ISOMORPHISMS BETWEEN SEMIGROUPS OF MAPS

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

MATHEMATICS

We accept this thesis as conforming
to the required standard

The University of British Columbia

April 1972

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Date April 26, 1972

ABSTRACT

Let X and Y be topological spaces and $\mathcal{C}$ and $\mathcal{D}$ semigroups under composition of maps from X to X and Y to Y respectively. Let H be an isomorphism from $\mathcal{C}$ to $\mathcal{D}$; it is shown that if both $\mathcal{C}$ and $\mathcal{D}$ contain the constant maps then there exists a bijection h from X to Y such that $H(f) = h \circ f \circ h^{-1}$, $\forall f \in \mathcal{C}$. We investigate this situation and find sufficient conditions for this h to be a homeomorphism. In this regard we study the familiar semigroups of continuous, closed, and connected maps.

An auxiliary problem is the case when $\mathcal{C} = \mathcal{D}$ and H is an automorphism of $\mathcal{D}$. We then ask when is every automorphism is inner. The question is answered for certain particular semigroups; e.g., the semigroup of differentiable maps on the reals has the property that all automorphisms are inner.

ACKNOWLEDGEMENTS

The Author is indebted to Dr. Sam B. Nadler, Jr. for first introducing him to the topics of this thesis, and for his continued encouragement throughout. He would also like to thank Dr. A.H. Cayford for his critical reading of the draft copy, and Dr. J.V. Whittaker for his enlightening comments as well as his reading of the draft copy. He is grateful for the financial assistance provided by the National Research Council of Canada. Finally he wishes to thank Miss Barbara Kilbray for her care and concern in typing this thesis.

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ISOMORPHISMS BETWEEN SEMIGROUPS OF MAPS

1. Introduction. If X is a topological space, then by X^X we denote the set of all maps from X to X. Then $(X^X, \circ)$ is a semigroup under composition; we write $f(g(x))$ as $(f \circ g)(x)$. In the following we discuss certain subsemigroups of X^X. Suppose X is a given topological space and $T(f)$ is a statement about $f \varepsilon X^X$. Then by $T(X)$ we denote $\{f \varepsilon X^X : T(f)\}$, and we consider only those T's sufficient to make $T(X)$ a subsemigroup of X^X. A class of topological spaces is said to be a T-admissible[1] class if for every pair X, Y in the class for which there is an isomorphism $H : T(X) \to T(Y)$, then $\exists\, h : X \to Y$, such that h is a homeomorphism and $H(f) = h \circ f \circ h^{-1}$, $\forall f \varepsilon T(X)$. We note that for our purposes by isomorphism we will always mean a one-to-one, onto homomorphism. We then have a criterion for two spaces being homeomorphic; i.e., if X and Y are both T-admissible and there exists some isomorphism from $T(X)$ onto $T(Y)$, then X and Y are homeomorphic. In fact, there are at least as many homeomorphisms from X to Y as isomorphisms from $T(X)$ to $T(Y)$. We will investigate T-admissibility for certain familiar semigroups; e.g., the semigroup of continuous maps, written as $S(X)$; and we will give examples of quite large T-admissible classes.

Another problem to be considered is the case of automorphisms; viz., when is every automorphism of $T(X)$ an inner automorphism. We note that an automorphism, H, on a semigroup, $T(X)$, is said to be inner iff $\exists\, h \varepsilon X^X$ such that h is a bijection (i.e., is invertible) with both h and h^{-1} in $T(X)$ and such that $H(f) = h \circ f \circ h^{-1}$, $\forall f \varepsilon T(X)$.

[1] This definition of T-admissibility is adapted from that of S-admissibility defined by Magill in [3].

2. Definitions. Let X be a topological space, then

Definition 2.1. If $x \varepsilon X$, then by $[x]$ we denote that map in X^X which is constantly x; i.e., $\forall x \varepsilon X$, $[x](y) = x$, $\forall y \varepsilon X$. Also let $K(X) = \{[x] : x \varepsilon X\}$.

Definition 2.2. $\Gamma(X) = \{f \varepsilon X^X : f^{-1}(\{x\})$ is closed, $\forall x \varepsilon X\}$.

