Embedding Theorems in Finite Soluble Groups

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ABSTRACT

By a group we will mean a finite soluble group. It is an interesting fact, (Pardoe [1]), that the subgroup closure of the class of groups $P^Q$, those with a unique complemented chief series, is all groups. Let $X$ be the class of groups with a complemented chief series. We investigate the action of closure operations $T$ such that $TX = X$ upon $P^Q$. The purpose of this is to find a collection of such closure operations whose join applied to $P^Q$ is $X$. In the course of this investigation we introduce a new closure operation $M$ defined by:

$$M^V = \{ G \mid G = \langle X_1, \ldots, X_n \rangle, X_i \in V, X_i \text{ sn } G, (|G : X_1|, \ldots, |G : X_n|) = 1 \}.$$
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Introduction

When given a group $G$ with a property $(\ast)$ it is often useful to know whether subgroups $H$ of $G$, or quotient groups $G/K$ of $G$, also possess this property. We can pose this question in a slightly different manner. Consider the class of groups $G$ with $(\ast)$, what properties characterise the subgroups, or quotient groups, of these groups? The answer to this question gives us a deeper understanding into the structure of the groups with $(\ast)$. Of course we may also ask questions about direct products, or subnormal subgroups, of groups with $(\ast)$. The object of this dissertation is to pose a question of this type in a formal manner and give some answers for certain properties $(\ast)$.

More explicitly, given a class of groups $X$ and a closure operation $T$, is there a simple description for the class $TX$? One way to tackle this problem is by looking at $T$-closed classes of groups $Y$ that contain $X$ and deciding whether or not, $TX = Y$. An obvious extension of this method is to look at the join of several such closure operations acting on $X$.

By a group we will always mean a finite soluble group and thus our universal class of groups will be the finite soluble groups. We will denote this class by $S$.

In Chapter 1 we will discuss some known results of the type discussed. One measure of the complexity of a class of groups $X$ relative to $S$ is to look at its subgroup closure. If $sX = S$ then we can say $X$
is "large" in as far as all groups can be embedded in an $X$-group. Chapter 1 contains several results of this special type for widely dissimilar classes $X$.

In Chapter 2 we discuss two similar classes of groups, those with a unique complemented chief series and those with a complemented chief series. We investigate whether the former can be extended to the latter by a judicious choice of closure operations that leave the latter fixed. As yet this question has not been answered, but some progress has been made towards selecting a suitable set of closure operations that may work.
§1.1. Definitions and Notation.

One of the fundamental tools used to generate new groups from known groups is the wreath product. There are several definitions for the wreath product. The version we will need the most is the regular wreath product and this is defined as follows.

Definition: The regular wreath product, $G \wr_r H$, of a group $G$ by a group $H$, is defined by:

$$G \wr_r H = \{ (f, h) \mid h \in H, f : G \to H \}$$

where multiplication is defined by:

$$(f_1, h_1)(f_2, h_2) = (g, h_1 h_2),$$

and

$$g(h) = f_1(h)f_2(\text{hh}_1) \text{ for all } h \text{ in } H.$$  

As we will usually be considering the regular wreath product we will drop the suffix 'r'.

We will need several results about wreath products. One used in (1.2.9) is concerned with extending an automorphism of $G$, (or $H$), to an automorphism of $G \wr_r H$. We also need two embedding properties that are used extensively in what follows.
Lemma : Huppert [1]. (a). If $\alpha$ is an automorphism of $G$, so is $\bar{\alpha}$ an automorphism of $G \wr H$ where;

$$(f,h)\bar{\alpha} = (g,h), \text{ and } g(k) = f(k)^\alpha \text{ for all } k \text{ in } H.$$ 

(b). If $\mu$ is an automorphism of $H$, so is $\bar{\mu}$ an automorphism of $G \wr H$ where;

$$(f,h)\bar{\mu} = (g,h^\mu), \text{ and } g(k) = g(k\mu^{-1}) \text{ for all } k \text{ in } H.$$ 

Lemma : Huppert [2]. (a). If $G_1 \leq G$ and $H_1 \leq H$ then $G_1 \wr H_1$ can be embedded in $G \wr H$. (b). Let $N \triangleleft G$, then $G$ can be embedded in $N \wr G/N$.

Theorem (1.2.11) provides not only a result of interest but also a method to construct groups that have a unique complemented chief series. This construction is later used in Chapter 2. In the proof we use two general results which we will quote here for completeness. The first result allows us to calculate the order of certain chief factors, while the second result is used to provide a contradiction in the proof of (1.2.11).

Theorem : Huppert [3]. Let $V$ be a faithful irreducible module of dimension $n$ over the field $K = GF(p^f)$, for an Abelian group $A$. Then $A$ is cyclic and there exists a group homomorphism;

$$\beta : A \longrightarrow GF(p^{nf})^\times,$$

and a $K$-isomorphism, $\alpha : V \longrightarrow GF(p^{nf})^+$

such that, $\alpha(va) = \beta(a)\alpha(v)$ for all $a \in A$ and $v \in V$. 

The integer \( n \) is the uniquely determined smallest integer such that
\[ |A| \mid p^{nf} - 1. \]

\textbf{(1.1.4) Theorem: } Huppert [4]. Let \( G \leq \text{GL}(n,p^f) \), \( K = GF(p^{nf}) \), and
\( A \) an Abelian normal subgroup of \( G \). Let \( V = V(n,p^{nf}) \), considered as a \( K[A] \)
module, be the direct sum of \( s = n/k \) irreducible isomorphic \( K[A] \) modules. Then \( G \), considered as a permutation group on \( V \), acts in the same way
as a group of \( GF(p^f) \) semilinear maps \( g \), such that:

\[
(u_1 + u_2)g = u_1g + u_2g, \quad u_i \in V(s,p^{kf}).
\]

\[
(uc)g = ugc^g, \quad u \in V(s,p^{kf}), \quad c \in GF(p^{kf}),
\]

where \( c \mapsto c^g \) is a field isomorphism. Also \( C_G(A) \) is the set of
\( GF(p^{kf}) \) - linear maps of \( V(s,p^{kf}) \).

\textbf{(1.1.5) Corollary: } \( G/C_G(A) \) is isomorphic to a subgroup of \( C_k \).

\textbf{Proof: } Consider the map \( \alpha : g \rightarrow \sigma_g \), where \( \sigma_g : c \mapsto c^g \). That is,

\[
\alpha : G \longrightarrow \text{Aut} [GF(p^{kf}) : GF(p^f)]
\]

\[
= \text{Galois group} [GF(p^{kf}) : GF(p^f)]
\]

\[
\cong C_k .
\]

Also \( \alpha \) is a homomorphism with kernel given by
\[ \ker \alpha = \{ g \in G \mid \sigma_g = 1\} \]

\[ = \{ g \in G \mid c^g = c \text{ for all } c \text{ in } GF(p^{kf}) \} \]

\[ = C_G(A). \]

Hence \( G/C_G(A) \cong \alpha(G) \), a subgroup of \( C_k \).

Another result used in the proof of (1.2.11) is a theorem of Dirichlet's concerning the distribution of prime numbers in a given sequence of numbers.

(1.1.6) Theorem: Dirichlet [1]. If \((m,n) = 1\) then the series \(\{ m + kn \mid k = 1, 2, \cdots \}\) contains an infinite number of prime numbers.

We will now introduce the concept of group classes and closure operations. These concepts are extremely useful and will be used throughout this exposition.

We say a set of groups \( X \) defined by some group theoretic property possessed by all its members is a **class of groups** if:

(a). If \( G \in X \), then all groups isomorphic with \( G \) are also in \( X \),

(b). The unit group \( 1 \) is in \( X \).

We will denote by (1) the trivial class of groups and by \( (G) \) the class of groups isomorphic to \( G \), (with 1).

A **closure operation** \( c \) is a map from classes of groups to classes of groups satisfying;
(C.1). $X \leq CX$, expanding,

(C.2). If $X \leq Y$ then $CX \leq CY$, monotonic,

(C.3). $CX = C(CX)$, idempotent.

We say $X$ is $C$-closed if $CX = X$. If $B$ and $C$ are closure operations then $C \leq B$ if and only if for all classes of groups $X$, $CX \leq BX$. The join $\{A, B\}$ of two closure operation $A$ and $B$ is defined by,

$$\{A, B\}X = \bigcap \{Y \mid X \leq Y, AV = BV = Y\}.$$

By $AB(X)$ we mean $A(BX)$ and we note that $AB$ is a closure operation if and only if $BA \leq AB$.

An alternative method to describe a closure operation $C$ is to specify the $C$-closed classes. Let $C$ be a closure operation and $C = \{X \mid CX = X\}$. Then the family $C$ satisfies;

(a) $S \in C$

(b) $C$ is closed under taking arbitrary intersections.

Conversely if $C$ is a family of classes of groups satisfying these two conditions then we may define a closure operation $T$ and its action on an arbitrary class of groups $X$ by;

$$TX = \bigcap \{Y \mid X \leq Y \in C\}.$$ 

Thus there is a one to one correspondance between closure operations and families of group classes that satisfy (a) and (b).
The following can be shown to be closure operations.

\[ SX = \{ H \mid H \leq G \in X \} . \]

\[ QX = \{ G/K \mid K \triangleleft G \in X \} . \]

\[ S_n X = \{ H \mid H \trianglelefteq G \in X \} . \]

\[ N_X = \{ G \mid G = \langle N_1, \ldots, N_r \rangle, \quad N_i \in X, \quad N_i \trianglelefteq G \} . \]

\[ D_X = \{ G \mid G = H_1 \times \cdots \times H_r, \quad H_i \in X \} . \]

\[ R_X = \{ G \mid \exists \quad N_1 \triangleleft G, \quad G/N_1 \in X, \quad \bigcap_{i=1}^{r} N_i = 1, \quad i = 1, \ldots, r \} . \]

Also the common classes of nilpotent and Abelian groups will be denoted by \( N \) and \( A \), respectively.

