CONNECTIONS

by

ROBERT ALEXANDER NICOLSON

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Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

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Supervisor: Dr. J. Gamst

ABSTRACT

The main purpose of this exposition is to explore the relations between the notions of covariant derivative, connection, and spray.

We begin by introducing the basic definitions and then use a method of Gromoll, Klingenberg and Meyer to show that covariant derivatives and connections on vector bundles are in a natural one-to-one correspondence.

We conclude by showing that on the tangent bundle of a manifold, sprays and "symmetric" connections are in a natural one-to-one correspondence. Although we use a different method, this re-establishes a result of Ambrose, Palais, and Singer.

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INTRODUCTION

One of the main objects of interest to the differential geometer are the geodesics of a Riemannian manifold.

Recall that, in local coordinates, geodesics are solutions of a system of second order differential equations:

$$\sigma_k''(t) + \sum_k \Gamma_{ij}^k(\sigma(t))\, \sigma_i'(t)\, \sigma_j'(t) = 0$$

where the Γ_{ij}^k are the classical Christoffel symbols. To simplify the notation, we introduce, for x in some coordinate domain, the bilinear

$$\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

given by $\Gamma_x(e_i, e_j) = \sum_k \Gamma_{ij}^k(x) e_k$, where $\{e_i\}$ denotes the standard basis of $\mathbb{R}^n$. In these terms, the above differential equation reads:

$$\sigma''(t) + \Gamma_{\sigma(t)}(\sigma'(t), \sigma'(t)) = 0.$$

Of course, the Γ_x 's depend on the local coordinates one uses. The basic difficulty encountered by the founders of the theory is that the Γ_{ij}^k's "do not transform as the components of a tensor". That is, the Γ_x 's do not define a bilinear map on tangent spaces.

It was Levi-Civita who saw that the bilinear maps

Γ_x do have an intrinsic meaning: they allow one to introduce absolute differentiation and parallel transport along curves. We recall the result: for curves $\sigma : I \longrightarrow M$ on a Riemannian manifold M one considers vector fields ω along σ, that is, "lifts" of σ to curves

$$\omega : I \longrightarrow TM$$

on the tangent bundle TM of M. Absolute differentiation associates with ω a further vector field $\nabla_t \omega$ along σ. In local coordinates:

$$\nabla_t \omega : \; t \longmapsto \omega'(t) + \Gamma_{\sigma(t)}(\sigma'(t), \omega(t)).$$

ω is called a parallel family along σ if $\nabla_t \omega = 0$. So, in local coordinates, parallel families are the solutions of a homogeneous linear differential equation. Therefore, for each curve σ on M, the parallel families along σ form a vector space P_σ, and evaluation at any t_0 in the domain of σ gives a linear isomorphism of P_σ with the tangent space of M at $\sigma(t_0)$. Put differently, parallel families give a specific way of propagating tangent vectors along curves.

In terms of parallel families, σ is a geodesic if and only if σ' is a parallel family along σ. Moreover, one can use "parallel transports" along integral curves of a vector field X to define a general covariant derivative $\nabla_X Y$ of vector fields Y with respect to X.

Considerable effort has been spent in the last fifty years to find intrinsic formulations of the various aspects of the foregoing theory.

The idea of absolute differentiation leads to the notion of a covariant derivative on a vector bundle: that is, an operator taking a vector field X on a manifold M and a section S of a vector bundle E over M to a further section $\nabla_X S$ of E. In Chapter One we review the basic facts about covariant derivatives and show how one extracts the analogue of the Christoffel symbols from the formal definition.

The problem of formulating the idea of parallel transport along curves turns out to be more subtle. One might be tempted to specify, for each curve σ on the manifold, the vector space of all parallel families along σ. However, this is not how parallel transport arises in practice. Moreover, it is technically difficult to formulate smoothness conditions in such a context. The way out is to consider the "infinitesimal" aspect of the situation. In other words, one prescribes for each tangent vector ξ at a point x on a manifold M, the vector space of all "initial velocity vectors" of parallel families along curves which pass through x with "velocity" ξ.

Note that parallel families are curves on the tangent bundle, hence their "velocity vectors" are in the tangent bundle of the tangent bundle! So, one is forced to consider the "double tangent bundle" of a manifold. Actually, one gains in clarity by generalizing to the case of an arbitrary vector bundle. Accordingly, in Chapter Two we start by analyzing the structure of the tangent bundle TE of a vector bundle E over a manifold. The difficulties with the failure of the Cristoffel symbols to "transform like a tensor" show up again: TE is not a vector bundle over M. TE does, however, carry two distinct vector bundle structures, one over E, and the other over the tangent bundle of M.

We then give the formal definition of a connection on a vector bundle E over a manifold M: it is a map which assigns to each tangent vector ξ of M a subspace of the fibre of TE over ξ with respect to the vector bundle structure over TM.

Finally, Ambrose, Palais and Singer showed how one can deal directly with geodesics by introducing the notion of a spray on a manifold M: it is a vector field on the tangent bundle bundle of M whose integral curves "look like geodesics".

In Chapter Four we recall the basic properties of sprays and show how one obtains the analogue of the Christoffel symbols.

The main purpose of this exposition is to explore the relations between the notions of covariant derivative, connection, and spray.

In Chapter Three we use a method of Gromoll, Klingenberg, and Meyer to show that covariant derivatives and connections on vector bundles are in a natural one-to-one correspondence.

In Chapter Four we show how, on the tangent bundle of a manifold, sprays and "symmetric" connections are in a natural one-to-one correspondence. Thus, we re-establish the main result of Ambrose, Palais and Singer by a different method.

Chapter 1: Covariant Derivatives

Section 1.1: Definition

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. For an open subset U of M, let V(U) denote the vector space of smooth vector fields defined on U, and $\Gamma_U(E)$ the vector space of smooth sections of E over U.

DEFINITION: A covariant derivative on E is an operator

$$\begin{aligned} \nabla : V(M) \times \Gamma_M(E) &\longrightarrow \Gamma_M(E) \\ (X, S) &\longmapsto \nabla_X S \end{aligned}$$

having the following properties:

D1) $$\nabla_{X+Y}S = \nabla_X S + \nabla_Y S,$$

D2) $$\nabla_{fX}S = f\nabla_X S,$$

D3) $$\nabla_X(S+T) = \nabla_X S + \nabla_X T, \text{ and}$$

D4) $$\nabla_X(fT) = (Xf)T + f\nabla_X T,$$

(where f is any smooth real-valued function defined on M).

Section 1.2: Local Coordinates

In order to exhibit local coordinates for covariant derivatives we first examine their restriction to open subsets of M.

§ 1.2.1: Restriction to Open Sets

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let ∇ be a covariant derivative on E, and let U be an open subset of M.

LEMMA:

(a) If $Y \in V(M)$ vanishes on U, then $\nabla_Y S$ vanishes on U for each $S \in \Gamma_M(E)$, and

(b) if $T \in \Gamma_M(E)$ vanishes on U, then $\nabla_X T$ vanishes on U for each $X \in V(M)$.

PROOF: We will prove only (b), for (a) is even more straightforward.

Fix some point $y \in U$, and let $f : M \longrightarrow \mathbb{R}$ be a smooth function, having support in U, such that f is identically 1 on some neighbourhood of y. Then by definition

$$fT = 0 \in \Gamma_M(E).$$

Thus

$$0 = \nabla_X fT = (Xf)T + f\nabla_X T,$$

so, at y, we have

$$0 = (Xf)(y)T(y) + f(y)(\nabla_X T)(y) = (\nabla_X T)(y).$$

Since $y \in U$ was arbitrarily chosen, the lemma is proved.

PROPOSITION: If ∇ is a covariant derivative on E, and U is an open subset of M, then there exists a covariant derivative $\tilde{\nabla}$ on $p^{-1}(U)$ such that, for any $(X, S) \in V(M) \times \Gamma_M(E)$,

$$(\nabla_X S)(y) = \tilde{\nabla}_{\tilde{X}}\tilde{S}(y)$$

for each $y \in U$, where $\tilde{X} \in V(U)$ and $\tilde{S} \in \Gamma_U(E)$ are the restrictions of X and S respectively.

PROOF: The problem is to define $\tilde{\nabla}$ on vector fields and sections which may not be extendable to global vector fields and sections. So, let X be in V(U) and S in $\Gamma_U(E)$, and fix some point $y \in U$. Let f be as in the proof of the lemma. Then $fX \in V(M)$ and $fS \in \Gamma_M(E)$, so we may set

$$(\tilde{\nabla}_X S)(y) = (\nabla_{fX} fS)(y).$$

The lemma ensures that $\tilde{\nabla}$ is well-defined, and by construction it is a covariant derivative on $p^{-1}(U)$ satisfying the conclusion of the proposition.

§ 1.2.2: Explicit Local Coordinates

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let ∇ be a covariant derivative on E, and let U be an open subset of M such that TM, the tangent bundle of M, and E are trivial over U. Note that V(U) may be identified with $C^\infty(U,\mathbb{R}^n)$, the vector space of smooth $\mathbb{R}^n$-valued functions defined on U, and $\Gamma_U(E)$ with $C^\infty(U,\mathbb{R}^k)$, by considering principal parts. The restricted covariant derivative $\tilde{\nabla}$ thus induces an operator

$$\delta^U : C^\infty(U,\mathbb{R}^n) \times C^\infty(U,\mathbb{R}^k) \longrightarrow C^\infty(U,\mathbb{R}^k).$$

According to the definition in [1.1], this operator δ^U is linear over $C^\infty(U,\mathbb{R})$ in the first argument but not in the second. If, however, for $F \in C^\infty(U,\mathbb{R}^n)$ and $G \in C^\infty(U,\mathbb{R}^k)$, we define

$$\Gamma^U(F,G) = \delta^U(F,G) - DG(F)$$

where $$DG : U \longrightarrow \mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^k)$$

is the derivative of G, then one checks easily that Γ^U is linear over $C^\infty(U,\mathbb{R})$ in both variables.

