C1

TORSION AND LOCALIZATION

by

Algis Richard Vilciauskas

B.Sc., Loyola of Montreal, 1970

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

July, 1972

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of MATHEMATICS

The University of British Columbia
Vancouver 8, Canada

Date July 25, 1972

ABSTRACT

The purpose of this thesis is to develop the machinery of noncommutative localization as it is being used to date, along with some fundamental results and examples. We are not concerned with a search for a "true torsion theory" for R-modules, but rather with a unification of previous generalisations in a more natural categorical setting.

In section 1, the generalisation of torsion for a ring R manifests itself as a kernel functor which is a left exact subfunctor of the identity functor on the category of R-modules. If a kernel functor σ also has the property $\sigma(M/\sigma(M)) = 0$ for any R-module M, we say that σ is idempotent. We treat the Gabriel correspondence which establishes a canonical bijection between kernel functors, filters of left ideals in R, and classes of R-modules closed under submodules, extensions, homomorphic images, and arbitrary direct sums. This result, which allows us to view torsion in several equivalent ways, is fundamental to the rest of the thesis.

Section 2 presents some positive and negative observations on when a kernel functor is idempotent.

In section 3 we begin by generalising the concept of injective module by defining σ-injectivity relative to an idempotent kernel functor σ. This yields a full coreflective subcategory of the category of R-modules. The localization functor relative to σ is then constructed as the composite of the coreflector with the embedding of the subcategory.

In section 4 we discuss the important "property T" which allows us to express the localization of an R-module as the module tensored with the localized ring, just as in the classical commutative case of localizing at a prime ideal.

Finally in section 5 we see that every idempotent kernel functor can be represented by a finitely cogenerating injective R-module V and the relative localization of R by the double centralizer of V .

Indications are that the generalised concept of torsion with its relative localization will prove itself increasingly valuable in the further study of rings and modules.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

The author wishes to express his thanks to his supervisor Dr. S. Page for suggesting the topic of this thesis and for his helpful advice and encouragement during its preparation.

The author also wishes to thank Dr. D. C. Murdoch for his careful reading of this thesis.

The most generous financial support of the National Research Council of Canada is gratefully acknowledged.

INTRODUCTION

To every abelian group G we can assign a torsion subgroup $T(G)$ consisting of all the elements of G with finite order. If $G' \subseteq G$ is a subgroup, it is clear that $T(G') = G' \cap T(G)$. Furthermore, any group homomorphism $G \longrightarrow H$ necessarily maps $T(G)$ into $T(H)$. Thus we may regard T as a left exact subfunctor of the identity functor on the category of abelian groups. This formulation of the usual torsion theory in abelian groups lends itself easily to a generalised concept of torsion for other "nice" abelian categories. In this thesis we shall be concerned only with the generalisation to categories of modules ${}_{R}\underline{M}$ over an arbitrary ring with unity. A left exact subfunctor of the identity on ${}_{R}\underline{M}$ is called a kernel functor. A kernel functor σ for which $\sigma(M/\sigma(M)) = 0$ for every R-module M will be called idempotent. Thus the usual torsion theory in abelian groups is a prototype for our idempotent kernel functors defined on more general module categories. However, we are not concerned with a search for a "true torsion theory" for R-modules, but rather with a unification of previous generalisations in a more natural categorical setting.

In section 1 we treat the Gabriel correspondence which establishes a canonical bijection between kernel functors, filters of left ideals in R , and classes of R-modules closed under submodules, extensions, homomorphic images, and arbitrary direct sums. In the case of idempotent kernel functors, this bijection restricts to a similar correspondence which enables us to view the generalised concept of torsion in several equivalent ways. The Gabriel correspondence is fundamental to the rest of this thesis.

Section 2 presents some positive and negative observations on when a kernel functor is idempotent. This is done mainly by investigating the associated filter of left ideals.

In section 3 we begin by generalising the concept of injective module. This is done relative to any idempotent kernel functor σ by remodeling the injective test lemma in the sense that a module A is called σ-injective if the extension property enunciated in the injective test lemma holds for at least the left ideals in the filter associated with σ . If the extensions are unique, we say that the module A is faithfully σ-injective. Now the localization functor relative to σ assigns to each R-module its faithfully σ-injective hull. In order to construct this functor explicitly, we consider the full subcategory of ${}_R\underline{M}$ consisting of the faithfully σ-injective R-modules. This subcategory is coreflective with exact coreflector, and the localization functor is the composite of the coreflector with the embedding of the subcategory. The localization of the ring R is again a ring, but now need no longer be a local ring in the sense of having a unique maximal ideal. Equivalent formulations of this localization process are also mentioned.

In section 4 we discuss the important "property T" which allows us to express the localization of any R-module as the module tensored with the localization of R , just as in the classical commutative case of localizing at a prime ideal.

In section 5 we see that every idempotent kernel functor σ

can be represented by a finitely cogenerating injective R-module V (where finitely cogenerating is the dual of finitely generated) in the sense that $\sigma(M)$ is the intersection of the kernels of all R-homomorphisms of M into V for any M in ${}_R\underline{M}$. Furthermore the localization of R relative to σ is the double centralizer of V .

All rings have unity 1 and all ring morphisms are unital. The category of left unitary modules over a ring R is denoted by ${}_R\underline{M}$ ($\underline{M}_R$ for right R-modules). Morphisms in ${}_R\underline{M}$ are called R-maps. Module always means left unless stated otherwise. For any R-module M with submodules M', M'' we use the notation $(M':M'')$ for the left ideal $\{ r \in R \mid rM'' \subseteq M' \}$. Thus in particular $(0:m)$ is the annihilator of $m \in M$, and $(0:M)$ the annihilator of M . $I(M)$ denotes the injective hull of M . The hom-functor in a category $\underline{C}$ is denoted by $\underline{C}(\underline{?}, \underline{?})$. The situation of a functor F being left adjoint to a functor G is denoted by $F \dashv G$. Proof of results are given either when they could not be found in the literature, or an alternate proof is offered. Otherwise a reference is given. The symbol ▮ indicates the end of a proof. Effort has been made to indicate as much as possible the source of terminology used in this thesis, and to mention other terminology used elsewhere. The basic references throughout are [7,9,16,17,22,33].

1. TORSION THEORIES

Let ${}_R\underline{M}$ be the category of left modules (written $\underline{M}$ for short) over a ring R with unity 1. A subfunctor of the identity on $\underline{M}$ is a covariant endofunctor

$$\sigma : \underline{M} \longrightarrow \underline{M}$$

such that $\sigma(M) \subseteq M$ is a submodule for every $M \in \underline{M}$ and $\sigma(f) : \sigma(N) \longrightarrow \sigma(M)$ is the restriction for any $f : N \longrightarrow M$ in $\underline{M}$.

(1.1) Definition: A subfunctor σ of the identity on $\underline{M}$ is called a kernel functor [9] if σ is left exact. Equivalently, σ is a kernel functor if σ is a subfunctor of the identity on $\underline{M}$ such that $\sigma(N) = N \cap \sigma(M)$ for any submodule N of $M \in \underline{M}$. An idempotent kernel functor [9] is a kernel functor σ satisfying $\sigma(M/\sigma(M)) = 0$.

We denote the fact that $\sigma : \underline{M} \longrightarrow \underline{M}$ is a kernel functor (idempotent kernel functor) by $\sigma \in KF(R)$ ($\sigma \in IKF(R)$) . We have the class inclusion $IKF(R) \subseteq KF(R)$ which will be shown to be set inclusion. For $\sigma \in KF(R)$ and any $M \in \underline{M}$ we call $\sigma(M)$ the σ-torsion submodule of M. Observe that if $\sigma \in KF(R)$ then $\sigma^2 = \sigma$ ie. $\sigma(\sigma(M)) = \sigma(M) \cap \sigma(M) = \sigma(M)$.

Various endofunctors of $\underline{M}$ having an assortment of different names appear in the literature: a subfunctor of the identity on $\underline{M}$ is called a preradical in [5,21]; a kernel functor is called a concordant in [33]; a subfunctor σ of the identity on $\underline{M}$ such that $\sigma(M/\sigma(M)) = 0$ for every $M \in \underline{M}$ is called a radical in [5,17] where also an idempotent kernel functor is called a torsion radical.

(1.2) Definition: A filter of left ideals [9,33] in a ring R is a set of left ideals $\underline{F}$ satisfying the following conditions:

i) if $U \in \underline{F}$ and I is a left ideal containing U then $I \in \underline{F}$

ii) if $U, V \in \underline{F}$ then $U \cap V \in \underline{F}$

iii) if $r \in R$ and $U \in \underline{F}$ then $(U:r) \in \underline{F}$.

These filters were considered by Gabriel [2,7] where such an object was called un ensemble d'ideaux a gauche topologisant.

If in addition a filter satisfies:

iv) if I is a left ideal for which there exists some $U \in \underline{F}$ with $(I:u) \in \underline{F}$ for every $u \in U$ then $I \in \underline{F}$

then the filter $\underline{F}$ is called strongly complete [33].

Gabriel [2,7] calls such an object un ensemble d'idéaux (à gauche) topologisant et idempotent.

(1.3) Definition: A Serre class [33] in ${}_R\underline{M}$ is a non-empty subclass $\underline{S}$ of $\underline{M}$ such that if

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is exact then $A', A'' \in \underline{S}$ if and only if $A \in \underline{S}$. Equivalently, a Serre class $\underline{S}$ is a non-empty subclass of ${}_R\underline{M}$ closed under submodules, homomorphic images, and extensions.

An additive class [33] of R-modules is a non-empty subclass of $\underline{M}$ closed under submodules, homomorphic images, and finite direct sums.

We say that a class of R-modules is strongly complete [33] if it is closed under arbitrary direct sums.

These Serre classes, strongly complete additive classes, and strongly complete Serre classes are exactly les sous-catégories épaisse, les sous-catégories fermées, and les sous-catégories localisante respectively, considered in [7].

In the following paragraphs we want to give explicitly the Gabriel correspondance which was first announced (partially) in [7, Chap.5]

Let us denote the set of filters on R by FIL(R). For any $\sigma \in KF(R)$, put

$$\underline{F}_\sigma = \{ I \in \text{left ideals of } R \mid \sigma(R/I) = R/I \}$$

Then the mapping $\sigma \longmapsto \underline{F}_\sigma$ defines a canonical bijection between KF(R) and FIL(R) by [9,Thms. 2.1,2.2] with the inverse mapping for any $\underline{F} \in FIL(R)$ given by

$$\underline{F} \longmapsto \tau \quad \text{where} \quad \tau(M) = \{ m \in M \mid (0:m) \in \underline{F} \} \quad \text{for any} \quad M \in \underline{M}$$

having the property that $\underline{F} = \underline{F}_\tau$ and $\sigma(M) = \{ m \in M \mid (0:m) \in \underline{F}_\sigma \}$. From this it follows immediately that KF(R) forms a set, as observed in [9,1.3], and that $IKF(R) \subseteq KF(R)$ is set inclusion.

(1.4) <u>Lemma</u>: $\sigma \in KF(R)$ *is idempotent if and only if* $\underline{F}_\sigma$ *is a strongly complete filter.*

<u>Proof</u>: This is exactly the content of [9,Thm 2.5].

Next let us denote the class of all strongly complete additive classes in ${}_R\underline{M}$ by CAD(R) . For any $\underline{F} \in FIL(R)$ put

$$\underline{S}_{\underline{F}} = \{ M \in \underline{M} \mid (0:m) \in \underline{F} \text{ for every } m \in M \}$$

Then the mapping $\underline{F} \longmapsto \underline{S}_{\underline{F}}$ defines a canonical bijection between FIL(R) and CAD(R) by [33,Thm 1.10], with the inverse mapping for any $\underline{S} \in CAD(R)$ given by

$$\underline{S} \longmapsto \underline{F}_{\underline{S}} = \{ I \in \text{left ideals of } R \mid R/I \in \underline{S} \}$$

having the property that $\underline{S} = \underline{S}_{(\underline{F}_{\underline{S}})}$ and $\underline{F} = \underline{F}_{(\underline{S}_{\underline{F}})}$.

