REMOTE POINTS IN βR AND P-POINTS IN $\beta R - R$

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

APRIL, 1971

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Date APRIL 14, 1971

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ABSTRACT

We are going to study the remote points in βR and the P-points in $\beta R - R$. A remote point in βR is a point which is not in the βR chosure of any discrete subset of R. A point $p \in \beta R - R$ is a P-point of $\beta R - R$ if every G_δ-set containing p is a neighbourhood of p.

ACKNOWLEDGEMENTS

I am deeply indebted to Professor J.V. Whittaker for suggesting the topic of this thesis and for rendering invaluable assistance and encouragement throughout the course of my work. I would like to thank Professor T.E. Cramer for reading the final form of this work.

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INTRODUCTION

As we know, every completely regular space X has a compactification βX such that every function f in $C^*(X)$ has an extension to a function f^β in $C(\beta X)$. This thesis is devoted to study the papers [1], [3], [4], [5].

In chapter II, we study the class of subalgebras of $C(X)$ called β-subalgebras. With each β-subalgebra A of $C(X)$, we define A-points in $\beta X - X$. Then we study the A-points in chapter III. In chapter IV, we turn our attention to the remote points in βR. Finally, we study the prime ideal structure of $C(X)$.

CHAPTER I

PRELIMINARIES

Throughout this thesis, all given spaces are assumed to be completely regular and Hausdorff. $C(X)$ will denote the collection of all real-valued continuous functions on X, and $C^*(X)$ will denote the subcollection of bounded functions. Under the pointwise operation, $C(X)$ and $C^*(X)$ are commutative rings with identity. All ideals in $C(X)$ or $C^*(X)$, unmodified, will always mean proper ideals. If S is a set, then $|S|$ will denote the cardinality of S. As is standard, let c denote the cardinality $2^{\aleph_0}$ of the continuum. Furthermore, we assume the continuum hypothesis $(c = \aleph_1)$. If $S \subset X$, then $Cl_X S$, $int_X S$, $\partial_X S$ will denote, respectively, the closure, interior and boundary of S in X. If f is a function, then we let $f^{\leftarrow}$ denote the inverse map.

Definition (1.1) For $f \in C(X)$, $Z(f) = f^{\leftarrow}(0) = \{ x \in X : f(x) = 0 \}$ is called a zero set in X while $X - Z(f)$ is called a cozero set in X. The family $Z[C(X)]$ of all zero sets in X will be denoted by $Z(X)$.

Remark (1.2)

(1) The family $Z(X)$ of all zero sets is a base for the closed sets.

(2) f is a unit of $C(X)$ if and only if $Z(f) = \phi$

(3) Every zero set is a G_δ set.

<u>Definition</u> (1.3) Two subsets A and B of X are said to be completely separated in X if there exists a function $f \in C^*(X)$ such that $0 \leq f \leq 1$. $f[A] = \{0\}$, $f[B] = \{1\}$.

<u>Definition</u> (1.4) A subspace S of X is said to be C-embedded in X if every function in $C(S)$ can be extended to a function in $C(X)$. S is C^*-embedded in X if every function in $C^*(S)$ can be extend to a function in $C^*(X)$.

<u>Definition</u> (1.5) A non-empty family $\mathcal{F}$ of $Z(X)$ is called a z-filter on X provided that

(a) $\phi \notin \mathcal{F}$

(b) if $Z(f)$, $Z(g) \in \mathcal{F}$, then $Z(f) \cap Z(g) \in \mathcal{F}$

(c) if $Z(f) \in \mathcal{F}$, $Z(g) \in Z(X)$ and $Z(f) \subset Z(g)$, then $Z(g) \in \mathcal{F}$.

If in addition, $\mathcal{F}$ is not contained in any other z-filter, then $\mathcal{F}$ is called a z-ultrafilter on X.

<u>Theorem</u> (1.6)

(a) If I is an ideal [resp. maximal ideal] in $C(X)$, then $Z[I]=\{Z(f):f\in I\}$ is a z-filter [resp. z-ultrafilter] on X.

(b) If $\mathcal{F}$ is a z-filter [resp. z-ultrafilter] on X, then $Z^{\leftarrow}[\mathcal{F}]=\{f:Z(f)\in\mathcal{F}\}$ is an ideal [resp. maximal ideal] in $C(X)$.

Hence the mapping Z is one-one from the set of all maximal ideals in C onto the set of all z-ultrafilters.

Definition (1.7) An ideal I in $C(X)$ is called a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$.

Definition (1.8) A z-filter $\mathcal{F}$ in X is called a prime z-filter if $\mathcal{F}$ has the following property : whenever the union of two zero sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Definition (1.9) An ideal I in $C(X)$ or $C^*(X)$ is said to be fixed if $\bigcap Z[I] \neq \phi$. Otherwise I is said to be free .

Theorem (1.10)

(a) The fixed maximal ideals in $C(X)$ are precisely the sets

$$M_p = \{ f \in C : f(p) = 0 \} \qquad (p \in X) .$$

The ideals M_p are distinct for distinct p . For each p, C/M_p is isomorphic with the real field R ; in fact, the mapping $M_p(f) \longrightarrow f(p)$ is the unique isomorphism of C/M_p onto R .

(b) The fixed maximal ideals in $C^*(X)$ are precisely the sets

$$M_p^* = \{ f \in C^* : f(p) = 0 \} \qquad (p \in X) .$$

The ideals M_p^* are distinct for distinct p . For each p, C^*/M_p^* is isomorphic with the real field R ; in fact, the mapping $M_p^*(f) \longrightarrow f(p)$ is the unique isomorphism of C^*/M_p^* onto R .

Definition (1.11) For $p \varepsilon X$, let O_p denote the set of all f in C for which $Z(f)$ is a neighbourhood of p. If $M_p = O_p$, then p is called a P-point of X.

Remark (1.12) $p \varepsilon X$ is a P-point of X if and only if every G_δ containing p is a neighbourhood of p.

Remark (1.13)

(a) For $p \varepsilon X$, M_p is the only maximal ideal (fixed or free) containing O_p.

(b) If P is a prime ideal in C, and $P \subset M_p$, then $P \supset O_p$.

Definition (1.14) By a compactification of a space X, we mean a compact space in which X is dense.

Theorem (1.15) Every space X has a Stone-Cech compactification βX with the following equivalent properties :

(1) (Stone) Every continuous mapping τ from X into any compact space Y has a continuous extension $\bar{\tau}$ from βX into Y.

(2) (Stone-Cech) Every function f in $C^*(X)$ has an extension to a function f^β in $C(\beta X)$.

(3) (Cech) Any two disjoint zero sets in X have disjoint closures in βX.

(4) For any two zero sets Z_1 and Z_2 in X,

$$Cl_{\beta X}(Z_1 \cap Z_2) = Cl_{\beta X} Z_1 \cap Cl_{\beta X} Z_2 .$$

(5) If X is dense and C^*-embedded in T, then $X \subset T \subset \beta X$.

(6) If X is dense and C^*-embedded in T, then $\beta T = \beta X$. Furthermore, βX is unique, in the following sense : if a compactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of βX onto T that leaves X pointwise fixed.

Remark (1.16)

(1) For $S \subset X$. S is C^*-embedded in X if and only if $Cl_{\beta X} S = \beta S$.

(2) The mapping $f \longrightarrow f^\beta$ is an isomorphism of $C^*(X)$ onto $C(\beta X)$.

Theorem (1.17) The maximal ideals in $C^*(X)$ are precisely the sets

$$M^{*p} = \{ f \in C^*(X) : f^\beta(p) = 0 \} \qquad (p \in \beta X) ,$$

and they are distinct for distinct p . The maximal ideals in $C(X)$ are precisely the sets

$$M^p = \{ f \in C(X) : p \in Cl_{\beta X} Z_X(f) \} \qquad (p \in \beta X) ,$$

and they are distinct for distinct p .

