a_2) are in the same path-connected component of $G(f,f)$. We have $f(x) \neq f(a_1)$ for $x > a_2$ and $f(a_2) \neq f(x)$ for $x < a_1$. Hence the set $\{(x,y) \mid y = a_2,\ x < a_1\}$ and the set $\{(x,y) \mid x = a_1,\ y > a_2\}$ are disjoint from $G(f,f)$. Therefore the points $(0,1)$ and (a_1,a_2) are in the same path-connected component of the set:

$$\{(x,y) \in G(f,f) \mid 0 \le x \le a_1, \quad a_2 \le y \le 1\}.$$

Define:

$$f_1(x) = f(a_1 x) \qquad \text{for } x \in [0,1]$$

$$f_2(x) = f\big((a_2-1)x + 1\big) \qquad \text{for } x \in [0,1].$$

Then by Theorem 3.18, $f_1|[0,1]$ is compatible with $f_2|[0,1]$. Hence we have

$$0 = x_{-n} < x_{-n+1} < \cdots < x_{-1} = a_1$$

$$a_2 = x_1 < x_2 < \cdots < x_n = 1$$

such that each interval $[x_i, x_{i+1}]$ is an α-interval of f and $f(x_i) = f(x_{-i})$. If we let $x_o = a_3$, the theorem is proved for the case $a_1 < a_3 \le a_4 < a_2$.

Similarly, we can prove the second case in which $a_3 < a_1 \le a_2 < a_4$. This completes the proof of Theorem 3.19.

<u>Theroem 3.20</u>:

Let f, g be continuous functions from $[0,1]$ to $S_1 = \{x \in R^2 \mid |x| = 1\}$ with $f(0) = g(0) = f(1) = g(1)$.

If (0,0) and (1,1) are in the same connected component of $G(f,g)$, then f is homotopic to g.

Proof:

For any f and g from $[0,1]$ to S_1, there exist $\tilde{f}$ and $\tilde{g}$ from $[0,1]$ to R such that $\tilde{f}(0) = \tilde{g}(0)$, $f = e\tilde{f}$ and $g = e\tilde{g}$, where $e(t) = (\cos t, \sin t)$. Let $H(x,y) = \{\tilde{f}(x) - \tilde{g}(y)\}/2\pi$. If $(x,y) \varepsilon G(f,g)$, then $f(x) = g(y)$, i.e., $e\tilde{f}(x) = e\tilde{g}(y)$. Hence $\tilde{f}(x) - \tilde{g}(y) = 2n\pi$, where n is an integer. Therefore H is a continuous function from the connected component G' of $G(f,g)$ containing (0,0), (1,1) to the set of integers. Since $H(0,0) = 0$ and G' is connected, we must have $H(x,y) = 0$ for any $(x,y) \varepsilon G'$. Hence $\tilde{f}(1) - \tilde{g}(1) = 0$. Therefore f, g have the same degree, i.e., f is homotopic to g.

Theorem 3.21:

Let f, $g \varepsilon F(a,b;Y)$, and let (0,0) and (1,1) be in the same path-connected component of $G(f,g)$. Then f is homotopic to g.

Proof:

Since (0,0) and (1,1) are pathwise connected in

$G(f,g)$, we have $\psi:[0,1] \to [0,1] \times [0,1]$ such that $\psi(x) = (\psi_1(x), \psi_2(x))$, $\psi_1(0) = 0 = \psi_2(0)$, $\psi_1(1) = \psi_2(1) = 1$ and $(\psi_1(x), \psi_2(x)) \in G(f,g)$ for any x, i.e., $f\psi_1(x) = g\psi_2(x)$. Define $H(x,t) = f(t\psi_1(x) + (1-t)x)$. Then H is a continuous function from $[0,1] \times [0,1]$ to Y with $H(x,0) = f(x)$ and $H(x,1) = f\psi_1(x)$. But we have $f\psi_1(x) = g\psi_2(x)$, hence f is homotopic to $g\psi_2$. Define $E(x,t) = g(t\psi_2(x) + (1-t)x)$. Then E is a continuous function from $[0,1] \times [0,1]$ to Y with $E(x,0) = g(x)$ and $E(x,1) = g(\psi_2(x))$. Hence g is homotopic to $g\psi_2$. It follows that f is homotopic to g.