**Notation:** If \( G \) is a group and \( H \) is a subgroup of \( G \) we define;

- \( H \leq G \), \( H \) is a subgroup of \( G \).
- \( H \triangleleft G \), \( H \) is a normal subgroup of \( G \).
- \( H \trianglelefteq G \), \( H \) is a minimal normal subgroup of \( G \).
- \( H \triangleleft G \), \( H \) is a maximal normal subgroup of \( G \).
- \( H \trianglelefteq G \), \( H \) is a subnormal subgroup of \( G \).
- \( \text{Core } H \), largest normal subgroup of \( G \) contained in \( H \).
- \( N_G(H) \), the normaliser of \( H \) in \( G \).
- \( C_G(H) \), the centraliser of \( H \) in \( G \).
- \( Z(G) \), the centre of \( G \).
F(G). \( \) the Fitting subgroup of \( G \).

\( \Phi(G) \). \( \) the Frattini subgroup of \( G \).

\( \text{Aut}(G) \). \( \) the group of automorphisms of \( G \).

\( \langle A \rangle \). \( \) the subgroup of \( G \) generated by a subset \( A \) of \( G \).

\( H^G = \langle H^g | g \in G \rangle \). \( \) the subgroup generated by the conjugates of \( H \) in \( G \).

\( [g,h] = g^{-1}h^{-1}gh \). \( \) the commutator of \( g \) and \( h \) in \( G \).

\( [H,K] = \langle [h,k] | h \in H, k \in K \rangle \).

\( G' = [G,G] \). \( \) the commutator subgroup of \( G \).

\( |G| \). \( \) the order of \( G \).

\( |G : H| \). \( \) the index of \( H \) in \( G \).

\( \pi(G) \). \( \) the set of prime divisors of \( G \).

\( G_\pi \). \( \) a Hall \( \pi \)-subgroup of \( G \), where \( \pi \) is a set of primes.

\( \) i.e. a largest subgroup of \( G \) whose order is a \( \pi \)-number.

\( C_p \). \( \) the cyclic group of order \( p \).

\( S(n) \). \( \) the symmetric group of degree \( n \).

If \( a \) and \( b \) are integers we define,

\( a \mid b \). \( \) a divides \( b \).

\( a \nmid b \). \( \) a does not divide \( b \).

\( (a,b) \). \( \) the greatest common divisor of \( a \) and \( b \).
The following explains the notation used in (1.1.3), (1.1.4).

$\mathbb{Z}_q$, the integers modulo $q$, the field of $q$ elements.

$\text{GF}(p^n)$, the Galois field of order $p^n$.

$\text{GF}(p^n)^+$, the additive group of the field.

$\text{GF}(p^n)^\times$, the multiplicative group of the field.

$\text{GL}(n, p^f)$, the group of invertible linear transformations of a vector space of dimension $n$ over $\text{GF}(p^f)$.

§1.2. Some Known Results.

In this section we will bring together some known results about closure operations acting on given group classes.

It is well known that the converse of Lagrange's theorem is false. There are groups, for example, $A_4$ the alternating group of degree 4, which do not possess subgroups of all possible orders dividing $|G|$. If $M$ denotes the class of groups which have subgroups of all possible orders it is clear from P. Hall's sufficient condition for solubility that $M \leq S$. The following theorem shows that, in the sense that any soluble group can be embedded in a group in $M$, that the class $M$ is large in relation to $S$.

(1.2.1) Theorem: Maclain [1]. $S = sM$. 
Proof: Let $G$ be a soluble group with order $\prod_{i=1}^{r} p_i^{a_i}$. Let $U$ be an Abelian group of order $\prod_{i=1}^{r} p_i^{a_i-1}$ then $G \times U$ is in $M$. For, let $d = p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$ be a divisor of $|G \times U|$. We can reorder the $d_i$'s such that $a_i \leq d_i$, $i = 1, 2, \ldots, s$; $d_i < a_i$, $i = s + 1, \ldots, r$. Let $H$ be a Hall $\pi$-subgroup of $G$ where $\pi = \{p_1, p_2, \ldots, p_s\}$, and $B$ a subgroup of $U$ of order $p_1^{d_1-a_1} \cdots p_s^{d_s-a_s} p_{s+1}^{d_{s+1}} \cdots p_r^{d_r}$. Then $|H \times D| = d$ and the proof is complete.

A subgroup $H$ of $G$ is called complemented if there is a subgroup $K$ of $G$ such that, $HK = G$ and $H \cap K = 1$. The class of groups $C$ in which every subgroup has a complement, is called the class of complemented groups. We have $C \leq S$, by P. Hall's characterisation of soluble groups, namely that $G$ is soluble if and only if it has Sylow $p$-complements for all primes $p | |G|$. The class $C$ is both $S$- and $D_0$-closed. Also if $G \in C$ then the Sylow $p$-subgroups of $G$ are elementary Abelian and the chief factors of $G$ are cyclic.

A subclass of $C$ is the class of groups $Q$, the groups of squarefree order. That these groups are soluble follows from Burnside's Theorem, namely that if a Sylow $p$-subgroup is contained in the centre of its normaliser then it has a normal Sylow $p$-complement. Further, if $H \leq G \in Q$ then $|G| = |H| \cdot m$ where $(|H|, m) = 1$. So, by Hall [1], $H$ has a complement in $G$.

We now turn to the connection between $Q$ and $C$. 
Theorem: P. Hall [3]. \[ \text{SD}_0(Q) = C. \]

Proof: By closure axiom (C.2) and the above remarks \( \text{SD}_0(Q) \subseteq \text{SD}_0(C) = C. \) Thus we need only show the opposite inclusion.

Let \( 1 \neq g \in G \) and let \( G_g \) be a normal subgroup of \( G \) maximal with respect to not containing \( g \). Then \( G \) is isomorphic to a subgroup of the direct product of the groups \( H_g = G/G_g \).

We know by the above remarks that a group \( G \) in \( C \) has the following properties.

1. Every Sylow subgroup of \( G \) is elementary Abelian.
2. Every chief factor of \( G \) is cyclic.

The groups \( H_g \) have the properties (1) and (2) and also the property.

3. of containing a unique minimal normal subgroup.

Thus it is sufficient to show that a group \( G \) satisfying (1), (2), and (3) is in \( Q \).

By (2) the unique minimal normal subgroup \( N \) of \( G \) is cyclic and has order \( p \). Let \( C = C_G(N) \). We show \( C = N \). Suppose not. Then there exists an Abelian normal subgroup \( K \) of \( G \), such that \( N \leq K \leq C \), and \( K/N \) is a chief factor of \( G \). By (2), \( K/N \) is cyclic of order a prime \( q \).

(a). Suppose \( p \neq q \). Then, \( K \) being Abelian, the Sylow \( q \)-subgroup of \( K \) is characteristic in \( K \), and \( G \) has a minimal normal subgroup different from \( N \), contradicting (3).
(b). Suppose \( p = q \). Then, as \( |K| = p^2 \), \( K \triangleleft G \), and as each Sylow \( p \)-subgroup of \( G \) is Abelian by (i), \( K \) is centralised by each Sylow \( p \)-subgroup of \( G \). So certainly \( N \) is centralised by each Sylow \( p \)-subgroup of \( G \) and thus \( F = G/C_G(N) \) is a \( p' \) group isomorphic to a subgroup of \( \text{Aut}(N) \). So we can form \( w = KF \), the semidirect product of \( K \) by \( F \), in the natural way. As \( N \triangleleft G \) and \( (|K|, |F|) = 1 \) and \( K \) is Abelian we can use Maschke's theorem to give \( K = N \times M \), where \( M \triangleleft G \). This again contradicts (3).

Thus \( N = C \) and \( G/N \) is isomorphic to a subgroup of \( \text{Aut}(C_p) \), which is cyclic of order \( p - 1 \). i.e. \( G \) is metacyclic of order \( p^k \), where \( k | p - 1 \). By (1), \( G \) contains no cyclic subgroups of order the square of a prime; so \( k \) is square free and \( G \in \mathcal{Q} \), completing the proof.

Nilpotent groups are characterised by being the direct product of their Sylow subgroups. Thus it is always possible to construct a normal series in which the factor groups have order the largest power of each prime divisor dividing the group order. Thus nilpotent groups form a subclass of the class of Sylow Tower groups \( T \), which is defined by,

\[
T = \{ G \mid G \text{ has a normal series } 1 = C_0 \triangleleft C_1 \triangleleft \cdots \triangleleft C_n = G \\
\text{where the normal factors } G_i/G_{i-1} \text{ are isomorphic to Sylow subgroups of } G \}
\]

A normal series for a group \( G \) may be defined by

\[
1 \triangleleft O_p(G) \triangleleft O_{p'}(G) \triangleleft O_{p''}(G) \triangleleft \cdots \triangleleft G \quad (1)
\]
where if \( \pi \) is a set of primes \( O_\pi(G) \) is the product of all the normal subgroups of \( G \) whose orders are divisible only by members of \( \pi \). We define \( O_{\pi_1 \pi_2 \cdots \pi_1}(G) \), where \( \pi_i \), \( i = 1, \ldots, r \) are sets of primes, inductively by

\[
\frac{O_{\pi_1 \pi_2 \cdots \pi_1}(G)}{O_{\pi_1}(G)} = O_{\pi_1} \left( \frac{G}{O_{\pi_2 \cdots \pi_1}(G)} \right).
\]

The set of primes \( \pi' \) is just the complementary set of primes to \( \pi \).

Clearly for a nilpotent group the series stops at \( O_p p(G) = G \). That is the number of factor groups in (1) divisible by \( p \) for each prime \( p \) is one. However this is also true for Sylow Tower groups as the series reaches \( G \) at most at \( O_{p_1 p_2 p_3}(G) \). Groups with this property of having only one factor group in the series (1) divisible by \( p \) for all primes \( p \) are called groups of \( p \) length 1, and form the class \( L_\omega(1) \). The class \( L_\omega(1) \) can also be locally defined by the formation function \( f : p \rightarrow S_{p'} \).