Definition (1.18) Let M be a maximal ideal of $C(X)$. [resp. $C^*(X)$] . M is said to be a real ideal if C/M [resp. C^*/M] is isomorphic to the real field R . If M is not real, then we call M hyper-real .

Definition (1.19)

(a) X is said to be realcompact if every real maximal ideal in $C(X)$ is fixed.

(b) By a realcompactification of X, we mean a realcompact space in which X is dense.

(c) X is said to be pseudocompact if $C(X) = C^*(X)$.

Theorem (1.20) M^p is hyper-real if and only if M^{*p} contains a unit of C.

Theorem (1.21) Let νX denote the set of all points $p \in \beta X$ such that M^p is real. Then

(a) νX is the largest subspace of βX in which X is C-embedded.

(b) νX is the smallest realcompact space between X and βX . In particular, X is realcompact if and only if $X = \nu X$.

Theorem (1.22) Every (completely regular) space X has a realcompactification νX, contained in βX, with the following equivalent properties.

(1) Every continuous mapping τ from X into any realcompact space Y has a continuous extension τ^o from νX into Y . (Necessarily, $\tau^o = \overline{\tau}|\nu X$, where $\overline{\tau}$ is the Stone extension of τ into βY.)

(2) Every function f in $C(X)$ has an extension to a function f^ν in $C(\nu X)$. (Necessarily $f^\nu = f^*|\nu X$.) Furthermore, the space νX is unique,

in the following sense : if a realcompactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of νX onto T that leaves X pointwise fixed.

Theorem (1.23) If $f \in C(X)$, and αR denotes the one-point compactification of R, then there is a (unique) continuous function $f^* : \beta X \longrightarrow \alpha R$ which agrees with f on X .

Theorem (1.24) In the ring $C(X)$, and also in $C^*(X)$, the prime ideals containing a given prime ideal form a chain. (A chain is a totally ordered sets.)

CHAPTER II

β-SUBALGEBRAS

Let A be a commutative ring with an identity. Let $\mathcal{F}$ be the set of prime ideals in A . For $E \subset A$, define

$$V(E) = \{ P \in \mathcal{F} : E \subset P \} .$$

Note that

(1) $V(\phi) = \mathcal{F}$

(2) $V(A) = \phi$

(3) $V(\bigcup_{i \in \ell} E_i) = \bigcap_{i \in \ell} V(E_i)$ $\quad E_i \subset A$, $i \in \ell$, where ℓ is an index set .

(4) $V(E \cap F) = V(E) \cup V(F)$ $\quad E \subset A$, $F \subset A$.

Therefore the V's determine a topology on $\mathcal{F}$. This topology is called the hull-kernel topology.

Now for $a \in A$, define

$$V(a) = \{ P \in \mathcal{F} : a \in P \}$$

and let

$$\mathcal{F}_a = \mathcal{F} - V(a) .$$

Theorem (2.1)

(i) $\{ \mathcal{F}_a : a \in A \}$ is a basis of open sets for $\mathcal{F}$ with the hull-kernel topology.

(ii) $\mathcal{F}$ is compact.

<u>Proof</u> : (i) Let $\mathcal{B}$ be a closed subset in $\mathcal{F}$, then $\mathcal{B} = V(E)$ for some $E \subset A$. Now $P \in \mathcal{F} - \mathcal{B}$ if and only if $P \notin \mathcal{B}$ if and only if $E \not\subset P$ if and only if there exists $a \in E$ such that $a \notin P$ if and only if there exists $a \in E$ such that $P \in \mathcal{F}_a$. Thus $\mathcal{F} - \mathcal{B} = \bigcup_{a \in E} \mathcal{F}_a$.

(ii) Suppose $\mathcal{F} = \bigcup_{a \in E} \mathcal{F}_a$, $E \subset A$. Let $I = (E)$ = ideal generated by E. We claim $I = A$. Suppose $I \neq A$, then by Zorn's lemma $I \subset P$ for some $P \in \mathcal{F}$, then $P \in \mathcal{F}_a$ for some $a \in E$. Hence $a \notin P$. But $a \in E \subset I \subset P$, contradicting $a \notin P$. Therefore we must have $I = A$. So $1 = \sum_{i=1}^{r} b_i a_i$, $a_i \in E$, $b_i \in A$. Now for $P \in \mathcal{F}$, since $1 \notin P$, there exists i, $1 \leq i \leq n$ such that $a_i \notin P$. It follows that $P \in \mathcal{F}_{a_i}$. This proves that

$$\mathcal{F} = \mathcal{F}_{a_1} \cup \cdots \cup \mathcal{F}_{a_n}$$

and $\mathcal{F}$ is compact.

<u>Notation</u> (2.2) Let $\mathcal{M}_A$ denote the collection of maximal ideals in A endowed with the hull-kernel topology.

<u>Definition</u> (2.3) By a subalgebra A of $C(X)$, we mean a subalgebra in the usual sense which contains the constant functions.

Given a subalgebra A of $C(X)$. Define for each $p \in \beta X$,

$$M_A^p = \{ f \in A : (fg)^*(p) = 0 \text{ for all } g \in A \}$$

where f^* maps βX into the one point compactification of R as stated in 1.23 . Let

$$G_A = \{ M_A^p : p \in \beta X \} .$$

<u>Theorem</u> (2.4) M_A^p is a prime ideal in A, $p \in \beta X$.

<u>Proof</u> : Since $0 \in M_A^p$ and $1 \notin M_A^p$, we see that $M_A^p \neq \phi$ and $M_A^p \neq A$. Obviously M_A^p is an ideal in A . To prove that M_A^p is prime, it suffices to show that if $f, g \in A$ with $f, g \notin M_A^p$, then $fg \notin M_A^p$. Now let $f, g \in A$, choose $h, k \in A$ such that $(fh)^*(p) \neq 0$ and $(gk)^*(p) \neq 0$. Then $(fghk)^*(p) \neq 0$. Thus $fg \notin M_A^p$.

<u>Definition</u> (2.5) Let $\tau_A : \beta X \longrightarrow G_A$ be such that $\tau_A(p) = M_A^p$. A is said to be a β-subalgebra of $C(X)$ if τ_A is a homeomorphism of βX onto M_A .

<u>Remark</u> (2.6) $C^*(X)$ and $C(X)$ are β-subalgebras of $C(X)$.

For $f \in A$, let

$$\begin{aligned} S_A(f) &= \overleftarrow{\tau_A} \{ p \in G_A : f \in P \} \\ &= \{ p \in \beta X : f \in M_A^p \} \\ &= \bigcap_{g \in A} Z((fg)^*) . \end{aligned}$$

Since $Z((fg)^*)$ is closed in βX, $S_A(f)$ is closed in βX. Note that τ_A is continuous, since $\{ \{ p \in \mathcal{G}_A : f \in P \} : f \in A \}$ is a base for the closed sets in $\mathcal{G}_A$.

<u>Definition</u> (2.7) A subalgebra A of $C(X)$ is said to be β-determining if $\{ Z(f^*) : f \in A \}$ forms a base for the closed sets in βX. A is said to be closed under bounded inversion if f is a unit of A whenever $f \in A$ with $f \geq 1$.

<u>Definition</u> (2.8) An ideal I in A is said to be absolutely convex if $f \in I$ whenever $f \in A$ and $g \in I$ satisfying $|f| \leq |g|$.

For convenience, we shall abbreviate $\mathcal{M}_A$, M_A^p, $\mathcal{G}_A$, τ_A and S_A to $\mathcal{M}$, M^p, $\mathcal{G}$, τ and S, respectively.

<u>Theorem</u> (2.9) Given a subalgebra A of $C(X)$, the following are equivalent.

