Finite subset spaces of the circle

A hyperbolic approach

by

Simon Rose

B.Sc., The University of Alberta, 2005

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

The Faculty of Graduate Studies
(Mathematics)

The University Of British Columbia
April, 2007
© Simon Rose 2007
Abstract

In this thesis we investigate a new and highly geometric approach to studying finite subset spaces of the circle. By considering the circle as the boundary of the hyperbolic plane, we are able to use the full force of hyperbolic geometry—in particular, its well-understood group of isometries—to determine explicitly the structure of the first few configuration spaces of the circle $S^1$. Once these are understood we then move onto studying their union—that is, $\exp_3(S^1)$—and in particular, we re-prove both an old theorem of Bott and a newer (unpublished) result of E. Шепин (E. Shchepin) about this space.
# Contents

Abstract ................................................................. ii

Contents ................................................................. iii

List of Figures ........................................................... v

Acknowledgements ........................................................ vi

1 Introduction and Preliminaries ........................................ 1
   1.1 Introduction ....................................................... 1
   1.2 Preliminaries ..................................................... 5
       1.2.1 A rapid introduction to $PSL_2(\mathbb{R})$ ..................... 5
       1.2.2 Seifert-Fibred spaces .................................... 9

2 The analysis of $\exp_3(S^1)$ ......................................... 15
   2.1 The topology and geometry of $C_1(S^1)$ ......................... 15
       2.1.1 Coordinate-based description of topology ................. 18
       2.1.2 Coordinate-independent descriptions of topology ....... 22
       2.1.3 Topology of the union $\exp_k(S^1)$ ................. 25
   2.2 $\exp_3(S^1)$ and the inclusion $\exp_1(S^1) \hookrightarrow \exp_3(S^1)$ 28

3 Further Research ...................................................... 35
List of Figures

1.1 Exceptional fibres of the $S^1$ action .............................................. 13
1.2 The inclusions of $C_t(S^1) \hookrightarrow C_0(S^1)$ ............................... 14

2.1 $\arg(x + iy)$ .................................................................................. 19
2.2 The homeomorphism between $C_2(S^1)$ and $\mathcal{M}$ ......................... 20
2.3 The action of $\gamma$ on the $S^1$ coordinate of $PSL_2(\mathbb{R})$ .............. 21
2.4 Geometry of $C_2(S^1)$ ..................................................................... 24
2.5 Choice of Fundamental Domain .......................................................... 27
2.6 A neighbourhood of $\{p, q\}$ ............................................................ 29
2.7 The covering used for the Seifert-Van Kampen theorem ......................... 30
2.8 The generator $b$ of $\pi_1(A \cap B)$ ...................................................... 31
2.9 The homotopy class of $i_\ast(b)$ .......................................................... 31
2.10 The homotopy class of $j_\ast(b)$ .......................................................... 32
2.11 Covering of $B'$ .................................................................................. 33

3.1 Isomorphisms due to exact sequences .................................................. 38
Acknowledgements

I would like to thank pretty much all of the people that I have worked with throughout both my undergraduate and fledgeling graduate careers, and in particular my supervisor, Dr. Denis Sjerve, who has provided me with many ideas and insights, and corrected my typos. Beyond that, I would like to thank all of the instructors for the courses I have taken both here and at the University of Alberta, since without their tutelage I would not have been able to understand any of the text contained herein.

I would also like to thank Dr. Volker Runde, who helped encourage me into higher studies, and without whom I never would have appreciated the true nature of that elusive $\epsilon$.

I would also like to dedicate this work to my parents, who have been unfailingly supportive of me throughout my education.
Chapter 1

Introduction and Preliminaries

1.1 Introduction

Since the mid 1900s, the study of finite subsets of topological spaces has cropped up in one guise or another—initially by Borsuk [1], later corrected by Bott [2] and more recently by the likes of Handel [3], Tuffley [11] and Mostovoy [9].

A few different definitions have been given in the literature (which are all easily seen to be equivalent), but the one that we will use for this thesis is the following.

Definition 1.1.1 Let \( X \) be a topological space, and let \( \exp_k(X) \) denote the set of all non-empty subsets of \( X \) of cardinality at most \( k \). That is

\[
\exp_k(X) := \{ S \subseteq X \mid 0 < |S| \leq k \}
\]

There is a natural surjective map \( \pi : X^k \twoheadrightarrow \exp_k(X) \) which takes \( (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\} \); we thus endow \( \exp_k(X) \) with the quotient topology induced by this map. It is of course clear that under this topology, \( \exp_1(X) \cong X \).

There are a number of results that can be proven about this space and its relation to \( X \). First of all, the map \( X \mapsto \exp_k(X) \) is a homotopy functor from \( U \), the category of topological spaces and continuous maps, to itself. More precisely, we have the following theorem.
Theorem 1.1.2 \ Let \( X, Y \) be topological spaces, let \( f, g : X \to Y \). If \( F : X \times I \to Y \) is a homotopy from \( f \) to \( g \), then the map

\[
\overline{F} : \exp_k(X) \times I \to \exp_k(Y) \quad (\{x_1, \ldots, x_k\}, t) \mapsto \{F(x_1, t), \ldots, F(x_k, t)\}
\]

is a homotopy from \( \exp_k(f) \) to \( \exp_k(g) \).

From which it immediately follows that

Corollary 1.1.3 \ The homotopy type of \( \exp_k(X) \) depends only on that of \( X \).

There are also natural inclusions \( \exp_k(X) \hookrightarrow \exp_m(X) \) for \( k \leq m \), taking the subset \( \{x_1, \ldots, x_k\} \hookrightarrow \{x_1, \ldots, x_k\} \); thus we have

\[
X \cong \exp_1(X) \hookrightarrow \exp_2(X) \hookrightarrow \cdots \hookrightarrow \exp_k(X) \hookrightarrow \cdots
\]

with each \( \exp_k(X) \) embedded as a closed subset of \( \exp_m(X) \).

The proofs of these results (as well as many other topological and homotopy-theoretic results) are in [3], which provides a good introduction to many of the basic properties of \( \exp_k(X) \) (which he denotes as \( Sub(X, k) \)).

It should also be noted that much of this is in marked contrast to the notion of a configuration space, which is not functorial (unless we suitably restrict our category); nor are there nice embeddings as above.

The most famous result in the study of these spaces, however, comes from Bott's correction to Borsuk's initial paper. That is:

Theorem 1.1.4 \ The space \( \exp_3(S^1) \) is homeomorphic to \( S^3 \), the three-sphere.

This has been proven so far in a number of different ways—Bott uses an elaborate "cut-and-paste" style argument (in effect showing that it the union of tori together with a calculation of its fundamental group to show that it is a simply connected lens
space), Tuffley finds a decomposition of it as a $\Delta$-complex (in the sense of [5]) and shows that it is a simply connected Seifert-fibered space, while Mostovoy ties it in with a result of Quillen about lattices in the plane.

There are of course many other results about similar such spaces, primarily in [11] and [7], most notably the following result.

