MACDONALD CHARACTERS OF WEYL GROUPS OF RANK ≤ 4

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the department

of

MATHEMATICS

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August 1973

Supervisor: Dr.B.Chang

ABSTRACT

In this thesis we obtain all the ordinary irreducible characters of Weyl groups of rank ≤ 4 by using MacDonald's method.This method enables us to find almost all the characters, and the remaining ones may be obtained by combining MacDonald characters and characters of exterior products of the reflection representation.

TABLE OF CONTENTS

LIST OF TABLES

ACKNOWLEDGMENT

I would like to express my gratitude to Dr. B.Chang for suggesting the topic of this thesis and for all the help he so readily offered during its preparation.I would also like to thank Dr.R.Ree for reading the thesis.

INTRODUCTION

In a paper entitled "Some Irreducible Representations of Weyl Groups" [5], I.G.MacDonald describes the following construction which gives many, but in general not all of the irreducible representations of a Weyl group.

Let V be a finite-dimensional vector space over the rational field, and suppose V has a positive-definite inner product. Let V* be the dual of V. Let R be a root system of a Weyl group (R) and let S be a subsystem of R. For a given ordering on R, let $\Pi(S)$ denote the product of all the positive roots of S. Then $\Pi(S)$ is a homogeneous rational-valued polynomial function on V. The space of all rational-valued polynomial functions on V is the symmetric algebra $\Sigma=\mathrm{Sym}(V^*)$. Let the Weyl group (R) act on Σ, and let P(S) denote the subspace of Σ spanned by the polynomial functions $g(\Pi(S))$ for all g in (R). MacDonald proves that P(S) is an absolutely irreducible (R)-module.

The purpose of this thesis is to use this construction to compute the irreducible characters of the Weyl groups of rank ≤ 4. We wish to find out which characters can be so obtained and whether the missing characters are related to those obtained. The characters that are obtained by this method will be called MacDonald characters.

Using this method we find all the irreducible characters in the case of the groups (A_i). We find that neither the subsystems of B_i alone nor the subsystems of C_i alone provide all the irreducible characters of the group $(B_i)=(C_i)$. However, using the subsystems of both B_i and C_i we obtain all the irreducible characters. In the case of (D_4), the method gives eleven of the thirteen irreducible characters.The exterior products of the reflection representation provide the missing characters.In the case of (F_4) we find seventeen characters.The remaining eight can be written as products of two MacDonald characters.

We describe below the notation used and the method of obtaining all the subsystems of a given root system of a Weyl group.

We use Dynkin diagrams to represent root systems and the set of positive roots in terms of orthonormal basis as given by N.Bourbaki [1].As in [2],in listing the subsystems of B_i and F_4,we distinguish between a diagram with long roots and one with shorter roots by using S to denote a Dynkin diagram with long roots and $\tilde{S}$ one with shorter roots.We do the opposite for subsystems of C_i.

The conjugacy classes and the orders of the centralizers for (A_i),(B_i) or (C_i) for $i=2,3,4$,and (D_4) are staightforward to find.For (F_4) we use the conjugacy classes obtained by R.W.Carter [2],and in all cases we use Carter's notation for conjugacy classes.For the sake of convenience,we also use permutation notation and sign changes.

In the tables,the entries under h_i denote the numbers of elements in the conjugacy classes.χ_i denotes a character. For any root α,w_α denotes the reflection defined by α.We use the notation $V(a_1,\ldots,a_n)$ to denote the Vandermonde determinant of order n,

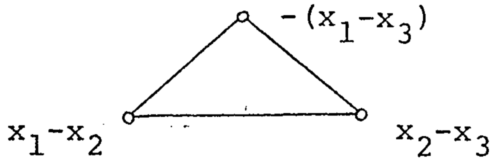
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}$$

Given a root system R,we use Dynkin's method [3] to obtain all the subsystems,We start by adjoining to the root system R the minimal root of (R).The diagram so obtained is called the extended Dynkin diagram for R.By deleting roots from the extended Dynkin diagram we obtain subsystems. Repeating the process with the subsystems obtained we eventually find all the subsystems of the given root system.

1.A_2.

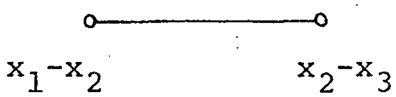
Subsystems of A_2:

The extended Dynkin diagram for A_2 is

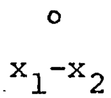


From this diagram we obtain the following subsystems:

(1) A_2:



(2) A_1:



We note that $\Pi(A_2)=-V(x_1,x_2,x_3)$ where $V(x_1,x_2,x_3)$ is the Vandermonde determinant of order 3.

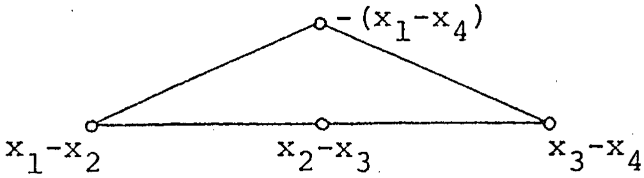
Table I: Character Table for (A_2)

				Φ	A_2	A_1
Conjugacy Class Representative		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2
Φ	(1)	x^2-2x+1	1	1	1	2
A_1	(12)	$x^2\quad -1$	3	1	-1	0
A_2	(123)	x^2+x+1	2	1	1	-1

2.A_3.

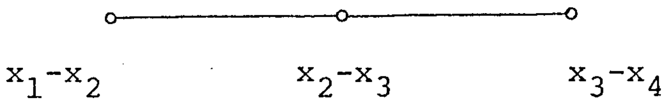
Subsystems of A_3:

The extended Dynkin diagram for A_3 is



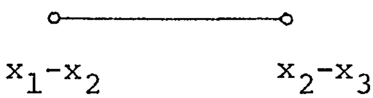
From this diagram we obtain the following subsystems:

(1) A_3:

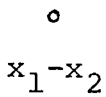


(2) $2A_1$:

(3) A_2:



(4) A_1:



$P(A_3)$: We have

$\Pi(A_3)=V(x_1,x_2,x_3,x_4)$, the Vandermonde determinant of order 4.

$P(2A_1)$: In this case, it is convenient to write the polynomials in terms of the roots α_i (where $\alpha_i=x_i-x_{i+1}$) rather than the x_i's. We have

$$\Pi(2A_1)=\alpha_1\alpha_3,$$

and so $P(2A_1)$ is spanned by:

$$b_1=\alpha_1\alpha_3,$$
$$b_2=\alpha_2(\alpha_1+\alpha_2+\alpha_3),$$
$$b_3=(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)=b_1+b_2.$$

$P(A_2)$: The group (A_3) acting on $\Pi(A_2)$ gives rise to the following polynomials:

$$b_1=V(x_1,x_2,x_3),$$
$$b_2=V(x_2,x_3,x_4),$$
$$b_3=V(x_1,x_3,x_4),$$
$$b_4=V(x_1,x_2,x_4).$$

Then

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix} =0$$

shows

$$b_1-b_2+b_3-b_4=0$$

and $\{b_1,b_2,b_3\}$ may be taken as a basis for $P(A_2)$.

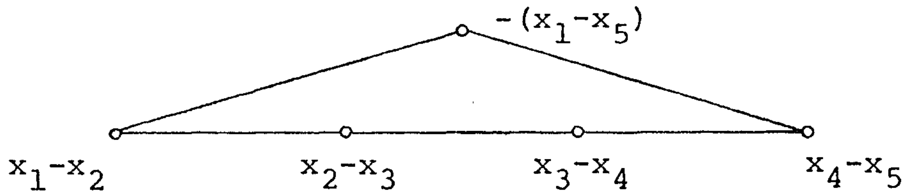
Table II:Character Table for (A_3)

				Φ	A_3	$2A_1$	A_2	A_1
Conjugacy Class Representative		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3	χ_4
Φ	(1)	x^3-3x^2+3x-1	1	1	1	2	3	3
A_1	(12)	x^3-x^2-x+1	6	1	-1	0	-1	1
A_2	(123)	x^3-1	8	1	1	-1	0	0
A_3	(1234)	x^3+x^2+x+1	6	1	-1	0	1	-1
$2A_1$	(12)(34)	x^3+x^2-x-1	3	1	1	2	-1	-1

3. A_4

<u>Subsystems of A_4:</u>

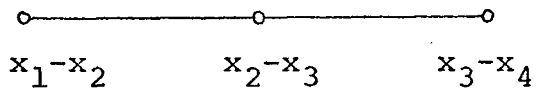
The extended Dynkin diagram for A_4 is



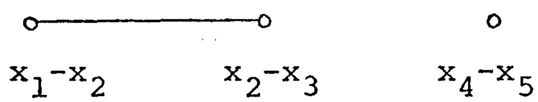
From this diagram we obtain the following subsystems:

(1) A_4:

(2) A_3:



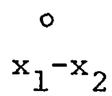
(3) A_2+A_1:



(4) $2A_1$:

(5) A_2:

(6) A_1:



$P(A_4)$: We have

$$\Pi(A_4)=V(x_1,x_2,x_3,x_4,x_5).$$

$P(A_3)$: The (A_4)-orbit of $\Pi(A_3)$ consists of $\pm b_i$ $(i=1,\ldots,5)$ where

$$b_1=V(x_1,x_2,x_3,x_4),$$
$$b_2=V(x_2,x_3,x_4,x_5),$$
$$b_3=V(x_1,x_3,x_4,x_5),$$
$$b_4=V(x_1,x_2,x_4,x_5),$$
$$b_5=V(x_1,x_2,x_3,x_5).$$

As in the case of $P(A_2)$ of (A_3),we have

$$b_1+b_2-b_3+b_4-b_5=0$$

and $\{b_1,b_2,b_3,b_4\}$ forms a basis for $P(A_3)$.

$P(A_2+A_1)$: We have

$$\Pi(A_2+A_1)=-(x_4-x_5)V(x_1,x_2,x_3).$$

Hence $P(A_2+A_1)$ is spanned by:

$$b_1=(x_4-x_5)V(x_1,x_2,x_3),$$
$$b_2=(x_3-x_5)V(x_1,x_2,x_4),$$
$$b_3=(x_2-x_5)V(x_1,x_3,x_4),$$
$$b_4=(x_1-x_5)V(x_2,x_3,x_4),$$
$$b_5=(x_1-x_4)V(x_2,x_3,x_5),$$
$$b_6=(x_2-x_4)V(x_1,x_3,x_5),$$
$$b_7=(x_3-x_4)V(x_1,x_2,x_5),$$
$$b_8=(x_1-x_3)V(x_2,x_4,x_5),$$
$$b_9=(x_2-x_3)V(x_1,x_4,x_5),$$
$$b_{10}=(x_1-x_2)V(x_3,x_4,x_5).$$

We have

$$b_1-b_2+b_3-b_4=0$$

which can be seen from the identity:

$$\begin{vmatrix} x_1-x_5 & x_2-x_5 & x_3-x_5 & x_4-x_5 \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix} =0$$

Similarly,we have

$$b_7=-b_1-b_5+b_6,$$

$$b_8=-b_1-b_3+b_6,$$

$$b_9=b_2-b_3+b_5,$$

$$b_{10}=-b_2-b_5+b_6.$$

b_1, b_2, b_3, b_5, b_6 are easily seen to be linearly independent, and hence form a basis for $P(A_2+A_1)$.

$P(2A_1)$: Writing the polynomials in terms of the roots α_i where $\alpha_i=x_i-x_{i+1}$, we have

$$\Pi(2A_1)=\alpha_1\alpha_3.$$

It is easy to see that $P(2A_1)$ is spanned by the following linearly independent elements:

$$\alpha_1\alpha_3,$$
$$\alpha_1\alpha_4,$$
$$\alpha_2\alpha_4,$$
$$\alpha_2(\alpha_1+\alpha_2+\alpha_3),$$
$$\alpha_3(\alpha_2+\alpha_3+\alpha_4).$$

$P(A_2)$: A basis for $P(A_2)$ is given by the polynomials:

$$-\Pi(A_2)=V(x_1,x_2,x_3),$$
$$V(x_1,x_2,x_4),$$
$$V(x_1,x_2,x_5),$$
$$V(x_1,x_3,x_4),$$
$$V(x_1,x_3,x_5),$$
$$V(x_1,x_4,x_5).$$

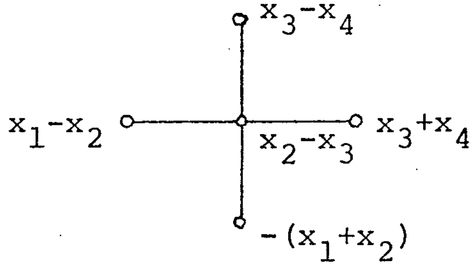
Table III: Character Table for (A_4)

				Φ	A_4	A_3	A_1	A_2+A_1	$2A_1$	A_2
Conjugacy Class Representative		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
Φ	(1)	$x^4-4x^3+6x^2-4x+1$	1	1	1	4	4	5	5	6
A_1	(12)	x^4-2x^3+2x-1	10	1	-1	-2	2	-1	1	0
A_2	(123)	x^4-x^3-x+1	20	1	1	1	1	-1	-1	0
A_3	(1234)	x^4-1	30	1	-1	0	0	1	-1	0
A_4	(12345)	$x^4+x^3+x^2+x+1$	24	1	1	-1	-1	0	0	1
$2A_1$	(12)(34)	x^4-2x^2+1	15	1	1	0	0	1	1	-2
A_2+A_1	(123)(45)	x^4+x^3-x-1	20	1	-1	1	-1	-1	1	0