CHAPTER IV

SOME PROPERTIES OF MOUNTAIN CLIMBING AND APPLICATIONS

If we have four continuous functions f, g, h and j from $[0,1]$ into $[0,1]$ with $fh = gj$ and these functions also satisfy some other conditions, then we shall discuss more fully the relationship among f, g, h and j in this chapter.

Lemma 4.1:

Suppose that f is a continuous function defined on $[0,1]$ that contains no constant piece and that there exists a continuous function h from $[0,1]$ to $[0,1]$ with $h(0) = 0$ and $h(1) = 1$ such that $f(x) = fh(x)$ for any x in $[0,1]$. Then $h(x) = x$ for any x in $[0,1]$.

Proof:

If there is an x_o such that $x_o < h(x_o)$, then we will construct a sequence inductively. Using the inequality $0 = h(0) < x_o < h(x_o)$, we know from the continuity of h that there is an x_1 in $(0,x_o)$ such that $h(x_1) = x_o$. Using the inequality $0 = h(0) < x_1 < x_o = h(x_1)$, we know that there is an x_2 in $(0,x_1)$ such that $h(x_2) = x_1$.

Assume that x_n has been constructed, and $h(x_n) = x_{n-1}$ and $x_n < x_{n-1}$. Using the inequality $0 = h(0) < x_n < x_{n-1} = h(x_n)$, and by continuity of h, there is an x_{n+1} in $(0, x_n)$ with $h(x_{n+1}) = x_n$. This completes the induction step to construct our sequence. Hence there is a decreasing sequence $\{x_n\}$ with the property $h(x_n) = x_{n-1}$ for any n. Since we have $h(x_1) = x_o$ and by assumption we have $fh(x_1) = f(x_1)$, hence $f(x_o) = f(x_1)$. In general, $f(x_n) = fh(x_n) = f(x_{n-1}) = fh(x_{n-1}) = f(x_{n-2}) = \ldots$ $\ldots = f(x_o)$. Since $\{x_n\}$ is a bounded decreasing sequence, it has a limit. Say $\{x_n\}$ converges to $\overline{x}$. By continuity of f, we have $f(x_n)$ converges to $f(\overline{x})$. Since $f(x_n) = f(x_o)$ for any n, so we have:

$$(1) \qquad f(x_n) = f(\overline{x}) = f(x_o)$$

for any n. Since f has no constant piece there is a t_1 in (x_1, x_o) such that

$$(2) \qquad f(x_o) \neq f(t_1).$$

Using the inequality $h(x_2) = x_1 < t_1 < x_o = h(x_1)$, there is a t_2 in (x_2, x_1) such that $h(t_2) = t_1$. We can start from here to construct a sequence inductively. Hence there is a decreasing sequence $\{t_n\}$ with $t_n \in (x_n, x_{n-1})$ and $h(t_n) = t_{n-1}$. Since x_n converges to $\overline{x}$ and

and $t_n \varepsilon (x_n, x_{n-1})$, so t_n must converge to $\bar{x}$. So we also have $f(t_n)$ converges to $f(\bar{x})$. By (1) we have

(3) $f(t_n)$ converges to $f(x_o)$.

On the other hand, $f(t_n) = fh(t_n) = f(t_{n-1}) = fh(t_{n-1}) = \ldots$ $\ldots = f(t_1)$. Therefore we have

(4) $f(t_n) = f(t_1)$ for any n.

By (3) and (4), we have

(5) $f(t_1) = f(x_o)$.

But this contradicts (2). Hence there is no x with $x < h(x)$. Using exactly the same method, we can show that there is no x with $x > h(x)$. Hence $h(x) = x$ for any x in $[0,1]$, i.e., h is the identity function.

Theorem 4.2:

Suppose that f is a continuous function defined on $[0,1]$ which contains no constant piece and that there exists a continuous function h from $[0,1]$ to $[0,1]$ with $h(0) = 1$, $h(1) = 0$ and $f(x) = fh(x)$ for any x in $[0,1]$. Then $hh(x) = x$, i.e., h is an involution, hence h is a one to one function and so h is a homeomorphism, and h is also unique.