Note: we are not being consistent here - $\Gamma(X)$ need not be a semigroup.

Definition 2.3. $S(X) = \{f \varepsilon X^X : f$ is continuous$\}$.

Definition 2.4. $C(X) = \{f \varepsilon X^X : f$ is closed; i.e., if A is a closed subset of X, then so is $f(A)\}$.

Definition 2.5. $U(X) = \{f \varepsilon X^X : f$ is connected; i.e., if A is connected, then so is $f(A)\}$.

3. Preliminary Lemmas. We now prove the following simple lemmas. First we note that an element, z, of a semigroup, $\mathcal{D}$, is called a left-zero of $\mathcal{D}$ iff $zd = z$, $\forall d \varepsilon \mathcal{D}$.

Lemma 3.1.[2] If $K(X) \subsetneq T(X)$, then $K(X)$ is precisely the set of left-zeroes of $T(X)$.

Proof. This follows from the following circle of arrows: f is a left-zero of $T(X) \Rightarrow f \circ [x] = f$, $\forall x \varepsilon X$

$\Rightarrow (f \circ [x])(y) = f(y)$, $\forall x,y \varepsilon X \Rightarrow f(x) = f(y)$, $\forall x,y \varepsilon X$

$\Rightarrow f \varepsilon K(X) \Rightarrow f \circ g = f$, $\forall g \varepsilon T(X)$

$\Rightarrow f$ is a left zero of $T(X)$.

[2] Lemmas 3.1, 3.2, and 3.3 are all straightforward generalizations of parts of the discussion and proof of theorem 2.1 of [4].

Lemma 3.2. If $K(X) \subsetneq T(X)$ and $K(Y) \subsetneq T(Y)$ and $H : T(X) \to T(Y)$ is an isomorphism, then H maps $K(X)$ bijectively onto $K(Y)$.

Proof. This follows from the following list of equivalent statements:

$$\begin{aligned} f \in K(X) &\iff f \circ g = f \ , \forall g \in T(X) && \text{, by Lemma 3.1} \\ &\iff H(f \circ g) = H(f) \circ H(g) = H(f) \ , \forall g \in T(X) && , \\ &\iff H(f) \in K(Y) && \text{, by Lemma 3.1} \end{aligned}$$

Lemma 3.3. Under the hypotheses of Lemma 3.2, there exists a unique bijection $h : X \to Y$ such that $H(f) = h \circ f \circ h^{-1} \ , \forall f \in T(X)$.

Proof. Define h by $h(x) = y$ iff $H([x]) = [y]$. Then, by Lemmas 3.1 and 3.2, h is a well-defined bijection. Now we simply note that $\forall y \in Y$ and $\forall f \in T(X)$ we have:

$$\begin{aligned} (h \circ f \circ h^{-1})(y) &= h(f(h^{-1}(y))) && \text{, by definition of "}\circ\text{"} \ , \\ &= [h(f(h^{-1}(y)))](y) \\ &= H([f(h^{-1}(y))])(y) && \text{, by definition of } h \ , \\ &= H(f \circ [h^{-1}(y)])(y) \\ &= (H(f) \circ H([h^{-1}(y)]))(y) && \text{, } H \text{ is a homomorphism,} \\ &= (H(f) \circ [y])(y) && \text{, by definition of } h \ , \\ &= H(f)(y) \ . \end{aligned}$$

Hence $H(f) = h \circ f \circ h^{-1} \ , \forall f \in T(X)$. Uniqueness follows easily since if k is another map from $X \to Y$ such that $H(f) = k \circ f \circ k^{-1} \ , \forall f \in T(X)$, then $\forall x \in X$ we would have: $H([x]) = h \circ [x] \circ h^{-1} = [h(x)] = k \circ [x] \circ k^{-1} = [k(x)] \Rightarrow h = k$.

Now because of Lemma 3.3 we make the following conventions:

a) From now on all $T(X)$ will be such that $K(X) \subseteq T(X)$, for every topological space X.

b) If $H : T(X) \to T(Y)$ is an isomorphism, then by h we denote the map from X to Y such that $H(f) = h \circ f \circ h^{-1}$, $\forall f \in T(X)$. That is, we tacitly use Lemma 3.3 and write the bijection that corresponds to an isomorphism as the lower case latin letter of the upper case latin letter which represents the isomorphism.