**Theorem 1.1.5** The homotopy type of $\exp_k(S^1)$ is that of an odd dimensional sphere; specifically,

$$\exp_{2k}(S^1) \cong \exp_{2k-1}(S^1) \cong S^{2k-1}$$

This is proven in [7] using the notion of truncated product spaces—these can be considered to be, in a certain sense, the free $\mathbb{Z}/2$ set on points in our space. More specifically, these are given by

$$TP_n(X) := SP_n(X)/\sim$$

where $SP_n(X)$ is the $n$-th symmetric product of $X$ defined to be the collection of unordered $n$-tuples in $X$—that is, $SP_n(X) = \{x_1 + \cdots + x_n | x_i \in X\}$—and we say that $x_1 + \cdots + x_{k-2} + 2x \sim x_1 + \cdots x_{k-2}$ (where we use the convention of writing an unordered collection of points as a sum), topologized accordingly. The reason that this is useful is that $TP_n(S^1) \cong \mathbb{RP}^n$ (see [8]) and so if we then note the homeomorphism

$$\exp_k(X)/\exp_{k-1}(X) \cong TP_k(X)/TP_{k-2}(X)$$

this then becomes

$$\exp_k(S^1)/\exp_{k-1}(S^1) \cong \mathbb{RP}^k/\mathbb{RP}^{k-2}$$

We are then able to use the long exact sequence of homology groups to show that $\exp_k(S^1)$ is a homology sphere—since it is simply connected it is then homotopic to a sphere (by a standard argument).
Moreover, the fact that the exp functor is a homotopy functor has also been used in [12] and [13] to begin the determination of the homotopy type of $\exp_k(\Sigma)$ for $\Sigma$ a closed surface. The idea is that if you consider a punctured surface, then this is homotopic to a wedge of copies of $S^1$—and from the understanding of $\exp_k(S^1)$ Tuffley is then able to prove results about $\exp_k(\Gamma)$, where $\Gamma$ is a graph consisting of wedges of $\ell$ circles. From this, he uses a Mayer-Vietoris argument, covering a surface with $k + 1$ copies of $X = \Sigma \setminus \{p_i\}$ (for distinct points $p_i$), to determine information about the homology of the space $\exp_k(\Sigma)$ (this works since $\bigcap_{i=1}^{k+1} X_i = \Sigma \setminus \{p_1, \ldots, p_{k+1}\}$, and so these cover $\exp_k(\Sigma)$).

In this thesis we intend to provide yet another proof Theorem 1.1.4, beginning with certain properties of $PSL_2(\mathbb{R})$. This method will generalize and provide methods of calculation not only of the homotopy type of higher dimensional analogues, but possibly even their homeomorphism type.

The main difference in this approach as compared to previous ones lies in its inherent geometric nature. This heavy emphasis on the underlying geometry of the circle (or later, of higher dimensional spheres) and its connection to hyperbolic space affords us a much more specific understanding of the topology of the resulting space. In contrast, previous methods (for higher dimensional analogues) were restricted to knowledge of the homology and homotopy groups of these spaces which permitted a knowledge of the homotopy type of these spaces and no more.

**Notation** For the duration of this thesis (unless otherwise noted) we will consider $S^1$ to be both the quotient of $[0, \pi]$ in the usual manner, as well as the quotient $\mathbb{C}^\times/\mathbb{R}_{>0}$, depending on context. The two views are simply matters of convenience due to our working with $PSL_2(\mathbb{R})$, depending on our choice of how we describe the homeomorphism type of $PSL_2(\mathbb{R})$, as will be seen later.

We will denote the upper half-plane, $\mathbb{R} \times \mathbb{R}_{>0}$ by $\mathbb{H}$, considered as the subset of
complex numbers whose imaginary parts are strictly positive.

We will also use the notation

\[ C_k(X) := \{ S \subseteq X \mid |S| = k \} \]

to denote the \( k \)-th unordered configuration space of \( X \). It is of course clear that

\[
\exp_k(X) = \bigcup_{j=1}^{k} C_j(X) \tag{1.1}
\]

Not so clear, however, is just how this union behaves topologically. If we consider \( \{x_1, \ldots, x_k\} \in C_k(X) \) and let \( x_i \to x_j \), then we will end up with a point in \( C_{k-1}(X) \). To properly understand the topology of \( \exp_k(X) \) in this context, we then need to study exactly how this gluing takes place—which will be the subject of most of this thesis.

### 1.2 Preliminaries

#### 1.2.1 A rapid introduction to \( PSL_2(\mathbb{R}) \)

We begin with the most basic definition of \( PSL_2(\mathbb{R}) \), the one most people will have learned of in a basic linear algebra setting.

**Definition 1.2.1** Let \( SL_2(\mathbb{R}) \) denote the set of 2x2 real matrices of determinant 1. Then we define

\[
PSL_2(\mathbb{R}) := SL_2(\mathbb{R})/\pm I
\]

Note that this is equivalent to choosing all matrices of positive determinant, and dividing by all scalar multiples of the identity; that is,

\[
PSL_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d, \in \mathbb{R}, ad - bc > 0 \right\} / \sim
\]
where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
for all \( \lambda \in \mathbb{R}^* \). This characterization tends to be more useful, as it no longer forces us to ensure that our matrices have determinant 1 (which makes for neater formulae, if nothing else).

There is yet another way of considering \( PSL_2(\mathbb{R}) \) which is, at its heart, the most useful of the views for our purposes (and follows fairly neatly from the previous comments)—that is, as fractional linear transformations of \( \mathbb{P}^1 \) and in particular as the group of orientation-preserving isometries of \( \mathbb{H} \), the upper half-plane.

Purely as sets this is because we have
\[
\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{\lambda (az + b)}{\lambda (cz + d)} = \frac{az + b}{cz + d}
\]
and so we have a clear bijection between the elements of \( PSL_2(\mathbb{R}) \) and the fractional linear transformations with real coefficients. However, this correspondence is yet stronger; a quick calculation shows that
\[
\frac{az + b}{cz + d} \circ \frac{az + \beta}{\gamma z + \delta} = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}
\]
and so our bijection is in fact a group isomorphism. This of course means that we can consider \( PSL_2(\mathbb{R}) \) as a subgroup of the full group of isometries of \( \mathbb{P}^1 \). However, what we find useful is the fact that since we insist on using real coefficients, it follows that \( PSL_2(\mathbb{R}) \) also acts on \( S^1 = \mathbb{R} \cup \infty \), the boundary of \( \mathbb{H} \subset \mathbb{P}^1 \). Even stronger, we have the following:

**Lemma 1.2.2** \( PSL_2(\mathbb{R}) \) acts doubly transitively on \( S^1 = \mathbb{R} \cup \infty \). Moreover given three points \( p, q, r \) in \( S^1 \) then there exists an element \( \gamma \in PSL_2(\mathbb{R}) \) such that they map under \( \gamma \) (preserving their cyclic ordering) to \( \{0, 1, \infty\} \), or any cyclic permutation thereof.
Proof Without loss of generality, we assume that our two distinct points \((p, q)\) are in \(\mathbb{R}\). Consider the element
\[
\gamma(p, q)(z) := \pm \frac{z - p}{z - q}
\]
where the sign is chosen such that \(\pm (p - q) > 0\). This maps \(p\) to \(0\) and \(q\) to \(\infty\). As \(p, q\) were arbitrary, it follows that \(\gamma_{(a,b)} \circ \gamma(p,q)\) takes the pair \((p, q)\) to \((a, b)\) as claimed.

For the second part, note that the fractional linear transformation
\[
\xi(z) = \left( \frac{q - r}{q - p} \right) \frac{z - p}{z - r}
\]
takes the triplet \((p, q, r) \mapsto (0, 1, \infty)\), and that post-composing this with
\[
\gamma = \frac{z - 1}{z}
\]
cyclically permutes \((0, 1, \infty)\) as claimed.

Q.E.D.

As it turns out, this last fact will be the key argument in our proof of Theorem 1.1.4. First we need to know a bit more about the topology of \(PSL_2(\mathbb{R})\).

Lemma 1.2.3 As a topological space, \(PSL_2(\mathbb{R}) \cong \mathbb{H} \times S^1\).

We will provide two proofs of this lemma which will both be used later as we examine two parallel proofs of the main theorem.

Proof For any \(\frac{az + b}{cz + d} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL_2(\mathbb{R})\), we can decompose it as the composition of a rotation about \(i\) (a fractional linear transformation of the form \(\frac{\alpha z - \beta}{\beta z + \alpha}\)) and a translation (one of the form \(\alpha z + \beta\)). That is,
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} ad - bc & \alpha c + bd \\ \alpha c + bd & c^2 + d^2 \end{array} \right) \left( \begin{array}{cc} d & -c \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} c & d \\ 0 & 1 \end{array} \right)
\] (1.3)
This yields us an explicit map $\Theta : PSL_2(\mathbb{R}) \to \mathbb{H} \times S^1$ given by

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} = \begin{pmatrix}
    \alpha & \beta \\
    0 & 1
\end{pmatrix} \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix} \mapsto (\beta + i\alpha, \theta)
$$

Similarly, we have the map $\mathbb{H} \times S^1 \to PSL_2(\mathbb{R})$ by

$$(x + iy, \theta) \mapsto \begin{pmatrix}
    y & x \\
    0 & 1
\end{pmatrix} \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}$$

and these maps can be checked to be inverses of each other; They are clearly both continuous, and so the conclusion follows.