4. D_4.

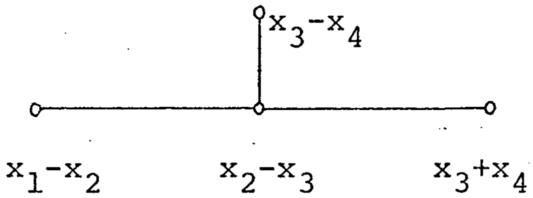
<u>Subsystems of D_4:</u>

The extended Dynkin diagram for D_4 is



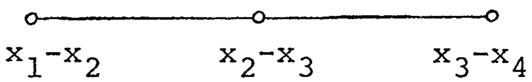
From this diagram we find the following subsystems:

(1) D_4:

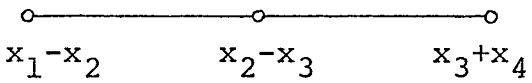


(2) $4A_1$:

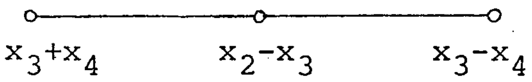
(3) A_3:



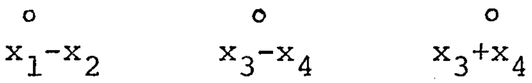
(4) A_3':



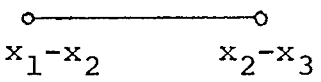
(5) A_3'':



(6) $3A_1$:



(7) A_2:



(8) $2A_1$:

(9) $2A_1'$:

(10) $2A_1''$:

$$\underset{x_1-x_2}{\circ} \qquad \underset{x_3-x_4}{\circ}$$

(11) A_1:

$$\underset{x_1-x_2}{\circ}$$

$P(D_4)$: We have

$$\Pi(D_4)=V(x_1^2,x_2^2,x_3^2,x_4^2).$$

$P(4A_1)$: A basis for $P(4A_1)$ consists of

$$\Pi(4A_1)=(x_1^2-x_2^2)(x_3^2-x_4^2),$$
$$(x_2^2-x_3^2)(x_1^2-x_4^2).$$

$P(A_3)$: The (D_4)-orbit of $\Pi(A_3)$ consists of $\pm b_i$ $(i=1,,..,4)$ where

$$b_1=V(x_1,x_2,x_3,x_4),$$
$$b_2=V(x_1,x_2,-x_3,-x_4),$$
$$b_3= V(x_1,-x_2,x_3,-x_4),$$
$$b_4=V(x_1,-x_2,-x_3,x_4).$$

A simple calculation of determinants shows that

$$b_1+b_2+b_3-b_4=0$$

and b_1,b_2,b_3 are linearly independent.

$P(A_3')$: We have

$$\Pi(A_3')=V(x_1,x_2,x_3,-x_4),$$

and so $P(A_3')$ is spanned by:

$$b_1=V(x_1,x_2,x_3,-x_4),$$
$$b_2= V(x_1,x_2,-x_3,x_4),$$
$$b_3=V(x_1,-x_2,x_3,x_4),$$

$$b_4=V(-x_1,x_2,x_3,x_4).$$

As above, a simple calculation shows that

$$b_1+b_2+b_3+b_4=0$$

and $\{b_1,b_2,b_3\}$ forms a basis for $P(A_3')$.

$P(A_3'')$: We find

$$\Pi(A_3'')=-V(x_2^2,x_3^2,x_4^2).$$

This case is analogous to that of $P(A_2)$ of (A_3), and we obtain a basis consisting of:

$$V(x_2^2,x_3^2,x_4^2),$$
$$V(x_1^2,x_3^2,x_4^2),$$
$$V(x_1^2,x_2^2,x_4^2).$$

$P(3A_1)$: We have

$$\Pi(3A_1)=(x_1-x_2)(x_3^2-x_4^2).$$

The element $(12)[-1\ 1\ -1\ 1]$ of (D_4) acting on $\Pi(3A_1)$ gives

$$(x_1+x_2)(x_3^2-x_4^2).$$

Then $\quad (x_1-x_2)(x_3^2-x_4^2)+(x_1+x_2)(x_3^2-x_4^2)=2x_1(x_3^2-x_4^2)$

is in $P(3A_1)$. Since $P(3A_1)$ is irreducible, we may take the (D_4)-orbit of $x_1(x_3^2-x_4^2)$ instead of that of $\Pi(3A_1)$ to span $P(3A_1)$. We can now easily write down a basis as follows:

$$x_1(x_2^2-x_3^2),$$
$$x_1(x_3^2-x_4^2),$$
$$x_2(x_1^2-x_3^2),$$
$$x_2(x_3^2-x_4^2),$$
$$x_3(x_1^2-x_2^2),$$
$$x_3(x_2^2-x_4^2),$$
$$x_4(x_1^2-x_2^2),$$

$$x_4(x_2^2-x_3^2).$$

$P(A_2)$: We have

$$\Pi(A_2)=-V(x_1,x_2,x_3).$$

We note that $\Pi(A_2)$ is in $P(3A_1)$ above, for

$$V(x_1,x_2,x_3)=-x_1(x_2^2-x_3^2)+x_2(x_1^2-x_3^2)-x_3(x_1^2-x_2^2).$$

Hence $P(A_2)\subseteq P(3A_1)$. But both spaces are irreducible, hence $P(A_2)=P(3A_1)$.

$P(2A_1)$: We have

$$\Pi(2A_1)=(x_1-x_2)(x_3+x_4).$$

Writing this in terms of the roots α_i of D_4, where $\alpha_i=x_i-x_{i+1}$ for $i=1,2,3$ and $\alpha_4=x_3+x_4$, we can easily find a basis for $P(2A_1)$, such as

$$\alpha_1\alpha_4,$$
$$\alpha_3(\alpha_1+2\alpha_2+\alpha_3+\alpha_4),$$
$$\alpha_2(\alpha_1+\alpha_2+\alpha_4).$$

$P(2A_1')$: A basis for $P(2A_1')$ is given by

$$x_1^2-x_2^2,$$
$$x_1^2-x_3^2,$$
$$x_1^2-x_4^2.$$

$P(2A_1'')$: We have

$$\Pi(2A_1'')=(x_1-x_2)(x_3-x_4).$$

Again, it is more convenient to use the α_i rather than the x_i. We may take

$$\alpha_1\alpha_3,$$

$$\alpha_4(\alpha_1+2\alpha_2+\alpha_3+\alpha_4),$$
$$\alpha_2(\alpha_1+\alpha_2+\alpha_3)$$

as a basis for $P(2A_1'')$.

In calculating the characters, the following fact enables us to simplify our computation.

The group (D_4) contains a central involution z (the element corresponding to $4A_1$). Then for any character χ and element g of the group, we have

$$\chi(zg)=\varepsilon\chi(g)$$

where

$$\varepsilon=\frac{\chi(z)}{\chi(1)} = 1 \text{ or } -1.$$

Furthermore, $\varepsilon=1$ or -1 according as $\Pi(S)$ is of even or odd degree.

Denote by $[X]$ an element in the conjugacy class designated by X in the table. Then we have the following conjugacy relations:

$$z=[4A_1],$$
$$[2A_1'] \sim z[2A_1'],$$
$$[A_1] \sim z[3A_1],$$
$$[D_4(a_1)] \sim z[D_4(a_1)],$$
$$[A_2] \sim z[D_4],$$
$$[2A_1''] \sim z[2A_1''],$$
$$[2A_1] \sim z[2A_1],$$
$$[A_3''] \sim z[A_3''],$$
$$[A_3] \sim z[A_3],$$
$$[A_3'] \sim z[A_3'].$$

Using the subsystems of D_4 we obtain all but two of the irreducible characters of (D_4).

Table IV: Character Table for (D_4).

				Φ	D_4	$4A_1$	$2A_1$	$2A_1'$	$2A_1''$
Conjugacy Class Representatives		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
Φ	(1) [1111]	$x^4-4x^3+6x^2-4x+1$	1	1	1	2	3	3	3
$2A_1'$	(1) [-1-111]	x^4-2x^2+1	6	1	1	2	-1	3	-1
$4A_1$	(1) [-1-1-1-1]	$x^4+4x^3+6x^2+4x+1$	1	1	1	2	3	3	3
A_1	(12) [1111]	x^4-2x^3+2x-1	12	1	-1	0	1	1	1
A_3'''	(12) [-11-11]	x^4-1	24	1	-1	0	-1	1	-1
$\left.\begin{matrix}3A_1\\A_3''\end{matrix}\right\}$	(12) [-1-1-1-1]	x^4+2x^3-2x-1	12	1	-1	0	1	1	1
A_2	(123) [1111]	x^4-x^3-x+1	32	1	1	-1	0	0	0
D_4	(123) [-1-1-1-1]	x^4+x^3+x+1	32	1	1	-1	0	0	0
$2A_1''$	(12)(34) [1111]	x^4-2x^2+1	6	1	1	2	-1	-1	3
$2A_1$	(12)(34)[-1-111]	x^4-2x^2+1	6	1	1	2	3	-1	-1
$D_4(a_1)$	(12)(34)[-11-11]	x^4+2x^2+1	12	1	1	2	-1	-1	-1
A_3	(1234) [1111]	x^4-1	24	1	-1	0	-1	-1	1
A_3'	(1234)[-1-111]	x^4-1	24	1	-1	0	1	-1	-1

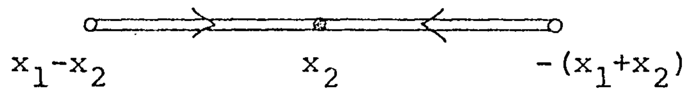
Table IV: Character Table for (D_4) Continued.

	A_3	A_3'	A_3''	A_1			$3A_1$ A_2
	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
Φ	3	3	3	4	4	6	8
$2A_1'$	-1	-1	3	0	0	-2	0
$4A_1$	3	3	3	-4	-4	6	-8
A_1	-1	-1	-1	2	-2	0	0
A_3'''	1	1	-1	0	0	0	0
$\left.\begin{matrix} 3A_1 \\ A_3'' \end{matrix}\right\}$	-1	-1	-1	-2	2	0	0
A_2	0	0	0	1	1	0	-1
D_4	0	0	0	-1	-1	0	1
$2A_1''$	3	-1	-1	0	0	-2	0
$2A_1$	-1	3	-1	0	0	-2	0
$D_4(a_1)$	-1	-1	-1	0	0	2	0
A_3	-1	1	1	0	0	0	0
A_3'	1	-1	1	0	0	0	0

5. B_2 and C_2

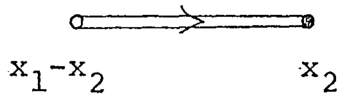
<u>Subsystems of B_2:</u>

The extended Dynkin diagram for B_2 is



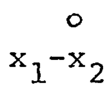
From this diagram we obtain the following subsystems:

(1) B_2:

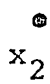


(2) $2A_1$:

(3) A_1:



(4) $\tilde{A}_1$:



$P(B_2)$: We have

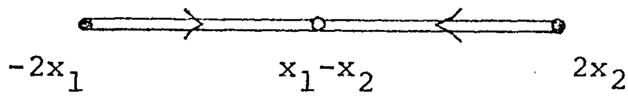
$$\Pi(B_2)=x_1x_2(x_1^2-x_2^2).$$

$P(2A_1)$: $P(2A_1)$ is spanned by

$$\Pi(2A_1)=-(x_1^2-x_2^2).$$

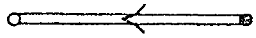
<u>Subsystems of C_2:</u>

The extended Dynkin diagram for C_2 is:



We obtain the following subsystems:

(1) C_2:



x_1-x_2 $\qquad$ $2x_2$

(2) $2\tilde{A}_1$:

$-2x_1$ $\qquad$ $2x_2$

(3) A_1:

x_1-x_2

(4) $\tilde{A}_1$:

$2x_2$

Combining the P(S) for the subsystems S of B_2 and of C_2, we obtain all the characters of $(B_2)=(C_2)$.

Table V: Character Table for $(B_2)=(C_2)$.

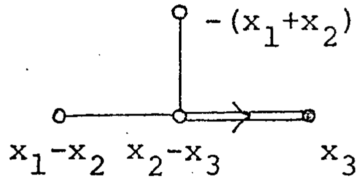
					Φ	B_2	$2A_1$		A_1 $\tilde{A}_1$
					Φ	C_2		$2\tilde{A}_1$	A_1 $\tilde{A}_1$
Conjugacy Class Representative B_2	C_2		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3	χ_4
Φ	Φ	(1) [11]	x^2-2x+1	1	1	1	1	1	2
$\tilde{A}_1$	$\tilde{A}_1$	(1) [-11]	$x^2 \quad -1$	2	1	-1	1	-1	0
$2A_1$	$2\tilde{A}_1$	(1) [-1-1]	x^2+2x+1	1	1	1	1	1	-2
A_1	A_1	(12) [11]	$x^2 \quad -1$	2	1	-1	-1	1	0
B_2	C_2	(12) [-11]	$x^2 \quad +1$	2	1	1	-1	-1	0

Note: The first line of the table gives the subsystems of B_2 and the second the subsystems of C_2.