The usefulness of Lemma 3.3, after we abide by the above conventions, then becomes clear. To determine whether a certain class is T-admissible we merely have to show that all such h's are homeomorphisms. Similarly, an automorphism, H, is inner iff h and h^{-1} are both in the domain of H.

4. <u>T-Admissibility</u>. We first concern ourselves with the space $S(X)$ and the question of S-admissibility. We define below and S-space and show that S-spaces are S-admissible. We then give examples to show how large the class of S-spaces is. Let X be a topological space, then.

<u>Definition 4.1</u>.[3] If $x \in X$ and $x \in G$, G an open set in X, then we say that G is an S-neighbourhood of x if either

a) $G = \{x\}$

or b) there exists a continuous $f : \overline{G} \to X$ such that $f(x) \neq x$ but $f(y) = y$, $\forall y \in \overline{G} \sim G$.

Note: By $\overline{G}$ we mean the closure of G, and by $\overline{G}_B$ we mean the closure of G relative to $B \subseteq X$. So $\overline{G} = \overline{G}_X$.

3 The discussion of S-spaces and hence results 4.1 through to 4.6.2 are due to Magill in [5].

We say that X is an <u>S-space</u> if X is Hausdorff and every point in X has a basis of S-neighbourhoods. This basis is called an <u>S-basis</u>.

<u>Theorem 4.3</u>. S-spaces are S-admissible.

<u>Proof</u>. Let X and Y be two S-spaces and suppose $H : S(X) \to S(Y)$ is an isomorphism, then we need only show that h is a homeomorphism. To do this we need only show that h and h^{-1} are closed, and to do this we only require that they be closed on basic closed sets. Because of this we first prove the following lemma.

<u>Lemma 4.4</u>. If X is an S-space and $\phi(f) = \{x \in X : f(x) = x\}$, $\forall f \in S(X)$, then $\{\phi(f) : f \in S(X)\}$ is a basis for the closed sets of X.

<u>Proof</u>. It is well known that the set of fixed points of a continuous function is closed, hence $\phi(f)$ is closed, $\forall f \in S(X)$. Now let F be a closed subset of X. Then $\forall x \in X \sim F$, there is an S-neighbourhood, G, of x such that $G \subseteq X \sim F$; and there is a continuous function $q_x : \overline{G} \to X$ such that $q_x(x) \neq x$ but $q_x(y) = y$, $\forall y \in \overline{G} \sim G$.

Define $g_x : X \to X$ by $g_x(y) = \begin{cases} q_x(y), & y \in \overline{G} \\ y, & y \in X \sim \overline{G} \end{cases}$.

Then clearly g_x is continuous. Also $F \subseteq \phi(g_x)$, $\forall x \in X \sim F$. Hence $F \subseteq \bigcap_{x \in X \sim F} \phi(g_x)$. But $x \notin \phi(g_x), \forall x \in X \sim F \Rightarrow (X \sim F) \cap (\bigcap_{x \in X \sim F} \phi(g_x)) = \emptyset \Rightarrow \bigcap_{x \in X \sim F} \phi(g_x) \subseteq F$. So, in fact, $F = \bigcap_{x \in X \sim F} \phi(g_x)$. Hence these sets form a basis for the closed sets.

Now back to the theorem. Let $f \in S(X)$, then $\phi(H(f))$ is closed. And $y \in \phi(H(f))$

$$\begin{aligned} &\Longleftrightarrow (h \circ f \circ h^{-1})(y) = y \\ &\Longleftrightarrow f(h^{-1}(y)) = h^{-1}(y) \\ &\Longleftrightarrow h^{-1}(y) \in \phi(f) \\ &\Longleftrightarrow y \in h(\phi(f)) . \end{aligned}$$

Hence $\phi(H(f)) = h(\phi(f))$, $\forall f \in S(X)$. So $h(\phi(f))$ is closed, $\forall f \in S(X)$. So Lemma 4.4 implies that h^{-1} is continuous. Similarly, $\phi(H^{-1}(g)) = h^{-1}(\phi(g))$, $\forall g \in S(Y)$. And so h is a homeomorphism.