The class \( S_{p'} \) of all groups not divisible by \( p \) is \( \{Q, S, R_Q, N_Q\} \)-closed and these closure properties are inherited by the locally defined class \( L_\omega(1) \). For a description of formations and formation functions see Huppert [5].

\( \text{(1.2.3) Theorem : Alperin [1]. } \) \( S_{N0}T = L_\omega(1) \).

\[ \text{Proof : It is clear that } T \leq L_\omega(1) \text{ and so by closure operation axiom C.2, } S_{N0}T \leq S_{N0}L_\omega(1) = L_\omega(1). \] To show the opposite inclusion we prove the stronger result namely:
If \( G \in L_\omega(1) \) then, (1) \( G \cong H \leq L_1 L_2 \cdots L_n = L \) where \( L_i \triangleleft L \)
and \( L_i \in T \), and (2) \( \pi(L_i) \leq \pi(G) \).

We prove this assertion by induction on \( |G| \).

(a) If \( N_1 \) and \( N_2 \) are two distinct minimal normal subgroups of \( G \) then, as is well known, \( G/N_1 \times G/N_2 \) contains a subgroup isomorphic with \( G \). So by the inductive hypothesis \( G \in SD_o(SN_o(T)) = SN_oT \), and \( \pi(G/N_1) \cup \pi(G/N_2) = \pi(G) \).

(b) Suppose \( G \) has a unique minimal normal subgroup \( N \). If \( \{p\} = \pi(N) \), then \( O_p'(G) = 1 \), and \( P \) the Sylow \( p \)-subgroup of \( G \) is normal in \( G \). So by the induction hypothesis:

(1) \( G/P \cong H \leq L_1 L_2 \cdots L_n \in N_0T \) and

(2) \( \pi(L_i) \leq \pi(G/P) \). So,

\[
P \text{ wr. } G/P \leq P \text{ wr. } L = BL = BL_1 BL_2 \cdots BL_n \in N_0T,
\]

where \( B \) is the base group of the wreath product and by (2) \( p \nmid |L| \). Thus \( \pi(BL_1) \leq \pi(G) \) and by (1.1.2), \( G \in SN_oT \) completing the proof.

Nilpotent groups have the property of being both \( S_n \) - and \( N_o \)-closed. In fact, classes of groups which are closed under these two operations are quite numerous and have been studied to determine properties common to them all. Thus we say a class \( X \) is a Fitting class if \( X = \{S_n, N_o\}X \). We will call a Fitting class \( F \) trivial if \( F = (1) \) or \( S \).
For any class of groups $X$ and any group $G$ we define a subgroup $V$ of $G$, called the $X$-injector; if for all $N$ in $G$, $N \cap V$ is $X$-maximal. i.e. $N \cap V \in X$ and is not contained properly in any $X$-subgroup of $N$. Of course such a $V$ may not exist, but what we can say is, if $F$ is a Fitting class, then every group has an $X$-injector. It was proved by Gaschütz, Fischer, and Hartley [1], that when $X$ is a Fitting class, every group has a unique conjugacy class of $X$-injectors. It now makes sense to call a Fitting class $F$ normal if for any group $G$ the $F$ injector $G_F$ of $G$ is normal in $G$.

In a recent paper, Gaschütz and Blessenohl [1], investigate normal Fitting classes and give a theorem of the type we are discussing. What follows is a summary of some of the results of this paper. First we will give an example of a non trivial normal Fitting class.

(1.2.4) Example: G-B ; Satz 33. Consider a group $G$ with chief series $C$ and let $M_1, M_2, \ldots, M_r$ be the $p$ chief factors in $C$. Let $A$ be a cyclic group of order $p - 1$. We define

$$d_1(g) = \text{determinant of the linear transformation}$$

induced on $M_i$ by conjugation with $g$.

Let $d_G(g) = \prod_{i=1}^{r} d_1(g)$, or $d_G(g) = 1$ if $G$ has no $p$-chief factors. Define

$$F_p = \{ G \mid d_G(g) = 1 \text{ for all } g \text{ in } G \}.$$  

Then the classes $F_p$, for each prime $p$, are non trivial normal Fitting classes.
If a non trivial Fitting class $F$ contains a group $G$ and $p$ a prime, then there exists some $n$ such that $mnG \wr C_p$ is in $F$, for all integers $m > 0$. For any class of groups $X$ we define the characteristic of $X$ by

$$\text{char } X = \{ p \mid C_p \in X \} .$$

If $F$ is a Fitting class this definition is equivalent to defining

$$\text{char } F = \{ p \mid \text{there is a } G \text{ in } F \text{ such that } p \mid |G| \} .$$

Thus a normal Fitting class $F \neq 1$, contains the class $S_p$ of $p$-groups for all primes $p$. But the nilpotent groups $N$ are simply direct products of $p$-groups, and hence $N \leq F$, for all normal Fitting classes $F \neq 1$.

A non trivial normal Fitting class is a large class of groups, in the sense that its subgroup closure is all soluble groups.

(1.2.5) Theorem: Let $1 \neq F$ be a normal Fitting class then $sF = S$.

Proof: We may assume that $F < S$. Let $G \in S$ and proceed by induction on $|G|$. Let $G_1 \triangleleft G$ and $|G : G_1| = p$. If $G_1 = 1$, then $G \in N \leq F$. If $G_1 \neq 1$ then there is a $m$ such that $mG_1 \wr C_p \in F$. But then there exist monomorphisms $\mu_1$ and $\mu_2$ such that

$$G \xrightarrow{\mu_1} G_1 \wr C_p \cong G_1 \wr C_p \xrightarrow{\mu_2} mG_1 \wr C_p .$$
In general normal Fitting classes are not $Q$-closed. This is shown in the following example.

(1.2.6) Example: Let $F^\_3$ be the non trivial normal Fitting class described in (1.2.4) with $p = 3$. Let $G$ be the semidirect product of an elementary Abelian group of order 4 with an inverting involution, i.e. $G = AX$ where $A = C_3 \times C_3$ and $|X| = 2$. We can consider elements of $G$ to be triples $(a, b, y)$ where $(a, b) \in A$, $y \in X$; and multiplication is defined by

$((a, b), y) \cdot ((u, v), z) = ((a, b) (u, v)y, z)$.

Then $G$ has a chief series

$1 \triangleleft ((a, 1), 1) > = B \triangleleft B \cdot ((1, b), 1) > = A \triangleleft G = AX$.

Let $M_1 = B$, $M_2 = A/B$, then the $M_i$ are the 3-chief factors. Now $d_1(g) = 1$ if $g$ is of the form $((u, v), 1)$ and $d_1(g) = 2$ if $g$ is of the form $((u, v), x)$. In all cases $d_G(g) = d_1(g) \times d_2(g) = 1$ and $G \in F^\_3$.

But $S_3 = G/B$ and the chief series for $S_3$ is $1 \triangleleft A_3 \triangleleft S_3$, with 3-chief factor $A_3 = M_1$. In this case $d_G(g) = d_1(g)$, and $d_1((12)) = 2$. So $S_3 \notin F^\_3$.

This example suggests the question, "what is $QF$ for $F$ a non trivial Fitting class?" It would be of interest to know whether, for such $F$, $QF$ is the universal class.
If $F$ is a Fitting Class define the class Radical-Quotient $(F) = \{ H \mid H = G/G_F \text{ for some } G \}$. Blessenohl and Gaschütz prove that if $F$ is a non trivial normal Fitting class then Radical-quotient$(F) \leq A$. In other words the $F$-injectors of $G$ always contain the derived group $G'$.

Moreover, they show that this structural restriction on Radical-quotient$(F)$ characterises normal Fitting class. Indeed, any structural restriction on Radical-quotient$(F)$ seems to imply normality of the Fitting class $F$ in the following precise sense:

\(\text{(1.2.7) Theorem : B-G. [1]}\) Let $X$ be a $S$-closed class of groups different from $S$. Suppose $F$ is a Fitting class and $F \leq X$. Then $F$ is a non trivial normal Fitting class.

\(\text{(1.2.8) Theorem : F.P. Lockett. [unpublished]}\). Let $X$ be a $Q$-closed class of groups different from $S$. Suppose $F$ is a Fitting class and $F \leq X$. Then $F$ is a non trivial normal Fitting class. In other words, if $F$ is a non normal Fitting class, then $C(\text{Radical-quotient}(F)) = S$, for $C = S$ or $C = Q$.

In any soluble group the Sylow subgroups always have Sylow complements and these are conjugate, and any subgroup of order prime to a Sylow $p$-subgroup is contained in a Sylow $p$-complement. P.Hall [4] has shown that these properties can be carried over to Sylow systems. Let $S_1, \ldots, S_r$ be a complete set of Sylow-complements in a group $G$ of order $p_1^{a_1} \cdots p_r^{a_r}$. i.e. One representative from each of the $r$ conjugacy
classes of complements of Sylow subgroups. A **Sylow system** consists of all $2^r$ intersections, including the empty intersection $G$, formed from these $r$ subgroups.

If $S$ is a Sylow system of a group $G$ we can define the **system normaliser**, $N = N(S)$, to be all those $g$ in $G$ that transform each member of $S$ into itself. Alternatively we can define $N$ by, $N = \bigcap_{i=1}^{r} N_G(S_i)$.