6. B_3 and C_3.

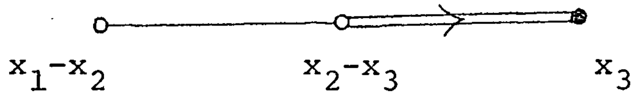
<u>Subsystems of B_3:</u>

The extended Dynkin diagram for B_3 is

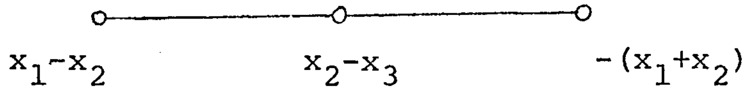


We obtain the following subsystems:

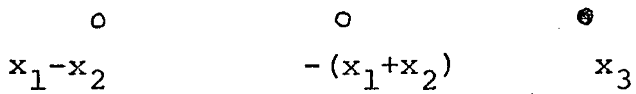
(1) B_3:



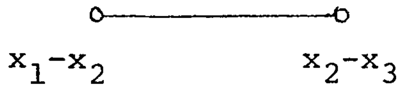
(2) A_3:



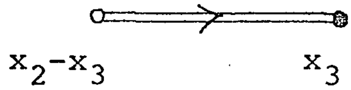
(3) $2A_1+\tilde{A}_1$:



(4) A_2:



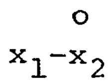
(5) B_2:



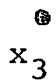
(6) $A_1+\tilde{A}_1$:

(7) $2A_1$:

(8) A_1:



(9) $\tilde{A}_1$:



$P(B_3)$: We have

$$\Pi(B_3)=-x_1x_2x_3V(x_1^2,x_2^2,x_3^2).$$

$P(A_3)$: It is easy to see that $P(A_3)$ is spanned by

$$\Pi(A_3)=V(x_1^2,x_2^2,x_3^2).$$

$P(2A_1+\tilde{A}_1)$: We have

$$\Pi(2A_1+\tilde{A}_1)=-x_3(x_1^2-x_2^2).$$

Hence $P(2A_1+\tilde{A}_1)$ is spanned by:

$$x_1(x_2^2-x_3^2),$$
$$x_2(x_1^2-x_3^2),$$
$$x_3(x_1^2-x_2^2),$$

which are linearly independent.

$P(A_2)$: We have

$$\Pi(A_2)=-V(x_1,x_2,x_3),$$

and

$$\begin{aligned} &V(x_1,x_2,x_3)-w_{x_3}(V(x_1,x_2,x_3)) \\ &=V(x_1,x_2,x_3)-V(x_1,x_2,-x_3) \\ &=2\begin{vmatrix} 1 & 1 & 0 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & 0 \end{vmatrix} \\ &=2x_3(x_1^2-x_2^2) \\ &=-2\Pi(2A_1+\tilde{A}_1). \end{aligned}$$

This shows that $P(A_2)=P(2A_1+\tilde{A}_1)$.

$P(B_2)$: We find

$$\Pi(B_2)=x_2x_3(x_2^2-x_3^2).$$

Thus $P(B_2)$ is spanned by the following linearly independent elements:

$$x_2x_3(x_2^2-x_3^2),$$

$x_1x_3(x_1^2-x_3^2)$,

$x_1x_2(x_1^2-x_2^2)$.

$P(A_1+\tilde{A}_1)$: The (B_3)-orbit of $\Pi(A_1+\tilde{A}_1)$ consists of $x_i(x_j-x_k)$ and $\pm x_i(x_j+x_k)$ where i,j,k are distinct elements of $\{1,2,3\}$. Therefore

x_1x_2,

x_1x_3,

x_2x_3

form a basis for $P(A_1+\tilde{A}_1)$.

$P(2A_1)$: We have

$$\Pi(2A_1)=-(x_1^2-x_2^2),$$

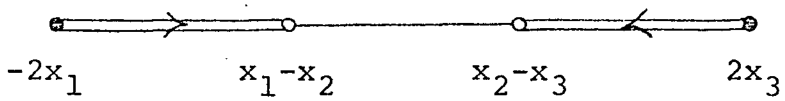
and a basis for $P(2A_1)$ is given by:

$x_1^2-x_2^2$,

$x_1^2-x_3^2$.

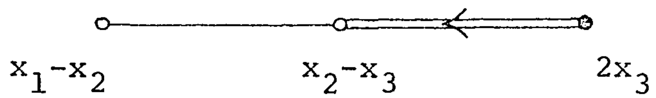
<u>Subsystems of C_3:</u>

The extended Dynkin diagram for C_3 is

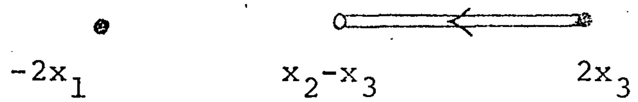


From this diagram we obtain the following subsystems:

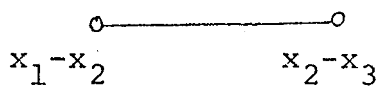
(1) C_3:



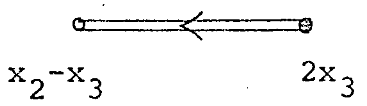
(2) $C_2+\tilde{A}_1$:



(3) A_2:



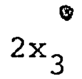
(4) C_2:



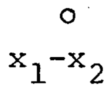
(5) $2\tilde{A}_1$:

(6) $A_1+\tilde{A}_1$:

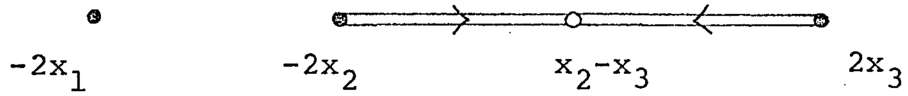
(7) $\tilde{A}_1$:



(8) A_1:

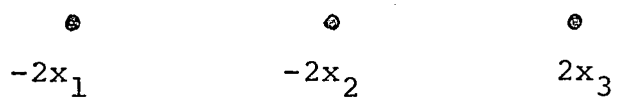


We next consider subsystems of the above. From the extended Dynkin diagram of $C_2+\tilde{A}_1$,



we obtain the subsystem

(9) $3\tilde{A}_1$:



$P(C_2+\tilde{A}_1)$: We have

$$\Pi(C_2+\tilde{A}_1)=-8x_1x_2x_3(x_2^2-x_3^2).$$

Hence a basis for $P(C_2+\tilde{A}_1)$ is given by:

$$x_1x_2x_3(x_1^2-x_2^2),$$
$$x_1x_2x_3(x_1^2-x_3^2).$$

$P(3\tilde{A}_1)$: $P(3\tilde{A}_1)$ is spanned by

$$\Pi(3\tilde{A}_1)=8x_1x_2x_3.$$

We note that χ_3 and χ_4, obtained from subsystems of C_3, cannot be obtained using subsystems of B_3, whereas χ_2 and χ_5, obtained from subsystems of B_3, cannot be obtained from C_3.

The group (B_3) contains a non-trivial central element, the element corresponding to $2A_1+\tilde{A}_1$. As in the case of (D_4), denoting this central element by z and an element in the conjugacy class X by $[X]$, we observe the following conjugacy relations:

$$[\tilde{A}_1] \sim z\,[2A_1]\,,$$
$$[A_1] \sim z\,[A_1+\tilde{A}_1]\,,$$
$$[B_2] \sim z\,[A_3]\,,$$
$$[A_2] \sim z\,[B_3]\,.$$

Table VI: Character Table for $(B_3)=(C_3)$

					Φ	B_3	A_3	
					Φ	C_3		$3\tilde{A}_1$
Conjugacy Class Representative			Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3
B_3	C_3							
Φ	Φ	(1) [111]	x^3-3x^2+3x-1	1	1	1	1	1
$\tilde{A}_1$	$\tilde{A}_1$	(1) [11-1]	x^3-x^2-x+1	3	1	-1	1	-1
$2A_1$	$2\tilde{A}_1$	(1) [1-1-1]	x^3+x^2-x-1	3	1	1	1	1
$2A_1+\tilde{A}_1$	$3\tilde{A}_1$	(1) [-1-1-1]	x^3+3x^2+3x+1	1	1	-1	1	-1
A_1	A_1	(12) [111]	x^3-x^2-x+1	6	1	-1	-1	1
B_2	C_2	(12) [-111]	x^3-x^2+x-1	6	1	1	-1	-1
$A_1+\tilde{A}_1$	$A_1+\tilde{A}_1$	(12) [11-1]	x^3+x^2-x-1	6	1	1	-1	-1
A_3	$C_2+\tilde{A}_1$	(12) [1-1-1]	x^3+x^2+x+1	6	1	-1	-1	1
A_2	A_2	(123) [111]	x^3-1	8	1	1	1	1
B_3	C_3	(123) [-111]	x^3+1	8	1	-1	1	-1

Table VI: Continued

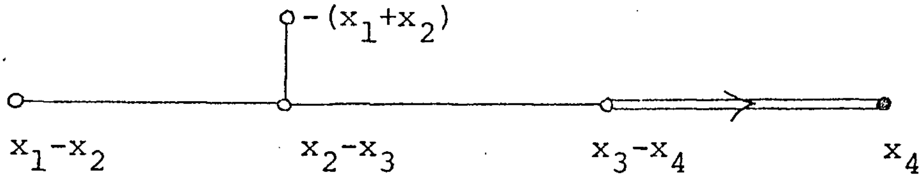
		$2A_1$	$\tilde{A}_1$ A_1	A_2 $2A_1+\tilde{A}_1$	B_2	$A_1+\tilde{A}_1$
	$C_2+\tilde{A}_1$		$\tilde{A}_1$ A_1	A_2	C_2	$2\tilde{A}_1$ $A_1+\tilde{A}_1$
	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
(1) [111]	2	2	3	3	3	3
(1) [11-1]	-2	2	1	1	-1	-1
(1) [1-1-1]	2	2	-1	-1	-1	-1
(1) [-1-1-1]	-2	2	-3	-3	3	3
(12) [111]	0	0	1	-1	-1	1
(12) [-111]	0	0	1	-1	1	-1
(12) [11-1]	0	0	-1	1	-1	1
(12) [1-1-1]	0	0	-1	1	1	-1
(123) [111]	-1	-1	0	0	0	0
(123) [-111]	1	-1	0	0	0	0

Note: The first line of the table gives the subsystems of B_3 and the second the subsystems of C_3.

7. B_4 and C_4.

Subsystems of B_4:

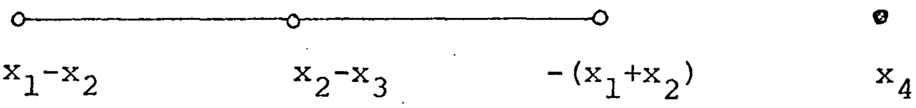
The extended Dynkin diagram for B_4 is:



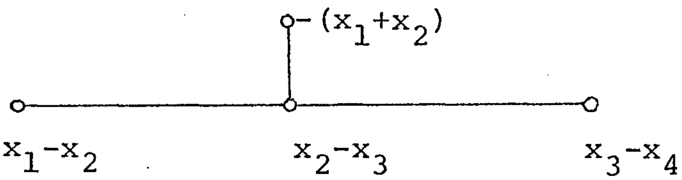
From this diagram we obtain the following subsystems:

(1) B_4:

(2) $A_3+\tilde{A}_1$:

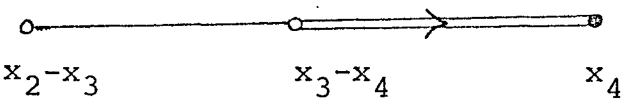


(3) D_4:

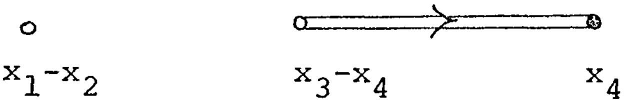


(4) B_2+2A_1:

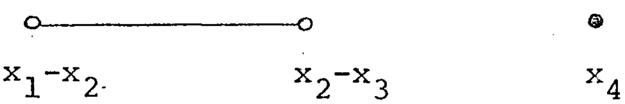
(5) B_3:



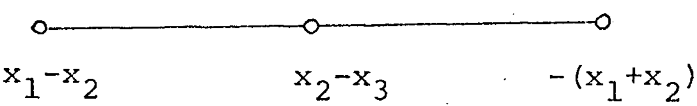
(6) B_2+A_1:



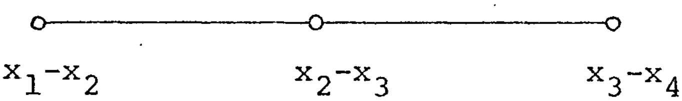
(7) $A_2+\tilde{A}_1$:



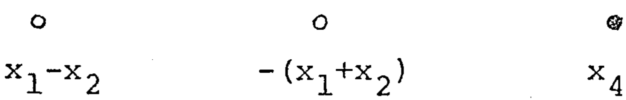
(8) A_3:



(9) A_3':



(10) $2A_1+\tilde{A}_1$:



(11) $3A_1$: $\quad x_1-x_2 \qquad -(x_1+x_2) \qquad x_3-x_4$

(12) A_2: $\quad x_2-x_3 \qquad x_3-x_4$

(13) $2A_1$: $\quad x_1-x_2 \qquad -(x_1+x_2)$

(14) $2A_1'$: $\quad x_1-x_2 \qquad x_3-x_4$

(15) $A_1+\tilde{A}_1$: $\quad x_2-x_3 \qquad x_4$

(16) B_2: $\quad x_3-x_4 \qquad x_4$

(17) A_1: $\quad x_1-x_2$

(18) $\tilde{A}_1$: $\quad x_4$

From the extended Dynkin diagram for B_2+2A_1,

$x_1-x_2 \qquad -(x_1+x_2) \qquad x_3-x_4 \qquad x_4 \qquad -(x_3+x_4)$

we obtain the subsystem:

(19) $4A_1$: $\quad x_1-x_2 \qquad -(x_1+x_2) \qquad x_3-x_4 \qquad -(x_3+x_4)$

$P(B_4)$: $P(B_4)$ is spanned by

$$\Pi(B_4)=x_1x_2x_3x_4V(x_1^2,x_2^2,x_3^2,x_4^2).$$

$P(A_3+\tilde{A}_1)$: We have

$$\Pi(A_3+\tilde{A}_1)=x_4V(x_1^2,x_2^2,x_3^2),$$

and so $P(A_3+\tilde{A}_1)$ is spanned by:

$$x_1V(x_2^2,x_3^2,x_4^2),$$
$$x_2V(x_1^2,x_3^2,x_4^2),$$
$$x_3V(x_1^2,x_2^2,x_4^2),$$
$$x_4V(x_1^2,x_2^2,x_3^2).$$

It is easy to see that these are linearly independent.