LEMMA: If an operator

$$\Gamma : C^\infty(U,\mathbb{R}^n) \times C^\infty(U,\mathbb{R}^k) \longrightarrow C^\infty(U,\mathbb{R}^k)$$

is bilinear over $C^\infty(U,\mathbb{R})$, then

(a) if $F \in C^\infty(U,\mathbb{R}^n)$ vanishes at $x \in U$, $\Gamma(F,G)$ vanishes at x for each $G \in C^\infty(U,\mathbb{R}^k)$, and

(b) if $G \in C^\infty(U,\mathbb{R}^k)$ vanishes at $x \in U$, $\Gamma(F,G)$ vanishes at x for each $F \in C^\infty(U,\mathbb{R}^n)$.

PROOF: We will prove only the first assertion. For each i from 1 to n, let $E_i \in C^\infty(U,\mathbb{R}^n)$ denote the constant function onto the i^{th} canonical basis vector of $\mathbb{R}^n$. Then we may write

$$F = \sum_i f_i E_i$$

where each $f_i \in C^\infty(U,\mathbb{R})$ vanishes at x. Then for any $G \in C \quad (U,\mathbb{R}^k)$ we have

$$\begin{aligned}\Gamma(F,G)(x) &= \Gamma(\textstyle\sum_i f_i E_i, G)(x) \\ &= \textstyle\sum_i f_i(x)\Gamma(E_i, G)(x) \\ &= 0.\end{aligned}$$

Returning to the earlier discussion, we can now see that for each $x \in U$, Γ^U induces a bilinear morphism

$$\Gamma_x : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k.$$

That is; if, for $(dx,u) \in \mathbb{R}^n \times \mathbb{R}^k$, we choose $F \in C^\infty(U,\mathbb{R}^n)$ such that $F(x) = dx$, and $G \in C^\infty(U,\mathbb{R}^k)$ such that $G(x) = u$, and set

$$\Gamma_x(dx,u) = \Gamma^U(F,G)(x),$$

then, by the lemma, Γ_x is well-defined; and, by construction, it is bilinear. Moreover, by definition the map

$$\Gamma : U \longrightarrow \mathrm{Bil}(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^k)$$

$$x \longmapsto \Gamma_x$$

is smooth.

So, over U, ∇ is represented by the smooth family of bilinear $\Gamma_x : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ in the sense that

$$\delta^U(F,G)(x) = DG\big|_x F(x) + \Gamma_x(F(x),G(x)).$$

Consequently, $\nabla_X S(x)$ is already determined by the value of X at x and the values of S along any smooth curve fitting X(x).

§ 1.3: Parallel Transport

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let ∇ be a covariant derivative on E, and let $\sigma : I \longrightarrow M$ be a smooth curve.

By a section of E along σ we mean a smooth

$$S : I \longrightarrow E$$

such that $p \circ S = \sigma$. <u>The covariant derivative of a section S of E along σ</u> is the new section

$$\nabla_\tau S \quad : \quad I \longrightarrow E$$

of E along σ defined as follows:

For fixed $t_0 \in I$, we note that the map

$$\alpha : t \longmapsto (t_0 + t)$$

is a smooth curve representing $\sigma'(t_0) = \xi \in T_{\sigma(t_0)}M$. Moreover, S is defined along α, so we may form

$$\nabla_\xi S(t_0),$$

which we define to be the value of $\nabla_\tau S$ at t_0.

In local coordinates, if ∇ is given by

$$\Gamma_x \quad : \quad \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k,$$

and S by

$$G \quad : \quad I \longrightarrow \mathbb{R}^k,$$

one finds that S is given by

$$t \longmapsto G'(t) + \Gamma_{\sigma(t)}(\sigma'(t), G(t)).$$

We say that a section of E along σ is <u>parallel along σ</u> if

$$\nabla_\tau S \equiv 0,$$

Locally, finding sections parallel along σ means solving

$$G'(t) + \Gamma_{\sigma(t)}(\sigma'(t), G(t)) \equiv 0$$

for G. Since this is a linear homogeneous differential equation, we see that parallel sections along any given curve exist, and are uniquely determined by any one of their values.

§ 1.4: The Difference Between Two Covariant Derivatives

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let ∇ and $\tilde{\nabla}$ be covariant derivatives on E.

Since $\Gamma_M(E)$ is a real vector space, the following is well-defined. Let $D(\nabla, \tilde{\nabla})$ denote the map

$$D(\nabla, \tilde{\nabla}) \quad : \quad V(M) \times \Gamma_M(E) \longrightarrow \Gamma_M(E)$$
$$(X, S) \longmapsto \nabla_X S - \tilde{\nabla}_X S.$$

Clearly $D(\nabla,\tilde{\nabla})$ satisfies conditions D1) to D3) of [1.1] and condition D4) yields:

$$\begin{aligned} D(\nabla,\tilde{\nabla})(X,fS) &= \nabla_X(fS) - \tilde{\nabla}_X(fS) \\ &= (Xf)S + f\nabla_X S - (Xf)S - f\tilde{\nabla}_X S \\ &= f\nabla_X S - f\tilde{\nabla}_X S \end{aligned}$$

Thus the difference $D(\nabla,\tilde{\nabla})$ is bilinear over $C^\infty(M,\mathbb{R})$. $D(\nabla,\tilde{\nabla})$ is called the <u>difference tensor</u> of ∇ and $\tilde{\nabla}$.

Suppose now that ∇ is any covariant derivative defined on E, and that

$$D \quad : \quad V(M) \times \Gamma_M(E) \longrightarrow \Gamma_M(E)$$

is bilinear (over $C^\infty(M,\mathbb{R})$). Setting $\tilde{\nabla} = \nabla - D$, it is trivial to check that $\tilde{\nabla}$ is then a covariant derivative of E, and that $D(\nabla,\tilde{\nabla}) = D$.

Locally, over a suitable coordinate domain U in M, we may represent ∇ by a bilinear map

$$\Gamma^U : C^\infty(U,\mathbb{R}^n) \times C^\infty(U,\mathbb{R}^k) \longrightarrow C^\infty(U,\mathbb{R}^k),$$

and $\tilde{\nabla}$ by a similar map $\tilde{\Gamma}^U$. The difference $D(\nabla,\tilde{\nabla})$ is then represented by $\Gamma^U - \tilde{\Gamma}^U$ which is again bilinear over $C^\infty(U,\mathbb{R})$.

Section 1.5: Covariant Derivatives on Manifolds

§ 1.5.1: Definition and Local Coordinates

Let M be a smooth manifold of dimension n, and let $\pi : TM \longrightarrow M$ denote its tangent bundle. Recall that this is a smooth vector bundle over M with fibre $\mathbb{R}^n$.

A covariant derivative on M is a covariant derivative on TM as defined in [1.1]. That is, an operator

$$\begin{aligned} \nabla : V(M) \times V(M) &\longrightarrow V(M) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

satisfying conditions D1) to D4).

Let U be an open subset of M such that $\pi^{-1}(U)$ is trivial. Then, as in [1.2.2], we may identify V(U) with $C^\infty(U,\mathbb{R}^n)$. Let X and Y be smooth vector fields on U with corresponding maps F and G $\in C^\infty(U,\mathbb{R}^n)$ respectively. Then, in the notation of [1.2.2], we have, for $x \in U$,

$$\delta(F,G)(x) = DG(F)(x) + \Gamma_x(F(x),G(x))$$

where the induced map

$$\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is bilinear. Since ∇ is determined locally by these Γ_x 's, we may derive an explicit representation of in terms of the coordinate system of U.

§ 1.5.2: Classical Notation

We will continue with the notation of the last paragraph. Let E_i denote the i^{th} canonical basis vector of $\mathbb{R}^n$. Then we may write

$$F = \sum_i f_i E_i$$

where each $f_i \in C^\infty(U,\mathbb{R})$, and similarly

$$G = \sum_j g_j E_j.$$

Then, since Γ_x is bilinear, we have

$$\begin{aligned}\Gamma_x(F(x),G(x)) &= \Gamma_x\left(\sum_i f_i(x)E_i, \sum_j g_j(x)E_j\right)\\ &= \sum_{i,j} f_i(x)g_j(x)\,\Gamma_x(E_i,E_j).\end{aligned}$$

Moreover, each map

$$\begin{aligned}\Gamma_{ij} : U &\longrightarrow \mathbb{R}^n\\ x &\longmapsto \Gamma_x(E_i,E_j)\end{aligned}$$

must be smooth, so it may be written

$$\Gamma_{ij} = \sum_k \Gamma_{ij}^k E_k,$$

where each $\Gamma_{ij}^k \in C^\infty(U,\mathbb{R})$.

Finally we have

$$\Gamma_x(F(x),G(x)) = \sum_{i,j} f_i(x) g_j(x) [\sum_k \Gamma_{ij}^k(x) E_k],$$

so Γ, and hence ∇, is given locally by specifying n^3 smooth real-valued functions on U.

Section 1.6: Geodesics

Let M be a smooth manifold of dimension n, and let ∇ be a covariant derivative on M (that is, on the tangent bundle of M). Let $\sigma : I \longrightarrow M$ be a smooth curve. By a vector field along σ we mean a section $\omega : I \longrightarrow TM$ of TM along σ. As in [1.3] we say that ω is parallel along σ if $\nabla_\tau \omega = 0$. Note that the canonical lift of σ is a particular vector field

$$\sigma' : I \longrightarrow TM$$

along σ. We say that σ is a geodesic with respect to ∇ if σ' is parallel along σ. So, in local coordinates, where ∇ is represented by

$$\Gamma_x : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

σ is a geodesic if and only if

$$\sigma''(t) + \Gamma_{\sigma(t)}(\sigma'(t),\sigma'(t)) \equiv 0.$$

In other words, to find geodesics means to solve an explicit second order differential equation which is quadratic in σ'. By the existence theory of ordinary differential equations, for each $x \in M$, and each $\xi \in T_xM$, there exists a unique geodesic $\sigma : I \longrightarrow M$ such that $\sigma(0) = x$, and $\sigma'(0) = \xi$.