Of some interest is the class $W$ defined by:

$$W = \{ G \mid \text{the system normalisers of } G \text{ are self normalising} \}.$$  

The groups in $W$ are called **S-C groups**. Huppert [6], gives many of the known properties of system normalisers and S-C groups. Carter [1], proved the existence of self normalising nilpotent subgroups in any group and showed these were all conjugate. For the next result we need two standard results about the system normalisers of a soluble group, namely:

(1) Every system normaliser covers each central chief factor, and

(2) Every Carter subgroup contains a system normaliser.

\[ (1.2.9) \text{ Theorem : } [\text{Alperin Thompson}] \quad S(W) = S, \]

**Proof :** Huppert [7]. Let $G \in S$ and

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n \lhd G$$
be a chief series for \( G \). Let \( \mathbb{K}_i = G_i/G_{i-1} \) \( i = 1, \cdots, m \) and \( |\mathbb{K}_i| = p_i^{k_i} \).

Choose a prime \( p \) different from all the \( p_i \)'s and choose \( n_i \) so that \( k_i < n \) and \( p_i^n \equiv 1(p) \). Let \( N_i = GF(p_i^{n_i})^+ \).

(a) By a simple induction, we show \( G \) can be embedded in the iterated regular wreath product \((\cdots(N_1 \text{ wr. } N_2) \cdots) \text{ wr. } N_m \). As

\[
G \longrightarrow G_{m-1} \text{ wr. } G/G_{m-1} \longrightarrow [(\cdots(N_1 \text{ wr. } N_2) \cdots) \text{ wr. } N_{m-1}] \text{ wr. } N_m.
\]

(b) Every \( N_i \) has an automorphism \( \alpha_i \) of order \( p \) that fixes no non trivial elements; namely that induced by left multiplication by a primitive \( p^{th} \) root of unity in \( GF(p_i^{n_i})^+ \).

Claim: The group \( H_k = (\cdots(N_1 \text{ wr. } N_2) \cdots) \text{ wr. } N_k \) has an elementary Abelian group of automorphisms \( U_k \) with \( |U_k| = p^k \), and such that no \( 1 \neq h \in H_k \) is fixed by all \( \alpha \) in \( U_k \). For \( k = 1 \) set \( U_1 = \langle \alpha_1 \rangle \).

Suppose that claim is true for \( H_k \) and consider the group \( H_{k+1} = H_k \text{ wr. } N_{k+1} \).

By (1.1.1) there corresponds to the automorphism \( \mu = \alpha_{k+1} \) of \( N_{k+1} \) an automorphism \( \overline{\mu} \) of order \( p \) of \( H_{k+1} \) defined by

\[
(f,n)^{\overline{\mu}} = (f',n^\mu) \text{ where } f'(n) = f(n^{\mu-1}).
\]

The identity, \( ( \text{ for suitable } g \text{ and } 1 \neq n \in N_{k+1}) \),

\[
(f,n)^{\overline{\mu}}(f,n)^{-1} = (g,n^{\mu}n^{-1})
\]

shows the automorphism \( \overline{\mu} \) acting on \( H_{k+1}/H_k \text{ wr. } 1 \), fixes no element. By (1.1.1) we can define the automorphism \( \alpha \in \text{Aut}(H_{k+1}) \) corresponding to
a ∈ U_k by \((f,n)\alpha = (f',n)\), where \(f'(x) = f(x)^\alpha\). Then \(\overline{U}_k = \{\alpha : \alpha \in U_k\}\) is a group of automorphisms of \(H_{k+1}\) isomorphic to \(U_k\). Let \(U_{k+1} = \langle \overline{U}_k, P \rangle\). By our induction hypothesis \(\overline{U}_k\) is elementary Abelian of order \(p^k\) and by definition \(\overline{u}\) and \(\alpha \in \overline{U}_k\) commute. So \(H_{k+1}\) is elementary Abelian of order \(p^{k+1}\). Also if \((f,n)^\tau\) for all \(\tau\) in \(U_{k+1}\), then by considering \(\tau = \overline{u}\) and \(\tau = \overline{u} \in \overline{U}_k\), we see that \(f = n = 1\). Thus the claim is proved.

(c) Let \(H = H_m U_m\) the semi direct product of \(H_m\) by \(U_m\). We have chosen \(p\) such that \((|H_m|, |U_m|) = 1\) and so can apply Huppert [8]. Let \(x \in H_m \cap N_H(U_m)\). Then as \(H_m \leq H\), for all \(a \in U_m\), \([x,a] \in H_m \cap U_m = 1\). So \(x \in C_H(U_m) \cap H_m = 1\). Hence \(N_H(U_m) = U_m\) and \(U_m\) being Abelian is thus a Carter subgroup of \(H\). A system normaliser of \(H\) covers all central chief factors and thus covers \(U_m\), as \(H/H_m \cong U_m\) is Abelian. Also a system normaliser is contained in each Carter subgroup. Thus \(U_m\) is a system normaliser for \(H\) which is self normalising making \(H\) a S-C-group.

By (a) \(G\) is isomorphic to a subgroup of \(H\), completing the proof.

A group is called primitive if it has a faithful primitive permutation representation. This is equivalent to saying the group has a complemented unique minimal normal subgroup. It turns out that this subgroup is the Fitting subgroup. Thus another characterisation of primitive groups is those that possess a self centralising minimal normal subgroup. We call the class of primitive groups \(P\). A subclass of \(P\), \(P^Q\) is defined by \(P^Q = \{ G : Q(G) \leq P \}\).

Alternatively \(P^Q\) can be defined as the class of groups with a unique complemented chief series. This is also equivalent to specifying
that each chief factor is self centralising.

Suppose $P$ is a $p$-group in $P^Q$. Then as $P$ is in $P$, we have $\phi(P) = 1$. Hence $P(P)/\phi(P) = P$, is an elementary Abelian $p$-group. But the only elementary Abelian $p$-groups in $P^Q$ are those of order $q$, $q$ a prime.

Let $X$ be the class of groups in which every chief factor is complemented. Then we have the following Lemma.

\textbf{(1.2.10) Lemma:} If $G \in X$ and $N \triangleleft G$, then $N$ is complemented in $G$.

\textbf{Proof:} We use induction on $|G|$. Let $H \triangleleft G$ and $N \triangleleft G$, such that $N \leq H$. In (2.1.1) we show $QX = X$, so $G/N \in X$ and $|G/N| < |G|$. As $H/N \triangleleft G/N$, by the induction hypothesis, $H/N$ is complemented in $G/N$. i.e. there exists $L/N$ such that $L/N \cdot H/N = G/N$ and $L/N \cap H/N = N/N$. The chief factor $N$ has a complement $M$ in $G$. We consider $M \cap L$. By Dedekind's modular law, $N(M \cap L) = NM \cap L = L$, and $N \cap (M \cap L) = 1$. Also $H(M \cap L) = HN(M \cap L) = HL = G$; and $(M \cap L) \cap H = M \cap N = 1$. So $M \cap L$ is a complement of $H$ in $G$.

Clearly $P^Q \leq X$, so if $N \triangleleft G \in P^Q$, then $N$ is complemented in $G$.

It is of interest to know what the subgroup closure of $P^Q$ is.

\textbf{(1.2.11) Theorem} Pardoe [1]. $S(P^Q) = S$. 
Proof: We will give a survey of the method. Let $G \in S$. We will use induction on $|G|$. Let $L \triangleleft G$ and $G/L \cong G_r$. By induction there is a monomorphism, $\mu : L \rightarrow M$, where $M \in P^Q$. As the chief factors of groups in $P^Q$ are self centralising, $H' \triangleleft M$ and $M/M' \cong C_q$. We choose a prime $p \not\equiv q$ or $r$. Let $M_i = M$, $i = 1, 2, \ldots, p$, and let $W = M \wr C_p = (M_1 \times \cdots \times M_p)C_p$. If $N_i = M_i$, $i = 1, \ldots, p$, and $N = N_1 \times \cdots \times N_p$, then $\overline{W} = W/N \cong C_q \wr C_p$.

Let $\overline{S}$ be a minimal normal eccentric subgroup of $\overline{W}$ contained in the base group. Let $S$ be the inverse image of $\overline{S}$ under the natural homomorphism $: W \rightarrow \overline{W}$. Define $T = SC_p$. We will call the construction of $T$, "Pardoes Construction", and will refer to it as such later on in this proof and in Chapter 2. It is then shown that,

1. $L$ can be embedded in $S$ and,
2. $T \in P^Q$, provided $p \parallel q - 1$.

The proof of (1) is routine and will be omitted. The proof of (2) contains the crux of the argument. Now $\overline{T} = \overline{SC_p}$ is clearly in $P^Q$.

We claim that any chief series of $G$ must pass through $N$. For let,

$$\overline{H}_i = \{(1, \ldots, h_i, \ldots, 1) \mid h_i \in H_i \leq M_1 \times \cdots \times M_p,$$

where $H_i \triangleleft M_i$. Then,

$$S/C_S(\overline{H}_i) \xrightarrow{\alpha} M_i/C_{M_i}(H_i)$$

for all $i = 1, \ldots, p$.

where $\alpha : sC_S(\overline{H}_i) = (s_1, \ldots, s_i, \ldots, s_p)C_S(\overline{H}_i) \rightarrow s_iC_{M_i}(H_i)$.
is the required isomorphism. Let \( K \triangleleft T \) such that, \( K \ntriangleleft N \) and \( N \ntriangleleft K \).

The following diagram of normal subgroups is then applicable.

\[
\begin{array}{c}
  \text{KN} \\
  \text{K} \\
  \text{N} \\
  \text{K/N} \\
  \text{1}
\end{array}
\]

As \( N \triangleleft KN \triangleleft T \), then \( KN \triangleright S \). Also \([K,N] \triangleleft K \cap N\), so \( K/(K \cap N) \)
centralises \( N/(K \cap N) \). Thus \( K \) centralises all chief factors of \( T \)
between \( K \cap N \) and \( N \). Let \( N/H \) be a chief factor of \( T \). As \( N/H \)
Abelian \( C_T(N/H) \triangleright N \) and combining these two results, \( C_T(N|H) \triangleright NK \triangleright S \).