Q.E.D.

Second proof  Given $T \in PSL_2(\mathbb{R})$, we note that as $T$ maps $\mathbb{H}$ to itself that $T(i) \in \mathbb{H}$. Also, note that if $T(z) = \frac{az + b}{cz + d}$ then

$$
\frac{dT}{dz}(i) = \frac{ad - bc}{(ci + d)^2}
$$

which up to scaling is an element of $S^1$; to see this, note that we have the liberty to choose $c, d$ above such that $ci + d$ is in the upper half-plane together with the (strictly) positive real axis, and so squaring it we see that we can indeed get any element in $S^1$.

Combining all of the above, we define the map $\Psi : PSL_2(\mathbb{R}) \to \mathbb{H} \times S^1$

$$
T \mapsto \left( T(i), \frac{dT}{dz}(i) \right)
$$

which is then easily seen to be a homeomorphism.

Q.E.D.
1.2.2 Seifert-Fibred spaces

For the purposes of this thesis, the simplest definition that we need of a Seifert-Fibred space is that of a space with a nice $S^1$-action on it. Equivalently, we can consider it as a space that can be foliated by circles. More precisely, we can define a Seifert-fibred space in the following manner (We mostly follow [6]).

A **model Seifert fibring** of the solid torus $S^1 \times D^2$ is a decomposition of $S^1 \times D^2$ into disjoint copies of $S^1$ (called fibres) in the following way. If we consider $S^1 \times D^2$ as a quotient of $[0, \pi] \times D^2$, then instead of the usual identification of $\{0\} \times D^2$ and $\{\pi\} \times D^2$ we identify them with a twist of $2\pi p/q$ (that is, we instead identify $(0, x) \sim (\pi, e^{2\pi ip/q}x)$). We then retain an $S^1$ action in the obvious way (simply moving a point along its respective fibre), but due to the twist we end up with an extra detail—if we take a point in $S^1 \times D^2$ (other than one in the central fibre) and move it along the fibre than it returns to its starting point after $q$ times the number of passes it would take a point in the central fibre to return to its starting point.

With that in mind, we come to the following central definition.

**Definition 1.2.4** A **Seifert-Manifold** is a triple $(M, F, \pi)$ where $M$ is a 3-manifold, $F$ is a 2-dimensional surface (oriented or not, closed or not) and $\pi : M \to F$ is a map such that:

(i) For each $x \in F$, the preimage of $x$ under $\pi$ is homeomorphic to $S^1$.

(ii) For each $x \in F$ there is a $D^2$ neighbourhood of $x$ such that $\pi^{-1}(D^2)$ is fibre-preserving diffeomorphic to a model Seifert fibring as described above.

That is, $M$ is an "almost" locally trivial $S^1$ bundle—if we stay away from certain exceptional fibres, then the resulting space is a locally trivial (in fact, trivial) $S^1$ bundle.
Remark Note that there is a more general notion of a Seifert-Manifold in which $F$ is allowed to have a boundary—this corresponds to allowing a more general sort of model Seifert-fibring in which reflections as well as rotations are allowed in the gluing process.

With the above definition an $S^1$ action becomes immediately apparent; simply move a point along its fibre. Moreover, we can consider the space $F$ to be a quotient of $M$ determined by collapsing each fibre to a point. Conversely, if we are given a sufficiently nice $S^1$ action on a space $M$, then we obtain a map $M \to M/\mathbb{S}^1$ which is a Seifert fibreing; $M$ is clearly foliated by circles as well, the circles being the orbits of $S^1$. Thus we can be justified in saying that Seifert Manifolds are in 1-1 correspondence with $S^1$ actions on 3-manifolds.

Now, there is no a priori reason why this definition should permit us to accurately classify 3-manifolds—however, it turns out that the imposed structure is sufficient to provide quite a thorough classification of a great many manifolds, as follows.

The first thing to do is to identify the types of fibres that occur in our manifold. If we consider the model Seifert fibring containing $x$ on its central fibre, we have two cases. The first is that this tubular neighbourhood was obtained with a $2\pi p/q$ twist (with $q \neq \pm 1$), and the second was that it wasn't. In the former case we call it an exceptional fibre, and in the latter, a regular fibre. Under the assumption that $F$ be compact (if and only if $M$ is also compact), it can be shown that there are only finitely many exceptional fibres in a space.

Now, if we remove all of these tubular neighbourhoods then we obtain a trivial $S^1$ bundle over $F\setminus (D_1 \cup \cdots \cup D_k)$—this is since principal $S^1$ bundles over a space $X$
are in one-to-one correspondence with the elements of

\[ [X, BS^1] \cong [X, \mathbb{CP}^\infty] \]

\[ \cong [X, K(\mathbb{Z}, 2)] \]

\[ \cong H^2(X, \mathbb{Z}) \]

which in our case can be easily seen to be zero. Thus to construct our manifold \( M \), we consider the following information. Let \( g \) be the genus of our surface \( F \), and let \( (p_1/q_1, \ldots, p_k/q_k) \) be a collection of integers with \( \gcd(p_i, q_i) = 1 \). The idea then is to glue in solid tori \( T_i \) along each boundary according to the homology relation

\[ H_i \sim p_i A_i + q_i B_i \]

(1.4)

where \( H_i \) is the meridian of \( T_i \) and \( A_i \) and \( B_i \) are the respective generating homology classes of the \( i \)-th boundary component of \( F \setminus (D_1 \cup \cdots \cup D_k) \times S^1 \).

We then call \( (g; p_1/q_1, \ldots, p_k/q_k) \) the Seifert invariants of \( M \). Conversely, given a collection of Seifert invariants \( (g; p_1/q_1, \ldots, p_k/q_k) \), we denote by \( M(g; p_1/q_1, \ldots, p_k/q_k) \) the Seifert manifold constructed from these invariants as described above. With this in mind, we have the following classification theorem for Seifert manifolds (cf [6]).

**Theorem 1.2.5** Let \( (M, F, \pi) \) be a Seifert manifold with \( F \) closed and orientable. Then

1. \( M \) is fibre-preserving diffeomorphic to some \( M(g; p_1/q_1, \ldots, p_k/q_k) \) for some \( g, p_i, q_i \).

2. \( M(g; p_1/q_1, \ldots, p_k/q_k) \) and \( M(g'; p_1'/q_1', \ldots, p_k'/q_k') \) are fibre-preserving diffeomorphic if and only if

\[ (a) \; g = g' \]
(b) Ignoring any of the \( \frac{p_i}{q_i} \) and \( \frac{p'_i}{q'_i} \) which are integers (and not equal to \( \infty \)), the remaining \( \frac{p_i}{q_i} \) (mod 1) are a permutation of the remaining \( \frac{p'_i}{q'_i} \) (mod 1).

\[
\sum_{i=1}^{k} \frac{p_i}{q_i} = \sum_{i=1}^{l} \frac{p'_i}{q'_i}, \text{ where we use the convention that } 1/0 = -1/0 = \infty \\
\text{and } a + \infty = \infty \text{ for any } a \in \mathbb{R} \cup \{\infty\}.
\]

Note that this is not a complete classification; there is still the case where \( F \) is either not closed or not orientable to deal with. There are also a number of manifolds (see [4], [6], and the remark below) which have multiple non-isomorphic fibrations. However, for the purposes of this thesis this case is sufficient.

In particular though, we have the following.

**Theorem 1.2.6** The 3-sphere \( S^3 \) is a Seifert-fibred manifold with Seifert invariants \((0; 1/2, -1/3)\).

**Remark** This is not the only fibreing of \( S^3 \)—in fact, any \( p, q \) relatively prime will provide a fibreing with invariants \((0; 1/p, \pm 1/q)\) in a similar manner.