Section 1.7: Difference and Torsion

Let M be a smooth manifold of dimension n, and let ∇ and $\tilde{\nabla}$ be covariant derivatives defined on M. As in [1.3], their difference tensor

$$D : V(M) \times V(M) \longrightarrow V(M)$$
$$(X, Y) \longmapsto \nabla_X Y - \tilde{\nabla}_X Y$$

is bilinear. In this case, moreover, we may decompose D into symmetric and alternating parts. Thus we write

$$D(X,Y) = S(X,Y) + A(X,Y)$$

where

$$S(X,Y) = (1/2)[D(X,Y) + D(Y,X)],$$

and

$$A(X,Y) = (1/2)[D(X,Y) - D(Y,X)].$$

PROPOSITION: The following are equivalent:

(1) ∇ and $\tilde{\nabla}$ have the same geodesics,

(2) $\nabla_X X = \tilde{\nabla}_X X$ for each $X \in V(M)$, and

(3) S, the symmetric part of D, vanishes.

PROOF:

Clearly, (3) implies (2). Conversely, if

$$0 = D(X,X) = S(X,X)$$

for all $X \in V(M)$, we obtain

$$0 = S(X+Y,X+Y) = 2S(X,Y)$$

for all $X,Y \in V(M)$, since S is symmetric. Thus, (2) and (3) are equivalent.

To see that (1) and (2) are equivalent, we work in local coordinates. By definition, a curve σ is a geodesic for ∇ if

$$\sigma''(t) + \Gamma_{\sigma(t)}(\sigma'(t), \sigma'(t)) = 0,$$

and a geodesic for $\tilde{\nabla}$ if

$$\sigma''(t) + \tilde{\Gamma}_{\sigma(t)}(\sigma'(t), \sigma'(t)) = 0$$

(where Γ and $\tilde{\Gamma}$ represent ∇ and $\tilde{\nabla}$). Since, for each $x \in M$ and $\xi \in T_xM$, there is a geodesic σ with $\sigma(0) = x$, and $\sigma'(0) = \xi$ we see that (1) means:

$$\Gamma_x(u,u) = \tilde{\Gamma}_x(u,u)$$

for all $u \in \mathbb{R}^n$. Consequently, (1) is equivalent to (2).

By the torsion tensor of a covariant derivative ∇ on M, we mean the map

$$T_\nabla : V(M) \times V(M) \longrightarrow V(M)$$

defined by

$$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[X,Y]$ is the usual bracket of smooth vector fields. It is easily verified that T_∇ is bilinear and alternating over $C^\infty(M,\mathbb{R})$.

A covariant derivative ∇ on M is said to be <u>torsion-free</u> if $T_\nabla \equiv 0$.

Locally, if X and Y in V(M) are represented by F and G in $C^\infty(U,\mathbb{R}^n)$ respectively, then $[X,Y]$ is represented by $DG(F) - DF(G)$. Since $\nabla_X Y$ is represented by $DG(F) + \Gamma^U(F,G)$, then we know $T_\nabla(X,Y)$ will be given by the map

$$\tau : U \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \Gamma_x(G(x),F(x)) - \Gamma_x(F(x),G(x)).$$

Note that ∇ is torsion-free if and only if each corresponding Γ_x is symmetric.

By straightforward computation, one may establish the following:

<u>LEMMA</u>: Let ∇ and $\tilde{\nabla}$ be covariant derivatives on M with torsion tensors T_∇ and $T_{\tilde{\nabla}}$ respectively. Then if A denotes the alternating part of the difference tensor of ∇ and $\tilde{\nabla}$, we have

$$T_\nabla - T_{\tilde{\nabla}} = 2A.$$

<u>PROPOSITION</u>: For any covariant derivative ∇ on M, there exists a unique torsion-free covariant derivative $\tilde{\nabla}$ on M having the same geodesics as ∇.

PROOF: If $D = S + A$ denotes the difference tensor of ∇ and $\tilde{\nabla}$, then the conditions we want are

$$T_{\tilde{\nabla}} = 0$$

and

$$D = A.$$

Therefore, by the lemma, we must set

$$\tilde{\nabla} = \nabla - (1/2)T_{\nabla} .$$

One may show by computation that $\tilde{\nabla}$ is the desired derivative.

Chapter 2: Bundle Connections

From the geometric viewpoint, it is desirable to characterize connections in terms of morphisms of vector bundles. A detailed digression on the vector bundles involved is required in order to accomplish this. We begin by considering smooth groups and their associated tangent bundles.

Section 2.1: Tangent Bundles of Smooth Groups

A smooth group is a smooth manifold having a compatible group structure. That is, a structure under which multiplication and the taking of inverses are smooth operations. We will show that the tangent bundle of a smooth group inherits a compatible group structure. The proof of this is greatly simplified if we express the definition of a smooth group in the language of diagrams.

§ 2.1.1: Definition of a Smooth Group

DEFINITION: A smooth group is a smooth manifold G together with smooth maps

$$m : G \times G \longrightarrow G$$

and

$$i : G \longrightarrow G$$

such that:

(1) the following diagram commutes (associativity):

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\ \Big\downarrow{\scriptstyle m \times \mathrm{id}_G} & & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

(2) there exists a smooth map $*$ of the one-point manifold into G such that the following diagram commutes (unit element):

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id}_G,\, *)} & G \times G \\ \Big\downarrow{\scriptstyle (*,\mathrm{id}_G)} & \searrow{\scriptstyle \mathrm{id}_G} & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

, and

(3) the following diagram commutes (inverses):

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id}_G,\, i)} & G \times G \\ \Big\downarrow{\scriptstyle (i,\mathrm{id}_G)} & \searrow{\scriptstyle *} & \Big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G. \end{array}$$

Note that all of the maps appearing in this definition are smooth. We may therefor apply the tangent functor T throughout the definition and thus gain information about the tangent bundle TG.

§ 2.1.2: The Tangent Group of a Smooth Group

Let G be a smooth group and TG the associated tangent bundle. The following theorem shows that TG has a natural group structure compatible with its manifold structure.

THEOREM: If multiplication and inverses for G are given by maps m and i respectively, then the tangent maps Tm and Ti induce a compatible group structure on TG.

Examining the diagrams of [2.1.1], we see that this is an easy consequence of the fact that the tangent functor T commutes with products. Thus we need only prove the following lemma.

LEMMA: If M and N are smooth manifolds, then $T(M \times N)$ and $TM \times TN$ are naturally diffeomorphic.

PROOF: Let pr_1 denote the canonical projection of $M \times N$ onto M, and pr_2 the corresponding projection onto N. The definition of $M \times N$ ensures that these are smooth maps. Thus they will have smooth tangents:

$$Tpr_1 \ : \ T(M \times N) \longrightarrow TM$$

and

$$Tpr_2 \ : \ T(M \times N) \longrightarrow TN.$$

Together, these induce a smooth morphism of

$$T(M \times N) \text{ onto } TM \times TN$$

Recall that a tangent vector ξ at $x \in M$ may be represented by a smooth curve $\sigma : I \longrightarrow M$ where I is an open interval of $\mathbb{R}$ containing 0, $\sigma(0) = x$, and $\sigma'(0) = \xi$.

Let α represent a tangent vector at $x \in M$, and β a tangent vector at $y \in N$. Then, the definition of $M \times N$ ensures that there exists a unique curve γ at $(x,y) \in M \times N$ such that

$$pr_1 \circ \gamma = \alpha$$

and

$$pr_2 \circ \gamma = \beta$$

This induces a morphism of $TM \times TN$ into $T(M \times N)$ which is easily seen to be smooth. This new morphism is inverse to the one introduced above, so the manifolds are indeed naturally diffeomorphic.

We may now examine more closely the structure of TG. Note that the map

$$Tm : TG \times TG \longrightarrow TG$$

is given locally by:

$$Tm: ((g,u), (h,v)) \longmapsto (m(g,h), T_{g,h}m(u,v)).$$

If α represents u, and β represents v, then $T_{g,h}m(u,v)$

may be represented by γ, where

$$\gamma(t) = m(\alpha(t), \beta(t)).$$

Thus on the tangent level, the multiplication comes from pointwise multiplication of curves. Consequently, the unit element of TG will be that vector in the fibre of TG over the unit element of G which represents the constant curve at that point. That is, the zero vector.

§ 2.1.3: Decomposition of TG

Let G be a smooth group, and TG the associated tangent bundle. Let $p : TG \longrightarrow G$ denote the canonical projection. The group structure of TG may be more explicitly viewed under the decomposition to follow.

Let m and i represent multiplication and inversion on G respectively, and let Tm and Ti be their associated tangent maps. From the definition of Tm as given in [2.1.2] we derive:

$$\begin{aligned} p \circ Tm((g,u),(h,v)) &= m(g,h) \\ &= m(p \times p)((g,u),(h,v)). \end{aligned}$$

Thus p is a smooth group homomorphism of TG onto G.

Let 0_G denote the canonical "zero-section" of TG. That is,

$$\begin{aligned} 0_G : G &\longrightarrow TG \\ g &\longmapsto (g,0). \end{aligned}$$

Since $0 \in T_g G$ is represented by the constant curve at g, and tangent multiplication is essentially pointwise, 0_G is also a smooth group homomorphism. Moreover, $p \circ 0_G$ is the identity map on G.

This situation may be neatly described algebraically in terms of a <u>semi-direct product</u>, defined as follows.

Let H and G be groups, and let p be a group homomorphism of H onto G. Let K denote the kernel of p, and i the inclusion of K into H. Let s be a group homomorphism of G into H such that $p \circ s = id_G$. The following diagram describes the situation:

$$K \xrightarrow{\ i\ } H \underset{p}{\overset{s}{\rightleftarrows}} G.$$

Note that for any $h \in H$,

$$p(h \cdot s \circ p(h^{-1})) = p(h) \cdot p(h)^{-1} = 1$$

so $h \cdot s \circ p(h^{-1}) = i(k)$ for some $k \in K$. Thus $h = i(k) s \circ p(h)$, and as sets, $H = i(K) \times s(G)$. In terms of this decomposition, multiplication is given by:

$$i(k)s(g) \cdot i(k')s(g') = i(k)[s(g)i(k')s(g^{-1})]s(g)s(g')$$

K is normal; so, $s(g)i(k')s(g^{-1}) \in i(K)$, and so the product is in the desired form.