Let \( H_i \triangleleft N_i \triangleleft M_i \) be the top portion of the chief series for \( M_i \). Let
\[
N/F = \frac{N_1 \times \cdots \times N_i \cdots \times N_p}{N_1 \times \cdots \times H_i \cdots \times N_p} \cong \frac{N_i}{H_i}.
\]
From what follows below, \( H \) can be
chosen as \( H_1 \times \cdots \times H_p \), so \( N/F \in Q(N/H) \) and hence \( C_T(N/F) \triangleright C_T(N/H) \triangleright S \).

Then \( 1 = S/C_S \left( \frac{N_i}{H_i} \right) \rightleftharpoons M_i/C_{M_i}(N_i/H_i) = \frac{M_i}{N_i} \not= 1 \). For, \( M_i \in P^Q \), giving
\( C_{M_i}(N_i/H_i) = N_i \). Thus no such \( K \) can exist. This concludes the proof that
if \( K \triangleleft T \) then either \( K \leq N \) or \( N \leq K \).

Next we show if,
\[
l = M_{i,k} \triangleleft M_{i,k-1} \triangleleft \cdots \triangleleft M_{i,2} \triangleleft N_i \triangleleft M_i
\]
is the chief series for \( M_i \), then \( M_{i,j} \times \cdots \times M_{p,j} / M_{i,j+1} \times \cdots \times M_{p,j+1} \)
is a chief factor of \( T \), when \( j \geq 2 \). For, by induction on \( |G| \) we can
assume that the chief series for \( M_i \) is, \( 1 \triangleleft A \triangleleft N_i \triangleleft M_i \), and it is
sufficient to show \( A = A_1 \times \cdots \times A_p \triangleleft T \). This is done by noting that,
if \( U \triangleleft T \) and \( U \leq A \), then \( A_i \leq U \) for some \( i \) and hence \( A \leq U \).

The essential point in the argument comes in showing that \( N_i \triangleleft T \), where we have assumed inductively that \( N_i \triangleleft M_i \). If \( \ell | |N_i| \), \( \ell \) a prime, then \( \ell \neq q \) as the only \( q \) groups in \( P^Q \) are cyclic of prime order. If we consider \( N \) as a \( S \)-module under conjugation, then the \( N_i \) are irreducible \( S \)-submodules and \( S/N \simeq \overline{S} \) is an elementary Abelian \( q \)-group. Let \( S = NQ \) where \( Q \) is a Sylow \( q \)-subgroup of \( S \). Choose a \( p \) such that \( p \mid q - 1 \).

Then \( \overline{S} \) is a faithful irreducible module for \( C_p \) and by (1.1.3) \( \dim_{Z_q}(\overline{S}) = \) smallest integer \( n \), such that \( p \mid q^n - 1 \). Hence \( \dim_{Z_q}(\overline{S}) > 1 \), and as \( \overline{S} \cong Q \) is an elementary Abelian \( q \)-group of order at least \( q^2 \), \( Q \) is not cyclic. Consider \( N \) as a \( Q \)-module. By Maschke's theorem \( N \) is a semisimple \( Q \)-module and,

\[
N = N_1 \oplus N_2 \cdots \oplus N_p,
\]

is the decomposition of \( N \) into irreducible \( Q \)-submodules. Let \( H_i = C_{Q}(N_i) \). Then \( N_i \) is a faithful irreducible \( Z_q(Q/C_{Q}(N_i)) \) module and by (1.1.3), \( Q/C_{Q}(N_i) \) is cyclic. Thus \( H_i \neq 1 \). Also \( H_i \neq Q \) by the remark that

\[
S/C_{S}(\overline{H_i}) \cong M_i/C_{M_i}(H_i) .
\]

Suppose \( N_i \cong N_j \) for some \( i \neq j \). Then \( H_i = H_j \). The action of \( C_p \) is to permute the \( N_i \), so there is a \( g \) in \( C_p \), such that \( N_i^g = N_j \) and \( C_p = \langle g \rangle \). But \( \overline{S} = QN/N \triangleleft T \) so, \( Q^g = Q^g/N \cap Q = Q/N \cap Q \cong Q \). Hence \( C_{Q}(N_i) = (C_{Q}(N_i))^g \) and \( NC_{Q}(N_i) \triangleleft T \), contradicting the fact that \( \overline{S} \triangleleft T \).

So the \( N_i \) are non isomorphic in pairs and hence the only submodules for
N are direct sums of the \( N_i \)'s, \( i = 1, \ldots, p \). Thus any non trivial submodule contains a \( N_i \) and by the action of \( C_p \) permuting the \( N_i \), contains \( N \). So \( N \triangleleft T \).

To show \( T \) is in \( P^Q \) it is sufficient to show that the chief series \( 1 \triangleleft \cdots \triangleleft M_1, 2 \times \cdots \times M_p, 2 \triangleleft N \triangleleft S \triangleleft T \) is complemented and each chief factor is self centralising. These are routine and will be omitted.

The final step is broken down into two cases.

(1) Suppose \( r \neq 2 \). We repeat the above construction with \( r \) instead of \( p \) and \( T \) instead of \( M \). Now \( T' = S \) is the maximal normal subgroup of \( T \) and \( p \) replaces \( q \). By (1.1.6) there are an infinite number of primes \( p \) such that \( p = 2 + nr \), (some \( n \)), so it is possible to choose a prime \( p \) such that \( p \mid q - 1 \) and \( r \mid p - 1 \). The new group \( U \) obtained contains \( S \) as a subgroup and by (1.1.2)

\[
G \rightarrow L \text{ wr. } G/L \rightarrow S \text{ wr. } C_r \rightarrow U \leq T \text{ wr. } C_r
\]

So \( G \) can be embedded in the group \( U \in P^Q \).

(2) Suppose \( r = 2 \). Since \( 2 \mid p - 1 \) for all odd primes \( p \), at this stage we will have to resort to a different method of proving that \( "N \triangleleft T" \). Without loss of generality we can assume \( 5 \mid q - 1 \) and apply Pardoe's construction to \( M \text{ wr. } C_5 \). Let \( X \) be the group in \( P^Q \) so formed.

For, suppose \( 5 \mid q - 1 \). Then by (1.1.6) there is a prime \( s \) such that \( s \mid q - 1 \) and \( 5 \mid s - 1 \). Then we would apply Pardoe's construction.
to \( M \) wr. \( C_5 \) to form a group \( A \in P^Q \). Then we would apply Pardoe's construction to \( A \) wr. \( C_5 \) to form \( B \in P^Q \). We would then complete the proof with \( B \) instead of \( X \).

Apply Pardoe's construction to \( X \) wr. and note that \( 31 \mid 5 - 1 \) and \( 31 \mid 5^3 - 1 \). If \( Y \) is the group in \( P^Q \) so formed apply Pardoe's construction to \( Y \) wr. \( C_2 \) to form the group \( Z \). These constructions give us a series of groups with normal series.

\[
\begin{array}{ccc}
(a) & (b) & (c) \\
& & \\
\text{TC}_5 & Y & Z \\
T & U & \\
q^\alpha & 5^3 & \\
1 & 1 & 5^6 \\
\end{array}
\]

Recall that for the chief series below \( S \) we just use the method of the theorem, as the fact that \( p \nmid q - 1 \) was only used to show \( "N \triangleleft T". \)

Let \( \overline{R} = R/S \) and \( \overline{Z} = Z/S \). Then it is sufficient to show \( \overline{R} \triangleleft \overline{Z} \).

We can consider \( \overline{R} \) as a vector space of dimension 6 over \( Z_5 \) and a \( D = Z/R \)-module by conjugation. Note \( \overline{R} \) considered as a \( A = V/R \), \( (\cong C_31) \), module is the direct sum of two isomorphic irreducible modules \( \overline{X} \) and \( \overline{Y} \), of order \( 5^3 \). So \( \overline{R}|_A = \overline{X} \oplus \overline{Y} \). Suppose \( \overline{R} \) is not an irreducible \( D \)-module. The only possible irreducible submodules are \( \overline{X} \) or \( \overline{Y} \). As \( A \triangleleft D \), \( A \) is Abelian and \( D \in P^Q \), then \( C_D(A) = A \). Thus by (1.1.5) \( C_2 \cong D/C_D(A) \) is isomorphic to a subgroup of \( C_3 \). This is impossible and thus \( \overline{R} \) is an irreducible module for \( D \). Thus \( \overline{R} \triangleleft \overline{Z} \). This completes the proof that
\[ \bar{Z} \text{ and hence } Z \text{ belongs to } \mathcal{P}^0. \text{ To complete (2) we note that there are monomorphisms} \]

\[ G \to L \text{ wr. } G/L \to S \text{ wr. } C_r \to Z \]

which completes the proof of the theorem.
CHAPTER 2

§2.1: Preliminaries.

In this section we will investigate the relationship between $P^Q$ and the larger class of groups $X$ defined by

$$X = \{ G \mid \text{every chief factor of } G \text{ is complemented} \}.$$ 

Carter, Fischer, and Hawkes [1] have shown that if a group $G$ has a chief series all of whose factors are complemented then every chief factor is complemented. Thus we may define

$$X = \{ G \mid G \text{ has a chief series in which the chief factors are complemented} \}.$$ 

If $\mathcal{V}$ is the class of groups with trivial Frattini subgroup then $X$ also may be characterised by

$$X = \{ G \mid Q(G) \leq \mathcal{V} \}$$

$$= \{ G \mid \text{For all } K \triangleleft G, \phi(G/K) = 1 \}.$$ 

Theorem (1.2.11) shows that the class $P^Q$ is large in the sense that every soluble group can be embedded in a $P^Q$ group. What can be said about the groups that can be subnormally embedded in a $P^Q$ group? As $P^Q \leq X$ we have $S_nP^Q \leq S_nX = X$. However we shall see that $S_nP^Q < X$. We are led to ask whether there is a natural collection of closure operations
whose join applied to $P^Q$ enlarges it to $X$. Clearly $X$ itself must be closed under each of the operations appearing in the collection. We first examine these.

$$(2.1.1) \text{ Lemma: } X = \{Q, S_n R_0, D_0\} X .$$

**Proof:** (i) From the alternative definition of $X$ clearly $QX = X$.

(ii) Let $H \triangleleft G \in X$. Suppose

$$1 = H_0 < H_1 < \cdots < H_r = H \triangleleft G$$

be part of a chief series of $G$ running through $H$. By Clifford's Theorem $H_i/H_{i-1}$ is completely reducible as a $H$-module. If $L$ complements $H_i/H_{i-1}$ in $G$ then $L \cap H$ complements $H_i/H_{i-1}$ in $H$. If $N/H_{i-1} \triangleleft H/H_{i-1}$, with $N \leq H_i$, then $H_i/H_{i-1} = N/H_{i-1} \times N^*/H_{i-1}$ with $N^* \leq H$. So that $(L \cap H)N^*$ is a complement of $N/H_{i-1}$ in $H$. Thus $H$ has a chief series in which every chief factor is complemented. Hence $H \in X$ and $S_n X = X$.