From [33,Lemma 1.18] we have

(1.5) Lemma: $\underline{F} \in FIL(R)$ *is strongly complete if and only if* $\underline{S}_{\underline{F}}$ *is a strongly complete Serre class.*

The correspondences given above induce a canonical bijection between KF(R) and CAD(R) which is easily computed to be the mapping $\sigma \longmapsto \underline{S}_\sigma = \{ M \in \underline{M} \mid \sigma(M) = M \}$ with inverse given by $\underline{S} \longmapsto \tau$ where $\tau(M) = \sum \{ Im(\psi) \mid \psi \in \underline{M}(S,M) , S \in \underline{S} \}$. The members of $\underline{S}_\sigma$ are called σ-torsion modules. Hence $\sigma(M)$ is the largest (necessarily unique) σ-torsion submodule of M. Combining Lemmas (1.4,1.5) we have

(1.6) Lemma: $\sigma \in KF(R)$ *is idempotent if and only if* $\underline{S}_\sigma$ *is a strongly complete Serre class.*

Collecting these results, we state

(1.7) Theorem: (Gabriel correspondence for ${}_R\underline{M}$)

i) *There is a canonical bijection between kernel functors, filters of left ideals, and strongly complete additive classes.*

ii) *Restriction of the bijection in* i) *yields a canonical bijection between idempotent kernel functors, strongly complete filters of left ideals, and strongly complete Serre classes.*

There is another object associated with every $\sigma \in IKF(R)$ that turns out to be important later when considering localizations, namely

(1.8) $$\underline{V}_\sigma = \{ M \in \underline{M} \mid \sigma(M) = 0 \} \quad .$$

The class of R-modules $\underline{V}_\sigma$ is closed under isomorphic images, submodules, direct products, and injective hulls by [17,Prop 0.3], and its members are called σ-torsion-free modules. Moreover, by the same Proposition we have

(1.9) $\qquad \underline{S}_\sigma = \{ M \in \underline{M} \mid \underline{M}(M,A) = 0 \text{ for every } A \in \underline{V}_\sigma \}$.

This coincides with the classical notion in abelian groups that we cannot map a torsion group into a torsion-free group in a non-trivial way.

(1.10) Proposition: *For any* $\sigma \in \mathrm{IKF}(R)$, $\underline{V}_\sigma$ *is a full coreflective subcategory of* $\underline{M}$ *with coreflector* $F : \underline{M} \longrightarrow \underline{V}_\sigma$ *whose object function is given by* $F(M) = M/\sigma(M)$. *Furthermore,* $\underline{V}_\sigma$ *is an abelian category, and the coreflector* F *is exact.*

Proof: Let $K : \underline{V} \longrightarrow \underline{M}$ be the embedding functor of $\underline{V}_\sigma$ considered as a full subcategory. It should first be remarked that monomorphisms in $\underline{V}_\sigma$ and in $\underline{M}$ coincide.

If $f : M \longrightarrow N$ in $\underline{M}$, then the diagram

(1.11)
$$\begin{tikzcd} 0 \arrow[r] & \sigma(M) \arrow[r] \arrow[d, "\sigma(f)"'] & M \arrow[r] \arrow[d, "f"'] & M/\sigma(M) = F(M) \arrow[r] \arrow[d, dotted, "\exists !"] & 0 \\ 0 \arrow[r] & \sigma(N) \arrow[r] & N \arrow[r] & N/\sigma(N) = F(N) \arrow[r] & 0 \end{tikzcd}$$

has exact rows. We define $F(f)$ as the unique factorization of f over the cokernels making (1.11) commute. This makes F into a functor. For any $M \in \underline{M}$, $C \in \underline{V}_\sigma$, and $\pi_M : M \longrightarrow M/\sigma(M)$ canonical

$$\pi_M^* : \underline{V}\ (FM,C) \longrightarrow \underline{M}(M,KC)$$

is clearly a natural isomorphism by (1.9). Thus $\pi_M : M \longrightarrow F(M)$ is a coreflection for M in $\underline{V}_\sigma$ [22,p.128]. Suppose f in (1.11) is a mono and that $F(f)(m + \sigma(M)) = 0$ for $m \in M$. As the diagram commutes, we have $f(m) \in \sigma(N)$. So there is a $U \in \underline{F}_\sigma$ such that $Uf(m) = 0$. This implies $f(Um) = 0$, and f mono implies $Um = 0$. Hence $m \in \sigma(M)$ and so $F(f)$ is a mono. Hence by [22,Prop 5.3,p.130] $\underline{V}_\sigma$ is an abelian category and by [22,Prop 12.1,p.67] F is exact. ∎

The Gabriel correspondence stated in (1.7) allows us to view the generalised concept of torsion in several equivalent ways. Given $\sigma \in IKF(R)$ we shall call the triple of associated objects $(\underline{F}_\sigma, \underline{S}_\sigma, \underline{V}_\sigma)$ a <u>torsion theory</u>. These ideas appear much more concrete after investigating a few special cases.

(1.12) <u>Example</u>: Let R be any ring, $S \subset R$ a multiplicatively closed system (ie. $s_1s_2 \in S$ if $s_1, s_2 \in S$). Let $\underline{G}$ be the set of left ideals I of R such that for any $r \in R$ there exists $s \in S$ with $sr \in I$. Equivalently,

$$\underline{G} = \{ I \in \text{left ideals of } R \mid (I:r) \cap S \neq \emptyset \text{ for any } r \in R \} .$$

The conditions for a strongly complete filter are easily seen to be satisfied by $\underline{G}$. For instance to see that condition iv) holds let J be a left ideal such that $(J:u) \in \underline{G}$ for every $u \in U$ with $U \in \underline{G}$. For any $r \in R$, we have that $s_2r \in U$ for some $s_2 \in S$. Then $(J:s_2r) \in \underline{G}$ so that $s_1s_2r \in J$ for some $s_1 \in S$. As $s_1s_2 \in S$ we have $J \in \underline{G}$.

Let σ be the idempotent kernel functor corresponding to $\underline{G}$. An R-module H is σ-torsion if and only if for every $h \in H$, $sh = 0$ for some $s \in S$, and $\sigma(M)$ is the largest σ-torsion submodule of M for any $M \in \underline{M}$. If $0 \in S$ everything is torsion, but assuming $1 \in S$ changes nothing.

Even though every $I \in \underline{G}$ meets S, $\sigma(M)$ is not the subset of M consisting of elements killed by some element of S. This set is not even a submodule in general. The reason is that not every $s \in S$ need be contained in an ideal belonging to $\underline{G}$. For this to happen we need a "common left multiple property" :

(1.13) $\quad \forall s \in S \quad \forall r \in R \quad \exists t \in S \quad \exists r' \in R \ni tr = r's$.

This implies that every left ideal that meets S is contained in $\underline{G}$. The condition (1.13) is trivially satisfied if S is central in R, which is certainly the case if R is commutative. In case S consists of all non-zero divisors of R, (1.13) is exactly the classical left Ore condition [16,p.109]. For an entire ring R, the set of all non-zero left ideals forms a strongly complete filter if and only if R is a left Ore ring.

Now if $S \subset R$ is a multiplicatively closed system satisfying (1.13) it is clear that

$$\sigma(M) = \{ m \in M \mid sm = 0 \text{ for some } s \in S \} \quad \text{for any } M \in \underline{M}$$

defines an idempotent kernel functor with torsion theory

$$\underline{F}_\sigma = \{ I \in \text{left ideals of } R \mid I \cap S \neq \emptyset \}$$

$$\underline{S}_\sigma = \{ H \in \underline{M} \mid (0:h) \cap S \neq \emptyset \text{ for any } h \in H \}$$

$$\underline{V}_\sigma = \{ C \in \underline{M} \mid (0:c) \cap S = \emptyset \text{ for all } 0 \neq c \in C \} .$$

For the remainder of this section we shall investigate two more special torsion theories.

For any R-module M define the <u>left singular submodule</u> $Z_1(M)$ as :

(1.14) $\quad Z_1(M) = \{ m \in M \mid (0:m) \text{ is essential left ideal in } R \}$

In case $M = R$, $Z_1(R)$ is a 2-sided ideal [16,p.106 ff].

The essential left ideals of a ring R (which play an important role in the Goldie Theory - see for example [11]) form a filter which is not in general strongly complete. This is because

$Z_1(R/Z_1R)$ need not be zero. But we can either shrink or enlarge this set of left ideals (apart from the obious extreems) so that we do obtain a strongly complete filter. This was basically the approach of Dlab [6].

We start by enlarging the set of essentials.

(1.15) Lemma: *For a left ideal* L *of* R, *the following conditions are equivalent:*

i) *there exists an essential left ideal* E *in* R *such that* (L:x) *is essential in* R *for every* $x \in E$

ii) *for every* $r \in R$ *with* $r \notin L$ *there exists* $s \in R$ *such that* (L:sr) *is proper (ie.* $\neq R$ *) essential in* R .

Proof: i) => ii) If $r \notin L$, $Rr \neq 0$ so $Rr \cap E \neq 0$ implies $0 \neq sr \in E$ for some $s \in R$, and we can pick it so that $sr \notin L$ (since otherwise (E:r) = (L:r) , making (L:r) essential already). Then (L:sr) is proper essential.

ii) => i) Let $S = \{ s \in R \mid (L:s) \text{ is essential} \}$. $S \neq 0$ by taking $r = 1$ in condition ii). Let E be the left ideal generated by L and S. If $I \neq 0$ is any left ideal in R such that $I \cap L = 0$, then for $0 \neq a \in I$ there exists $b \in R$ such that (L:ba) is proper essential. Then $ba \neq 0$, $ba \in I$ and $ba \in S \subset E$. Hence E is essential. If (L:x) and (L:y) are essential, then (L:x + y) is essential as it contains $(L:x) \cap (L:y)$. For any $r \in R$ and $s \in S$ (L:rs) = ((L:s):r) is essential. Hence (L:x) is essential for every $x \in E$. ∎

Clearly any essential satisfies the conditions of the Lemma.

It was shown by Alin (found in [31]) that condition i) of the Lemma characterises the left ideals in the strongly complete filter of the socalled Goldie Torsion Theory [8], where the torsion submodule of any $M \in \underline{M}$ is given by $Z_2(M) = \pi^{-1}(\, Z_1(M/Z_1M)\,)$ with $\pi : M \longrightarrow M/Z_1M$ canonical. Equivalently,

(1.16) $Z_2(M) = \{\, m \in M \mid (Z_1M{:}m)$ is essential left ideal in R$\}$.

Clearly all quotients M/N of M by an essential submodule N are Z_2-torsion. An R-module is Z_2-torsion-free if and only if it has zero singular submodule. Notice that for any $M \in \underline{M}$ $Z_2(M)$ is essential over $Z_1(M)$. In fact $Z_2(M)$ is the maximal essential extension of $Z_1(M)$ in M in the sense that if N is a submodule of M which is essential over $Z_1(M)$, then $N \subsetneq Z_2(M)$.

Left ideals L satisfying ii) of Lemma (1.15) were called "maxi" in [6]. In section 5 we will be able to give another characterization of these maxi ideals as being " $Z_2(R)$-dense " .

Next let us shrink the set of essential left ideals in R.

(1.17) Lemma: *For a left ideal* D *of* R, *the following conditions are equivalent:*

i) $\forall\; 0 \neq r_1 \in R$ *and* $\forall\; r_2 \in R$ *there exists* $r \in R$ *such that* $rr_1 \neq 0$ *and* $rr_2 \in D$

ii) *for any* $r \in R$ *there is no* $0 \neq s \in R$ *such that* $(D{:}r)s = 0$.

Proof: i) => ii) Condition i) says that for any $r \in R$ and $0 \neq s \in R$ there exists $x \in (D{:}r)$ such that $xs \neq 0$. This is exactly condition ii).

ii) => i) Take any $r_2 \in R$. Then saying that for every $0 \neq r_1 \in R$

$(D:r_2)r_1 \neq 0$ means there is an $r \in R$ such that $rr_1 \neq 0$ and $rr_2 \in D$, as required. ▯

Any left ideal satisfying the conditions of the Lemma must be essential. Condition i) is the same as the condition of [16, Prop 4,p.96] and characterizes the <u>dense</u> left ideals which do form a strongly complete filter and lead to the complete ring of left quotients of R introduced by Utumi. The torsion theory corresponding to the dense left ideals will be referred to as the <u>Lambek Torsion Theory</u>. Left ideals D satisfying ii) of Lemma (1.17) were called "strong" in [6] where the following were proven to be equivalent:

(1.18)

i)	$Z_1(R) = 0$		
ii)	essential	<=>	dense
iii)	essential	<=>	maxi
iv)	dense	<=>	maxi

Thus if a ring has zero singular ideal, the essential, dense, and maxi left ideals all coincide, and the essentials form a strongly complete filter [7,Lemme 1,p.416].

On the other hand suppose the essential left ideals in a ring R already form a strongly complete filter. Let M be any R-module. If $m \in Z_2(M)$ then there is an essential E such that $Em \subset Z_1(M)$. So for each $x \in E$ there is an essential $E^{(x)}$ with $E^{(x)}xm = 0$; ie. $E^{(x)} \subset (0:xm) = ((0:m):x)$. This says $((0:m):x)$ is essential for every $x \in E$, which under the hypothis implies $(0:m)$ is essential. Hence $m \in Z_1(M)$ and we have $Z_2(M) = Z_1(M)$ for all $M \in \underline{M}$. This means that the idempotent kernel functor induced by the essentials under this hypothis is exactly Z_2. From the Gabriel correspondence we conclude that the essentials coincide with the maxi left ideals, and so by (1.18) essential $\Leftrightarrow$ dense and R has zero singular ideal. In particular, this is true for any semiprime Goldie ring R [11] ; here the multiplicatively closed system $S \subset R$ consisting of all non-zero divisors has the common left multiple property (1.13) and the set of essentials is precisely the strongly complete filter associated with S as in Example (1.12).

