SEPARATION AXIOMS AND MINIMAL TOPOLOGIES

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Abstract

A hierarchy of separation axioms can be obtained by considering which axiom implies another. This thesis studies the properties of some separation axioms between T_o and T_1 and investigates where each of the axioms belongs in this hierarchy. The behaviours of the axioms under strengthenings of topologies and cartesian products are considered.

Given a set X, the family of all topologies defined on X is a complete lattice. A study of topologies which are minimal in this lattice with respect to a certain separation axiom is made. We consider certain such minimal spaces, obtain some characterizations and study some of their properties.

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Introduction

We shall say that a separation axiom implies another if every topological space which satisfies the first axiom also satisfies the second. Separation axioms between T_o and T_1, that is, separation axioms which imply T_o and are implied by T_1, were first studied extensively by Aull and Thron [1]. They introduced a hierarchy of separation axioms between T_o and T_1, namely T_{UD}, T_D, $T(\gamma)$, T_F, T_Y, T_{YS}, T_{DD} and T_{FF}, gave characterizations of them, and studied their properties. Later Robinson and Wu [10] defined $T^{(m)}$, strong T_o and strong T_D spaces. The first chapter of this thesis is devoted to a survey of these separation axioms. Their relative positions are studied, and examples are given. We introduce a new axiom, namely a $T_{UD}^{(m)}$ space. At the end of the chapter we obtain a diagram which shows the positions of these axioms.

In the first part of chapter II we shall study the behaviours of the separation axioms under a strengthening of the topology, following the pattern of Park [9]. It will be found that with the exceptions of $T(\gamma)$, strong T_o and strong T_D, our axioms are preserved when the topology is strengthened. Product spaces of a family of T_D or $T^{(m)}$ (m a cardinal) spaces are considered in the second part of this chapter, as Robinson and Wu did in [10]. The major result will be that it is not possible to define a separation axiom between $T^{(m)}$ and T_1 which is preserved under arbitrary products.

Chapter III is devoted to the study of minimal topologies on a set. Given a set X, the family of all topologies defined on X is a complete lattice. We shall consider topologies in this lattice which are minimal with respect to a certain topological property. Minimal T_o, minimal T_D, minimal T_1, minimal T_2 and minimal regular spaces are considered. Characterizations of minimal T_o and minimal T_D spaces are obtained by Larson [6], while that of minimal regular spaces is obtained by Berri and Sorgenfrey [3]. In the last section we produce a characterization of order topologies on a set by means of minimal T_o topologies by Thron and Zimmerman [11].

Terminology and Notation

The terminology and notation used in this thesis follow those of Kelley [5]. In chapter III, where results on filter bases are used, one may refer to Bourbaki [4].

A mention of the following terminologies is in order :

(1) A set is said to be degenerate iff it consists of at most one element.

(2) For the closure of a point x, or more precisely, of the set $\{x\}$, we shall write $\{\overline{x}\}$. For the associated derived set we shall write $\{x\}'$.

CHAPTER I

SEPARATION AXIOMS BETWEEN T_o AND T_1

1. Introduction

In this chapter we shall describe various separation axioms intermediate in strength between T_o and T_1. Emphasis will be given on those not included in [1]. We shall describe the axioms, and for those not found in [1] examples and characterizations will be given. The reader is referred to [1] and [7] for examples and equivalent forms for separation axioms introduced in [1]. It will be observed that all the axioms can be described in terms of the behaviour of derived sets of points.

2. Separation Axioms Between T_o and T_1

To characterize separation axioms between T_o and T_1 it is convenient to introduce the concept of weak separation in a topological space.

Definition 2.1 A set A in a topological space $(X,\mathcal{T})$ is said to be weakly separated from another set B iff there exists an open set $G \supset A$ such that $G \cap B = \phi$. We shall write $A \vdash B$ in this case. When $A = \{x\}$ or $B = \{y\}$, we write $x \vdash B$ and $A \vdash y$ instead of $\{x\} \vdash B$ and $A \vdash \{y\}$, respectively.

The following axioms are introduced by Aull and Thron in [1] :

Definition 2.2. A topological space $(X,\mathcal{T})$ is called a

(a) T_{UD}-space iff for every $x \in X$, $\{x\}'$ is the union of disjoint closed sets ;

(b) T_D-space iff for every $x \in X$, $\{x\}'$ is a closed set ;

(c) T_{DD}-space iff it is T_D and in addition for all $x, y \in X$, $x \neq y$, we have $\{x\}' \cap \{y\}' = \phi$;

(d) T_F-space iff given any point x and any finite set F such that $x \notin F$, either $x \vdash F$ or $F \vdash x$;

(e) T_{FF}-space iff given two arbitrary finite sets F_1 and F_2 with $F_1 \cap F_2 = \phi$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$;

(f) T_Y-space iff for all $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is degenerate;

(g) T_{YS}-space iff for all $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either ϕ or $\{x\}$ or $\{y\}$; and

(h) $T(Y)$-space iff for every $x \in X$ $\{x\}'$ is the union of disjoint point closures.

The following three separation axioms are due to Robinson and Wu [10] :

Definition 2.3. Let m be an infinite cardinal. A topological space $(X, \mathcal{T})$ is called a $T^{(m)}$ space iff for every $x \in X$, $\{x\}=F\cap(\cap\{O_i : i \in I\})$ where F is closed, each O_i is open, and card $(I) = m$.

<u>Definition 2.4.</u> A topological space $(X,\mathcal{T})$ is called a strong T_D space iff for each $x \in X$, $\{x\}'$ is either empty or is a union of a finite family of non-empty closed sets, such that the intersection of this family is empty.

<u>Definition 2.5.</u> A topological space $(X,\mathcal{T})$ is called a strong T_o space iff for each $x \in X$, $\{x\}'$ is either empty or a union of non-empty closed sets, such that the intersection of this family is empty and at least one of the non-empty members is compact.

In view of the fact that in a T_1 space $\{x\}' = \phi$ for every x and that in a T_o space $\{x\}'$ is the union of closed sets, it is immediate that strong T_D and strong T_o lie between T_o and T_1. The following theorem shows that this is also true for $T^{(m)}$ spaces.

<u>Theorem 2.1.</u> The following are equivalent.:

(a) $(X,\mathcal{T})$ is a $T^{(m)}$ space ;

(b) For each $x \in X$, $\{x\}'$ is the union of m closed sets , i.e., $\{x\}' = \cup \{C_i : i \in I\}$ where each C_i is closed and card $(I) =$ $= m$; and

(c) For each $A \subset X$ such that card $A \leq m$, A' is the union of m closed sets.

<u>Proof</u> : (a) => (b) For $x \in X$, if $\{x\} = F \cap (\cap \{0_i : i \in I\})$, then

$$\begin{aligned} \{x\}' &= \overline{\{x\}} - \{x\} \\ &= \overline{\{x\}} - (F \cap (\cap \{0_i : i \in I\})) \\ &= (\overline{\{x\}} - F) \cup (\cup \{ \overline{\{x\}} - 0_i : i \in I \}) \\ &= \cup \{ \overline{\{x\}} - 0_i : i \in I \} \end{aligned}$$

since $\overline{\{x\}} \subset F$, where card $(I) = m$. Hence $\{x\}'$ is the union of m closed sets.

(b) => (c) Suppose $A = \{x_i : i \in I\}$ where card $I \leq m$. Denote by C the set of ω-limit points of A, i.e.,

$$C = \{x \in X : \text{every neighborhood of } x \text{ contains infinitely many points of } A\} .$$

Then C is closed, and

$$A' = C \cup (\cup \{ \{x_i\}' : i \in I \})$$

because A' contains the right hand side and if $x \in A'$ is not an ω-limit point, then $x \in \{x_i\}'$ for some $x_i \in A$. Now by (b) each $\{x_i\}'$ is the union of m closed sets and since card $I \leq m$, A' is the union of m closed sets.

(c) => (b) obvious.

(b) =>(a) If $\{x\}' = \cup \{F_i : i \in I\}$ where each F_i is closed and $\text{card } I = m$, then

$$\begin{aligned}\{x\} &= \{\overline{x}\} - \{x\}' \\ &= \{\overline{x}\} - \cup \{F_i : i \in I\} \\ &= \{\overline{x}\} \cap \{X - F_i : i \in I\}\end{aligned}$$

Hence $(X,\mathcal{T})$ is a $T^{(m)}$ space.

<u>Corollary</u>. (a) Each $T^{(m)}$-space is a T_o space.

(b) Each T_D space is a $T^{(m)}$ space for any m.

(c) Each T_o space is a $T^{(m)}$ space for some m.

A combination of T_{UD} and $T^{(m)}$ yields the following separation axiom.

<u>Definition 2.6</u>. Let m be an infinite cardinal. A topological space $(X,\mathcal{T})$ is called a $T_{UD}^{(m)}$ space iff for every $x \in X$, $\{x\}'$ is the union of m disjoint closed sets.