**Proof** We consider \( S^3 \) as the unit sphere in \( \mathbb{C}^2 \); Then we have the action of \( S^1 \) (seen as a subset of \( \mathbb{C} \)) via

\[
\lambda \cdot (z_1, z_2) = (\lambda^2 z_1, \lambda^{-3} z_2)
\]

which has exceptional fibres given by the orbits of \((1, 0)\) and \((0, 1)\) with multiplicity 2 and \(-3\) as claimed.

Q.E.D.

The reason this interests us is that \( \exp_k(S^1) \) has a natural \( S^1 \) action on it given by

\[
z \cdot \{x_1, \ldots, x_k\} = \{\zeta x_1, \ldots, \zeta x_k\}
\]
That is, rotation of our collection of points in a counter-clockwise direction about the circle. If one will take for granted for a moment that \( \exp_3(S^1) \) is a 3-manifold, we see that it is in particular a Seifert manifold and that the above classification theorem can be used to describe its homeomorphism type.

Now, if we have an \( S^1 \) action, then we should examine the exceptional fibres. It should be fairly easy to see that the collection of points of the form \( \{z, -z\} \)—antipodal points—will form an exceptional fibre of multiplicity 2, while the points of the form \( \{z, \zeta_3 z, \zeta_3^2 z\} \)—those that form the vertices of an equilateral triangle—will form one of multiplicity 3, as shown in figure 1.1.

Now, there is no real reason to stop here—if we have exceptional fibres in \( \exp_3(S^1) \), we should have analogous objects in \( \exp_k(S^1) \) for higher values of \( k \) (although \( \exp_k(S^1) \) is no longer a manifold past \( k = 3 \)). If nothing else, consider \( k \) equally spaced points; then this is certainly analogous to the two situations above.

However, the situation is more interesting than that. In fact, we have the following.

**Proposition 1.2.7** For each \( d \) which divides \( k \), \( C_k(S^1) \) contains a homeomorphic copy of \( C_d(S^1) \) such that rotation about \( S^1 \) by \( \frac{2\pi}{k} \) fixes this subspace. Moreover, \( C_{d_1}(S^1) \cap C_{d_2}(S^1) = C_{\gcd(d_1, d_2)}(S^1) \).
This can alternately be worded as saying that $C_k(S^1)$ contains exceptionally fibred subspaces $C_d(S^1)$ of multiplicity $\frac{k}{d}$ for every $d$ dividing $k$.

How can we see this? Well, the simplest non-trivial examples are $C_4(S^1)$ and $C_6(S^1)$; we will examine the latter, being the more interesting of the two.

For the copy of $C_3(S^1)$, consider figure 1.2. If we let $\Upsilon_3$ be the collection of pairwise antipodal points, then we have a map $\Upsilon_3 \to C_3([0, \pi] / 0 \sim \pi / 2)$ which is easily seen to be a homeomorphism.

![Figure 1.2: The inclusions of $C_4(S^1) \hookrightarrow C_6(S^1)$](image)

For the copy of $C_2(S^1)$, we instead consider $\Upsilon_2$ to be the subset of points that make up the vertices of equilateral triangles. Yet again there is a clear map $\Upsilon_2 \to C_2([0, \pi / 3] / 0 \sim \pi / 3)$ which is a homeomorphism.

Having seen these two cases, it is easy to see how to generalize this idea for arbitrary inclusions $C_d(S^1) \hookrightarrow C_k(S^1)$ whenever $d$ divides $k$. Note also that this means that we can learn quite a bit about these spaces by studying $C_p(S^1)$ for $p$ prime.
Chapter 2

The analysis of \( \exp_3(S^1) \)

2.1 The topology and geometry of \( C_3(S^1) \)

The main structure of the proof of Theorem 1.1.4 is as follows. For any three points of \( S^1 = \mathbb{R} \cup \infty \) there is an element of \( PSL_2(\mathbb{R}) \) (by lemma 1.2.2) which takes the points to \( \{0, 1, \infty\} \), preserving cyclic ordering. This element is not unique; post-composing it with any element which cyclically permutes \( (0, 1, \infty) \) yields another possible choice. If we let \( \Gamma \) be the subgroup of \( PSL_2(\mathbb{R}) \) generated by all elements cyclically permuting \( (0, 1, \infty) \), then this subgroup acts on \( PSL_2(\mathbb{R}) \) by left multiplication; if we quotient out by this action (choosing, in effect, the space of left cosets of \( \Gamma \) in \( PSL_2(\mathbb{R}) \)) then we obtain the following result:

**Lemma 2.1.1** The subgroup of \( PSL_2(\mathbb{R}) \) which cyclically permutes \( (0, 1, \infty) \) is simply the cyclic group \( \Gamma = \mathbb{Z}/3 \), generated by \( \gamma = \frac{z - 1}{z} \); thus

\[
C_3(S^1) \cong PSL_2(\mathbb{R})/\Gamma
\]

By similar reasoning we have that for any two points of \( S^1 \) there is an element of \( PSL_2(\mathbb{R}) \) taking those points to \( (0, \infty) \); if we then let \( \Xi \) be the subgroup of \( PSL_2(\mathbb{R}) \) which fixes the set \( \{0, \infty\} \) we similarly have the following:

**Lemma 2.1.2** The subgroup of \( PSL_2(\mathbb{R}) \) which setwise fixes \( \{0, \infty\} \) is generated by the elements \( \tau = -\frac{1}{z} \) and \( \sigma_\lambda = \lambda z \) (for \( \lambda \in \mathbb{R}_{>0} \)). In matrix form these are the
elements of the form \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \), and they generate the subgroup

\[ \Xi = \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \begin{pmatrix} 0 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix} \mid \lambda > 0 \right\rangle \]

and so we have

\[ C_2(S^1) \cong \text{PSL}_2(\mathbb{R}) / \Xi \]

And of course, \( C_1(S^1) \) is simply the circle \( S^1 \).

This tells us less than half of the story, however. We also need to understand how these quotients behave topologically, as well as determine how these three pieces glue together to form \( \exp_3(S^1) \) (cf. eq (1.1)).

We first need a quick lemma, however.

**Lemma 2.1.3** If \( \alpha \in \text{PSL}_2(\mathbb{R}) \) fixes three points, then \( \alpha(z) = z \) for all \( z \).

**Proof** This follows since if we have

\[ \alpha(z) = \frac{az + b}{cz + d} = z \]

then this is equivalent to

\[ cz^2 + (d - a)z - b = 0 \]

which of course has only two (or less) solutions unless \( c = b = 0, a = d \).

Q.E.D.

**Proof of lemmas 2.1.1, 2.1.2** Let \( \alpha \neq 1 \) be any element permuting the set \( \{0, 1, \infty\} \) (and of course preserving the cyclic ordering); note that as \( \alpha^3 \) has to fix this set, it must be the identity by the previous lemma (if \( \alpha^2 \) were the identity, then it would reverse orientations). Choose an appropriate power \( k \) of \( \alpha \) such that \( (0, 1, \infty) \) maps under \( \alpha^k \) to \( (1, \infty, 0) \)—thus \( \alpha^k \circ \gamma \) fixes \( (0, 1, \infty) \) and so by the lemma above, it must
be the identity. That is, either \( \alpha = \gamma^{-1} \) or \( \alpha^2 = \gamma^{-1} = \gamma^2 \). It then follows that \( \Gamma = \langle \gamma \rangle \cong \mathbb{Z}/3 \).

For the case of \( \{0, \infty\} \), it is clear that any element permuting these two points either fixes or swaps them. If it fixes them, then it has to be of the form \( \lambda z \); otherwise it has to be of the form \(-\frac{A}{z}\) (in both cases, \( \lambda > 0 \)), which proves the lemmas.

Q.E.D.

Before we move onto a topological description of these quotients, a quick word about orientation should be mentioned. It may appear in the above like we are only quotienting out cyclic permutations of \( \{0, 1, \infty\} \), and so we shouldn’t get the full configuration space, which is in fact \(((S^1)^3 - \Delta)/\Sigma_3\), where \( \Delta \) is the fat diagonal. This is, however, not the case.

There are two ways to see this. The first is that there is an implicit ordering of points around a circle, and that we simply choose those isometries which preserve that ordering.