Collecting a few of the above facts, we have

(1.19) Proposition: *For any ring* R, *the following are equivalent:*

i) *if* I *is a left ideal in* R *such that* $(I:x)$ *is essential for every* $x \in E$ *with* E *essential, then* I *is essential*

ii) R *has zero singular ideal*

2. KERNEL FUNCTORS

All kernel functors are certainly not idempotent. for example take a commutative ring A (always with 1) and $a \in A$ such that $a \neq 0$, $a \neq 1$. Define for any A-module M a submodule

$$\alpha_a(M) = \{ m \in M \mid am = 0 \}$$

Clearly α_a is a kernel functor, but we do not have to look very far for a ring in which such an α is not idempotent. In fact, take $A = \mathbb{Z}$ $n > 1$. Let $G = \mathbb{Z}/n^3\mathbb{Z}$. Then $\alpha_n(G)$ is a proper subgroup of G since $n^2 \in \alpha_n(G)$ whereas $n \notin \alpha_n(G)$. But $\alpha_n(G/\alpha_n G) \neq 0$ because $0 \neq n + \alpha_n(G)$ is a member.

With the aim of localization in mind, the main interest in kernel functors is to determine whether or not the ones that arise naturally are indeed idempotent.

The set KF(R) has an obious partial ordering $\leqslant$ given by

(2.1) $\qquad \sigma \leqslant \rho \quad \Longleftrightarrow \quad \sigma(M) \subseteq \rho(M) \quad$ for every $M \in \underline{M}$

There is a smallest and a largest member with respect to this partial ordering: namely 0 such that $0(M) = 0$ for every M and ∞ such that $\infty(M) = M$ for every M respectively. Clearly $0, \infty \in$ IKF(R). These are the trivial torsion theories which exist for any ring.

(2.2) Example: The Goldie Torsion Theory is the smallest non-trivial torsion theory for which all modules of the form R/E are torsion, where E is any essential left ideal in the ring R. To see this, suppose $\sigma \in$ IKF(R) such that $\sigma(R/E) = R/E$ for every essential E. This means that $\underline{F}_\sigma$ contains all the essentials. Let L be a maxi left ideal in R. Then there exists an essential left ideal E such that

$(L:x)$ is essential for every $x \in E$. Now E essential implies $E \in \underline{F}_\sigma$ and $\sigma \in IKF(R)$ implies $L \in \underline{F}_\sigma$. Hence $\underline{F}_\sigma$ also contains all the maxi left ideals and so $Z_2 \leqslant \sigma$.

(2.3) <u>Example</u>: The Lambek Torsion Theory is the largest non-trivial torsion theory for which the ring R is torsion-free. To prove this, let $\underline{D}$ be the set of dense left ideals in R. Clearly R is torsion-free with respect to the Lambek Torsion Theory since $D \in \underline{D}$ implies $(D:1) = D$ has no right annihilators by Lemma (1.17). Now suppose $\sigma \in IKF(R)$ such that $\sigma(R) = 0$. If $U \in \underline{F}_\sigma$, then $(U:r) \in \underline{F}_\sigma$ for any $r \in R$. Hence $(U:r)s = 0$ implies $s = 0$ for any $r,s \in R$, which shows by Lemma (1.17) that $\underline{F}_\sigma \subsetneq \underline{D}$.

Any $\sigma \in KF(R)$ satisfies $\sigma^2 = \sigma$, so already appears to be " idempotent ". In order to justify our terminology for idempotent kernel functors, we introduce a product on filters (following [2]).

Let $\underline{F}_\sigma$, $\underline{F}_\rho$ be two filters of left ideals in R. Define:

(2.4) $\underline{F}_\sigma * \underline{F}_\rho = \{ I \in \text{left ideals of } R \mid \exists\ J \in \underline{F}_\rho \text{ with } \sigma(J/I) = 0 \}$.

$\underline{F}_\sigma * \underline{F}_\rho$ is a filter, therefore has a uniquely associated kernel functor which we denote by $\sigma * \rho$ in order to write $\underline{F}_\sigma * \underline{F}_\rho = F_{\sigma * \rho}$. An R-module M is $\sigma*\rho$-torsion if and only if M has a σ-torsion submodule M' such that M/M' is ρ-torsion. This star product is associative. From the definition of strongly complete filter it is clear that $\underline{F}_\sigma$ is strongly complete if and only if for left ideals $I \subsetneq J$ in R such that $J \in \underline{F}$ and J/I σ-torsion we have that $I \in \underline{F}_\sigma$. Now Goldman [9,Thm 2.5] states that $\sigma \in KF(R)$ is idempotent if and only if $\sigma * \sigma = \sigma$.

Another way of forming new kernel functors from given ones is by intersection. More precisely, let $\{ \sigma_i \mid i \in I \}$ be a family of kernel functors in KF(R). Define $\sigma = \inf \{ \sigma_i \mid i \in I \}$ by $\sigma(M) = \bigcap_I \sigma_i(M)$ for any $M \in \underline{M}$. Trivially $\sigma \in KF(R)$, $\sigma \leqslant \sigma_i$ for all $i \in I$, and if $\rho \leqslant \sigma_i$ for all $i \in I$ then $\rho \leqslant \sigma$. Furthermore it is trivial that if $\sigma = \inf \{ \sigma_i \mid i \in I \}$ then $\underline{F}_\sigma = \bigcap \underline{F}_{\sigma_i}$.

This concept of inf gives rise to a closure operation $^c : KF(R) \longrightarrow IKF(R)$ defined by

(2.5) $$\sigma \longmapsto \sigma^c = \inf \{ \rho \in IKF(R) \mid \sigma \leqslant \rho \}$$

More explicitly $\sigma^c(M) = \bigcap \{ N \subsetneq M \text{ submodule} \mid \sigma(M/N) = 0 \}$. This definition does indeed make σ^c an idempotent kernel functor by [9,Prop. 1.1 and Thm. 1.6] and $\sigma \in KF(R)$ is idempotent if and only if $\sigma = \sigma^c$. Via the Gabriel correspondence, the mapping $\sigma \longmapsto \sigma^c$ extends to a closure operation on filters:

(2.6) $$\underline{F} = \underline{F}_\sigma \longmapsto \underline{F}_{\sigma^c} = \underline{F}^c$$

and on strongly complete additive classes:

(2.7) $$\underline{S} = \underline{S}_\sigma \longmapsto \underline{S}_{\sigma^c} = \underline{S}^c$$

Given any subclass $\underline{C}$ of ${}_R\underline{M}$, there is certainly at least one $\sigma \in KF(R)$ for which every R-module in $\underline{C}$ is σ-torsion, namely $\sigma = \infty$. Finding the smallest such is equivalent to finding the smallest strongly complete additive class containing $\underline{C}$. Of course we obtain this class by intersection, and the desired kernel functor is $\inf\{ \sigma \in KF(R) \mid \underline{C} \subseteq \underline{S}_\sigma \}$. To obtain the smallest idempotent one, simply apply the closure operation, and thereby also obtain the smallest strongly complete Serre class containing $\underline{C}$. This was exactly the situation for the Goldie Torsion Theory where we closed the class $\{ R/E \mid E \text{ essential left ideal in } R \}$ of R-modules.

We turn to another concept that will be useful later.

(2.8) Definition: A kernel functor $\sigma \in KF(R)$ is called noetherian [9] if for every ascending chain $I_1 \subset I_2 \subset \ldots$ of left ideals in R whose union is in $\underline{F}_\sigma$, $I_n \in \underline{F}_\sigma$ for some n.

In particular, every kernel functor in KF(R) for a left noetherian ring R enjoys this property. We shall investigate the behavior of noetherian kernel functors with respect to the formation of inf.

(2.9) Observation: If $\rho_1,\ldots,\rho_n$ are finitely many noetherian kernel functors, then $\rho = \inf \{ \rho_i \mid i = 1,\ldots,n \}$ is also noetherian. The proof follows immediately from the remark that some member of any acsending chain of left ideals whose union is in $\underline{F}_\rho$ must be in every $\underline{F}_{\rho_i}$. This is however not the case for infinitely many memners of KF(R) for arbitrary R. A counterexample will be given in (2.13) after the following discussion which helps us to determine when a kernel functor is noetherian.

(2.10) Definition: A filter $\underline{F}$ is said to have a cofinal subset of finitely generated left ideals if every $U \in \underline{F}$ contains a finitely generated left ideal which is also in $\underline{F}$.

(2.11) Observation: If a filter $\underline{F}_\sigma$ has a cofinal subset of finitely generated left ideals then σ is noetherian. To see this, let $I_1 \subset I_2 \subset \ldots$ be any ascending chain of left ideals in R such that $U = \bigcup I_i$ is in $\underline{F}_\sigma$. Then U contains a finitely generated left ideal $(x_1,\ldots,x_n) \in \underline{F}_\sigma$, and so some I_i must contain all the $x_1,\ldots,x_n$. This says that some $I_i \in \underline{F}_\sigma$.

The converse of this is not yet clear as remarked in [9]. However we make the following:

(2.12) Observation: Suppose every $U \varepsilon \underline{F}_\sigma$ is at most countably generated. If σ is noetherian then $\underline{F}_\sigma$ has a cofinal subset of finitely generated left ideals. The proof is obtained simply by taking chains where we keep adding on generators of left ideals in $\underline{F}_\sigma$.

(2.13) Example: Let $A = k[x_1,x_2,\ldots]$ be the commutative polynomial ring in infinitely many indeterminants over a field k. Let $S_i = \{1,x_i,x_i^2,x_i^3,\ldots\}$ be the multiplicatively closed system in A defined by x_i , and let σ_i be the idempotent kernel functor associated with S_i as in Example (1.12) . Since A is commutative, for any $M \varepsilon {}_A\underline{M}$. $m \varepsilon \sigma_i(M) \iff x_i^n m = 0$ for some n , and $\underline{F}_{\sigma_i}$ consists of those ideals in A that meet S_i . This means $\underline{F}_{\sigma_i}$ has a cofinal subst of finitely generated ideals of the form As , $s \varepsilon S_i$ and so by (2.11) each σ_i is noetherian. Let $\sigma = \inf \{\sigma_i \mid i = 1,\ldots,\infty\}$. Consider the chain of ideals

$$(2.14) \qquad (x_1) \subset (x_1,x_2) \subset (x_1,x_2,x_3) \subset \ldots$$

and let U be their union. Since $\sigma_i(A/U) = A/U$ for every i, $U \varepsilon \underline{F}_\sigma$. But $\sigma(A/(x_1,\ldots,x_n)) \neq A/(x_1,\ldots,x_n)$ for any n because if $m > n$, $x_m + (x_1,\ldots,x_n)$ cannot be a σ_m-torsion element and consequently cannot be a σ-torsion element. Hence no member of the chain (2.14) is in $\underline{F}_\sigma$, and σ is not noetherian.

(2.15) Definition: A filter $\underline{F}$ is called multiplicative if $U,V \varepsilon \underline{F}$ implies $UV \varepsilon \underline{F}$.

(2.16) Remark: A strongly complete filter is always multiplicative since $U \subset (UV:v) \varepsilon \underline{F}$ for every $v \varepsilon V$. This is the same as saying $\underline{F}_\sigma$ is multiplicative for any $\sigma \varepsilon IKF(R)$. The converse of this remark is however not true in general. The following counterexample is indicated in [2,p.158].

(2.17) Example: Let $k[x_1,x_2,\ldots]$ be the commutative polynomial ring in infinitely many indeterminants over a field k . Let $(x_ix_j)_{i\neq j}$ be the ideal generated by all the x_ix_j for $i\neq j$ and put

$$A = k[x_1,x_2,\ldots]/(x_ix_j)_{i\neq j}$$

Let $\xi_k = x_k + (x_ix_j)_{i\neq j}$ and $I = (\xi_k)_{k=1}^{\infty}$ be the ideal in A generated by all the ξ_k . Consider the set $\underline{F}$ of ideals in A containing a power of I . Clearly $\underline{F}$ is a multiplicative filter of ideals in A. Denote the corresponding kernel functor by τ such that $\underline{F} = \underline{F}_\tau$. Let $J = (\xi_i^{\,i})_{i=1}^{\infty}$ be the ideal in A generated by all the $\xi_i^{\,i}$ (i^{th} powers of the ξ_i's). Then for any $\alpha \in I$, $I^s\alpha \subset J$ for large enough s. So $I^s \subset (J:\alpha) \in \underline{F}_\tau$ for every $\alpha \in I$ (s depending on α). But J can contain no power of I. Hence $\underline{F}_\tau$ is not strongly complete, and we conclude that τ is not idempotent.

By imposing some restrictions we do get a partial converse to Remark (2.16) in the case of commutative rings.