From the definition it is immediate that every T_D space is a $T_{UD}^{(m)}$ space for all m, and every $T_{UD}^{(m)}$ space is a T_{UD} space as well as a $T^{(m)}$ space. Moreover every T_{UD} space is a $T_{UD}^{(m)}$ space for some cardinal m. Thus $T_{UD}^{(m)}$ lies between T_o and T_1 .

3. Relations of the Axioms

The following diagram is obtained by Aull and Thron in [1]. In this diagram $T_\alpha \longrightarrow T_\beta$ means that every T_α space is a T_β-space.

$$
\begin{tikzcd}
T_1 \arrow[r] \arrow[dd] & T_{DD} \arrow[rrr] \arrow[d] & & & T_D \arrow[dd] & \\
& T_{YS} \arrow[d] & & & & \\
T_{FF} \arrow[r] & T_Y \arrow[r] & T_F \arrow[r] & T(\gamma) \arrow[r] & T_{UD} \arrow[r] & T_o
\end{tikzcd}
$$

We shall now attempt to place the $T^{(m)}$ and $T_{UD}^{(m)}$ spaces into this chart and show that they are new axioms intermediate in strength between T_o and T_1. By virtue of Corollary to Theorem 2.1 and the remark following Definition 2.6 we have

$$
\begin{tikzcd}
T_1 \arrow[r] \arrow[dd] & T_{DD} \arrow[rrr] \arrow[d] & & & T_D \arrow[d] & & \\
& T_{YS} \arrow[d] & & & T_{UD}^{(m)} \arrow[d] \arrow[r] & T^{(m)} \arrow[dr] & \\
T_{FF} \arrow[r] & T_Y \arrow[r] & T_F \arrow[r] & T(r) \arrow[r] & T_{UD} \arrow[rr] & & T_o
\end{tikzcd}
$$

We introduce some examples.

Example 1. X = real numbers.

closed sets : ϕ, X and $[a,\infty)$, where $a \in X$.

Example 2. X = real numbers

closed sets : ϕ, X and $\{x\}$ for $x \neq 0$ plus finite unions of these sets.

Example 3. X = integers

closed sets : ϕ, X and $\{n\}$, $n \neq 0$ plus finite unions of these sets.

The reader is referred to [7] for detailed description of these examples.

Theorem 3.1. $T^{(m)}$ and T_{UD} are unrelated.

Proof : (1) $T^{(m)} \not\Rightarrow T_{UD}$: Example 1 is a $T^{(m)}$ space where $m = c$, but not a T_{UD} space.

(2) $T_{UD} \not\Rightarrow T^{(m)}$: Example 2 is a T_{UD} space but not a $T^{(\omega)}$ space.

Theorem 3.2. $T^{(m)}$ and $T(\Upsilon)$ are unrelated.

Proof : (1) $T^{(m)} \not\Rightarrow T(\Upsilon)$: Theorem 3.1 Proof (1).

(2) $T(\Upsilon) \not\Rightarrow T^{(m)}$: Example 2 is a $T(\Upsilon)$ space but not a $T^{(\omega)}$ space.

For $T_{UD}^{(m)}$ we have the following theorem :

<u>Theorem 3.3</u>. (a) $T_o \not\Rightarrow T_{UD}^{(m)}$.

(b) $T_{UD}^{(m)} \not\Rightarrow T_D$.

(c) $T(\gamma) \not\Rightarrow T_{UD}^{(m)}$.

<u>Proof</u> : (a) Example 1 is a T_o space which is not a $T_{UD}^{(m)}$ space for any m.

(b) Example 2 is a $T_{UD}^{(c)}$ space but not a T_D space.

(c) Example 2 is a $T(\gamma)$ space which is not a $T_{UD}^{(\omega)}$ space.

For strong T_D and strong T_o spaces we can make the following observations.

<u>Theorem 3.4</u>. (a) Every strong T_D space is a T_D space.

(b) Every strong T_o space is a T_o space.

<u>Proof</u> : This follows from the definitions.

<u>Example 4</u>. $X = \{a,b\}$.

$\mathcal{T} = \{\phi, \{a\}, X\}$.

This is a T_{FF} space, because $\{x\}' = \phi$ for all but at most one $x \in X$ (cf. [1] Theorem 3.3), and also a T_{DD} space because $\{x\}'$

is closed for every $x \in X$ and for $x \neq y$, $\{x\}' \cap \{y\}' = \phi$. However, it is not a strong T_D nor a strong T_o space since $\{a\}' = \{b\}$ cannot be expressed as the union of a family of non-empty closed sets whose intersection is empty.

We conclude this section by the following chart :

$$
\begin{tikzcd}
& & \text{Strong } T_D \arrow[drr] & & & \\
T_1 \arrow[urr] \arrow[r] \arrow[dd] \arrow[dddrr] & T_{DD} \arrow[rrr] \arrow[d] & & & T_D \arrow[d] & \\
& T_{YS} \arrow[d] & & & T_{UD}^{(m)} \arrow[r] \arrow[d] & T^{(m)} \arrow[d] \\
T_{FF} \arrow[r] & T_Y \arrow[r] & T_F \arrow[r] & T(\gamma) \arrow[r] & T_{UD} \arrow[r] & T_o \\
& & \text{Strong } T_o \arrow[urrr] & & &
\end{tikzcd}
$$

CHAPTER II

PROPERTIES OF THE SEPARATION AXIOMS

1. Introduction

In this chapter an attempt is made to investigate various properties of the separation axioms we have introduced. Properties like whether a separation axiom is preserved under strengthening of the topology or under product are considered.

2. Strengthening of Topologies

It is known that the property of being T_o or T_1 is preserved under strengthening of the topology. In this section we shall study the axioms between T_o and T_1. It will be found that the same is true for most of our axioms. The following lemma, which follows directly from the definition, will be useful. In this lemma and throughout the remainder of this section, $(X,\mathcal{T}_1)$ will denote a strengthening of a topological space $(X,\mathcal{T})$, where $\mathcal{T}$ and $\mathcal{T}_1$ are the two families of closed sets.

Lemma 2.1. (a) Let $\overline{\{x\}}$, $\{x\}'$ and $\overline{\{x\}}_1$, $\{x\}'_1$ be the closures and derived sets of the point x in $\mathcal{T}$ and $\mathcal{T}_1$ respectively. Then $\overline{\{x\}}_1 \subset \overline{\{x\}}$ and $\{x\}'_1 \subset \{x\}'$.

(b) Suppose $\mathcal{T}_1 = \mathcal{T} \cup \{A_\alpha\}$ where A_α is a closed set in $\mathcal{T}_1$ for each α. Then if $x \notin \cup A_\alpha$, then $\overline{\{x\}} = \overline{\{x\}}_1$, $\{x\}' = \{x\}'_1$.

Proof : (a) is clear since we have more closed sets in $\mathcal{T}_1$ than in $\mathcal{T}$.

(b) since $x \notin \cup A_\alpha$, the closed sets in $\mathcal{T}_1$ containing x are precisely those in $\mathcal{T}$ containing x, and the equality follows.

Theorem 2.2. If $(X,\mathcal{T})$ is a T_{UD} space, then so is $(X,\mathcal{T}_1)$.

Proof : Let x be an arbitrary point in X. We shall show that $\{x\}_1'$ is the union of disjoint closed sets in $\mathcal{T}_1$. Since $(X,\mathcal{T})$ is T_{UD}, $\{x\}' = \cup C_\alpha$ where $C_\alpha \in \mathcal{T}$ for each α and $C_\alpha \cap C_{\alpha'} = \phi$ if $\alpha \neq \alpha'$. Now by Lemma 2.1 (a), $\{x\}_1' \subset \{x\}$, hence

$$\begin{aligned}\{x\}_1' &\subset \overline{\{x\}}_1 \cap \{x\}' \\ &= \overline{\{x\}}_1 \cap (\cup C_\alpha) \\ &= \cup (\overline{\{x\}}_1 \cap C_\alpha) \\ &\subset \{x\}_1'\end{aligned}$$

where the last inclusion follows from the fact that each $\overline{\{x\}}_1 \cap C_\alpha$ is a subset of $\overline{\{x\}}_1$ not containing x. Since $(\overline{\{x\}}_1 \cap C_\alpha) \cap (\overline{\{x\}}_1 \cap C_{\alpha'}) = \phi$ if $\alpha \neq \alpha'$, $\{x\}_1'$ is the union of disjoint closed sets. Therefore $(X,\mathcal{T}_1)$ is a T_{UD} space.

Theorem 2.3. If $(X,\mathcal{T})$ is a T_D space, so is $(X,\mathcal{T}_1)$.

Proof : We have seen in the Proof of Theorem 2.2. that the equality

$$\{x\}'_1 = \overline{\{x\}}_1 \cap \{x\}'$$

holds for every $x \in X$. Thus if $(X,\mathcal{T})$ is T_D, then $\{x\}'$ is closed in $\mathcal{T}$ and so $\{x\}'_1$ is closed in $\mathcal{T}_1$. Hence $(X,\mathcal{T}_1)$ is also T_D.

Theorem 2.4. $T(\Upsilon)$ is not preserved under the strengthening of the topology.