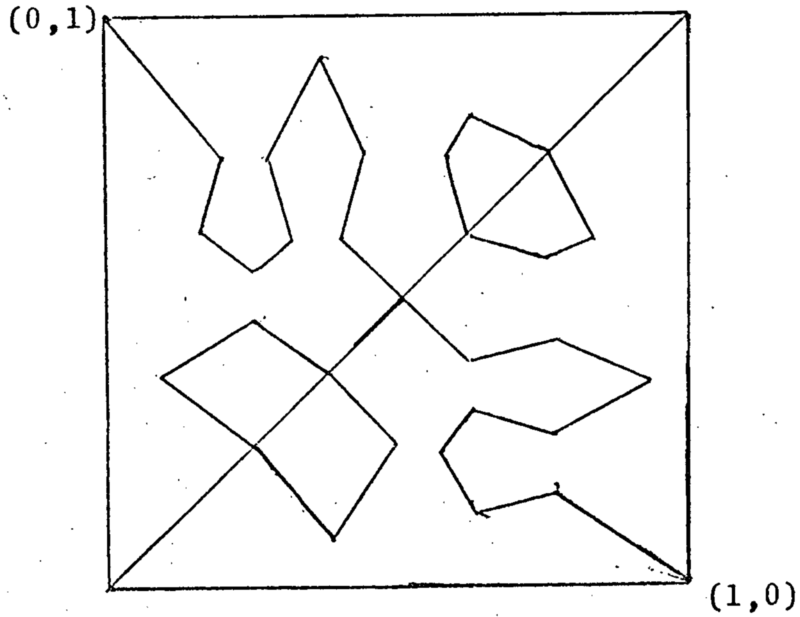
Proof:

Consider hh. We have $hh(0) = 0$ and $hh(1) = 1$, and we also have $fhh(x) = fh(x) = f(x)$ for any x in $[0,1]$. Hence by Lemma 4.1, $hh(x) = x$. Now we want to show that h is one-to-one. Let $h(x) = h(y)$. Then we have $x = hh(x) = hh(y) = y$. Hence $x = y$. Now we want to show that h is unique. Assume there is another continuous function j from $[0,1]$ to $[0,1]$ with $fj(x) = f(x)$ for any x in $[0,1]$ and $j(0) = 1$, $j(1) = 0$. Since $fh(j(x)) = fj(x) = f(x)$ for any x in $[0,1]$, and $hj(0) = 0$, $hj(1) = 1$, so by Lemma 4.1, we have $hj(x) = x$. We have already proved that $hh(x) = x$ and h is one-to-one, hence $h(x) = hhj(x) = j(x)$, i.e. $h = j$. Therefore h is unique.

Theorem 4.2 has the following simple interpretation in terms of mountain-climbing. Two men stand at opposite sides of a mountain range (graph of $f(x)$). The condition $f(x) = fh(x)$ with $h(0) = 1$ and $f(1) = 0$ means that they can climb the mountain range from one side to the other in such a way that their elevations remain equal all the time, and one man always moves forward along the way. The conclusion of Theorem 4.2 tells us that the other man must also move forward all the time.

It is surprising that even if we know that two men can climb the mountain range from one side to the other, maintaining a common elevation, there is no guarantee that they can climb properly such that at any instant there is at least one man moving forward. The following example shows that we will have two men both going backwards at some instant. Define:

Let $f(0) = f(1) = 0$, $f(\frac{1}{4}) = \frac{3}{4}$, $f(\frac{3}{8}) = \frac{1}{4}$, $f(\frac{1}{2}) = 1$, $f(\frac{5}{8}) = \frac{1}{2}$ and $f(\frac{3}{4}) = \frac{5}{8}$, and define f to be linear in all the remaining intervals. If we look at the graph of $G(f,f)$ (See the figure), then there is essentially one path connecting $(0,1)$ and $(1,0)$, and this path shows that at some instant both men go backwards.



Corollary 4.3:

Suppose that f is a continuous function defined on $[0,1]$ which contains no constant piece and that there exist continuous functions h and g from $[0,1]$ to $[0,1]$ with $h(0) = 1$, $h(1) = 0$, $g(0) = 0$ and $g(1) = 1$ such that $fg(x) = fh(x)$ for all x in $[0,1]$. If g is a one-to-one function, then h is a homeomorphism and is unique.