(2.18) Proposition: *Let* A *be a commutative ring. If a filter* $\underline{F}_\sigma$ *has a cofinal subset of finitely generated ideals, then* $\underline{F}_\sigma$ *is multiplicative if and only if the associated kernel functor* σ *is idempotent.*

Proof: If σ is idempotent, the conclusion follows from Remark (2.16). Conversely, let M be any A-module. For any $\theta = m + \sigma(M) \in \sigma(M/\sigma(M))$ there is a finitely generated ideal $I = (a_1,\ldots,a_n) \in \underline{F}_\sigma$ such that $I\theta = 0$; ie. $a_im \in \sigma(M)$ for $i = 1,\ldots,n$. Then for each i there is a $U_i \in \underline{F}_\sigma$ with $U_ia_im = 0$. Let $J = \bigcap U_i \in \underline{F}$. Then $JIm = 0$ and $JI \in \underline{F}_\sigma$ so that $m \in \sigma(M)$ and $\sigma(M/\sigma(M)) = 0$. ∎

(2.19) Corollary: *If* $\underline{F}$ *is a filter of ideals in a commutative ring* A *then the following assertions are equivalent:*

i) $\underline{F}$ *is multiplicative*

ii) $\underline{F}$ *is strongly complete*

iii) *the associated kernel functor is idempotent* ∎

3. LOCALIZATION FUNCTORS

In this section a localization functor is constructed for each $\sigma \in IKF(R)$ and some of the basic properties are obtained. The following theorem characterises a class of modules which turns out to be quite important.

(3.1) Theorem: *For any* $\sigma \in IKF(R)$ *with torsion theory* $(\underline{F}_\sigma, \underline{S}_\sigma, \underline{V}_\sigma)$ *and* $A \in \underline{M}$ *the following are equivalent in* $\underline{M}$:

i) *if* $M/N \in \underline{S}_\sigma$ *for* $N \subsetneq M$ *submodule then any* R-*map* $N \longrightarrow A$ *extends to an* R-*map* $M \longrightarrow A$ *; ie.* $\underline{M}(M,A) \longrightarrow \underline{M}(N,A) \longrightarrow 0$ *is exact whenever* $N \subsetneq M$ *and* $M/N \in \underline{S}_\sigma$

ii) $I(A)/A$ *is* σ-*torsion-free ; ie.* $I(A)/A \in \underline{V}_\sigma$

iii) *if* $U \in \underline{F}_\sigma$ *and* $g : U \longrightarrow A$ *is any* R-*map then there exists* $a \in A$ *such that* $g(u) = ua$ *for every* $u \in U$

iv) *if* $E \in \underline{F}_\sigma$ *is an essential left ideal in* R *, then any* R-*map* $E \longrightarrow A$ *extends to* R

v) *any* R-*map* $U \longrightarrow A$ *with* $U \in \underline{F}_\sigma$ *extends to* R *; ie. for every* $U \in \underline{F}_\sigma$ $\underline{M}(R,A) \longrightarrow \underline{M}(U,A) \longrightarrow 0$ *is exact*

vi) $\mathrm{Ext}_R^{\,1}(S,A) = 0$ *for every* $S \in \underline{S}$

vii) *for any essential extension* M *of* N *with* $(A:m) \in \underline{F}_\sigma$ *for every* $m \in M$ *any* R-*map* $N \longrightarrow A$ *extends to an* R-*map* $M \longrightarrow A$

Proof of equivalence of these conditions can be found scattered through the literature: i) <=> v) by [9,Prop 3.2]; i) <=> ii) <=> iii) by [17,Prop 0.5] where such an $A \in \underline{M}$ is called divisible; iv) <=> v) by [32,Thm 11]; i) <=> ii) <=> v) <=> vi) by [33,Prop 2.4]; i) <=> vii) by [20,Prop 1.2] ∎

(3.2) Definition: An R-module A is called σ-injective [9] if A satisfies any (hence all) of the conditions of Theorem (3.1). If in addition the extension in i) in the Theorem is unique , ie. $\underline{M}(M,A) \simeq \underline{M}(N,A)$ for $N \subseteq M$ and $M/N \in \underline{S}_\sigma$, then A is called faithfully σ-injective.

From [9,Prop 3.1] we have

(3.3) Proposition: *$A \in \underline{M}$ is faithfully σ-injective if and only if A is σ-injective and $A \in \underline{V}_\sigma$.*

For any $\sigma \in IKF(R)$ let $\underline{A}_\sigma$ denote the full subcategory of ${}_R\underline{M}$ consisting of the faithfully σ-injective R-modules. Regarding $\underline{V}_\sigma$ as a full subcategory of ${}_R\underline{M}$ the above Proposition (3.3) gives an embedding functor $J : \underline{A} \longrightarrow \underline{V}$ (subscripts have been dropped as we are considering σ fixed for now). Composing with the embedding $K : \underline{V} \longrightarrow \underline{M}$ we get the embedding

$$E : \underline{A} \xrightarrow{J} \underline{V} \xrightarrow{K} \underline{M} \quad . \tag{3.4}$$

For $C \in \underline{V}$ let $D(C) \subseteq I(C)$ be the extension of $\sigma(I(C)/C)$ by C (ie. $0 \longrightarrow C \longrightarrow D(C) \longrightarrow \sigma(I(C)/C) \longrightarrow 0$ exact in ${}_R\underline{M}$) such that $D(C)/C = \sigma(I(C)/C)$ and $C \subseteq D(C)$.

(3.5) Proposition: *For any $C \in \underline{V}$ we have $D(C) \in \underline{A}$.*

Proof: As $C \subsetneq D(C) \subseteq I(C)$, $D(C) \in \underline{V}$ by (1.8)ff. Since $I(C)$ is clearly an essential extension of $D(C)$ (as it is already essential over C) $I(C)$ is the injective hull of $D(C)$ by [16,Prop 10,p.92]; ie. $I(D(C)) = I(C)$. Now

$$\begin{aligned} I(C)/D(C) &\simeq (I(C)/C)/(D(C)/C) \\ &= (I(C)/C)/\sigma(I(C)/C) \end{aligned}$$

is σ-torsion-free because σ is idempotent. Hence $D(C)$ is σ-injective by Theorem (3.1,ii), and together with Proposition (3.3) this means $D(C) \in \underline{A}$. ▮

Here it should be remarked that monomorphisms in $\underline{A}$ and in $\underline{V}$ coincide, and hence also coincide with monomorphisms in $\underline{M}$.

Now $D : \underline{V} \longrightarrow \underline{A}$ is easily made into a functor: since $D(B)/B = \sigma(I(B)/B)$ is σ-torsion, for any $f : B \longrightarrow C$ in $\underline{V}$ there is a unique $D(f)$ by the faithful σ-injectivity of $D(C)$ that makes the diagram

(3.6)
$$\begin{tikzcd} B \arrow[r, "i_B"] \arrow[d, "f"'] & D(B) \arrow[d, dotted, "D(f)"] \\ C \arrow[r, "i_C"] & D(C) \end{tikzcd}$$

commute

where i_B , i_C are inclusion. If f in (3.6) is a monomorphism, $b \in B$ such that $D(f)(i_B(b)) = 0$, then $i_C f(b) = 0$ and so $b = 0$ since $i_C f$ is a mono. Hence $\ker D(f) \cap B = 0$. But $D(B)$ essential over B implies $\ker D(f) = 0$. This shows that D preserves monomorphisms.

Furthermore, for any $A \varepsilon \underline{A}$, $C \varepsilon \underline{V}$ with $i_C : C \longrightarrow D(C)$ inclusion

(3.7)
$$i_C^* : \underline{A}(DC,A) \longrightarrow \underline{V}(C,JA)$$

is clearly a natural isomorphism. Composing with the adjoint of Proposition (1.10) and putting $Q = DF$ we have that Q is left adjoint to the embedding functor E . Since both F and D preserve monos, [22,Prop 5.1,p.129] again gives us that $\underline{A}$ is an abelian category, and by [22,Prop 12.1,p.67] Q is exact. Notice that $\underline{A}$ is not in general an abelian subcategory of $\underline{M}$ as the embedding functor E need not be exact. Since the category $\underline{M}$ is complete, [22,Prop 5.1,p.129] gives us that $\underline{A}$ is complete. Collecting these results (see also [17,Prop 0.8]) yields the following :

(3.8) Theorem: *For any* $\sigma \in \mathrm{IKF}(R)$, *the full subcategory* $\underline{A}$ *consisting of the faithfully* σ*-injective* R*-modules is a coreflective subcategory of* $\underline{M}$ *with coreflector* $Q = DF$. *Furthermore* $\underline{A}$ *is a complete abelian category and the coreflector* Q *is exact.*

(3.9) Definition: The endofunctor $EQ : {}_R\underline{M} \longrightarrow {}_R\underline{M}$ will be called the localization functor relative to $\sigma \in \mathrm{IKF}(R)$.

In effect, the functor EQ provides a σ-injective hull for every $M \in \underline{M}$ as in [33,Prop 4.2] and can be computed by:

(3.10) Proposition: *For any* $M \in \underline{M}$

$$EQ(M) = \{\ x \in I(M/\sigma(M)) \mid (M/\sigma(M):x) \in \underline{F}_\sigma\ \} .$$

Proof: Immediate by our definition of D which is

$$EQ(M)/F(M) = EDF(M)/F(M) = \sigma(I(FM)/F(M)) ,$$

and that $\sigma(M) = \{\ m \in M \mid (0:m) \in \underline{F}_\sigma\ \}$. ∎

This Proposition (3.10) in the particular case $M = R$ appears as [27,Prop 1.7].

More information regarding this localization can be obtained by investigating the unit and counit of the adjunction $\eta : Q \dashv E$. The unit is given by $\phi_M = \eta(1_{Q(M)}) : M \longrightarrow EQ(M)$ and is easily computed to be:

(3.11)
$$\begin{tikzcd} M \arrow[r, "\phi_M"] \arrow[d, "\pi_M"'] & EQ(M) \\ M/\sigma(M) \arrow[ur, "i_{FM}"'] & \end{tikzcd}$$

Hence there is a natural mapping $\phi_M : M \longrightarrow EQ(M)$ for every $M \in \underline{M}$ whose kernel is exactly $\sigma(M)$.

(3.12) Proposition: *An R-module* M *is faithfully* σ*-injective if and only if* $EQ(M) = M$.

Proof: If $M = EQ(M)$, then M is faithfully σ-injective by Proposition (3.5).

Conversely, M faithfully σ-injective implies $\sigma(M) = 0$, so that $\phi_M : M \longrightarrow EQ(M)$ is inclusion. Since $EQ(M)$ is already faithfully σ-injective and $EQ(M)/M$ is σ-torsion, there exists a unique α such that the diagram

(3.13)
$$\begin{tikzcd}
M \arrow[r, "\phi_M"] \arrow[d, "1_M"'] & EQ(M) \arrow[dl, "\alpha"'] \arrow[ddl, "1_{EQ(M)}"] \\
M \arrow[d, "\phi_M"'] & \\
EQ(M) &
\end{tikzcd}$$
(uniquely) commutes

From $\phi_M \cdot 1_M = 1_{EQ(M)} \cdot \phi_M = \phi_M \cdot \alpha \cdot \phi_M$ we have $\phi_M \cdot \alpha = 1_{EQ(M)}$ by uniqueness, which together with $\alpha \cdot \phi_M = 1_M$ implies that the inclusion ϕ_M is an isomorphism. Hence $EQ(M) = M$. ∎

(3.14) Corollary: i) $(EQ)^2 = EQ$

ii) *the counit of the adjunction* $\eta : Q \dashv E$ *is the identity.*

Proof: i) $EQ(M)$ is faithfully σ-injective.

ii) for any $A \in \underline{A}$ the counit of η is given by

$$\eta^{-1}(1_{EA}) : QE(A) \longrightarrow A \quad .$$

But $EQE(A) = E(A)$ and E being an embedding implies $QE(A) = A$. Furthermore 1_A works!, so we have it by uniqueness. ∎

We write the counit of η as $1 : QE \longrightarrow Id_{\underline{A}}$.

Here we can mention a few more simple facts:

(3.15) Observation: M is σ-torsion <=> $Q(M) = 0$ <=> $EQ(M) = 0$.

(=>) by construction of the functor

(<=) by the fact that $\sigma(M) = \ker \phi_M$

(3.16) Observation: If $U \in \underline{F}_\sigma$ then $Q(U) \simeq Q(R)$ and hence $EQ(U) \simeq EQ(R)$. This is because $0 \longrightarrow U \longrightarrow R \longrightarrow R/U \longrightarrow 0$ is exact in $\underline{M}$, consequently $0 \longrightarrow Q(U) \longrightarrow Q(R) \longrightarrow Q(R/U) \longrightarrow 0$ is exact in $\underline{A}$. But R/U is σ-torsion, so $Q(R/U) = 0$ and we have $Q(U) \simeq Q(R)$ in $\underline{A}$. Since E preserves isomorphisms (always !) $EQ(U) \simeq EQ(R)$ in $\underline{M}$.