<u>Theorem 4.5</u>. If X is Hausdorff and if $\forall x \in X$, x has a basis, $\mathcal{B}_x$, of open sets such that $\overline{G} \sim G$ is at most a singleton, $\forall G \in \mathcal{B}_x$; then X is an S-space.

<u>Proof</u>. If X is a singleton, nothing to prove, so we assume X has more than one point. Let $x \in X$ and $x \in G \in \mathcal{B}_x$, and chose any $y \in X$ such that $x \neq y$. Define $f : \overline{G} \to X$ by

$$f = \begin{cases} [y] , & \text{if } \overline{G} \sim G = \emptyset \\ [z] , & \text{if } \overline{G} \sim G = \{z\} \end{cases}.$$

Then f is continuous with the desired properties, and $\mathcal{B}_x$ is an S-basis. Hence X is an S-space.

If a space has a basis of sets which are both open and closed, then we say that X is 0-dimensional.

<u>Corollary 4.5.1</u>. A 0-dimensional Hausdorff space is an S-space.

A space is said to be locally Euclidean if each point has an open neighbourhood about it that is homeomorphic to Euclidean n-space, E^n, for some $n \geq 1$ (n depends on the point in question).

<u>Theorem 4.6.</u> A locally Euclidean space is an S-space.

<u>Proof.</u> First we note that every locally Euclidean space is regular, Hausdorff, and a union of homeomorphic images of E^n, possibly for many different n's. Also we note that homeomorphic images of S-spaces are again S-spaces. So the proof is complete with the following two lemmas.

<u>Lemma 4.6.1.</u> E^n is an S-space, $\forall n \geq 1$.

<u>Lemma 4.6.2.</u> If X is a regular Hausdorff space which is the union of a collection of open subspaces which are S-spaces, then X is an S-space.

<u>Proof of Lemma 4.6.1.</u> If $x \in E^n$ and d the usual Euclidean metric, then $\forall \varepsilon > 0$ we define $N_\varepsilon(x) = \{y \in E^n : d(x,y) < \varepsilon\}$. Also let x_i denote the i^{th} coordinate of x, $\forall i = 1,2,\ldots,n$. Now define $f_i : \overline{N_\varepsilon(x)} \to E^1$ as follows: $f_i(y) = y_i + d(x,y) - \varepsilon, \forall y \in \overline{N_\varepsilon(x)}$, $\forall i = 1,2,\ldots,n$. Then each f_i is continuous, so the function $f : \overline{N_\varepsilon(x)} \to E^n$ defined by $f(y) = (f_1(y),\ldots,f_n(y))$, $\forall y \in \overline{N_\varepsilon(x)}$, is continuous, since each projection map is continuous. And f has the property that $f(y) = y$, $\forall y \in \overline{N_\varepsilon(x)} \sim N_\varepsilon(x)$ and $f(x) = (x - \varepsilon,\ldots,x_n-\varepsilon) \neq x$. Hence neighbourhoods of this type form an S-basis for E^n. So E^n is an S-space.

Proof of Lemma 4.6.2. Let $x \in X$. Let H be an open subset of X which is itself an S-space containing x . Let $\mathcal{B}_x$ be an S-basis for x in H . Since X is regular, there exists an open subset V of X and a closed subset F of X such that $x \in V \subsetneq F \subsetneq H$. Let $\mathcal{B}_x^* = \{G \in \mathcal{B}_x : G \subseteq V\}$, then $\mathcal{B}_x^*$ is a basis for x in X . We want to show that it is an S-basis. Let $G \in \mathcal{B}_x^*$. If $G = \{x\}$, we are done. Otherwise we know that there is a continuous function $f : \overline{G_H} \to H$ such that $f(x) \neq x$, but $f(y) = y$, $\forall y \in \overline{G_H} \sim G$. (Note $\overline{G_H}$ means the closure of G relative to H , so $\overline{G} = \overline{G_X}$.) But note that $\overline{G_H} \subseteq F \subseteq H$, and F is closed in X , so $\overline{G_H} = \overline{G_X} = \overline{G}$. So G is in fact an S-neighbourhood of x in X . So X is an S-space.