The second view is somewhat more explicit, however. In principle we should begin with the collection of all isometries of \( \mathbb{H} \)—that is, the group \( G \) generated by \( PSL_2(\mathbb{R}) \) and \( f(z) = 1/z \)—and quotient out as before the subgroup of isometries which permute the set \( \{0, 1, \infty\} \). This subgroup can be shown to be isomorphic to \( \Sigma_3 = \langle a, b \mid a^3 = b^2 = bab = 1 \rangle \), the permutations on three letters. Now, this group \( G \) is topologically the disjoint union of two copies of \( PSL_2(\mathbb{R}) \)—one copy representing those elements which reverse orientation (ie those involving \( \bar{z} \)), and the other those that preserve it. Thus when we quotient out by the action of \( b = 1/\bar{z} \), it simply identifies the two copies of \( PSL_2(\mathbb{R}) \), and we are left with quotienting out the resultant by \( a = \gamma \) (as above). Thus our initial choice of \( PSL_2(\mathbb{R}) \), despite its restriction to orientation preserving isometries, is in fact the correct one.
We will now begin with a topological description of these configuration spaces. However, we will present two different methods of looking at them, one involving the natural coordinates obtained from $PSL_2(\mathbb{R})$ and the second more coordinate independent, involving the second proof of lemma 1.2.3.

2.1.1 Coordinate-based description of topology

We begin with $C_2(S^1)$, the simplest of the two.

**Proposition 2.1.4** $C_2(S^1)$, described as the quotient $PSL_2(\mathbb{R})/\Xi$ above, is homeomorphic to the open Möbius band $M$.

**Proof** Recall that $\Xi$ is the group generated by $\tau$ and the $\sigma_\lambda$ (for $\lambda > 0$), subject to the relations

\[
\tau \sigma_\lambda \tau = \sigma_\lambda^{-1} = \sigma_\lambda^{-1}
\]

\[
\sigma_\lambda \sigma_\mu = \sigma_{\lambda \mu}
\]

\[
\tau^2 = 1
\]

and so in particular from the first of those it suffices to consider the actions of $\sigma_\lambda$ and $\tau$ separately.

Writing both of these as matrices we have

\[
\sigma_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and so examining the action of the former on an element of $PSL_2(\mathbb{R})$ we have that

\[
\sigma_\lambda \cdot \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \lambda y & \lambda x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]
and thus via our action we have that, as elements of \( \mathbb{H} \times S^1 \), \((x + iy, \theta) \sim (\lambda(x + iy), \theta)\) for any \( \lambda > 0 \). Thus if we quotient out by this action we find that \( PSL_2(\mathbb{R})/\langle \sigma \lambda \rangle \cong (0, \pi) \times S^1 \) via the map (see figure 2.1)

\[
[x + iy, \theta] \mapsto (\phi, \theta)
\]

where the square brackets denote the equivalence class, and \( \phi = \arg(x + iy) \).

![Figure 2.1: arg(x + iy)](image)

As for the action of \( \tau \), we first note that by (1.3) we have

\[
\tau \cdot \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix}
\]

and so combining these two actions, and noting that \( \begin{pmatrix} y & -x \\ x & y \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \) corresponds to the angle \( \theta - \phi \) we find that the whole quotient is simply

\[
(0, \pi) \times S^1/(\phi, \theta) \sim (\pi - \phi, \theta - \phi)
\]

which can be represented pictorially as in figure 2.2 (where the numbered triangles are identified with one another), showing that \( PSL_2(\mathbb{R})/\Sigma \cong \mathbb{RP}^2 \setminus \{*\} \cong \mathcal{M} \) as claimed.

Q.E.D.
Proposition 2.1.4 can also be proven in any number of ways; there is a quite simple pictoral proof of the matter in [11].

We now move onto the more complicated case (which, unfortunately, does not have such a simple proof as the one cited above). To aid in its description we will use the machinery of Seifert-fibred spaces described in section 1.2.2—this turns out to be a very natural sort of mechanism to describe $\exp_k(S^1)$ for $k \leq 3$.

**Proposition 2.1.5** $C^*_S(1)$, described as a the quotient $PSL_2(\mathbb{R})/\Gamma$ above, is homeomorphic to the (open) model Seifert fibreing with twist $2\pi/3$.

**Proof** Recall that $\Gamma = \langle \gamma \rangle$ where $\gamma = \frac{z-1}{z} = (1 -1 \; 1 \; 0)$. We begin by computing the action of $\gamma$, and note that

$$
\gamma \cdot \begin{pmatrix} y \\ x \\ 0 \\ 1
\end{pmatrix} = \begin{pmatrix} \frac{y}{x^2+y^2} & 1 - \frac{x}{x^2+y^2} \\ 0 & 1 \\ y & x
\end{pmatrix} \begin{pmatrix} x \\ -y \\ 0 \\ 1
\end{pmatrix}
$$

and thus if we let $\phi$ be as before (cf. figure 2.1) our isomorphism yields that

$$(x + iy, \theta) \sim \left(1 - \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}, \theta - \phi \right)$$
We end up then with the following description of the action of $\gamma$ on $PSL_2(\mathbb{R})$. In the $\mathbb{H}$ coordinate, it is the composition of two reflections—first around the geodesic $x^2 + y^2 = 1$, and then across the line $x = \frac{1}{2}$—and thus is a rotation about the point $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ by an angle of $\frac{4\pi}{3}$.

In the $S^1$ coordinate, $\gamma$ shifts a point $(x + iy, \theta)$ back by the angle between $x + iy$ and the positive $x$-axis, as seen in figure 2.3 (recall that $S^1 = [0, \pi]/\sim$).

![Figure 2.3: The action of $\gamma$ on the $S^1$ coordinate of $PSL_2(\mathbb{R})$](image)

To see that this is then homeomorphic to the model Seifert-fibreing stated above, we first choose an explicit description of that model, which will be

$$Q := \mathbb{H} \times [0, 1]/\sim \quad (z, 1) \sim \left(1 - \frac{1}{z}, 0\right)$$

That is, a model Seifert-fibre with twist $\frac{4\pi}{3}$.

Now, I first claim that the set $F = \{(z, \theta) \in PSL_2(\mathbb{R}) \mid 0 \leq \theta < \arg(z)\}$ forms a fundamental domain for the action of the group $\Gamma$. To see this we begin by noting that under the action of the group $\Gamma$ we have the identifications

$$(z, \theta) \sim \left(\frac{z - 1}{z}, \theta - \arg(z)\right) \quad (2.2a)$$

$$\sim \left(\frac{-1}{z - 1}, \theta - \arg(z - 1)\right) \quad (2.2b)$$

Thus noting that $0 \leq \theta < \pi$ we have two cases to deal with, assuming that a pair $(z, \theta) \notin F$. 


(i) \( \arg(z) \leq \theta < \arg(z - 1) \)

In this case we look at (2.2a). Since \( \arg(z) \leq \theta \) we have that \( \theta - \arg(z) \geq 0 \). Next, we see that

\[
\arg\left(\frac{z - 1}{z}\right) = \arg(z - 1) - \arg(z) > \theta - \arg(z)
\]

and so this element here is in \( F \) as claimed.

(ii) \( \arg(z - 1) \leq \theta < \pi \)

In this case we look at (2.2b). Similar to the first case, we have that \( \theta - \arg(z - 1) \geq 0 \). Moreover, we have that

\[
\arg\left(\frac{-1}{z - 1}\right) = \pi - \arg(z - 1) > \theta - \arg(z - 1)
\]

and thus the claim follows.

With \( F \) identified as a fundamental domain, it is easy to see that this space is homeomorphic to \( Q \) above. Simply define a map \( C^3(S^1) \rightarrow Q \) by mapping the appropriate \((z, \theta) \in F\) to \((z, \frac{1}{\arg(z)} \theta)\); this is clearly a homeomorphism.

Q.E.D.

2.1.2 Coordinate-independent descriptions of topology

In this section we simply re-prove propositions 2.1.4 and 2.1.5 using more geometric rather than coordinate based methods.