(1) A is β-determining

(2) $\mathcal{G}$ is Hausdorff and τ is one-to-one

(3) τ is a homeomorphism

<u>Proof</u> : (1) implies (2). Suppose A is β-determining and let $p,q \in \beta X$ with $p \neq q$. By [2, 6.5(b)], there exists Z_1, $Z_2 \in Z(X)$ such that $Z_1 \cup Z_2 = X$ and $p \notin Cl_{\beta X}Z_1$, $q \notin Cl_{\beta X}Z_2$. Since A is β-determining, $\{ Z(f^*) : f \in A \}$ is a base for the closed sets in βX. So we can choose

$f, g \in A$ such that $p \notin Z(f^*) \supset Cl_{\beta X} Z_1$ and $q \notin Z(g^*) \supset Cl_{\beta X} Z_2$. By the choice above, $f \notin M^p$. Thus $M^p \in G - \{ M^s \in G : f \in M^s \}$ which is an open set in G . Similarly $M^q \in G - \{ M^s \in G : g \in M^s \}$ which is an open set in G . Furthermore by the choice of f, g, we see that $fg = 0$. Thus $\{ M^s \in G : f \in M^s \} \cup \{ M^s \in G : g \in M^s \} = G$. So $G - \{ M^s \in G : f \in M^s \}$ and $G - \{ M^s \in G : g \in M^s \}$ are disjoint open sets in G . Since p, q are arbitrary, G is Hausdorff. Since $M^p \neq M^q$, τ is one-to-one .

(2) implies (3) . It suffices to prove that τ is closed. Let F be a closed set in βX . Since βX is compact, F is compact . Since τ is continuous, $\tau[F]$ is compact. Since G is Hausdorff, $\tau[F]$ is closed.

(3) implies (1) . Let F be a closed set in βX and $p \in \beta X$ with $p \notin F$. Since τ is a homeomorphism, $\{ S(f) : f \in A \}$ is a base for the closed sets in βX . Thus there exists $f \in A$ such that $p \notin S(f)$ and $F \subset S(f)$. Since $S(f) = \bigcap_{g \in A} Z((fg)^*)$, $(fg)^*(p) \neq 0$ for some $g \in A$. Thus $p \notin Z(f^*)$; but $F \subset S(f) \subset Z((fg)^*)$. This proves that $\{Z(f^*) : f \in A\}$ is a base for the closed sets in βX .

<u>Theorem</u> (2.10) Given a subalgebra A of $C(X)$, the following are equivalent.

(1) A is closed under bounded inversion.

(2) If I is an ideal in A, then $\bigcap_{f \in I} Z(f^*) \neq \phi$.

(3) Every ideal in A is contained in some M^p .

(4) $M_A \subset G_A$.

(5) Every $M \in \mathcal{M}_A$ is absolutely convex .

Proof : (1) implies (2). Let I be an ideal in A . Let $\mathcal{F}=\{Z(f^*): f \in I\}$. To prove (2), by the compactness of βX , it suffices to show that $\mathcal{F}$ has the finite intersection property. Let $f_1, \cdots, f_n \in I$ and let $g=f_1^2+\cdots+f_n^2 \in I$. Then $Z(g^*) = \bigcap_{i=1}^{n} Z(f_i^*)$. Suppose $Z(g^*) = \phi$. Then $|g^*(p)| > 0$ for all $p \in \beta X$. Since βX is compact, there exists $r > 0$ such that $|g^*(p)| \geq r > 0$. So $g \geq r$, and g is a unit of A . Since $g \in I$ and since I is proper, this is a contradiction. So we must have $Z(g^*) = \phi$.

(2) implies (3) . Let I be an ideal in A . Let $p \in \bigcap_{f \in I} Z(f^*)$. We claim that $I \subset M^p$. For if $f \in I$, then $fg \in I$ for all $g \in A$. So $(fg)^*(p) = 0$, for all $g \in A$. So $f \in M^p$.

(3) implies (4) . Obvious.

(4) implies (5). It suffices to show that M^p is absolutely convex. Let $f \in A$ and $g \in M^p$ satisfying $|f| \leq |g|$. Then $|fh| \leq |gh|$ for all $h \in A$. Since X is dense in βX , $|(fh)^*| \leq |(gh)^*|$ for all $h \in A$. So $f \in M^p$.

(5) implies (1). Since 1 does not belong to any maximal ideal, it follows that f is a unit of A whenever $f \in A$ with $f \geq 1$.

This completes the proof .

<u>Theorem</u> (2.11) Given a subalgebra A of C(X), the following are equivalent.

(1) A is a β-subalgebra of C(X) .

(2) A is β-determining and closed under bounded inversion.

<u>Proof</u> : Suppose A is a β-subalgebra of C(X) . By 2.9, A is β-determining. By 2.10, A is closed under bounded inversion.

Conversely suppose (2) holds. By 2.9, τ is a homeomorphism of βX onto $\mathcal{G}$. By 2.10, $\mathcal{M} \subset \mathcal{G}$. Since $\mathcal{G}$ is T_2 , no two ideals of $\mathcal{G}$ are comparable. So $\mathcal{M} = \mathcal{G}$. This proves that A is a β-subalgebra of C(X) .

CHAPTER III

THE A-POINTS OF $\beta X - X$

Let A be a β-subalgebra of $C(X)$. By 2.9, the family $\{S(f): f\varepsilon A\}$ forms a base for the closed sets in βX. Let X^* denote $\beta X - X$. For $f \varepsilon A$, let $S^*(f) = S(f) \cap X^*$. Then $\{ S^*(f) : f \varepsilon A \}$ is a base for the closed sets in X^*. For convenience, let us agree that the symbols "Cl", "int" and "∂", without subscripts, refer to the topology of X^*.

<u>Definition</u> (3.1) A space X is said to have the G_δ-property if every nonvoid G_δ subset of X has a nonvoid interior.

<u>Remark</u> (3.2) Since in a completely regular space X, every G_δ containing a compact set S contains a zero set containing S, it follows that X has the G_δ-property if and only if every nonempty zero set in X has a nonempty interior.

The following theorem will be used several times throughout this thesis : Let Y be a nonvoid locally compact Hausdorff space with the G_δ-property. If $\mathcal{D}$ is a family of at most $\aleph_1$ dense open subsets of Y, then $\cap \mathcal{D}$ is dense in Y. If, in addition, Y has no isolated points, then $|\cap \mathcal{D}| \geq 2^{\aleph_1}$. ([5, 3.2]).

<u>Definition</u> (3.3) Given a β-subalgebra A of $C(X)$, a point $p \varepsilon X^*$ is said to be an A-point of X^* if, for all $f \varepsilon A$, $p \notin \partial S^*(f)$.

Remark (3.4)

(1) A point $p \in X^*$ is an A-point of X^* if and only if $S^*(f)$ is a neighbourhood of p whenever $f \in A$ and $p \in S^*(f)$.

(2) The set of A-points of X^* is precisely $\bigcap_{f \in A} (X^* - \partial S^*(f))$.

Theorem (3.5) X is realcompact if and only if for every $p \in X^*$, there is a $Z \in Z(\beta X)$ such that $p \in Z \subset X^*$.

Proof : Suppose X is realcompact and $p \in X^*$. Then M^p is hyperreal by [2, 8.4] . By 1.20, M^{*p} contains a unit f of $C(X)$. Since f is a unit of $C(X)$, it follows that $Z(f^\beta) \subset X^*$. By 1.17, $p \in Z(f^\beta)$. This proves the necessity.

Conversely, let $p \in X^*$. By assumption, there exists $Z(g) \in Z(\beta X)$ such that $p \in Z(g) \subset X^*$. Then $g(x) \neq 0$ for all $x \in X$. So the restriction of g on X is a unit of $C(X)$. Since $g(p) = 0$, $g \in M^{*p}$. By 1.20, M^p is hyperreal . This proves that X is real compact .

Theorem (3.6) Suppose X is a locally compact and realcompact space, then X^* has the G_δ property.