The objective of such localization is to study the ring R by means of $EQ(R)$ which should also be a ring. In order to give $EQ(R)$ a suitable ring structure, notice that as $\underline{A}$ is an abelian category on its own right, the adjunction $\eta : \underline{A}(Q(M),A) \simeq \underline{M}(M,E(A))$ is an isomorphism of abelian groups. Putting $M = R$ there is an isomorphism $E(A) \simeq \underline{A}(Q(R),A)$ of abelian groups which says that E is representable (in the sense of [17]). It is clear from the general theory of representable functors that ($Q(R)$, $\phi(1)$) is a representing pair for E , where we write $\phi = \phi_R$ for the canonical map $R \longrightarrow EQ(R)$. The rest of the story now follows immediately from [17,Prop 1.1] which is stated here in its entire generality.

(3.17) Proposition: (Lambek) *If* $\underline{C}$ *is an additive category,* $U : C \longrightarrow {}_R\underline{M}$ *a representable functor with representing pair* (A_o, s_o) , $s_o \in U(A_o)$, *such that* $U(C) \simeq C(A_o,C)$ *is an abelian group isomorphism for every* $C \in \underline{C}$ *, then the following are true:*

i) $U(A_o)$ *can be made into a ring* $S \simeq C(A_o, A_o)$ *with underlying abelian group same as that of* $U(A_o)$

ii) *the map* $R \longrightarrow U(A_o)$ *given by* $r \longmapsto rs_o$ *is a ring homomorphism* $\beta : R \longrightarrow S$

iii) *for each* $C \in \underline{C}$ $U(C)$ *is a left* S*-module, call it* $T(C)$

iv) *for each* $f : C \longrightarrow C'$ *in* $\underline{C}$ $U(f)$ *is an* S*-map, call it* $T(f)$

v) $T : \underline{C} \longrightarrow {}_S\underline{M}$ *is a functor such that* $U^\beta T = U$ (*where* $U^\beta : {}_S\underline{M} \longrightarrow {}_R\underline{M}$ *is change of rings functor via* β *- see [12]*)

vi) T *is representable with representing pair* (A_o, s_o) *and* $S \longrightarrow T(A_o)$ *by* $x \longmapsto xs_o$ *is an isomorphism.*

From this proposition it now follows that $EQ(R) = Q(R)$ is a ring with unit $\phi(1)$ such that $Q(R) \simeq \underline{A}(Q(R),Q(R))$ and $\phi : R \longrightarrow EQ(R)$ is a ring homomorphism such that the induced R-structure by change of rings coincides with the original structure as in [9,Thm 4.1]. There is a functor $T : A \longrightarrow {}_{EQ(R)}\underline{M}$ such that $U^\phi T = E$ and $T \simeq \underline{A}(Q(R),\underline{?})$ giving each faithfully σ-injective R-module an $EQ(R)$-module structure (see also [9,Cor 4.2]). In Lambeck's terminology [17], $(Q(R), T)$ is called the completion of (R, E). Putting together the facts thus far, we have the following commutative diagram of categories and functors:

(3.18)
$$\begin{tikzcd}
& EQ(R)\underline{M} \arrow[dr, "U^\phi"] & \\
\underline{A} \arrow[ur, "\underline{A}(EQ(R)\text{,}\underline{?})"] \arrow[rr, shift right, "E"'] \arrow[dr, "J"] & & {}_R\underline{M} \arrow[ll, shift right, "Q"'] \arrow[dl, shift left, "F"] \\
& \underline{V} \arrow[ul, shift left, "D"] \arrow[ur, "K"] &
\end{tikzcd}$$

It should be pointed out here that this functor Q cannot be considered the same as Goldman's localization functor Q which is actually EQ . This is most dramatically illustrated by the fact that Q is exact while EQ is not in general right exact. It will be easy to give a counterexample in Section 4 after a discussion of the important "property T", but the basic reason is that $\underline{A}$ need not be an abelian subcategory of ${}_R\underline{M}$. In fact, it turns out to be an abelian subcategory if and only if EQ is exact - see [33,Thm 3.13].

Since the endofunctor EQ arose from a pair of adjoints $Q \dashv E$, it gives rise to a monad [25] which by Corollary (3.14) is particularly simple. Explicitly it is given by the commutative diagrams:

(3.19)
$$\begin{tikzcd} EQ \arrow[r, "\phi EQ"] \arrow[d, "EQ\phi"'] \arrow[dr, equal] & (EQ)^2 \arrow[d, equal, "\iota"] \\ (EQ)^2 \arrow[r, equal, "\iota"] & EQ \end{tikzcd} \qquad \begin{tikzcd} (EQ)^3 \arrow[r, "EQ\iota"] \arrow[d, "\iota EQ"'] & (EQ)^2 \arrow[d, equal, "\iota"] \\ (EQ)^2 \arrow[r, equal, "\iota"] & EQ \end{tikzcd}$$

written (EQ, ϕ, ι) where $\phi : Id_{\underline{M}} \longrightarrow EQ$ is the unit of the adjunction $\eta : Q \dashv E$ and $\iota = E1Q : (EQ)^2 \longrightarrow EQ$ is the identity natural transformation since the counit 1 of η is the identity and $(EQ)^2 = EQ$.

Having Theorem (3.8) at our disposal, we can apply [25,Thm 2,p.75] to conclude that E is a monadic functor (caution: our coreflective as in [22] corresponds to reflexive in [25]). This means the following: the endofunctor EQ gives rise to a category $\underline{M}^{EQ}$ of socalled EQ algebras. The objects of $\underline{M}^{EQ}$ are pairs (A, α) where $A \varepsilon \underline{M}$ and $\alpha : EQ(A) \longrightarrow A$ is a morphism in $\underline{M}$ such that the following diagrams commute:

(3.20)
$$\begin{tikzcd} A \arrow[d, "\phi_A"'] \arrow[dr, equal] & \\ EQ(A) \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} (EQ)^2(A) \arrow[r, "EQ(\alpha)"] \arrow[d, equal, "\iota_A"'] & EQ(A) \arrow[d, "\alpha"] \\ EQ(A) \arrow[r, "\alpha"] & A \end{tikzcd}$$

Morphisms from (A, α) to (B, β) are morphisms $f : A \longrightarrow B$ in $\underline{M}$ such that the diagram

(3.21)
$$\begin{tikzcd} EQ(A) \arrow[r, "EQ(f)"] \arrow[d, "\alpha"'] & EQ(B) \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$
commutes.

We get a functor $L : \underline{A} \longrightarrow \underline{M}^{EQ}$ by defining $L(A) = (E(A), E(1_A))$ on objects and the obvious one on morphisms. That $\underline{M}^{EQ}$ is indeed a category and L a functor has been shown in great generality by Pareigis [25,Thm 1,p.62]. Now E monadic means that $L : \underline{A} \longrightarrow \underline{M}^{EQ}$ is an isomorphism of categories. Furthermore by Theorem (3.8), $\underline{M}^{EQ}$ is abelian. Hence we can regard the category $\underline{A}$ as an abelian category of EQ algebras. This was done by Heinicke [10,Thm 4.3] under somewhat more general circumstances using the Eilenberg-Moore construction directly. In [10], localization functors are defined via natural transformations and made to correspond with certain monads deemed localizing. However, the functor EQ and its monad (EQ, ϕ, ι) are the canonical choices as seen by [10,Thm 2.4].

Several other equivalent descriptions of this localization process can be found in the literature. In [9] Goldman gives an explicit elementwise characterization of $EQ(M)$ which for any σ-torsion-free R-module C is essentially given by

$$EQ(C) = \underset{U\varepsilon \underline{F}_\sigma}{} \underline{M}(U,C) \;/\; \equiv$$

where $\equiv$ is an equivalence relation such that $f \,\varepsilon\, M(U,C)$ and

$g \in \underline{M}(V,C)$ are related when there exists some $W \in \underline{F}$ with $W \subseteq U \cap V$ on which f and g agree. We can translate this description of EQ into

$$(3.22) \qquad EQ(M) = \varinjlim_{U \in \underline{F}_\sigma} \underline{M}(U, M/\sigma(M)) \quad \text{for any } M \in \underline{M}$$

by making the observation that inclusion in $\underline{F}$ induces a direct system of the abelian hom-groups with the required R-module structure defined on the direct limit by letting $r[f]$ be the equivalence class of

$$\begin{aligned} (U:r) &\longrightarrow M \\ x &\longmapsto f(xr) \end{aligned}$$

for any $r \in R$, and any equivalence class $[f]$, $f \in \underline{M}(U,M)$. This coincides with the construction in [2] as well as that in [7,33] using the quotient category of $\underline{M}$ with respect to the strongly complete Serre class $\underline{S}_\sigma$.

4. PROPERTY T

Let EQ be the localization functor relative to $\sigma \in IKF(R)$ and ϕ the unit of the adjunction $\eta : Q \dashv E$. Denote the localised ring $EQ(R)$ by Q (all other notations will be the same as in the previous section).

(4.1) Lemma: *There is a natural transformation* $\kappa : Q \otimes_R \underline{?} \longrightarrow EQ$ *of endofunctors on* ${}_R\underline{M}$ *given by* $\kappa_M(q\otimes m) = q(\phi_M(m))$ *for any* $M \in \underline{M}$ *and* $m \in M$, $q \in Q$.

Proof: The set map

$$\begin{aligned} Q \times M &\longrightarrow EQ(M) \\ (q,m) &\longrightarrow q(\phi_M(m)) \end{aligned}$$

is bilinear and R-balanced for any R-module M, thus extends uniquely to an R-map $\kappa_M : Q \otimes_R M \longrightarrow EQ(M)$ by the universal property of the tensor product. Let $f : M \longrightarrow N$ be any R-map and consider the diagram:

$$(4.2)\qquad \begin{array}{ccccc} M & \xrightarrow{\phi_M} & EQ(M) & \xleftarrow{\kappa_M} & Q\otimes M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle EQ(f)} & & \downarrow{\scriptstyle 1\otimes f} \\ N & \xrightarrow{\phi_N} & EQ(N) & \xleftarrow{\kappa_N} & Q\otimes N \end{array}$$

Now for any generator $q\otimes m \in Q \otimes M$

$$\begin{aligned} EQ(f)\cdot\kappa_M(q\otimes m) &= EQ(f)(q(\phi_M(m))) \\ &= q(EQ(f)\cdot\phi_M(m)) \\ &= q(\phi_N\cdot f(m)) \qquad \text{by naturality of } \phi \\ \kappa_N\cdot(1\otimes f)(q\otimes m) &= \kappa_N(q\otimes f(m)) \\ &= q(\phi_N\cdot f(m)) \end{aligned}$$

Hence extending linearly, we have naturality of κ. ∎

(4.3) Lemma: *If* $Q\phi(I) = Q$ *then* $I \varepsilon \underline{F}$.

Proof: Exactness of $I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ implies the exactness of $Q \otimes I \xrightarrow{\alpha} Q \otimes R \longrightarrow Q \otimes R/I \longrightarrow 0$. The image of α is $Q\phi(I)$. But $Q\phi(I) = Q \simeq Q \otimes R$. Thus $Q \otimes R/I = 0$, which implies $\kappa_{R/I} = 0$ and so $\phi_{R/I} = 0$. Hence $R/I = \ker \phi_{R/I} = \sigma(R/I)$ which implies that $I \varepsilon \underline{F}_\sigma$. ∎

(4.4) Lemma: $Q \otimes_R (R/U)$ *is σ-torsion for every* $U \varepsilon \underline{F}_\sigma$.

Proof is given in [27,Lemme 1.3]

The three Lemmas above provide some information about the general case of localizing relative to an arbitrary idempotent kernel functor. They also raise the following questions:

~~i) when is the natural transformation in Lemma (4.1) a natural isomorphism?~~

ii) when is the converse of Lemma (4.3) true ?

iii) when is $Q \otimes_R (R/U) = 0$ (ie. "really" torsion) for every U in the strongly complete filter $\underline{F}$.

We shall see from the answers in Theorem (4.5) below that these questions are intimately related.

It is also of interest to know when the functor $\underline{A}(Q(R),\underline{?}) : \underline{A} \longrightarrow {}_Q\underline{M}$ (see diagram 3.18) is a natural isomorphism. For if this is the case, every $X \varepsilon {}_Q\underline{M}$ is isomorphic to $\underline{A}(Q(R),A)$ for some $A \varepsilon \underline{A}$. Then regarding X as an R-module by change of rings U^ϕ via ϕ , $U^\phi(X) \simeq U^\phi \underline{A}(Q(R),A) = E(A)$ is faithfully σ-injective. On the other hand, if for every $X \varepsilon {}_Q\underline{M}$, $U^\phi(X)$ is faithfully σ-injective, then $QU^\phi \underline{A}(Q(R),\underline{?}) = \mathrm{Id}_{\underline{A}}$ and $\underline{A}(Q(R),QU\ \underline{?}) = \mathrm{Id}_{{}_Q\underline{M}}$ by diagram (3.18) which implies that $\underline{A}(Q(R),\underline{?})$ is a natural isomorphism .

The following Theorem establishes the connection between the foregoing remark and the above questions.