Proof:

Since g is on-to-one, it has an inverse function g^{-1}. From $fg(g^{-1}x) = fh(g^{-1}(x))$, we have $f(x) = fhg^{-1}(x)$ and $hg^{-1}(0) = 1$, $hg^{-1}(1) = 0$. By Theorem 4.2, we have $hg^{-1}hg^{-1}(x) = x$, i.e., $hg^{-1}h(x) = g(x)$. Assume we have $h(x) = h(y)$, then $hg^{-1}h(x) = hg^{-1}h(y)$ and hence $g(x) = g(y)$. But g is one-to-one, so we get $x = y$, i.e., h is one-to-one. Now we want to show that h is unique. Suppose j is another function that has all the required properties. Then $fg(x) = fh(x)$ and $fg(x) = fj(x)$ give us $fj(x) = fh(x)$. Now h has inverse h^{-1}, so $fjh^{-1}(x) = f(x)$ for all x, and $jh^{-1}(0) = 0$, $jh^{-1}(1) = 1$. By Lemma 4.1, $jh^{-1}(x) = x$. Hence $j(x) = h(x)$ for all x. That means $h = j$ and h is unique.

Lemma 4.4:

Suppose that f is a continuous function defined on R, $f(x) = 0$ for $x \varepsilon R-(0,1)$ and $f(x) \geq 0$ for all x in $(0,1)$. Then given any $d > 0$, there exist a, b such that $(a+b)/2 \varepsilon (0,1)$, $|a-b| = d$ and $f(a) = f(b)$.

Proof:

If $d \geq 1$, it is clear that we can choose $a = (1-d)/2$, $b = (1+d)/2$. Suppose $d < 1$. Let $\{f_n\}$ be a sequence of continuous functions which converges to f and f_n consists of a finite number of monotone pieces. We may choose $f_n(x) = 0$ for $x \varepsilon R-(0,1)$ and $f_n(x) \geq 0$ for x in $[0,1]$. Now from Whittaker's result [2], for each n there exist continuous functions j_n, h_n from $[0,1]$ to $[0,1]$ such that

(1) $\quad j_n(0) = 0, \quad j_n(1) = 1, \quad h_n(0) = 1, \quad h_n(1) = 0,$

and $f_n j_n(x) = f_n h_n(x)$ for all x in $[0,1]$. From (1) and the continuity of j_n and h_n, it follows that their graphs must cross at some point $t_o \varepsilon [0,1]$. Then

(2) $$j_n(t_o) = h\ (t_o).$$

Define $H_n(x) = |j_n(x) - h_n(x)|$. From (1) and (2), we have $H_n(0) = 1$ and $H_n(t_o) = 0$. From the continuity of H_n, it follows that there exists a point $t_n \in (0,t_o)$ such that $H_n(t_n) = d$. Let $j_n(t_n) = a_n$ and $h_n(t_n) = b_n$. Then $|a_n - b_n| = d$ and $f_n(a_n) = f_n j_n(t_n) = f_n h_n(t_n) = f_n(b_n)$. Without loss of generality, we may assume that the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b respectively. Then $f_n(a_n)$ converges to $f(a)$ and $f_n(b_n)$ converges to $f(b)$. Hence $f(a) = f(b)$. It is clear that $|a - b| = d$. Since a_n, $b_n \in [0,1]$, we have $(a + b)/2 \in (0,1)$.

<u>Lemma 4.6</u>:

Let D be a unit disc in the plane R^2. Let f be a continuous function defined on R^2 such that

$$f(x) \geq 0 \qquad \text{if } x \in D$$

and

$$f(x) = 0 \qquad \text{if } x \notin D.$$

Let the number $d > 0$ be given, then there are three points in R^2 forming a triangle with area (or perimeter) equal to d and with the center of the triangle in D such that f takes the same value on the vertices of the triangle.

Proof:

We shall get a triangle with area d. The case of a triangle with given perimeter is treated in the same way. Actually we can use this kind of method to get other similar results.

$3\sqrt{3}/4$ is the maximum area we can get from a triangle inside D. If $d > 3\sqrt{3}/4$, it is clear that we can choose three points outside D forming a triangle with area d. and center in D. As a matter of fact we can even choose this triangle to be equilateral with center at the center of the unit disc. If $d < 3\sqrt{3}/4$, then take any three points p, q and r from the boundary of the disc such that the triangle pqr is equilateral, so its area is $3\sqrt{3}/4$. Let m be a point where $f(x)$ attains its maximum value. Take three arcs inside D joining p, q and r to m. Denote these three arcs by $\overline{pm}$, $\overline{qm}$ and $\overline{rm}$. Let h_1, h_2, and h_3 be homeomorphisms from $[0,1]$ onto $\overline{pm}$, $\overline{qm}$ and $\overline{rm}$ respectively such that $h_1(0) = p$, $h_2(0) = q$, $h_3(0) = r$ and $h_1(1) = h_2(1) = h_3(1) = m$. Let $f_1 = f|_{\overline{pm}}$, $f_2 = f|_{\overline{qm}}$ and $f_3 = f|_{\overline{rm}}$. Then f_1h_1, f_2h_2, and f_3h_3 are continuous functions defined on $[0,1]$. Let $\{f_{1n}\}$, $\{f_{2n}\}$, $\{f_{3n}\}$ be sequences of continuous functions which converge to f_1h_1, f_2h_2, f_3h_3 respectively such that those functions $\{f_{1n}\}$, $\{f_{2n}\}$, $\{f_{3n}\}$ consist of a finite number of monotone pieces.