Proof : By remark 3.2, it suffices to prove that every nonempty zero set Z in X^* has nonempty interior. Since X is locally compact, by [2, 6.9(d)], X is open in βX. So X^* is closed. Since βX is compact and Hausdorff, βX is normal. So X^* is C^*-embedded in βX by [2, 3D]. Therefore

$Z = Z(f) \cap X^*$ for some $f \in C(\beta X)$. Let $p \in Z$. By 3.5, there exists a function $g \in C(\beta X)$ such that $g(p) = 0$ but $g(x) \neq 0$ for all $x \in X$. Define $h = |f| + |g|$, then $p \in Z(h) \subset Z \cap X^*$. Now let $\{x_\alpha\}$ be a set in X converging to p. By continuity of h, $\{h(x_\alpha)\}$ converges to $h(p) = 0$. Obviously we can choose a subsequence $\{x_{\alpha_i}\}$ of distinct points of $\{x_\alpha\}$ such that $h(x_{\alpha_i}) \longrightarrow 0$. By induction, choose disjoint compact neighbourhood V_i of x_{α_i} such that $|h(x) - h(x_{\alpha_i})| < \frac{1}{i}$ for $x \in V_i$. By complete regularity of X, there exists a function w_i such that $0 \leq w_i \leq 1$, $w_i(x_{\alpha_i}) = 1$, $w_i[X - V_i] = 0$. Let $w = \sum_{i=1}^{\infty} w_i$, w is well defined provided that $\{x_{\alpha_i}\}$ has no limit point in X; but in fact, $\{X_{\alpha_i}\}$ cannot has a limit point in X by the fact that h is not zero at any point of X. Note that $w(x_{\alpha_i}) = 1$ for each i and $w(x) = 0$ for $x \in X - \bigcup_{i=1}^{\infty} V_i$. Now suppose $w^\beta(q) \neq 0$ for some $q \in X^*$, we see that every neighbourhood of q meets infinitely many V_i's. Thus $h(q) = 0$. This proves that $X^* - (Z(w^\beta) \cap X^*) \subset Z(h)$. Since βX is compact, $\{x_{\alpha_i}\}$ has a limit point q in βX. As proved already $q \in X^*$. Thus there exists a subsequence $\{x_{\alpha_{i_n}}\}$ of $\{x_{\alpha_i}\}$ such that $w^\beta(x_{\alpha_{i_n}}) \longrightarrow w^\beta(q)$. But $w^\beta(x_{\alpha_{i_n}}) = 1$ for all n, it follows that $w^\beta(q) = 1$. So $X^* - (Z(w^\beta) \cap X^*) \neq \phi$. Since $Z(h) \subset Z$ and $X^* - (Z(w^\beta) \cap X^*)$ is open, this proves the theorem.

<u>Theorem</u> (3.7) If X is realcompact, then X^* has no isolated points.

Proof : Suppose p is an isolated point in X^*. Then there exists a zero set neighbourhood $Z(f)$ of p in βX such that $Z(f) \cap X^* = \{p\}$. By 3.5, there exists $Z(h) \in Z(\beta X)$ such that $p \in Z(h) \subset X^*$. So $\{p\} = Z(f) \cap Z(h) \in Z(\beta X)$. So $\{p\}$ is a zero set in βX. Since $\{p\}$ is disjoint from X, by [2, 9.5], $\{p\}$ contains a copy of N. This leads to a contradiction.

Theorem (3.8) Let X be a locally compact and real compact metric space. Let A be a β-subalgebra of $C(X)$ with $|A| = C$. If, in addition, X is not compact, then X^* has a dense subset of 2^c A-points.

Proof : Let $\mathcal{D} = \{ X^* - \partial S^*(f) : f \in A \}$. Obviously, for each $f \in A$, $X^* - \partial S^*(f)$ is an open dense subset of X^*. By 3.6, X^* has the G_δ property. By 3.7, X^* has no isolated points. Now apply [5, 3.2], we see that $\cap \mathcal{D}$ is dense in X^* and $|\cap \mathcal{D}| \geq 2^c$. Since A is a β-subalgebra of $C(X)$, $|X^*| \leq 2^{|A|} = 2^c$. So $|\cap \mathcal{D}| = 2^c$. By remark 3.4 (b), $\cap \mathcal{D}$ is precisely the set of A-points of X^*.

Theorem (3.9) Let X be a locally compact and realcompact but not compact metric space. Let $\{ A_\alpha : \alpha \in \Delta \}$ be a family of β-subalgebras of $C(X)$ with $|A_\alpha| = C$ for each $\alpha \in \Delta$ and $|\Delta| \leq C$, then X^* has a dense subset of 2^c points which are simultaneously A_α-points for all $\alpha \in \Delta$.

Proof : Let $\mathcal{D} = \{ X^* - \partial S^*_{A_\alpha}(f) : f \in A_\alpha, \alpha \in \Delta \}$. Then $\cap \mathcal{D} = \bigcap_{\alpha\in\Delta} \bigcap_{f\in A_\alpha} (X^* - \partial S^*_{A_\alpha}(f))$ is precisely the set of points of X^* that are similtaneously A_α-points for all $\alpha \in \Delta$. Applying [5, 3.2] again, $\cap \mathcal{D}$ is dense in X^* and $|\cap \mathcal{D}| \geq 2^c$. Since A_α is a β-subalgebra of $C(X)$, $|X^*| \leq 2^{|A_\alpha|} = 2^c$. So $|\cap \mathcal{D}| = 2^c$.

<u>Theorem</u> (3.10) A point in X^* is a $C^*(X)$-point if and only if it is a P-point of X^* .

<u>Proof</u> : Since $M^{*p} = \{ f \in C^*(X) : f^\beta(p) = 0 \}$, we see that $S_{C^*}(f) = Z(f^\beta)$, $f \in C^*(X)$. So $S^*_{C^*}(f) = X^* \cap Z(f^\beta)$. Now by definition, a point in X^* is a P-point of X^* if and only if it is not an element of the X^*-boundary of any zero set of X^* , and is a $C^*(X)$-point if and only if $p \notin \partial S^*_{C^*}(f) = \partial(X^* \cap Z(f^\beta))$ for all $f \in C^*(X)$. Obviously a P-point is a $C^*(X)$-point.

Conversely suppose p is not a P-point. Then there exists $Z_1 \in Z(X^*)$ such that $p \in \partial Z_1$. Let S be a G_δ-set of βX such that $S \cap X^* = Z_1$. By [2, 3.11 (b)], there exists a $Z_2 \in Z(\beta X)$ such that $p \in Z_2 \subset S$. Then $p \in \partial(Z_2 \cap X^*)$. This proves that p is not a $C^*(X)$-point.

<u>Corollary</u> (3.11)

(1) $\beta N - N$ has a dense subset of 2^c P-points .

(2) $\beta R - R$ has a dense subset of 2^c P-points .

Proof : (1) Obviously N is locally compact and realcompact but not compact. Furthermore $|C^*(N)| = C$. Applying 3.8, $\beta N - N$ has a dense subset of 2^c $C^*(N)$ points. By 3.10, $\beta N - N$ has a dense subset of 2^c P-points.

(2) R is obviously locally compact and realcompact but not compact. Since R is separable, $|C^*(R)| = C$. Applying 3.8 and 3.10, p has a dense subset of 2^c P-points.

CHAPTER IV

REMOTE POINTS IN βR

In this chapter, we shall turn our attention to the remote points in the space βR, the Stone Čech compactification of the space R of real numbers. As in [2], we associate with each maximal ideal M^p in $C(R)$ the z-ultrafilter

$$A^p = \{ Z(f) : f \in M^p \} = \{ Z \in Z(R) : p \in Cl_{\beta R} Z \} .$$

For $p \in \beta R$, we denote by O^p the set of all $f \in C(R)$ for which $Cl_{\beta R} Z(f)$ is a neighbourhood of p, i.e.

$$O^p = \{ f \in C(R) : p \in int_{\beta R} Cl_{\beta R} Z(f) \} .$$

<u>Definition</u> (4.1) A point $p \in \beta R$ is said to be a remote point in βR if p is not in the βR closure of any discrete subset of R.

