ORDERABLE TOPOLOGICAL SPACES

by

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ABSTRACT

Let $(X , \mathfrak{J})$ be a topological space. If $<$ is a total ordering on X, then $(X , \mathfrak{J} , <)$ is said to be an ordered topological space if a subbasis for $\mathfrak{J}$ is the collection of all sets of the form $\{x \in X \mid x < t\}$ or $\{x \in X \mid t < x\}$ where $t \in X$. The pair $(X , \mathfrak{J})$ is said to be an orderable topological space if there exists a total ordering, $<$, on X such that $(X , \mathfrak{J} , <)$ is an ordered topological space.

Definition: Let T be a subspace of the real line $\mathbb{R}$. Let Q be the union of all non-trivial components of T, both of whose end points belong to $Cl_{\mathbb{R}}(Cl_{\mathbb{R}}(T) - T)$.

The following characterization of orderable subspaces of $\mathbb{R}$ is due to M. E. Rudin.

Theorem: Let T be a subspace of $\mathbb{R}$ with the relativized usual topology. Then T is orderable if and only if T satisfies the following two conditions:

(1) If $T - Q$ is compact and $(T-Q) \cap Cl_{\mathbb{R}}(Q) = \emptyset$ then either $Q = \emptyset$ or $T - Q = \emptyset$.

(2) If I is an open interval of $\mathbb{R}$ and p is an end point of I and if $\{p\} \cup (I \cap (T-Q))$ is compact and $\{p\} = Cl_{\mathbb{R}}(I \cap Q) \cap Cl_{\mathbb{R}}(I \cap (T-Q))$, then $p \notin T$ or $\{p\}$ is a component of T.

This theorem enables us to prove a conjecture of I. L. Lynn, namely

Corollary: if T contains no open compact sets then T is totally orderable.

If T is a subspace of an arbitrary ordered topological space a generalization of the theorem can be made. The generalized theorem is stated and some examples are given.

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INTRODUCTION

In this thesis we examine a result by Mary Ellen Rudin which gives a characterization of orderable topological spaces. Dr. Rudin won the 1965 Netherlands Mathematical Society prize for this work.

Dr. Rudin has stated and proved the result for the special case when the topological space is a subspace of the real line. Since the proof of the general result follows almost identically that of the special case and since an attempt to prove the main theorem resulted in complicated unreadable notation, Dr. Rudin has omitted the proof of this theorem. We have pointed out, in the text of the thesis, that the general result cannot be used in many applications. In the discussion following Theorem (2) we give our reasons.

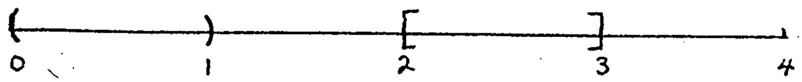
The paper by Dr. Rudin was not read by a referee and there were some mistakes which we have corrected. We have been in touch with Dr. Rudin and she has acknowledged those corrections made to her paper. The minor errors in Lemma (8) and the more serious error in Lemma (12) of Dr. Rudin's paper will be described in the text of the thesis. There is also an error in the statement of the theorem as it appeared in the "Transactions of the American Mathematical Society" and Dr. Rudin has pointed this out.

As the proof of the result for a subspace of the real line is a constructive one, we have included sufficient examples in this thesis to give the reader a good idea of the construction involved. Thus the reader can achieve a good grasp of the proof by reading the sections labeled "Discussion".

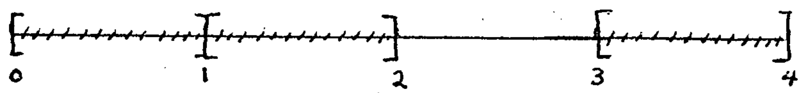
In addition, we have related Dr. Rudin's paper to work done by I. L. Lynn and shown how this follows from Dr. Rudin's work. Also, some applications for the main theorem are given.

NOTATION

Given a topological space, T and a subset, $X \subset T$, then the closure of X in T is denoted $Cl_T(X)$. The set of real numbers is denoted by $\mathbb{R}$ and the natural numbers are defined by $\mathbb{N} = \{1,2,3\ldots\}$. If $X \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ define $aX + b = \{ax + b \mid x \in X\}$. The usual conventions are used in denoting intervals of $\mathbb{R}$. For example, $(0,1) = \{x \in R \mid o < x < 1\}$ and $[2,3] = \{x \in \mathbb{R} \mid 2 \leq x \leq 3\}$ and $(0,1) \cup [2,3]$ is indicated in diagrams as follows.



It will often be usefull to indicate on a diagram a set whose elements are intervals. We use shading to indicate such a set. For example, $\{[0,1]\ , [1,2]\ , [3,4]\}$ is denoted in diagrams as follows



Definition: Let $(X,\mathfrak{J})$ be a topological space. If $<$ is a total ordering on X, then $(X,\mathfrak{J},<)$ is said to be an ordered topological space if a subbasis for $\mathfrak{J}$ is the collection of all sets of the form

$$\{x \in X \mid x < t\} \text{ or } \{t \in X \mid t < x\} \text{ where } t \in X .$$

$(X,\mathfrak{J})$ is said to be an orderable topological space if there exists a total ordering, Δ , on X such that $(X,\mathfrak{J},\Delta)$ is an ordered topological space.

Example(1): Let $T = (0,1) \cup [2,3]$. Let $\mathfrak{J}$ be the usual topology for $\mathbb{R}$ relativized to T . Then T is not orderable.
proof: Suppose $f\colon T \to S$ is an homeomorphism of T onto an ordered topological space, S . Without loss of generality, f is order preserving on both $(0,1)$ and $[2,3]$. For example, assume f is neither order preserving nor order reversing on $(0,1)$. Then, without loss of generality, there exist $a, b, c \in (0,1)$ such that $a < c < b$ and $f(a) < f(b) < f(c)$. Let $B = \{x \in (a,b) \mid f(x) > f(b)\}$. Since $c \in B$, $B \neq \emptyset$ and let $y = \inf(B)$. There are 3 possibilities.

(1) If $y = a$ then $y \in Cl_T(B)$ but $f(y) \notin Cl_S(f(B))$.
(2) If $y \neq a$ and $y \in B$ then $y \in Cl_T((a,y))$ but $f(y) \notin$
(3) $Cl_S(f(a,y))$
(3) If $y \neq a$ and $y \notin B$ then $y \in Cl_T(B)$ but $f(y) \notin Cl_S(f(B))$.

Hence we can assume that f is order preserving on $(0,1)$ and on $[2,3]$. $f[(0,1)]$ is an interval, for otherwise $f[(0,1)]$ is not connected. Similarly for $f[[2,3]]$. Without loss of generality, $f(a) < f(b)$ for all $a \in (0,1)$ and $b \in [2,3]$. Hence, since $f[(0,1)]$ has no largest element, $f(2)$ is an accumulation point of $f[(0,1)]$ which contradicts the facts that f is an homeomorphism and that 2 is not an accumulation point of $(0,1)$

Example (2): Define

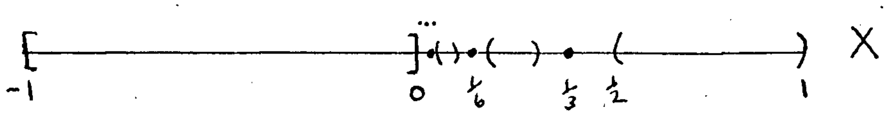
$I = [-1,0]$,

$P = \{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots\}$,

$G_1 = (\frac{1}{2}, 1)$, $G_2 = (\frac{1}{5}, \frac{1}{4})$, $G_3 = (\frac{1}{8}, \frac{1}{7})$, $G_4 = (\frac{1}{11}, \frac{1}{10})$, ...

Let $X = I \cup P \bigcup_{i=1}^{\infty} G_i$. With the relativized topology, X is not orderable.

proof: Suppose $g: X \to S$ is an homeomorphism of X onto S, an ordered topological space. Choose from each G_i, $Z_i \in G_i$. Then $\{g(Z_i)\}_{i=1}^{\infty}$ converges to $g(0)$ as does $\{g(\frac{1}{3n+3})\}_{n=0}^{\infty}$. Now $g[I]$ is an interval, for otherwise $g[I]$ is not connected and since g, without loss of generality, preserves order on I, $g(0)$ is the maximum of $g[I]$. Now since the two sequences $\{g(\frac{1}{3n+3})\}_{n=0}^{\infty}$ and $\{g(Z_i)\}_{i=1}^{\infty}$ both converge to $g(0)$ and only finite numbers of each sequence are less than $g(0)$, we can choose $m, r \in \mathbb{N}$ such that $g(Z_m) < g(\frac{1}{3r+3})$ and no point of $\{g(\frac{1}{2n+3})\}_{n=0}^{\infty}$ is between $g(Z_m)$ and $g(\frac{1}{2r+3})$. Now any open set containing $g(\frac{1}{2r+3})$ contains a point of $g[G_m]$ but $\frac{1}{2r+3}$ is not accumulation point of G_m and this contradiction establishes that X is not linearly orderable.



Definition: Let T be a subspace of $\mathbb{R}$. Denote by Q the union of all non-trivial components of T, both of whose end points belong to $Cl_{\mathbb{R}}(Cl_{\mathbb{R}}(T)-T)$.

The following theorem gives a characterization of orderable topological subspaces of $\mathbb{R}$. The remainder of this chapter will be devoted to the proof.

Theorem (1): Let T be a subspace of $\mathbb{R}$ with relativized usual topology. Then T is orderable if and only if T satisfies the following 2 conditions:

(1) If $T - Q$ is compact and $(T-Q) \cap Cl_{\mathbb{R}}(Q) = \emptyset$ then either $Q = \emptyset$ or $T - Q = \emptyset$.

(2) If I is an open interval of $\mathbb{R}$ and p is an endpoint of I and if $\{p\} \cup (I \cap (T-Q))$ is compact and $\{p\} = Cl_{\mathbb{R}}(I \cap Q) \cap Cl_{\mathbb{R}}(I \cap (T-Q))$, then $p \notin T$ or $\{p\}$ is a component of T.

Notes:

1. In example (1) above, $Q = (0,1)$, $T - Q = [2,3]$ and hence $T - Q$ is compact and $(T-Q) \cap Cl_{\mathbb{R}}(Q) = \emptyset$. But $Q \neq \emptyset$ and $(T-Q) \neq \emptyset$ and so condition (1) is not satisfied.

2. In example (2) above, $Q = (\frac{1}{2}, 1) \cup (\frac{1}{5}, \frac{1}{4}) \cup (\frac{1}{8}, \frac{1}{7}) \cup \ldots$, $Cl_{\mathbb{R}}(Q) = [\frac{1}{2}, 1] \cup [\frac{1}{5}, \frac{1}{4}] \cup \ldots \cup \{0\}$, and $T - Q = [-1, 0] \cup \{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots\}$.

Thus $T - Q$ is compact and $(T-Q) \cap Cl_{\mathbb{R}}(Q) = \{0\}$ and so condition (1) holds vacuously. However, consider the interval $I = (0,1)$ and let $p = 0$. Then $\{0 , \frac{1}{3} , \frac{1}{6} , \frac{1}{9} , \ldots\} = \{p\} \cup (I \cap (T-Q))$ is compact and $\{0\} = Cl_{\mathbb{R}}(I \cap Q) \cap Cl_{\mathbb{R}}(I \cap (T-Q))$ but $0 \in T$ and $\{0\}$ is not a component of T hence condition (2) does not hold.

We prove now that conditions (1) and (2) are necessary

Necessity of condition (1):

Suppose that $T - Q$ is compact, that $(T-Q) \cap Cl_{\mathbb{R}}(Q) \neq \emptyset$, that $q \in Q$ and $p \in T - Q$, and that Δ is a total order of T giving T interval topology. Without loss of generality, assume $p \Delta q$. Let $p' = \sup \{x \in T - Q \mid x \Delta q\}$. Then since $T - Q$ is compact, $p' \in T - Q$. Since $p' \notin Cl_{\mathbb{R}}(Q)$ and T has interval topology, there exists $r , q' \in T$ such that $p' \in \{x \in T \mid r \Delta x \Delta q'\} \subset T - Q$. By definition of p' , $q \in Q$ and q' is an end point of a component of Q . But $q' \notin Cl_{\mathbb{R}}(Cl_{\mathbb{R}}(T) - T)$ and this contradiction establishes the result.

Necessity of condition(2):

Suppose I is an open interval of $\mathbb{R}$, p is an endpoint of I , $\{p\} \cup (I \cap (T-Q))$ is compact, and $\{p\}$ is the intersection of $Cl_{\mathbb{R}}(I \cap Q)$ and $Cl_{\mathbb{R}}(I \cap (T-Q))$. Suppose, also, that

$p \in T$, $t \neq p$, and t is a member of C, the component of T to which p belongs. Suppose that Δ is a total ordering on T for which T has interval topology. There is no point $i \in I \cap T$ such that $i \in C$ for $C \subset T - Q$ or else $C \subset Q$. If we assume there exists $i \in I$, $i \in C$ and $C \subset T - Q$ then p is not an accumulation point of $I \cap Q$. Similarly for $C \subset Q$, p is not an accumulation point of $I \cap (T-Q)$. Hence there is no point $i \in I \cap T$ between p and t in Δ. For otherwise $\{x \in C \mid x \,\Delta\, i\}$ and $\{x \in C \mid i \,\Delta\, x\}$ is a separation for C. Without loss of generality, assume $t < p$ and $t \,\Delta\, p$. Choose $i \in I \cap T$ where $p \,\Delta\, i$. Then $\{x \in T \mid t < x < i\}$ is an open set containing p and so there exist c and i_0 such that c, $i_0 \in \{x \in T \mid t < x < i\}$ and $p \in \{x \in T \mid c \,\Delta\, x \,\Delta\, i_0\} \subset \{x \in T \mid t < x < i\}$, and further, $\{x \in T \mid p \,\Delta\, x \,\Delta\, i_0\} \subset I$. Since p is an accumulation point of $I \cap Q$, there exists $q \in I \cap Q$ such that $q \in \{x \in T \mid c \,\Delta\, x \,\Delta\, i_0\}$ and $p \,\Delta\, q$. Similarly, since p is an accumulation point of $I \cap (T-Q)$ and $\{p\} \cup \{I \cap (T-Q)\}$ is compact, there exists $p' \in I \cap (T-Q)$ where $p \,\Delta\, p' \,\Delta\, q$ and $\{x \in T \mid p' \,\Delta\, x \,\Delta\, q\}$ contains no point of $T - Q$. Since $p' \neq p$, $p' \notin Cl_{\mathbb{R}}(I \cap Q)$ and hence there exist q' and r where $p' \in \{x \in T \mid q' \,\Delta\, x \,\Delta\, r\} \subset T - Q$. Then $q' \in Q$ and q' is an end point of a component of Q but $q' \notin Cl_{\mathbb{R}}(Cl_{\mathbb{R}}(T) - T)$. This contradiction establishes the result.