By Theorem 1.2, there exist continuous functions g_{1n}, g_{2n}, g_{3n} in F such that $f_{1n}g_{1n} = f_{2n}g_{2n} = f_{3n}g_{3n}$. Define $H_n(t)$ to be the area of the triangle with vertices $h_1g_{1n}(t)$, $h_2g_{2n}(t)$, $h_3g_{3n}(t)$. We have $H_n(0) = 3\sqrt{3}/4$ and $H_n(1) = 0$. By continuity of H_n, there is a t_n in $[0,1]$ such that $H_n(t_n) = d$, i.e., the triangle with vertices $h_1g_{1n}(t_n)$, $h_2g_{2n}(t_n)$, $h_2g_{3n}(t_n)$ has area d. Without loss of generality, we may assume that $g_{1n}(t_n)$, $g_{2n}(t_n)$, $g_{3n}(t_n)$ converge to $h_1^{-1}(a)$, $h_2^{-1}(b)$, $h_3^{-1}(c)$. respectively, where $a \in \overline{pm}$, $b \in \overline{qm}$, $c \in \overline{rm}$. The area of triangle abc is equal to d by the above conditions. We have $f_{1n}g_{1n}(t_n) = f_{2n}g_{2n}(t_n) = f_{3n}g_{3n}(t_n)$ for all n. Therefore

$f_1h_1h_1^{-1}(a) = f_2h_2h_2^{-1}(b) = f_3h_3h_3^{-1}(c)$, i.e., $f_1(a) = f_2(b) = f_3(c)$. Hence $f(a) = f(b) = f(c)$. This completes the proof of Lemma 4.7.

CHAPTER V

CHORD SLIDING ALONG A PATH

R. Fenn [3] considered the question of sliding a chord of length d from one end of a path to the other, i.e., if $I = [0,1]$ is the unit interval and $h: I \to X$ a path in a metric space (X,ρ), then do there exist maps p and q from I to I such that $\rho(hp(s),hq(s)) = d$ for all $s \in I$, $p(0) = 0$, and $q(1) = 1$? He proved that the answer is yes if the path is reasonable, say piecewise analytic, and d is not greater than the distance between the end points, $\rho(h(0),h(1))$. We wish to discuss a similar question from the point of view of point set topology. And we also wish to give an example to show that there are paths for which sliding cannot take place for certain values of d.

The question we asked is completely similar to the mountain climbing problems. All we asked is whether there exist x and y such that $(x,0)$ and $(1,y)$ are pathwise connected in the set

$$E_d = \{(s,t) \in I^2 | \rho(h(s),h(t) = d\},$$

where $0 \leq d \leq \rho(h(0),h(1))$. We are able to show that there are x and y such that $(x,0)$ and $(1,y)$ are in the same component C of E_d by means of elementary point set topology. And if the path satisfies property (*) which we will define later, then we are able to show that C is locally connected. It is clear that C is a compact metric space. Then by the Arcwise Connectedness Theorem [4,p.36], $(x,0)$ and $(1,y)$ can be joined in C by a simple arc. That means we will have the maps p and q. Before we do anything, we wish to quote a theorem from [4,p.109].

Zoretti Theorem:

If K is a component of a compact set M in the plane, and ε is any positive number, then there exists a simple closed curve J which encloses K and is such that $J \cap M = \phi$, and every point of J is at a distance less than ε from some point of K.

Lemma 5.1:

Let h be a continuous function from I to a metric space (X,ρ). Then there exist x and y in I such that (x,o) and $(1,y)$ are in the same connected component C of E_d.