<u>Theorem</u> (4.2) $\beta R - R$ has a dense subset of 2^c C-points.

<u>Proof</u> : Since R is separable, $|C(R)| = C$. By 3.8, it is immediate that $\beta R - R$ has a dense subset of 2^c C-points.

<u>Lemma</u> (4.3) If Z is a closed nowhere dense set in R, then there exists a discrete subset D of R such that $D \cap Z = \phi$, $D \cup Z = Cl_R D$.

Proof : Since Z is closed, $R - Z$ is open . As an open set in R, $R - Z$ is a union of disjoint open intervals I_α . For each I_α , choose a discrete subset $D_\alpha \subset I_\alpha$ such that the endpoints of I are the only limit points of D_α . Put $D = \bigcup_\alpha D_\alpha$. Obviously $D \cap Z = \phi$ and $D \cup Z = Cl_R D$.

Theorem (4.4) For $p \in \beta R$, the following are equivalent :

(1) p is a remote point in βR .

(2) A^p has no nowhere dense member .

(3) $M^p = O^p$.

(4) p is a C-point of $\beta R - R$.

(5) M^p is a minimal prime ideal .

(6) O^p is prime .

Proof : (1) => (2) . Suppose that A^p has a nowhere dense member Z . By 4.3, there is a discrete subset D of R such that $Z \cap D = \phi$ and $Z \cup D = Cl_R D$, so that $Cl_{\beta R} Z \subset Cl_{\beta R} D$. Hence $p \in Cl_{\beta R} Z \subset Cl_{\beta R} D$. Therefore p is not a remote point in βR .

(2) =>(1) . Suppose p is not a remote point in βR . Then there is a discrete subset D of R such that $p \in Cl_{\beta R} D$. Clearly $Cl_{\beta R} D \in A^p$. We claim $int_R Cl_R D = \phi$. Suppose, on the contrary, that $int_R Cl_R D \neq \phi$. Then $(int_R Cl_R D) \cap D \neq \phi$. Let $q \in (int_R Cl_R D) \cap D$. Since D is discrete, q is open in D. So $\{q\} = D \cap G$ for some open set G in R. Obviously $\{q\} \subset G \cap (int_R Cl_R D)$. Conversely, let $r \in G \cap (int_R Cl_R D)$. Then r is

either a point of D or a limit point of D. If r is a point of D, then $r \in D \cap G$. Hence $r = q$. If r is a limit point of D, then $G \cap D$ contains infinitely many points of D. This contradicts the fact that $D \cap G$ is a singleton set. So this cannot be the case, and $\{q\} = G \cap (\mathrm{int}_R \mathrm{Cl}_R D)$. This proves that $\{q\}$ is open in R, i.e. q is an isolated point in R. But this cannot be true. So we must have the fact that $\mathrm{int}_R \mathrm{Cl}_R D = \phi$. So A^p has a nowhere dense member.

(2) => (3). Suppose that A^p has no nowhere dense member. Let $f \in M^p$. Since $\mathrm{Cl}_R(R - Z(f))$ is a closed set in R, by Urysohn's lemma there exists a function $g \in C(R)$ such that $Z(g) = \mathrm{Cl}_R(R - Z(f))$. Thus $R = Z(f) \cup Z(g)$. We claim $p \notin \mathrm{Cl}_{\beta R} Z(g)$. Suppose not, then $p \in \mathrm{Cl}_{\beta R} Z(f) \cap \mathrm{Cl}_{\beta R} Z(g)$. By theorem 1.15, (4), $p \in \mathrm{Cl}_{\beta R} Z(f) \cap \mathrm{Cl}_{\beta R} Z(g) =$ $= \mathrm{Cl}_{\beta R}(Z(f) \cap Z(g)) = \mathrm{Cl}_{\beta R} \partial_R Z(f)$. This proves that $\partial_R Z(f) \in A^p$. Since $\partial_R Z(f)$ is nowhere dense, this contradicts our hypothesis that A^p has no nowhere dense member. So $p \notin \mathrm{Cl}_{\beta R} Z(g)$. So $p \in \beta R - \mathrm{Cl}_{\beta R} Z(g) \subset \mathrm{Cl}_{\beta R} Z(f)$. Since $\mathrm{Cl}_{\beta R} Z(g)$ is closed, $\beta R - \mathrm{Cl}_{\beta R} Z(g)$ is open. This proves that $\mathrm{Cl}_{\beta R} Z(f)$ is a neighbourhood of p. Thus $f \in O^p$.

(3) => (4). Suppose that $O^p = M^p$. For any $f \in C(R)$ and $p \in S_c^*(f) = S_c(f) \cap (\beta R - R) = (\mathrm{Cl}_{\beta R} Z(f)) \cap (\beta R - R)$, then $f \in M^p$, whence $f \in O^p$. Thus $p \in \mathrm{int}_{\beta R} \mathrm{Cl}_{\beta R} Z(f)$. Thus p is in the interior of $S_c^*(f)$ in $\beta R - R$. By remark 3.4, (1), this proves that p is a C-point of $\beta R - R$.

(4) => (2). Suppose that p is a C-point of $\beta R - R$, and let $Z \in A^p$. We shall show that Z is not nowhere dense. Since $Z \in A^p$,

$p \in Cl_{\beta R}Z$. So $p \in (Cl_{\beta R}Z) \cap (\beta R - R) = S_c(f) \cap (\beta R - R) = S_c^*(f)$. Since p is a C-point, by remark 3.4, (1), p is in the interior of $S_c^*(f)$ in $\beta R - R$. Thus $p \in int_{\beta R}Cl_{\beta R}Z$. Obviously $(int_{\beta R}Cl_{\beta R}Z) \cap R \neq \phi$ and is a subset of Z . This proves that Z is not nowhere dense.

(2) => (5) . Assume (2) . Suppose, on the contrary, that M^p is a nonminimal prime ideal. Let I be a prime ideal properly contained in M^p . Choose $Z \in Z[M^p] - Z[I] = A^p - Z[I]$. Since $R = Z \cup Cl(R - Z)$ and $Z \notin Z[I]$, it follows that $Cl(R - Z) \in Z[I]$. So $Cl(R - Z) \in M^p$. Thus $\partial_R Z = Z \cap Cl(R - Z) \in M^p$. Obviously $\partial_R Z$ is nowhere dense. This contradicts our hypothesis. So M^p is a minimal prime ideal.

(5) => (3) . Assume (5). By [2, 2.8], O^p is the intersection of all the prime ideals contained in M^p . Since M^p is a minimal prime ideal, it follows that $M^p = O^p$.

(3) => (6) . Obvious .

(6) => (5) . Suppose M^p is not a minimal prime ideal. Since (5) and (2) are equivalent, it follows that A^p has a nowhere dense member Z . Choose disjoint discrete subsets D_1, D_2 of R such that $D_i' = Z$, $i = 1, 2$, where D_i' denotes the derived set of D_i in R . Let $G_i = Cl_R D_i$, $i = 1, 2$. Obviously $Cl_R(G_i - Z) \in A^p$. By [4, 4.2], A^p has a prime z-filter $\mathcal{F}_i$ containing G_i but not Z, for $i = 1, 2$. Since $G_1 \cap G_2 = Z$, we see that $\mathcal{F}_1$ and $\mathcal{F}_2$ are incomparable. Thus $Z^{\leftarrow}[\mathcal{F}_1]$ and $Z^{\leftarrow}[\mathcal{F}_2]$ are incomparable . Since $\mathcal{F}_i$ is a prime z-filter in A^p, $Z^{\leftarrow}[\mathcal{F}_i]$ is a prime ideal contained in M^p , $i = 1, 2$. By [2, 7.5], $Z^{\leftarrow}[\mathcal{F}_i]$ contains O^p , $i = 1, 2$. By 1.23 , we see that O^p is not prime.