Proof 2 of proposition 2.1.4 As before, we have that \( \Xi \) is generated by \( \tau \) and the \( \sigma_\lambda \), subject to the relations given in (2.1), and so it suffices to consider the actions of \( \sigma_\lambda \) and \( \tau \) separately.
We again consider the action of $\sigma_\lambda$ first. Now we have both

$$ (\sigma_\lambda \circ T)(i) = \lambda(T(i)) $$

$$ \frac{d(\sigma_\lambda \circ T)}{dz}(i) = \lambda \frac{dT}{dz}(i) $$

and so the action of $\sigma_\lambda$ is given by $\sigma_\lambda \cdot (z, \theta) = (\lambda z, \theta)$. If we then quotient out by the action of the subgroup of $E$ generated by the $\sigma_\lambda$'s we find

$$ PSL_2(\mathbb{R})/\langle \sigma_\lambda \rangle \cong (0, \pi) \times S^1 $$

where the angle $\phi \in (0, \pi)$ is once again as in figure 2.1.

Moving on to the action of $\tau$, we note that

$$ (\tau \circ T)(i) = -\frac{1}{T(i)} $$

$$ \frac{d(\tau \circ T)}{dz}(i) = \frac{1}{T^2(i)} \frac{dT}{dz}(i) $$

and so since up to scaling (as an element of $(0, \pi)$) $-\frac{1}{T(i)} = \pi - \phi$, and similarly $\frac{1}{T^2(i)} = -2\phi$, we find that

$$ PSL_2(\mathbb{R})/\Xi \cong (0, \pi) \times S^1/(\phi, \theta) \sim (\pi - \phi, \theta - 2\phi) $$

which can be pictorially represented as in figure 2.2 (subject to suitable rescaling), whence follows the conclusion.

Q.E.D.

We also have the following succinct proof.

Proof 3 of proposition 2.1.4. Given two ordered points $(p, q)$ in $S^1$, let $z$ be the point halfway between the two as shown in figure 2.4. Let $\theta$ be the angle from $p$ to $q$. This yields a homeomorphism between the pairs of distinct ordered points in $S^1$ to
$S^1 \times (0,2\pi)$. Now, when we quotient out by the action of $\Sigma_2$, then we find that $(z, \theta) \sim (-z, 2\pi - \theta)$. Thus

$$C_2(S^1) \cong S^1 \times (0,2\pi)/(\psi, \theta) \sim (\pi + \psi, 2\pi - \theta).$$

which is easily seen to be the Möbius band.

![Diagram of C_2(S^1)](image)

Figure 2.4: Geometry of $C_2(S^1)$

Lastly, we move onto the case of $C_3(S^1)$.

**Proof 2 of proposition 2.1.5** The proof of this theorem is identical in method to the second proof of proposition 2.1.4. Since $\gamma = 1 - \frac{1}{z}$, we have immediately that

$$(\gamma \circ T)(i) = 1 - \frac{1}{\Lambda(i)}$$

$$\frac{d(\gamma \circ T)}{dz}(i) = \frac{1}{T^2(i)} \frac{dT}{dz}(i)$$

and so $\gamma$ does the same to the $S^1$ coordinate as $\tau$. Now, in the $\mathbb{H}$ coordinate, it turns out that $1 - \frac{1}{z}$ can be written as $R_1 \circ R_2$ where

$$R_1(z) = 1 - \overline{z} \quad R_2(z) = \frac{z}{|z|^2}$$
and so it is the composition of two reflections, first about the circle \(|z| = 1\) and then about the line \(\Re z = \frac{1}{2}\)—it is thus a rotation of the angle \(\frac{4\pi}{3}\). As with the previous proof, it follows that this is indeed homeomorphic to our model Seifert fibreing as claimed.

Q.E.D.

2.1.3 Topology of the union \(\exp_k(S^1)\)

At this point we have a solid grasp (via a variety of methods) as to exactly what the three pieces \(C_i(S^1), i = 1, 2, 3\) are. Remaining still is to describe the topology of their union—that is, what happens when points begin to coalesce.

We begin by looking at \(\exp_2(S^1)\). First, in \(\mathcal{M} \cong C_2(S^1)\), we find that a pair \(\{p, r\}\) maps to the element

\[
\xi(z) = \frac{z - p}{z - r}
\]

in \(PSL_2(\mathbb{R})\), and so using the homeomorphism \(\Psi\) from lemma 1.2.3 we see first that

\[
\xi(i) = \frac{i - p}{i - r} \quad \frac{d\xi}{dz}(i) = \frac{r - p}{(i + r)^2}
\]

and so that as \(p \to r, \xi(i) \to 1\), while \(\frac{d\xi}{dz}(i) = \frac{r - p}{(i + r)^2}\), since we can ignore scaling, stays constant.

We can use this to prove the first part of our gluing process.

**Proposition 2.1.6** The space \(\exp_2(S^1) = C_1(S^1) \cup C_2(S^1)\) is homeomorphic to the closed Möbius band \(\mathcal{M}^*\).

**Proof** We consider the closed Möbius band to be the quotient of \([0, \pi] \times S^1\) modulo the action of \(\mathbb{Z}/2\) via \((\phi, \theta) \sim (\pi - \phi, \theta - 2\phi)\) (cf. the second proof of proposition 2.1.4). We then have a clear embedding of \(C_2(S^1) \hookrightarrow \mathcal{M}^*\).

Now, from the comments above it is easy to see that as \(p \to r, \phi \to 0\) (see figure 2.1) and \(\theta\) stays constant—and thus that \(C_1(S^1) \cup C_2(S^1) \cong \mathcal{M}^*\) as claimed.
We now need to move on to the situation with three points. Recalling back to lemma 1.2.2 (and in particular, equation (1.2)) our triple \{p, q, r\} (with \(p < q < r\), without loss of generality) maps to the element

\[
\xi(z) = \left( \frac{q-r}{q-p} \right) \frac{z-p}{z-r}
\]

which, described in terms of equation (1.3) becomes

\[
\xi = \begin{pmatrix}
\frac{(q-r)(p-r)}{(q-p)(1+r^2)} & \frac{(q-r)(1+pr)}{(q-p)(1+r^2)} \\
0 & 1
\end{pmatrix} \begin{pmatrix} r & 1 \\ -1 & r \end{pmatrix}
\]

(2.3)

We note again that as in the case of two points, the position in the \(S^1\) coordinate only depends on a single one of these points—in this case, \(r\).

Now, the gluing together of \(\exp_2(S^1)\) and \(C_3(S^1)\) is somewhat of a surprise and is strongly reminiscent of the notion of a blowup from Algebraic Geometry. Recall that the blowup of a point \(p\) in an \(n\)-dimensional variety \(V\) is a space \(BL_p(V)\) together with a map \(\pi : BL_p(V) \to V\) which separates the lines through \(V\); that is, away from \(\pi^{-1}(p)\) this map is a homeomorphism, while \(\pi^{-1}(p)\) is a copy of \((n-1)\)-dimensional projective space. The key idea, of course, is that the distinct lines through a point become separated in the resulting space, and this is what we will see when we glue \(\exp_2(S^1)\) onto \(C_3(S^1)\).

Before we move on though, we should first once and for all select a region of \(PSL_2(\mathbb{R})\) to use as our fundamental domain for the action of \(\Gamma\); that region will be the one shown in figure 2.5, crossed with \(S^1\) and with top and bottom edges identified as per the identifications given in proposition 2.1.5. Note also that the round corners are exaggerated for reasons which will momentarily become clear.

To aid in our description, we will assume that it is the point \(q\) which tends towards either \(p\) or \(r\). That being the case, it should be noted that as \(q\) varies, the quantity

Q.E.D.
Figure 2.5: Choice of Fundamental Domain

\( \theta \) varies between \(-\infty\) and 0. As such (see equation (2.3)), by fixing both \( p \) and \( r \) we trace out a path in a particular \( \mathbb{H} \) slice which happens to be a straight line from the point \( 0 + i0 \) of slope \( \frac{p-r}{1+pr} \). Letting \( q \to r \) (for simplicity's sake) we approach either \( 0 + i0 \) or \( 1 + i0 \), depending on the sign of \( \frac{p-r}{1+pr} \) (Consider our choice of fundamental domain). Then the Möbius band is glued on to \( C_3(S^1) \) entirely at the points \( 0 + i0 \) and \( 1 + i0 \)—however, the point which we end up at on the Möbius band depends on the slope of the path we take towards these points, and so much like a blowup we find that these points are in fact separated by the set of lines passing through them, which are then glued to the corresponding point on the open Möbius band.