Now we try to broaden our knowledge of which spaces are T-admissible, and for what T's. So we define two more classes of spaces.

Definition 4.7.1.[4] X is a T^*-space if it is T_1 and for every closed subset, F , of X and $\forall y \in X \sim F$, there exists a $k_y \in T(X)$ and an $x_y \in X$ such that $y \notin k_y^{-1}(\{x_y\}) \supseteq F$.

Definition 4.7.2. X is a T_*-space if it is T_1 and for every closed subset, F , of X there exists a $k_F \in T(X)$ such that $k_F(F)$ is finite (i.e., k_F assumes only a finite number of values on F) and such that $k_F^{-1}(k_F(F)) = F$.

In [6, p. 295, Theorem 1] Magill proves that S^*-spaces are S-admissible. We generalize this to:

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4 The definition of T^*-space is adapted from Magill's definition of S^*-space in [6]. The condition of T_*-'ness was found by the author.

Theorem 4.8. If $T(X) \subseteq \Gamma(X)$, for every T_*- and T^*-space X, then T^*- and T_*-spaces are both T-admissible classes.

Proof. Let F be any closed subset of X and suppose $H : T(X) \to T(Y)$ is an isomorphism.

Case i) X and Y are T^*-spaces, then $\forall y \in X \sim F$ let $k_y \in T(X)$ and $x_y \in X$ be as in definition 4.7.1. Then clearly $F \subseteq \bigcap_{y \in X \sim F} k^{-1}(\{x_y\})$. But $\forall y \in X \sim F$, $y \notin k_y^{-1}(\{x_y\}) \Rightarrow (X \sim F) \cap (\bigcap_{y \in X \sim F} k_y^{-1}(\{x_y\})) = \emptyset$. So $F = \bigcap_{y \in X \sim F} k_y^{-1}(\{x_y\})$. Hence the class $\{f^{-1}(\{x\}) : x \in X$ and $f \in T(X)\}$ forms a basis for the closed sets of X (each member of this class is closed since $T(X) \subseteq \Gamma(X)$). Similarly $\{g^{-1}(\{y\}) : y \in Y$ and $g \in T(Y)\}$ is a basis for the closed sets of Y.

So now to show h^{-1} is continuous we need only show that $h(f^{-1}(\{x\}))$ is closed, $\forall x \in X$ and $\forall f \in T(X)$. So let $f \in T(X)$ and $x \in X$, then $H(f) = g \Leftrightarrow h \circ f \circ h^{-1} = g$

$$\Rightarrow h(f^{-1}(\{x\})) = g^{-1}(h(\{x\})) .$$

But h a bijection $\Rightarrow h(\{x\})$ is a singleton, and $g \in \Gamma(x)$ implies that $g^{-1}(h(\{x\}))$ is closed. Hence h^{-1} is continuous. Similarly we get that h is continuous.

Case ii) X and Y are T_*-spaces. Let $k_F \in T(X)$ be as in definition 4.7.2, then if $g = H(k_F)$ we have $h \circ k_F \circ h^{-1} = g \Rightarrow h(F) = h(k_F^{-1}(k_F(F))) = g^{-1}(h(k_F(F)))$. But $k_F(F)$ is finite $\Rightarrow h(k_F(F))$ is finite. And $g \in \Gamma(X)$ now says in fact that $h(F)$ is closed. So h^{-1} is continuous. In the same manner h is continuous as well.

It is easy to see that some T_*-spaces are T^*-spaces, and vice-versa. Whether one is class is bigger or how largely the classes intersect is unknown as yet. Below we show some idea of how large the class of T^*-spaces is.

Theorem 4.9. Suppose X is a 0-dimensional Hausdorff space and suppose $T(X)$ is such that whenever $X = A \cup (X \sim A)$, where A is a non-empty, open, and closed subset of X with $X \sim A$ also non-empty, then for some $a \in A$ the function $f : X \to X$ defined by

$f(z) = \begin{cases} z , & z \in X \sim A \\ a , & z \in A \end{cases}$ belongs to $T(X)$. Then X is a T^*-space.