Define $\mathbb{R}^*$ to be the compactification of $\mathbb{R}$ by adding the points $-\infty$ and ∞ where $-\infty < x < \infty$ for all $x \in \mathbb{R}$. Now we replace $\mathbb{R}$ by S, which is obtained by substituting an interval homeomorphic to $[0, 1]$ for each trivial component of $\mathbb{R}^* - T$. When this is done T is a subspace of S and S can be given a total order Δ which preserves the order of T in R. Clearly if the theorem is shown with $\mathbb{R}$ replaced by S, it can be shown as stated.

Definition: Let H be the union of all open intervals I of S such that $Cl_S(I) \cap (T-Q)$ is compact and $I \cap (T-Q) \cap Cl_S(Q) = \emptyset$.

Lemma (0): If $S = H$ then $T - Q$ is compact and $(T-Q) \cap Cl_S(Q) = \emptyset$.

Proof: Since H is compact there exist $I_1, I_2, \ldots, I_n$, open intervals of H such that $H \subset \bigcup_{i=1}^{n} I_i$ and $Cl_S(I_i) \cap (T-Q)$ is compact and $I_i \cap (T-Q) \cap Cl_S(Q) = \emptyset$ for $i = 1, 2, \ldots, n$

Then

$$T - Q = (T-Q) \cap H$$
$$= \bigcup_{i=1}^{n} (T-Q) \cap Cl_S(Ii) \text{ and hence } T - Q \text{ is compact .}$$

Also, $(T-Q) \cap Cl_S(Q) = \bigcup_{i=1}^{n} [(T-Q) \cap Cl_S(Q) \cap I_i]$

$$= \emptyset \ .$$

Lemma (1): If $S = H$ and condition (1) holds, then T is totally orderable.

Proof: Suppose $S = H$. Then condition (1) and the previous proposition imply $T \subset Q$ or $Q = \emptyset$. If $Q = \emptyset$, then since $S = H$, $T - Q = T$ is compact. This clearly implies that T is linearly orderable under the order Δ . For let $x_0 \in T$ be given and $x_0 \in \{x \in T \mid a \, \Delta \, x \, \Delta \, b\}$ where $a, b \in S$. Then since T is compact, $a' = \sup\{x \in T \mid x \, \Delta \, a$ or $x = a\}$ and $b' = \inf\{x \in T \mid x \, \Delta \, b$ or $x = b\}$ are both in T and

$$\{x \in T \mid a \, \Delta \, x \, \Delta \, b\} = \{x \in T \mid a' \, \Delta \, x \, \Delta \, b'\} \ .$$

If $T \subset Q$ then also T is linearly orderable under Δ . For let $x_0 \in T$ be given and $x_0 \in \{x \in T \mid a \, \Delta \, x \, \Delta \, b\}$ where a , $b \in S$. If x_0 is an interior point of Q , clearly there exist a' , $b' \in T$ where

$x_0 \in \{x \in T \mid a' \, \Delta \, x \, \Delta \, b'\} \subset \{x \in T \mid a \, \Delta \, x \, \Delta \, b\}$.

If x_0 is not an interior point of Q then x_0 is an end point of a component of T and since $x_0 \in Cl_S(Cl_S(T) - T)$ there exist a', $b' \in T$ such that

$$x_0 \in \{x \in T \mid a' \, \Delta \, x \, \Delta \, b'\} \subset \{x \in T \mid a \, \Delta \, x \, \Delta \, b\}.$$

We now complete the proof of the theorem by showing that if $S - H \neq \emptyset$, condition (2) implies that T is linearly orderable.

Lemma (1): Suppose condition (2) holds and let I and J be two open intervals of S such that $I \subset H$, $J \subset H$, $I \cap J = \emptyset$, and x is a common end point of I and J where $x \notin H$. If $J \subset T$ then $J \subset T - Q$ and under this assumption, $x \notin Cl_S(I \cap (T-Q))$

Discussion: A method of proof can be seen if one tries to construct a counterexample with the required conditions. In all cases let $x = 0$ $I = (-1, 0)$, $J = (0, 1)$.

Case (1): Suppose $x \notin T$. Since the components of $S - T$ are non-trivial, our example might be the following.

$T = (-1, \frac{1}{2}) \cup (0, 1)$. Here, clearly, $x \in H$.

Case (2): Suppose $x \in T$. There are 3 distinct ways to make x satisfy the condition $x \in Cl_S(Cl_S(T) - T)$

(a) Let $A = [\frac{1}{2}, 1)$

$$T = (\tfrac{1}{4} A - 1) \cup (\tfrac{1}{8} A - \tfrac{1}{2}) \cup (\tfrac{1}{16} A - \tfrac{1}{4}) \cup \ldots \cup [0, 1)$$

In this case there are points of I not in H.

(b) Let $A = (\frac{1}{2}, 1)$, and

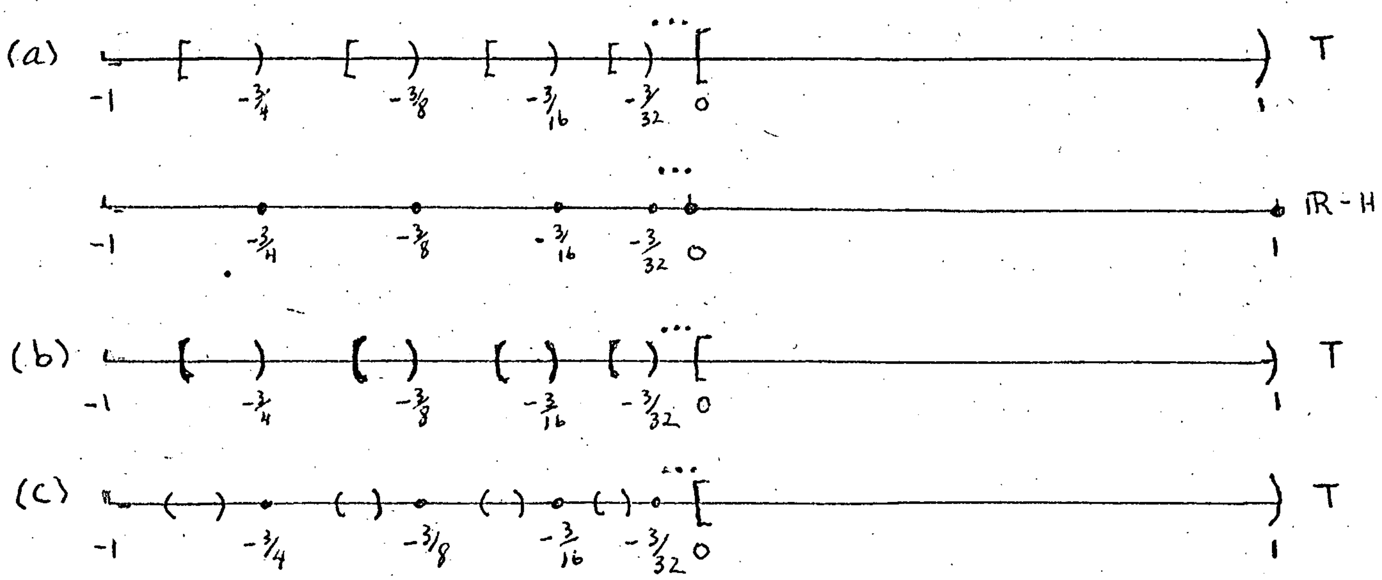
$$T = (\tfrac{1}{4} A - 1) \cup (\tfrac{1}{8} A - \tfrac{1}{2}) \cup (\tfrac{1}{16} A - \tfrac{1}{4}) \cup \ldots \cup [0, 1).$$

In this case $x \in H$.

(c) Let $A = (\frac{1}{2}, \frac{3}{4}) \cup \{1\}$, and

$$T = (\frac{1}{4} A - 1) \cup (\frac{1}{8} A - \frac{1}{2}) \cup (\frac{1}{16} A - \frac{1}{4}) \cup \ldots \cup [0, 1) .$$

In this case, condition (2) does not hold. The other part of the lemma is easily seen.



Proof: We will now formalize the arguement in the discussion. Since $J \subset T$, either $J \subset Q$ or $J \subset T - Q$. Assume to the contrary that $J \subset Q$. Then there exists an open interval $I' \subset I$ such that x is an endpoint of I' and $\{x\} \cup (I' \cap (T-Q))$ is compact. For otherwise there is a point of I which is also in H . Hence if $x \in T$, by condition (2) , x is not in both $Cl_S(I \cap Q)$ and $Cl_S(I \cap (T-Q))$. If $x \notin T$, then $x \notin Cl_S(I \cap (T-$ since the component of $S - T$ containing x is not trivial. Hence if $x \notin T$ or $x \notin Cl_S(I \cap (T-Q))$ then clearly $x \in H$, a contradiction. If $x \notin T$ and $x \notin Cl_S(I \cap Q)$ then , by definition of Q , there is a sequence $\{x_n\}_{n=1}^{\infty}$ outside of H .

and converging to x . This contradiction establishes that $J \subset T - Q$.

Now suppose $x \in Cl_S(I \cap (T-Q))$. Then $x \in T$ since the components of $S - T$ are non-trivial and from above, $x \notin Cl_S(I \cap Q)$. Hence there exists an open interval, I' containing x such that $I' \cap (T-Q) \cap Cl_S(Q) = \emptyset$ and $Cl_S(I') \cap (T-Q)$ is compact. (For, otherwise, I' contains a sequence of points not in H and converging to x) Hence $x \in H$ and this contradiction completes the proof.

Lemma (2): Suppose condition (2) holds and let X be an open interval of S whose end points are not in T . Suppose $Cl_S(X) - H = \{x\}$. Then there is a total ordering, θ for $X \cap T$ such that $X \cap T$ has interval topology. Further, $\theta(X \cap T)$ has no first element but has a last element.

Discussion: We consider 2 cases.

Case (1): Suppose x is the closure of some non-trivial component of T .

Example (3): $x = 0$, $T = (0 , \frac{1}{2}]$, $X = (-1 , 1)$.

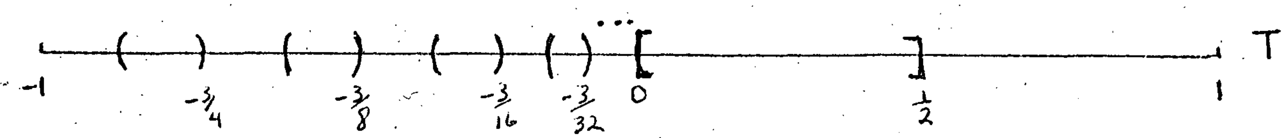
In this example $x \notin T$ and $x \notin H$ because for any open interval, I containing x , $Cl_S(I) \cap (T-Q)$ is not compact. There can be no increasing sequence from T converging to x because $S - T$ contains no non-trivial components, hence locally about x this example is typical for $x \notin T$.

Obviously the induced order from $\mathbb{R}$ satisfies the hypothesis.

Example (4): Let $A = (\frac{1}{2}, 1)$, $x = 0$. Define

$T = (\frac{1}{4} A - 1) \cup (\frac{1}{8} A - \frac{1}{2}) \cup (\frac{1}{16} A - \frac{1}{4}) \cup \ldots \cup [0, \frac{1}{2}]$, $X = (-1, 1)$.

In this example $x \in T$ and $x \notin H$ because x is in the closure of $T - Q$ and in the closure of Q. There can be no increasing sequence from $T - Q$ converging to x because of condition (2), hence, locally about x this example is typical for $x \in T$. Again the induced order from $\mathbb{R}$ satisfies the hypothesis.



Case (2): Suppose x is not in the closure of some non trivial component of T.

Example (5): let $x = 0$,

$$T = (\tfrac{1}{3},\tfrac{1}{2}) \cup (-\tfrac{1}{2},-\tfrac{1}{3}) \cup (\tfrac{1}{6},\tfrac{1}{5}) \cup (-\tfrac{1}{5},-\tfrac{1}{6}) \cup (\tfrac{1}{9},\tfrac{1}{8}) \cup (-\tfrac{1}{8},-\tfrac{1}{9})$$

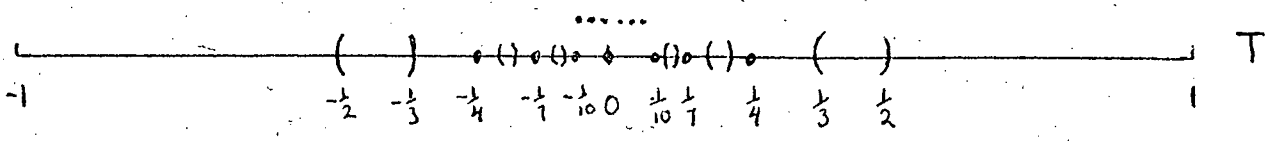
$\cup \ldots \cup \{0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{7}, -\frac{1}{7}, \frac{1}{10}, -\frac{1}{10}, \ldots\}$, $X = (-1,1)$.

Here, $x \in T$ and $x \notin H$ because x is in the closure of

$T - Q$ and in the closure of Q. An order, θ which satisfies the hypothesis is as follows:

$$(\tfrac{1}{3},\tfrac{1}{2})\ \theta\,(-\tfrac{1}{2},-\tfrac{1}{3})\ \theta\,(\tfrac{1}{6},\tfrac{1}{5})\ \theta\,(-\tfrac{1}{5},-\tfrac{1}{6})\ \theta\ \ldots\ \theta\{0\}\ \ldots\ \theta\{-\tfrac{1}{7}\}\theta\{\tfrac{1}{7}\}\theta\{-\tfrac{1}{4}\}\ \theta\{\tfrac{1}{4}\}\ .$$

To complete the definition of θ, we add that θ orders each interval which lies to the left of 0 as the induced order from $\mathbb{R}$ and each interval which lies to the right of 0 as the reverse of the induced order from $\mathbb{R}$.



Proof: Let $X = (p,q)$, Let $I = (p,x)$, $J = (x,q)$. Then the order for $X \cap T$ is as follows:

$J \cap (T-Q)$ has the total ordering induced by Δ. We define

$$\begin{aligned}
\theta(J\cap(T-Q)) &= \Delta(J\cap(T-Q)) . \quad \text{Similarly,}\\
\theta(I\cap(T-Q)) &= -\Delta(I\cap(T-Q)) ,\\
\theta(J\cap Q) &= -\Delta(J\cap Q) ,\\
\theta(I\cap Q) &= \Delta(I\cap Q) .
\end{aligned}$$

Hence to describe a total ordering, θ for $X \cap T$ we need only specify an order θ for the set $A = \{ (J\cap(T-Q)), I\cap(T-Q) , J\cap Q , I\cap Q , \{x\} \cap T \}$.

Case (1): There exists a component, J' of $J\cap T$ such that

$x \in Cl_S(J')$. Then we define the following order for A ,

$(J \cap Q)\ \theta\ (I \cap Q)\ \theta\ (\{x\} \cap T)\ \theta\ (J \cap (T-Q))\ \theta\ (I \cap (T-Q))$

We now show that $X \cap T$ has an interval topology .