Proof : Let $X = \{0, 1, 2, 3,\cdots\}$

$$\mathcal{T} = \{\phi\} \cup \{\ \{n, n+1, n+2,\cdots\} : n = 0, 1,\cdots\}$$

Then $(X,\mathcal{T})$ is a $T(\Upsilon)$ space because for each n,

$$\begin{aligned}\{n\}' &= \{n+1, n+2, \cdots\} \\ &= \overline{\{n+1\}}\end{aligned}$$

Now let

$$\mathcal{T}_1 = \mathcal{T} \cup \{\ \{n, n+2, n+4, \cdots\} : n = 0,2,\cdots\}$$

Then $(X,\mathcal{T}_1)$ is a strengthening of $(X,\mathcal{T})$ but $(X,\mathcal{T}_1)$ is not a $T(\Upsilon)$ space since for each n

$$\begin{aligned}\{2n - 1\}'_1 &= \overline{\{2n - 1\}}_1 \cap \{2n - 1\}' \\ &= \{2n - 1, 2n, 2n + 1,\cdots\} \cap \{2n, 2n + 1,\cdots\} \\ &= \{2n, 2n + 1,\cdots\}\end{aligned}$$

cannot be written as a union of disjoint point closures.

<u>Theorem 2.5</u>. If $(X,\mathcal{T})$ is a T_F space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : A space is a T_F space iff for every $x \in X$, $y \in \{x\}'$ implies $\{y\}' = \phi$. (Cf. [1] Theorem 3.2). Now if $y \in \{x\}_1'$, then $y \in \{x\}'$ and hence $\{y\}' = \phi$. But $\{y\}_1' \subset \{y\}'$. Thus $\{y\}_1' = \phi$.

<u>Theorem 2.6</u>. If $(X,\mathcal{T})$ is a T_Y space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : We have, for $x, y \in X$, $x \neq y$,

$$\overline{\{x\}}_1 \cap \overline{\{y\}}_1 \subset \overline{\{x\}} \cap \overline{\{y\}} .$$

Since $(X,\mathcal{T})$ is T_Y, $\overline{\{x\}} \cap \overline{\{y\}}$ is degenerate, so that $\overline{\{x\}}_1 \cap \overline{\{y\}}_1$ is also degenerate.

<u>Theorem 2.7</u>. If $(X,\mathcal{T})$ is a T_{YS} space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : For $x, y \in X$, $x \neq y$, since $\overline{\{x\}} \cap \overline{\{y\}}$ is either ϕ, $\{x\}$, or $\{y\}$, the same holds for $\overline{\{x\}}_1 \cap \overline{\{y\}}_1$.

<u>Theorem 2.8</u>. If $(X,\mathcal{T})$ is a T_{FF} space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : $(X,\mathcal{T})$ is T_{FF} iff $\{x\}' = \phi$ for all but at most one $x \in X$ (cf. [1] Theorem 3.3). Since $\{x\}_1' \subset \{x\}'$, the same is true for $\{x\}_1'$.

<u>Theorem 2.9</u>. If $(X,\mathcal{T})$ is a T_{DD} space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : Since $(X,\mathcal{T})$ is T_D , by Theorem 2.3 $(X,\mathcal{T}_1)$ is also T_D . Now for $x, y \in X, x \neq y$, $\{x\}' \cap \{y\}' = \phi$, hence $\{x\}_1' \cap \{y\}_1' = \phi$. Thus $(X,\mathcal{T}_1)$ is also T_{DD} .

<u>Theorem 2.10</u>. If $(X,\mathcal{T})$ is a $T^{(m)}$ space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : Let $x \in X$. Then

$$\{x\}' = \cup \{C_i : i \in I\}$$

where each $C_i \in \mathcal{T}$ and card $I = m$. But

$$\begin{aligned} \{x\}_1' &= \{\overline{x}\}_1 \cap \{x\}' \\ &= \{\overline{x}\}_1 \cap (\cup C_i) \\ &= \cup (\{\overline{x}\}_1 \cap C_i) \end{aligned}$$

where $\{\overline{x}\}_1 \cap C_i \in \mathcal{T}_1$. Hence $(X,\mathcal{T}_1)$ is also $T^{(m)}$.

<u>Theorem 2.11</u>. If $(X,\mathcal{T})$ is a $T_{UD}^{(m)}$ space, so is $(X,\mathcal{T}_1)$.

<u>Proof</u> : Since $(X,\mathcal{T})$ is $T_{UD}^{(m)}$, for each $x \in X$

$$\{x\}' = \cup C_i , \quad i \in I$$

where each $C_i \in T$, card $I = m$ and $C_j \cap C_k = \phi$ if $j \neq k$. Hence

$$\begin{aligned} \{x\}_1' &= \overline{\{x\}}_1 \cap \{x\}' \\ &= \overline{\{x\}}_1 \cap (\cup C_i) \\ &= \cup (\overline{\{x\}}_1 \cap C_i) \end{aligned}$$

where the family $\{ \overline{\{x\}}_1 \cap C_i\}$ is disjoint. Hence (X,T_1) is a $T_{UD}^{(m)}$ space.

<u>Theorem 2.12.</u> Strong T_D is not preserved under the strengthening of the topology.

<u>Proof</u> : Consider the example :

$$X = \{a, b, c\}$$

$$T = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\} .$$

Here

$$\{a\}' = \{b,c\} = \{b\} \cup \{c\}$$

$$\{b\}' = \phi \qquad \{c\}' = \phi$$

Hence (X,T) is a strong T_D space. Now let $T_1 = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. Then (X,T_1) is not a strong T_D space because $\{a\}_1' = \{c\}$.

<u>Theorem 2.13.</u> Strong T_o is not preserved under the strengthening of the topology.

Proof : In the example in the proof of Theorem 2.12, $(X,\mathcal{T})$ is a strong T_o space but $(X,\mathcal{T}_1)$ is not.

3. Product Spaces

An interesting question concerning the separation axioms is whether they are preserved under arbitrary products. The product space of a family of T_o (or T_1) spaces is again T_o (or T_1). In this section we shall consider products of T_D and $T^{(m)}$ spaces, m a cardinal.

Theorem 3.1. Let $\{(X_i,\mathcal{T}_i) : i = 1, 2,\cdots,n\}$ be a finite family of T_D spaces. Then $\prod_{i=1}^{n} X_i$ is also a T_D space.

Proof : A space $(X,\mathcal{T})$ is a T_D space iff for every $x \in X$, $\{x\} = G \cap C$, where G is an open set and C is a closed set in X (cf. [1] Theorem 3.1). Now take $x = \{x_1, x_2,\cdots,x_n\} \in \prod_{i=1}^{n} X_i$. Then each $\{x_i\} = G_i \cap C_i$, where G_i and C_i are open and closed respectively in X_i . Hence

$$\begin{aligned}
\{x\} &= \bigcap_{i=1}^{n} p_i^{-1}(G_i \cap C_i) \\
&= \bigcap_{i=1}^{n} (p_i^{-1}(G_i) \cap p_i^{-1}(C_i)) \\
&= \bigcap_{i=1}^{n} p_i^{-1}(G_i) \cap \bigcap_{i=1}^{n} p_i^{-1}(C_i)
\end{aligned}$$

Thus $\{x\}$ is the intersection of an open set with a closed set. Hence $\prod_{i=1}^{n} X_i$ is also T_D .

The following theorem shows that T_D is not preserved under arbitrary products.

<u>Theorem 3.2.</u> Let $\{X_i : i \varepsilon I\}$ be an infinite family of T_D spaces, where each X_i is not T_1 . Then $\prod_{i\varepsilon I} X_i$ is not a T_D space.

<u>Proof</u> : For every $i \varepsilon I$, since X_i is not T_1 , there is a point $\alpha_i \varepsilon X_i$ such that $\{\alpha_i\}' \neq \phi$. Let $\alpha = \{\alpha_i : i \varepsilon I\} \varepsilon \prod_{i\varepsilon I} X_i$, and let $Y = \prod_{i\varepsilon I} \{\overline{\alpha}_i\}$. Let $\{\alpha\}'_Y$ be the derived set of α in the subspace Y. We shall show that $\{\alpha\}'_Y$ is not closed in Y. To this end we first observe that if $x \varepsilon Y - \{\alpha\}$ and 0 is a basic open neighborhood of x, then $p_i(x) \varepsilon \{\overline{\alpha}_i\}$ implies that $\alpha_i \varepsilon p_i(0)$ for every i. Hence $x \varepsilon \{\alpha\}'_Y$. Thus we have $\{\alpha\}'_Y = Y - \{\alpha\}$. It therefore suffices to show that $Y - \{\alpha\}$ is not closed (in Y). We observe that if 0 is a basic open neighborhood of α in Y, then

$$0 = Y \cap \pi\{0_i : i \varepsilon I\}$$

where each 0_i is an open neighborhood of $\alpha_i \varepsilon X_i$ and $0_i \neq X_i$ for only a finite number of $i \varepsilon I$. Since I is infinite, there exists a non-empty subset $I' \subset I$ such that $0_i = X_i$ for all $i \varepsilon I'$. Since we have $\{\overline{\alpha}_i\} - \{\alpha\} \neq \phi$ for all $i \varepsilon I$, we can choose a point $\beta \varepsilon Y$

as follows :

$$\beta_i = \begin{cases} \alpha_i & \text{if } i \in I - I' \\ \{\overline{\alpha}_i\} - \{\alpha\} & \text{if } i \in I' \end{cases}$$

Then $\beta = \{\beta_i : i \in I\} \in 0$. This means that every open neighborhood of α in Y contains points of $Y - \{\alpha\}$. Thus $Y - \{\alpha\}$ is not closed in Y.