(iii) Suppose $G$ is a group of minimal order in $R_0X - X$. Then there exist $N_1, N_2 \triangleleft G$, $N_1 \cap N_2 = 1$, and $G/N_1, G/N_2 \in X$. We may assume $N_1 \triangleleft G$. For, let $N \triangleleft G$ such that $N \leq N_1$. Now consider $G/N$. It has normal subgroups $N_1/N$ and $N_2N/N$ which have trivial intersection. Also $G/N_1/N \cong G/N_1 \in X$ and $G/N_2N/N \cong G/N_2/N_2N/N_2 \in QX = X$. Thus $G/N \in R_0X$ and as $|G/N| < |G|$ it is also in $X$.

As $G/N_1 \in X$ then $\phi(G/N_1) = 1$, $i = 1,2$; and $\phi(G) \leq N_1 \cap N_2 = 1$. Now $G/N_1$ as a member of $X$ has a chief series in which every chief factor
is complemented and $N_1$ is complemented in $G$. So $G$ has a chief series in which every chief factor is complemented. Thus $G \in \mathcal{X}$ and $R_0 \mathcal{X} = \mathcal{X}$.

(iv) If $H$ and $K$ are in $\mathcal{X}$ and

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = H$$

$$1 = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_s = K$$

are respective chief series for $H$ and $K$ in which each chief factor is complemented then,

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r \triangleleft H_r \times K_1 \triangleleft \cdots \triangleleft H_r \times K_s = H \times K$$

is a chief series for $H \times K$ in which each chief factor is complemented. The case for the direct product of a finite number of groups in $\mathcal{X}$ follows by a simple inductive step and hence $D_0 \mathcal{X} = \mathcal{X}$.

The class $\mathcal{X}$ is not $S$- or $N_0$-closed. The symmetric group of degree 4 is in $\mathcal{X}$ but it contains a subgroup isomorphic to the dihedral group $D$ of order 8; a group which has a non-trivial Frattini subgroup. Also $D$ is the normal product of two Klein 4-groups and so $D \in N_0 \mathcal{X} - \mathcal{X}$.

Consider the semidirect product $G$ of $C_7$ by its automorphism group $C_6$. Then $G$ is the product of two normal subgroups, $N_1 = C_7 \cdot C_2$ and $N_2 = C_7 \cdot C_3$ which are in $P^0$ and which have coprime index in $G$. In §2.3 we show $G \in \mathcal{X}$. This group $G$ offers on insight into the investigation of a closure operation less general than $N_0$ and under which $\mathcal{X}$ may be closed.
§2.2 Two Closure Operations.

The example of the holomorph of $C_7$ suggests we might define an operation $T$ on group classes by

$$TV = \{ G \mid G = N_1N_2; N_1, N_2 \triangleleft G; N_1, N_2 \in \mathcal{X}; (|G : N_1|, |G : N_2|) = 1 \}.$$  

This definition, however, does not give us a closure operation as it is not idempotent. But recall the alternative method of defining a closure operation in §1.1. It is sufficient to specify all the $T$-closed classes and that the collection $\mathcal{C}$ of $T$-closed classes satisfy; (1) $S \in \mathcal{C}$; (2) $\mathcal{C}$ is closed under taking arbitrary intersections.

We will say the class $\mathcal{V}$ is $T$-closed if and only if all groups $G$, which are the normal product of two $\mathcal{V}$-groups of coprime index in $G$, are in $\mathcal{V}$. Clearly the $T$-closed classes satisfy the conditions (1) and (2), and so associated with this collection of $T$-closed classes is a closure operation which we will denote by $T$. This description is in fact identical to one given by Kappe [1] when we restrict our attention to the universal class $S$.

The next question to ask is what is the action of $T$ on an arbitrary class of groups $\mathcal{V}$. This question has not yet been answered but its investigation led to the description of a new closure operation which we call $M$. The definition of the operation given at the start of this paragraph failed to be a closure operation because it was not idempotent. To overcome this we define $M$ by;
This operation is clearly expanding and monotonic. It remains to show it is idempotent and hence a closure operation.

If \( G \in M(\mathcal{V}) \) then \( G = \langle Y_1, \ldots, Y_n \rangle \) where the \( Y_i \) are subnormal in \( G \) and belong to \( \mathcal{V} \). Also if we let \( y_1 = |G : Y_1| \) then \( (y_1, \ldots, y_n) = 1 \).

Each \( Y_i = \langle X_{i1}, \ldots, X_{ir_i} \rangle \) where the \( X_{ij} \) are subnormal in \( Y_i \) and belong to \( \mathcal{V} \). Also if \( x_{ij} = |Y_i : X_{ij}| \) then \( (x_{i1}, \ldots, x_{ir_i}) = 1 \). But the \( X_{ij} \) are in \( X \) and subnormal in \( G \) and if \( x_{ij} = |G : X_{ij}| \), we claim that \( (x_{i1}, \ldots, x_{ir_n}) = 1 \). For suppose \( p | x_{ij} \) for all relevant \( i \) and \( j \). Then \( p| |G : X_{ij}| = |G : Y_i| |Y_i : X_{ij}| = y_i x_{ij} |. \) For a given \( i \) there is at least one \( x_{ij} \) not divisible by \( p \), so \( p|y_i \) for all \( i \). This contradiction proves our contention and \( M \) is idempotent.

Philip Hall [2] has remarked that if \( H_1, H_2, \ldots, H_r \) are subgroups of a soluble group \( G \) such that \( (|G : H_1|, \ldots, |G : H_r|) = 1 \) then \( G = H_1 H_2 \cdots H_r \). Let \( P_1, P_2, \ldots, P_s \) be a representative set of sylow \( p_i \) subgroups of \( G \) for each distinct prime \( p_i | |G|, \) \( i = 1, \ldots, s \). Then there is for each \( i \) and \( j \) such that \( P_i \leq H_j \). Thus the remark is equivalent to saying that \( P_1 P_2 \cdots P_s = G \), where the ordering of the product is not specified. This follows by induction on \( |G| \) as \( G \) being soluble means one of the \( P_i \) contains a proper normal subgroup. Thus in the definition of \( M \) we may say \( G \) is the product of subnormal subgroups and not merely generated by subnormal subgroups.
The closure operation $M$ is strictly less than $N_0$ as the dihedral group of order 8 is in $N_0A$ but not in $MA$.

We can generalise $M$ to a family of closure operations $M_\pi$, where $\pi$ is a set of primes, by defining:

$$M_\pi = \{ G \mid G = X_1X_2\cdots X_n ; X_1 \text{ sn } G ; X_1 \in \mathcal{V} ; (|G : X_1|, \ldots, |G : X_n|) \text{ is a } \pi\text{-number} \} ;$$

when $\pi$ is the empty set we get $M$ and when $\pi = \{\text{all primes}\}$ we get $N_0$.

It is clear from the definition that $T < M$, and in fact this inequality is strict. For consider:

(2.2.1) Example. Let $A$ and $B$ be groups in $P^Q$ formed by Pardoe's construction (1.2.11) from $C_2 \text{ wr. } C_3$ and $C_7 \text{ wr. } C_5$. That is $A$ has a unique minimal normal subgroup of order $2^2$ with factor group cyclic of order 3 and $B$ has a unique minimal normal subgroup of order $7^4$ with factor group cyclic of order 5. Let $C$ be a subgroup of $A$ of order 2 and let $D$ be a subgroup of $B$ of order 7. Let $N_1 = A \times D$ and $N_2 = C \times B$ be subgroups of $G = A \times B$, and set $\mathcal{V} = \{1, N_1, N_2\}$. Then $G \in M_\pi - \mathcal{V}$. On the other hand $\mathcal{V}$ is $T$-closed. For if $K \in T\mathcal{V}$ then $K$ is the normal product of two members of $\mathcal{V}$ of coprime index in $K$. We may suppose that neither of these is 1 as in this case $K = N_i$, $i = 1, 2$, and $K \in \mathcal{V}$. Then $K = N_1N_2 = A \times B = G$. But $N_1$ and $N_2$ are not normal in $G$ so this case is excluded. But this has exhausted all the possibilities for $T\mathcal{V}$ and our claim stands true. Hence $T < M$. 
(2.2.2) Lemma: If $V = S_n^V$ and $G \leq MV$ then we may choose the generating $X_i$ normal in $G$.