$P(D_4)$: $P(D_4)$ is spanned by one element:

$$\Pi(D_4)=V(x_1^2,x_2^2,x_3^2,x_4^2).$$

$P(B_2+2A_1)$: We find

$$\Pi(B_2+2A_1)=-x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2).$$

It can be seen that the following elements are linearly independent and span $P(B_2+2A_1)$;

$$x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2),$$
$$x_1x_2(x_1^2-x_2^2)(x_3^2-x_4^2),$$
$$x_1x_3(x_1^2-x_3^2)(x_2^2-x_4^2),$$
$$x_2x_4(x_1^2-x_3^2)(x_2^2-x_4^2),$$
$$x_1x_4(x_1^2-x_4^2)(x_2^2-x_3^2),$$
$$x_2x_3(x_1^2-x_4^2)(x_2^2-x_3^2).$$

$P(B_3)$: We find

$$\Pi(B_3)=x_2x_3x_4V(x_2^2,x_3^2,x_4^2),$$

hence $P(B_3)$ is spanned by:

$$x_2x_3x_4V(x_2^2,x_3^2,x_4^2),$$
$$x_1x_3x_4V(x_1^2,x_3^2,x_4^2),$$
$$x_1x_2x_4V(x_1^2,x_2^2,x_4^2),$$

$$x_1x_2x_3V(x_1^2,x_2^2,x_3^2).$$

We observe that these are linearly independent.

$P(B_2+A_1)$: We have

$$\Pi(B_2+A_1)=x_3x_4(x_3^2-x_4^2)(x_1-x_2).$$

Since

$$(I-w_{x_2})(x_3x_4(x_3^2-x_4^2)(x_1-x_2)=2x_1x_3x_4(x_3^2-x_4^2),$$

$P(B_2+A_1)$ is spanned by the (B_4)-orbit of $x_1x_3x_4(x_3^2-x_4^2)$.

We can now easily write down a basis as follows:

$$x_1x_2x_3(x_1^2-x_2^2),$$
$$x_1x_2x_3(x_1^2-x_3^2),$$
$$x_1x_2x_4(x_1^2-x_2^2),$$
$$x_1x_2x_4(x_1^2-x_4^2),$$
$$x_1x_3x_4(x_1^2-x_3^2),$$
$$x_1x_3x_4(x_1^2-x_4^2),$$
$$x_2x_3x_4(x_2^2-x_3^2),$$
$$x_2x_3x_4(x_2^2-x_4^2).$$

$P(A_2+\tilde{A}_1)$; We have

$$\Pi(A_2+\tilde{A}_1)=x_4V(x_1,x_2,x_3).$$

In this case it is easier to consider the (B_4)-orbit of

$$(I-w_{x_1})(\Pi(A_2+\tilde{A}_1))=2x_1x_4(x_2^2-x_3^2)$$

rather than that of $\Pi(A_2+\tilde{A}_1)$. Then a basis for $P(A_2+\tilde{A}_1)$ is given by:

$$x_1x_2(x_3^2-x_4^2),$$
$$x_1x_3(x_2^2-x_4^2),$$
$$x_1x_4(x_2^2-x_3^2),$$
$$x_2x_3(x_1^2-x_4^2),$$

$$x_2x_4(x_1^2-x_3^2),$$
$$x_3x_4(x_1^2-x_2^2).$$

$P(A_3)$: We find

$$\Pi(A_3)=V(x_1^2,x_2^2,x_3^2).$$

The (B_4)-orbit in this case is the same as the (A_3)-orbit of $\Pi(A_2)$ with each x_i replaced by x_i^2. Therefore, a basis for $P(A_3)$ is given by:

$$V(x_1^2,x_2^2,x_3^2),$$
$$V(x_2^2,x_3^2,x_4^2),$$
$$V(x_1^2,x_3^2,x_4^2).$$

$P(A_3')$: We have

$$\Pi(A_3')=V(x_1,x_2,x_3,x_4).$$

Let

$$Q=(I-w_{x_3})(\Pi(A_3')),$$
$$R=(I-w_{x_4})(Q),$$
$$S=(I+w_{x_2})(R).$$

Writing these in determinant form we see that

$$S=8\Pi(B_2+2A_1).$$

Therefore $P(A_3')=P(B_2+2A_1)$.

$P(2A_1+\tilde{A}_1)$: It can be seen that the following elements are linearly independent and that they span $P(2A_1+\tilde{A}_1)$:

$$-\Pi(2A_1+\tilde{A}_1)=x_4(x_1^2-x_2^2),$$
$$x_4(x_1^2-x_3^2),$$
$$x_1(x_2^2-x_3^2),$$
$$x_1(x_2^2-x_4^2),$$

$$x_2(x_1^2-x_3^2),$$
$$x_2(x_1^2-x_4^2),$$
$$x_3(x_1^2-x_2^2),$$
$$x_3(x_1^2-x_4^2).$$

$P(3A_1)$: We have

$$\Pi(3A_1)=-(x_1^2-x_2^2)(x_3-x_4),$$

and

$$\begin{aligned}(I-w_{x_4})(\Pi(3A_1))&=2x_4(x_1^2-x_2^2)\\&=-2\Pi(2A_1+\tilde{A}_1).\end{aligned}$$

Therefore $P(3A_1)=P(2A_1+\tilde{A}_1)$.

$P(A_2)$: We have

$$\Pi(A_2)=-V(x_2,x_3,x_4),$$

and

$$\begin{aligned}(I-w_{x_4})(w_{x_1-x_3}(\Pi(A_2)))&=2x_4(x_1^2-x_2^2)\\&=-2\Pi(2A_1+\tilde{A}_1).\end{aligned}$$

Hence $P(A_2)=P(2A_1+\tilde{A}_1)$.

$P(2A_1)$: A basis for $P(2A_1)$ is given by

$$-\Pi(2A_1)=x_1^2-x_2^2,$$
$$x_1^2-x_3^2,$$
$$x_1^2-x_4^2.$$

$P(2A_1')$: Let

$$c_1=\Pi(2A_1')=(x_1-x_2)(x_3-x_4),$$
$$c_2=w_{x_2}(c_1)=(x_1+x_2)(x_3-x_4),$$
$$c_3=w_{x_4}(c_1)=(x_1-x_2)(x_3+x_4),$$

$$c_4 = w_{x_2}(c_3) = (x_1+x_2)(x_3+x_4).$$

We have

$$c_1+c_2+c_3+c_4 = 4x_1x_3,$$

and we obtain the following basis for $P(2A_1')$:

$$x_1x_2,$$
$$x_1x_3,$$
$$x_1x_4,$$
$$x_2x_3,$$
$$x_2x_4,$$
$$x_3x_4.$$

$P(A_1+\tilde{A}_1)$: We have

$$\Pi(A_1+\tilde{A}_1) = x_4(x_2-x_3),$$

showing that $P(A_1+\tilde{A}_1) = P(2A_1')$.

$P(B_2)$: We find

$$\Pi(B_2) = x_3x_4(x_3^2-x_4^2),$$

and $P(B_2)$ is spanned by:

$$x_1x_2(x_1^2-x_2^2),$$
$$x_1x_3(x_1^2-x_3^2),$$
$$x_1x_4(x_1^2-x_4^2),$$
$$x_2x_3(x_2^2-x_3^2),$$
$$x_2x_4(x_2^2-x_4^2),$$
$$x_3x_4(x_3^2-x_4^2).$$

It is clear that these are linearly independent.

$P(4A_1)$: We have

$$\Pi(4A_1) = (x_1^2-x_2^2)(x_3^2-x_4^2),$$

and the (B_4)-orbit of $\Pi(4A_1)$ consists of $\pm b_i$ $(i=1,2,3)$, where

$$b_1=(x_1^2-x_2^2)(x_3^2-x_4^2),$$

$$b_2=(x_2^2-x_3^2)(x_1^2-x_4^2),$$

$$b_3=(x_1^2-x_3^2)(x_2^2-x_4^2).$$

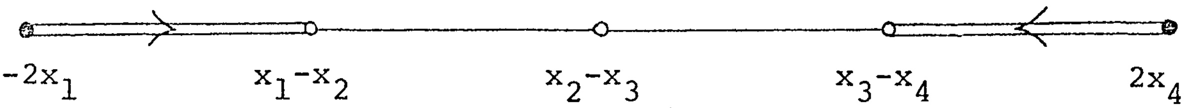
As in $P(2A_1)$ of A_3,

$$b_3=b_1+b_2$$

and $\{b_1,b_2\}$ forms a basis for $P(4A_1)$.

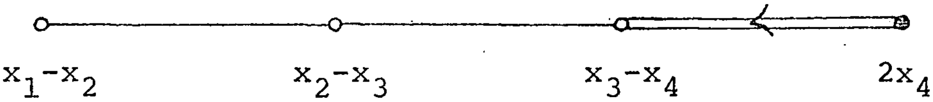
<u>Subsystems of C_4:</u>

The extended Dynkin diagram for C_4 is



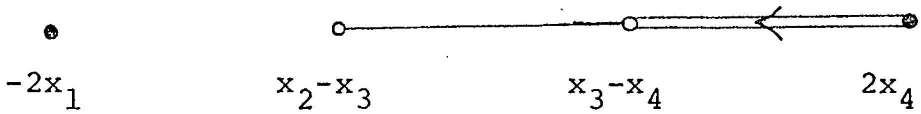
From this diagram we obtain the following subsystems:

(1) C_4:

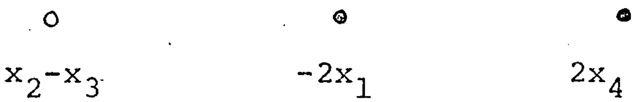


(2) $2C_2$:

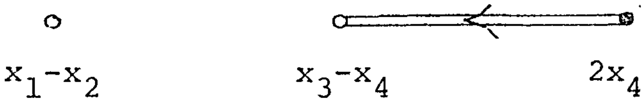
(3) $C_3+\tilde{A}_1$:



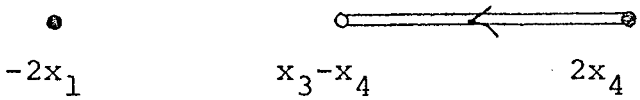
(4) $A_1+2\tilde{A}_1$:



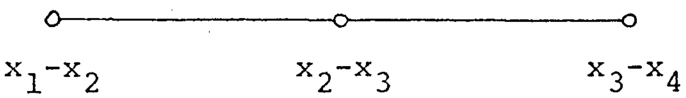
(5) C_2+A_1:



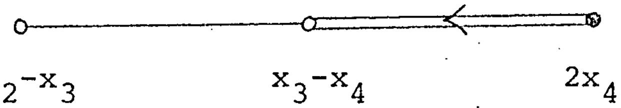
(6) $C_2+\tilde{A}_1$:



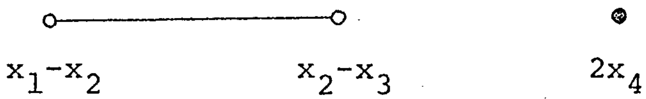
(7) A_3:



(8) C_3:



(9) $A_2+\tilde{A}_1$:

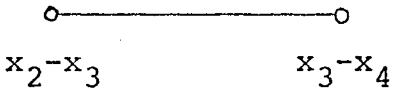


(10) $2A_1$:

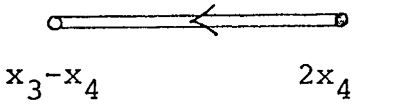
(11) $A_1+\tilde{A}_1$:

(12) $2\tilde{A}_1$:

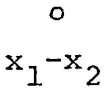
(13) A_2:



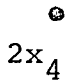
(14) C_2:



(15) A_1:



(16) $\tilde{A}_1$:

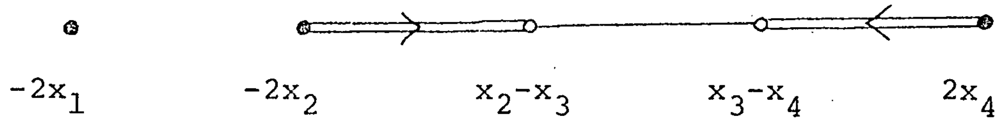


Next,we consider subsystems of the above.From the extended Dynkin diagram for $2C_2$

we obtain

(17) $4\tilde{A}_1$:

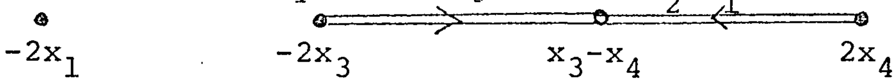
From the extended Dynkin diagram for $C_3+\tilde{A}_1$



we obtain

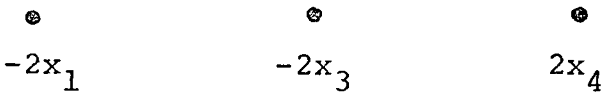
(18) $C_2+2\tilde{A}_1$:

Finally, from the extended Dynkin diagram for $C_2+\tilde{A}_1$



we obtain

(19) $3\tilde{A}_1$:



$P(2C_2)$: We have

$$\Pi(2C_2)=-16x_1x_2x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2).$$

As in $P(4A_1)$ of (B_4),

$$x_1x_2x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2),$$

$$x_1x_2x_3x_4(x_2^2-x_3^2)(x_1^2-x_4^2)$$

is a basis for $P(2C_2)$.

$P(C_3+\tilde{A}_1)$: We find

$$\Pi(C_3+\tilde{A}_1)=16x_1x_2x_3x_4V(x_2^2,x_3^2,x_4^2).$$

As in $P(A_3)$ of (B_4), a basis for $P(C_3+\tilde{A}_1)$ is given by:

$$x_1x_2x_3x_4V(x_1^2,x_2^2,x_3^2),$$

$$x_1x_2x_3x_4V(x_2^2,x_3^2,x_4^2),$$

$$x_1x_2x_3x_4V(x_1^2,x_3^2,x_4^2).$$

$P(A_1+2\tilde{A}_1)$: We have

$$\Pi(A_1+2\tilde{A}_1)=-4x_1x_4(x_2-x_3),$$

and we find

$$(I+w_{x_3})(\Pi(A_1+2\tilde{A}_1))=-8x_1x_2x_4.$$

Thus

$$x_1x_2x_3,$$
$$x_1x_2x_4,$$
$$x_1x_3x_4,$$
$$x_2x_3x_4$$

form a basis for $P(A_1+2\tilde{A}_1)$.