We therefore have proved that the subspace Y of $\prod_{i\in I} X_i$ is not a T_D space. Since the property of being T_D is hereditary, $\prod_{i\in I} X_i$ is not a T_D space.

Theorem 3.3. Let $(X,\mathcal{T})$ be a T_D space. Then all powers (that is, products of X by itself) of X are T_D spaces iff $(X,\mathcal{T})$ is T_1 .

Proof : This follows from Theorem 3.2.

Theorem 3.4. A product of m $T^{(m)}$ spaces is again a $T^{(m)}$ space.

Proof : Let $\prod_{i\in I} X_i$ be a product of the $T^{(m)}$ spaces X_i , where card $(I) = m$. Let $x = \{x_i : i \in I\}$ be an arbitrary point in $\prod_{i\in I} X_i$. Then for every $i \in I$, $\{x_i\} = G_{i,m} \cap F_i$ where F_i is closed in X_i and $G_{i,m}$ is an intersection of m open sets in X_i . Thus

$$\{x\} = \cap\, \{p_i^{-1}(x_i) : i \in I\}$$

$$= \cap\, \{p_i^{-1}(G_{i,m} \cap F_i) : i \in I\}$$

$$= \cap\, \{p_i^{-1}(G_{i,m}) : i \in I\} \cap \cap\, \{p_i^{-1}(F_i) : i \in I\} \quad .$$

Now by the continuity of p_i, each $p_i^{-1}(G_{i,m})$ is the intersection of m open sets in πX_i, and also $\cap\, \{p_i^{-1}(F_i) : i \in I\}$ is closed in πX_i. Since card $(I) = m$, we have expressed $\{x\}$ as the intersection of a closed set with m open sets. Thus πX_i is $T^{(m)}$.

Theorem 3.5. Let $\{X_i : i \in I\}$ be a family of $T^{(m)}$ spaces none of which is T_1, and let card $(I) = n$. Then πX_i is a $T^{(m)}$ space iff $n \leq m$.

Proof : If $n \leq m$, then by Theorem 3.4, πX_i is again a $T^{(m)}$ space.

If $n > m$, then by an analogous argument as in that used in the proof of Theorem 3.2, πX_i is not a $T^{(m)}$ space.

Theorem 3.6. There does not exist a separation axiom between $T^{(m)}$ and T_1 which is inherited by arbitrary products.

Proof : Let T_α be a separation axiom between $T^{(m)}$ and T_1. Fix an infinite cardinal m, and let $\{X_i : i \in I\}$ be a family of T_α spaces, none of which is T_1 and suppose card $(I) = n > m$. Now each X_i is a

$T^{(m)}$ space, so by Theorem 3.5 πX_i cannot be a $T^{(m)}$ space since $n > m$. Therefore πX_i is not a T_α space.

CHAPTER III

Minimal Topologies

1. Introduction

Given a set X, the family of all topologies defined on X is a complete lattice. Of great interest are topologies which are minimal in this lattice with respect to a certain topological property, in the sense of the following definition.

Definition 1.1. Let P be a topological property. A topology $\mathcal{T}$ defined on a set X is called a minimal P space iff $\mathcal{T}$ has property P and every strictly weaker topology on X does not have property P.

Thus if P stands for T_o, T_D, T_1, T_2, regular, completely regular, normal or locally compact space, the topology $\mathcal{T}$ will be called a minimal T_o , minimal T_D , minimal T_2 , minimal regular, minimal completely regular, minimal normal or minimal locally compact topology accordingly.

It is the purpose of this chapter to investigate some of these minimal topologies, obtain their characterizations and arrive at some of their properties.

2. Minimal T_o and Minimal T_D Spaces

For the characterizations of minimal T_o and minimal T_D spaces the following Lemmas will be useful.

Lemma 2.1. Let $(X,\mathcal{T})$ be a T_o (T_D) space and let B be an open subset of X. Let

$$\mathcal{T}(B) = \{G \in \mathcal{T} : G \subset B \text{ or } B \subset G\} .$$

Then $(X,\mathcal{T}(B))$ is a T_o (T_D) space.

Proof : We first show that $\mathcal{T}(B)$ is indeed a topology. Since $\phi \subset B$ and $B \subset X$, we have ϕ, $X \in \mathcal{T}(B)$. Now if G_1, $G_2 \in \mathcal{T}(B)$, then either $B \subset G_1$, $B \subset G_2$, in which case $B \subset G_1 \cap G_2$, or B contains one of the two sets in which case $G_1 \cap G_2 \subset B$. Hence $G_1 \cap G_2 \in \mathcal{T}(B)$. Finally if $G_\alpha \in \mathcal{T}(B)$ for every $\alpha \in A$, then either every $G_\alpha \subset B$ or $B \subset G_\alpha$ for some α, so that $\bigcup_{\alpha\in A} G_\alpha \subset B$ or $B \subset \bigcup_{\alpha\in A} G_\alpha$, whence $\bigcup_{\alpha\in A} G_\alpha \in \mathcal{T}(B)$.

Now suppose $\mathcal{T}$ is a T_o topology. To show that $\mathcal{T}(B)$ is also T_o, let $x, y \in X$, $x \neq y$ and suppose that $G \in \mathcal{T}$, $x \in G$, $y \notin G$. We consider three cases.

(1) If $x \in B$ and $y \notin B$, then B is an open set in $\mathcal{T}(B)$ containing x but not y. Similarly if $y \in B$ and $x \notin B$.

(2) If $x, y \varepsilon B$, then $G \cap B$ is an open set in $\mathcal{T}(B)$ containing x but not y.

(3) If $x, y \notin B$, then $G \cup B$ is an open set in $\mathcal{T}(B)$ containing x but not y.

Thus $(X,\mathcal{T}(B))$ is also a T_o space.

Suppose next $(X,\mathcal{T})$ is a T_D space. To prove that $\mathcal{T}(B)$ is also T_D, take arbitrary $x \varepsilon X$, and consider $\{x\}'_B$, the derived set of x in the topology $\mathcal{T}(B)$. Again we consider two cases.

(i) If $x \notin B$, then $x \varepsilon X - B$ which is closed in $\mathcal{T}(B)$. Thus $\{x\}'_B \subset X - B$ and so $B \subset X - \{x\}'_B$. Hence $X - \{x\}'_B$ is open, and so $\{x\}'_B$ is closed in $\mathcal{T}(B)$.

(ii) If $x \varepsilon B$, we can prove that if $y \varepsilon B, y \neq x$, then $y \notin \{x\}'_B$. Indeed, in this case $B - \{x\} \varepsilon \mathcal{T}(B)$, $y \varepsilon B - \{x\}$, but $x \notin B - \{x\}$. Hence $\{x\}'_B \cap B = \phi$, which means that $B \subset X - \{x\}'_B$. Hence $\{x\}'_B$ is closed in $\mathcal{T}(B)$.

<u>Lemma 2.2</u>. In a topological space $(X,\mathcal{T})$, the following are equivalent:

(1) The open sets in the topology are rested, i.e., for $A, B \varepsilon \mathcal{T}$, either $A \subset B$ or $B \subset A$.

(2) The closed sets in the topology are rested.

(3) Finite unions of point closures are point closures.

Proof : The equivalence of (1) and (2) is clear. That (2) implies (3) is also obvious, since the union of a finite number of point closures is the largest one.

To show that (3) implies (2), let C, D be two non-empty closed sets in $(X,\mathcal{T})$, and suppose $C \neq D$. Then either $C - D \neq \phi$ or $D - C \neq \phi$ must hold. Assume $C - D \neq \phi$. Take $y \varepsilon D$ and choose $x \varepsilon C - D$. Then by (3) $\overline{\{x\}} \cup \overline{\{y\}} = \overline{\{z\}}$ for some $z \varepsilon X$. It follows that $z \varepsilon \overline{\{x\}}$ or $z \varepsilon \overline{\{y\}}$, i.e., $\overline{\{z\}} \subset \overline{\{x\}}$ or $\overline{\{z\}} \subset \overline{\{y\}}$. But $\overline{\{x\}} \subset \overline{\{z\}}$ and $\overline{\{y\}} \subset \overline{\{z\}}$. Hence $\overline{\{x\}} = \overline{\{z\}}$ or $\overline{\{y\}} = \overline{\{z\}}$. But $\overline{\{y\}} = \overline{\{z\}}$ is impossible since this would imply $x \varepsilon \overline{\{z\}} = \overline{\{y\}} \subset D$, a contradiction because $x \notin D$. Therefore $\overline{\{x\}} = \overline{\{z\}}$ and so $y \varepsilon \overline{\{z\}} = \overline{\{x\}} \subset C$. We have thus proved that $D \subset C$. Similarly if $D - C \neq \phi$, then $C \subset D$. Hence (2) holds.