If $x_0 \in T$, let us denote by $\mathfrak{J}_{x_0}$ the family of sets containing x_0 which are open in the topology of S relativized to T . Let $\mathfrak{L}_{x_0}$ denote sets containing x_0 which are open in the interval topology determined by the order θ . Let $y \in X \cap T$ be given.

(a) Suppose $y = x$. By lemma (1) , $J' \subset T - Q$ and $x \notin Cl_S(I \cap (T-Q))$. Clearly, since $x \in T$ and $x \in Cl_S(J')$ then $x \in T - Q$. Since $x \in T - Q$, and because $x \notin H$ we must have $x \in Cl_S(I \cap Q)$. By assumption, $x \in Cl_S(J \cap (T-Q))$ and so $\mathfrak{J}_x \subset \mathfrak{L}_x$. Since $x \in Cl_S(I \cap (T-Q))$ and $x \notin Cl_S(J \cap Q)$ we have $\mathfrak{L}_x \subset \mathfrak{J}_x$.

(b) Suppose $y \in Q$ and $y \neq x$. If y is an interior point of Q , clearly $\mathfrak{L}_y = \mathfrak{J}_y$ so assume without loss of generality that y is a right hand end point of a component of Q . Since $y \in H$, let I' be an open interval containing y such that $Cl_S(I') \cap (T-Q)$ is compact and $I' \cap (T-Q) \cap Cl_S(Q) = \emptyset$. Then $y \notin Cl_S(T-Q)$ for otherwise $Cl_S(I') \cap (T-Q)$ is not compact. Hence $\mathfrak{L}_y \subset \mathfrak{J}_y$. Also, y is a left hand end point of a monotone sequence from Q since $y \in Cl_S(Cl_S(T)-T)$ and y is clearly a right hand end point of such a sequence and so $\mathfrak{I}_y \subset \mathfrak{L}_y$.

(c) Suppose $y \in T - Q$ and $y \neq x$. Note that $\{t \in T - Q \mid t \,\theta\, y\} \neq \emptyset$ since $J' \subset T - Q$. Let $a = \sup\{t \in T - Q \mid t \,\theta\, y\}$. Then $a \in T - Q$ since $a \neq x$ and $a \in H$. If $\{t \in T - Q \mid y \,\theta\, t\} \neq \emptyset$, let $b = \inf\{t \in T - Q \mid y \,\theta\, t\}$. Then $b \neq x$ since $x \notin I \cap Cl_S(T-Q)$ and so $b \in (T-Q)$.

Hence (1) $a \in T - Q$ and

(2) $b \in T - Q$ when $\{t \in T - Q \mid y \,\theta\, t\} = \emptyset$.

imply $\mathfrak{I}_y \subset \mathfrak{L}_y$. Since $y \in T - Q$ and $y \in H$, $y \notin Cl_S(Q)$ and so $\mathfrak{L}_y \subset \mathfrak{I}_y$. Hence by (a) (b) and (c) , $X \cap T$ has interval topology.

Now from the argument in (b) above, if $(I \cup J) \cap T$ contains a point of Q , $\theta(X \cap T)$ has no first element. So assume $(I \cup J) \cap Q = \emptyset$. Then $x \notin Q$ and since $x \notin H$, $x \notin T - Q$. Hence, by assumption, J' has no first element and so $\theta(X \cap T)$ has no first element. To show that $\theta(X \cap T)$ has a last element, let $y \in J'$ and $b = \inf\{t \in T - Q \mid y \,\theta\, t\}$. Then $b \neq x$ and so $b \in T - Q$ and b is the largest element of $\theta(X \cap T)$.

Case (2): For each component J' of $T \cap J$, $x \notin Cl_S(J')$ and, without loss of generality for each component I' of $I \cap T$, $x \notin Cl_S(I')$. Then there exist monotone sequences,

$\{i_n\}_{n=0}^{\infty}$ and $\{j_n\}_{n=0}^{\infty}$ such that

(1) $i_0 = p$ and $\{i_n\}_{n=1}^{\infty} \subset I - T$, ($\{i_n\}$ is increasing)

(2) $j_0 = q$ and $\{j_n\}_{n=1}^{\infty} \subset J - T$, ($\{j_n\}$ is decreasing)

(3) $\lim_{n\to\infty} i_n = x = \lim_{n\to\infty} j_n$

Define the following order, θ for $X \cap T$.

$[(i_0,i_1) \cap Q] \;\theta\; [(j_1,j_0) \cap Q] \;\theta\; [(i_1,i_2) \cap Q] \;\theta\; [(j_2,j_1) \cap Q] \;\theta\; \ldots$

$\theta[\{x\} \cap T] \ldots \theta\; [(i_1,i_2) \cap (T-Q)] \;\theta\; [(j_1,j_0) \cap (T-Q)] \;\theta\; [(i_0,i_1) \cap Q]$

To show $\theta\,(X \cap T)$ has interval topology, let $y \in X \cap T$ be given.

(a) Suppose $x = y$. Clearly $x \notin Q$ and so $x \in T - Q$ and since $x \notin H$, either $Cl_S(I') \cap (T-Q)$ is not compact for some open interval I' containing x or $x \in Cl_S(Q)$. If $Cl_S(I') \cap (T-Q)$ is not compact then X contains a point not in H other than x. Hence $x \in Cl_S(Q)$. If $A = \{t \in T \mid x \;\theta\; t\} \neq \emptyset$, let $b = \inf A$. If $b = x$, then since $x \in Cl_S(T-Q)$, $\mathfrak{I}_x = \mathfrak{L}_x$. If $b \neq x$, $b \in T - Q$ since $b \in H$ and similarly $\mathfrak{I}_x = \mathfrak{L}_x$. The same holds if $A = \emptyset$.

(b) Suppose $y \in Q$. The argument for (b) above holds.

(c) Suppose $y \in T - Q$. The same argument for (c) above applies since $\{t \in T - Q \mid t \;\theta\; y\} \neq \emptyset$. For if $x \notin T - Q$ then $x \in Cl_S(T-Q)$ since $x \notin H$.

Definition: Denote by H' the union of all open intervals of S which contain not more than one point outside H. Denote by G the set whose elements are the closures of components from H'. Let G' be the subset of G consisting of elements of G which have an interior point not in H.

We will want to divide the terms of G' into intervals, each of which has one point outside H and then use the previous lemma to order each of the intervals. In ordering the terms of G' it will be convenient if the points of division are not in T. Hence the following lemma.

Lemma (3): Assume condition (2) holds and that $X \in G'$ and J is a component of $H \cap X$. Then there exists $x \in J$ such that $x \notin T$.

Proof: Since $X \in G'$, there exists an interval, I of S such that $I \subsetneq H$, $I \cap J = \emptyset$, and x is a common end point of I and J where $x \notin H$. Assume that $J \subset T$. Then by lemma (2), $J \subset T - Q$ and has an end point j such that $j \notin Cl_S(Cl_S(T)-T)$. Hence there exists an open bounded interval, J', containing j and such that J' contains no points of $Cl_S(T)-T$. Hence $Cl_S(J') \cap T$ is closed since it contains all its accumulation points and, being bounded is compact. Since J' contains no points of $Cl_S(T)-T$, it contains no end points of Q and hence no points of Q. Hence $J' \cap (T-Q) \cap Cl_S(Q) = \emptyset$ and $J' \subset H$. This is clearly a contradiction since J is a component of H.

We describe now, a total ordering for a subset, $Y^* \cap T$, of K for each $K \in G'$, for each component, J of $K \cap H$, by lemma (3), select a point $z_J \notin T$. Let Z be the set consisting of all such z_J for J a component of $K \cap H$. Note that by definition of G', the accumulation points of Z occur only at the end points of K if Z has any accumulation points. Note, also, that the components of $K - Z$ have a natural order induced from Δ, the order in S. So, define Y to be the collection of all components of $K - Z$ excluding the first component, if it exists and excluding the last component, if it exists. Let $Y^* = \cup \{Cl_S(X) \mid X \in Y\}$. Then for each $X \in Y$, assuming condition (2) holds, the hypothesis of Lemma (2) holds so there is a total ordering θ of $X \cap T$ where $X \cap T$ has interval topology with respect to the order θ and $X \cap T$ has no first element but does have a last element. Clearly $X \cap T$ will have a total ordering, namely $-\theta$, which yields interval topology and for which $X \cap T$ has a first element but no last element.

Now $\{X \cap T \mid X \in Y\}$ has a natural order induced by S and so $Y^* \cap T = \cup \{X \cap T \mid X \in Y\}$ has a number of total orderings where for each $X \in Y$, $X \cap T$ has interval topology. The following orders, λ and λ' will assure that, in fact, $Y^* \cap T$ has the order topology. If Y has a first element in the order induced by S, Let Y_0 be the first element. Otherwise, chose Y_0 arbitrarily. Then set:

(a) $\lambda\ (Y_0 \cap T) = \theta\ (Y_0 \cap T)$,

(b) For a given $X \in Y$, $\lambda\ (X \cap T) = \theta\ (X \cap T)$ if there is an odd number of terms of Y between X and Y_0,

(c) $\lambda\ (X \cap T) = -\theta\ (X \cap T)$ otherwise.

(Since the accumulation points of Z occur only at the end points of K, there is only a finite number of terms of Y between any such terms). To complete the description of the total ordering on $Y^* \cap T$, for $X_1, X_2 \in Y$, set

$$(X_1 \cap T)\ \lambda\ (X_2 \cap T) \text{ if and only if } (X_1 \cap T)\ \Delta\ (X_2 \cap T) .$$

The description of the total ordering λ' is identical to the above with θ replaced by $-\theta$ and $-\theta$ replaced by θ.

It is an immediate verification that $Y^* \cap T$ has interval topology under λ and λ' when it is noted that:

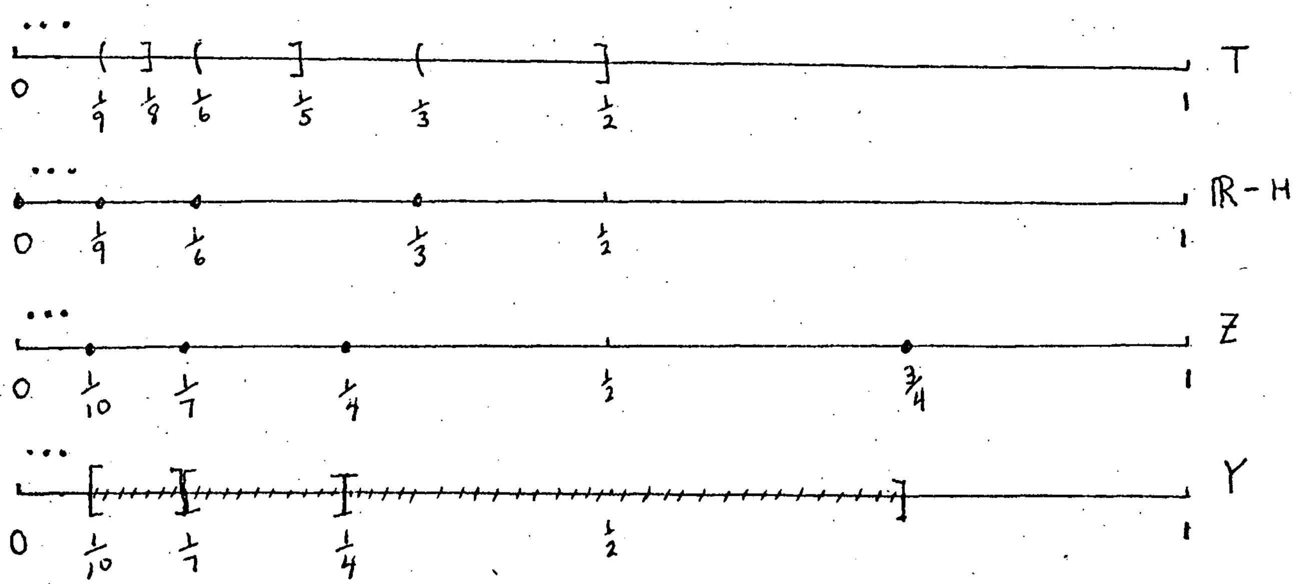
(1) under λ and λ' if x is a last point of $X \cap T$ for $X \in Y$, then x has an immediate successor and similarly if

x is a first point of $X \cap T$ then x has an immediate predecessor and (2) under λ and λ', $X \cap T$ has interval topology for each $X \in Y$.

Example (6): Let $T = (\frac{1}{3}, \frac{1}{2}] \cup (\frac{1}{6}, \frac{1}{5}] \cup (\frac{1}{9}, \frac{1}{8}] \cup \ldots$. Then $G = \{[0, \infty], [-\infty, 0]\}$ and $G' = \{[0, \infty]\}$. Choose $Z = \{\frac{3}{4}\} \cup \{\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \ldots\}$ and then $Y = \{[\frac{1}{4}, \frac{3}{4}], [\frac{1}{7}, \frac{1}{4}], [\frac{1}{10}, \frac{1}{7}], \ldots\}$. Since Y has no first element in the order induced by $\mathbb{R}$, choose $Y_0 = [\frac{1}{4}, \frac{3}{4}]$. Then λ and λ' are defined as follows:

(1) $\ldots (\frac{1}{9}, \frac{1}{8}] \,\lambda\, (\frac{1}{6}, \frac{1}{5}] \,\lambda\, (\frac{1}{3}, \frac{1}{2}]$ where λ orders $(\frac{1}{3}, \frac{1}{2}]$, $(\frac{1}{9}, \frac{1}{8}]$, $(\frac{1}{15}, \frac{1}{14}]$, ... as the induced order from $\mathbb{R}$ and λ orders other intervals as the reverse of the induced order from $\mathbb{R}$.

(2) $\ldots (\frac{1}{9}, \frac{1}{8}] \,\lambda'\, (\frac{1}{6}, \frac{1}{5}] \,\lambda'\, (\frac{1}{3}, \frac{1}{2}]$ where λ orders $(\frac{1}{3}, \frac{1}{2}]$, $(\frac{1}{9}, \frac{1}{8}]$, $(\frac{1}{15}, \frac{1}{14}]$, ... as the reverse of the induced order from $\mathbb{R}$ and λ' orders other intervals as the induced order from $\mathbb{R}$.



Example (7): For $n = 1, 2, \ldots$, let f_n be the order preserving homeomorphism of $(-1, 1)$ onto $(\frac{1}{2n}, \frac{1}{2n-1})$. Let T' be the subspace of example (5) and let $T = f_1(T') \cup f_2(T') \cup f_3(T') \cup \ldots$. Let θ_i be the order of $f_i(T')$ such that $\theta_i(f_i(T'))$ is order isomorphic to $\theta(T')$. Then G, G', Z, and Y_0 are as in the previous example. The orders, λ and λ' are defined as follows:

(1) $\ldots\ f_3(T')\ \lambda\ f_2(T')\ \lambda\ f_1(T')$

where $\lambda(f_i(T')) = \theta_i(f_i(T))$ if i is odd

$\lambda(f_i(T') = -\theta_i(f_i(T))$ if i is even

(2) λ' is as in (1) above with odd and even interchanged.