$P(4\tilde{A}_1)$: $P(4\tilde{A}_1)$ is spanned by

$$\Pi(4\tilde{A}_1)=16x_1x_2x_3x_4.$$

$P(C_2+2\tilde{A}_1)$: We find

$$\Pi(C_2+2\tilde{A}_1)=16x_1x_2x_3x_4(x_3^2-x_4^2).$$

A basis for $P(C_2+2\tilde{A}_1)$ is given by:

$$x_1x_2x_3x_4(x_1^2-x_2^2),$$
$$x_1x_2x_3x_4(x_1^2-x_3^2),$$
$$x_1x_2x_3x_4(x_1^2-x_4^2).$$

$P(3\tilde{A}_1)$: We have

$$\Pi(3\tilde{A}_1)=8x_1x_3x_4$$

and $P(3\tilde{A}_1)=P(A_1+2\tilde{A}_1)$.

We note that χ_3, χ_5, χ_8, χ_9, and χ_{13} obtained from subsystems of C_4, cannot be obtained using subsystems of B_4, while χ_2, χ_4, χ_6, χ_7, and χ_{10}, obtained from subsystems of B_4, cannot be obtained from C_4.

Again, in calculating the characters we use the following relations:

$$z = [4A_1] ,$$
$$[\tilde{A}_1] \sim z [2A_1 + \tilde{A}_1] ,$$
$$[2A_1] \sim z [2A_1] ,$$
$$[A_1] \sim z [3A_1] ,$$
$$[B_2] \sim z [B_2 + 2A_1] ,$$
$$[A_1 + \tilde{A}_1] \sim z [A_1 + \tilde{A}_1] ,$$
$$[A_3'] \sim z [A_3'] ,$$
$$[A_2] \sim z [D_4] ,$$
$$[B_3] \sim z [A_2 + \tilde{A}_1] ,$$
$$[2A_1'] \sim z [2A_1'] ,$$
$$[B_2 + A_1] \sim z [B_2 + A_1] ,$$
$$[D_4(a_1)] \sim z [D_4(a_1)] ,$$
$$[A_3] \sim z [A_3] ,$$
$$[B_4] \sim z [B_4] .$$

Table VII: Character Table for $(B_4)=(C_4)$.

					Φ	B_4	D_4	
					Φ	C_4		$4\tilde{A}_1$
Conjugacy Class Representative B_4	C_4		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3
Φ	Φ	(1) [1111]	$x^4-4x^3+6x^2-4x+1$	1	1	1	1	1
$\tilde{A}_1$	$\tilde{A}_1$	(1) [-1111]	x^4-2x^3+2x-1	4	1	-1	1	-1
$2A_1$	$2\tilde{A}_1$	(1) [-1-111]	x^4-2x^2+1	6	1	1	1	1
$2A_1+\tilde{A}_1$	$3\tilde{A}_1$	(1) [-1-1-11]	x^4+2x^3-2x-1	4	1	-1	1	-1
$4A_1$	$4\tilde{A}_1$	(1) [-1-1-1-1]	$x^4+4x^3+6x^2+4x+1$	1	1	1	1	1
A_1	A_1	(12) [1111]	x^4-2x^3+2x-1	12	1	-1	-1	1
B_2	C_2	(12) [-1111]	$x^4-2x^3+2x^2-2x+1$	12	1	1	-1	-1
$A_1+\tilde{A}_1$	$A_1+\tilde{A}_1$	(12) [11-11]	x^4-2x^2+1	24	1	1	-1	-1
A_3	$C_2+\tilde{A}_1$	(12) [-11-11]	x^4-1	24	1	-1	-1	1
B_2+2A_1, $A_3+\tilde{A}_1$	$C_2+2\tilde{A}_1$	(12) [-11-1-1]	$x^4+2x^3+2x^2+2x+1$	12	1	1	-1	-1
$3A_1$	$A_1+2\tilde{A}_1$	(12) [-1-1-1-1]	x^4+2x^3-2x-1	12	1	-1	-1	1
A_2	A_2	(123) [1111]	x^4-x^3-x+1	32	1	1	1	1
B_3	C_3	(123) [-1111]	x^4-x^3+x-1	32	1	-1	1	-1
$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	(123) [111-1]	x^4+x^3-x-1	32	1	-1	1	-1
D_4	$C_3+\tilde{A}_1$	(123) [-1-1-1-1]	x^4+x^3+x+1	32	1	1	1	1
$2A_1'$	$2A_1'$	(12)(34) [1111]	x^4-2x^2+1	12	1	1	1	1
B_2+A_1	C_2+A_1	(12)(34) [-1111]	x^4-1	24	1	-1	1	-1
$D_4(a_1)$	$2C_2$	(12)(34) [-11-11]	x^4+2x^2+1	12	1	1	1	1
A_3'	A_3	(1234) [1111]	x^4-1	48	1	-1	-1	1
B_4	C_4	(1234) [-1111]	x^4+1	48	1	1	-1	-1

Table VII: Continued

	$4A_1$		$2A_1$	A_3			$A_3+\tilde{A}_1$	B_3	$\tilde{A}_1$ A_1	
		$2C_2$			$C_3+\tilde{A}_1$	$C_2+2\tilde{A}_1$		C_3	$\tilde{A}_1$ A_1	$3\tilde{A}_1$ $A_1+2\tilde{A}_1$
	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}
(1) [1111]	2	2	3	3	3	3	4	4	4	4
(1) [-1111]	2	-2	3	3	-3	-3	2	-2	2	-2
(1) [-1-111]	2	2	3	3	3	3	0	0	0	0
(1) [-1-1-11]	2	-2	3	3	-3	-3	-2	2	-2	2
(1) [-1-1-1-1]	2	2	3	3	3	3	-4	-4	-4	-4
(12) [1111]	0	0	1	-1	-1	1	-2	-2	2	2
(12) [-1111]	0	0	1	-1	1	-1	-2	2	2	-2
(12) [11-11]	0	0	1	-1	1	-1	0	0	0	0
(12) [-11-11]	0	0	1	-1	-1	1	0	0	0	0
(12) [-11-1-1]	0	0	1	-1	1	-1	2	-2	-2	2
(12) [-1-1-1-1]	0	0	1	-1	-1	1	2	2	-2	-2
(123) [1111]	-1	-1	0	0	0	0	1	1	1	1
(123) [-1111]	-1	1	0	0	0	0	1	-1	1	-1
(123) [111-1]	-1	1	0	0	0	0	-1	1	-1	1
(123) [-1-1-1-1]	-1	-1	0	0	0	0	-1	-1	-1	-1
(12)(34) [1111]	2	2	-1	-1	-1	-1	0	0	0	0
(12)(34) [-1111]	2	-2	-1	-1	1	1	0	0	0	0
(12)(34) [-11-11]	2	2	-1	-1	-1	-1	0	0	0	0
(1234) [1111]	0	0	-1	1	1	-1	0	0	0	0
(1234) [-1111]	0	0	-1	1	-1	1	0	0	0	0

Table VII: Continued

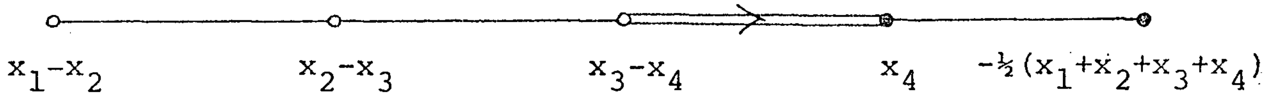
	A_3' B_2+2A_1	$A_2+\tilde{A}_1$	$A_1+\tilde{A}_1$ $2A_1'$	B_2	B_2+A_1	A $3A_1^2$ $2A_1+\tilde{A}_1$
	A_3	$A_2+\tilde{A}_1$	$2\tilde{A}_1$ $A_1+\tilde{A}_1$ $2A_1'$	C_2	$C_2+\tilde{A}_1$ C_2+A_1	A_2
	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
(1) [1111]	6	6	6	6	8	8
(1) [-1111]	0	0	0	0	-4	4
(1) [-1-111]	-2	-2	-2	-2	0	0
(1) [-1-1-11]	0	0	0	0	4	-4
(1) [-1-1-1-1]	6	6	6	6	-8	-8
(12) [1111]	-2	0	2	0	0	0
(12) [-1111]	0	-2	0	2	0	0
(12) [11-11]	0	2	0	-2	0	0
(12) [-11-11]	2	0	-2	0	0	0
(12) [-11-1-1]	0	-2	0	2	0	0
(12) [-1-1-1-1]	-2	0	2	0	0	0
(123) [1111]	0	0	0	0	-1	-1
(123) [-1111]	0	0	0	0	1	-1
(123) [111-1]	0	0	0	0	-1	1
(123) [-1-1-1-1]	0	0	0	0	1	1
(12)(34) [1111]	2	-2	2	-2	0	0
(12)(34) [-1111]	0	0	0	0	0	0
(12)(34) [-11-11]	-2	2	-2	2	0	0
(1234) [1111]	0	0	0	0	0	0
(1234) [-1111]	0	0	0	0	0	0

Note: The first line in the table gives the subsystems of B_4 and the second the subsystems of C_4.

8. F_4.

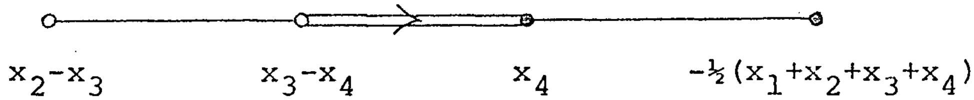
Subsystems of F_4:

The extended Dynkin diagram for F_4 is

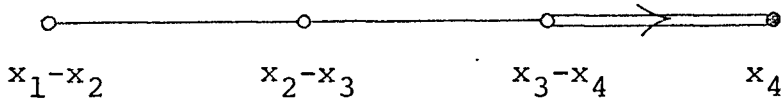


From this diagram we obtain the following subsystems:

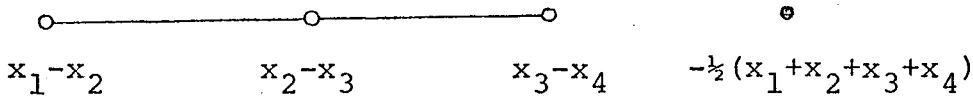
(1) F_4:



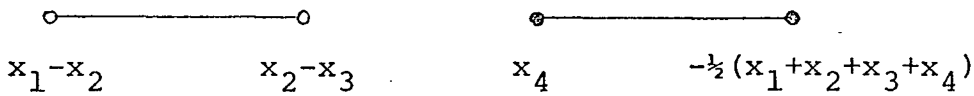
(2) B_4:



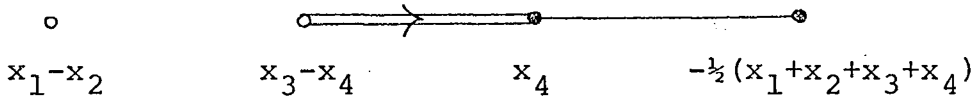
(3) $A_3+\tilde{A}_1$:



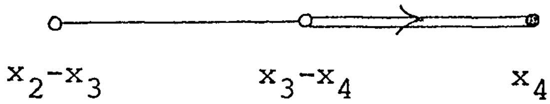
(4) $A_2+\tilde{A}_2$:



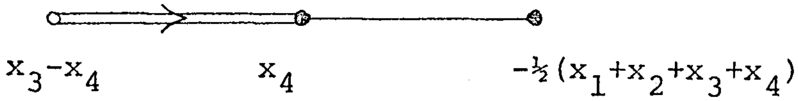
(5) C_3+A_1:



(6) B_3:

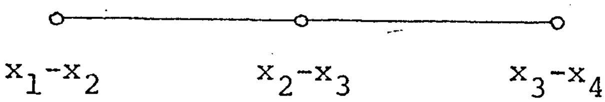


(7) C_3:



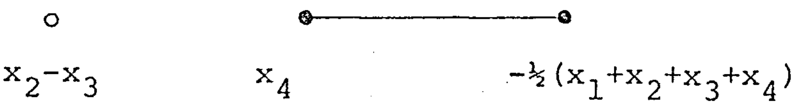
(8) $2A_1+\tilde{A}_1$:

(9) A_3:

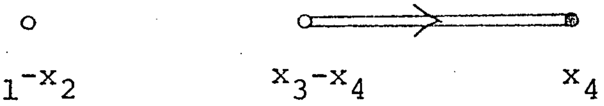


(10) $A_2+\tilde{A}_1$:

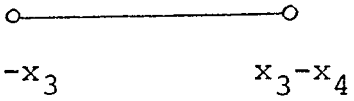
(11) $\tilde{A}_2+A_1$:



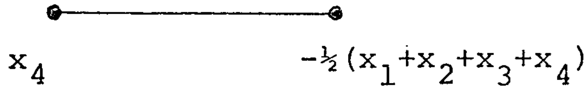
(12) B_2+A_1:



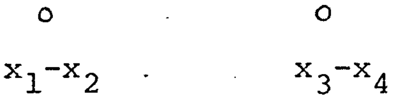
(13) A_2:



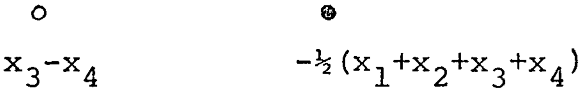
(14) $\tilde{A}_2$:



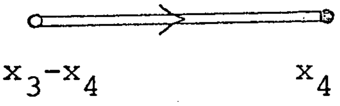
(15) $2A_1$:



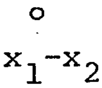
(16) $A_1+\tilde{A}_1$:



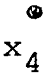
(17) B_2:



(18) A_1:

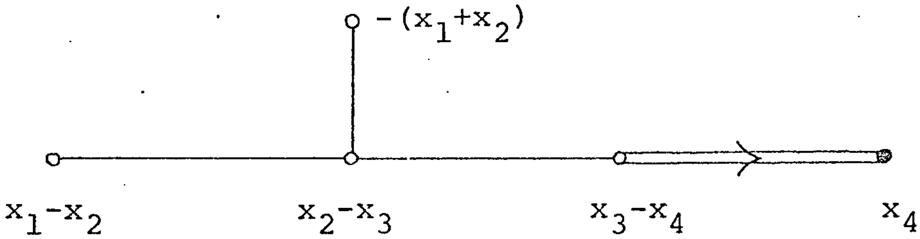


(19) $\tilde{A}_1$:



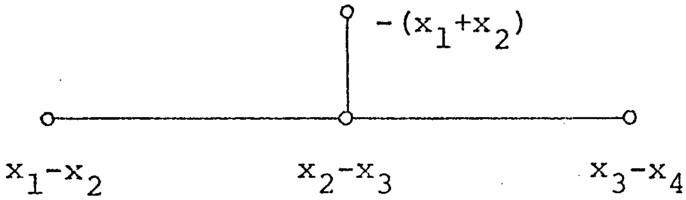
Repeating the process with the above systems of roots we find four subsystems that are not congruent under (F_4) to any of the above, as seen below.