The following Theorem gives a characterization of minimal T_o spaces.

Theorem 2.3. A T_o topological space $(X,\mathcal{T})$ is minimal T_o iff the family $\beta = \{X - \overline{\{x\}} : x \varepsilon X\} \cup \{X\}$ is a base for $\mathcal{T}$ and finite unions of point closures are point closures.

Proof : ($\Longrightarrow$) Let A, B be open sets in $\mathcal{T}$. If $A \not\subset B$, $B \not\subset A$, then by Lemma 2.1, $\mathcal{T}(B)$ is a T_o topology on X such that $\mathcal{T}(B) \subset \mathcal{T}$ and $A \notin \mathcal{T}(B)$. This contradicts the minimality of $\mathcal{T}$. Thus either $A \subset B$ or $B \subset A$. Hence by Lemma 2.2, finite unions of point closures are point closures. Now since $\mathcal{T}$ is a nested family of open sets, the subfamily $\{X - \overline{\{x\}} : x \in X\} \cup \{X\}$ is closed under finite intersections, and so is a base for some topology $\mathcal{T}_1$ on X. Clearly $\mathcal{T}_1 \subset \mathcal{T}$. Also $\mathcal{T}_1$ is a T_o topology, because for $x \neq y$, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$ since $\mathcal{T}$ is T_o. By the minimality of $\mathcal{T}$, we have $\mathcal{T}_1 = \mathcal{T}$. So β is a base for $\mathcal{T}$.

($\Longleftarrow$) Let $(X,\mathcal{T})$ be T_o where $\mathcal{T}$ is nested and β is a base for $\mathcal{T}$. Suppose $\mathcal{T}^* \subset \mathcal{T}$, where $\mathcal{T}^*$ is a T_o topology. Suppose there is a point $x \in X$ such that $\overline{\{x\}} \neq \overline{\{x\}}^*$, where $\overline{\{x\}}^*$ is the closure of $\{x\}$ in $\mathcal{T}^*$. Since $\overline{\{x\}} \subset \overline{\{x\}}^*$, we can choose a point $y \in \overline{\{x\}}^*$, $y \notin \overline{\{x\}}$. Since $\mathcal{T}$ is nested, we must have $\overline{\{x\}} \subset \overline{\{y\}}$. But $\overline{\{y\}} \subset \overline{\{y\}}^*$, hence $x \in \overline{\{x\}} \subset \overline{\{y\}}^*$. Thus $x \in \overline{\{y\}}^*$ and $y \in \overline{\{x\}}^*$, which is impossible in a T_o space. Hence for every $x \in X$, $\overline{\{x\}} = \overline{\{x\}}^*$. Since $\beta = \{X - \overline{\{x\}} : x \in X\} \cup \{X\}$ is a base for $\mathcal{T}$, we must have $\mathcal{T} = \mathcal{T}^*$. Thus $\mathcal{T}$ is minimal T_o.

<u>Theorem 2.4</u>. A T_D topological space $(X,\mathcal{T})$ is minimal T_D iff finite unions of point closures are point closures.

<u>Proof</u> : ($\Longrightarrow$) By a proof identical to the one in Theorem 2.3.

($\Longleftarrow$) Suppose $(X,\mathcal{T})$ is T_D and $\mathcal{T}$ is nested. Let $\mathcal{T}^* \subset \mathcal{T}$, where $\mathcal{T}^*$ is a T_D space. We shall show that $\mathcal{T}^* = \mathcal{T}$. Since a T_D space is a T_o space, we can apply the argument in Theorem 2.3 and arrive at the conclusion that $\overline{\{x\}} = \overline{\{x\}}^*$, $\{x\}' = \{x\}'^*$ for every $x \in X$. Now assume that $\mathcal{T}^* \neq \mathcal{T}$. Then there is a set $C \subset X$ closed in $\mathcal{T}$ but not closed in T^*. If C^* denotes the closure of C in T^*, then $C \subsetneqq C^*$. But T^* is T_0, therefore $C^* - C$ consists of exactly one point x. Since there does not exist a closed set in T^* smaller than C^* which contains x, we have $C^* = \{\overline{x}\}^* = C \cup \{x\} = \{x\}'^* \cup \{x\}$. Thus it follows that $C = \{x\}'^*$ because $x \notin C$, $x \notin \{x\}'^*$. But this contradicts the fact that C is not closed in T^*. Thus we must have $T^* = T$ and T is minimal T_D.

The following two examples show that the two conditions in Theorem 2.3 cannot be relaxed and that they are independent of each other.

Example 2.1. X = real numbers

$$T = \{(-\infty,x) : x \in X\} \cup \{(-\infty,x] : x \in X\} \cup \{\phi,X\}.$$

(X,T) is a T_0 space in which the open sets are nested, but is not a minimal T_0 space because, for example, the proper subfamily $T' = \{(-\infty,x) : x \in X\} \cup \{\phi,X\}$ is a T_0 topology on X.

Example 2.2. $X = \{a,b,c\}$

$$T = \{\phi, \{a\}, \{b\}, \{a,b\}, X\} .$$

(X,T) is a T_0 space. Moreover, since $\overline{\{a\}} = \{a,c\}$, $\overline{\{b\}} = \{b,c\}$, $\overline{\{c\}} = \{c\}$, the complements of these sets are $\{b\}, \{a\}, \{a,b\}$ which form a base for the topology T. However, the open sets are obviously not nested.

The next example shows that minimal T_o is not hereditary.

Example 2.3. X = real numbers.

$\mathcal{T} = \{(-\infty,x) : x \in X\} \cup \{\phi,X\}$

$A = (-\infty,0] \cup (1,\infty)$

$(X,\mathcal{T})$ is a minimal T_o space since the open sets are clearly nested and for every $x \in X$, $\{\overline{x}\} = [x,\infty)$ so that the family $\{X - \{\overline{x}\} : x \in X\}$ is precisely $\mathcal{T}$ itself. However, the subspace A is not minimal T_o because although the open sets are again nested, the complements of point closures do not form a base. Indeed, we first observe that $(-\infty,0]$ is open in A and $0 \in (-\infty,0]$. Now if $x \in (-\infty,0]$, then $0 \in [x,\infty) \cap A =$ $= \{\overline{x}\} \cap A = \{\overline{x}\}_A$ and so $0 \notin A - \{\overline{x}\}_A$. On the other hand, if $x \in (1,\infty)$, then $A - \{\overline{x}\}_A \not\subset (-\infty,0]$. Hence $\{A - \{\overline{x}\}_A : x \in A\}$ is not a base for the relative topology.

For minimal T_D spaces we have the following result.

Theorem 2.5. Every subspace of a minimal T_D space is again minimal T_D.

Proof : Each subspace of a T_D space is T_D. By the definition of relative topology the nestedness of open sets is inherited. Hence by Theorem 2.4 and Lemma 2.2 the result follows.

3. Minimal T_1 spaces

For minimal T_1 space we have the following neat theorem.

Theorem 3.1. A topological space $(X,\mathcal{T})$ is minimal T_1 iff the non-trivial closed sets are precisely the finite sets.

Proof : Given any set X let

$$\mathcal{T}^* = \{A \subset X : X - A \text{ is finite}\} \cup \{\phi\} .$$

Then it is well-known that $(X,\mathcal{T}^*)$ is a T_1 space. Moreover, $\mathcal{T}^*$ is the weakest T_1 topology on X because if $\mathcal{T}$ is another T_1 topology on X, then all finite sets are closed in $\mathcal{T}$ and so $\mathcal{T}^* \subset \mathcal{T}$. It follows that $\mathcal{T}$ is a minimal T_1 topology iff $\mathcal{T} = \mathcal{T}^*$. The theorem follows.

Corollary 3.2. Any subspace of a minimal T_1 space is minimal T_1 .

4. Minimal Regular Spaces

For subsequent discussion we shall make use of the notion of a filter base. The reader is referred to [4] for definitions and results concerning filter bases in a topological space. For our present arguement we introduce some definitions.

<u>Definition 4.1</u>. A filter base $\mathcal{F}$ on a set X is said to be weaker than a filter base $\mathcal{G}$ on X iff for each $F \in \mathcal{F}$, there exists some $G \in \mathcal{G}$ such that $G \subset F$.

<u>Definition 4.2</u>. A filter base $\mathcal{F}$ on a set X is said to be equivalent to a filter base $\mathcal{G}$ on X iff $\mathcal{F}$ is weaker than $\mathcal{G}$ and $\mathcal{G}$ is weaker than $\mathcal{F}$.

It is readily checked that the relation of equivalence between filters is an equivalence relation.

<u>Definition 4.3</u>. A filter base $\mathcal{F}$ on a topological space $(X,\mathcal{F})$ is called an open (closed) filter base iff for every $F \in \mathcal{F}$, F is an open (closed) set.

<u>Definition 4.4</u>. A filter base $\mathcal{F}$ on a topological space is called a regular filter base iff it is open and is equivalent to a closed filter base.