(4.5) <u>Theorem</u>: *For any* $\sigma \in \mathrm{IKF}(R)$ *, the following statements are equivalent:*

i) *the functor* $\underline{A}(Q(R),\underline{?}) : \underline{A} \longrightarrow {}_{Q}\underline{M}$ *is a natural isomorphism*

ii) $U^{\phi}(X)$ *is* σ*-torsion-free for every* $X \in {}_{Q}\underline{M}$

iii) $Q\phi(U) = Q$ *for every* $U \in \underline{F}_{\sigma}$

iv) $U^{\phi}(X)$ *is faithfully* σ*-injective for every* $X \in {}_{Q}\underline{M}$

v) $\kappa : Q \otimes_{R} \underline{?} \longrightarrow EQ$ *is a natural isomorphism*

vi) *the functor* EQ *is right exact and commutes with direct sums*

vii) $Q \otimes_{R} (R/U) = 0$ *for every* $U \in \underline{F}_{\sigma}$

viii) *the functor* EQ *is right exact and* $\underline{F}_{\sigma}$ *has a cofinal subset of finitely generated left ideals*

ix) *the functor* EQ *preserves colimits* .

<u>Proof</u>: The initial remark establishes i) <=> iv) ; vi) <=> ix) is an immediate consequence of the dual statement to [22,Cor 6.3,p.55] since ${}_{R}\underline{M}$ is a conormal category with direct sums. In fact ix) is equivalent to the weaker statement: *the functor* EQ *preserves direct limits* ; ii) <=> iii) <=> iv) <=> v) <=> vi) by [9,Thm 4.3] with naturality following by Lemma (4.1) ; vii) <=> v) <=> iii) <=> ii) by [33,Thm 3.2] ; viii) => i) and viii) => v) by [7,Cor 2,p.414] ; vii) <=> ii) <=> viii) <=> v) <=> i) by [27,Prop 2.8] where a new proof is offered. ∎

(4.6) <u>Definition</u>: We say that $\sigma \in \mathrm{IKF}(R)$ has <u>property T</u> [9] (for tensor product) if any (hence all) of the conditions of the above Theorem (4.5) are satisfied.

In [9] Goldman proves two more interesting equivences which are useful in determining property T :

(4.7) *the functor* EQ *commutes with direct sums if and only if* σ *is noetherian*

(4.8) *the functor* EQ *is right exact if and only if every* $U \in \underline{F}_\sigma$ *is "σ-projective" in the sense of the following:*

Definition: An R-module P is called σ-<u>projective</u> if for any epimorphism $C \longrightarrow C''$ of σ-torsion-free modules in $\underline{M}$, any R-map $P \longrightarrow C''$ can be lifted to an R-map $P' \longrightarrow C$ on a submodule P' of P with P/P' σ-torsion making the diagram

$$\begin{array}{ccccc} 0 \longrightarrow & P' & \longrightarrow & P & \\ & \downarrow & & \downarrow & \\ & C & \longrightarrow & C'' & \longrightarrow 0 \end{array}$$

commute.

(4.9) <u>Example</u>: Since any projective R-module is clearly σ-projective for any $\sigma \in$ IKF(R) , it is evident that for a left noetherian hereditary ring R , every $\sigma \in$ IKF(R) has property T .

(4.10) <u>Example</u>: If R is a left semisimple artinian ring, then R is (left) hereditary by Wedderburn's Theorem [13] and also left noetherian [16,p.69]. Hence by Example (4.9) every $\sigma \in$ IKF(R) for a left semisimple artinian ring R has property T .

(4.11) <u>Example</u>: Let $S \subset R$ be a multiplicatively closed system with associated idempotent kernel functor σ as in Example (1.12). Suppose S has the common left multiple property (1.13). Then $\underline{F}_\sigma$ contains Rs for every $s \in S$ and $\underline{F}_\sigma$ has a cofinal subset of principal left ideals. Let $U \in \underline{F}$, $s \in S$ U , $p : C \longrightarrow C''$ an epimorphism of σ-torsion-free R-modules, and $f : U \longrightarrow C''$ any R-map. For some $c \in C$, $p(c) = f(s)$. The R-map $Rs \longrightarrow C$ defined by $rs \longmapsto rc$ makes the diagram

$$\begin{tikzcd} 0 \arrow[r] & Rs \arrow[r] \arrow[d] & U \arrow[d, "f"] & \\ & C \arrow[r, "p"] & C'' \arrow[r] & 0 \end{tikzcd}$$

commute

Also $U/Rs \subseteq R/Rs$ and so U/Rs is σ-torsion. This means that every $U \in \underline{F}_\sigma$ is σ-projective which by (4.8) implies EQ is right exact. Hence by Theorem (4.5,viii), σ has property T .

The converse of the above need not hold in general, ie. the common left multiple property is not a necessary condition for σ to have property T . For a counterexample, consider the ring R of 2×2 matricies over a division ring D and the multiplicativly closed system S consisting of matricies of the form $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$ with $0 \neq d \in D$. ~~Let ρ be the associated idempotent kernel functor. Since R is~~ simple artinian, ρ has property T by Example (4.10), but S does not have the common left multiple property because taking

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S$, $\begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = (d_{ij})_{2\times 2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ implies $d = 0$.

Returning to the general situation of a multiplicatively closed system S in a ring R , the common left multiple property does insure that every $\phi(s)$ for $s \in S$ under the canonical map $\phi : R \longrightarrow EQ(R)$ has a left inverse in EQ(R) . To see this, pick any $s \in S$ and define $k_s : R \longrightarrow R$ by $r \longmapsto rs$. This induces the exact sequence

$$(4.12) \qquad 0 \longrightarrow K \longrightarrow R \xrightarrow{k_s} R \longrightarrow R/Rs \longrightarrow 0$$

where $K = \ker k_s$, $R/Rs = \operatorname{cok} k_s$. Since σ has property T , we have an exact sequence

$$EQ(R) \xrightarrow{\mu} EQ(R) \longrightarrow EQ(R/Rs) \longrightarrow 0$$

where $\mu = EQ(k_s)$ is the unique extension by faithful σ-injectivity of EQ(R) defined by $q \longmapsto q\ (s)$. Now $R/Rs \in \underline{F}$ implies $EQ(R/Rs) = 0$ and so μ is an epimorphism. Hence $1 = q(\phi(s))$ for some $q \in EQ(R)$ and we obtain the desired left inverse.

If we demand that each $\phi(s)$ is also to have a right inverse in EQ(R) then $rs = 0$ implies $\phi(r)\phi(s) = \phi(rs) = 0$ and consequently $\phi(r) = 0$ ie. $r \in \sigma(R)$ which means there is some $t \in S$ such that $tr = 0$. Hence a necessary condition for right invertibility of $\phi(s)$ is :

(4.13) if $rs = 0$ with $r \in R$ then $tr = 0$ for some $t \in S$.

This condition is also sufficient because it implies $K \subsetneq \sigma(R)$ in (4.12) so $EQ(K) = 0$ making $\mu = EQ(k_s)$ a monomorphism (hence an isomorphism) . Again letting q be a left inverse of $\phi(s)$, from $\mu(\phi(s)q - 1) = (\phi(s)q - 1)\phi(s) = 0$ we have $\phi(s)q = 1$ and hence $\phi(s)$ has a 2-sided inverse in EQ(R) . We have thus arrived at the full generalization of the classical Ore condition:

(4.14) <u>Definition</u>: A multiplicatively closed system $S \subset R$ is called a <u>left denominator set</u> [4] if S satisfies both the common left multiple property (1.13) and condition (4.13) above.

(4.15) <u>Definition</u>: Let $S \subset R$ be a multiplicatively closed system. <u>A ring of left fractions for R with denominators in S</u> [2,7] is a pair (Q, ψ) where Q is a ring and $\psi : R \longrightarrow Q$ is a ring homomorphism satisfying the following three conditions:

i) if $\psi(r) = 0$ then $sr = 0$ for some $s \in S$

ii) $\psi(s)$ is invertable in Q for every $s \in S$

iii) every element of Q has the form $\psi(s)^{-1}\psi(r)$ $r \in R$, $s \in S$.

(4.16) <u>Theorem</u>: *For a multiplicatively closed system* S *in a ring* R *with associated idempotent kernel functor* σ *the following are* *equivalent:*

i) R *has a ring of left fractions with denominators in* S

ii) S *is a left denominator set*

iii) *for every* $s \in S$, $\phi(s)$ *via the canonical* $\phi : R \longrightarrow EQ(R)$ *is invertable in* $EQ(R)$.

<u>Proof</u>: ii) => iii) follows from the preceeding discussion.

iii) => i) follows from Proposition (3.10) and the fact that every $U \in \underline{F}_{\sigma}$ meets S . The pair ($EQ(R)$, ϕ) becomes the (unique up to isomorphism) ring of left fractions for R which in this case is also the universal S-inverting object in [4].

i) => ii) is easily proven as in [7,Prop 5,p.415]. ∎

If S contains only the non-zero divisors, a ring of left fractions with denominators in S (if it exists) is called a <u>classical ring of left quotients</u> for R . From the Theorem we can immediately deduce [16,Prop 1,p.109] :

(4.17) <u>Corollary</u>: *A ring* R *has a classical ring of left quotients if and only if it satisfies the classical Ore condition.*

For a commutative ring A , every multiplicatively closed system S is a denominator set. Therefore the ring of fractions of A with denominators in S always exists and can be constructed in the classical way as in [1,Chap 3]. Because of property T , this construction extends to every A-module M by $EQ(A) \otimes_A M$ exactly as in [1,Prop 3.5] which gives the module of fractions with denominators in S .

(4.18) Proposition: *If* $\sigma \in IKF(R)$ *has property* T *, then the canonical map* $\phi : R \longrightarrow EQ(R)$ *is a ring epimorphism and* $EQ(R)$ *is flat as a right* R-*module.*

Proof: The isomorphism $EQ(R) = (EQ)^2(R) \simeq EQ(R) \otimes_R EQ(R)$ is induced by multiplication, so $\phi : R \longrightarrow EQ(R)$ is a ring epimorphism by [28,Prop 1.1]. Since $EQ \simeq EQ(R) \otimes_R \underline{?}$ is a natural isomorphism, $EQ(R)$ is flat in $\underline{M}_R$ ∎

In particular, the results of [28] apply to any $\sigma \in IKF(R)$ with property T in which case $\phi : R \longrightarrow EQ(R)$ is a left localization in the sense of Silver. However, the converse of Proposition (4.18) is not true in general.

(4.19) Example: Let $A = k[x,y]$ be the commutative polynomial ring in two indeterminants over a field k . Let $M = (x,y)$ be the maximal ideal of A , and take $\underline{F}$ to be the filter consisting of ideals containing a power of M . Since $\underline{F}$ is clearly multiplicative, $\underline{F}$ is a strongly complete filter by Corollary (2.19). Let ρ be the associated idempotent kernel functor. Clearly A is ρ-torsion-free (as A is an integral domain) so the canonical map $\phi : A \longrightarrow EQ(A)$ is inclusion. If we let K denote the field of fractions $k(x,y)$ of A we have $K = I(A)$ and $\rho(K/A) = 0$. Hence A is faithfully ρ-injective and hence $A = EQ(A)$. By Observation (3.16) $EQ(M) = EQ(A) = A$. Consequently $EQ(M)$ is not isomorphic to $A \otimes_A M \simeq M$ and thus ρ does not have property T . Nevertheless $\phi : A \longrightarrow EQ(A) = A$ is a ring epimorphism and A is certainly flat as a right A-module.

This example shows more:

(4.20) Since A is noetherian, ρ is also noetherian. Hence EQ cannot be right exact relative to this ρ . But Q is still exact as always.

(4.21) Not every torsion theory for a commutative ring arises from a multiplicatively closed system since all these do have property T .

(4.22) The product of torsion modules need not be a torsion module. In the present example, $\prod_{n=1}^{\infty} A/M^n$ is not ρ-torsion (the element $\{ 1 + M^n \}_{n=1}^{\infty}$ cannot be killed by any single power of M) but each A/M^n is ρ-torsion by construction of the filter. A similar example of this can be found in abelian groups. Consider the group $\mathbb{Z}(p^{\infty})$ (written additively) with generators $c_1, c_2, \ldots, c_n, \ldots$ and relations $pc_1 = 0$, $pc_2 = c_1$,..., $pc_{n+1} = c_n$,... . The groups $\mathbb{Z}(p^{\infty})/\mathbb{Z}c_i$ $i = 1,2,\ldots$ are all torsion in the usual torsion theory for $\mathbb{Z}^M$, but their product is not a torsion group.

The above Example (4.19) is a concrete version of one indicated in [9,Ex 2,p.45].

Even though the converse of Proposition (4.18) is not true in general, a partial converse in this connection can be found in the literature.

(4.23) <u>Theorem:</u> *A ring map* $\psi : R \longrightarrow Q$ *is an epimorphism and* Q *is flat as a right* R*-module if and only if the set of left ideals* I *in* R *such that* $Q\psi(I) = Q$ *is a strongly complete filter and for the localization relative to this torsion theory there is an isomorphism* $Q \simeq EQ(R)$ *making the diagram*

$$\begin{tikzcd} & Q \arrow[d, "\simeq"] \\ R \arrow[ur, "\psi"] \arrow[r, "\phi"'] & EQ(R) \end{tikzcd}$$

commute .