The extended Dynkin diagram for B_4 is

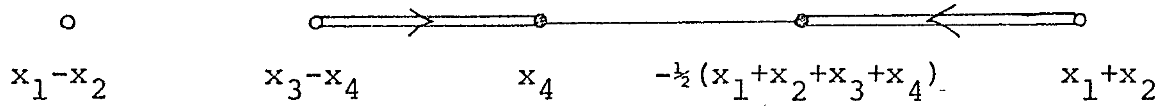


from which we obtain

(20) D_4:



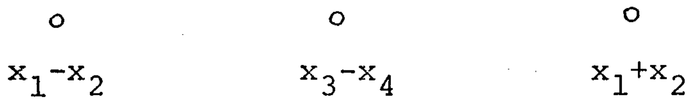
The extended Dynkin diagram for C_3+A_1 is



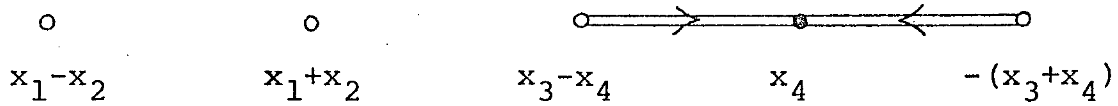
From this diagram we obtain the following two new subsystems:

(21) B_2+2A_1:

(22) $3A_1$:



Finally, from the extended Dynkin diagram for B_2+2A_1



we obtain

(23) $4A_1$:

In the group (F_4), we have the following coset decomposition:

$$(F_4)=(B_4)\cup(B_4)w_r\cup(B_4)w_{x_4}w_r$$

where $r=-\frac{1}{2}(x_1+x_2+x_3+x_4)$. Consequently the (F_4)-orbit of a polynomial $\Pi(S)$ is the union of the (B_4)-orbits of

$$\Pi(S),\ w_r(\Pi(S)),\text{and } w_{x_4}w_r(\Pi(S)).$$

In some cases we find that $\pm w_r(\Pi(S))$ and $\pm w_{x_4}w_r(\Pi(S))$ are polynomials that are already in the (B_4)-orbit of $\Pi(S)$. Then the (F_4)-orbit of $\Pi(S)$ is the same as its (B_4)-orbit. We find this to be the case for the subsystems $A_2+\tilde{A}_2$, C_3+A_1, C_3, $\tilde{A}_2+A_1$, A_2, $\tilde{A}_2$, D_4, $3A_1$ and $4A_1$. In all the other cases it turns out that either

$$w_r(\Pi(S))=\pm\Pi(S)$$

or

$$w_r(\Pi(S))=\pm w_{x_4}w_r(\Pi(S)),$$

and so a basis for $P(S)$ can be found in the union of the (B_4)-orbits of two of

$$\Pi(S), w_r(\Pi(S)), w_{x_4}w_r(\Pi(S)).$$

$P(F_4)$:: We have

$$\Pi(F_4)=\frac{1}{4096}V(y_1^2,y_2^2,y_3^2,y_4^2)V(x_1^2,x_2^2,x_3^2,x_4^2)$$

where

$$y_1=x_1+x_2,$$
$$y_2=x_1-x_2,$$
$$y_3=x_3+x_4,$$
$$y_4=x_3-x_4.$$

$P(B_4)$: We have

$$\Pi(B_4)=x_1x_2x_3x_4V(x_1^2,x_2^2,x_3^2,x_4^2).$$

We find that

$$x_1x_2x_3x_4V(x_1^2,x_2^2,x_3^2,x_4^2),$$
$$-16w_r(\Pi(B_4))=16w_{x_4}w_r(\Pi(B_4))$$
$$=\left[(x_1-x_2)^2-(x_3+x_4)^2\right]\left[(x_1+x_2)^2-(x_3-x_4)^2\right]V(x_1^2,x_2^2,x_3^2,x_4^2)$$

form a basis for $P(B_4)$.

$P(A_3+\tilde{A}_1)$: We have

$$\Pi(A_3+\tilde{A}_1)=-\tfrac{1}{2}(x_1+x_2+x_3+x_4)V(x_1,x_2,x_3,x_4).$$

We find

$$w_r(\Pi(A_3+\tilde{A}_1))=-\Pi(A_3+\tilde{A}_1),$$
$$w_{x_4}w_r(\Pi(A_3+\tilde{A}_1))=x_4V(x_1^2,x_2^2,x_3^2).$$

As in (B_4), we obtain from $x_4V(x_1^2,x_2^2,x_3^2)$ the following four linearly independent polynomials:

$$b_1=x_1V(x_2^2,x_3^2,x_4^2),$$
$$b_2=x_2V(x_1^2,x_3^2,x_4^2),$$
$$b_3=x_3V(x_1^2,x_2^2,x_4^2),$$
$$b_4=x_4V(x_1^2,x_2^2,x_3^2).$$

We have

$$-2(\Pi(A_3+\tilde{A}_1))=\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{vmatrix}$$
$$=-b_1+b_2-b_3+b_4.$$

We note that any other polynomial in the (B_4)-orbit of $\Pi(A_3+\tilde{A}_1)$ can be obtained from this by an appropriate change in signs. We can, therefore, express each of these polynomials as a linear combination of $b_1,\ldots,b_4$ by making the corresponding change in signs in $b_1,\ldots b_4$. Thus $\{b_1,b_2,b_3,b_4\}$ is a basis for $P(A_3+\tilde{A}_1)$.

$P(A_2+\tilde{A}_2)$: We find

$$\Pi(A_2+\tilde{A}_2)=-\frac{1}{4}x_4\left[(x_1+x_2+x_3)^2-x_4{}^2\right]V(x_1,x_2,x_3),$$

and

$$-w_r(\Pi(A_2+\tilde{A}_2))=w_{x_4}w_r(\Pi(A_2+\tilde{A}_2))=\Pi(A_2+\tilde{A}_2).$$

Therefore the (F_4)-orbit of $\Pi(A_2+\tilde{A}_2)$ is the same as its (B_4)-orbit. Let

$$Q=(I-w_{x_1})(\Pi(A_2+\tilde{A}_2)),$$
$$R=(I+w_{x_3})(Q).$$

Then

$$R=x_1x_4(x_2^2-x_3^2)(x_1^2-x_2^2-x_3^2+x_4^2).$$

The (B_4)-orbit of R gives us the following linearly independent polynomials spanning $P(A_2+\tilde{A}_2)$:

$$x_1x_2(x_3^2-x_4^2)(x_1^2+x_2^2-x_3^2-x_4^2),$$
$$x_1x_3(x_2^2-x_4^2)(x_1^2-x_2^2+x_3^2-x_4^2),$$
$$x_1x_4(x_2^2-x_3^2)(x_1^2-x_2^2-x_3^2+x_4^2),$$
$$x_2x_3(x_1^2-x_4^2)(-x_1^2+x_2^2+x_3^2-x_4^2),$$
$$x_2x_4(x_1^2-x_3^2)(-x_1^2+x_2^2-x_3^2+x_4^2),$$
$$x_3x_4(x_1^2-x_2^2)(-x_1^2-x_2^2+x_3^2+x_4^2).$$

$P(C_3+A_1)$: We have

$$\Pi(C_3+A_1)=-\frac{1}{16}\ x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2)\left[(x_1-x_2)^2-(x_3+x_4)^2\right]\left[(x_1-x_2)^2-(x_3-x_4)^2\right]$$

We find

$$-w_r(\Pi(C_3+A_1))=w_{x_4}w_r(\Pi(C_3+A_1))$$
$$=w_{x_1-x_3}w_{x_2-x_4}(\Pi(C_3+A_1)).$$

Therefore we only need to consider the (B_4)-orbit of $\Pi(C_3+A_1)$.

We find that $P(C_3+A_1)$ is spanned by:

$b_1=x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2)\left[(x_1-x_2)^2-(x_3+x_4)^2\right]\left[(x_1-x_2)^2-(x_3-x_4)^2\right],$

$b_2=x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2)\left[(x_1+x_2)^2-(x_3+x_4)^2\right]\left[(x_1+x_2)^2-(x_3-x_4)^2\right],$

$b_3=x_1x_2(x_1^2-x_2^2)(x_3^2-x_4^2)\left[(x_1+x_2)^2-(x_3-x_4)^2\right]\left[(x_1-x_2)^2-(x_3-x_4)^2\right],$

$b_4=x_1x_2(x_1^2-x_2^2)(x_3^2-x_4^2)\left[(x_1+x_2)^2-(x_3+x_4)^2\right]\left[(x_1-x_2)^2-(x_3+x_4)^2\right],$

$b_5=x_1x_4(x_2^2-x_3^2)(x_1^2-x_4^2)\left[(x_1+x_4)^2-(x_2-x_3)^2\right]\left[(x_1-x_4)^2-(x_2-x_3)^2\right],$

$b_6=x_1x_4(x_2^2-x_3^2)(x_1^2-x_4^2)\left[(x_1+x_4)^2-(x_2+x_3)^2\right]\left[(x_1-x_4)^2-(x_2+x_3)^2\right],$

$b_7=x_2x_3(x_2^2-x_3^2)(x_1^2-x_4^2)\left[(x_1-x_4)^2-(x_2+x_3)^2\right]\left[(x_1-x_4)^2-(x_2-x_3)^2\right],$

$b_8=x_2x_3(x_2^2-x_3^2)(x_1^2-x_4^2)\left[(x_1+x_4)^2-(x_2+x_3)^2\right]\left[(x_1+x_4)^2-(x_2-x_3)^2\right],$

$b_9=x_2x_4(x_1^2-x_3^2)(x_2^2-x_4^2)\left[(x_1-x_3)^2-(x_2+x_4)^2\right]\left[(x_1-x_3)^2-(x_2-x_4)^2\right],$

$b_{10}=x_2x_4(x_1^2-x_3^2)(x_2^2-x_4^2)\left[(x_1+x_3)^2-(x_2+x_4)^2\right]\left[(x_1+x_3)^2-(x_2-x_4)^2\right],$

$b_{11}=x_1x_3(x_1^2-x_3^2)(x_2^2-x_4^2)\left[(x_1+x_3)^2-(x_2-x_4)^2\right]\left[(x_1-x_3)^2-(x_2-x_4)^2\right],$

$b_{12}=x_1x_3(x_1^2-x_3^2)(x_2^2-x_4^2)\left[(x_1+x_3)^2-(x_2+x_4)^2\right]\left[(x_1-x_3)^2-(x_2+x_4)^2\right].$

If we substitute

$$y_1=x_2+x_2$$
$$y_2=x_1-x_2$$
$$y_3=x_3+x_4$$
$$y_4=x_3-x_4$$

in b_1,b_2,b_3,b_4 we obtain the simpler forms

$$b_1=-\tfrac{1}{4}y_1y_2y_3y_4V(y_2^2,y_3^2,y_4^2),$$
$$b_2=-\tfrac{1}{4}y_1y_2y_3y_4V(y_1^2,y_3^2,y_4^2),$$
$$b_3=-\tfrac{1}{4}y_1y_2y_3y_4V(y_1^2,y_2^2,y_4^2),$$
$$b_4=-\tfrac{1}{4}y_1y_2y_3y_4V(y_1^2,y_2^2,y_3^2).$$

From this we can see that

$$b_4=b_1-b_2+b_3.$$

Similarly,we find

$$b_8=b_5-b_6+b_7,$$
$$b_{12}=b_9-b_{10}+b_{11}.$$

It can be seen that $\{b_1,b_2,b_3,b_5,b_6,b_7,b_9,b_{10},b_{11}\}$ forms a linearly independent set.