<u>Theorem</u> (4.5) $\beta R - R$ has a dense subset of 2^c remote points in βR .

<u>Proof</u> : Follows immediately from 4.2 and 4.4 .

<u>Theorem</u> (4.6) $\beta R - R$ has a dense subset of 2^c points which are simultaneously remote points in βR and P-points of $\beta R - R$.

<u>Proof</u> : Apply 3.9 to the family $\{ C(R), C^*(R) \}$ of β-subalgebras of $C(R)$. Then $\beta R - R$ has a dense subset of 2^c points which are simultaneously C^*-points and C-points of $\beta R - R$. By 3.10, C^*-points of $\beta R - R$ are precisely the P-points of $\beta R - R$. By 4.4, C-points of $\beta R - R$ are precisely the remotes points in βR .

<u>Theorem</u> (4.7) $\beta R - R$ has a dense subset of 2^c points which are P-points of $\beta R - R$ but not remote points in βR .

<u>Proof</u> : Let V be a closed neighbourhood in βR of any point in $\beta R - R$. Obviously $V \cap R$ is not pseudocompact. Since $V \cap R$ is closed, by [2, 1.18], it is C-embedded in R . Thus by [2, 1.20], $V \cap R$ contains a copy D of N which is C-embedded in R . Since D is C^*-embedded in R , by 1.16, $\beta D = Cl_{\beta R} D$. Since V is closed in βR , we see that $D^* = \beta D - D = Cl_{\beta R} D - D \subset V \cap R^*$. Since $\beta D - D$ is homeomorphic with $\beta N - N$, by 3.11, $\beta D - D$ has 2^c P-points of $\beta D - D$. By [2, 9 M.2] , we see that a point in $\beta D - D$ is a P-point of $\beta D - D$ if and only if it is a P-point of $\beta R - R$, that $\beta D - D$ has 2^c P-points of $\beta R - R$. Since

D is discrete, no point of $\beta D - D$ is a remote point of βR. Since V is arbitrary, this proves the theorem.

Definition (4.8) A space X is said to be an F-space if every cozero set in X is C^*-embedded in X.

Remark (4.9) By [2, 14.27], $\beta N - N$ is a compact F-space and so is $\beta R - R$.

Lemma (4.10) Every infinite compact F-space has at least 2^c non P-points.

Proof : Let X be an infinite compact F-space. Since X is infinite, there is a countable discrete subset $D = \{ d_n : n \in N \}$. By [2, 14 N.5], D is C^*-embedded in X. So $Cl_X D = \beta D$ by 1.16. Let $f \in C^*(X)$ be such that $f(d_n) = n^{-1}$, $n \in N$. Then for any $p \in D^* = \beta D - D = Cl_X D - D$, $p \in Z(f)$, but obviously $Z(f)$ is not a neighbourhood of p. Thus p is not a P-point. Since $|\beta D - D| = 2^c$, this proves the lemma.

Theorem (4.11) $\beta R - R$ has a dense subset of 2^c points which are neither remote points in βR nor P-points of $\beta R - R$.

Proof : Let V be a closed neighbourhood in βR of any point in $\beta R - R$. As in the proof of 4.7, $V \cap R^*$ contains a copy $D^* = \beta D - D$ of $\beta N - N$. By remark 4.9, D^* is a compact F-space. By 4.10, D^* has at least 2^c non P-points of D^*. So by [2, 9M.2], D^* has at least 2^c non P-points of $\beta R - R$. Since D is discrete, no point of $\beta D - D$ is a remote point of βR. This proves the theorem.

<u>Theorem</u> (4.12) $\beta R - R$ has a dense subset of 2^c points which are remote points in βR but not P-points of $\beta R - R$.

<u>Proof</u> : Let V be a closed neighbourhood in βR of any point in $\beta R - R$. By [5, 5.5], there exists an infinite compact set Δ of remote points in βR such that $\Delta \subset V \cap (\beta R - R)$. Since $\beta R - R$ is an F-space by 4.9, the C^*-embedded subset Δ is also an F-space by [2, 14.26]. By 4.10, Δ has 2^c non P-points. By [2, 4L.2], each of these points is a non P-point of $\beta R - R$. This proves the theorem .

CHAPTER V

PRIME IDEAL STRUCTURE AND REMOTE POINTS

Definition (5.1) Let $P(X)$ denote the family of all prime z-filters on X. A prime z-filter is said to be minimal if it is a minimal element of $P(X)$. For $\mathcal{A}, \mathcal{B} \in P(X)$, if $\mathcal{A} \subset \mathcal{B}$, we say that $\mathcal{A}$ is a predecessor of $\mathcal{B}$ and that $\mathcal{B}$ is a successor of $\mathcal{A}$. If in addition there is no prime z-filter between them, we use the term immediate predecessor and immediate successor.

Theorem (5.2) Let $\mathcal{A}$ be a prime z-filter on X. Suppose there exists $Z \in \mathcal{A}$ such that for any zero set $W \notin \mathcal{A}$, $Z \cup W \neq X$. Then $\mathcal{A}$ is non-minimal.

Proof : For any $E \subset X$, let

$$z(E) = \{ Z \in Z(X) : E \subset Z \} .$$

By assumption, we have $z(X - Z) \subset \mathcal{A}$. Now let

$$\mathcal{B} = \{ W \in Z(X) : z(W - Z) \subset \mathcal{A} \} .$$

Since $X \in \mathcal{B}$, $\mathcal{B} \neq \phi$. Furthermore $\mathcal{B}$ has the following properties :

(i) $\mathcal{B}$ is closed under supersets : Let $W \in \mathcal{B}$ and let $V \in Z(X)$ such that $W \subset V$. Obviously $z(V - Z) \subset z(W - Z)$ and hence $z(V - Z) \subset \mathcal{A}$.

Thus $V \in \mathcal{B}$.

(ii) for any $W_1, W_2 \in Z(X)$, if $W_i \notin \mathcal{B}$ for $i = 1, 2$, then $W_1 \cup W_2 \notin \mathcal{B}$: choose $V_i \in z(W_i - Z) - \mathcal{A}$ for $i = 1, 2$. Since $\mathcal{A}$ is prime, $V_1 \cup V_2 \notin \mathcal{A}$. On the other hand, it is obvious that $V_1 \cup V_2 \in z(W_1 \cup W_2 - Z)$ and by definition of $\mathcal{B}$, $W_1 \cup W_2 \notin \mathcal{B}$.

Now applying Zorn's lemma, there exists a z-filter $\mathcal{F}$ which is maximal among the z-filters contained in $\mathcal{B}$. Note that $Z \notin \mathcal{F}$. Furthermore, for any $W \in \mathcal{F}$, $W \in z(W - Z) \subset \mathcal{A}$, so that $W \in \mathcal{A}$. Thus $\mathcal{F} \subset \mathcal{A}$, $\mathcal{F} \neq \mathcal{A}$. Finally we shall prove that $\mathcal{F}$ is prime. Let Z_1 , $Z_2 \in Z(X)$ with $Z_1 \cup Z_2 \in \mathcal{F}$. Suppose $Z_i \notin \mathcal{F}$ for $i = 1, 2$. By the maximality of $\mathcal{F}$, there is $W_i \in \mathcal{F}$ such that $W_i \cap Z_i \notin \mathcal{B}$, for $i = 1, 2$. Setting $W = W_1 \cap W_2$, obviously $W \cap (Z_1 \cup Z_2) \in \mathcal{F}$. Since $\mathcal{B}$ is closed under supersets, $W \cap Z_i \notin \mathcal{B}$, $i = 1, 2$. By property (ii) of $\mathcal{B}$, we see that $W \cap (Z_1 \cup Z_2) \notin \mathcal{B}$. Thus $W \cap (Z_1 \cup Z_2) \notin \mathcal{F}$, and this leads to a contradiction . Thus we must have that $\mathcal{F}$ is prime, and hence $\mathcal{F}$ is an immediate predecessor of $\mathcal{A}$. So $\mathcal{A}$ is non-minimal .