Proof is given in [17,Thm 2.7] and [27,Prop 2.7].

In the light of the above Theorem we may regard the left localizations of Silver [28] as arising relative to an idempotent kernel functor which appears very close to having property T .

5. REPRESENTATIONS OF IDEMPOTENT KERNEL FUNCTORS AND THEIR RELATIVE LOCALIZATIONS

Let $\sigma \in \mathrm{IKF}(R)$ be arbitrary and let $(\underline{F}, \underline{S}, \underline{V})$ be its corresponding torsion theory. Again notations from previous sections will be retained.

The σ-torsion modules $\underline{S}$ are generated by the cyclic modules R/U with $U \in \underline{F}$, in the sense that for any $M \in \underline{S}$ there is an epimorphism

$$\bigoplus_{U \in \underline{F}} R/U \longrightarrow M \quad .$$

The σ-torsion-free modules $\underline{V}$ are cogenerated by the modules $I(R/I)$ where I is a left ideal in R such that $\sigma(R/I) = 0$ (called closed left ideals in [17]) , in the sense that for any $C \in \underline{V}$ there is a monomorphism

$$C \longrightarrow \prod \{ I(R/I) \mid \sigma(R/I) = 0 \} = V$$

via a factorization through the injective hull of C and then by a construction of Janš (described in [17,p.6]) .

Let M be any R-module and put

$$k(M) = \bigcap \{ \ker(f) \mid f \in \underline{M}(M,V) \} \quad .$$

Clearly $\sigma(M) \subsetneq k(M)$ by (1.9) as V is σ-torsion-free. On the other hand, for any $m \in k(M)$, $\underline{M}(Rm,V) = 0$ because we can always fill the diagram

$$\begin{tikzcd} 0 \arrow[r] & Rm \arrow[r] \arrow[d] & M \arrow[dl, dotted, "\exists"] \\ & V & \end{tikzcd}$$

by injectivity of V . Now if C is any σ-torsion-free module, we have a monomorphism $C \longrightarrow V$. Hence $0 \longrightarrow \underline{M}(Rm,C) \longrightarrow \underline{M}(Rm,V) = 0$ is exact which implies $\underline{M}(Rm,C) = 0$ for every $C \in \underline{V}$. Consequently by (1.9) $Rm \in \underline{S}$ and therefore $m \in \sigma(M)$. Hence V completely determines the torsion theory as:

$$\sigma(M) = \bigcap \{ \ker(f) \mid f \in \underline{M}(M,V) \} \tag{5.1}$$

By reversing the procedure, it is clear that any R-module S determines a kernel functor if we take $V = I(S)$. We denote the kernel functor which arises in this way by τ_S and observe that τ_S is idempotent and in fact is the largest idempotent kernel functor for which S is torsion-free [9,Thm 5.1].

From the above discussion we have:

(5.2) <u>Proposition</u>: *For every* $\sigma \in IKF(R)$ *there exists* $M \in \underline{M}$ *such that* $\sigma = \tau_M$.

Notice that an R-module X is τ_M-torsion

$\iff$ $\underline{M}(X,I(M)) = 0$

$\iff$ $\forall x \in X \quad \forall m \in M \quad \exists r \in R \ni rx = 0$ and $rm \neq 0$

Clearly the R-module M whose existance was asserted in Proposition (5.2) is not unique - not even up to injective hull. But it is unique up to a relation that is manufactured to do precisely that job.

(5.3) <u>Definition</u>: i) We say that a module M is <u>cogenerated</u> by a module G if M can be embedded in a product of coppies of G .

ii) Two modules are called <u>similar</u> [22] if they cogenerate eachother.

(5.4) <u>Lemma</u>: *Two modules* M, N *give rise to the same torsion theory (ie.* $\tau_M = \tau_N$ *) if and only if* M *and* N *have similar injective hulls.*

<u>Proof</u>: (Storrer [17, Appendix]) We first remark that a necessary

and sufficient condition for a module X to be τ_S-torsion-free is that X be cogenerated by $I(S)$. This condition is obiously sufficient because products and submodules of torsion-free monules are again torsion-free. Necessity follows from the fact that $\underline{M}(Rx, I(S)) \neq 0$ for any $0 \neq x \in X$ since $Rx \subsetneq X$ is τ_S-torsion-free By injectivity we can fill the diagram

$$\begin{tikzcd} 0 \arrow[r] & Rx \arrow[r] \arrow[d] & X \arrow[dl, dotted, "\exists\ f \neq 0"] \\ & I(S) & \end{tikzcd}$$

which by the universal property of direct products gives a map $X \longrightarrow I(S)^X$ into the X-fold product of $I(S)$ which must be mono.

Now if $I(M)$ amd $I(N)$ are similar, then $\underline{V}_{\tau_M} = \underline{V}_{\tau_N}$ because cogenerating is transitive, and hence $\tau_M = \tau_N$.

Conversely if $\tau_M = \tau_N$ then M is τ_N-torsion-free and we have an embedding $e : M \longrightarrow I(N)^J$ for some J-fold product of $I(N)$. By injectivity we can fill the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "e"'] & I(M) \arrow[dl, dotted, "\exists"] \\ & I(N)^J & \end{tikzcd}$$

with a mono since $I(M)$ is essential over M . Therefore $I(M)$ is cogenerated by $I(N)$. By symmetry, $I(N)$ is cogenerated by $I(M)$ and hence M and N have similar injective hulls. $\blacksquare$

(5.5) <u>Proposition</u>: *An idempotent kernel functor* $\sigma = \tau_S$ *has property T if and only if the localization* $EQ(R)$ *of the ring relative to* σ *is flat as a right* R-*module and* $I(S)$ *is similar to the injective left* R-*module* $W = {}_{\mathbb{Z}}\underline{M}(EQ(R), \mathbb{Q}/\mathbb{Z})$.

Proof: If σ has property T , it has already been shown in Proposition (4.18) that $EQ(R)$ is flat as a right R-module. It remains to show $\sigma = \tau_W$. Now

$$_{R}\underline{M}(M,W) \simeq {}_{\mathbb{Z}}\underline{M}(EQ(R) \otimes_R M, \mathbb{Q}/\mathbb{Z})$$
$$\simeq {}_{\mathbb{Z}}\underline{M}(EQ(M), \mathbb{Q}/\mathbb{Z})$$

for any $M \in {}_R\underline{M}$. If $m \in \sigma(M)$ then $EQ(Rm) = 0$ and so $_R\underline{M}(Rm,W) = 0$. This means $\tau_W(Rm) = Rm$ and so $m \in \tau_W(M)$. On the other hand if $m \in \tau_W(M)$ then $_R\underline{M}(Rm,W) = 0$ and $_{\mathbb{Z}}\underline{M}(EQ(Rm),\mathbb{Q}/\mathbb{Z}) = 0$. By [16,p.89] we have $EQ(Rm) = 0$. Consequently Rm is σ-torsion. Hence $\sigma = \tau_W$ and we get the desired similarity by Lemma (5.4) .

Conversely, let $U \in \underline{F}_\sigma$. Then R/U is τ_W-torsion which means $_R\underline{M}(R/U,W) = 0$. But $0 = {}_R\underline{M}(R/U,W) \simeq {}_{\mathbb{Z}}\underline{M}(EQ(R) \otimes_R R/U, \mathbb{Q}/\mathbb{Z})$ implies $EQ(R) \otimes_R R/U = 0$. Hence by Theorem (4.5,vii) σ has property T . ∎

Tachikawa [30] mentions the following result:

(5.6) Proposition: *The localization of any* M *in* $\underline{M}$ *relative to* $\tau = \tau_S$ *is given by*

$$EQ(M) = \{ x \in I(M/\tau M) \mid \phi(x) = 0 \text{ for all } \phi \in \Phi \}$$

where $\Phi = \{ \phi \in \underline{M}(I(M/\ M),V) \mid \phi(M/\tau M) = 0 \}$ *and* V *is any injective similar to* $I(S)$.

Proof: Putting $C = M/\tau M$, the Proposition follows from

$$ED(C)/C = \tau_V(I(C)/C) = \bigcap \{ \ker(f) \mid f \in \underline{M}(I(C)/C,V) \}$$

and that $\pi^* : \underline{M}(I(C)/C,V) \longrightarrow \Phi$, induced by the canonical projection $\pi : I(C) \longrightarrow I(C)/C$, is a bijection since we can fill the diagram

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] & I(C) \arrow[r] \arrow[d, "\phi"'] & I(C)/C \arrow[r] \arrow[dl, dotted, "\exists g"] & 0 \\ & & V & & \end{tikzcd}$$

with a unique g for any $\phi \in \Phi$. ▮

If V is any injective in $_R\underline{M}$, it is tempting to try and construct injective resolutions from products of copies of V as far as possible because the further we can push such a resolution, the more closely the module being resolved is pinned down by the torsion theory associated with V .

(5.7) Definition: Let $U \in {_R\underline{M}}$. We say that an R-module X has U-dominant dimension $\geqslant n$ [23] (notation: U-dom.dim(X) $\geqslant n$) if there is an exact sequence

$$0 \longrightarrow X \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n$$

such that each X_i is a product of copies of U .

(5.8) Lemma: *If* V *and* W *are similar injective* R-*modules, then* V-dom.dim(X) $\geqslant n$ *if and only if* W-dom.dim(X) $\geqslant n$ *for every* $X \in \underline{M}$.

Proof: Suppose V-dom.dim(X) $> n$. Then there is an exact sequence $0 \longrightarrow X \longrightarrow X_1 \rightarrow \ldots \rightarrow X_n$ such that each X_i is a product of copies of V . Then each X_i is cogenerated by W because of the similarity and the fact that a product of monos is mono in $_R\underline{M}$. From [17,Lemma A.4,p.87] we conclude that W-dom.dim(X) $> n$. The coverse implication follows by symmetry . ▮

Out of the similarity class of an injective, we wish to pick out a distinguished representative having the finiteness condition:

(5.9) Definition: An R-module X is called finitely cogenerating [23] if there is a finite number of elements $f_i \in \underline{M}(R,X)$ $i = 1,\ldots,n$ such that $\bigcap \{ \ker(g) \mid g \in \underline{M}(R,X) \} = \bigcap \{ \ker(f_i) \mid i = 1,\ldots,n \}$.

Notice that this is the dual of finitely generated.

Since $(o:x) = \ker(g)$ for $g \in \underline{M}(R,X)$ defined by $r \longmapsto rx$ $x \in X$ and $\underline{M}(R,X) \simeq X$ by $f \longmapsto f(1)$, X is finitely cogenerating if and only if there exists elements $x_1,\ldots,x_n \in X$ such that

(5.10) $$(0:X) = \bigcap \{ (0:x_i) \mid i = 1,\ldots,n \} \quad .$$

(5.11) Lemma: *Every injective in* ${}_R\underline{M}$ *is similar to a finitely cogenerating injective.*

Proof: Suppose V is any injective R-module. Let $W = V^V$ be the V-fold product of copies of V . Obiously W and V are similar. If ξ is the element of W whose v^{th} coordinate is v for any $v \in V$, then $(0:W) = (0:\xi)$ and W is finitely cogenerating. ▮

Since the dominant dimension of a module is uniform over a similarity class of injectives in the sense of Lemma (5.8), we make the following:

(5.12) Definition: For $\sigma \in IKF(R)$ we say that an R-module X has σ-dominant dimension $\geqslant n$ (notation: $\sigma\text{-dom.dim}(X) \geqslant n$) if $V\text{-dom.dim}(X) \geqslant n$ for any injective V such that $\sigma = \tau_V$.

Using this terminology and keeping Lemma (5.8) in mind, we state:

(5.13) Proposition: *For any* $\sigma \in IKF(R)$ *and* $M \in \underline{M}$

i) M *is σ-torsion-free if and only if* $\sigma\text{-dom.dim}(M) \geqslant 1$

ii) M *is faithfully σ-injective if and only if* $\sigma\text{-dom.dim}(M) \geqslant 2$,

<u>Proof</u> is given in [17,Prop A.6,p.88] and [23,Lemmas 5.1 & 5.2] .

This Proposition provides yet another characterization of the faithfully σ-injective modules. Thus by Lemma (5.11) the full subcategory $\underline{D}(V)$ of $\underline{M}$ consisting of modules with V-dom.dim $\geqslant 2$ for a finitely cogenerating injective V considered by Morita [23] is exactly our category $\underline{A}_\sigma$ associated with $\sigma = \tau_V$. The fact that $\underline{A}$ is an abelian subcategory of $\underline{M}$ if and only if EQ is exact, a condition which does not hold in general, caused trouble in Section 3. Property T is a good attempt at patching up this difficulty and also provides some fringe benifits. In [23,Thm 6.1] Morita gives another answer to this problem: for $\sigma \in$ IKF(R) , the category $\underline{A}$ is an abelian subcategory of $\underline{M}$ if and only if σ-dom.dim(A) $= \infty$ for every $A \in \underline{A}$.