$P(B_3)$: We find

$$\Pi(B_3)=-x_2x_3x_4V(x_2^2,x_3^2,x_4^2),$$
$$-w_r(\Pi(B_3))=w_{x_4}w_r(\Pi(B_3))$$
$$=\frac{1}{8}(x_1-x_2+x_3+x_4)(x_1+x_2-x_3+x_4)(x_1+x_2+x_3-x_4)V(-x_1,x_2,x_3,x_4).$$

Four linearly independent polynomials are found from the (B_4)-orbit of $\Pi(B_3)$;

$$b_1=x_2x_3x_4V(x_2^2,x_3^2,x_4^2),$$
$$b_2=x_1x_3x_4V(x_1^2,x_3^2,x_4^2),$$
$$b_3=x_1x_2x_4V(x_1^2,x_2^2,x_4^2),$$
$$b_4=x_1x_2x_3V(x_1^2,x_2^2,x_3^2).$$

Let

$$
\begin{aligned}
Q&=8w_{x_1}(w_r(\Pi(B_3)))\\
&=(x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)V(x_1,x_2,x_3,x_4)\\
&=\Big(\sum_{i=1}^{4}x_i^3-\sum_{\substack{i,j=1\\ i\neq j}}^{4}x_i^2x_j+2\sum_{\substack{i,j,k=1\\ i<j<k}}^{4}x_ix_jx_k\Big)V(x_1,x_2,x_3,x_4)
\end{aligned}
$$

This can be written in terms of the following determinants:

$$
A=\begin{vmatrix} 1 & 1 & 1 & 1\\ x_1 & x_2 & x_3 & x_4\\ x_1^2 & x_2^2 & x_3^2 & x_4^2\\ x_1^6 & x_2^6 & x_3^6 & x_3^6 \end{vmatrix}=\sum_{i=1}^{4}x_i^3+\sum_{\substack{i,j=1\\ i\neq j}}^{4}x_i^2x_j+\sum_{\substack{i,j,k=1\\ i<j<k}}^{4}x_ix_jx_k)V(x_1,x_2,x_3,x_4),
$$

$$
B=\begin{vmatrix} 1 & 1 & 1 & 1\\ x_1 & x_2 & x_3 & x_4\\ x_1^3 & x_2^3 & x_3^3 & x_4^3\\ x_1^5 & x_2^5 & x_3^5 & x_4^5 \end{vmatrix}=\Big(\sum_{\substack{i,j=1\\ i\neq j}}^{4}x_i^2x_j+2\sum_{\substack{i,j,k=1\\ i<j<k}}^{4}x_ix_jx_k\Big)V(x_1,x_2,x_3,x_4),
$$

$$
C=\begin{vmatrix} 1 & 1 & 1 & 1\\ x_1^2 & x_2^2 & x_3^2 & x_4^2\\ x_1^3 & x_2^3 & x_3^3 & x_4^3\\ x_1^4 & x_2^4 & x_3^4 & x_4^4 \end{vmatrix}=\Big(\sum_{\substack{i,j,k=1\\ i<j<k}}^{4}x_ix_jx_k\Big)V(x_1,x_2,x_3,x_4).
$$

We have

$$Q=A-2B+3C.$$

Let

$$
\begin{aligned}
R&=(I+w_{x_2})(Q),\\
S&=(I+w_{x_3})(R),\\
T&=(I+w_{x_4})(S).
\end{aligned}
$$

Then

$$T=8\left(-x_1\begin{vmatrix}1&1&1\\x_2^2&x_3^2&x_4^2\\x_2^6&x_3^6&x_4^6\end{vmatrix}+3x^3\begin{vmatrix}1&1&1\\x_2^2&x_3^2&x_4^2\\x_2^4&x_3^4&x_4^4\end{vmatrix}\right)$$

$$=-8\left[x_1(x_2^2+x_3^2+x_4^2)V(x_2^2,x_3^2,x_4^2)-3x_1^3V(x_2^2,x_3^2,x_4^2)\right]$$

$$=-8x_1\left[(x_1^2+x_2^2+x_3^2+x_4^2)-4x_1^2\right]V(x_2^2,x_3^2,x_4^2).$$

We can now complete the basis for $P(B_3)$ by applying $w_{x_i-x_j}$ to T:

$$b_5=x_1\left[(x_1^2+x_2^2+x_3^2+x_4^2)-4x_1^2\right]V(x_2^2,x_3^2,x_4^2),$$
$$b_6=x_2\left[(x_1^2+x_2^2+x_3^2+x_4^2)-4x_2^2\right]V(x_1^2,x_3^2,x_4^2),$$
$$b_7=x_3\left[(x_1^2+x_2^2+x_3^2+x_4^2)-4x_3^2\right]V(x_1^2,x_2^2,x_4^2),$$
$$b_8=x_4\left[(x_1^2+x_2^2+x_3^2+x_4^2)-4x_4^2\right]V(x_1^2,x_2^2,x_3^2).$$

$P(C_3)$: In this case it is simpler to write $\Pi(C_3)$ in terms of the y_i as defined in $P(F_4)$. We have

$$\Pi(C_3)=-\frac{1}{64}\,y_1y_3y_4V(y_1^2,y_3^2,y_4^2).$$

We note that the elements $w_{x_i}, w_{x_i\pm x_j}, w_r$, and $w_{x_4}w_r$ of (F_4) can be expressed in terms of $w_{y_i}, w_{y_i\pm y_j}, w_{-\frac{1}{2}(y_1+y_2+y_3+y_4)}$, and $w_{y_4}w_{-\frac{1}{2}(y_1+y_2+y_3+y_4)}$, and conversely. Therefore, we can find the (F_4)-orbit of $\Pi(C_3)$ by applying $w_{y_i}, w_{y_i\pm y_j}, w_{-\frac{1}{2}(y_1+y_2+y_3+y_4)}$, and $w_{y_4}w_{-\frac{1}{2}(y_1+y_2+y_3+y_4)}$ to

$$y_1y_3y_4V(y_1^2,y_3^2,y_4^2).$$

We observe that this is the same as $-w_{x_1-x_2}(\Pi(B_3))$ with y_i replacing the x_i. Hence a basis for $P(C_3)$ is given by the polynomials in the basis for $P(B_3)$ with y_i replacing the x_i.

$P(2A_1+\tilde{A}_1)$: We have

$$\Pi(2A_1+\tilde{A}_1)=-\tfrac{1}{2}(x_1-x_2)(x_3-x_4)(x_1+x_2+x_3+x_4).$$

We find

$$w_r(\Pi(2A_1+\tilde{A}_1))=-\Pi(2A_1+\tilde{A}_1),$$
$$w_{x_4}w_r(\Pi(2A_1+\tilde{A}_1))=-x_4(x_1^2-x_2^2).$$

As in (B_4),

$$x_4(x_1^2-x_2^2),$$
$$x_4(x_1^2-x_3^2),$$
$$x_1(x_2^2-x_3^2),$$
$$x_1(x_2^2-x_4^2),$$
$$x_2(x_1^2-x_3^2),$$
$$x_2(x_1^2-x_4^2),$$
$$x_3(x_1^2-x_2^2),$$
$$x_3(x_1^2-x_4^2)$$

are linearly independent. It can be seen that $\Pi(2A_1+\tilde{A}_1)$ and other polynomials in its (B_4)-orbit are all linear combinations of these elements.

$P(A_3)$: We find

$$\Pi(A_3)=V(x_1,x_2,x_3,x_4),$$
$$w_r(\Pi(A_3))=\Pi(A_3),$$
$$w_{x_4}w_r(\Pi(A_3))=V(x_1^2,x_2^2,x_3^2).$$

As shown for $P(A_3')$ of (B_4), to find the span of the (B_4)-orbit of $V(x_1,x_2,x_3,x_4)$, we may consider the orbit of

$$x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2).$$

We have seen in $P(B_2+2A_1)$ of (B_4) that the following polynomials are linearly independent:

$$b_1=x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2),$$
$$b_2=x_1x_2(x_1^2-x_2^2)(x_3^2-x_4^2),$$

$b_3=x_1x_3(x_1^2-x_3^2)(x_2^2-x_4^2)$,

$b_4=x_2x_4(x_1^2-x_3^2)(x_2^2-x_4^2)$,

$b_5=x_1x_4(x_1^2-x_4^2)(x_2^2-x_3^2)$,

$b_6=x_2x_3(x_1^2-x_4^2)(x_2^2-x_3^2)$.

The (B_4)-orbit of $V(x_1^2,x_2^2,x_3^2)$ gives us three more linearly independent polynomials,completing the basis for $P(A_3)$:

$b_7=V(x_1^2,x_2^2,x_3^2)$,

$b_8=V(x_1^2,x_2^2,x_4^2)$,

$b_9=V(x_1^2,x_3^2,x_4^2)$.

$P(A_2+\tilde{A}_1)$: We find

$\Pi(A_2+\tilde{A}_1)=\frac{1}{2}(x_1+x_2+x_3+x_4)V(x_2,x_3,x_4)$,

$w_r(\Pi(A_2+\tilde{A}_1))=-\Pi(A_2+\tilde{A}_1)$,

$w_{x_4}w_r(\Pi(A_2+\tilde{A}_1))=-x_4V(-x_1,x_2,x_3)$.

We have seen in $P(A_2+\tilde{A}_1)$ of (B_4) that the span of the (B_4)-orbit of $x_4V(x_1,x_2,x_3)$ contains the following linearly independent polynomials:

$b_1=x_1x_2(x_3^2-x_4^2)$,

$b_2=x_1x_3(x_2^2-x_4^2)$,

$b_3=x_1x_4(x_2^2-x_3^2)$,

$b_4=x_2x_3(x_1^2-x_4^2)$,

$b_5=x_2x_4(x_1^2-x_3^2)$,

$b_6=x_3x_4(x_1^2-x_2^2)$.

We can complete this to a basis of $P(A_2+\tilde{A}_1)$ by considering the (B_4)-orbit of $\Pi(A_2+\tilde{A}_1)$.Let

$Q=2w_{x_1-x_4}(\Pi(A_2+\tilde{A}_1))=(x_1+x_2+x_3+x_4)V(x_1,x_2,x_3)$,

$R=w_{x_4}(Q)=(x_1+x_2+x_3-x_4)V(x_1,x_2,x_3)$.

Then

$$Q+R=2(x_1+x_2+x_3)V(x_1,x_2,x_3)$$

$$=2\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix}$$

and

$$(I+w_{x_3})(Q+R)=-4x_1x_2(x_1^2-x_2^2).$$

Therefore, a basis for $P(A_2+\tilde{A}_1)$ is given by $b_1,\ldots b_6$ above and

$$b_7=x_1x_2(x_1^2-x_2^2),$$
$$b_8=x_1x_3(x_1^2-x_3^2),$$
$$b_9=x_1x_4(x_1^2-x_4^2),$$
$$b_{10}=x_2x_3(x_2^2-x_3^2),$$
$$b_{11}=x_2x_4(x_2^2-x_4^2),$$
$$b_{12}=x_3x_4(x_3^2-x_4^2).$$

$P(\tilde{A}_2+A_1)$: We have

$$\Pi(\tilde{A}_2+A_1)=\frac{1}{4}x_4(x_2-x_3)\left[(x_1+x_2+x_3)^2-x_4^2\right],$$

and

$$(I-w_{x_1})(\Pi(\tilde{A}_2+A_1))=x_1x_4(x_2^2-x_3^2),$$

showing that

$$P(\tilde{A}_2+A_1)=P(A_2+\tilde{A}_1).$$

$P(B_2+A_1)$: We find

$$\Pi(B_2+A_1)=x_3x_4(x_1-x_2)(x_3^2-x_4^2),$$
$$-w_r(\Pi(B_2+A_1))=w_{x_4}w_r(\Pi(B_2+A_1)$$
$$=\tfrac{1}{4}\left[(x_1+x_2)^2-(x_3-x_4)^2\right](x_1^2-x_2^2)(x_3-x_4).$$

Therefore, the (B_4)-orbits of $\Pi(B_2+A_1)$ and $w_{x_4}w_r(\Pi(B_2+A_1))$ span $P(B_2+A_1)$. We have already seen that the following polynomials form a basis for $P(B_2+A_1)$ of (B_4):

$$b_1=x_1x_2x_3(x_1^2-x_2^2),$$
$$b_2=x_1x_2x_3(x_1^2-x_3^2),$$
$$b_3=x_1x_2x_4(x_1^2-x_2^2),$$
$$b_4=x_1x_2x_4(x_1^2-x_4^2),$$
$$b_5=x_1x_3x_4(x_1^2-x_3^2),$$
$$b_6=x_1x_3x_4(x_1^2-x_4^2),$$
$$b_7=x_2x_3x_4(x_2^2-x_3^2),$$
$$b_8=x_2x_3x_4(x_2^2-x_4^2).$$

Let

$$Q=(x_1^4-x_2^4)(x_3-x_4)-(x_1^2-x_2^2)(x_3-x_4)^3.$$

We have

$$4w_{x_4}w_r(\Pi(B_2+A_1))=Q+2b_1-2b_3.$$

Therefore we may consider Q instead of $w_{x_4}w_r(\Pi(B_2+A_1))$.