If we abuse notation, and let $g \in G$ denote the action

$$g : i(k) \longmapsto s(g)i(k)s(g^{-1}),$$

then our formula becomes

$$[i(k)s(g)]\cdot[i(k')s(g')] \;=\; [i(k)gi(k')]\cdot[s(g)s(g')]$$

Algebraically, then, H is said to be the semi-direct product of G and K relative to the action of G on i(K) defined above.

Returning to the earlier discussion and notation, we have that TG is the semi-direct product of G with ker(p) relative to the action on ker(p) given by conjugation with elements of 0_G.

§ 2.1.4: Reinterpretation of TG

Let G be a smooth group with multiplication given by a smooth map m. Let e denote the unit element of G. Let TG be the tangent bundle of G, and let p and 0_G be as defined in [2.1.3]. We now identify ker(p) and the action of 0_G mentioned in the last section.

Clearly, as a manifold, ker(p) is just $T_e(G)$. Note that, as a vector space, T_eG has the structure of an additive group, and that there is a canonical action of G on T_eG given as follows:

For $g \in G$, let

$$\mathrm{int}(g) : G \longrightarrow G$$

be the inner automorphism

$$h \longmapsto ghg^{-1}$$

given by g. Since int(g) leaves e fixed, its tangent map at e,

$$\mathrm{ad}(g) = T_e(\mathrm{int}(g)) \;:\; T_eG \longrightarrow T_eG$$

must be linear. The resulting homomorphism

$$\mathrm{ad} : G \longrightarrow \mathrm{Lin}(T_eG, T_eG)$$

is known as the <u>adjoint</u> <u>representation</u>.

<u>THEOREM</u>: In the above notation:

(a) the group structure on ker(p) induced by Tm is vector space addition in T_eG, and

(b) the action of G on ker(p) induced by the semi-direct decomposition of TG is the adjoint representation.

<u>PROOF</u>:

To see that (a) holds, we recall that the unit element of TG is the zero-vector in T_eG. Therefore, since

$$T_em : T_eG \times T_eG \longrightarrow T_eG$$

is linear,

$$T_em(\xi, \eta) = T_em(\xi, 0) + T_em(0, \eta) = \xi + \eta .$$

(b) follows from the fact that if $\eta \in T_eG$ is represented by a curve σ, then $\mathrm{ad}(g)(\eta)$ is represented by the curve

$$t \longmapsto g\sigma(t)g^{-1}.$$

Thus, if α is the constant curve with image g, we may rewrite the above map as

$$t \longmapsto \alpha(t)\sigma(t)[\alpha(t)]^{-1}$$

But, by definition, this is the curve representing $0_G(g)\eta\, 0_G(g^{-1})$, so we are done.

Note that ker(p), with this structure, is the additive group of the Lie-Algebra of G, denoted L(G).

To sum up: <u>if G is a smooth group, then TG is the semi-direct product of G with the additive group of L(G) relative to the adjoint representation.</u>

<u>§ 2.1.5</u> : <u>Actions of Smooth Groups on Smooth Manifolds</u>

We define here what is meant by the action of a smooth group on a smooth manifold.

Let G be a smooth group with multiplication given by m. Let M be a smooth manifold. We say that G acts smoothly on M if there exists a smooth map

$$\alpha : G \times M \longrightarrow M$$

such that:

(1) the following diagram commutes:

$$\begin{array}{ccc} G \times G \times M & \xrightarrow{m \times id_M} & G \times M \\ \Big\downarrow {\scriptstyle id_G \times \alpha} & & \Big\downarrow {\scriptstyle \alpha} \\ G \times M & \xrightarrow[\alpha]{} & M, \end{array}$$

and

(2) the following diagram also commutes:

$$\begin{array}{ccc} M & \xrightarrow{(*, id_M)} & G \times M \\ & {\scriptstyle id_M} \searrow & \Big\downarrow {\scriptstyle \alpha} \\ & & M \end{array}$$

Clearly, if G acts smoothly on M, then TG will act smoothly on TM with the tangent action.

§ 2.1.6: Special Case: GL_k

The smooth group that we will be interested in is GL_k, the group of all linear automorphisms of $\mathbb{R}^k$.

As an open subset of $\mathbb{R}^{k^2}$, GL_k has a smooth structure which clearly is compatible with the group structure.

By definition, GL_k acts linearly (and hence smoothly) on $\mathbb{R}^k$ by:

$$GL_k \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

$$(A\ ,\ u) \longmapsto Au.$$

As a manifold, the tangent bundle TGL_k simply is $GL_k \times M_k$, where M_k is the vector space of all $(k \times k)$-matrices. To identify the group structure of TGL_k, we look at the "tangent action"

$$TGL_k \times \mathbb{R}^k \times \mathbb{R}^k \longrightarrow \mathbb{R}^k \times \mathbb{R}^k,$$

where we have identified $T\mathbb{R}^k$ with $\mathbb{R}^k \times \mathbb{R}^k$.
Clearly the action is given by:

$$(A,M,u,v) \longmapsto (Au, Mu + Av).$$

Therefore, TGL_k can be identified with the subgroup of GL_{2k} consisting of all matrices of the form

$$\left\{ \begin{bmatrix} A & 0 \\ M & A \end{bmatrix} \middle|\ A \in GL_k, M \in M_k \right\}.$$

Note then that p is given by

$$\begin{bmatrix} A & 0 \\ M & A \end{bmatrix} \longmapsto A,$$

and O_G by

$$A \longmapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

so the semi-direct decomposition is

$$\begin{bmatrix} A & 0 \\ M & A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ MA^{-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = (MA^{-1}, A).$$

Section 2.2: The Double Structure of TE

The tangent bundle of a smooth vector bundle inherits two smooth vector bundle structures.

§ 2.2.1: The Standard Structure

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Since E is a smooth manifold, it has a tangent bundle which we denote

$$\pi_E : TE \longrightarrow E.$$

We call this <u>the standard vector bundle structure of TE over E</u>, or simply the E-structure.

If the transition function between two intersecting coordinate domains of M is given by

$$x \longmapsto h(x)$$

where h is a diffeomorphism between open subsets of

$\mathbb{R}^n$, then the corresponding transition for the tangent bundle TM is given by:

$$(x,dx) \longmapsto (h(x),h'(x)dx).$$

Then since local trivializations of E have transition functions of the form

$$(x,u) \longmapsto (h(x),t(x)u),$$

where t is a smooth mapping of an open subset of $\mathbb{R}^n$ into GL_k, the corresponding transitions with respect to the E-structure of TE will be given by

$$\boxed{(x,u,dx,du) \longmapsto (h(x),t(x)u,h'(x)dx,t(x)du+t'(x)(dx)u).}$$

Note that "fibrewise", this is linear in (dx,du).

§ 2.2.2: The Tangent Structure

Again let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. We will show that the tangent map

$$Tp : TE \longrightarrow TM$$

gives TE a smooth vector bundle structure over TM.

It suffices to exhibit a system of local trivializations of TE over TM in such a way that the transition functions act linearly on the fibres of Tp. To do this, fix some point in TE, and suppose that $(U,\varphi,\emptyset)$ is a vector bundle chart at its image under $\pi_E : TE \longrightarrow E$.

Then since

$$\emptyset : p^{-1}(U) \longrightarrow \varphi(U) \times \mathbb{R}^k$$

is a diffeomorphism,

$$T\emptyset : T(p^{-1}(U)) \longrightarrow T(\varphi(U) \times \mathbb{R}^k)$$

is also. But we know that the tangent functor T commutes with products, and from the definition of Tp we get that $T(p^{-1}(U)) = Tp^{-1}(TU)$, so $(TU, T\varphi, T\emptyset)$ gives a local trivialization of TE.

In [2.2.1] we saw that the transition functions for this system of local coordinates have the form

$$(x,u,dx,du) \longmapsto (h(x),t(x)u,h'(x)dx,t(x)du+t'(x)(dx)u),$$

where h is a diffeomorphism of open subsets of $\mathbb{R}^n$, and t is a smooth map from an open subset of $\mathbb{R}^n$ into GL_k. If we restrict our attention to fibre of Tp over (x,dx), then we induce a morphism of the form

$$(u,du) \longmapsto (t(x)u, t(x)du + t'(x)dxu).$$

Note that the matrix representation

$$\begin{bmatrix} t(x) & 0 \\ t'(x)dx & t(x) \end{bmatrix}$$

of this morphism is identical to the image of the tangent map of t at (x,dx). This lies in TGL_k which was established in [2.1.6] to be a subgroup of GL_{2k}.

So, the transition functions do act linearly on the fibres of Tp, and we have a vector bundle structure on TE. We call this <u>the tangent structure of TE over TM</u>, or simply the TM-structure.

Note that, although the transition functions are linear in (u,du) and in (dx,du) separately, they are not linear in (u,dx,du), and so we <u>do not</u> get a vector bundle structure for TE over M.

<u>Section 2.3: Definition of Υ</u>

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let $\pi_M : TM \longrightarrow M$ denote the tangent bundle of M.

We denote by $E \underset{M}{\times} TM$ the submanifold of $E \times TM$ consisting of all points (ξ, ϱ) such that

$$p(\xi) = \pi_M(\varrho).$$

Note that if p^* denotes the projection of $E \underset{M}{\times} TM$ onto E, then we have a smooth vector bundle over E with fibre $\mathbb{R}^n$, and if π_M^* denotes the projection of $E \underset{M}{\times} TM$ onto TM, we have a smooth vector bundle over TM with fibre $\mathbb{R}^k$.

Local coordinates for $E \underset{M}{\times} TM$ come from the product structure. In particular, points are denoted locally by triples (x,u,dx) and the projections by

$$p^* : (x,u,dx) \longmapsto (x,u)$$

and

$$\pi_M^* : (x,u,dx) \longmapsto (x,dx).$$

In [2.2] we saw that

$$Tp : TE \longrightarrow TM$$

and

$$\pi_E : TE \longrightarrow E$$

each establish a smooth vector bundle structure on TE. Since the morphisms are smooth, we may combine them to obtain a smooth map

$$(\pi_E, T_p) : TE \longrightarrow E \times TM$$

Moreover, by definition, these morphisms agree in the first coordinate, so the image of (π_E, T_p) will be $E \underset{M}{\times} TM$.