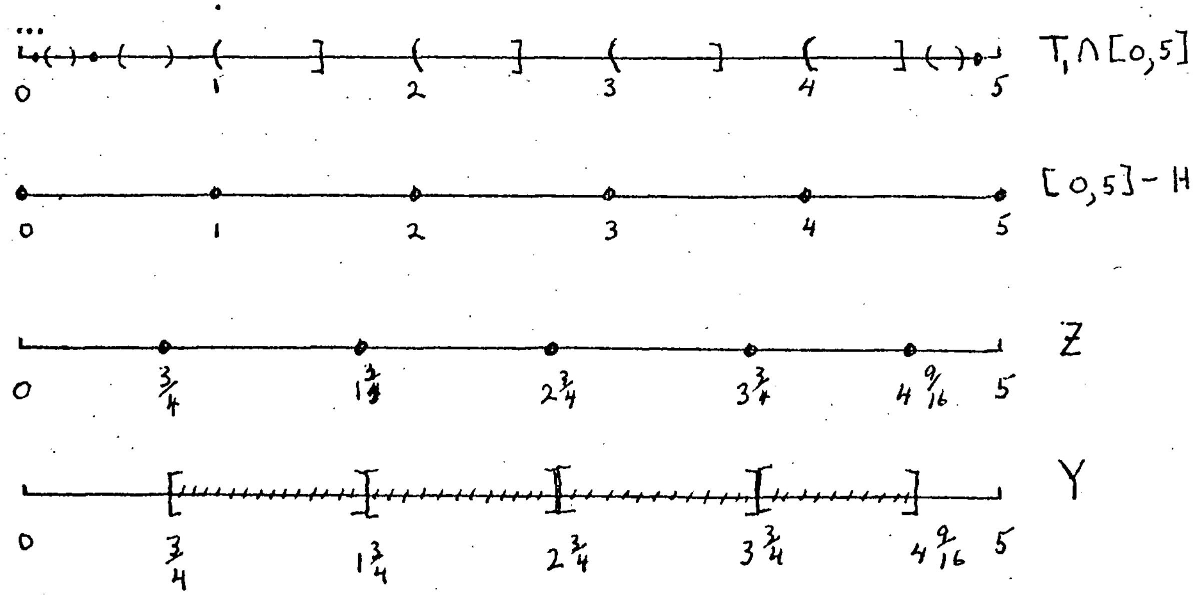
Lemma (4): Let $K = [p, q] \in G'$. Assume condition (2) holds and that $p \in P_K$ but $q \notin P_K$. Then there exists $x \in (p, q)$ and a total ordering, π on $[x, q] \cap T$ such that $(p, x] \subset H$ and π orders T nicely in $[x, q]$.

Discussion: In the following examples let $[p, q] = [0, 5]$. Let $\overrightarrow{(x, y)}$ indicate the ordered set $\{r \in R \mid x < r < y\}$ with order induced from $\mathbb{R}$. Let $\overleftarrow{(x, y)}$ indicate the ordered set $\{r \in R \mid x < r < y\}$ with the reverse of the order induced from $\mathbb{R}$. Similarly for $\overrightarrow{(x, y]}$ etc.

Example (7): Define $A = (\frac{1}{3},\frac{1}{2}) \cup (\frac{1}{6},\frac{1}{5}) \cup (\frac{1}{9},\frac{1}{8}) \cup \ldots \cup \{\frac{1}{4},\frac{1}{7},\frac{1}{10},\ldots\} \cup (1,1\frac{1}{2}] \cup (2,2\frac{1}{2}] \cup (3,3\frac{1}{2}] \cup (4,4\frac{1}{2}] \cup (4\frac{10}{16},4\frac{11}{16}) \cup \{4\frac{12}{16},0\}$. Let T_1 be the set A together with an increasing sequence of disjoint semi-closed intervals converging to 0 and a descending sequence of disjoint semi-closed intervals converging to 5 .

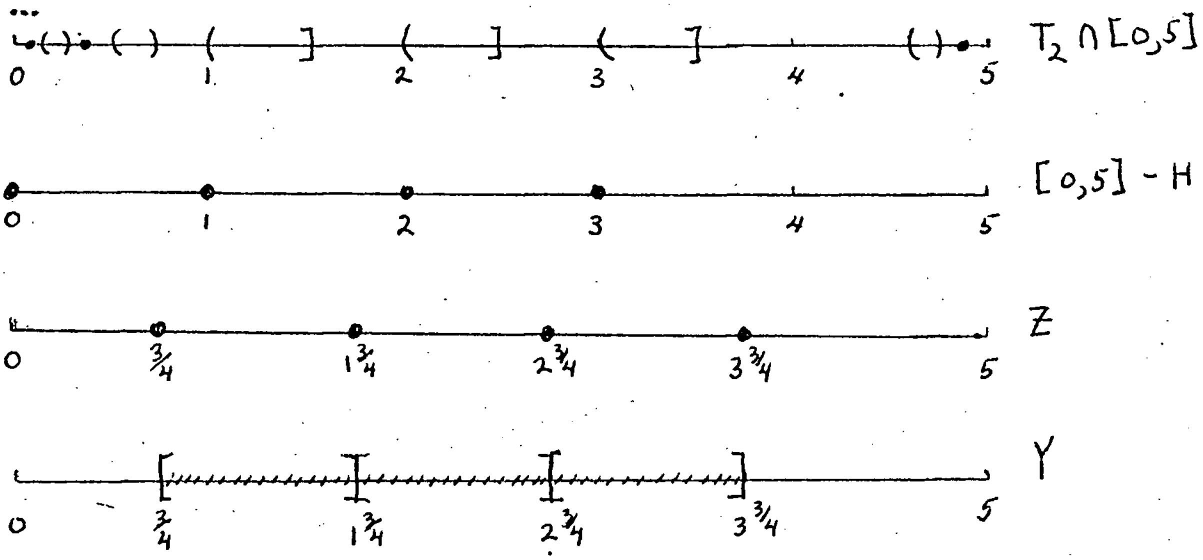
Then $[0, 5] - H = \{0,1,2,3,4,5\}$. Choose $Z = \{\frac{3}{4}, 1\frac{3}{4}, 2\frac{3}{4}, 3\frac{3}{4}, 4\frac{9}{16}\}$. Then $Y = \{[\frac{3}{4}, 1\frac{3}{4}], [1\frac{3}{4}, 2\frac{3}{4}], [3\frac{3}{4}, 4\frac{9}{16}]\}$. Now we have a procedure for ordering T_1 in the union of the elements of Y which is $Y^* = [\frac{3}{4}, 4\frac{9}{16}]$. If $x = \frac{3}{4}$ (in general, if x is the first element of Z) then we may disregard the interval $[0, \frac{3}{4}]$. If π_1 is to order T_1 nicely in $[\frac{3}{4}, 5]$ it will have no first element and no last element. Let $X = [4\frac{3}{4}, 5]$. (In general, $X = (z_n, q]$ where z_n is the last element of Z) . The problem is to order $X \cap T_1$ and to insert this ordered set into the ordered set, $\lambda(y^* \cap T_1)$, in such a way that the resulting order has neither first nor last element. Since Y has an even number of elements define π_1, on $[\frac{3}{4}, 5] \cap T_1$ as follows:

$$\overrightarrow{(1, 1\tfrac{1}{2}]}\ \pi_1\ \{4\tfrac{12}{16}\}\ \pi_1\ \overleftarrow{(2, 2\tfrac{1}{2}]}\ \pi_1\ \overrightarrow{(3, 3\tfrac{1}{2}]}\ \pi_1\ \overleftarrow{(4, 4\tfrac{1}{2}]}\ \pi_1\ \overrightarrow{(4\tfrac{10}{16}, \tfrac{11}{16})}$$



To see the procedure when Y has an odd number of elements let $T_2 = T_1 - (4 , 4\frac{1}{2}]$. Then $[0 , 5] - H = \{0,1,2,3,5\}$. Choose $Z = \{\frac{3}{4} , 1\frac{3}{4} , 2\frac{3}{4} , 3\frac{3}{4}\}$. Then $Y = \{\frac{3}{4} , 1\frac{3}{4}] , [1\frac{3}{4} , 2\frac{3}{4}] , [2\frac{3}{4} , 3\frac{3}{4}]\}$ and X is as in T_1 . Now it is not possible to choose $x = \frac{3}{4}$ as before, to order $X \cap T_2$, and to insert this ordered set into the ordered set $\lambda\ (Y^* \cap T_2)$ in such a way that the resulting order has neither first nor last element. Hence we choose $x = \frac{1}{4}$ (x is the last element in $(T_2 - Q) \cap [0 , \frac{3}{4}])$ and attempt to order the sets $(\frac{1}{3} , \frac{1}{2})$ and $X \cap T_2$ and insert these sets into the ordered set $\lambda'\ (Y^* \cap T_2)$ in such a way that the resulting ordered set has a first element and no last element. Define $\pi_2([\frac{1}{4} , 5] \cap T_2)$ as follows:

$\{\frac{1}{4}\}\ \pi_2\{4\frac{12}{16}\}\ \pi_2\overleftarrow{(1,1\frac{1}{2}]}\ \pi_2\overrightarrow{(2,2\frac{1}{2}]}\ \pi_2\overleftarrow{(3,3\frac{1}{2}]}\ \pi_2\overrightarrow{(4\frac{10}{16},4\frac{11}{16})}\ \pi_2\overrightarrow{(\frac{1}{3},\frac{1}{2})}$.



The above two examples are typical of the case when Y has a finite number of elements and when $q \notin T$ or $q \in Cl_S(X \cap Q)$. Those which follow are two examples which are typical of the case when Y has a finite number of elements and when $q \in T$ and $q \notin Cl_S(X \cap Q)$.

Example (8): Let $\{t_i\}_{i=1}^{\infty}$ be an increasing sequence in $[4\frac{12}{16}, 5)$ converging to 5 and let $t_0 = 4\frac{12}{16}$. Define $T_3 = T_1 \cup \{t_i \mid i = 0,1,2,\ldots\} \cup \{5\}$. As the considerations in constructing $\pi_3(T_3 \cap [x , q])$ are the same as those in the previous examples, we will define $\pi_3(T_3 \cap [x , q])$ directly. The number of terms of Y is even so let $x = \frac{1}{4}$ and define π_3 as follows:

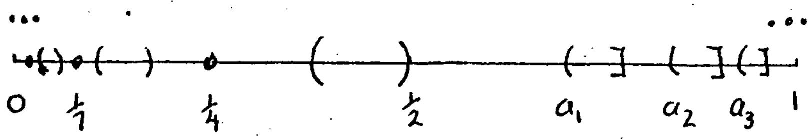
$\{\frac{1}{4}\}\ \pi_3\overleftarrow{(1,1\frac{1}{2}]}\ \pi_3\overrightarrow{(\frac{1}{3},\frac{1}{2})}\ \pi_3\overrightarrow{(4\frac{10}{16},4\frac{12}{16})}\ \pi_3\overrightarrow{(2,2\frac{1}{2}]}\ \pi_3\overleftarrow{(3,3\frac{1}{2}]}$
$\pi_3\overrightarrow{(4,4\frac{1}{2}]}\ \pi_3\{t_0\}\ \pi_3\{t_1\}\ \pi_3\{t_2\}\ .\ .\ .\ \pi_3\{5\}$.

To see the construction when the number of terms of Y is odd, define $T_4 = T_3 - (4\ ,\ 4\frac{1}{2}]$. Let $x = \frac{3}{4}$. Define $\pi_4([x,q] \cap T_4)$ as follows:

$\overrightarrow{(4\frac{10}{16},4\frac{11}{16})}\ \pi_4\overrightarrow{(1,1\frac{1}{2}]}\ \pi_4\overleftarrow{(2,2\frac{1}{2}]}\ \pi_4\overrightarrow{(3,3\frac{1}{2}]}\ \pi_4\{t_0\}\ \pi_4\{t_1\}\ \pi_4\{t_2\}$
$.\ .\ .\ \pi_4\{5\}$.

Finally, we consider an example typical of the case when Y has an infinite number of terms.

Example (9): Let $A = (\frac{1}{3},\frac{1}{2}) \cup (\frac{1}{6},\frac{1}{5}) \cup (\frac{1}{9},\frac{1}{8}) \cup \ldots \cup\{\frac{1}{4},\frac{1}{7},\frac{1}{10},\ldots\}$ $\cup\{0\}$. Let A' be A together with an increasing sequence of semi-closed intervals, $(a\ ,\ b]$, each contained in $(\frac{3}{4}\ ,\ 1)$ and converging to 1 . Let T be the set A' together with an increasing sequence of semi-closed intervals converging to 0 and a decreasing sequence of semi-closed intervals converging to 1 . In this case, let $x = \frac{5}{8}$, the first point of Z and then $\lambda(T \cap [\frac{3}{4}\ ,\ 1])$, as described in example (6) orders T nicely in $[\frac{3}{4}\ ,\ 1]$.



$T \cap [0,1]$

Proof of Lemma (4): Let Z, Y, Y^*, Y_0, λ and λ' be as defined previously for the element K of G'. Since $p \in P_k$, there is a first component, I of $K \cap H$ and hence in the previous construction, Y_0 is the first element of Y. Ley y be the first point in Δ of Y^* which is also the first point of Z.

Case(1): Suppose Y has a last element. Then there is a component X of $K - Z$ whose end point is q. Since the accumulation points of Z exist only at the end points of K, Z has no accumulation points and there are only a finite number of components of $K \cap H$ between I and X and hence only a finite number of terms of Y. Let $\pi(X \cap(T - Q)) = \Delta(X \cap(T-Q))$ and $\pi(X \cap Q) = \Delta(X \cap Q)$

(a) Suppose $q \notin T$ or $q \in Cl_S(X \cap Q)$. If the number of terms of Y is even, let $x = y$, $\pi(Y^* \cap T) = \lambda(Y^* \cap T)$ and $\pi(Y_0 \cap T)$ $\pi(X \cap(T-Q))$ $\pi((Y^* - Y_0) \cap T)$ $\pi(X \cap Q)$ $\pi(\{q\} \cap T)$.

Now we verify that π orders T nicely in $[y, q]$. As in Lemma (2), it is verified that if $X \cap(T-Q)$ has 2 elements, each point of $\pi(X \cap(T-Q))$ and each point of $\pi(X \cap Q)$ has interval topology. To show that $\pi([x, q] \cap T)$ has interval topology, we need only verify the following:

(1) $\pi(Y_0 \cap T)$ has a last point and if $X \cap(T-Q)$ is non void, $\pi(X \cap(T-Q))$ has a first and last element. Since $X \subset H$ and the left hand end point of X is in H and not in T,

$\pi(X \cap (T-Q))$ has a first element. If $q \in Cl_S(X \cap Q)$, since $q \notin P_K$, $q \notin Cl_S(X \cap (T-Q))$ and again, since $X \subset H$, $\pi(X \cap (T-Q))$ has a last element. If $q \notin T$ and $q \in Cl_S(X \cap (T-Q))$ then q is a trivial component of $S - T$ for any interval containing q contains a point not in H. Hence $q \notin Cl_S(X \cap (T-Q))$ and so $\pi(X \cap (T-Q))$ has a last element.