Lemma (5.11) tells us that for any idempotent kernel functor σ there is sone finitely cogenerating injective V such that $\sigma = \tau_V$. This method of picking a distinguished representative from the similarity class of an injective is exploited in [23,Thm 5.6] as follows:

(5.14) <u>Theorem</u>: (Morita) *Let* $\sigma \in$ IKF(R) *and* V *a finitely cogenerating injective such that* $\sigma = \tau_V$. *If* DC *is the double centralizer of* V , *then* EQ(R) = DC *where* EQ *is the localization functor relative to* σ .

(5.15) <u>Observation</u>: From this Theorem and Proposition (3.12) we get very cheaply a result of Kato [15,Thm 2] which says that a faithful finitely cogenerating injective V has the double centralizer property if and only if R is faithfully τ_V-injective. However this is a particular case of a more general result [23,Thm 3.4] for which we have to work considerably harder.

The Theorem (5.14) no longer holds in general if we drop the condition that V be finitely cogenerating. However if W is any injective such that $\sigma = \tau_W$ and DC the double centralizer of W , we do in fact obtain a ring homomorphism $EQ(R) \longrightarrow DC$ which is a monomorphism ! (see [18]) .

(5.16) <u>Example</u>: For any ring R , the injective hull I(R) is always finitely cogenerating injective. Then Theorem (5.14) says that EQ(R) relative to τ_R is the double centralizer if I(R) . This is exactly the definition of the complete ring of quotients given in [16,p.94].

We shall return to this Example again later, but first a generalization of the notion of dense left ideal.

(5.17) <u>Definition</u>: Let I be a left ideal in R . A left ideal J in R is called <u>I-dense</u> if

$$\forall r_1 \notin I \quad \forall r_2 \in R \quad \exists\, r \in R \ni rr_1 \notin I \text{ and } rr_2 \in J .$$

(5.18) <u>Proposition</u>: (Popescu) *For any left ideal* I *in* R *the strongly complete filter* $\underline{F}$ *associated with* τ_N *for* $N = R/I$ *consists exactly of the* I-dense *left ideals.*

<u>Proof</u> is given in [26] of this Proposition and of the following:

(5.19) <u>Corollary</u>: *If* A *is a commutative ring,* I *an ideal,* $N = A/I$ *and* $\underline{F}$ *the strongly complete filter associated with* τ_N *, then* $U \in \underline{F}$ *if and only if* $a \in A$ *such that* $Ua \subseteq I$ *implies* $a \in I$.

(5.20) Example: Recall that in the Goldie Torsion Theory Z_2 the R-module $N = R/z_2(R)$ has zero singular ideal and so is Z_2-torsion-free. This means $Z_2 \leqslant \tau_N$. In order to prove that the converse implication also holds, suppose U is a Z_2-dense left ideal in R, and let $r \notin U$. If $r \varepsilon Z_2(R)$, then $Er \subsetneq Z_1(R)$ for some essential E.
If $Er = 0$ then $(U:r) \supsetneq (0:r) \supsetneq E$ is essential.
If $Er \neq 0$ then $0 \neq xr \varepsilon Z_1(R)$ for some $x \varepsilon E$ which we can pick so that $xr \notin U$ (otherwise $(U:r) \supsetneq E$ is essential already). Now there is an essential E' such that $E'xr = 0 \varepsilon U$ so $(U:xr) \supsetneq E'$ is also essential. Suppose $r \notin Z_2(R)$. If $(U:r)$ is not essential, there is a left ideal $B \neq 0$ such that $(U:r) \cap B = 0$. Let $0 \neq b \varepsilon B$, so $br \notin U$. If $br \notin Z_2(R)$ then because U is $Z_2(R)$-dense there exists x such that $xbr \notin Z_2(R)$ and $xbr \varepsilon U$ ie. $xb \neq 0$, $xb \varepsilon B$ and $xb \varepsilon (U:r)$ - impossible. Hence $br \varepsilon Z_2(R)$. This means $Ebr \subsetneq Z_1(R)$ for some essential E, which is what we want, for if $Ebr = 0$ then $(U:br) \supsetneq (0:br) \supsetneq E$ is essential already.
If $Ebr \neq 0$ then $0 \neq xbr \varepsilon Z_1(R)$ for some $x \varepsilon E$ and we can pick x such that $xbr \notin U$ (otherwise $(U:br) \supsetneq E$ is essential).
Now there is an essential E' such that $E'xbr = 0$ so $(U:xbr) \supsetneq E'$ is essential. In any case, we can always find $s \varepsilon R$ such that $(U:sr)$ is proper essential. Hence by Lemma (1.15,ii) every $Z_2(R)$-dense left ideal is maxi ie. $\tau_N \leqslant Z_2$. This shows $\tau_N = Z_2$ is the largest torsion theory for which $R/Z_2(R)$ is torsion-free, and we obtain another characterization of the maxi left ideals as being $Z_2(R)$-dense .

(5.21) Example: The 0-dense left ideals are just the usual dense left ideals. Hence the strongly complete filter of dense left ideals is exactly the filter associated with τ_R. For the case $\tau = \tau_R$, comparing Proposition (5.6) with [16,Prop 1,p.94] it is clear that $EQ(R)$ is the complete ring of left quotients, which we denote here by Q_m. This agrees with the conclusion of Example (5.16). Since R is τ_R-torsion-free, we consider R as a subring of Q_m.

Let $\sigma \in IKF(R)$ such that $\sigma(R) = 0$. The resulting localised ring $EQ(R)$ relative to such a σ is called a <u>faithful (left) quotient ring</u> of R. The importance of the complete ring of quotients Q_m of R comes from the fact that it exists for every ring R (which was seen not to be the case for the classical ring of quotients) and that any faithful quotient ring of R is a subring of Q_m. This is because $\sigma(R) = 0$ implies $\sigma \leqslant \tau_R$ so that every left ideal in $\underline{F}$ must be dense. Now by Proposition (3.10)

$$EQ(R) = \{\, x \in I(R) \mid (R:x) \in \underline{F}_\sigma \,\} \subseteq \{\, x \in I(R) \mid (R:x) \text{ is dense}\} = Q_m .$$

(5.22) <u>Proposition</u>: *The idempotent kernel functor* τ_R *has property T if and only if* Q_m *has no proper dense left ideals.*

<u>Proof</u>: Suppose τ_R has property T. Let D be a dense left ideal in Q_m. Then $R \cap D$ is dense in R. To see this, take $r_1 \neq 0$ and r in R. Now there exists $q \in Q_m$ such that $qr_1 \neq 0$ and $qr_2 \in D$. For some dense U in R, Uq R by Proposition (3.10) and $Uqr_1 \neq 0$ as Q_m is torsion-free. Hence for some $u \in U$, $uqr_1 \neq 0$ and $uqr_2 \in R$ D which shows that $R \cap D$ is dense.

Now
$$\begin{aligned} Q_m &= EQ(R \cap D) = Q_m \otimes_R (R \cap D) \\ &= Q_m(R \cap D) \qquad \text{as } Q_m \text{ is flat in } \underline{M}_R \\ &\subseteq Q_m D \subseteq D \end{aligned}$$

and so Q_m has no proper dense left ideals.

Conversely, let D be dense in R . Then $Q_m D$ is dense in Q_m : take $q_1 \neq 0$, q_2 in Q_m . For some dense J in R we have $Jq_1 \subsetneq R$ and $Jq_2 \subsetneq R$ with $Jq_1 \neq 0$. So $0 \neq aq_1$ and aq_2 are in R for some $a \in J$. As D is dense, there exists $r \in R$ such that $raq_1 \neq 0$ and $raq_2 \in D \subset Q_m D$. This shows $Q_m D$ is dense in Q_m and so $Q_m D = Q_m$. By Theorem (4.5,iii) we conclude that τ_R has property T . ∎

(5.23) <u>Remark</u>: The above Proposition has an obvious generalization: if $\sigma \in IKF(R)$ with $\phi : R \longrightarrow EQ(R)$ canonical then σ has property T if and only if $EQ(R)$ has no proper left ideals J such that $\phi^{-1}(J) \in \underline{F}_\sigma$. The same argument as in the Proposition shows this statement is equivalent to Theorem (4.5,iii) .

One of the objectives of such a localization process is of course to determine "local properties". By this we mean the following: let SUB(R) be a subset of IKF(R) and suppose $\underline{\Omega}$ is a property satisfied by the ring R (or by an R-module M). Then $\underline{\Omega}$ is called a SUB(R)-local property when R (M) has $\underline{\Omega}$ if and only if EQ(R) (EQ(M)) has $\underline{\Omega}$ for every localization relative to a member of SUB(R) . Now the advantage of having every idempotent kernel functor in the form τ_S allows us to distinguish a subset of IKF(R) by means of a distinguished class of modules, like simples or indecomposable injectives for example. This is done in [9] and in [26] by an equivalent method.

Another objective of this localization is to obtain information about the structure of R via the structure of its localizations $EQ(R)$ by imposing conditions on the rings R and $EQ(R)$ - see for example [24]. A classical example of this is of course the Goldie Theory (as in [11,Chap 7]).

In this thesis the machinery of localization has been developed as it is being used to date, along with some fundamental results and of course examples where, as usual, the real action of the theory is taking place. Indications are that the generalised concept of torsion with its relative localization will prove itself increasingly valuable in the further study of rings and modules.

BIBLIOGRAPHY

1. M. F. ATIYAH and I. G. MacDONALD, "Introduction to Commutative Algebra", Addison-Westley, Reading, Mass., 1969.

2. N. BOURBAKI, "Éléments de Mathématique, Fac XXVII", Hermann, Paris, 1961.

3. K. L. CHEW, Extensions of Rings and Modules, Thesis, University of British Columbia, 1965.

4. P. M. COHN, Rings of Fractions, *Amer. Math. Monthly* 78 (1971), 596-615.

5. S. E. DICKSON, A Torsion Theory for Abelian Categories, *Trans. Amer. Math. Soc.* 21 (1966), 223-235.

6. V. DLAB, The Concept of a Torsion Module, *Amer. Math. Monthly* 75 (1968), 973-976.

7. P. GABRIEL, Des Catégories Abéliennes, *Bull. Soc. Math. France* 90 (1962), 326-448.

8. A. W. GOLDIE, Torsion-free Modules and Rings, *J. Algebra* 1 (1964) 268-287.

9. O. GOLDMAN, Rings and Modules of Quotients, *J. Algebra* 13 (1969), 10-47.

10. A. G. HEINICKE, Triples and Localization, to appear

11. I. N. HERSTEIN, "Noncommutative Rings", Carus Monograph 15, 1968.

12. P. J. HILTON and U. STAMMBACH, "A Course in Homological Algebra", Springer-Verlag, New York, 1971.

13. J. P. JANS, "Rings and Homology", Holt Rinehart and Winston, New York, 1964.

14. __________, Sone Aspects of Torsion, *Pac. J. Math.* 15 (1965), 1249-1259.

15. T. KATO, Rings of U-dominant dimension $\geqslant 1$, *Tohoku Math J.* 21 (1969), 321-327.

16. J. LAMBEK, "Lectures on Rings and Modules", Gin(Blaisdelle), Waltham, Mass., 1966.

17. __________, Torsion Theories, Additive Semantics, and Rings of Quotients, Lecture Notes in Mathematics 177, Springer-Verlag, New York, 1971.

18. _________, Bicommutators of Nice Injectives, *J. Algebra* 21 (1972), 60-73.

19. _________, Noncommutative Localization, Alg. Symposium, Carleton University, 1972 (unpublished).

20. J. K. LUDEMAN, A Generalization of the Concept of a Ring of Quotients, *Can. Math. Bull.* 14 (1971), 517-529.

21. J.-M. MARANDA, Injective Structures, *Trans. Amer. Math. Soc.* 110 (1964), 98-135.

22. B. MITCHELL, "Theory of Categories", Academic Press, New York, 1965.

23. K. MORITA, Localizations in Categories of Modules I, *Math Z.* 114 (1970), 121-144.

24. S. PAGE, Properties of Quotient Rings, to appear.

25. B. PAREIGIS, "Categories and Functors", Academic Press, New York, 1970.

26. N. POPESCU, Le Spectre à Gauche d'un Anneaux, *J. Algebra* 18 (1971), 213-228.

27. __________ and T. SPIRCU, Épimorphisma D'Anneaux, *J. Algebra* 16 (1970), 40-59.

28. L. SILVER, Noncommutative Localization and Applications, *J. Algebra* 7 (1967), 44-76.

29. B. STENSTRÖM, Rings and Modules of Quotients, Lecture Notes in Mathematics 237, Springer-Verlag, New York, 1971.

30. H. TACHIKAWA, Double Centralizers and Dominant Dimensions, *Math. Z.* 116 (1970), 79-88.

31. M. L. TEPLY, Some Aspects of Goldie's Torsion Theory, *Pac. J. Math.* 29 (1969), 447-459.

32. C. T. TSAI, Report on Injective Modules, Queen's Papers No. 6, Queen's University, 1965.

33. C. L. WALKER and E. WALKER, Quotient Categories and Rings of Quotients, to appear.