Denote this new morphism

$$\Pi : TE \longrightarrow E \underset{M}{\times} TM$$

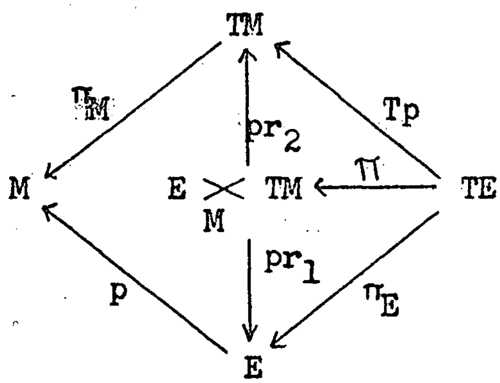
As an immediate consequence of the definitions, we see that locally

$$\Pi : (x,u,dx,du) \longmapsto (x,u,dx).$$

Π has two interpretations. Considering TE and $E \underset{M}{\times} TM$ as vector bundles over E, Π is fibre-preserving and acts linearly fibrewise. Such maps are called <u>E-morphisms.</u> Similarly, considering TE and $E \underset{M}{\times} TM$ as vector bundles over TM, Π is a <u>TM-morphism</u>.

<u>Section 2.4:</u> <u>The Kernels of</u> .

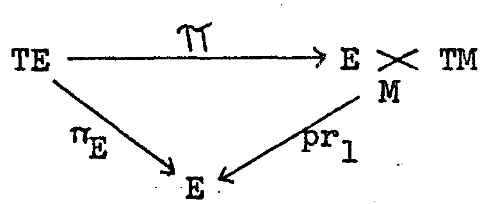
Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let $\Pi : TE \longrightarrow E \underset{M}{\times} TM$ be the "double-morphism" introduced in [2.3]. The following commutative diagram summarizes the relationships of the preceding section.



Since Π is both an E-morphism and a TM-morphism, it has two distinct kernels.

<u>§ 2.4.1</u>: <u>The E-Kernel of Π</u>.

Here we consider Π as a vector bundle morphism over E. That is, we concentrate on the following part of the diagram:



<u>PROPOSITION</u>: There exists a smooth vector bundle morphism

$$i : E \underset{M}{\times} E \longrightarrow TE$$

such that

$$0 \longrightarrow E \underset{M}{\times} E \xrightarrow{\ i\ } TE \longrightarrow E \underset{M}{\times} TM \longrightarrow 0$$

is a short exact sequence of vector bundles over E.

Before giving a formal proof of this proposition, we analyze the situation "geometrically".

Let V^E denote the kernel of π over E, and fix some $e \in E$. If $p(e) = x \in M$, the fibres of TE and $E \times_M TM$ over e are T_eE and T_xM respectively. So, by definition of π, the fibre V^E_e of V^E over e will be the kernel of the tangent map of p at e. That is,

$$V^E_e = \text{kernel of } T_ep : T_eE \longrightarrow T_xM$$

In other words, V^E_e is the tangent space at e to $p^{-1}(x)$. But $p^{-1}(x) = E_x$ is a vector space, so V^E_e may be identified with E_x.

We now exhibit the formal proof of the proposition.

<u>PROOF</u>: Define

$$i : E \times_M E \longrightarrow TE$$

locally by

$$i : (x,u,v) \longmapsto (x,u,0,v).$$

Examining the form of the transition maps of the E-structure of TE, we conclude that i is a vector bundle morphism over E. Moreover, the local description of π ensures that i satisfies the conclusion of the proposition.

<u>§ 2.4.2</u>: <u>The TM-Kernel of π</u>.

Here we consider π as a vector bundle morphism over TM. That is, we concentrate on the diagram

$$\begin{array}{ccc} TE & \xrightarrow{\pi} & E \times_M TM \\ & {}_{Tp}\searrow \quad \swarrow_{pr_2} & \\ & TM & \end{array}$$

PROPOSITION: There exists a vector bundle morphism

$$j : TM \times_M E \longrightarrow TE$$

such that

$$0 \longrightarrow TM \times_M E \xrightarrow{j} TE \xrightarrow{\pi} E \times_M TM \longrightarrow 0$$

is an exact sequence of vector bundles over TM.

Again, before giving the formal proof, we look at the situation fibrewise.

Fix $\xi \in TM$, and suppose $\pi_M(\xi) = x \in M$. We first determine the fibre $(TE)_\xi$ of TE over ξ. By definition

$$(TE)_\xi = Tp^{-1}(\xi).$$

Thinking of TE as the disjoint union of its fibres over E we have:

$$(TE)_\xi = \bigcup_{e \in E} (TE)_\xi \cap T_eE = \bigcup_{e \in E} Tp^{-1}(\xi) \cap T_eE$$

But $Tp^{-1}(\xi) \cap T_eE$ is the inverse image of ξ under T_ep. In particular,

$$Tp^{-1}(\xi) \cap T_eE = \emptyset$$

unless e is over x: that is, unless e is in the fibre E_x of E over x.

Suppose e is in E_x. Then $Tp^{-1}(\xi) \cap T_eE$ is the inverse image of ξ under T_ep and hence a translate of the kernel of $T_e p$. In [2.4.1] we identified the latter with E_x. Thus, as a set,

$$(TE)_\xi = \bigcup_{e \in E} (TE)_\xi \cap T_e E = \bigcup_{e \in E_x} E_x = E_x \times E_x .$$

Note that, after this identification,

$$\pi : TE \longrightarrow E \underset{M}{\times} TM$$

acts on $(TE)_\xi = E_x \times E_x$ by projection onto the second coordinate. In other words, the fibre of the kernel of π over ξ may be identified with E_x.

Now, the formal proof of the proposition:

PROOF: Define

$$j : TM \underset{M}{\times} E \longrightarrow TE$$

locally by

$$j : (x, dx, u) \longmapsto (x, 0, dx, u),$$

and check as before that j has the desired properties.

Section 2.5: Connections on Vector Bundles

§ 2.5.1: Connection Maps

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let $\pi : TE \longrightarrow E \underset{M}{\times} TM$ be the "double morphism" of [2.3].

DEFINITION: A connection map on E is a smooth section

$$C : E \underset{M}{\times} TM \longrightarrow TE$$

of π which is both an E-morphism and a TM-morphism.

In terms of local coordinates, a connection map C must be of the form

$$C : (x,u,dx) \longmapsto (x,u,dx,f(x,u,dx))$$

where, since C is an E-morphism, f acts linearly in dx, and since C is a TM-morphism, f acts linearly in u, when the corresponding fibres are fixed.

For each $x \in M$ then, C induces a bilinear map which we denote

$$-\Gamma_x : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

by

$$-\Gamma_x : (dx,u) \longmapsto f(x,u,dx).$$

The negative sign appearing in the notation is introduced so as to ensure later agreement with classical notation.

Locally then, a connection map is given by

$$\boxed{C : (x,u,dx) \longmapsto (x,u,dx, -\Gamma_x(u,dx))}$$

where $x \longmapsto -\Gamma_x$ is a smooth mapping of an open subset of $\mathbb{R}^n$ into the bilinear maps from $\mathbb{R}^n \times \mathbb{R}^k$ to $\mathbb{R}^k$.

§ 2.5.2: Connection Forms

There is an equivalent formulation of [2.5.1] in terms of connection forms.

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let

$$C : E \times_M TM \longrightarrow TE$$

be a connection map as defined in [2.5.1]. Recall the two short exact sequences of [2.4]. That is,

$$E \times_M E \xrightarrow{i} TE \xrightarrow{\pi} E \times_M TM$$

over E, and

$$TM \times_M E \xrightarrow{j} TE \xrightarrow{\pi} E \times_M TM$$

over TM. We may associate with C the two retractions:

$$K_E : TE \longrightarrow E \times_M E$$

via i, and

$$K_{TM} : TE \longrightarrow TM \times_M E$$

via j. These are given locally by

$$K_E : (x,u,dx,du) \longmapsto (x,u,du + \Gamma_x(u,dx))$$

and

$$K_{TM} : (x,u,dx,du) \longmapsto (x,dx,du + \Gamma_x(u,dx)).$$

These agree in the third coordinate. Thus K_E and K_{TM} induce the same map

$$K : TE \longrightarrow E$$

given locally by

$$\boxed{K : (x,u,dx,du) \longmapsto (x,du + \Gamma_x(u,dx)).}$$

Note that K has the following properties:

CF 1 : K is a vector bundle morphism over $p : E \longrightarrow M$ whose restriction to V^E is projection onto the second coordinate, and

CF 2 : K is a vector bundle morphism over $\pi_M : TM \longrightarrow M$ whose restriction to V^{TM} is projection onto the second coordinate,

where V^E and V^{TM} denote the "E-kernel" and "TM-kernel" of $\mathcal{M}$ respectively.

<u>DEFINITION</u>: A <u>connection form</u> on E is a smooth map

$$K : TE \longrightarrow E$$

satisfying conditions CF 1 and CF 2 above.

The one-to-one correspondence between connection maps and connection forms is clear from the construction, so we are done except for some remarks on notation.

We denote the connection form corresponding to a connection map C by K_C; and, alternatively, the connection map corresponding to a connection form K by C_K. Locally then, if C is given by

$$C : (x,u,dx) \longmapsto (x,u,dx, -\Gamma_x(u,dx)),$$

then K_C is given by

$$K_C : (x,u,dx,du) \longmapsto (x, du + \Gamma_x(u,dx)),$$

and if K is given by

$$K : (x,u,dx,du) \longmapsto (x, du + \Gamma_x(u,dx)),$$

then C_K is given by

$$C_K : (x,u,dx) \longmapsto (x,u,dx, -\Gamma_x(x,dx)).$$

§ 2.5.3: Alternative Definition of Connection Forms

In Chapter Four we will be talking about geodesic sprays, and there a slightly different definition of connection forms will be used.