(2) $\pi((Y^* - Y_0) \cap T)$ Has a first element and no last element and $\pi(X \cap Q)$ has no first element. Since the number of terms of Y is even, $(Y^* - Y_0) \cap T$ has a first element and no last element. If $\pi(X \cap Q)$ has a first element t, then t is a left hand end point of a closed component C, of Q and hence is an accumulation point of $T - Q$. Hence $t \notin H$ and this is clearly a contradiction. Similarly we prove the last fact

(3) $\pi(X \cap Q)$ has no last element.

Now, that π orders $[x, q] \cap T$ nicely is immediate. For $y \notin T$ and $[x, q] \cap T$ has no first element. If $q \in T$ then q is the last point of $[x, q] \cap T$, otherwise $[x, q] \cap T$ has no last element.

If the number of terms of Y is odd, then let z be the last point of $(p, y) \cap (T-Q)$ in Δ. Since $(p, y) \subset H$ and $y \notin T$ and $p \in P_K$, the existence of z is guaranteed. Let $x = z$ and $\pi(Y^* \cap T) = \lambda' (Y^* \cap T)$ and let $\pi((z, y) \cap Q) = \Delta((z, y) \cap Q)$. As in Lemma (2) it is shown that $\pi((z, y) \cap Q)$

has interval topology. Then we set

$$\{z\}\ \pi(X \cap (T-Q))\ \pi(Y^{*} \cap T)\ \pi((z\ ,\ y) \cap Q)\ \pi(X \cap Q)\ \pi(\{q\} \cap T)\ .$$

We can see by the same reasoning as above that $\pi(X \cap (T-Q))$ has a last and a first point if it is non void, $\pi(Y^* \cap T)$ has a first but no last point, $(z,y) \cap Q$ has neither first nor last point and $\pi(X \cap Q)$ has neither first nor last point.

(b) Suppose $q \in T$ but $q \notin Cl_S(X \cap Q)$. If the number of terms of Y is odd, let $\pi(Y^* \cap T) = \lambda(Y^* \cap T)$ and let $x = y$. Let

$$(X \cap Q)\ \pi\ (Y^* \cap T)\ \pi\ (X \cap (T-Q))\ \pi\ \{q\}\ .$$

It is verified, as above, that π orders T nicely in $[x\ ,\ q]$ but in this case, $\pi(X \cap (T-Q))$ may have a last element. If the number of terms of Y is even, let $\pi(Y^* \cap T) = \lambda'(Y^* \cap T)$ and let $x = z$. Let

$$\{z\}\ \pi(Y_0 \cap T)\ \pi((z\ ,\ y) \cap Q)\ \pi(X \cap Q)\ \pi((Y^* - Y_0) \cap T)\ \pi(X \cap (T-Q))\pi\{q\}\ .$$

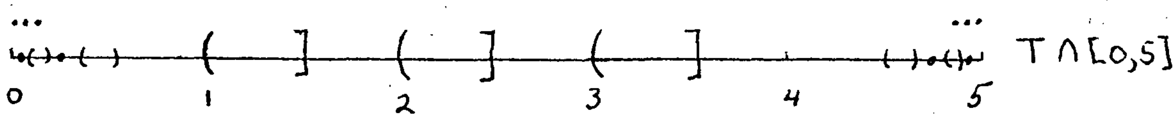
Case (2) Y has no last element. Then let $\pi(Y^* \cap T) = \lambda(Y^* \cap T)$ and let $x = y$. Then clearly, π orders T nicely in $[x\ ,\ q]$.

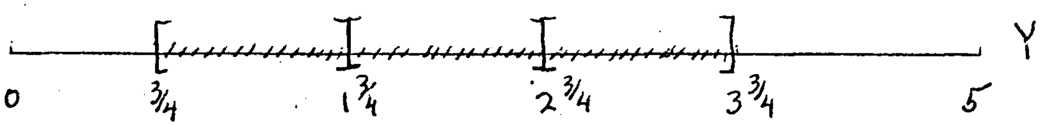
Lemma (5): Suppose condition (2) holds and that $[p\ ,\ q] = K \in G$. Suppose, also, that $p\ ,\ q \in P_K$. Then there are points, $x\ ,\ y \in (p\ ,\ q)$ such that $x \Delta y$, $(p\ ,\ x] \subset H$, and $[y\ ,\ q) \subset H$ and there is an order π on $[x\ ,\ y] \cap T$ that orders T nicely in $[x\ ,\ y]$.

Discussion: Since p, $q \in P_K$ we are concerned with cases where the number of elements of Y is finite. Again we consider 2 cases.

Example (10): The number of elements of Y is odd. Let $p = 0$ and $q = 5$. Let $A = (\frac{1}{3},\frac{1}{2}) \cup (\frac{1}{6},\frac{1}{5}) \cup (\frac{1}{9},\frac{1}{8}) \cup \ldots \cup \{\frac{1}{4},\frac{1}{7},\frac{1}{10},\ldots\}$ Let $A' = A \cup (5 - A) \cup (1,1\frac{1}{2}] \cup (2,\frac{1}{2}] \cup (3,3\frac{1}{2}] \cup \{0,5\}$. Finally, define T to be the set A together with an increasing sequence of disjoint semi-closed intervals converging to 0 and a decreasing sequence of disjoint semi-closed intervals converging to 5. Then $[0,5] - H = \{0,1,2,3,5\}$. Choose $Z = \{\frac{3}{4},1\frac{3}{4},2\frac{3}{4},3\frac{3}{4}\}$. Let $x = \frac{3}{4}$ and $y = 4\frac{3}{4}$. (In general, x is the first point of Z and y is the first point of T-Q in $[z_n,q]$ where z_n is the last point of Z). Then define $\pi([x, y] \cap T)$ as follows:

$$\overrightarrow{(4\tfrac{1}{2},4\tfrac{2}{3})}\ \pi\overrightarrow{(1,1\tfrac{1}{2}]}\ \pi\overleftarrow{(2,2\tfrac{1}{2}]}\ \pi\overrightarrow{(3,3\tfrac{1}{2}]}\ \pi\{4\tfrac{3}{4}\}.$$





Example (11): The number of elements of Y is even.

Define $T' = T - (3, 3\frac{1}{2}]$ where T is as in example (10) above.

Then $[0,5] - H = \{0,1,2,5\}$. Choose $Z = \{\frac{3}{4}, 1\frac{3}{4}, 2\frac{3}{4}\}$. Let $x = \frac{3}{4}$ and $y = 2\frac{3}{4}$. (In general, x is the first element of Z and y is the last element of Z. Define $\pi([x, y] \cap T)$ as follows:

$$\overrightarrow{(1, 1\tfrac{1}{2}]}\ \pi\ \overleftarrow{(2, 2\tfrac{1}{2}]}$$. Note that $\pi([x, y] \cap T)$ is just $\lambda([x, y] \cap T)$.

Proof of Lemma (5): If $K \notin G'$ i.e. $(p, q) \subset H$, then since $p \in P_K$, $(p, q) - T \neq \emptyset$ and we may choose points $x, y \in (p, q)$ so that x and y belong to the same components of $(p, q) - T$. Then the lemma is trivially satisfied since $[x, y] \cap T = \emptyset$.

Suppose $(p, q) \not\subset H$; then $K \in G'$ and define Z, Y, Y^*, Y_0, λ, and λ'. Since $p, q \in P_K$, Z is finite and hence Y has a finite number of terms. Let w be the first point (with respect to Δ) of Z and let z be the last point of Z. Then w is the first point of Y^* and z is the last point of Y^*. Since $w, z \notin T$, if the number of terms of Y is even, $Y^* \cap T$ has no first or last element and if $x = w$, $y = z$, then λ orders T nicely in $Y^* = [x, y]$. If the number of terms of Y is odd, let y be the first point of $T-Q$ in $[z, q)$. Such a y exists since $q \in P_K$. Let $\pi(Y^* \cap T) = \lambda(Y^* \cap T)$,

$\pi([z, y) \cap Q) = \Delta([z, y) \cap Q)$ and let

$$([z, y) \cap Q)\ \pi(Y^* \cap T)\ \pi\{y\} .$$

As shown previously, $\pi([z, y) \cap Q)$ has the interval topology and if $[z, y) \cap Q \neq \emptyset$, $\pi([z, y) \cap Q)$ has neither first nor last element. Then $\pi([x, y] \cap T)$ has no first element, but has a last element and $x \notin T$ but $y \in T$ and so T is ordered nicely in $[x, y]$.

Lemma (6): Suppose condition (2) holds and that $K \in G$. Suppose one end point of K is not in P_K and the other is not the end point of any component of $K \cap H$. Then there is a total ordering, π of $T \cap K$ such that π orders T nicely in K.

Discussion: Since one end point of K is not the end point of any component of $K \cap H$, it is clear that Y has an infinite number of elements. We consider 2 cases. In each case let $K = [0, 5]$.

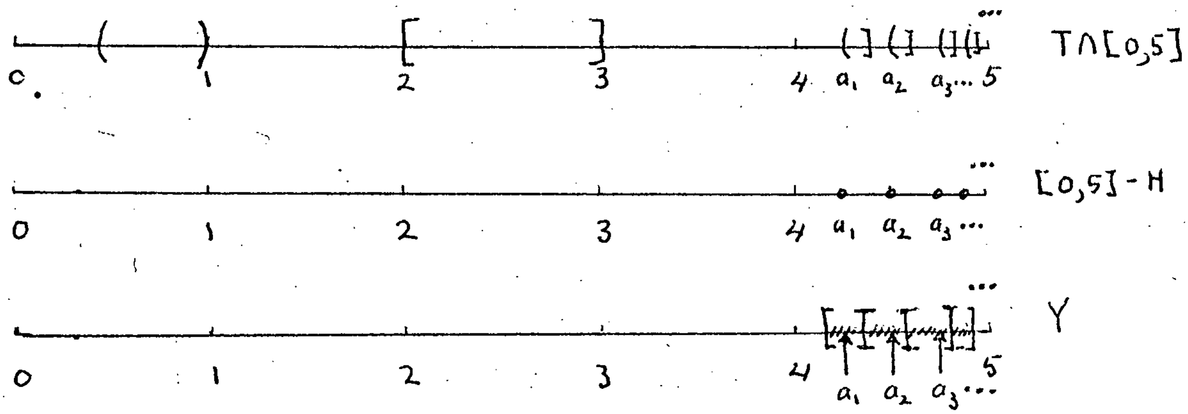
Case (1): The set Y has a first or last element. Without loss of generality we may assume Y has a first element.

Example (12): Let T be the union of $\{0, 5\}$, $(\frac{1}{2}, 1)$, and $[2, 3]$ together with an increasing sequence of disjoint semi-closed intervals in $(4, 5)$ converging to 5, say $\{(a_i, b_i] \mid i \in \mathbb{N}\}$ and together with any increasing sequence of disjoint semi-closed intervals converging to 0.

Then $(\frac{1}{2}, 1) \subset Q$, $[2, 3] \subset T - Q$, and $[0, 5] - H =$ $\{0, 5\} \cup \{a_i \mid i \in \mathbb{N}\}$. A total order, π that orders T nicely in $[0, 5]$ can be defined as follows:

$$\{0\} \;\pi\; \overrightarrow{(\tfrac{1}{2},1)} \;\pi\; \overrightarrow{(a_1,b_1]} \;\pi\; \overrightarrow{[2,3]} \;\pi\; \overleftarrow{(a_2,b_2]} \;\pi\; \overrightarrow{(a_3,b_3]} \;\pi\; \overleftarrow{(a_4,b_4]}$$
$$\ldots \;\pi\; \{5\}$$

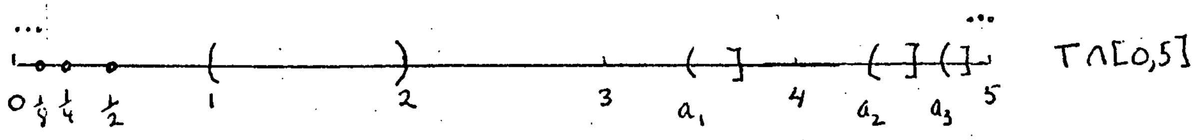
In this example, $p \in Cl_S(K \cap Q)$.



Example (13): Let $A = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} \cup (1, 2)$. Let T be the union of $\{0, 5\}$ and A together with an increasing sequence of disjoint, semi-closed intervals in $(2, 5)$ converging to 5 , say $\{(a_i, b_i] \mid i \in \mathbb{N}\}$ and together with any increasing sequence of disjoint semi-closed intervals converging to 0 . In this case $Q = (1, 2)$ and $[0, 5] - H =$ $\{a_i \mid i \in \mathbb{N}\}$. A total order, π that orders T nicely in $[0, 5]$ can be defined as follows:

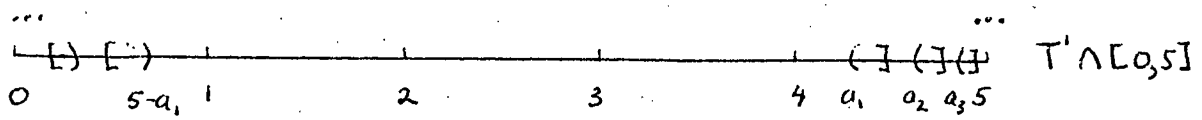
$$\{0\} \;\pi\; \ldots \;\pi\; \{\tfrac{1}{8}\} \;\pi\; \{\tfrac{1}{4}\} \;\pi\; \{\tfrac{1}{2}\} \;\pi\; \overleftarrow{(a_1, b_1]} \;\pi\; \overrightarrow{(1, 2)} \;\pi\; \overrightarrow{(a_2, b_2]} \;\pi\; \overleftarrow{(a_3, b_3]}$$
$$\pi\; \overrightarrow{(a_4, b_4]} \;\ldots\; \pi\; \{5\}$$

In this example, $p \notin Cl_S(K \cap Q)$.



Case (2): The set Y has neither first nor last element.

Example (14): Define $T' = (T-A) \cup \{(5-a_i, 5-b_i] \mid i \in \mathbb{N}\}$ where a_i, b_i, A, and T are defined in example (12). Clearly the order $\lambda([0, 5] \cap T)$ satisfies the hypothesis of the lemma.



Proof of Lemma (6): Assume q is not the end point of any component of $K \cap H$, and that $p \notin P_K$ and $K = [p, q]$. For the given K, since $K \in G'$, we can define Z which has an infinite number of elements and also define Y_0, Y, Y^*, λ, λ' for the given K.

Case (1): The set Y has a first or last element, so assume without loss of generality that Y has a first element. Then denote by X the component of $(p, q) - Z$ whose end point is p. Let $\pi(X \cap (T-Q)) = \Delta(X \cap (T-Q))$ and $\pi(X \cap Q) = \Delta(X \cap Q)$ and verify immediately that $\pi(X \cap (T-Q))$

and $\pi(X \cap Q)$ has the interval topology.