Definition 4.4 is suggested by the following theorem.

<u>Theorem 4.1</u>. In a regular topological space $(X,\mathcal{T})$, the filter base of open neighborhoods of a point is regular.

<u>Proof</u> : Let $\mathcal{B}(x)$ be the filter base of open neighborhoods of the point x and let $\mathcal{C}(x)$ be the filter base of closed neighborhoods of x.

Obviously for every $C \in \mathcal{C}(x)$, there is a $B \in \mathcal{B}(x)$ such that $B \subset C$ Since $\mathcal{T}$ is regular, for every $B \in \mathcal{B}(x)$, we can also find $C \in \mathcal{C}(x)$ such that $C \subset B$. Hence $\mathcal{B}(x)$ is equivalent to $\mathcal{C}(x)$ and so is regular.

We shall be interested in the following conditions in a topological space :

(α) Every regular filter base which has a unique cluster point is convergent to this point.

(β) Every regular filter base has a cluster point.

<u>Theorem 4.2</u>. A regular space $(X,\mathcal{T})$ is minimal regular iff (α) holds.

<u>Proof</u> : Necessity : Suppose β is a regular filter base which has the unique cluster point p, and assume that β does not converge to p. We shall construct a topology on X which is regular but strictly weaker than $\mathcal{T}$. For each $x \in X$, let $\mathcal{U}(x)$ be the filter base of open neighborhoods of x. Define

$$\mathcal{U}'(x) = \begin{cases} \mathcal{U}(x) & \text{if } x \neq p \\ \{U \cup B : U \in \mathcal{U}(p),\ B \in \mathcal{B}\} & \text{if } x = p\ . \end{cases}$$

Under this definition there is defined on X a topology $\mathcal{T}'$ with $\mathcal{U}'(x)$ as an open neighborhood base at each $x \in X$. Now since $\mathcal{B}$ does not converge to p, there is a $U \in \mathcal{U}(p) - \mathcal{B}$ which does not contain any set in $\mathcal{U}'(p)$. Hence $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$. To show that $\mathcal{T}'$ is regular, first it is clear that $\mathcal{T}'$ is regular at each point $x \neq p$. At the point p, since $\mathcal{B}$ is regular, $\mathcal{B}$ is equivalent to some closed filter base $\mathcal{C}$. Now if $p \varepsilon U \cup B$, where $U \varepsilon \mathcal{U}(x)$, $B \varepsilon \mathcal{B}$, there exists closed sets V, C such that $p \varepsilon V \subset U$, $C \varepsilon \mathcal{C}$, $C \subset B$, so that $p \varepsilon V \cup C$. Thus $\mathcal{T}'$ is also regular at p. This shows that $\mathcal{T}$ is not a minimal regular topology.

Sufficiency : Suppose $(X,\mathcal{T})$ is regular and satisfies (α). Let $\mathcal{T}'$ be a regular topology on X such that $\mathcal{T}' \subset \mathcal{T}$. For each $x \varepsilon X$, denote by $\mathcal{U}(x)$ and $\mathcal{U}'(x)$ the open neighborhood systems of x in $\mathcal{T}$ and $\mathcal{T}'$ respectively. Since $\mathcal{T}'$ is regular, the filter base $\mathcal{U}'(x)$ is, by Theorem 4.1, $\mathcal{T}'$-regular. Moreover, x is the only cluster point of $\mathcal{U}'(x)$. Since $\mathcal{T}' \subset \mathcal{T}$, it follows that $\mathcal{U}'(x)$ is regular in $\mathcal{T}$ and has x as its unique cluster point. By (α), $\mathcal{U}'(x)$ must converge to x in $\mathcal{T}$, so that by definition $\mathcal{U}(x) \subset \mathcal{U}'(x)$. But $\mathcal{T}' \subset \mathcal{T}$ implies $\mathcal{U}'(x) \subset \mathcal{U}(x)$. Thus $\mathcal{U}(x) = \mathcal{U}'(x)$ and we have $\mathcal{T}' = \mathcal{T}$. Hence $\mathcal{T}$ is minimal regular.

<u>Lemma 4.3</u>. If the subspace A of the regular space $(X,\mathcal{T})$ satisfies (β), then A is closed in X.

<u>Proof</u> : Suppose $\overline{A} \neq A$ and let $p \varepsilon \overline{A} - A$. Let $\mathcal{U}$ and $\mathcal{V}$ be the open and closed neighborhood systems of p in X, respectively ($\mathcal{V}$ is a neighborhood system since $\mathcal{T}$ is regular). Let

$$\mathcal{B} = \{A \cap U : U \in \mathcal{U}\}$$

$$\mathcal{C} = \{A \cap V : V \in \mathcal{V}\}$$

Then $\mathcal{B}$ is an open filter base in A, while $\mathcal{C}$ is a closed filter base in A. Moreover, since $\mathcal{T}$ is regular, $\mathcal{U}$ is equivalent to $\mathcal{V}$, and hence $\mathcal{B}$ is equivalent to $\mathcal{C}$. Thus $\mathcal{B}$ is a regular filter base on A. But $\mathcal{B}$ is also a filter base on X, and $\mathcal{U}$ is weaker than $\mathcal{B}$. Since p is the only cluster point of $\mathcal{U}$ in X, it is also the only cluster point of $\mathcal{B}$ in X. This means that, since $p \notin X$, β has no cluster point in X, which contradicts (β). Hence A must be closed.

<u>Theorem 4.4</u>. In a regular space, (α) implies (β).

<u>Proof</u> : Let $\mathcal{B}$ be a regular filter base on a regular space $(X,\mathcal{T})$. Assume $\mathcal{B}$ has no cluster point. Let $\mathcal{C}$ be a closed filter base equivalent to $\mathcal{B}$. Fix $p \in X$. Let $\mathcal{U}$, $\mathcal{V}$ be the open and closed neighborhood systems of p respectively. Then $\mathcal{U}$ and $\mathcal{V}$ are equivalent, by Theorem 4.1. Let $\mathcal{F} = \{B \cup U : B \in \mathcal{B},\ U \in \mathcal{U}\}$ and $\mathcal{G} = \{C \cup V : C \in \mathcal{C},\ V \in \mathcal{V}\}$. Then $\mathcal{F}$ is an open filter base, and $\mathcal{G}$ is a closed filter base on X. Moreover, $\mathcal{F}$ is equivalent to $\mathcal{G}$, which follows from the equivalences of $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{U}$ to $\mathcal{V}$. Thus $\mathcal{F}$ is regular. Now p is a cluster point of $\mathcal{F}$, and since $\mathcal{B}$ has no cluster point, $\mathcal{F}$ has no cluster point other than p. However, $\mathcal{F}$ does not converge to p. This contradiction to (α) shows that (α) implies (β).

Theorem 4.5. A minimal regular subspace of a regular space is closed.

Proof : By Theorem 4.2, a minimal regular subspace of a topological space satisfies (α). By Theorem 4.4, it also satisfies (β), so that by virtue of Lemma 4.3 it is closed.

Corollary 4.6. Minimal regularity is not hereditary.

5. Minimal Hausdorff Spaces

For the characterization of minimal Hausdorff spaces we consider the following two properties of a topological space :

(1) Every open filter base has a cluster point.

(2) Every open filter base which has a unique cluster point converges to this point.

Theorem 5.1. In a Hausdorff space, (2) implies (1).

Proof : Suppose (1) does not hold, and let $\mathcal{B}$ be an open filter base which has no cluster point. Fix $p \in X$. Let $\mathcal{U}$ be the open neighborhood system of p. Define

$$\mathcal{G} = \{V \cup B \mid V \in \mathcal{U} \text{ and } B \in \mathcal{B}\}$$

Then $\mathcal{G}$ is an open filter base and p is its only cluster point. By (2), $\mathcal{G}$ converges to p. But $\mathcal{B}$ is weaker than $\mathcal{B}$, so $\mathcal{B}$ also converges to p. But this contradicts the assumption that $\mathcal{B}$ has no cluster point. Hence (1) holds.

Theorem 5.2. A Hausdorff space $(X,\mathcal{T})$ is minimal Hausdorff iff (2) holds in $\mathcal{T}$.