Proof: Let $G = X_1X_2 \cdots X_n$ where the $X_i$ are in $V$, subnormal in $G$, and if $|G : X_i| = x_i$ then $(x_1, \ldots, x_n) = 1$. Let $\sigma(x_i) = \{p \mid p \nmid x_i\}$ and $\pi_i = \sigma(x_i)'$, the complementary set of primes. Let $H_i$ be a Hall $\pi_i$ subgroup of $x_i$. Then by the definition of $\pi_i$, $H_i$ is a Hall $\pi_i$ subgroup of $G$, i.e. $H_i = G_{\pi_i}$. It is a well known fact that if a Hall $\pi$-subgroup is contained in a subnormal subgroup then so is the join of all its conjugates. Hence $<G_{\pi_i}^G> \leq X_i$. Thus $<G_{\pi_i}^G> \leq \text{Core } X_i$ and $\sigma(|G : \text{Core } X_i|) = \sigma(x_i)$.

Thus if $\overline{x_i} = |G : \text{Core } X_i|$, we have $(\overline{x}_1, \ldots, \overline{x_n}) = 1$. We now show $G = \text{Core } X_1 \cdot \text{Core } X_2 \cdot \cdots \cdot \text{Core } X_n$. Let $p_i$, $i = 1, 2, \ldots, r$, be the distinct prime divisors of $|G : G'|$. Then there is an $x_i$ such that $p_i \nmid x_i$. If $G_{p_i}$ is a Sylow $P_i$-subgroup of $G$, then $P_i = <G_{p_i}^G> \leq x_i$. By the same reasoning that $<G_{\pi_i}^G> \leq x_i$. Then $P_i \triangleleft G$ and $P_i \leq \text{Core } X_i$. We claim $P = P_1P_2 \cdots P_r = G$. If not, then $P < G$ and hence there is an $M \triangleleft G$ such that $P \leq M$. If $(G : M) = q$, then $q = p_j$; some $j \leq r$, since $G' \leq M$. But $<G_{q}^G > = P_j \leq P \leq M$ and therefore $q \nmid |G : M|$. This is a contradiction and hence $P = G$. Thus $G = <\text{Core } X_i \mid i = 1, \ldots, r >$ and as $V = S_n^V$, the subgroups $\text{Core } X_i$ are in $V$, normal in $G$, and satisfy $(\overline{x}_1, \ldots, \overline{x}_n) = 1$. This complete, the proof.

(2.2.3) Corollary: If $V = \{S_n, Q\}^V$ then $QM^V = MV$. 
Proof: If $H \in QMV$, then $H \cong G/K$ where $K \triangleleft G$ and $G \in MV$. As $\mathcal{Y} = S_n^{\mathcal{Y}}$ we can assume $G = X_1 X_2 \cdots X_n$, where the $X_i$ are normal in $G$, belong to $\mathcal{Y}$, and $(x_1, \ldots, x_n) = 1$. In $G/K$ we have $X_iK/K \triangleleft G/K$ and $X_iK/K$ in $\mathcal{Y}$ as $Q\mathcal{V} = \mathcal{Y}$. Also $|G/K : X_iK/K| = |G : X_i|$, so if $\bar{x}_i = \bar{G/K : X_iK/K}$, then $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = 1$. Thus $G/K \in MV$.

§2.3. A Further investigation of $X$.

The holomorph of $C_7$ was the motivation for investigating the closure operation $M$. At the end of §2.1 we stated it was in $X$. We now prove that claim.

(2.3.1) Lemma: $M X = X$.

Proof: We have shown $S_n X = X$ so by (2.2.3) we may assume if $G \in M X$, then, $G = X_1 X_2 \cdots X_n$, $X_i \in X$, $X_i \triangleleft G$, and if $x_i = |G : X_i|$ then $(x_1, \ldots, x_n) = 1$.

Let $G$ be a group of minimum order in $M X - X$. Let $N \triangleleft G$ and as $|G/N| < |G|$ and $Q X = X$ by the induction hypothesis and (2.2.3) then $G/N \in X$ and has a complemented chief series. If $N = p^a$, there is an $x_i$ such that $p \mid x_i$ and thus $X_i$ contains all the $p$-subgroups of $G$. Hence $N \leq X_i$. Then $(|N|, |G : X_i|) = 1$ and $N$ is an Abelian normal subgroup of $G$ contained in $X_i$. But (1.2.10) shows that every normal subgroup of an $X$ group $G$ is complemented in $G$. $X_i \in X$ implies that $N$ is complemented in $X_i$ and therefore by Gaschütz's theorem [1], $N$
is complemented in G. So G has a complemented chief series and thus
MX = X.

We will now investigate the action on PQ of the joins of some
of those closure operations that leave X fixed. But first we will
introduce a useful normal series.

The Upper Fitting Series of a group G,

\[ 1 = F_0(G) \triangleleft F_1(G) \triangleleft \cdots \triangleleft F_\ell(G) = G \]

is defined inductively by \( F_0(G) = 1 \) and \( F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G)) \).
We call the length \( \ell(G) = \ell \) of this series the Fitting length of G.
For a group in PQ the upper Fitting series coincides with the unique
chief series.

(2.3.2) Lemma: If \( H \triangleleft G \in PQ \) then

\[ F_i(H)/F_{i-1}(H) \in S(F_i(G)/F_{i-1}(G)), \quad i = 1, \ldots, \ell(G). \]

Proof: The proof follows by induction on \( \ell(G) \). If \( \ell(G) = 1 \) the
result is clearly true. So suppose the hypothesis is true for all groups
K such that \( \ell(K) = \ell - 1 \). Let \( \ell(G) = \ell \). Then \( \ell(G/F_1(G)) = \ell - 1 \) and
\( G/F_1(G) \in PQ = X \). Also if \( H \triangleleft G \) then \( HF_1(G)/F_1(G) \triangleleft G/F_1(G) \). But
as \( F_1(H) = F_1(G) \cap H \) we have \( HF_1(G)/F_1(G) \cong H/F_1(H) \). Thus by the
induction hypothesis \( F_i(H/F_1(H))/F_{i-1}(H/F_1(H)) \in S[F_i(G/F_1(G))/F_{i-1}(G/F_1(G))] \)
for \( i = 1, \ldots, \ell - 1 \). But for any group K,
and the lemma is satisfied for \( i = 2, \ldots, \ell \). The case \( i = 1 \) is covered by the remark that \( F_1(H) = H \cap F_1(G) \).

**Corollary**: If \( H \) is an Abelian subnormal subgroup of a group in \( P_0 \) then \( H \) is an elementary Abelian \( p \)-group for some prime \( p \).

**Proof**: If \( H \trianglelefteq G \in P_0 \) and \( H \) Abelian then \( F_1(H) = H \). By (2.3.2) \( F_1(H) \trianglelefteq F_1(G) \) and \( F_1(G) \) is elementary Abelian for some prime \( p \) as it is the unique minimal normal subgroup of \( G \).

**Example**: \( S_n(P^O) \subset X \). The Abelian group \( C_2 \times C_3 \) is in \( X \) but by (2.3.3) it cannot be a subnormal subgroup of a group in \( P^O \).

We know \( X \) is \( R_0 \)- and \( S_n \)-closed, and, in fact, we can show \( R_0 S_n \) is a closure operation. For, if \( \mathcal{V} \) is a class of groups and \( G \in S_n R_0 \mathcal{V} \), then \( G \trianglelefteq H \) where \( H \) has \( r \) normal subgroups \( H_i \) with trivial intersection such that each \( H/H_i \) is in \( \mathcal{V} \). Consider the \( r \) normal subgroups \( H \cap G \) of \( G \). They have trivial intersection and \( G/(H \cap G) \cong G H_i/H_i \cap G \) is in \( \mathcal{V} \). Thus \( G \) is in \( R_0 S_n \mathcal{V} \) and \( S_n R_0 \leq R_0 S_n \). This is a sufficient condition for \( R_0 S_n \) to be a closure operation.

**Example**: \( R_0 S_n(P^O) \subset X \). First observe that, if \( K \) is a group in \( P \cap R_0 \mathcal{V} \), then \( K \) is in \( \mathcal{V} \). This follows from the definition of \( P \) and \( R_0 \)-closure. Consider the holomorph \( G \) of \( C_7 \). Then \( G \) is in \( P \cap X \).
and the upper Fitting series for $G$ is $1 \unlhd C_7 \unlhd G$. Thus $G$ is not in $S_n(P^Q)$ by (2.3.2) as the factors in its upper Fitting series are not all elementary Abelian $p$-groups for various primes $p$. Hence $G$ is in $X$ but, by the initial observation, not in $R_oS_n(P^Q)$.

Another line of attack could be to consider the closure operation $D_oS_n$ applied to $P^Q$. But this is not sufficient as shown in the following example.

(2.3.6) Example: $D_oS_n(P^Q) < X$. Let $G$ be the semidirect product of $C_3 \times C_5$ by $C_2$, where the action of the generator $x$ of $C_2$ is to transform $(a,b)$ in $C_3 \times C_5$ to $(a^{-1},b^{-1})$. Then $G$ is in $X$ and is not the direct product of any two proper subgroups. So, if $G$ were in $D_oS_n(P^Q)$, it would be in $S_n(P^Q)$, which by (2.3.2) is not possible.

Another line of investigation is the closure operation $QS_n$ applied to $P^Q$. To decide whether $X \leq QS_n(P^Q)$ we can begin by showing certain subclasses of $X$ are in $QS_n(P^Q)$. One such subclass is $X \cap A$. Clearly the simple Abelian groups are in $X$, and as $D_oX = X$, so are the elementary abelian groups. But these are all. For if $A$ is an Abelian group, then $F(A)/\phi(A) = A/\phi(A)$ is elementary Abelian. Hence, if $A$ is not elementary Abelian, then $\phi(A)$ is non trivial and $A$ is not in $X$.

In what follows we will make use of the following non standard notation. If $G$ is a group and $n$ a positive integer define, $nG = G_1 \times G_2 \times \cdots \times G_n$, where $G_i = G$, $i = 1, 2, \cdots , n$. 
Let \( K \) be a group in \( p^Q \) and suppose \( |\bar{K}| = p \), where \( \bar{K} = K/K' \).

Let \( q \) be a prime such that \( q \mid p - 1 \). Consider \( A = \bar{K} \wr C_q \equiv C_p \wr C_q \).