(a) If $p \notin T$ or $p \in Cl_S(X \cap Q)$ define the following order, π on $K \cap T$. Let $\pi(Y^* \cap T) = \lambda(Y^* \cap T)$ and

$$(\{p\} \cap T)\ \pi\ (X \cap Q)\ \pi\ (Y_0 \cap T)\ \pi\ (X \cap (T-Q))\ \pi\ ((Y^* - Y_0) \cap T$$

$\pi(\{q\} \cap T)$. From an argument identical to that of Lemma (4), π orders T nicely in K.

(b) If $p \in T$ and $p \notin Cl_S(K \cap Q)$ define the following order, π on $K \cap T$. Let $\pi(Y^* \cap T) = \lambda'(Y^* \cap T)$ and

$$(X \cap (T-Q))\ \pi(Y_0 \cap T)\ \pi\ (X \cap Q)\ \pi((Y^* - Y_0) \cap T)\ \pi(\{q\} \cap T).$$

Similarly, it is clear that π orders T nicely in K.

Case (2): If Y has neither first nor last element, then an order, π which orders T nicely in K is:

$$(\{p\} \cap T)\ \pi(Y^* \cap T)\ \pi(\{q\} \cap T) \quad \text{where}$$

$$\pi(Y^* \cap T) = \lambda\ (Y^* \cap T).$$

In proving that condition (2) implies that T is totally orderable, we can prove the result easily for the case when $S \in G$ and $S - H \neq \emptyset$.

Proposition: If condition (2) holds and $S \in G$, $S - H \neq \emptyset$, then T is a totally orderable topological space.

Proof: Since $S - H \neq \emptyset$ but $S \in G$, also $S \in G'$ and so we can define Z, Y, Y_0, Y^*, and λ for $S \in G$. Since $+\infty$ and $-\infty$ are contained in non-trivial components of $S - T$ and the accummulation points of Z occur only at $+\infty$ and $-\infty$, Z has no accumulation points and so Y has a finite number of terms. Let $\pi\,(Y^* \cap T) = \lambda\,(Y^* \cap T)$ and $\pi((T-Q) - Y^*) = \Delta((T-Q) \cap Y^*)$, $\pi(Q-Y^*) = \Delta(Q - Y^*)$.
Define a total order π on $S \cap T$ by:

$$(Q-Y^*)\ \pi(Y_0 \cap T)\ \pi((T-Q) - Y^*)\ \pi((Y^* - Y_0) \cap T)\ .$$

Since $Q - Y^* \subset H$ it follows immediately that $(Q-Y^*)$ has the order topology and has neither first nor last element if it is non-void. It follows also that $((T-Q) - Y^*)$ has order topology and has first and last element if it is non-void. Hence T is a totally orderable topological space.

Definition: If I is a closed interval of S, $\Phi(J)$ is a replacement of I if

(1) Φ is a total order on the set J and J is a topological space with the order topology,

(2) $T \cap I \subset J$ and $J \cap T$ is homeomorphic to $I \cap T$,

(3) J is compact and connected, the end points of J are the end points of I and $J - T$ has no trivial components. We say $\Phi(J)$ is a nice replacement of I if $\Phi(J)$ is a replacement of I and Φ restricted to $J \cap T$ orders T nicely in I.

Proposition: Let I be a closed interval of S and assume π orders T nicely in I. Then there is a nice replacement, $\Phi(J)$ of I.

Proof: Let $I = [p, q]$ and let $T' = (T \cap I) \cup \{p, q\}$. Then π induces a natural order π' on T', namely

$$x \, \pi' y \text{ if and only if } x = p \text{ or } y = q \text{ or } x \, \pi \, y .$$

Let T'' be the set obtained from T' by inserting between each $a, b \in T'$ where b is an immediate successor of a, a set homeomorphic to $(0, 1)$ and let π'' be the order induced from π' on T'. Then let J be the set of all ordered pairs (A, B) where

(1) $\emptyset \neq A \subset T''$, $\emptyset \neq B \subset T''$, $A \cup B = T''$, and $A \cap B = \emptyset$,

(2) $a \, \pi'' b \ \forall a \in A, \forall b \in B$,

(3) B has no smallest element.

Then define an order, Φ on J by:

$$(A, B) \, \Phi \, (A', B') \text{ if and only if } A \subset A' .$$

When J is equipped with the order topology, $T \cap I$ is homeomorphic to $\{(A, B) \mid \sup(A) \in T \cap I\}$ and J is compact and connected and Φ restricted to $J \cap T$ orders T nicely in I. ∎

Now for each element K of G, define a replacement $\Phi(K_0)$ and a set of subsets of K, M_K as follows. Assume condition (2) holds.

(1) If a nice replacement of K exists, let $\Phi(K_0)$ be the nice replacement and let $M_K = \emptyset$. In (2) through (5) assume no nice replacement of K is possible.

(2) If one endpoint of K belongs to P_K and $K \notin G'$, let $\Phi(K_0) = \Delta(K)$ and $M_K = \{K\}$.

(3) If $K \in G'$ and $K = [p , q]$, $p \in P_K$ and $q \notin P_K$ then by lemma (4) there exists $x \in (p , q)$ and an order, π, of $[x , q] \cap T)$ such that $(p , x] \subset H$ and π orders T nicely in $[x , q]$. By the previous proposition there is a nice replacement, $\Phi(B)$ of $[x , q]$. Let $K_0 = [p , x] \cup B$ and define a total order Φ on K_0 by $\Phi\,[p , x] = \Delta\,[p , x]$ and $[p , x]\ \Phi\ (B)$. Let $M_K = \{[p , x]\}$.

(4) If $K \in G'$ and $K = [p , q]$ and $p , q \in P_K$, by Lemma (5) there exists $x , y \in (p , q)$ such that $[p , x) \subset H$, $(y , q] \subset H$ and there exists a total order π on $[x , y] \cap T$ that orders T nicely in $[x , y]$. Let $\Phi(B)$ be a nice replacement of $[x , y]$ and let $K_0 = [p , x) \cup B \cup (y , q]$. Define an order, Φ on K_0 by $\Phi\,[p , x] = \Delta\,[p , x]$, $\Phi\,[y , q] = \Delta\,[y , q]$ and $[p , x)\ \Phi\ B\ \Phi\ [y , q]$. Let $M_K = \{[p , x] , [y , q]\}$.

(5) If neither end point of K is in P_K, we first show that there exists an order on $K \cap T$ for which only a finite number of points of $K \cap T$ do not have interval topology. Assume that $K \in G'$ and assume that there

are an infinite number of points of $\Delta(K \cap T)$ that do not have interval topology. Then there exists $y \in K$ such that every open interval about y contains a point $x \in K \cap T$ where x does not have interval topology. If y is an end point of K, clearly $y \notin Cl_S(X \cap Q)$ or else $y \notin Cl_S(K \cap (T-Q))$ and if y is an end point of K, since $y \in H$, $y \notin Cl_S(K \cap Q)$ or else $y \notin Cl_S(K \cap (T-Q))$. Now if $y \notin Cl_S(K \cap Q)$, by an argument identical to that in Lemma (4), there is an open interval, I, about y such that every point of $(T-Q) \cap I$ has the interval topology. This is a contradiction to the definition of y. Similarly if $y \notin Cl_S(K \cap (T-Q))$ there is an open interval, I, about y such that every point of $Q \cap I$ has interval topology. Again this contradicts the definition of y. If $K \in G'$ then we can define Y^* and λ for the given K and $\lambda(Y^* \cap T)$ has interval topology and by an argument similar to the above, there are only finite number of members of $(K \cap T) - Y^*$ that do not have interval topology. Hence we have shown that there exists an order on $K \cap T$ for which only a finite number of points of $K \cap T$ do not have interval topology.

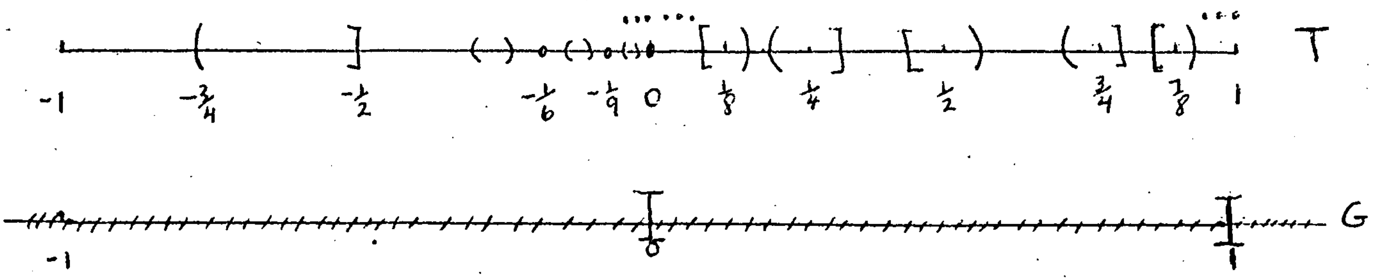
Now using the construction in the previous proposition, there exists a replacement of K, $\Phi(K_0)$, such that $\Phi(K_0 \cap T)$ fails to have interval topology at at most finitely many points of $K_0 \cap T$.

Example (15): Let $A = (-\frac{3}{4}, -\frac{1}{2}] \cup \{-\frac{1}{6}, -\frac{1}{9}, -\frac{1}{12}, \ldots\} \cup (-\frac{1}{4}, -\frac{1}{5}) \cup (-\frac{1}{7}, -\frac{1}{8}) \cup (-\frac{1}{10}, -\frac{1}{11}) \cup \ldots$. Let B be the set consisting of $\{0 , 1\}$ together with the union of a set of disjoint semi-closed intervals, each containing one and only one element of the set $B' = \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \ldots\}$. For convenience we will assume that if $[a , b)$ is a component of B then a has an immediate predecessor in B and b has no immediate successor in B . Also, assume that the interval containing $\frac{1}{2}$ is closed on the left and denote this interval $[a_{\frac{1}{2}} , b_{\frac{1}{2}})$. Define $T = A \cup B$.

Then $G = \{[-\infty , 0] , [0 , 1] [1 , \infty]\}$ and $G' = \{[-\infty , 0] , [0 , 1]\}$ For each element $K \in G$, $\Phi(K_0)$ is K with order induced from $\mathbb{R}$. Also $M_{[-\infty,0]} = [\frac{1}{2} , 0]$,

$$M_{[0,1]} = \emptyset ,$$

$$M_{[1,\infty]} = \emptyset .$$



Let $\Phi(N)$ be the replacement of S obtained by replacing each term K of G by $\Phi(K_0)$. Let $M = \{M_K | K \in G\}$.

Lemma (7): Suppose condition (2) holds and $S \notin G$. Then $\Phi(N)$ and M satisfy the following.

(a) Every point at which $\Phi(T)$ fails to have interval topology is a member of some element of M.

(b) Let H^* be the union of all open intervals of $\Phi(N)$ whose closures intersect $T - Q$ in a compact set disjoint from $Cl_N(Q)$.
If $I \in M$, there exists $\ell_I \in I$ such that $\ell_I \in Cl_N(N-(I \cup H^*))$ and if the set of points of $(I \cap T)$ at which $\Phi(T)$ fails to have interval topology is infinite, it can be ordered in a simple sequence, monotonic in Φ and with limit point ℓ_I.

(c) The set M is countable and for all $\epsilon > 0$ and $I \in M$ there exists an open interval of $\Phi(N)$ containing ℓ_I such that any two points of $T - I$ are no further apart than ϵ on the original line $\mathbb{R}$.

Proof: We show first that M is a disjoint family. Let $I, J, \in M$. $I \in M_{I'}$, $J \in M_{J'}$ and there are 2 possibilities,

(1) $I' = J'$ and in this case clearly $I \cap J = \emptyset$ or else $I = J$,

(2) $I' \neq J'$. Assume $I \cap J \neq \emptyset$. Then since $I' \subset I$ and $J' \subset J$, $I' \cap J' \subset I \cap J$. Further, I' and J' intersect in at most one point so assume $I \cap J = \{p\}$ and p is an end point of I' and J'. By definition of

G , p is not the end point of non-trivial components of $I' \cap H$ and $J' \cap H$ so assume p is not the end point of a non-trivial component of $I' \cap H$. By Lemma (6) , the other end point of I' is in $P_{I'}$. Hence I was chosen by rules (2) , (3) , (4) above and in this case $p \notin I$. This contradiction establishes the result.

(a) If $t \in T$ and $t \in K \in G$ then $\Phi(T)$ clearly has interval topology at t or else t is in some term of M_K . If t is not in some term K of G then t is a 2 sided accumulation point of points outside of H and hence a 2 sided accumulation point of points of T , and also of points of $\mathbb{R} - T$ and so $\Phi(T)$ has interval topology at t .

(b) Let $I \in M$ and assume $I \in M_K$ for $K \in G$. If there exists $p \in (P_K \cap I)$ let $\ell_I = p$. Otherwise, I is a replacement of K and since $S \notin G$, $K \neq S$ and there exists an end point of I , different from $-\infty$ or $+\infty$. Let ℓ_I be such an end point. Again by Lemma (6) , $\ell_I \in Cl_N(S - (K \cup H))$ and hence $\ell_I \in Cl_N(N - (I \cup H^*))$. If the number of points at which $\Phi(K)$ fails to have interval topology is infinite, then by definition of $\Phi(N)$, $\ell_I \in P_K$. Assume ℓ_I is a left end point of I and let x be a point of N such that $[\ell_I, x] = I$ and $(\ell_I, x) \subset H$ and $x \notin P_K$. Let V be the set consisting of those points v of $[\ell_I, x]$ such that

$v \in T - Q$ or v is between two points t_1 and t_2 of $T - Q$ and no points of Q are between t_1 and t_2. Then $V \neq \emptyset$ since $\ell_I \in P_K$ and also, there is a point of Q between every two components of V. Now since $x \notin P_K$, and $(\ell_I, x) \subset H$, ℓ_I is the limit point of a simple sequence which is monotonic in Φ consisting of components of V. But the points at which $\Phi(T)$ fails to have the interval topology are the endpoints of components of V hence the proof of (b) is complete.

(c) To show that M is countable, we notice that each term of M corresponds to a non trivial closed interval on the real line and hence the set M is countable. To show the second part of (c), note that Δ preserves the order of T in $\mathbb{R}$ and that Φ preserves the order of T in Δ except within terms of G. Hence if the only term of G which ℓ_I belongs to is K then the result is clear. If ℓ_I belongs to some other term, L, of G then $\ell_I \in P_L$ and $\ell_I \notin (H \cap L)$ by definition of G and in Lemma (4) or Lemma (6) the order π on L is described by case (2). Now by definition of a replacement, the result clearly holds.

Example (16): Let T be defined as in example (15). Then $M = \{[-\frac{1}{2}, 0]\}$ and $\Phi(N) = \mathbb{R}$. The lemma clearly holds and in this case $\ell_{[-\frac{1}{2},0]} = 0$.