<u>Theorem</u> (5.3) For each $p \in \beta X$, every prime ideal P of $C^*(X)$ contained in M^{*p} is comparable with $M^p \cap C^*$.

<u>Proof</u> : Obviously $M^p \cap C^*$ is a prime ideal contained in C^* . Choose a minimal prime ideal J such that $J \subset P$. By 1.24, it suffices to show that $J \subset M^p \subset C^*$. To show this, we first pass to the ring $C(\beta X)$ by means of the canonical isomorphism $f \longrightarrow f^\beta$ of $C^*(X)$ onto $C(\beta X)$, and

then we pass to the family of prime z-filters on βX .

Since $M^p = \{ f \in C(X) : p \in Cl_{\beta X} Z_X(f) \}$, the prime ideal in $C(\beta X)$ corresponding to $M^p \cap C^*$ is given by

$$(M^p \cap C^*)^\beta = \{ g \in C(\beta X) : p \in Cl_{\beta X} Z_X(g|X) \}$$

we claim $(M^p \cap C^*)^\beta$ is a z-ideal. Let $Z_{\beta X}(f) \in Z_{\beta X}((M^p \cap C^*)^\beta)$, then $Z_{\beta X}(f) = Z_{\beta X}(g)$ for some $g \in C(\beta X)$. Hence $Z_X(f|X) = Z_{\beta X}(f) \cap X =$ $= Z_{\beta X}(g) \cap X = Z_X(g|X)$, whence $p \in Cl_{\beta X} Z_X(f|X)$. This proves that $f \in (M^p \cap C^*)^\beta$ and hence $(M^p \cap C^*)^\beta$ is a z-ideal. Now let us denote the corresponding prime z-filter on βX by K^p ; obviously

$$K^p = \{ Z \in Z(\beta X) : p \in Cl_{\beta X}(Z \cap X) \} \quad .$$

Also by [2, 14.7], the minimal prime ideal $\tilde{J}^\beta$ of $C(\beta X)$ corresponding to J is a z-ideal ; let $\mathcal{B}$ denote the corresponding minimal prime z-filter on βX . Now we are going to show that $\mathcal{B} \subset K^p$. Let $Z \in \mathcal{B}$. To show that $Z \in K^p$, it suffices to show that $p \in Cl_{\beta X}(Z \cap X)$. Now let V be any zero set neighbourhood of p . By [2, 7.15], $V \in \mathcal{B}$ and hence $V \cap Z \in \mathcal{B}$. Since $\mathcal{B}$ is minimal, applying theorem 5.2, we can choose a zero set W not in $\mathcal{B}$ such that $(V \cap Z) \cup W = \beta X$. If $int_{\beta X}(V \cap Z) = \phi$, then W is dense in βX and hence $W = \beta X$. Thus $W \in \mathcal{B}$, but this is impossible. So we see that $int(V \cap Z) \neq \phi$, and $(V \cap Z) \cap X \neq \phi$, whence $p \in Cl_{\beta X}(Z \cap X)$ and $Z \in K^p$. Thus $\mathcal{B} \subset K^p$ and hence $J \subset M^p \cap C^*$.

Definition (5.4) If $Y \subset X$ and $\mathcal{F}$ is a z-filter on Y, it is clear that

$$\mathcal{F}^{\#} = \{ Z \in Z(X) : Z \cap Y \in \mathcal{F} \}$$

is a z-filter on X ; it is called the z-filter induced on X by $\mathcal{F}$.

If $Y \subset X$ and $\mathcal{F}$ is a z-filter on X , then $\mathcal{F}|Y = \{Z \cap Y : Z \in \mathcal{F}\}$ is called the trace of $\mathcal{F}$ on Y .

Definition (5.5) A z-ideal in C^* is an ideal I that contains any function that belongs to the same maximal ideals as some function in I .

Theorem (5.6) If Y is C^*-embedded in X and $\mathcal{F}$ is a prime z-filter on X such that every member of $\mathcal{F}$ meets Y, then $\mathcal{F}|Y$ is a prime z-filter on Y .

Proof : It is clear that $\mathcal{F}|Y$ is a z-filter on Y . To show that $\mathcal{F}|Y$ is prime, it suffices to show that for any Z, $W \in Z(Y)$ with $Z \cup W = Y$, at least one of them belongs to $\mathcal{F}|Y$. Since Y is C^*-embedded in X, we can choose $S, T \in Z(X)$ such that $Z = S \cap Y$, $W = T \cap Y$. Since $\mathcal{F}$ is prime and $\mathcal{F} \subset (\mathcal{F}|Y)^{\#}$, it follows that $(\mathcal{F}|Y)^{\#}$ is prime. Since $(S \cup T) \cap Y = Z \cup W = Y$, by definition of $(\mathcal{F}|Y)^{\#}$ we see that $S \cup T \in (\mathcal{F}|Y)^{\#}$. Thus at least one of S, T belongs to $(\mathcal{F}|Y)^{\#}$, and whence at least one of Z, W belongs to $\mathcal{F}|Y$. Hence $\mathcal{F}|Y$ is prime.

Review (5.7) In the rest of this chapter, we consider the real line R only. By the Stone-Čech compactification theorem and [2, 2.12], we see that the prime z-ideals contained in M^{*p} are in order preserving correspondence with the prime z-filters on βX contained in $A^p_{\beta R}$, by means of the mapping $P \longrightarrow Z[P^\beta]$. Under this mapping $M^p \cap C^* \longrightarrow K^p$ (see theorem 5.3), where

$$K^p = \{ Z \in Z(\beta R) : p \in Cl_{\beta R}(Z \cap R) \}$$

Since R is locally compact, it follows that $\beta R - R$ is a zero set in βX and is C^*-embedded in βR. Obviously there is a bounded unit of C(R) that belongs to M^{*p} for every $p \in \beta R - R$. Thus $M^p \cap C^* \neq M^{*p}$ if and only if $p \in \beta R - R$.

Theorem (5.8) For any $p \in \beta R$, the family of prime z-filters on βR contained in K^p is in one-to-one corresponding with the family of prime z-filters on R contained in A^p.

Proof : Let $\mathcal{P}$ be a prime z-filter contained in K^p, then every member of $\mathcal{P}$ meets R. By theorem 5.6, $\mathcal{P}|R = \{ Z \cap R : Z \in \mathcal{P} \}$ is a prime z-filter on R. Since $\mathcal{P} \subset K^p$, it follows that $\mathcal{P}|R \subset A^p$. If $\mathcal{B}$ is a prime z-filter on R contained in A^p, obviously the induced prime z-filter

$$\mathcal{B}^{\#} = \{ Z \in Z(\beta R) : Z \cap R \in \mathcal{B} \}$$

is contained in K^p and $\mathcal{B}^{\#}|X = \mathcal{B}$. Hence the mapping $\mathcal{P} \longrightarrow \mathcal{P}|X$ for $\mathcal{P} \subset K^p$ is onto the family of prime z-filters of C(R) contained in A^p.

To prove that the mapping is one to one, it suffices to show that $\mathcal{P} = (\mathcal{P}|X)^{\#}$. Obviously $\mathcal{P} \subset (\mathcal{P}|R)^{\#}$. Conversely for any $Z \in (\mathcal{P}|R)^{\#}$, there is $W \in \mathcal{P}$ such that $Z \cap R = W \cap R$. Obviously $W \subset Z \cup (\beta R - R)$, so that $Z \cup (\beta R - R) \in \mathcal{P}$. By definition of K^p, we see that $\beta R - R \notin \mathcal{P}$. Since $\mathcal{P}$ is prime, we have $Z \in \mathcal{P}$. This proves that $(\mathcal{P}|R)^{\#} \subset \mathcal{P}$ and hence $\mathcal{P} = (\mathcal{P}|R)^{\#}$.