Proof : Necessity : Let $(X,\mathcal{T})$ be Hausdorff and suppose that (2) does not hold, so that there exists an open filter base $\mathcal{B}$ having the unique cluster point p but $\mathcal{B}$ does not converge to p. For each $x \in X$, let $\mathcal{U}(x)$ denote the open neighborhood system of x. Define for every x a family of subsets of X as follows : $\mathcal{U}'(x) = \mathcal{U}(x)$ if $x \neq p$, and $\mathcal{U}'(p) = \{U \cup B : U \in \mathcal{U}(p), B \in \mathcal{B}\}$. With this definition there exists a topology $\mathcal{T}'$ on X with $\mathcal{U}'(x)$ as an open base at each x. That $\mathcal{T}' \subset \mathcal{T}$ is clear. Moreover since $\mathcal{B}$ does not converge to p, there is $U \in \mathcal{U}(p) - \mathcal{B}$ and U does not contain any set in $\mathcal{U}'(p)$. Thus $\mathcal{T}' \subsetneq \mathcal{T}$. We now show that $\mathcal{T}'$ is Hausdorff. Indeed, for $x, y \in X$, $x \neq y$, if $x \neq p$, $y \neq p$, then the existence of disjoint open neighborhoods is guaranteed. For $x \neq p$, by Hausdorffness of $\mathcal{T}$ there are $A \in \mathcal{U}(p)$, $D \in \mathcal{U}(x)$ such that $A \cap D = \phi$. Since p is the only cluster point of $\mathcal{B}$, x cannot be a cluster point and so there is $E \in \mathcal{U}(x)$, $B \in \mathcal{B}$ such that $E \cap B = \phi$. It follows that $D \cap E \in \mathcal{U}(x)$, $A \cup B \in \mathcal{U}'(p)$ and $(A \cup B) \cap (D \cap E) = \phi$. Thus $\mathcal{T}'$ is Hausdorff and $\mathcal{T}$ is not minimal Hausdorff.

Sufficiency : Let $(X,\mathcal{T})$ be Hausdorff satisfying (2) and let $\mathcal{T}'$ be a Hausdorff topology on X with $\mathcal{T}' \subset \mathcal{T}$. Let $x \in X$, and let $\mathcal{U}(x)$ and $\mathcal{U}'(x)$ be the open neighborhood systems of x in $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{U}'(x) \subset \mathcal{U}(x)$. The open filter base $\mathcal{U}'(x)$ has x as its only cluster point, because $\mathcal{T}'$ is Hausdorff. Since $\mathcal{T}' \subset \mathcal{T}$, x is the only cluster point of $\mathcal{U}'(x)$ in $(X,\mathcal{T})$. By (2), $\mathcal{U}'(x)$ converges to x. Thus $\mathcal{U}(x) \subset \mathcal{U}'(x)$. Thus the two topologies $\mathcal{T}$ and $\mathcal{T}'$ are identical, and so $\mathcal{T}$ is minimal Hausdorff.

Theorem 5.3. Let X be a subspace of the Hausdorff space $(Y,\mathcal{T})$. If X satisfies (1), then X is closed in Y.

Proof : If $X \neq \overline{X}$, let $p \in \overline{X} - X$. Let $\mathcal{U}$ be the open neighborhood system of p in Y. Then $\mathcal{B} = \{X \cap U : U \in \mathcal{U}\}$ is an open filter base on X. Moreover, as a filter base on Y, $\mathcal{B}$ is stronger than $\mathcal{U}$ and since p is the only cluster point of $\mathcal{U}$, $\mathcal{B}$ cannot have any other cluster point than p in Y. This means that $\mathcal{B}$ has no cluster point in X, since $p \notin X$. But this contradicts the hypothesis that X satisfies (1).

That the property of being minimal Hausdorff is not hereditary is shown by the following theorem.

Theorem 5.4. A minimal Hausdorff subspace X of a Hausdorff space $(Y,\mathcal{T})$ is closed.

Proof : Since X is minimal Hausdorff, Theorems 5.1 and 5.2 tell us that X satisfies property (1). Theorem 5.3 then concludes that X is closed.

Theorem 5.5. If a subspace of a minimal Hausdorff space is both open and closed, then it is minimal Hausdorff.

Proof : Let A be an open and closed subset of the minimal Hausdorff space $(X,\mathcal{T})$. Let $\mathcal{B}$ be an open filter base on A with only one cluster point $p \in A$. Since A is open in X, $\mathcal{B}$ is also an open filter base on X. Since A is also closed, the closure of $B \in \mathcal{B}$ in A is closed in X, and hence p is the only cluster point of $\mathcal{B}$ on X. But now X is minimal Hausdorff, hence $\mathcal{B}$ converges to p on X, by Theorem 5.2. Since $p \in A$, $\mathcal{B}$ also converges to p on A. Again invoking Theorem 5.2, A is minimal Hausdorff.

6. A Characterization of Order Topologies by Minimal T_0 Topologies

In this section we shall give a characterization of order topologies on a set X by means of minimal T_0 topologies on X. We recall that a topology $\mathcal{T}$ on X is an order topology on X iff there exists a linear order $\leq$ on X such that the sets of the forms $\{y : y < x\}$ and $\{y : x < y\}$, where $x \in X$, form a subbase for $\mathcal{T}$, where $a < b$ means that $a \leq b$ but $a \neq b$. We prove that a topology $\mathcal{T}$ on a set X is an order topology iff $(X,\mathcal{T})$ is T_1 and $\mathcal{T}$ is the least upper bound of two minimal topologies on X, in the sense of the following definition.

<u>Definition 6.1</u>. A topology $\mathcal{T}$ on a set X is the least upper bound of two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X iff $\mathcal{T}$ is the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$. We shall write $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ in this case.

<u>Lemma 6.1</u>. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X and $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a subbase for $\mathcal{T}_1 \vee \mathcal{T}_2$.

<u>Proof</u> : It is clear that $\mathcal{B}_1 \cup \mathcal{B}_2$ is a subbase for some topology, say $\mathcal{T}$, on X. Also $\mathcal{T}_1 \subset \mathcal{T}$ and $\mathcal{T}_2 \subset \mathcal{T}$. Hence $\mathcal{T}_1 \vee \mathcal{T}_2 \subset \mathcal{T}$. On the other hand any topology on X containing $\mathcal{T}_1$ and $\mathcal{T}_2$ must contain $\mathcal{B}_1 \cup \mathcal{B}_2$ and hence contains unions of finite intersections of members of $\mathcal{B}_1 \cup \mathcal{B}_2$, so that it contains $\mathcal{T}$. Thus $\mathcal{T} \subset \mathcal{T}_1 \vee \mathcal{T}_2$.

<u>Definition 6.2</u>. Let $\mathcal{T}$ be a topology on a set X. Define a relation $\leq_{\mathcal{T}}$ on X as follows :

$$a \leq_{\mathcal{T}} b \quad \text{iff} \quad b \in \overline{\{a\}} .$$

It is immediate that $\leq_{\mathcal{T}}$ as defined above is reflexive and transitive. In a T_0 space it is also anti-symmetric, as the following lemma shows.

<u>Lemma 6.2</u>. A topology $\mathcal{T}$ on a set X is T_o iff $\leq_{\mathcal{T}}$ is a partial order.

<u>Proof</u> : If $\mathcal{T}$ is T_o and $a \leq_{\mathcal{T}} b$, $b \leq_{\mathcal{T}} a$, then $b \in \{\overline{a}\}$ and $a \in \{\overline{b}\}$, so that $a = b$ since $\mathcal{T}$ is T_o. Conversely, if $\leq_{\mathcal{T}}$ is anti-symmetric, and if $a \neq b$ and $b \in \{\overline{a}\}$, then $a \notin \{\overline{b}\}$. Hence $\mathcal{T}$ is T_o.

<u>Lemma 6.3</u>. The topology $\mathcal{T}$ is nested iff any two elements are comparable, i.e., for $a, b \in X$, either $a \leq_{\mathcal{T}} b$ or $b \leq_{\mathcal{T}} a$.

<u>Proof</u> : If $\mathcal{T}$ is nested, then for $a, b \in X$, either $\{\overline{a}\} \subset \{\overline{b}\}$ or $\{\overline{b}\} \subset \{\overline{a}\}$, and so either $a \leq_{\mathcal{T}} b$ or $b \leq_{\mathcal{T}} a$.

Conversely, suppose the condition holds. Let $A, B \in \mathcal{T}$ and assume $A \not\subset B$. Choose $a \in A - B$. Now for each $x \in B$, $x \notin \{\overline{a}\}$ since B is an open set containing x but not a. Thus $a \not\leq_{\mathcal{T}} x$ and so $x \leq_{\mathcal{T}} a$. Therefore $a \in \{\overline{x}\}$ and since $a \in A \in \mathcal{T}$, we must have $x \in A$. Hence $B \subset A$ and so $\mathcal{T}$ is nested.

<u>Theorem 6.4</u>. A topological space $(X,\mathcal{T})$ is minimal T_o iff $\leq_{\mathcal{T}}$ is a linear order and $\{\{y : y <_{\mathcal{T}} x\} : x \in X\} \cup \{X\}$ is a base for $\mathcal{T}$.

<u>Proof</u> : If $\mathcal{T}$ is minimal T_o, then by Theorem 2.3 and Lemma 2.2 $\mathcal{T}$ is nested and $\{X - \{\overline{x}\} : x \in X\} \cup \{X\}$ is a base for $\mathcal{T}$. It follows from Lemma 6.2 and Lemma 6.3 that $\leq_{\mathcal{T}}$ is a linear order. Moreover for each $x \in X$,

$$\begin{aligned} X - \{\overline{x}\} &= X - \{y : y \varepsilon \{\overline{x}\}\} \\ &= X - \{y : x \leq_{\mathcal{T}} y\} \\ &= \{y : y <_{\mathcal{T}} x\} \end{aligned}$$

since $\leq_{\mathcal{T}}$ is linear. Thus $\{\{y : y <_{\mathcal{T}} x\} : x \varepsilon X\} \cup \{X\}$ is a base for $\mathcal{T}$.