Example (17): Let T be defined as above. Define $f_i: [-1, 1] \to [-\frac{1}{2^i}, -\frac{1}{2^{i+1}}]$ for $i = 0,1,2,\ldots$ to be the order preserving homeomorphism of $[-1, 1]$ onto $[-\frac{1}{2^i}, -\frac{1}{2^{i+1}}]$

Let $T' = \cup \{f_i(T) \mid i = 0,1,2,\ldots\}$.

Then $M = \{f_i[-\frac{1}{2}, 0] \mid i = 0,1,2,\ldots\}$ and $\ell_{f_i(T)} = f_i(0)$ for $i = 0,1,2,\ldots$.

Lemma (8): If condition (2) holds and $S \notin G$, then T is totally orderable.

Proof: Let Φ , M , N be as defined in Lemma (7) . Since M has a countable number of elements, let $M = \{M_1, M_2, \ldots\}$. For each $n \in \mathbb{N}$, we will construct N_n , a collection of compact intervals of $\Phi(N)$ and a total ordering, $\Delta_n(T)$ satisfying the following conditions:

1. The terms of N_n do not intersect in T and if $n > 0$ then every element of N_n is a subset of some element of N_{n-1} .
2. If $n > 0$ and $\Delta_n(T)$ fails to have interval topology at a point, $t \in T$, then $\Delta_{n-1}(T)$ fails to have interval topology at t and $t \notin M_n$.
3. If $\Delta_n(T)$ fails to have interval topology at a point, $t \in T$, then there is a term, $I \in M$, containing t and there is a term of N_n containing t and ℓ_I in its interior.

4. (a) If $X \in N_n$, then $\Delta_n(X \cap T) = \Phi(X \cap T)$ or $\Delta_n(X \cap T) = -\Phi(X \cap T)$

(b) No point of $T - X$ is between two points of $X \cap T$ in Δ_n

5. If $n > 0$ and $x \, \Delta_n \, y$ and $y \, \Delta_{n-1} x$ then x and y are in the same term of N_{n-1}. If $X \in N_n$ and X contains x or X contains y, then no two points of $X \cap T$ are at a distance apart, on the original line, greater than $\frac{1}{n}$.

6. If $n > 0$ and $x, y \in T$ and no point is between x and y in Δ_{n-1} then no point of T is between x and y in Δ_n.

We first verify that the sequences $\{\Delta_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$ can be defined. Let $\Delta_0 = \Phi$, $N_0 = \{N\}$ and these clearly satisfy the hypothesis. Assume we have defined $\Delta_n(T)$ and N_n for all $n < k$. Then we define $\Delta_k(T)$ and N_k as follows.

Clearly if $\Delta_{k-1}(T)$ has interval topology at every point of $M_k \cap T$ then set $\Delta_k(T) = \Delta_{k-1}(T)$ and $N_k = N_{k-1}$ and the conditions are satisfied.

So assume there exists a point of $M_k \cap T$ where $\Delta_{k-1}(T)$ fails to have interval topology. Let ℓ denote ℓ_{M_k} and by hypothesis (3), let X be the term of N_{k-1} containing, in its interior, ℓ and all points where

$\Delta_{k-1}(T \cap M_k)$ fails to have interval topology. Assume, without loss of generality, that ℓ is a right hand end point of M_k. By Lemma 7(b) and (c) there exists $t \in X \cap T$ such that $A = \{y \in T \mid \ell \,\Phi\, y \,\Phi\, t\}$ satisfies $t_1, t_2 \in (T \cap A) \Rightarrow |t_1 - t_2| < \frac{1}{k}$. Denote by d the endpoint of $M_k \cap X$. Denote by (a, b) the set $\{y \in N \mid a \,\Phi\, y \,\Phi\, b\}$ and similarly denote $[a, b]$. Now let V' be the set of half open intervals of N which are components of $X - T$ which have one and only one end point in $[d, \ell] \cap T$. If there is an element $v \in V'$ such that there exists $x \in X \cap T$ where x is the only end point of v in the interior of X and there exists $y \in (T-X)$ such that there are no points of T between x and y in $\Delta_{k-1}(T)$, then define $V = V' - \{v\}$. Otherwise, $V = V'$. Now it is clear from the definition of V that $\Delta_{k-1}(M_k \cap T)$ fails to have interval topology at x if and only if x is the end point of some element of V. Hence, from Lemma 7(b), if V has an infinite number of members, these members can be ordered in a sequence and this sequence is monotonic in Φ and has limit point ℓ. If V has a finite number of members, let $V = \{v_1, v_2, \ldots, v_j\}$ where $v_i \,\Phi\, v_{i+1}$ for $i = 1, \ldots, j-1$. Let d_i be any point of v_i and let b_i be the end point of v_i which is in $[d, \ell] \cap T$. We will define an order, Ω, on a superset of $[d, \ell] \cap T$. At present, let $\Omega([d, \ell] \cap T) = \Phi([d, \ell] \cap T)$ Let U be the set of all components of $(\ell, t) - T$ having one and only one end point in T and let U' be the union of all such components.

Case (1): Suppose $\ell \in Cl_N(U')$. If $u \in U$ let $b(u)$ be the one end point of u not in T . Given $u \in U$, $b(u)$ belongs to some term I of M and again by Lemma 7(b) , of all terms $w \in U$ satisfying $b(w) \in I$, there is a first with respect to order Φ , say u^* . Define $U^* = \{u^* \mid u \in U\}$ and since $\ell \in Cl_N(U')$ and ℓ belongs to only one term of G , let $\{u_0, u_1, \ldots\}$ be a sequence of terms of U^* , monotonic in Φ and having ℓ as a limit point. Let p_i be any point of u_i and let $\hat{U}$ be the subsequence of U^* that satisfies, without loss of generality, $b(u_n) \Phi p_n$. Define $L = \{[p_n, p_{n-1}] \mid n = 1,2,\ldots\}$. Let L' be L together with components of $X - (d, p_0)$ and define $N_k = (N_{k-1} - \{X\}) \cup L'$.

(a) Suppose V has only a finite number of elements. We now extend the definition of Ω to $[d , p_0] \cap T$. Let $((p_1, p_0)\cap T) \,\Omega\, ((d_1, d_2)\cap T) \,\Omega\, ((p_2, p_1)\cap T) \,\Omega\, ((d_2, d_3)\cap T)$ $\Omega \ldots \Omega\, ((d_{j-1})\cap T) \,\Omega\, ((p_j, p_{j-1})\cap T) \,\Omega\, ((p_{j+1}, p_j)\cap T)$ $\Omega \ldots$

If $x , y \in (p_n, p_{n-1})$ let $y \,\Omega\, x$ iff $y \Phi x$ and

(1). $n \leq j$ and $d_n \Phi b_n$ and $b(u_n) \Phi p_n$ or

(2) $n > j$ and $n - j$ is even.

Now define $\Delta_k(T)$ as follows,

$x\ \Delta_k\ y$ iff $x\ \Delta_{k-1}\ y$ and not both of x and y are in $[d\ ,\ p_0] \cap T$ and

$\Delta_k([d\ ,\ p_0] \cap T) = \Omega([d\ ,\ p_0] \cap T)$ if $\Delta_k(X \cap T)$ and $\Delta_k([d\ ,\ p_0] \cap T) = -\Omega([d\ ,\ p_0] \cap T)$ if $\Delta_k(X \cap T) = -\Phi(X \cap T)$. We now check the hypotheses for Δ_k and N_k . Clearly (1) and (5) are satisfied. Since if $\Delta_{k-1}(T)$ fails to have interval topology at $t \in M_k$ then $t = b_n$ for some n and by definition of $\Delta_k(T)$, hypotheses (2) is satisfied. By dêfinition of $\hat{U}$, hypothesis (3) holds and (4) and (6) are immediate.

(b) Suppose V is infinite. Define the following order, Ω on $[d\ ,\ p_0] \cap T$,

$((p_1, p_0) \cap T)\ \Omega\ ((d_1, d_2) \cap T)\ \Omega\ ((p_3, p_2) \cap T)$ $\Omega((d_2, d_3) \cap T)\ \Omega((p_5, p_4) \cap T) \ldots\ \ldots\ ((p_4, p_3) \cap T)$ $\Omega((p_2, p_1) \cap T)$. We complete the definition of Ω in the usual way. Suppose $x\ ,\ y \in (p_n, p_{n-1}) \cap T$ and n is odd. Then $y\ \Omega\ x$ iff $y\ \Phi\ x$ and $d_i\ \Phi\ b_i$ where $n = 2i + 1$.

If $x\ ,\ y \in (p_n, p_{n-1})$ and n is even, then $y\ \Omega\ x$ iff $y\ \Phi\ x$ and $\frac{n}{2}$ is even or $x\ \Phi\ y$ and $\frac{n}{2}$ is odd. Also, $\Delta_k(T)$ and N_k are defined as before and the hypotheses are similarly verified.

Case (2): Suppose $\ell \notin Cl_N(U')$. Then there exists a point t' of $(\ell\ ,\ t)$ such that $t' \in T$ and no component of $(\ell, t') - T$ has an end point belonging to T . For assume the contrary.

Since $\ell \notin Cl_N(U')$ and every point at which $\Phi(N)$ fails to have interval topology is in some term of M , ℓ is the end point of $M_\ell \in M$ where $\ell \neq k$. In the definition of $\Phi(N)$, ℓ is the left hand end point of a component of H^* and also the right hand end point of a component of H^* . This clearly contradicts Lemma 7 (b) . Hence choose a sequence, $\{p_n\}_{n=0}^{\infty} \subset ((\ell, t') - T)$ decreasing to ℓ . Between p_n and p_{n-1} there is a point not in H^* . Hence between p_n and p_{n-1} there is a point and since no point of T is the end point of a component of $((\ell, t') - T)$, choose t_n so that the component of T containing t_n is trivial. So let $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ be sequences such that $r_i \notin T$, $s_i \notin T$, $i = 0, 1, \ldots$ and $\{r_n\}$ increases to t_n , $\{s_n\}$ decreases to t_n and $r_0 = p_{n-1}$, $s_0 = p_n$.

Then define a total order Φ' on $((p_n, p_{n-1}) \cap T)$ by:

$\Phi'((s_0, t_n) \cap T) = \Phi((s_0, t_n) \cap T)$,

$\Phi'((t_n, r_0) \cap T) = -\Phi((t_n, r_0) \cap T)$, and

$((s_0, s_1) \cap T)\ \Phi'((r_1, r_0) \cap T)\ \Phi'\ ((s_1, s_2) \cap T)\ \Phi'((r_2, r_1) \cap T) \ldots$

$\Phi'\{t_n\}$. Note $\Phi'((p_n, p_{n-1}) \cap T)$ has interval topology with no first element and a last element. With Φ' replacing Φ , we can define U' , $\hat{U}$ and we have the same case as case (1) . Define Ω as in case (1) . We note, however, that hypothesis (4) does not hold if N_k is defined as in case 1 . But since $\Delta_k((\ell, t') \cap T)$ has interval topology at every point, we can define $L = \emptyset$ and then N_k as before and hypothesis (4) holds.

Now that we have defined $\Delta_n(T)$ and N_n for each n, we will show how to define a total order, $\Delta_\infty(T)$, such that $\Delta_\infty(T)$ has interval topology.

First notice that if $t \in T$, and for each k, there exusts $X_k \in N_k$ such that $t \in X_k \in N_k$ then there is an integer m such that $\Delta_n(X_n) = \Delta_m(X_m)$ for all $n \geq m$. For, if the above it not true, then by construction of N_k, the component of T containing t is trivial and this contradicts the fact that $\Phi(N)$ is a replacement of S.
Now define $\Delta_\infty(T)$ as follows:

(1) If there exists a (smallest) integer, n, such that x and y are in different elements of N_n, let $x \, \Delta_\infty \, y$ if and only if $x \, \Delta_n \, y$.

(2) If x and y are in the same element, X_n of N_n for each n, then there exists m such that for all $n \geq m$, $\Delta_n(X_n \cap T) = \Delta_m(X_n \cap T)$. Let $x \, \Delta_\infty \, y$ if and only if $x \, \Delta_m \, y$.

We first show that Δ_∞ is a total ordering of T.

(a) If $x \, \Delta_\infty \, y$ by (1) and $y \, \Delta_\infty \, z$ by (1) then by hypothesis (5), x and z are in different elements of N_p where $p = \max(n, m)$ and $x \, \Delta_p \, z$ so that $x \, \Delta_\infty \, z$.

(b) If $x \, \Delta_\infty \, y$ by (1) and $y \, \Delta_\infty \, z$ by (2) then by hypothesis (5) and 4(b), $x \, \Delta_\infty \, z$.

(c) If $x \Delta_\infty y$ by (2) and $y \Delta_\infty z$ by (2) then clearly, $x \Delta_\infty z$.

We now show that $\Delta_\infty(T)$ has interval topology. Suppose $\Delta_\infty(T)$ fails to have interval topology at $t \in T$. Then if $\Delta_n(T)$ fails to have interval topology at t for all n, there exists $M_n \in M$ such that $t \in M_n$ by (3). But by (2), $t \notin M_n$ because $\Delta_n(T)$ fails to have interval topology. Hence there exists n such that $\Delta_n(T)$ has interval topology at t. By (2), $\Delta_m(T)$ has interval topology at t for all $m > n$. To show that $\Delta_\infty(T)$ has interval topology at t, let $t \in 0 = \{z \in \mathbb{R} \mid a < z < b\}$ where $a, b \in \mathbb{R}$ and 0 is a basic open set of $\mathbb{R}$. Choose n_0 such that $n_0 > n$, $\frac{1}{n_0} < \frac{1}{2} \min \{\mid t-a \mid , \mid t-b \mid\}$ and $n_0 > k$ where k is an integer that satisfies

(1) t is not in any element of N_k or

(2) for all $m \geq k$, there exists $X_m \in N_m$ such that $t \in X_m$ and $\Delta_m(X_m \cap T) = \Delta_k(X_k \cap T)$. Then choose $x, y \in T$ such that $t \in \{u \in T \mid x \Delta_{n_0} u \Delta_{n_0} y\} \subset 0 \cap T$. Then assume $\mid x-t \mid < \frac{1}{n_0}$ if t has no immediate predecessor in Δ_{n_0} and $\mid y-t \mid < \frac{1}{n_0}$ if t has no immediate successor in Δ_{n_0}. If t has an immediate successor and predecessor, say y and x then by (6), $t \in \{u \in T \mid x \Delta_\infty u \Delta_\infty y\} \subset 0 \cap T$. So assume that x and y are chosen so that $\mid x-t \mid < \frac{1}{n_0}$

and $|\ y-t\ | < \frac{1}{n_0}$. By (5) and the definition of n_0 , $t \in \{u \in T \mid x\ \Delta_\infty\ u\ \ \Delta_\infty\ y\} \subset O \cap T$.