But what about \( C_1(S^1) = \partial(\mathcal{M}^*) \), and what of the edge running from \( 0 + i0 \) to \( 1 + i0 \)? The first thing to note is that as the union \( \text{exp}_3(S^1) \) must be compact, any sequence \( \{p_i, q_i, r_i\} \) which tends to that edge must necessarily converge to an element of \( C_1(S^1) \) in the union. The only question remaining is what that element is.

Let \( \{p_i, q_i, r_i\} \) be a sequence in \( \text{exp}_3(S^1) \) which converges (regarded as a point
in the quotient $\mathbb{H} \times S^1/\Gamma$) to some point $P = (\lambda + i0, \theta)$ where $\lambda \in (0, 1)$, $\theta \in [0, \pi)$. Choose a neighbourhood $N$ of $P$ such that $N$ is completely contained in the fundamental domain of figure 2.5. As such, each $\{p_i, q_i, r_i\}$ is eventually contained in this neighbourhood, and thus in our particular choice of fundamental domain.

However, by our choice of fundamental domain there is an explicit ordering now on these points, and thus the $\mathbb{H}$ slice that any $\{p_i, q_i, r_i\} = [(p_i, q_i, r_i)]$ finds itself in is determined by a particular one of these points, say $r_i$. As this converges towards $(\lambda + i0, \theta)$, we must have that $r_i \to \theta$ (as well as $p_i, q_i$). That is, the point in $C_1(S^1)$ that we converge to is simply the point that all of $p_i, r_i, q_i$ are converging to. Thus in $\exp_3(S^1)$, the edge $(0 + i0, 1 + i0) \times \{\theta\}$ simply collapses to $\{\ast\} \times \{\theta\}$.

### 2.2 $\exp_3(S^1)$ and the inclusion $\exp_1(S^1) \hookrightarrow \exp_3(S^1)$

We will now prove (after a quick lemma) the main result of this thesis, a stronger version of Theorem 1.1.4 (proven also in [11]).

**Lemma 2.2.1** The space $\exp_3(S^1)$ is a compact 3-manifold without boundary.

**Proof** Compactness is immediate as $\exp_3(S^1)$ is a quotient of $(S^1)^3$. As for it being a manifold, the only place where this might fail is on $\exp_2(S^1) \subset \exp_3(S^1)$, so we simply need to verify that each point therein has a euclidean neighbourhood.

For points in $C_2(S^1)$ this is rather easy. A point in a neighbourhood of $\{p, q\}$ for $p \neq q$ will be in one of the configurations shown in figure 2.6, and so it is fairly easy to see that there is a neighbourhood of $\{p, q\}$ which is homeomorphic to two copies of $(-\epsilon, \epsilon) \times C_2((-\epsilon, \epsilon))$ glued along their common boundary—which is then homeomorphic to $(-\epsilon, \epsilon)^3$ as required.

For a point in $C_1(S^1)$, it is similarly easy to see that it has a neighbourhood homeomorphic to $\exp_3((-\epsilon, \epsilon))$ which considered as a quotient of the space $X =$
Figure 2.6: A neighbourhood of \( \{p, q\} \)

\[ \{(x, y, z) \mid -\epsilon < x \leq y \leq z < \epsilon\} \] is also readily seen to be homeomorphic to a euclidean ball.

Q.E.D.

**Theorem 2.2.2** The space \( \exp_3(S^1) \) is homeomorphic to \( S^3 \), and the inclusion \( S^1 \cong \exp_1(S^1) \hookrightarrow \exp_3(S^1) \cong S^3 \) is the trefoil knot.

**Proof** The majority of this proof will rely on calculating the fundamental groups of \( \exp_3(S^1) \) and \( \exp_3(S^1) \setminus \exp_1(S^1) \), relying on classification theorems to show the above result. The main tool will be the Seifert-Van Kampen theorem.

From the covering shown in figure 2.7, the Seifert-Van Kampen theorem yields the following pushout of groups.

\[
\begin{array}{cc}
\pi_1(A \cap B) & \overset{i_*}{\longrightarrow} & \pi_1(B) \\
\downarrow j_* & & \downarrow \\
\pi_1(A) & \longrightarrow & \pi_1(\exp_3(S^1))
\end{array}
\]

where \( A \) deformation retracts onto a circle (and so \( \pi_1(A) \cong \langle s \rangle \)), and \( B \) deformation retracts onto \( M^* \cong S^1 \) (hence \( \pi_1(B) \cong \langle t \rangle \)); thus we have that \( \pi_1(\exp_3(S^1)) \cong \langle s, t \mid R \rangle \). It remains to determine what the relations \( R \) are.
Figure 2.7: The covering used for the Seifert-Van Kampen theorem

Now, $A \cap B$ is simply the "boundary" of $C_3(S^1)$; that is, up to homotopy it is simply a torus $T^2$. Thus its fundamental group is $\langle a \rangle \oplus \langle b \rangle$ where $a$ is the generator in the $S^1$ direction, and $b$ is the meridional generator.

We can explicitly describe this homotopy torus in terms of points on the underlying circle $S^1$ in the following manner. The longitudinal direction (ie the one corresponding to $a$ above) is given, as expected, simply by rotation of points along $S^1$; as we are avoiding the exceptional fibre, there is nothing unusual here and we need a full rotation at any given point to return to the starting point.

Now, we obtain the other generator $b$ (demonstrated in figure 2.8) simply by rotating each point in a counter-clockwise manner to the next point along. It is easy to see that this commutes with $a$, and that together these two paths make a torus.

So to determine the relations $R$, we first examine $i_*(a)$ and $j_*(a)$, the simplest of the two to deal with. Now, as the generator of $\pi_1(A)$ is the path along the exceptional fibre from $\theta = 0$ to $\theta = \frac{\pi}{3}$, it is easy to see that $j_*(a) = s^3$. The exact same reasoning shows that $i_*(a) = t^2$, and so we have the relation that $s^3 = t^2$.

So for the meridional generator, $b$, we have the situation shown in figure 2.9 (cf.
Figure 2.8: The generator $b$ of $\pi_1(A \cap B)$

figure 2.2) which can easily be seen to be homotopic to the generator $t$ of $\pi_1(B)$; thus it only remains to see what happens to $j_*(b)$ (shown in figure 2.10) to fully understand what $\pi_1(\exp_3(S^1))$ is. While it would be tempting to suggest that it simply collapses to a homotopically trivial path, this is not indeed the case.

If we restrict the front end of this path to have $S^1$ coordinate of 0, then recalling the proof of Proposition 2.1.5 we find that as we contract our path towards the center (that is, the exceptional fibre) of $A$ the $S^1$ coordinate of our back end begins to recede—until finally, when our path has been deformed completely to the center we find that it has become a path from $-\frac{\pi}{3}$ to 0; that is, the generator $s$ of $\pi_1(A)$.

As an aside, it is worth noting that figure 2.8 gives an alternate geometric idea as
to why $j_*(b)$ is the generator of $\pi_1(A)$. If we perturb that diagram so that our three points are equally spaced about the circle, then it is easy to see that the description of the path $b$ is exactly the path which generates $\pi_1(A)$.

Combining all of this together, we find that

$$\pi_1(\exp_3(S^1)) \cong \langle s, t \mid s^3 = t^2, s = t \rangle \cong 1$$

and so $\exp_3(S^1)$ is simply connected. From [10] it follows immediately that as a simply connected Seifert fibred space, this must be homeomorphic to $S^3$.

For brevity we will now define $X := \exp_3(S^1) \setminus \exp_1(S^1)$. Now, for the calculation of $\pi_1(X)$ the majority of the details above still hold through—the only difference is that what was labelled as $B$ above (now to be denoted $B'$) has become slightly different. In the calculation of $\pi_1(\exp_3(S^1))$, we claimed that $B$ deformation retracts onto $M^*$; this fails for $B'$.