Let M be a smooth manifold of dimension n. Let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. For any $s \in \mathbb{R}$, let s also denote the smooth map $s : E \longrightarrow E$ given locally by:

$$s : (x,u) \longmapsto (x, su)$$

Let s_* denote the corresponding tangent map

$$s_* : TE \longrightarrow TE$$
$$(x,u,dx,du) \longmapsto (x, su, dx, sdu).$$

Then:

<u>PROPOSITION</u>: A smooth map $K : TE \longrightarrow E$ is a connection form on E if and only if CF 1 is satisfied, and:

CF 3 : $\quad K \circ s_* = s \circ K$

for each $s \in R$. (Refer to [2.5.2] for the definition of connection forms on E, and CF 1.)

<u>PROOF</u>: Suppose $K : TE \longrightarrow E$ is a smooth map satisfying CF 1 and CF 3. Then, as we have seen, CF 1 implies that locally K has the form

$$K : (x,u,dx,du) \longmapsto (x, du + \Gamma_x(u,dx))$$

where $\Gamma_x(u,dx)$ is linear in dx.

By CF 3,

$$\Gamma_x(su,dx) = s\Gamma_x(u,dx),$$

so, $\Gamma_x(u,dx)$ is also linear in u (cf Appendix). Thus CF 2 of [2.5.2] is satisfied, so K is a connection form on E.

The converse is also easily established, so the proposition stands.

Section 2.6: The Difference Field of Two Bundle Connections

The two characterizations of bundle connections in [2.5] yield two corresponding interpretations for the difference between two connections.

§ 2.6.1: The Difference Field of Two Connection Maps

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let C and $\tilde{C}$ be two connection maps on E as defined in [2.5.1] and suppose they are given locally by

$$C : (x,u,dx) \longmapsto (x,u,dx,-\Gamma_x(u,dx))$$

and

$$\tilde{C} : (x,u,dx) \longmapsto (x,u,dx,-\tilde{\Gamma}_x(u,dx)).$$

Considering C and $\tilde{C}$ as E-morphisms, their difference

$$D_E(C,\tilde{C}) : E \underset{M}{\times} TM \longrightarrow TE$$

is given locally by

$$D_E(C,\tilde{C}) : (x,u,dx) \longmapsto (x,u,0,-\Gamma_x(u,dx) + \tilde{\Gamma}_x(u,dx)).$$

The image of $D_E(C,\tilde{C})$ thus lies in V^E, the E-kernel of π, so under the identification of [2.2.4], we arrive at an induced morphism given locally by

$$(x,u,dx) \longmapsto (x,u,-\Gamma_x(u,dx) + \tilde{\Gamma}_x(u,dx)) \in E \times_M E.$$

Similarly, considering C and $\tilde{C}$ as TM-morphisms, $D_{TM}(C,\tilde{C})$ induces a map given locally by

$$(x,u,dx) \longmapsto (x,-\Gamma_x(u,dx) + \tilde{\Gamma}_x(u,dx),dx) \in E \times_M TM.$$

These maps are both E-morphisms, and agree on the corresponding projections, so they both induce the same map

$$D(C,\tilde{C}) \;:\; E \times_M TM \longrightarrow E$$

$$(x,u,dx) \longmapsto (x,-\Gamma_x(u,dx) + \tilde{\Gamma}_x(u,dx)).$$

This new morphism we call the <u>difference field of C and $\tilde{C}$</u>. Note that the linearity requirements on Γ and $\tilde{\Gamma}$ ensure $D(C,\tilde{C})$ acts linearly fibrewise.

Suppose now that C is any connection map on E, and that

$$D \;:\; E \times_M TM \longrightarrow E$$

is a morphism as above. Then if C is given locally by

$$C \;:\; (x,u,dx) \longmapsto (x,u,dx,-\Gamma_x(u,dx)),$$

and D by

$$D \;:\; (x,u,dx) \longmapsto (x,f_x(u,dx)),$$

we can define a new connection map $\tilde{C}$ on E by

$$\tilde{C} \;:\; (x,u,dx) \longmapsto (x,u,dx,-\Gamma_x(u,dx) + f_x(u,dx)\;).$$

It is easily verified that $\tilde{C}$ is well-defined, and is the unique connection map satisfying:

$$D(C,\tilde{C}) \;=\; D.$$

§ 2.6.2: The Difference Field of Two Connection Forms

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let K and $\tilde{K}$ be two connection forms on E as defined in [2.5.2], and suppose they are given locally by

$$K \;:\; (x,u,dx,du) \longmapsto (x,du + \Gamma_x(u,dx)\;)$$

and

$$\tilde{K} \;:\; (x,u,dx,du) \longmapsto (x,du + \tilde{\Gamma}_x(u,dx)\;).$$

There are two possible interpretations for the difference between K and $\tilde{K}$.

Considering them as morphisms over $p : E \longrightarrow M$, their difference

$$D_E(K,\tilde{K}) \;:\; TE \longrightarrow E$$

$$(x,u,dx,du) \mapsto (x, \Gamma_x(u,dx) - \tilde{\Gamma}_x(u,dx)\;)$$

is a vector bundle morphism which annihilates V^E, since

$$\Gamma_x(u,0) \;=\; u \;=\; \tilde{\Gamma}_x(u,0)$$

by definition.

Passing to the quotient space under the identifications of [2.5.2] we have an induced morphism given locally by

$$(x,u,dx) \longmapsto (x, \Gamma_x(u,dx) - \tilde{\Gamma}_x(u,dx))$$

where $(x,u,dx) \in E \underset{M}{\times} TM$.

Similarly, seen as vector bundle morphisms over $\pi_M : TM \longrightarrow M$ the difference $D_{TM}(K,\tilde{K})$ annihilates V^{TM}. This induces a morphism given locally by

$$(x,u,dx) \longmapsto (x, \Gamma_x(u,dx) - \tilde{\Gamma}_x(u,dx)).$$

That is, under our identifications, $D_E(K,\tilde{K})$ and $D_{TM}(K,\tilde{K})$ induce the same morphism. We call this the difference field of K and $\tilde{K}$, and denote it simply $D(K,\tilde{K})$.

Finally, note that if C and $\tilde{C}$ are connection maps, and K_C and $K_{\tilde{C}}$ are the corresponding connection forms, then

$$D(K_C, K_{\tilde{C}}) \quad = \quad - D(C, \tilde{C})$$

where the latter is as defined in [2.6.1]. The minus sign appearing in the formula is the result of the earlier decision to define the Γ_x 's to agree with more classical results.

Chapter 3: Covariant Derivatives Versus Bundle Connections

Section 3.1: From Connection Forms to Covariant Derivatives

§ 3.1.1: Tangent Maps of Smooth Sections

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ be a smooth vector bundle on M with fibre $\mathbb{R}^k$. For a smooth section

$$S : M \longrightarrow E$$

of p, we denote by

$$S_* : TM \longrightarrow TE$$

the section TS of Tp.

In local coordinates, S is of the form

$$S : x \longmapsto (x, u(x))$$

where u is a smooth mapping of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^k$. It follows that S_* in the corresponding coordinate domains will be given by

$$S_* : (x, dx) \longmapsto (x, u(x), dx, u'(x)dx).$$

The following property of such morphisms is trivial

P1 : $(S + T)_* = S_* + T_*$

There is, however, another characteristic property. Let f be any smooth real-valued function defined on M. Note that for a smooth section S of p, fS is another smooth section. Locally, if S is given by

$$S : x \longmapsto (x, u(x))$$

then fS is given by

$$fS : x \longmapsto (x, f(x)u(x)).$$

The tangent map, $(fS)_*$, must then be of the form

$$(fS)_* : (x,dx) \longmapsto (x, f(x)u(x), dx, f(x)u'(x) + f'(x)(dx)u(x)).$$

This represents a section of TE as a vector bundle <u>over TM</u>, and so the right-hand side may be decomposed to give:

$$(x, f(x)u(x), dx, f(x)u'(x)) + (x, 0, dx, f'(x)(dx)u(x)).$$

Clearly, the first term of this decomposition comes from applying f to S_*. Examine the second term. Recall from [2.4.2] that the kernel of

$$\pi : TE \longrightarrow E \times_M TM,$$

considered as a TM-vector bundle morphism is

$$i : TM \times_M E \longrightarrow TE$$
$$(x, dx, u) \longmapsto (x, 0, dx, u).$$

The second term of the decomposition lies in the image of this map. This means there exists a global section of

$$pr_1 : TM \times_M E \longrightarrow TM,$$

which we denote dfS, such that locally, for a suitable choice of coordinates,

$$i \circ dfS : (x,dx) \longmapsto (x, 0, dx, f'(x)dx, u(x)).$$

If one abuses notation to identify S with the section

$$(x,dx) \longmapsto (x, dx, u(x))$$

of pr_1, (df)S simply is the product of

$$df : TM \longrightarrow \mathbb{R}$$

with S.

This establishes the following property:

P2 : $(fS)_* = fS_* + i(dfS)$

in the foregoing notation.

§ 3.1.2: The Action of S_* on Vector Fields

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let S and T be smooth sections of p, and denote their tangent maps as in [3.1.1]. Then if X and Y are smooth vector fields on M, and f is a smooth real-valued function defined on M, the following hold:

i) $S_* \circ (X + Y) = S_* \circ X + S_* \circ Y$,

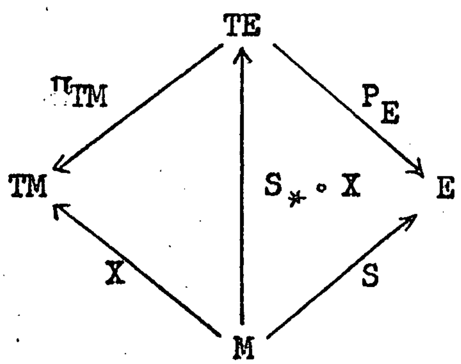
ii) $S_* \circ (fX) = fS_* \circ X$,

iii) $(S + T)_* \circ X = S_* \circ X + T_* \circ X$, and

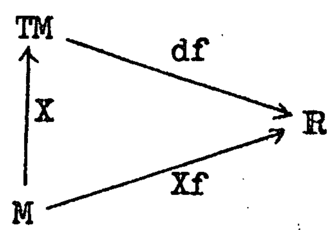
iv) $(fS)_* \circ X = fS_* \circ X + i(XfS)$

where i is the injection defined in [2.4.2].