If \( B = qK \) and \( \bar{B} = qK/K' \), then \( A = \bar{B}C_q \). By Parades' construction (1.2.11) we can form \( T = L \wr C_q \) in \( p^Q \) where \( \bar{L} = L/qK' \) is a minimal normal eccentric subgroup of \( A \) contained in \( \bar{B} \). Let \( b \) be the smallest positive integer such that \( q \mid p^b - 1 \). By (1.1.3), \( |\bar{L}| = p^b \). Since \( qK' \triangleleft T \in p^Q \), by (1.2.10), it has a complement \( H \) in \( T \). Let \( L^* \) be the complement of \( qK' \) in \( L \). i.e. \( L^* = H \cap L \). We note that \( L^* \cong \bar{L} \), and the following holds.

\[ (2.3.7) \text{ Lemma :} \quad \text{There exists a subgroup } W^* \text{ of } L^* \text{ such that} \]
\[
\begin{align*}
(\text{a}) & \quad |w^*| = p^{b-1}. \\
(\text{b}) & \quad [K'_q, W^*] = 1.
\end{align*}
\]

\textbf{Proof :} Consider \( \bar{B}C_q \cong C_p \wr C_q \). \( \bar{B} \) is an elementary Abelian p-group of order \( p^q \). We can characterise \( \bar{B} \) as a \( q \)-dimensional vector space \( V \) over the field of \( p \)-elements. For if \( \{u_1, \ldots, u_q\} \) is a suitable basis for \( V \) and \( C_p \cong \langle x_i \mid x_i^p = 1, i = 1, \ldots, q \rangle \), then the isomorphism is given by,

\[
\alpha : (x_1^I, \ldots, x_q^I) \longrightarrow \sum_{i=1}^{q} a_i u_i^I \ ; \ 1 \leq a_i \leq p, \ i = 1, 2, \ldots, q.
\]

Let \( \beta \) be the map,

\[
\beta : \sum_{i=1}^{q} a_i u_i^I \longrightarrow a_q.
\]

Restrict the map \( \alpha \) to \( L^* \cong \bar{L} \) and consider the map \( \beta \alpha \) on \( L^* \). The
kernel of this map is \( \{ (x_1, \ldots, x_q) \mid a_q = p \} \) and

\[
\dim \alpha(L^*) = \dim \beta\alpha(L^*) + \dim (\ker \beta\alpha).
\]

Thus there is a subspace of dimension \( b - 1 \) of \( \alpha(L^*) \) in which all elements have \( a_q = p \) (=0).

When considering \( L^* \) as a subgroup of \( L \) this means there is a subgroup \( W^* \) of \( L^* \) of order \( p^{b-1} \), in which every element is of the form \( (y_1, \ldots, y_{q-1}, 1) \). Now \( K_q' = \{(1, \ldots, 1, k_q) \mid k_q \in K_q' \} \), so clearly \([W^*, K_q'] = 1\).

(2.3.8) Theorem: \( A \cap X \leq \text{QS}_n(P^Q) \).

Proof: Let \( nC_p = C_p \times \cdots \times C_p \), (n times). Then it is sufficient to show \( m_1C_p \times \cdots \times m_rC_p \) is in \( \text{QS}_n(P^Q) \) where \( p_1, p_2, \ldots, p_r \) are distinct primes. When do this by induction on \( r \).

When \( r = 1 \), consider \( mC_p \). Choose a prime \( q > p^{m+1} \) and look at \( C_p \) wr. \( C_q = BC_q \). Let \( A \) be a minimal normal eccentric subgroup of \( BC_q \) contained in \( B \). Form \( W = AC_q \). Then \( W \) is in \( P^Q \) as it has a unique complemented chief series, \( 1 \triangleleft A \triangleleft W \). Let \( a \) be the smallest integer such that \( q \mid p^a - 1 \). Then \( a > m \) by the choice of \( q \). Also \( A = aC_p \) by (1.1.3), and \( mC_p \triangleleft C_p \triangleleft W \). So \( mC_p \in \text{S}_n(P^Q) \).

Thus we may assume \( \text{QS}_n(P^Q) \) contains all elementary Abelian groups with \( r - 1 \) distinct prime divisors. Now consider the case of \( r \) distinct prime divisors \( p_1, p_2, \ldots, p_r \). We can order these such that \( p_r \neq 2 \).
Let \( G = m_1C_{p_1} \times \cdots \times m_rC_{p_r} \) and \( G_1 = m_1C_{p_1} \times \cdots \times m_{r-1}C_{p_{r-1}} \).

Let \( G_1 \cong A/B \) where, \( B \triangleleft A \) and \( K \in P^Q \). We can always choose \( A, (or \ K) \), such that \( A \leq K' \). For, if \( A \not\leq K' \), then \( A = K \) as \( K' \) is the unique maximal normal subgroup of \( K \) and a subnormal subgroup passes through a normal one. In this case \( G_1 \in P^Q \) and thus must be cyclic of prime order. Then as in the case for \( r = 1 \) a \( K \) can be chosen such that \( A \leq K' \).

Let \( K/K' = \overline{K} \) be a cyclic group of order \( p \) for some prime \( p \). As \( p_r \neq 2 \) we can use Dirichlet's result that the sequence \( \{2 + np_r\}_{n=1}^\infty \) contains an infinite number of primes \( r \).

Thus it is possible to choose a prime \( r \) such that \( p_r \nmid r - 1 \) and \( r \nmid p - 1 \).

Consider \( \overline{K} \) wr. \( C_r = C \) \( C_r \cong C^p \) wr. \( C_r \). As \( r \nmid p - 1 \) we can use Pardoe's construction to form a group, \( L = R \) \( C_r \in P^Q \), where \( \overline{R} \) is a minimal normal eccentric subgroup of \( CC_r \) contained in \( C \). Let \( \overline{L} = L/L' \) and consider \( \overline{L} \) wr. \( C_r \) \( C^p_r \cong C_r \) wr. \( C^p_r \). Again, by Pardoe's construction we can form a group \( H = SC^p_r \) where \( \overline{S} \) is a minimal normal eccentric subgroup of \( DC^p_r \) contained in \( D \). Since \( p_r \nmid r - 1 \), we have \( H \in P^Q \).

Let \( s \) be a prime such that \( s > p^m_r + 1 \). Let \( \overline{H} = H/H' \), and consider \( \overline{H} \) wr. \( C_s = EC_s \cong C^p_r \) wr. \( C_s \). By Pardoe's construction once more we can choose \( \overline{T} \) a minimal normal eccentric subgroup of \( EC_s \) contained in \( E \) and form \( J = TC_s \). As \( s \nmid p_r - 1 \) then \( J \in P^Q \). We claim \( G \in Q S_n(J) \).

The group \( J \) has been constructed in the following steps.
By (1.1.3) the integers $a, b, c$ are the smallest such that $r|p^a - 1$, $p_r|r^b - 1$, $s|p_r^c - 1$, respectively. Let $SH' = H'_1 \times \cdots \times H'_s$, where $H'_1 \cong H'$. Now there exists a subgroup $U$ of $T$ that complements $SH'$ which, by (2.3.7), has a subgroup $V$, such that $|V| = p_r^{c-1}$ and $[V,H'_s] = 1$. By the choice of $s$, $c > m_r$, so we may choose a subgroup $W$ of $V$ such that $|W| = p_r^m$ and $[W,H'_s] = 1$. Consider the subgroup $(H'_1 \times \cdots \times H'_{s-1} \times A).W$ where $A \cap K' \cong K'_r$. Then,
\[(H_1' \times \cdots \times H_{s-1}' \times A)_W^s \approx (H_1' \times \cdots \times H_{s-1}' \times K_L)_W^s \approx (H_1' \times \cdots \times H_{s-1}' \times H_s)_W^s \approx J\]

Also \((H_1' \times \cdots \times H_{s-1}' \times B) \trianglelefteq (H_1' \times \cdots \times H_{s-1}' \times A)_W^s\) as \(B \trianglelefteq A\) and \(W\) centralises \(H_s'\). Finally,

\[
\frac{(H_1' \times \cdots \times H_{s-1}' \times A)_W^s}{H_1' \times \cdots \times H_{s-1}' \times B} \equiv \frac{A/B \times \overline{W}}{m_1c_{p_1} \times \cdots \times m_rc_{p_r}}.
\]

where \(\alpha\) is given by,

\[
\alpha[(h_1, \cdots, h_{s-1}, a)_W^s H_1' \times \cdots \times H_{s-1}' \times B]
\]

\[= (aB, WH_1' \times \cdots \times H_s') .\]

This completes the proof.
We conclude with some unanswered questions that follow from the above Chapter.

First is $\mathcal{Q}_n^p \mathcal{P} = X$? It would seem that the answer to this is no. For, let $L$ be the class of groups whose upper Fitting factors are $p$-groups for various primes $p$, and $\mathcal{V}$ be the class of groups whose chief factors are $L$-central i.e. satisfy $G/C_G(H/K) \in L$ for all chief factors $H/K$ of $G$. If we can show $\mathcal{V} = \mathcal{Q}_n \mathcal{V}$ then we are done. For $p^Q \leq \mathcal{V}$ and the holomorph of $C_7$ is contained in $X - \mathcal{V}$.

Let $K \triangleleft G \in \mathcal{V}$ and $H/K \triangleright L/K$ be a chief factor of $G/K$. Then $H/L$ is a chief factor of $G$ and as

$$C_{G/K}(H/K \triangleright L/K) \cong C_G(H/L)/K$$

we have $Q\mathcal{V} = \mathcal{V}$.

Thus we need only show $S_n \mathcal{V} = \mathcal{V}$ to complete the proof of the existence of the counter example. This is at the moment unsettled.

Finally, given that the first question is settled, we may then ask is $\{Q, S_n, M, R_0\}^p \mathcal{P} = X$?
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