Claim: $\pi_1(B') \cong \langle t, u \mid [t^2, u] = 1 \rangle$, where $t$ is the generator of $\pi_1(M)$ and $u$ is simply the image of the meridional generator in $B'$. 
From this claim it follows that

$$\pi_1(X) \cong \langle s, t, u \mid [t^2, u] = 1, s^3 = t^2, s = u \rangle$$

$$\cong \langle s, t \mid [t^2, s] = 1, s^3 = t^2 \rangle$$

$$\cong \langle s, t \mid s^3 = t^2 \rangle$$

Now, to see that this implies that the inclusion $\exp_1(S^1) \hookrightarrow \exp_3(S^1)$ is the trefoil knot, we proceed as follows. The first thing is to note that the center of the Möbius band—its exceptional fibre—is unkotted in $\exp_3(S^1)$. This is due to the fact that $\pi_1(C_3(S^1)) \cong \mathbb{Z}$. Thus if we consider a tubular neighbourhood of this subset we obtain a torus in $\exp_3(S^1)$—and the intersection of the boundary of this torus with the Möbius band is thus a torus knot which is isotopic to $\exp_1(S^1)$. We can now use the fundamental group to say that it is a $(2,3)$ torus knot, or a trefoil knot as claimed.

Proof of claim: Let us examine $B'$ a little more closely. Figure 2.11 shows a slice of $B'$, separated into open sets $U, V$.

![Figure 2.11: Covering of $B'$](image)

Now, $U \approx T^2$, and $V \approx M$. Lastly, $U \cap V \approx S^1$, and so we end up with the following pushout to calculate $\pi_1(B')$: 
from which it follows that

\[ \pi_1(B') \cong \langle a, b, c \mid [a, b] = 1, c^2 = a \rangle \cong \langle b, c \mid [c^2, b] = 1 \rangle \]

as claimed; Clearly, \( c \) is the generator of \( \pi_1(M) \), and \( b \) is the meridional generator and so the full claim follows.

Q.E.D.
Chapter 3

Further Research

Having determined the homeomorphism type of $\exp_k(S^1)$ for $k = 1, 2, 3$, the next obvious goal is either to determine the homeomorphism type of these spaces for higher and higher values of $k$, or alternately, to move on to see if the method generalizes for higher dimensional spheres.

The former generalization would conceivably involve considering certain bundles over $PSL_2(\mathbb{R})$—recall that in $PSL_2(\mathbb{R})$ we are limited to choosing fractional linear transformations which take at most three arbitrary points to a select group of points, and as such we cannot directly continue the process used above.

However, if we let $\{p < q < r < s\} \in C_4(S^1)$, then there remains still a fractional linear transformation which takes three of these points to $\{0, 1, \infty\}$ as considered before. What we can then do is consider what happens to that fourth point. If we let $T$ be this given transformation, then $T(s) \in \mathbb{H}$ and in particular, it lies on the geodesic between (say) 0 and $\infty$. Thus for fixed $p < q < r$ there is an $\mathbb{R}$ degree of freedom in choosing $s$. If we consider these points as ordered, then we have an $\mathbb{R}$ bundle over the ordered configuration spaces $F_3(S^1)$, and so we simply have to consider what happens when we quotient out by the action of the group of isometries which permute these points.

This works in general; in fact for sufficiently nice spaces $X$, if we let $F_k(X)$ denote the $k$-th ordered configuration space of $X$, then we obtain a sequence of fibrations
given by

\[ X \setminus \{p_1, \ldots, p_{k-1}\} \hookrightarrow F_k(X) \xrightarrow{\pi} F_{k-1}(X) \]

where the projection map \( \pi \) is given by \((x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k-1})\). It should be noted that for \( X \cong S^n \), this is in fact a fibre bundle.

This being said, the generalization that we intend to follow is that of considering higher dimensional spheres first. If we consider \( S^2 \) in particular, then \( S^2 \) is the boundary of \( \mathbb{H}^3 \) and more importantly, the group of isometries of this spaces is then going to be given by \( PSL_2(\mathbb{C}) \)—as such the general methodology implemented above, as well as a number of the calculations used, will still be valid for this case, making it a more natural generalization to approach first.

This leads us to examine yet again the first step. What is the topology of \( PSL_2(\mathbb{C}) \)?

We push this question aside for the moment and see what we can determine about the spaces \( C_i(S^2) \cong F_i(S^2)/\Sigma_i \). We can begin with at least obtaining a rough idea using the fibrations given above. From the first one

\[ S^2 \setminus \{\ast\} \hookrightarrow F_2(S^2) \rightarrow F_1(S^2) = S^2 \]

we obtain that \( F_2(S^2) \cong S^2 \); the second fibration is then (up to homotopy)

\[ S^1 \rightarrow F_3(S^2) \rightarrow S^2 \]

and so we can use either the Serre spectral sequence to compute the homology or the long exact sequence of homotopy groups to help understand this space.

In the latter approach, since \( \pi_i(S^1) \cong 0 \) for \( i > 1 \), we immediately find that

\[ \pi_i(F_3(S^2)) \cong \pi_i(S^2) \]

for all \( i > 2 \), as well as a (somewhat short) exact sequence

\[ 0 \rightarrow \pi_2(F_3(S^2)) \rightarrow \pi_2(S^2) \rightarrow \pi_1(S^1) \rightarrow \pi_1(F_3(S^2)) \rightarrow 0 \]
or

\[ 0 \to \pi_2(F_3(S^2)) \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to \pi_1(F_3(S^2)) \to 0 \]

and so \( \pi_2(F_3(S^2)) \cong \ker(k) \) and \( \pi_1(F_3(S^2)) \cong \text{coker}(k) \). So determining the nature of this map \( k \) tells us all of the homotopy groups of \( F_3(S^2) \).

So what is \( k \)? Well, to track it down we have to examining a chain of isomorphisms; namely

\[
\pi_2(F_3(S^2), S^2 \setminus \{0, \infty\}) \cong \pi_2(F_2(S^2), (0, \infty)) \cong \pi_2(F_2(S^2), S^2 \setminus \{0\}) \cong \pi_2(F_1(S^2), 0) \cong \pi_2(S^2)
\]

all of which come from the homotopy exact sequences and isomorphisms in the diagram in figure 3.1, where the vertical column comes from the exact sequence of relative homotopy groups, and the horizontal isomorphisms come from the isomorphism \( \pi_n(X, A) \cong \pi_n(B) \) for a fibration, and from the homeomorphism \( F_1(X) \cong X \).

If we then trace the generator of \( \pi_2(S^2) \) back through all of these isomorphisms, we find that \( k \) must simply be multiplication by 0. This of course implies that \( \pi_1(F_3(S^2)) \cong \mathbb{Z} \cong \pi_2(F_3(S^2)) \).

Remark This can also be seen by noting that it is necessary that \( \pi_1(F_3(S^2)) \cong \mathbb{Z} \).

In fact, we can describe a generator explicitly. Let \( (0, 1, \infty) \in F_3(\mathbb{P}^1) \) be the chosen basepoint; then the path \( t \mapsto (0, e^{2\pi it}, \infty) \) is a generator of an infinite cyclic subgroup of \( \pi_1(F_3(S^2)) \). As this must also be a quotient of \( \mathbb{Z} \), it follows that the map \( k \) in the exact sequence above must be 0.
\[ \pi_2(S^2 \setminus \{0\}) \]

\[ \pi_2(F_3(S^2), S^2 \setminus \{0, \infty\}) \xrightarrow{\cong} \pi_2(F_2(S^2), (0, \infty)) \]

\[ \cong \]

\[ \pi_2(F_2(S^2), S^2 \setminus \{0\}) \xrightarrow{\cong} \pi_2(F_1(S^2), 0) \cong \pi_2(S^2) \]

\[ \pi_1(S^2 \setminus \{0\}) \]

\[ \vdots \]

Figure 3.1: Isomorphisms due to exact sequences
Bibliography