Proof:

Our result is trivial if $d = \rho(h(1),h(0))$ or $d = 0$. Assume that $0 < d < \rho(h(1),h(0))$. Since E_d is a compact set which does not meet the diagonal nor contain the points $(0,1)$ and $(1,0)$, then there exists a set

$$L_1 = \{(x,0) \mid 0 < a \leq x \leq b < 1\}$$

for some a and b such that

$$E_d \cap \{(x,0) \mid 0 < x < 1\} \subset L_1.$$

Now we want to show that any component of E_d which does not intersect L_1 will be a component of $E_d \cup L_1$. Let E be a component of E_d such that $E \cap L_1 = \phi$. Let $L_2 = \{(x,0) \mid -1 \leq x \leq 2\}$. Since $E \subset I \times I$, and $E \cap L_2 = \phi$, then by the Zoretti Theorem, there exists a simple closed curve J which encloses E and $J \cap E_d = \phi$, and every point of J is at a distance less than ε from some point of E. If ε is small, say ε is less than the distance between L_2 and E, then J cannot be far away from $I \times I$ because $E \subset I \times I$. This means that L_2 cannot be completely inside J. But if some of L_2 is in the interior of J, then by the connectedness of L_2, we must have some $z \in J \cap L_2$. From $z \in J$, we infer that the distance between z and E is less than ε, from $z \in L_2$, we infer that the distance between z and E is is greater than ε. This is impossible. Hence L_2 is completely in the exterior of J. Let E' be the component of $E_d \cup L_1$ such that $E \subset E'$. If $E' \cap L_1 = \Phi$, i.e., $E' \subset E_d$, then since E is a component of E_d, we have $E = E'$. If $E' \cap L_1 \neq \phi$, Then E' cannot be connected because E' contains E which is in the interior of J and L_1 is in the exterior of J and $J \cap (E_d \cup L_1) = \phi$. Hence $E = E'$ and E is a component of $E_d \cup L_1$.

Now let A be the union of L_1 and all the components of E_d which intersect L_1. We shall use the above result to prove that A is a component of $E_d \cup L_1$. Suppose A is not a component of $E_d \cup L_1$. Then there exists a point $r \notin A$, but r and A are in the same component of $E_d \cup L_1$. By definition of A, it is clear that r is in a component B of E_d, which does not intersect L_1. Hence B is a component of $E_d \cup L_1$. The fact that $B \cap L_1 = \phi$ gives us a contradiction since we assume that r is in a component of $E_d \cup L_1$ which contains L_1. Therefore A is a component of $E_d \cup L_1$.

Assume that $A \cap L_3 = \phi$, where $L_3 = \{(1,y) \mid 0 < y < 1\}$. By the Zoretti Theorem, there exists a simple closed curve J which encloses A and $J \cap (E_d \cup L_1) = \phi$. If ε is small enough, then J is in the interior of $[0,1] \times [-1,1]$ and $J \cap A_1 \neq \phi$,

$J \cap A_2 \neq \phi$, where

$$A_1 = \{(x,y) | y = 0, 0 \leq x < a\}$$

and

$$A_2 = \{(x,y) | y = 0, b < x \leq 1\}.$$

We want to show that there is an arc $N \subset J$ and $N \subset I^2$ such that N is not disjoint from A_1 and A_2. It is clear that each component C_i of $J \cap I^2$ is not disjoint from $A_1 \cup A_2$. Let

$$c_i' = \inf \{(x,0) | (x,0) \in C_i \cap (A_1 \cup A_2)\}$$

$$c_i^* = \sup \{(x,0) | (x,0) \in C_i \cap (A_1 \cup A_2)\}.$$

Let $\overline{C}_i$ be the closed region bounded by C_i and the line segment between c_i' and c_i^*. If A is disjoint from $\overline{C}_i$, then we can deform the arc C_i to a line segment (the line between c_i' and c_i^*) without touching A. If we can do this for any C_i, then we can shrink J to a single point without touching A. This contradicts our choice of J which encloses A. Hence $A \cap \overline{C}_i \neq \phi$ for some i. Since A is connected and does not meet C_i, we have that $A \subset \overline{C}_i$ for some i. This C_i is the arc N which we want. Let $(x_1,0) \in N \cap A_1$ and $(x_2,0) \in N \cap A_2$, Let

$$R = N \cup \{(x,0) | 0 \leq x \leq x_1\} \cup \{(x,0) | x_2 \leq x \leq 1\}.$$

Define $H(x,y) = \rho(h(x),h(y))$. Then $H(0,0) = 0$ and $H(1,0) > d$. Since R is a connected set and H is a continuous function, hence there is a point $(s,t) \varepsilon R$ such that $H(s,t) = d$. This means that $(s,t) \varepsilon E_d$, but $R \cap E_d = \phi$. This is a contradiction. Hence we must have $A \cap L_3 \neq \phi$. Therefore there is a component C of E_d which contains $(x,0)$ and $(1,y)$ for some x and y.