The proof of the properties is based on the fact that the following two diagrams commute:



, and



.

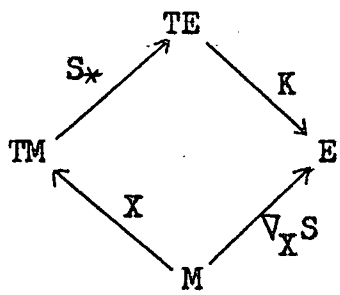
where π_{TM} and p_E are the vector bundle structural maps of TE over TM and E respectively, and df is the second component of $Tf : TM \longrightarrow T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$.

§ 3.1.3: The Construction

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Let $K : TE \longrightarrow E$ be a smooth connection form on E as defined in [2.5.2]. For a smooth vector field X on M, and a smooth section S of p, define

$$\nabla_X S = K \circ S_* \circ X$$

where $S_* = TS$. That is, the following diagram commutes:



By conditions CF 1 and CF 2 of [2.5.2], and i) to iv) of [3.1.2], ∇ is a well-defined covariant derivative on E.

Note that over a fixed coordinate domain in M, suitable choices for charts give the following formulas:

$$K : (x,u,dx,du) \longmapsto (x, du + \Gamma_x(u,dx)),$$

$$X : x \longmapsto (x, \xi(x)), \text{ and}$$

$$S : x \longmapsto (x,u(x)):$$

so $$S_* \circ X : x \longmapsto (x,u(x), \xi(x), u'(x)\xi(x)),$$

and $$\nabla_X S : x \longmapsto (x, u'(x)\xi(x) + \Gamma_x(u(x), \xi(x))).$$

Section 3.2: From Covariant Derivatives to Connection Forms

In [3.1] we established that connection forms induce covariant derivatives. We now show that one may go the other way.

§ 3.2.1: Uniqueness

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle over M with fibre $\mathbb{R}^k$. Fix $e \in E$, and suppose $p(e) = x \in M$. Let S be a smooth section of p such that $S(x) = e$. Then, in the notation of [3.1], for any smooth vector field X on M we have

$$(S_* \circ X)(x) \quad \in \quad T_e E .$$

Let U_e denote the set of elements of T_eE of this form. That is,

$$U_e = \left\{ (S_* \circ X)(x) \;\middle|\; S(x) = e, \text{ X arbitrary} \right\} .$$

Recall, from [2.4.1], that the double morphism

$$\pi : TE \longrightarrow E \times_M TM$$

is an E-morphism having an E-kernel V^E. We claim that

$$U_e \cup V_e^E = T_e E .$$

In local coordinates, points of TE are of the form (x,u,dx,du); and, for $e = (x_o, u_o)$, V_e^E is given by:

$$V_e^E = \left\{ (x \;, u \;, 0, du) \quad du \in \mathbb{R}^k \right\} .$$

So, we must establish that U_e contains all points of the form (x_o, u_o, dx, du) with $dx \neq 0$. Note that, for u_o and du in $\mathbb{R}^k$, and $dx \neq 0$ in $\mathbb{R}^n$, we can find a smooth map

$$F : U \longrightarrow \mathbb{R}^k,$$

where U is some neighbourhood of x , such that

$$F(x_o) = u,$$

and

$$F'(x_o)dx = du.$$

Now, there exists a vector field X on M with $X(x_o) = dx$, and a section S of E which agrees with F on a neighbourhood of x_o, so we are done.

<u>APPLICATION</u>: If K and $\tilde{K}$ are connection forms on E inducing the same covariant derivative, then $K = \tilde{K}$.

<u>PROOF</u>: By definition, K and $\tilde{K}$ agree on V^E, and by hypothesis they agree on each U_e as well. Therefore, by our earlier claim, the result.

<u>§ 3.2.2</u>: <u>Existence</u>

Let M be a smooth manifold of dimension n, and let $p : E \longrightarrow M$ denote a smooth vector bundle on M with fibre $\mathbb{R}^k$. Let ∇ be a covariant derivative on sections of E. We wish to define a connection form on E inducing ∇ as in [3.1]. By [3.2.1] it suffices to find K locally.

Recall that in local coordinates we may write:

$$\nabla_X S : x \longmapsto (x, u'(x)\xi(x) + \Gamma_x(u(x), \xi(x)))$$

where $\Gamma_x : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is bilinear and depends smoothly on x.

We define K locally by

$$K : (x,u,dx,du) \longrightarrow (x, du + \Gamma_x(u,dx)).$$

It is easily verified that the global morphism K so defined satisfied the conditions of a connection form.

Section 3.3: Differences

Note, in the foregoing notation, that if K and $\tilde{K}$ are connection forms on E inducing covariant derivatives ∇ and $\tilde{\nabla}$ respectively, then the difference field $D(K,\tilde{K})$ defined in [2.6.2], and the difference tensor $D(\nabla,\tilde{\nabla})$ of [1.3.1] are identical after the usual identification of operators

$$V(M) \times \Gamma_M(E) \longrightarrow \Gamma_M(E)$$

with smooth morphisms

$$TM \underset{M}{\times} E \longrightarrow E$$

which are linear fibrewise.

Chapter 4: Sprays

Section 4.1: Preliminaries

§ 4.1.1: The Double Tangent Bundle

Let M be a smooth manifold of dimension n. Denote the tangent bundle of M by

$$\pi_M \; : \; TM \longrightarrow M$$

and recall that it is a smooth vector bundle over M with fibre $\mathbb{R}^n$. Since TM is a smooth manifold of dimension n^2, it has a tangent bundle, called the double tangent bundle of M, which we denote by

$$\pi_M \; : \; TTM \longrightarrow TM.$$

Recall that if a point of M lies within the domain of a chart (U, φ), then we denote it by its image $x \in \varphi(U)$ in $\mathbb{R}^n$. Also, if (U, φ) and (V, ψ) are charts having a non-empty intersection, W, then there is a coordinate change on W given by a diffeomorphism

$$h \; : \; \varphi(W) \longrightarrow \psi(W).$$

At the tangent level, points of the corresponding domain are denoted by pairs $(x, dx) \in \varphi(U) \times \mathbb{R}^n$, and the coordinate change is given by the diffeomorphism

$$\begin{array}{ccc} \varphi(W) \times \mathbb{R}^n & \longrightarrow & \psi(W) \times \mathbb{R}^n \\ (x, dx) & \longmapsto & (h(x), h'(x)dx). \end{array}$$

Since TTM is the tangent bundle of TM, we will denote points by quadruples $(x, dx, \dot{x}, d\dot{x}) \in \varphi(U) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, and the corresponding coordinate change here is given by

$$\varphi(W) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \psi(W) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

$$(x, dx, \dot{x}, d\dot{x}) \longmapsto (h(x), h'(x)dx, h'(x)\dot{x}, h''(x)\dot{x}dx + h'(x)d\dot{x}).$$

§ 4.1.2: The Double Structure of TTM

Let M be a smooth manifold of dimension n. Denote by

$$\pi_M : TM \longrightarrow M$$

the tangent bundle of M, and by

$$\pi_{TM} : TTM \longrightarrow TM$$

the double tangent bundle. Since TM is a vector bundle over M, TTM will have the double structure discussed in [2.2].

In particular,

$$\pi_{TM} : TTM \longrightarrow TM$$

gives the standard structure of TTM, and we have the following exact sequence of smooth vector bundles over TM:

$$0 \longrightarrow TM \times_M TM \xrightarrow{\ i\ } TTM \xrightarrow{\ \Pi\ } TM \times_M TM \longrightarrow 0,$$

where, in local coordinates,

$$\pi_{TM} : (x, dx, x, dx) \longmapsto (x, dx),$$

$$i : (x, dx, x) \longmapsto (x, dx, 0, x), \text{ and}$$

$$\Pi : (x, dx, x, dx) \longmapsto (x, dx, x).$$

On the other hand, the tangent structure of TTM is given by: $\quad T\pi_M : TTM \longrightarrow TM,$

and another exact sequence over TM is given by:

$$0 \longrightarrow TM \underset{M}{\times} TM \xrightarrow{j} TTM \xrightarrow{\pi} TM \underset{M}{\times} TM \longrightarrow 0,$$

where, again in local coordinates,

$$T\pi_M : (x,dx,\dot{x},d\dot{x}) \longmapsto (x,\dot{x}), \text{ and}$$

$$j : (x,dx,\dot{x}) \longmapsto (x,0,\dot{x},dx).$$

It is important to note that these two structures are isomorphic. That is, if I_{TTM} denotes the involution of TTM given locally by

$$I_{TTM} : (x,dx,\dot{x},d\dot{x}) \longmapsto (x,\dot{x},dx,d\dot{x}),$$

and I_{TM} denotes the canonical involution of $TM \underset{M}{\times} TM$, then the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TM \underset{M}{\times} TM & \xrightarrow{i} & TTM & \longrightarrow & TM \underset{M}{\times} TM & \longrightarrow & 0 \\ & & \downarrow I_{TM} & & \downarrow I_{TTM} & & \downarrow I_{TM} & & \\ 0 & \longrightarrow & TM \underset{M}{\times} TM & \xrightarrow{j} & TTM & \longrightarrow & TM \underset{M}{\times} TM & \longrightarrow & 0. \end{array}$$

Section 4.2: Bundle Connections on Manifolds

§ 4.2.1: Definition

Let M be a smooth manifold of dimension n, with tangent bundle $\pi_M : TM \longrightarrow M$. We define a connection map on M to be a connection map on the vector bundle TM as in [2.5.1]. That is, a smooth section

$$C : TM \underset{M}{\times} TM \longrightarrow TTM$$

of π (as defined in [4.1.2]) which is a vector bundle morphism with respect to both the standard and tangent structures of TTM. Similarly, a connection form on M is defined to be a connection form on TM. That is, a smooth morphism

$$K \ : \ TTM \longrightarrow TM$$

satisfying:

CF 1 : K is a vector bundle morphism whose restriction to $i(TM \underset{M}{\times} TM)$ is projection onto the second coordinate, and

CF 2 : K is a vector bundle morphism over whose restriction to $j(TM \underset{M}{\times} TM)$ is also projection onto the second coordinate,

where i and j are as in [4.1.2].