In the other direction suppose $\leq_{\mathcal{T}}$ is linear and $\{\{y : y < x\} : : x \varepsilon X\} \cup \{X\}$ is a base for $\mathcal{T}$. Then by Lemma 6.3, $\mathcal{T}$ is nested. Again, because $\leq_{\mathcal{T}}$ is linear we have for each $x \varepsilon X$, $X - \{\overline{x}\} = \{y : y <_{\mathcal{T}} x\}$. Thus $\{X - \{\overline{x}\} : x \varepsilon X\} \cup \{X\}$ is a base for $\mathcal{T}$ and by Theorem 2.3 $\mathcal{T}$ is a minimal T_o topology.

<u>Theorem 6.5</u>. Given any set X, there exists a 1-1 conespondence between the set of all minimal T_o topologies on X and the set of all linear orders on X.

<u>Proof</u> : Let $\mathcal{M}$ be the set of all minimal topologies on X, and let $\mathcal{L}$ be the set of all linear orders on X. Define $\phi : \mathcal{M} \longrightarrow \mathcal{L}$ by $\phi(\mathcal{T}) = \leq_{\mathcal{T}}$ for each $\mathcal{T} \varepsilon \mathcal{M}$. By Theorem 6.4 $\leq_{\mathcal{T}}$ is indeed a linear order on X, and so ϕ is a well-defined map from $\mathcal{M}$ to $\mathcal{L}$. Also ϕ is 1-1 because if $\mathcal{T}_1, \mathcal{T}_2 \varepsilon \mathcal{M}$ and $\leq_{\mathcal{T}_1} = \leq_{\mathcal{T}_2}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ have bases $\{\{y : y <_{\mathcal{T}_1} x\} : x \varepsilon X\} \cup \{X\}$ and $\{\{y : y <_{\mathcal{T}_2} x\} : x \varepsilon X\} \cup \{X\}$, respectively. Since the two basis are the same, $\mathcal{T}_1 = \mathcal{T}_2$.

Now let $\leq$ be a linear order on X. Let $\mathcal{B} = \{\{y : y < x\} : x \in X\} \cup \{X\}$. For any $x, z \in X$, $\{y : y < x\} \cap \{y : y < z\}$ is either $\{y : y < x\}$ or $\{y : y < z\}$, since $\leq$ is linear. Thus $\mathcal{B}$ is a base for some topology on X, say $\mathcal{T}$. Let $\leq_{\mathcal{T}}$ be the relation on X defined by $\mathcal{T}$. We shall show that $\leq_{\mathcal{T}} = \leq$. If $a \leq b$ and $b \in N \in \mathcal{T}$ then there is a $B \in \mathcal{B}$ such that $b \in B \subset N$. If $B = X$ then $a \in B \subset N$. If $B \neq X$ then $B = \{y : y < c\}$ for some $c \in X$. Then since $a \leq b$ and $b < c$ we have $a < c$ and so $a \in B \subset N$. This means that every open set containing b contains a. Hence $b \in \overline{\{a\}}$ and $a \leq_{\mathcal{T}} b$. On the other hand, if $a \leq_{\mathcal{T}} b$ but $b < a$, then we have $b \in \{y : y < a\} \in \mathcal{T}$ but $a \notin \{y : y < a\}$, so that $b \notin \overline{\{a\}}$ and this contradicts $a \leq_{\mathcal{T}} b$. Hence $a \leq b$ since $\leq$ is a linear order. We have thus prove that $\leq_{\mathcal{T}} = \leq$. Hence by Theorem 6.4 $\mathcal{T}$ is a minimal topology on X. Consequently $\phi(\mathcal{T}) = \leq_{\mathcal{T}} = \leq$ and so ϕ is onto.

The following theorem gives the main result of this section.

<u>Theorem 6.6</u>. A topology $\mathcal{T}$ on a set X is an order topology iff $\mathcal{T}$ is T_1 and $\mathcal{T}$ is the least upper bound of two minimal T_o topologies.

<u>Proof</u> : Let $\mathcal{T}$ be an order topology and let $\leq$ be the associated linear order. Then the sets of the forms $\{y : y < x\}$ and $\{y : x < y\}$ form a subbase for $\mathcal{T}$. Clearly $\mathcal{T}$ is T_1. Let

$$\mathcal{B}_1 = \{\{y : y < x\} : x \in X\} \cup \{X\}$$

and

$$\mathcal{B}_2 = \{\{y : y > x\} : x \in X\} \cup \{X\} .$$

As in the proof of Theorem 6.5, $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively which are minimal T_o on X, and we have $\leq_{\mathcal{T}_1} = \leq$ and $\leq_{\mathcal{T}_2} = \leq^{-1}$ where $a \leq^{-1} b$ iff $b \leq a$. By Lemma 6.1 $\mathcal{B}_1 \cup \mathcal{B}_2$ is a subbase for $\mathcal{T}_1 \vee \mathcal{T}_2$. But as mentioned above $\mathcal{B}_1 \cup \mathcal{B}_2$ is also a subbase for $\mathcal{T}$. Hence $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ is the least upper bound of two minimal T_o topologies.

Conversely, assume that $\mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are minimal T_o and that $\mathcal{T}$ is T_1. Then we know that $\leq_{\mathcal{T}_1}$ and $\leq_{\mathcal{T}_2}$ are linear. We shall show that $\leq_{\mathcal{T}_1} = \leq^{-1}_{\mathcal{T}_2}$, i.e., $a \leq_{\mathcal{T}_1} b$ iff $b \leq_{\mathcal{T}_2} a$. For this purpose suppose $a \leq_{\mathcal{T}_1} b$, $a \leq_{\mathcal{T}_2} b$ and $a \neq b$, and let $G \in \mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2$ be such that $b \in G$. Then there exist $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$ such that $b \in G_1 \cap G_2 \subset G$. Since $a \leq_{\mathcal{T}_1} b$, $b \in \{\overline{a}\}_{\mathcal{T}_1}$ and so $a \in G_1$. Similarly $a \in G_2$. Thus $a \in G$. This means that every open set in $\mathcal{T}$ containing b also contains a, which is impossible since $\mathcal{T}$ is T_1. Hence for $a, b \in X$, $a \neq b$ and $a \leq_{\mathcal{T}_1} b$, we have $a \not\leq_{\mathcal{T}_2} b$ and since $\leq_{\mathcal{T}_2}$ is linear, we must have $b \leq_{\mathcal{T}_2} a$. Similarly $b \leq_{\mathcal{T}_2} a$, $b \neq a$ implies $a \leq_{\mathcal{T}_1} b$. Hence $\leq_{\mathcal{T}_1} = \leq^{-1}_{2}$. Now since $\mathcal{T}_1$ and $\mathcal{T}_2$ are minimal T_o, by Theorem 6.4 they have bases

$$\mathcal{B}_1 = \{\{y : y <_{\mathcal{T}_1} x\} : x \in X\} \cup \{X\}$$

and

$$\mathcal{B}_2 = \{\{y : y >_{\mathcal{T}_1} x\} : x \in X\} \cup \{X\}$$

respectively. By Lemma 6.1, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a subbase for $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}$. But since $\leq_{\mathcal{T}_1}$ is linear, $\mathcal{B}_1 \cup \mathcal{B}_2$ is the subbase for an order topology on X, which in this case must therefore be $\mathcal{T}$. Thus $\mathcal{T}$ is an order topology.

BIBLIOGRAPHY

1. Aull, C.E. and Thron, W.J., "Separation axioms between T_o and T_1", Indag. Math. 24 (1962), 26-37.

2. Berri, M.P., "Minimal topological spaces", Trans. Amer. Math. Soc. 108 (1963), 97-105.

3. Berri, M.P. and Sorgenfrey, R.H., "Minimal regular spaces", Proc. Amer. Math. Soc. 14 (1963), 454-458.

4. Bourbaki, N., "Elements of mathematics — general topology", Addison-Wesley Publishing Co., Inc., Mass. U.S.A.

5. Kelley, J.L., "General topology", D. Van Nostrand Co. Inc., Princeton, 1955.

6. Larson, R.E., "Minimal T_o-spaces and minimal T_D-spaces", Pacific Journ. Math. 31 (1969), 451-458.

7. Mah, P.F., "On some separation axioms", M.A. Thesis, U.B.C. (1965).

8. Pahk, Ki-Hyun, "Note on the characterization of minimal T_o and T_D spaces", Kyungpook Math. J. 8 (1968), 5-10.

9. Park, Young Sik, "The strengthening of topologies between T_o and T_1", Kyungpook Math. J. 8 (1968), 37-40.

10. Robinson, S.M. and Wu, Y.C., "A note on separation axioms weaker than T_1", J. Austral. Math. Soc. 9 (1969), 233-236.

11. Thron, W.J. and Zimmerman, S.J., "A characterization of order topologies by means of minimal T_o topologies", Proc. Amer. Math. Soc. Vol. 27 No. 1 (1971), 161-167.