Property (*) :

There is a dense subset D of I such that for any $e \varepsilon D$, the line segment $\{(x,e) | 0 \leq x \leq 1\}$ intersects C in a finite number of connected components, and the line segment $\{(e,y) | 0 \leq y \leq 1\}$ intersects C in a finite number of connected components.

Suppose h is a function from I into (X,ρ). Then we draw a circle of radius d, center at $h(t)$ for $t \varepsilon I$. If each such circle intersects $h(I)$ in a finite number of connected components, then we will have Property (*) provided the path is simple.

Theorem 5.2:

Let h be a continuous function from I to a metric space (X,ρ) such that Property (*) holds. Then there is a chord of length d which can be slid from one end of the path to the other provided that

$$0 \leq d \leq \rho(h(0),h(1)).$$

Proof:

We have proved that there exist x and y such that $(x,0)$ and $(1,y)$ are in the same connected component C of E_d. Now we want to use Property (*) to show that C is locally connected. By definition of locally connectedness, all we have to prove is that for any point (s_o,t_o) in C, every neighborhood of (s_o,t_o) in C contains a connected neighborhood of (s_o,t_o) in C. For any neighborhood of (s_o,t_o), we can find a smaller neighborhood $V = C \cap G'$, where

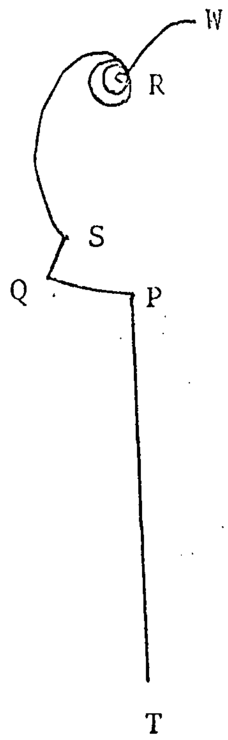
$$G' = \{(s,t) \mid s_1-\delta < s < s_2+\delta, \quad t_1-\delta < t < t_2+\delta\}$$

and (s_o,t_o) is in

$$T_1 = \{(s,t) \mid s_1 < s < s_2, \quad t_1 < t < t_2\}$$

for s_1, s_2, t_1, t_2 in D. Let T_2 be the closure of T_1. Then $T_2 \cap C \subset V$. $(T_2 - T_1) \cap C$ has a finite number of components by Property (*). Now we want to show that $T_2 \cap C$ has only a finite number of components. Assuming that this is not true, then there exists at least one component K with $K \cap (T_2 - T_1) = \phi$. Since K is a component of the compact set $T_2 \cap C$, we know that if we choose ε small enough, then by the Zorette Theorem, we can have a simple closed curve J enclosing K such that $J \cap T_2 \cap C = \phi$ and J is inside T_1. Evidently $(x,0)$ and $(1,y)$ are in the exterior of J. Hence C is separated by J. This is impossible. So we have that $T_2 \cap C$ has only a finite number of components. Hence every component of $T_2 \cap C$ is both closed and open in $T_2 \cap C$. Let G be the component of $T_2 \cap C$ such that $(s_o,t_o) \varepsilon G$. Since G is open in $T_2 \cap C$, we have $G = (T_2 \cap C) \cap U$ where U is open in C, (s_o,t_o) is in $(T_1 \cap C) \cap U \subset G$ and $T_1 \cap C \cap U$ is open in C. Hence G is a connected neighborhood of (s_o,t_o) in C, and G is contained in V. Hence C is locally connected. This implies that C is arcwise connected and proves the theorem.

The following example shows that if the curve is not nice, then for some d, $0 < d < \rho(h(0),h(1))$, sliding cannot take place. This example is a curve (Figure I) in the plane. Let h be the function from $[0,1]$ to the plane with $h(0) = T$, $h(1) = W$, where $h(x)$ goes from T to R, then returns to R via smaller and smaller circles. In fact $h(x)$ goes through R infinitely many times, and those circles converge to the point R. Any two points in the



(Figure I)

arc PQ will have the same distance from R. We call this distance d. This example shows that we can not slide any chord of length d from one end of the path to the other. Suppose there exist mappings p and q with the required properties. If hp(s) is near R and has not yet passed R, then we will have hq(s) near Q and in the arc SQ. If hp(s) is near R and has just passed R, then we will have hq(s) near P and in the arc PT. Hence there exists an increasing sequences $\{s_n\}$ such that $hp(s_n)$ converges to R, but $hq(s_{2n})$ converges to P and $hq(s_{2n+1})$ converges to Q. This means that q cannot be continuous.