In [2.5.3], we saw that condition CF 2 is equivalent to

CF 3 : $K \circ s_* = s \circ K$

where $s \in \mathbb{R}$ denotes fibrewise multiplication by s:

$$s \ : TM \longrightarrow TM$$
$$(x, dx) \longmapsto (x, sdx),$$

and s_* the tangent map.

§ 4.2.2: Connection Maps versus Connection Forms

In [3.5] it was shown that connection maps and connection forms are in a one-to-one correspondence. Recall that locally, connection maps have the form

$$\boxed{C \quad : \quad (x, dx, \dot{x}) \longmapsto (x, dx, \dot{x}, -\Gamma_x(dx, \dot{x}))}$$

where Γ_x is a bilinear mapping of $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n$ depending smoothly on x. Corresponding to each such C we have a connection form K_C given locally by

$$\boxed{K_C \quad : \quad (x, dx, \dot{x}, d\dot{x}) \longmapsto (x, d\dot{x} + \Gamma_x(dx, \dot{x})).}$$

Since the two bundle structures of TTM are isomorphic, we have a notion of symmetry. That is, we say that a connection map C on M is symmetric if it commutes with the involutions of TTM and $TM \times_M TM$ introduced in [4.1.2]. That is, when

$$C \circ I_{TM} = I_{TTM} \circ C.$$

Correspondingly, a connection form K on M is symmetric when

$$K \circ I_{TTM} = K.$$

In local coordinates, this means K and C are symmetric if and only if each of the corresponding Γ_x's is symmetric. Thus C is symmetric if and only if K_C is symmetric.

Finally, recall that for any two connection maps C and $\tilde{C}$ on M there is a morphism

$$D(C, \tilde{C}) \quad : \quad TM \times_M TM \longrightarrow TM$$

called the difference field of C and $\tilde{C}$, and that given any connection map C, and a morphism D as stated, there is a unique connection map $\tilde{C}$ on M such that

$$D(C,\tilde{C}) = D.$$

Note that if C and $\tilde{C}$ are symmetric, then so is $D(C,\tilde{C})$.

Section 4.3: Second Order Differential Equations and Sprays

§ 4.3.1: Second Order Differential Equations

Let M be a smooth manifold of dimension n, $\pi_M : TM \longrightarrow M$ its tangent bundle, and $\pi_{TM} : TTM \longrightarrow TM$ its double tangent bundle (with standard structure). Recall that $T\pi_M : TTM \longrightarrow TM$ also gives TTM a smooth vector bundle structure (the tangent structure) and that the canonical involution $I : TTM \longrightarrow TTM$ is an isomorphism of these two structures.

A second order differential equation on M is defined to be a smooth vector field

$$X : TM \longrightarrow TTM$$

which is a section of both the standard and tangent structures of TTM. That is, such that

$$\pi_{TM} \circ X = id_{TM} = T\pi_M \circ X,$$

or equivalently,

$$I \circ X = X.$$

In local coordinates, then, X must have the form

$$X : (x,dx) \longmapsto (x,dx,dx, \varsigma(x,dx))$$

where ξ is a smooth $\mathbb{R}^n$-valued map.

By definition, a smooth curve

$$\beta : I \longrightarrow TM$$

is an integral curve of a vector field Y on TM if and only if

$$\beta'(t) = Y(\beta(t)) \text{ for each } t \text{ in } I,$$

where $\quad \beta' : I \longrightarrow TTM$

is the canonical lift of β to the tangent bundle of TM. In local coordinates, β may be written

$$t \longmapsto (\sigma(t), \tau(t)) \in U \times \mathbb{R}^n.$$

Therefore, if X is a second order differential equation with local representation

$$(x, dx) \longmapsto (x, dx, dx, \xi(x, dx)),$$

β is an integral curve of X if and only if:

$$\sigma'(t) = \tau(t),$$

and

$$\tau'(t) = \xi(\sigma(t), \tau(t))$$

In other words, the condition is that $\tau = \sigma'$, and $\sigma'' = \xi(\sigma, \sigma')$. Therefore, integral curves of second order differential equations are equal to the canonical lifts of their projections onto M.

§ 4.3.2: Sprays

We continue with the notation of the last subsection. Let $\quad C : TM \times_M TM \longrightarrow TTM$

be a connection map on M as defined in [4.2.2].

If we restrict C to the diagonal of $TM \underset{M}{\times} TM$, then the resulting map induces a second order differential equation

$$X_C \;:\; TM \longrightarrow TTM$$

called the spray of C. In local coordinates, if C is given by

$$C : (x,dx,\dot{x}) \longmapsto (x,dx,\dot{x},-\ {}_x(dx,\dot{x}))$$

then X_C is given by

$$\boxed{X_C \;:\; (x,dx) \longmapsto (x,dx,dx,-\Gamma_x(dx,dx)).}$$

We may characterize those second order differential equations which arise from connection maps as follows: For $s \in \mathbb{R}$, let

$$s \;:\; TM \longrightarrow TM$$

and

$$\tilde{s} \;:\; TTM \longrightarrow TTM$$

denote fibrewise multiplication by s, and let s_* denote the tangent map of s. Then if X_C is the spray of a connection map C on M, the following condition holds:

(Sp) $$s_* \circ \tilde{s} \circ X_C \;=\; X_C \circ s,$$

since each Γ_x is quadratic. In general, a smooth morphism

$$X \;:\; TM \longrightarrow TTM$$

is called <u>a spray on M</u> whenever it is a second order differential equation on M satisfying condition (Sp).

Locally then, if X is given by

$$X \;:\; (x,dx) \longmapsto (x,dx,dx,\xi(x,dx)),$$

condition (Sp) implies

$$\xi(x, s\,dx) = s^2 \xi(x, dx),$$

so a second order differential equation is a spray if and only if ξ is a quadratic form in dx.

Section 4.4: The Connection Map of a Spray

Let M be a smooth manifold of dimension n, and let $X : TM \longrightarrow TTM$ be a spray on M as defined in the last section.

THEOREM: There is a unique symmetric connection map

$$C : TM \underset{M}{\times} TM \longrightarrow TTM$$

on M generating the spray X.

PROOF: Suppose first that C and $\tilde{C}$ are two symmetric connection maps generating the same spray X. Consider their difference as in [2.6.1]. By hypothesis, it will vanish on the diagonal of $TM \underset{M}{\times} TM$, and so must be alternating. On the other hand, it is symmetric. Therefore, it must vanish everywhere. Thus $C = \tilde{C}$.

Since uniqueness is established, we need only exhibit a suitable map C locally. Suppose X is given locally by

$$X : (x, dx) \longmapsto (x, dx, dx, \xi_x(dx))$$

where $\xi_x(dx) = \xi(x, dx)$ is quadratic in dx. Since ξ_x is homogeneous of degree two, it is the restriction to the diagonal of some bilinear mapping of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ (cf Appendix). The family of all such maps inducing ξ_x

contains a unique symmetric member. Denote this map by $-\Gamma_x$. The following is then a well-defined connection map:

$$C : TM \times_M TM \longrightarrow TTM$$

$$(x, dx, \dot{x}) \longmapsto (x, dx, \dot{x}, - {}_x(dx, \dot{x}))$$

and satisfies the hypothesis of the proposition.

As a consequence of this theorem, note that given any connection map C on M, there exists a unique symmetric connection map C^S on M such that C and C^S induce the same spray. Setting, for an arbitrary connection map C on M,

$$T(C) = D(C, C^S) : TM \times_M TM \longrightarrow TM,$$

we obtain an alternating tensor field T, called the <u>torsion field of C</u>.

Thus, although sprays are not in one-to-one correspondence with bundle connections or covariant derivatives, the explicit formulas in local coordinates do imply the following relationship.

Let ∇ be a covariant derivative on M, let K be the corresponding connection form, and C the corresponding connection map, and let X be the corresponding spray. Then:

(1) ∇ is torsion-free if and only if K and C are symmetric,

(2) a smooth curve on M is a geodesic of ∇ if and only if it is the projection to M of an integral curve of X, and

(3) the torsion tensor of ∇ is equal to the torsion field of C.

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APPENDIX Homogeneous Smooth Functions

Recall that a function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is homogeneous of degree $k \geqslant 0$, if and only if, for each $u \in \mathbb{R}^n$,

$$\text{(H)} \qquad f(su) = s^k f(u)$$

for any $s \in \mathbb{R}$. Note that if f is homogeneous of degree 0, then (H) implies that f is a constant function, and if f is homogeneous of degree $k > 0$, then $f(0) = 0$.

LEMMA: If f is a differentiable homogeneous function of degree $k > 0$, then

$$Df : \mathbb{R}^n \longrightarrow \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$$

is homogeneous of degree $k-1$.

PROOF: Differentiating (H) with respect to u gives:

$$sDf\Big|_{(su)} = s^k Df\Big|_u ,$$

so

$$Df\Big|_{(su)} = s^{k-1} Df\Big|_{(u)} .$$

LEMMA: (Euler's Relation) If f is a homogeneous differentiable function of degree $k > 0$, then

$$Df\Big|_u (u) = kf(u).$$

PROOF: Differentiating (H) with respect to s gives

$$Df\Big|_{(su)}(u) = ks^{k-1} f(u).$$

But by the above lemma,

$$Df\Big|_{(su)}(u) = s^{k-1}Df\Big|_{u}(u),$$

so we are done.

As a consequence of these two lemmas, we have the following proposition which may be proved by induction on k.

<u>PROPOSITION</u>: A smooth function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is homogeneous of degree $k > 0$ if and only if there exists a k-linear function

$$\hat{f} : \mathbb{R}^n \times \mathbb{R}^n \times \ldots\ldots \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that for each $u \in \mathbb{R}^n$,

$$\hat{f}(u,u, \ldots\ldots,u) = f(u).$$