Now let $t \in O' = \{u \in T \mid x\ \Delta_\infty\ u\}$ where $x \in T$ and O' is a subbasic open set of $\Delta_\infty(T)$. Let k be an integer such that

(1) x and t are in different elements of N_k if x and t are each in an element of N_k or

(2) for all $m \geq k$, there exist $X_m \in N_m$ such that x , $t \in X_m$ and $\Delta_m(X_m \cap T) = \Delta_k(X_k \cap T)$. Let n_0 be such that $n_0 > k$, $\Delta_{n_0}(T)$ has interval topology at t and $\frac{1}{n_0} < \frac{1}{2}\ |\ t-x\ |$. Since $\Delta_{n_0}(T)$ has interval topology at t , let $a \in \mathbb{R}$ be such that $t \in \{z \in \mathbb{R} \mid z > a\} \cap T \subset \{u \in T \mid x\ \Delta_{n_0}\ u\}$. Then by definition of n_0 , $t \in \{z \in \mathbb{R} \mid z > a\} \cap T \subset O'$.

Example (18): If T is defined as in example (15) then we could define as follows an order Φ on T such that T has the interval topology:

$$\overrightarrow{(-\tfrac{3}{4}, -\tfrac{1}{2}]}\ \Phi\ \overleftarrow{(a\tfrac{1}{4}, b\tfrac{1}{4}]}\ \Phi\ \overrightarrow{(-\tfrac{1}{4}, -\tfrac{1}{5})}\ \Phi\ \overleftarrow{[a\tfrac{1}{8}, b\tfrac{1}{8})}\ \Phi\ \{-\tfrac{1}{6}\}\ \Phi\ \overleftarrow{(a\tfrac{1}{16}, b\tfrac{1}{16}]}$$

$$\Phi\ \overrightarrow{(-\tfrac{1}{7}, -\tfrac{1}{8})}\ \Phi\ \overleftarrow{[a\tfrac{1}{32}, b\tfrac{1}{32})}\ \Phi\ \{-\tfrac{1}{9}\}\ \ldots\ \Phi\ \{0\}\ \Phi\ \overrightarrow{[a\tfrac{1}{2}, b\tfrac{1}{2})}\ \Phi\ \overrightarrow{(a\tfrac{3}{4}, b\tfrac{3}{4}]}$$

$\Phi\ \overrightarrow{[a\tfrac{7}{8}, b\tfrac{7}{8})}\ \ldots\ \Phi\{1\}$. Clearly $\Phi\ (f_i(T))$ could be defined similarly.

Example (19): If T' is defined as in example (17) then it should be clear from the above example that an order Φ' for T' can be constructed so that T' has the interval topology. For $\Phi'(T')$ is defined as follows:

$$\Phi'(f_i(T)) = \Phi\ (f_i(T)) \quad \text{for} \quad i = 0,1,2,\ldots$$

and $f_1(T)\ \Phi'\ f_2(T)\ \Phi'\ f_3(T)\ \ldots$. In fact, such an order, Φ' is constructed in the proof of lemma (8) and is denoted by Δ_∞ . In the notation of lemma (8) we have:

$$M = \{f_i[-\tfrac{1}{2}\ ,\ 0]\ |\ i = 0,1,2\ldots\ \}\ ,$$

$$N_0 = \{\mathbb{R}^*\}\ ,$$

$$N_i = \{R - (-\frac{1}{2^{i-1}}\ ,\ -\frac{1}{2^i})\ |\ i = 1,2,\ldots\}\ .$$

Also, Δ_0 is the induced order from $\mathbb{R}$ and $\Delta_i(T')$ is defined by

$$\Delta_i(f_j(T)) = \Phi\ (f_j(T)) \text{ if } j \leq i\ ,$$

and $\quad \Delta_i(f_j(T)) = \Delta_0(f_j(T))$ otherwise ,

and $\quad (f_i(T))\ \Delta_i\ (f_2(T))\ \Delta_i (f_3(T))\ \ldots$.

The following result was shown by I. L. Lynn. Let T be a subspace of $\mathbb{R}$ and $\eta(T)$ denote the end points of the open ends of components of $\mathbb{R} - T$ which are half open intervals. If no open subset of T is compact and $Cl_{\mathbb{R}}(\eta(T)) \cap T$ is countable then T is linearly orderable.

Lynn also conjectured the following result which is easily proven using the main theorem.

Corollary: If T contains no open compact sets, then T is linearly orderable

Proof: Assume T contains no open compact sets and we will show that condition (1) and condition (2) hold.

(1) First assume that Q is defined as in the main theorem and that $T - Q$ is compact and $(T-Q) \cap Cl_{\mathbb{R}}(Q) = \emptyset$. Then $T - Q = T \cap (\mathbb{R} - Cl_{\mathbb{R}}(Q))$ and so $T - Q$ is open in T since T contains no open compact sets, $T - Q = \emptyset$ so condition (1) holds.

(2) Assume I is an open interval of $\mathbb{R}$ and p is an end point of I and $\{p\} \cup (I \cap (T-Q))$ is compact and

$$\{p\} = Cl_{\mathbb{R}}(I \cap (T-Q)) \cap Cl_{\mathbb{R}}(I \cap Q) .$$

Since $p \in Cl_{\mathbb{R}}(I \cap (T-Q))$, $I \cap (T-Q) \neq \emptyset$ and choose $x \in I \cap (T-Q)$.

Let $A = \{t \in I \cap (T-Q) \mid t \leq x \text{ and } (x , t) \cap Q = \emptyset\}$,

$B = \{t \in I \cap (T-Q) \mid t \geq x \text{ and } (t , x) \cap Q = \emptyset\}$.

A is bounded below by p and $A \neq \emptyset$ so let $t_1 = \inf (A)$. B is bounded above since $\{p\} \cup (I \cap (T-Q))$ is compact and $B \neq \emptyset$ hence let $t_2 = \sup (B)$. Clearly $t_1 \neq p$ and $t_1, t_2 \in I \cap (T-Q)$ since $\{p\} \cup (I \cap (T-Q))$ is compact. Hence $t_1, t_2 \notin Cl_{\mathbb{R}}(I \cap Q))$ and so $[t_1, t_2] \cap T$ is open and is compact. This contradiction establishes condition (2) vacuously.

Definitions: Let $\Phi(R)$ be an ordered topological space and let T be a subspace.

(1) If $p \in T$, let $A(p)$ denote the set of all sequences of points of T, of type α for α an ordinal, approaching p which are monotonic in Φ and which have no subsequence of smaller cardinality approaching p.

(2) Let Q denote the set of all points $q \in T$ such that:

(a) If p is the first or last point of R or any point of R not in the component of T containing q, then the interval, $[p, q]$ of $\Phi(R)$ does not intersect T in a compact set and

(b) either (1) the component of T containing q is trivial or

(2) there are terms of $A(q)$ of different cardinality or

(3) there are terms, A', $A \in A(q)$ such that every subsequence of A' approaching q has a limit point not in $Cl_R(T)$.

(3) Define $\Delta(S)$, to be the Dedikind compactification of $\Phi(R)$; that is, let p be in S if (a) $p \in R$ or (b) p is an initial interval of $\Phi(R)$ having no last point and such that no first point of R follows this interval in R. Let the topology of S be the interval topology induced by Δ defined as follows:

Let $x , y \in S$. Then $x \Delta y$ if

(a) $x , y \in R$ and $x \Phi y$ or

(b) $x \in R$ and $y \in S - R$ and an element of y follows x in Φ or

(c) $x , y \in S - R$ and some term of y follows every term of x in Φ or

(d) $y \in R$ and $x \in S - R$ and some term of x follows y in Φ .

(4) The sets H , H' and G can be defined as before, preceding Lemma (0) and Lemma (3) . Let F denote the set of all terms $[p , q]$ of G where $p \Delta q$ and there is no total ordering Γ of $([p , q] \cap T)$ such that

(a) Γ $([p , q] \cap T)$ has the interval topology at each point, and

(b) if $p \in T$, p is the first point of $\Gamma([p , q] \cap T)$ and

(c) if $q \in T$, q is the last point of $\Gamma([p , q] \cap T)$ and

(d) if $p \notin T$, there is no first point of $\Gamma([p , q] \cap T)$ and if $q \notin T$ there is no last point of $\Gamma([p , q] \cap T)$

Theorem (2) Let $\Phi(R)$ be an ordered topological space and T a subspace. The following conditions are necessary and sufficient to insure that T is an orderable topological space.

(1) If $T - Q$ is compact and $(T-Q) \cap Cl_T(Q) = \emptyset$ then either $Q = \emptyset$ or $T - Q = \emptyset$.

(2) If I is an open interval of $\Phi(R)$ and $p \in (T-I)$ and $I \cap (T-Q) \cup \{p\}$ is compact and $Cl_T(I \cap (T-Q)) \cap Cl_T(I \cap Q) = \{p\}$ then (a) the component of T containing p is trivial, and

(b) no term of $A(p)$ is uncountable.

(3) If $S \notin G$, then, for each term I of F , there is a point $f(I) \in T - I$ such that if $X \subset F$ and X^* is the intersection of T and the union of terms of X and $p \in (T-(X^* \cup f(X)))$, then $p \in Cl_T(X^*)$ if and only if $p \in Cl_T(f(X))$.

Discussion: Because of the way F is defined, the theorem cannot easily be used to prove that a subspace, T , is an orderable topological space. Essentially, if $[p , q]$ is a term of G , it is as hard to determine whether $[p , q] \cap T$ is orderable as it is to determine whether T is orderable.

Let Ω be the first uncountable ordinal. We consider the set, $2^{\Omega} = R$ ordered lexicographically. We first show that R is order complete. Let A be a non-void subset of R and A clearly has an upper bound since R has a last element. If α is an ordinal and $\alpha < \Omega$ define A_α and y_α as follows:

(1) $A_0 = \{a \in A \mid a_0 = 1\}$ and $y_0 = 1$ if $\{a \in A \mid a_0 = 1\} \neq \emptyset$ and $A_0 = A$ and $y_0 = 0$ otherwise,

(2) if $\alpha = \beta + 1$ for β an ordinal
$A_\alpha = \{a \in A_\beta \mid a_\alpha = 1\}$ and $y_\alpha = 1$ if $\{a \in A_\beta \mid a_\alpha = 1\} \neq \emptyset$ and $A_\alpha = A_\beta$ and $y_\alpha = 0$ otherwise,

(3) if α is a limit ordinal,
$A_\alpha = (\bigcap_{B<\alpha} A_\beta) \cap \{a \in A \mid a_\alpha = 1\}$ and $y_\alpha = 1$ if $A_\alpha \neq \emptyset$ but $y_\alpha = 0$ if $A_\alpha = \emptyset$.

Now clearly, either there exists α_0, an ordinal, such that $A_\alpha = \{a\}$ for some $a \in A$ and for all $\alpha > \alpha_0$ or ~~else~~there exists a limit ordinal α_0 such that $A_{\alpha_0} = \emptyset$. In this first case, a is the largest element of A and $a = \sup(A)$. If $A_{\alpha_0} = \emptyset$ for some limit ordinal α_0 then $y = \{y_\alpha\}$ a $< \Omega$ is the supremum of A. Hence R is order-complete and so any closed bounded subset of R is compact. Then by definition of Q, if a, b $\in$ R and [a,b] is a component of T then $[a, b] \subset T - Q$. If a, b $\in$ R and (a,b) is a component of T then $(a, b) \subset Q$.

Also, if $p \in R$ then A(p) consists of those sequences converging to p of type α where α is an ordinal and $\overline{\overline{\Omega}} = \overline{\overline{\alpha}}$. Hence condition (2) is false unless it holds vacuously.

Further, $\Delta(S) = \Phi(R)$ where Φ is the lexicographical ordering on R.

Example (20): Let $T = [a , b] \cup (c , d)$ where $a,b,c,d \in R$ and $a < b < c < d$. Then $[a , b] \subset T - Q$ and $(c , d) \subset Q$ and condition (1) does not hold so that T is not orderable.

Example (21): Choose $p \in R$ and let $\{a_\alpha\}_{\alpha<\Omega}$ be a descending sequence of type Ω contained in R converging to p. For each ordinal $\alpha < \Omega$, define the set $T_\alpha \subset R$ as follows. First write $\alpha = \lambda + n$ where λ is a limit ordinal and n is a finite ordinal.

Define $T_\alpha = (a_{\alpha+1}, a_\alpha)$ if $n \in \{2,5,8,\ldots\}$ and $T_\alpha = \{a_\alpha\}$ if $n \in \{1,4,7,\ldots\}$ and $T_\alpha = \emptyset$ otherwise. Define $T = \bigcup_{\alpha<\Omega} T_\alpha$. Then T is not orderable for condition (2) does not hold since every term of $A(p)$ is uncountable.

We now examine an ordered topological space, R in which every subspace, T satisfies condition (3). Let γ be an infinite ordinal. Let β be the first ordinal number such that $\overline{\overline{\beta}} = 2^{\overline{\overline{\gamma}}}$ and consider the set $R = 2^\beta$ ordered lexicographically. Let Φ be the lexicographical order for R. Then define, for each $\alpha < \beta$, $A_\alpha = \{x \in R \mid x_\gamma = 0$ for all $\gamma > \alpha\}$. Define $A = \bigcup_{\alpha<\beta} A_\alpha$. Then clearly A is dense in R. Also, assuming the Generalized continuum Hypothesis, $\overline{\overline{A}} = \overline{\overline{\bigcup_{\alpha<\beta} A_\alpha}} \leq \sum'_{\alpha<\beta} 2^{\overline{\overline{\alpha}}} \leq \beta \cdot \beta = \beta$ and so clearly $\overline{\overline{A}} = \beta$.

Since R has a dense subset of cardinality β, $A(p)$ consists precisely of those sequences of cardinality β converging to p and hence $A(p)$ is uncountable if and only if $\beta > \omega$. Hence if $\beta > \omega$ condition (2) holds if and only if it holds vacuously. We have, of course, already examined $R = \mathbb{R}^* = 2^{\omega}$ in detail.

Also, we can show that $\Phi(R)$ is order-complete in the same way that we have shown that 2^{Ω} is ordered lexiocgraphically is order complete. Hence $\Delta(S) = \Phi(R)$ and if $[a, b]$ is a component of T, then $[a, b] \subset T - Q$ and if (a, b) is a component of T, then $(a, b) \subset Q$.

Hence condition (3) holds for all subspaces T of R. For assume, at first, that $\bar{\bar{F}} = \beta$. Then $F = \{I_\alpha | \alpha < \beta\}$ and assume $I_\alpha = [a_\alpha, b_\alpha]$ where $a_\alpha, b_\alpha \in R$. For $I_\alpha \in F$, define $f(I_\alpha) = x$ where $x_\gamma = b_\gamma$ for all $\gamma \neq \alpha'$ and $x_{\alpha'} = 1$ where α' is the first ordinal greater than α such that $b_{\alpha'} = 0$. Clearly condition (3) holds. If $\bar{\bar{F}} < \beta$ then condition (3) holds vacuously.

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