In the above example, if we take any chord of length $b \neq d$, then we are still able to slide the chord from one end of the path to the other.

Now we want to extend this example by adjoining other copies of h to get a path H on which sliding cannot take place for any chord of length $b \in Y$, where Y is a countable sequence converging to 0 and

$$Y \subset [0,\rho(H(0),H(1)].$$

The construction of H is made by first choosing an arbitrary positive number α. Then it is clear that there is a sequence $\{s_i\}$ converging to 0, which lies in $(0,\alpha)$. We make a copy of h in which s_1 is the forbidden chord length. Then we extend the arc P_1T_1 to O_1 such that the distance between T_1 and O_1 is α, and connect W_1 with T_1(Figure II). We call this new path g_{s_1} which we consider defined on $[0,\frac{1}{2}]$ with $g_{s_1}(0) = O_1$ and $g_{s_1}(\frac{1}{2}) = T_1$. For each s_i $(i \neq 1)$, we can get a copy of h in which s_i is the forbidden chord length. Then we connect W_i with T_i to get a new closed path g_{s_i} which we consider defined on $[(i-1)/i,i/(i+1)]$ with

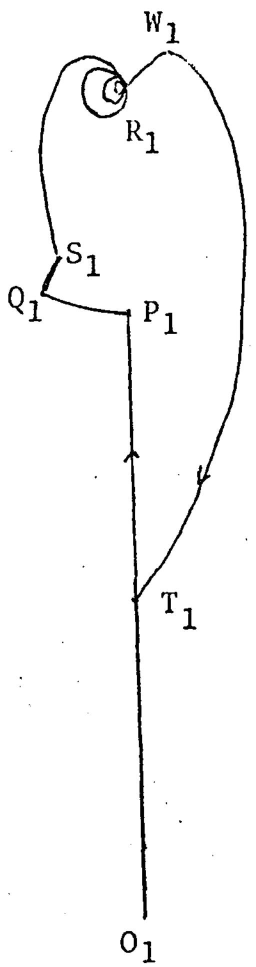
$$g_{s_i}(\frac{i-1}{i}) = T_i = g_{s_i}(\frac{i}{i+1}).$$

We join the g_{s_i} for all i in such a way that the points T_i all coincide, and the new copies g_{s_i} with $i \neq 1$ are in any position as long as they do not meet the interior of the circle with center at 0_1 and radius α (Figure III).

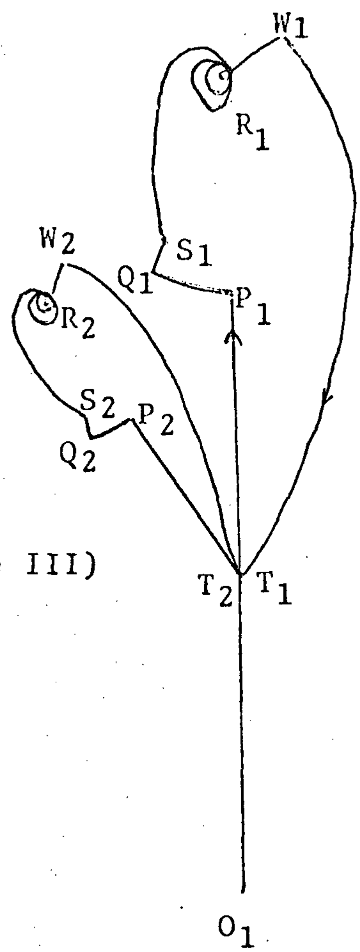
Let

$$H(x) = g_{s_i}(x) \quad \text{if} \quad x \in [(i-1)/i, i/(i+1)].$$

Then $H(0) = 0_1$ and $H(1) = T_1$. Since $\{s_i\}$ converges to 0, H must be continuous at 1. For any s_i, there is a unique chord of length s_i to start sliding. It is clear that we cannot slide the chord from one end of the path H to the other.



(Figure II)



(Figure III)

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