<u>Corollary</u> (5.9) The family of prime z-ideals of $C^*(R)$ contained in $M^p \cap C^*$ is order isomorphic with the family of prime z-ideals of $C(R)$ contained in M^p.

<u>Proof</u> : It follows immediately from 5.8, the Stone-Čech compactification theorem and [2, 2.12].

<u>Corollary</u> (5.10) M^p is a minimal prime ideal of C if and only if $M^p \cap C^*$ is a minimal prime ideal of C^*.

<u>Corollary</u> (5.11) p is a remote point in βR if and only if $M^p \cap C^*$ is a minimal prime ideal of C^*.

<u>Theorem</u> (5.12) For any $p \in \beta R - R$.. The family of prime z-filters on βR properly containing K^p is in one-to-one correspondence with the family of prime z-filters on $\beta R - R$ contained in $A^p_{\beta R - R}$.

<u>Proof</u> : Let $\mathcal{P}$ be a prime z-filter on βR properly containing K^p. Obviously every member of $\mathcal{P}$ meets $\beta R - R$. So by theroem 5.6, we see that the trace $\mathcal{P}|(\beta R - R)$ is a prime z-filter on $\beta R - R$. Since $\mathcal{P} \subset A^p_{\beta R}$, it follows that $\mathcal{P}|(\beta R - R) \subset A^p_{\beta R - R}$. Let $\mathcal{B}$ be a prime z-filter on $\beta R - R$ contained in $A^p_{\beta R - R}$. The induced z-filter

$$\mathcal{B}^{\#} = \{ Z \in Z(\beta R) : Z \cap (\beta R - R) \in \mathcal{B} \}$$

is clearly prime and $\mathcal{B}^{\#}|(\beta R - R) = \mathcal{B}$. Since $\beta R - R \notin K^p$ and $\beta R - R \in \mathcal{B}^{\#}$, it follows from theorem 5.3 that $\mathcal{B}^{\#}$ properly contains K^p. This proves that the mapping $\mathcal{P} \longrightarrow \mathcal{P}|(\beta R - R)$, for $K^p \subset \mathcal{P}$ is onto the family of prime z-filters on $\beta R - R$ contained in $A^p_{\beta R - R}$. Finally we are going to show that it is one-to-one. It suffices to show that $\mathcal{P} = (\mathcal{P}|(\beta R - R))^{\#}$. Obviously $\mathcal{P} \subset (\mathcal{P}|(\beta R - R))^{\#}$. Now let $Z \in (\mathcal{P}|(\beta R - R))^{\#}$, then there exists $W \in \mathcal{P}$ such that $Z \cap (\beta R - R) = W \cap (\beta R - R)$. We claim $\beta R - R \in \mathcal{P}$. Suppose not, then the z-ideal P in $C^*(R)$ corresponding to $\mathcal{P}$ contains no unit of $C(R)$. Let $f \in P$ and let V be a zero set neighbourhood of p in βR. Since P^β is prime and is contained in $A^p_{\beta R}$, by [2, 4I.4], it follows that $V \in Z[P^\beta]$. Thus $V \cap Z(f^\beta) \in Z[P^\beta]$ and hence $V \cap Z(f) \in Z[P]$. Since P contains no unit of $C(R)$, $V \cap Z(f) \neq \phi$. Hence $p \in Cl_{\beta R} Z(f)$ and therefore $f \in M^p$. This proves that $P \subset M^p \cap C^*$, i.e. P is contained in K^p, but this is impossible. So we must have $\beta R - R \in \mathcal{P}$. Thus $Z \cap (\beta R - R) = W \cap (\beta R - R) \in \mathcal{P}$ and hence $Z \in \mathcal{P}$. This proves that $(\mathcal{P}|(\beta R - R))^{\#} \subset \mathcal{P}$, and hence the mapping is one to one.

<u>Definition</u> (5.13) The z-filter generated by a z-filter $\mathcal{F}$ and a zero set Z that meets every member of $\mathcal{F}$ is denoted by $(\mathcal{F}, Z)$. Obviously

$$(\mathcal{F}, Z) = \{ W \varepsilon Z(X) : \text{ for some } F \varepsilon \mathcal{F}, F \cap Z \subset W \}.$$

<u>Remark</u> (5.14) In the last part of the proof of 5.12, we showed that for any $p \varepsilon \beta R - R$, a prime z-filter contained in $A^p_{\beta R}$ properly contains K^p if and only if it contains the zero set $\beta R - R$. This means that K^p has an immediate successor $(K^p)^+$ in the family of prime z-filters on βR , generated by K^p and the zero set $\beta R - R$, i.e. $(K^p)^+ = (K^p, \beta R - R)$. Furthermore, according to the construction of the one to one onto mapping in theorem 5.12, we note that $(K^p)^+ = (Z[O^p_{\beta R - R}])^{\#}$.

<u>Theorem</u> (5.15) $(Z[O^p_{\beta R}] , \beta R - R) = (Z[O^p_{\beta R - R}])^{\#}$. Hence $(Z[O^p_{\beta R}, \beta R - R) =$ $= (K^p)^+$, and the immediate successor of $M^p \cap C^*$ in the family of prime z-ideals of $C^*(R)$ consists of all functions f such that f^β vanishes on a neighbourhood of p in $\beta R - R$.

<u>Proof</u> : For any $Z \varepsilon (Z[O^p_{\beta R}], \beta R - R)$, there exists $W \varepsilon Z[O^p_{\beta R}]$ such that $W \cap (\beta R - R) \subset Z$. Since $W \cap (\beta R - R) \varepsilon Z[O^p_{\beta R - R}]$, it follows that $Z \cap (\beta R - R) \varepsilon Z[O^p_{\beta R - R}]$. Thus $Z \varepsilon (Z[O^p_{\beta R - R}])^{\#}$. Conversely for any $Z \varepsilon (Z[O^p_{\beta R - R}])^{\#}$, then $Z \cap (\beta R - R) \varepsilon Z[O^p_{\beta R - R}]$. This means that $Z \cap (\beta R - R)$ is a zero set neighbourhood of $p \varepsilon \beta R - R$ in $\beta R - R$. So there is $W \varepsilon Z[O^p_{\beta R}]$ such that $W \cap (\beta R - R) \subset Z \cap (\beta R - R)$. Thus

$W \cap (\beta R - R) \subset Z$, and $Z \in (Z[O^p_{\beta R}] , \beta R - R)$.

<u>Corollary</u> (5.16) For any $p \in \beta R - R$, p is a P-point of $\beta R - R$ if and only if M^{*p} is the immediate successor of $M^p \cap C^*$ in the family of prime z-ideals of $C^*(X)$.

<u>Corollary</u> (5.17) For any $p \in \beta R - R$, the family of prime z-ideals of $C^*(R)$ contained in M^{*p} consists of just the two ideals M^{*p} and $M^p \cap C^*$ if and only if p is both a remote point in βR and a P-point of $\beta R - R$.

<u>Theorem</u> (5.18) p is a remote point in βR if and only if the prime ideals contained in M^p form a chain .

<u>Proof</u> : If p is a remote point in βR , then M^p is a minimal prime ideal and hence the necessity follows immediately.

Conversely, suppose that the prime ideals contained in M^p form a chain $\mathcal{C}$. By [2, 2.8] $O^p = \cap\, \mathcal{C}$. To show that p is a remote point of $\beta R - R$, it suffices to show that $O^p = \cap\, \mathcal{C}$ is prime. Now let $a \notin \cap\, \mathcal{C}$, $b \notin \mathcal{C}$. Then there exists $P, J \in \mathcal{C}$ such that $a \notin P$, $b \notin J$. Since $\mathcal{C}$ is a chain, it follows that $P \subset J$, say. Thus $b \notin P$. Since P is prime, $ab \notin P$. Hence $ab \notin \cap\, \mathcal{C}$. This proves that O^p is prime.

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