We find

$$\tfrac{1}{2}(Q+w_{x_4}(Q))=x_3(x_1^4-x_2^4)-(x_1^2-x_2^2)(x_3^3+3x_3x_4^2)$$
$$=x_3(x_1^2-x_2^2)(x_1^2+x_2^2-x_3^2-3x_4^2).$$

We can now complete the basis for $P(B_2+A_1)$ by the following linearly independent polynomials:

$$b_9=x_1(x_2^2-x_3^2)(x_1^2-x_2^2-x_3^2-3x_4^2),$$
$$b_{10}=x_1(x_2^2-x_4^2)(x_1^2-x_2^2-3x_3^2-x_4^2),$$
$$b_{11}=x_2(x_1^2-x_3^2)(-x_1^2+x_2^2-x_3^2-3x_4^2),$$
$$b_{12}=x_2(x_1^2-x_4^2)(-x_1^2+x_2^2-3x_3^2-x_4^2),$$
$$b_{13}=x_3(x_1^2-x_2^2)(-x_1^2-x_2^2+x_3^2-3x_4^2),$$
$$b_{14}=x_3(x_1^2-x_4^2)(-x_1^2-3x_2^2+x_3^2-x_4^2),$$
$$b_{15}=x_4(x_1^2-x_2^2)(-x_1^2-x_2^2-3x_3^2+x_4^2),$$
$$b_{16}=x_4(x_1^2-x_3^2)(-x_1^2-3x_2^2-x_3^2+x_4^2).$$

$P(A_2)$: As in the case of (B_4), we have

$$\Pi(A_2)=-V(x_2,x_3,x_4)$$

and

$$P(A_2)=P(2A_1+\tilde{A}_1).$$

$P(\tilde{A}_2)$: We find

$$\Pi(\tilde{A}_2)=\frac{1}{4}x_4\left[(x_1+x_2+x_3)^2-x_4^2\right],$$

$$-w_r(\Pi(\tilde{A}_2))=w_{x_4}w_r(\Pi(\tilde{A}_2))=\Pi(\tilde{A}_2).$$

Therefore the (F_4)-orbit of $\Pi(\tilde{A}_2)$ is the same as its (B_4)-orbit. Let

$$Q=(I-w_{x_1})(\Pi(\tilde{A}_2))=x_1x_4(x_2+x_3).$$

Hence

$$(I+w_{x_3})(Q)=2x_1x_2x_4.$$

Also,

$$(I+w_{x_1}+w_{x_2}+w_{x_3})(\Pi(\tilde{A}_2))=x_4(x_1^2+x_2^2+x_3^2-x_4^2).$$

It can be seen that $\Pi(\tilde{A}_2)$ and any polynomial in its (B_4)-orbit may be expressed as a linear combination of polynomials of the form

$$x_m(x_i^2+x_j^2+x_k^2-x_m^2) \text{ and } x_ix_jx_k$$

Therefore a basis for $P(\tilde{A}_2)$ is given by:

$$x_1(x_2^2+x_3^2+x_4^2-x_1^2),$$
$$x_2(x_1^2+x_3^2+x_4^2-x_2^2),$$
$$x_3(x_1^2+x_2^2+x_4^2-x_3^2),$$
$$x_4(x_1^2+x_2^2+x_3^2-x_4^2),$$
$$x_1x_2x_3,$$
$$x_1x_2x_4,$$
$$x_1x_3x_4,$$
$$x_2x_3x_4.$$

$P(2A_1)$: We have

$$\Pi(2A_1)=(x_1-x_2)(x_3-x_4),$$
$$w_r(\Pi(2A_1))=\Pi(2A_1),$$
$$w_{x_4}w_r(\Pi(2A_1))=-(x_1^2-x_2^2).$$

We observe that $(x_1-x_2)(x_3-x_4)$ is $\Pi(2A_1')$ of (B_4) and $-(x_1^2-x_2^2)$ is $\Pi(2A_1)$ of (B_4). We find that the union of the bases for $P(2A_1)$ and $P(2A_1')$ of (B_4) provides a basis for $P(2A_1)$ of (F_4):

$$x_1x_2,$$
$$x_1x_3,$$
$$x_1x_4,$$
$$x_2x_3,$$
$$x_2x_4,$$
$$x_3x_4,$$
$$x_1^2-x_2^2,$$
$$x_1^2-x_3^2,$$
$$x_1^2-x_4^2.$$

$P(A_1+\tilde{A}_1)$: We have

$$\Pi(A_1+\tilde{A}_1)=-\tfrac{1}{2}(x_3-x_4)(x_1+x_2+x_3+x_4),$$
$$w_r(\Pi(A_1+\tilde{A}_1))=-\Pi(A_1+\tilde{A}_1),$$
$$w_{x_4}w_r(\Pi(A_1+\tilde{A}_1))=-x_4(x_1+x_2).$$

Since $-x_4(x_1+x_2)$ is contained in $P(2A_1)$,

$$P(A_1+\tilde{A}_1)=P(2A_1).$$

$P(B_2)$: We find

$$\Pi(B_2)=x_3x_4(x_3^2-x_4^2).$$

This is one of the elements in the basis of $P(A_2+\tilde{A}_1)$. Therefore

$$P(B_2)=P(A_2+\tilde{A}_1).$$

$P(D_4)$: As in (B_4), $P(D_4)$ is spanned by one element

$$\Pi(D_4)=V(x_1^2,x_2^2,x_3^2,x_4^2).$$

$P(B_2+2A_1)$: We find

$$\Pi(B_2+2A_1)=x_3x_4(x_1^2-x_2^2)(x_3^2-x_4^2).$$

But this is in $P(A_3)$, hence

$$P(B_2+2A_1)=P(A_3).$$

$P(3A_1)$: We have

$$\Pi(3A_1)=(x_1^2-x_2^2)(x_3-x_4).$$

We observe that this is in $P(2A_1+\tilde{A}_1)$, hence

$$P(3A_1)=P(2A_1+\tilde{A}_1).$$

$P(4A_1)$: We have

$$\Pi(4A_1)=-(x_1^2-x_2^2)(x_3^2-x_4^2)$$

and

$$w_r(\Pi(4A_1))=w_{x_4}w_r(\Pi(4A_1))=\Pi(4A_1).$$

Therefore, as in (B_4),

$$(x_1^2-x_2^2)(x_3^2-x_4^2),$$
$$(x_2^2-x_3^2)(x_1^2-x_4^2)$$

form a basis for $P(4A_1)$.

Using the above (F_4)-modules, we obtain seventeen of the irreducible characters of (F_4). The characters that cannot be obtained by the above method can be found by using the following relations:

$$\chi_3=\chi_1\chi_2,$$
$$\chi_6=\chi_1\chi_5,$$

$\chi_7=\chi_1\chi_4$,

$\chi_{10}=\chi_4\chi_5$,

$\chi_{11}=\chi_1\chi_8$,

$\chi_{12}=\chi_1\chi_9$,

$\chi_{14}=\chi_2\chi_{13}$,

$\chi_{22}=\chi_1\chi_{20}$.

The following conjugacy relations are used in calculating the characters of (F_4):

$z=[4A_1]$,

$[A_1]\sim z[3A_1]$,

$[\tilde{A}_1]\sim z[2A_1+\tilde{A}_1]$,

$[2A_1]\sim z[2A_1]$,

$[A_1+\tilde{A}_1]\sim z[A_1+\tilde{A}_1]$,

$[A_2]\sim z[D_4]$,

$[\tilde{A}_2]\sim z[C_3+A_1]$,

$[B_2]\sim z[A_3+\tilde{A}_1]$,

$[A_3]\sim z[A_3]$,

$[B_2+A_1]\sim z[B_2+A_1]$,

$[C_3]\sim z[\tilde{A}_2+A_1]$,

$[B_3]\sim z[A_2+\tilde{A}_1]$,

$[A_2+\tilde{A}_2]\sim z[F_4(a_1)]$,

$[D_4(a_1)]\sim z[D_4(a_1)]$,

$[B_4]\sim z[B_4]$,

$[F_4]\sim z[F_4]$.

Table VIII: Character Table for (F_4).

				Φ	F_4	D_4		B_4
Conjugacy Class Representative		Characteristic Polynomial	h_i	χ_0	χ_1	χ_2	χ_3	χ_4
Φ	(1) [1111]	$x^4-4x^3+6x^2-4x+1$	1	1	1	1	1	2
A_1	(12) [1111]	x^4-2x^3+2x-1	12	1	-1	-1	1	-2
$\tilde{A}_1$	(1) [-1111]	x^4-2x^3+2x-1	12	1	-1	1	-1	0
$2A_1$	(1) [-1-111]	x^4-2x^2+1	18	1	1	1	1	2
$A_1+\tilde{A}_1$	(12) [11-11]	x^4-2x^2+1	72	1	1	-1	-1	0
A_2	(123) [1111]	x^4-x^3-x+1	32	1	1	1	1	2
$\tilde{A}_2$		x^4-x^3-x+1	32	1	1	1	1	-1
B_2	(12) [-1111]	$x^4-2x^3+2x^2-2x+1$	36	1	1	-1	-1	0
$3A_1$	(12) [-1-1-1-1]	x^4+2x^3-2x-1	12	1	-1	-1	1	-2
$2A_1+\tilde{A}_1$	(1) [-1-1-11]	x^4+2x^3-2x-1	12	1	-1	1	-1	0
A_3	(1234) [1111]	x^4-1	72	1	-1	-1	1	-2
B_2+A_1	(12)(34) [-1111]	x^4-1	72	1	-1	1	-1	0
C_3		x^4-x^3+x-1	96	1	-1	-1	1	1
B_3	(123) [-1111]	x^4-x^3+x-1	96	1	-1	1	-1	0
$\tilde{A}_2+A_1$		x^4+x^3-x-1	96	1	-1	-1	1	1
$A_2+\tilde{A}_1$	(123) [111-1]	x^4+x^3-x-1	96	1	-1	1	-1	0
$4A_1$	(1) [-1-1-1-1]	$x^4+4x^3+6x^2+4x+1$	1	1	1	1	1	2
$A_2+\tilde{A}_2$		$x^4+2x^3+3x^2+2x+1$	16	1	1	1	1	-1
$A_3+\tilde{A}_1$	(12) [-11-1-1]	$x^4+2x^3+2x^2+2x+1$	36	1	1	-1	-1	0
C_3+A_1		x^4+x^3+x+1	32	1	1	1	1	-1
D_4	(123) [-1-1-1-1]	x^4+x^3+x+1	32	1	1	1	1	2
$D_4(a_1)$	(12)(34) [-11-11]	x^4+2x^2+1	12	1	1	1	1	2
B_4	(1234) [-1111]	x^4+1	144	1	1	-1	-1	0
$F_4(a_1)$		$x^4-2x^3+3x^2-2x+1$	16	1	1	1	1	-1
F_4		x^4-x^2+1	96	1	1	1	1	-1

	$4A_1$			$\tilde{A}_1$ A_1	$A_3+\tilde{A}_1$				$A_2+\tilde{A}_2$		B_3	C_3
	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
Φ	2	2	2	4	4	4	4	4	6	6	8	8
A_1	0	0	2	2	-2	0	-2	2	0	0	-4	0
$\tilde{A}_1$	2	-2	0	2	2	0	-2	-2	0	0	0	-4
$2A_1$	2	2	2	0	0	4	0	0	-2	-2	0	0
$A_1+\tilde{A}_1$	0	0	0	0	0	0	0	0	2	-2	0	0
A_2	-1	-1	2	1	1	-2	1	1	0	0	2	-1
$\tilde{A}_2$	2	2	-1	1	1	-2	1	1	0	0	-1	2
B_2	0	0	0	2	-2	0	2	-2	-2	2	0	0
$3A_1$	0	0	2	-2	2	0	2	-2	0	0	4	0
$2A_1+\tilde{A}_1$	2	-2	0	-2	-2	0	2	2	0	0	0	4
A_3	0	0	2	0	0	0	0	0	0	0	0	0
B_2+A_1	2	-2	0	0	0	0	0	0	0	0	0	0
C_3	0	0	-1	1	-1	0	-1	1	0	0	1	0
B_3	-1	1	0	1	1	0	-1	-1	0	0	0	1
$\tilde{A}_2+A_1$	0	0	-1	-1	1	0	1	-1	0	0	-1	0
$A_2+\tilde{A}_1$	-1	1	0	-1	-1	0	1	1	0	0	0	-1
$4A_1$	2	2	2	-4	-4	4	-4	-4	6	6	-8	-8
$A_2+\tilde{A}_2$	-1	-1	-1	-2	-2	1	-2	-2	3	3	2	2
$A_3+\tilde{A}_1$	0	0	0	-2	2	0	-2	2	-2	2	0	0
C_3+A_1	2	2	-1	-1	-1	-2	-1	-1	0	0	1	-2
D_4	-1	-1	2	-1	-1	-2	-1	-1	0	0	-2	1
$D_4(a_1)$	2	2	2	0	0	4	0	0	2	2	0	0
B_4	0	0	0	0	0	0	0	0	0	0	0	0
$F_4(a_1)$	-1	-1	-1	2	2	1	2	2	3	3	-2	-2
F_4	-1	-1	-1	0	0	1	0	0	-1	-1	0	0

	A_2 $3A_1$ $2A_1+\tilde{A}_1$	$\tilde{A}_2$	$2A_1$ $A_1+\tilde{A}_1$	A_3 B_2+2A_1	C_3+A_1		B_2 $A_2+\tilde{A}_1$ $\tilde{A}_2+A_1$	B_2+A_1
	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
Φ	8	8	9	9	9	9	12	16
A_1	0	4	3	-3	-3	3	0	0
$\tilde{A}_1$	4	0	3	3	-3	-3	0	0
$2A_1$	0	0	1	1	1	1	-4	0
$A_1+\tilde{A}_1$	0	0	1	-1	1	-1	0	0
A_2	-1	2	0	0	0	0	0	-2
$\tilde{A}_2$	2	-1	0	0	0	0	0	-2
B_2	0	0	1	-1	1	-1	0	0
$3A_1$	0	-4	3	-3	-3	3	0	0
$2A_1+\tilde{A}_1$	-4	0	3	3	-3	-3	0	0
A_3	0	0	-1	1	1	-1	0	0
B_2+A_1	0	0	-1	-1	1	1	0	0
C_3	0	-1	0	0	0	0	0	0
B_3	-1	0	0	0	0	0	0	0
$\tilde{A}_2+A_1$	0	1	0	0	0	0	0	0
$A_2+\tilde{A}_1$	1	0	0	0	0	0	0	0
$4A_1$	-8	-8	9	9	9	9	12	-16
$A_2+\tilde{A}_2$	2	2	0	0	0	0	-3	-2
$A_3+\tilde{A}_1$	0	0	1	-1	1	-1	0	0
C_3+A_1	-2	1	0	0	0	0	0	2
D_4	1	-2	0	0	0	0	0	2
$D_4(a_1)$	0	0	-3	-3	-3	-3	4	0
B_4	0	0	-1	1	-1	1	0	0
$F_4(a_1)$	-2	-2	0	0	0	0	-3	2
F_4	0	0	0	0	0	0	1	0

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