Proof. Let F be a closed subset of X and $y \in X \sim F$, then there exists a subset, G, of X which is both open and closed such that $y \in G \subseteq X \sim F$. Then by assumption $\exists\, x \in X \sim G$ such that the function $f \in X^X$ defined by $f(z) = \begin{cases} z , & z \in G \\ x , & z \notin G \end{cases}$ belongs to $T(X)$. But now $y \in f^{-1}(\{x\}) \supseteq F$. So X is a T^*-space.

In [6, Theorem 2] Magill proves that 0-dimensional Hausdorff spaces are S^*-spaces. Using our Theorem 4.9 we generalize this to:

Corollary 4.9.1. If X is a 0-dimensional Hausdorff space, then X is a T^*-space if $S(X) \cap C(X) \subseteq T(X)$.

Proof. Any function f defined as in the statement of Theorem 4.9 is continuous and closed. We note here that if $T_1(X) \subseteq T_2(X)$ and X is a T_1^*-space, then X is a T_2^*-space as well. The same holds for T_*-spaces as well. So we get the following restatement for theorem 3 of [6].

Theorem 4.9.2.[5] If X is a completely regular Hausdorff space containing at least one non-degenerate path between two distinct points, and if $S(X) \subsetneq T(X)$, then X is a T^*-space.

Proof. Let F be a closed subset of X and let $y \in X \sim F$. Then complete regularity implies that there exists a continuous function $f : X \to [0,1] = I$ such that $f(F) = 0$ and $f(y) = 1$. By assumption there are two distinct points $x, z \in X$ and a continuous function $g : I \to X$ such that $g(0) = x$ and $g(1) = z$.

Then $g \circ f \in S(X) \Rightarrow g \circ f \in T(X)$. But also $y \notin (g \circ f)^{-1}(\{x\}) \supseteq F$ So x is a T^*-space.

If X and Y are two topological spaces, then by $X \cup Y$ we denote the space with a basis of open sets consisting of those which are open in either X or Y. We assume X and Y are disjoint.

Corollary 4.9.3. If X is a completely regular Hausdorff space then X is a subspace of an S^*-space; viz., of $Y = X \cup I$.

Proof. Y certainly satisfies the hypothesis of Theorem 4.9.2.

There exist completely regular Hausdorff spaces which are not S^*-spaces. For example in [1] Cook gives an example of a compact, metric, one-dimensional, indecomposable continuum, Z, such that $S(Z)$ consists entirely of $K(Z)$ and the identity map. Therefore Z cannot be an S^*-space; and we have

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5 The following discussion of S^*-space, hence 4.9.2 to 4.10, is due to Magill in [6]. Where possible we make the obvious generalizations to T^*-space.

<u>Corollary 4.9.4</u>. The property of being an S^*-space is not hereditary.

<u>Proof</u>. Let $Y = Z \cup I$, then by Corollary 4.9.3 Y is an S^*-space with Z as a subspace.

Now continuing with Cook's example we get:

<u>Theorem 4.10</u>. There is an S^*-space which is not an S-space; viz., $Y = Z \cup I$ is such a space.

<u>Proof</u>. Let $x \varepsilon Z$ and G any open subset of Z such that $Z \sim G$ has more than one point. If Y is an S-space, then there would exist a continuous function $f : \overline{G} \to Y$ such that $f(x) \neq x$ and $f(y) = y$, $\forall y \varepsilon \overline{G} \sim G$. Define $g \varepsilon Y^Y$ by $g(y) = \begin{cases} f(y), & y \varepsilon \overline{G} \\ y, & y \varepsilon Y \sim G \end{cases}$. Then g is continuous if f is. But since Z is connected, then $g(Z)$ would be connected. But $Z \sim G$ is not empty, so $\exists w \varepsilon Z \sim G$ such that $g(w) = w \Rightarrow g(Z) \cap Z \neq \emptyset$. Then Z connected implies that $g(Z) \subseteq Z \Rightarrow g \varepsilon S(Z)$. But g is neither a constant (since $Z \sim G$ has more than one point) nor the identity (since $g(x) \neq x$), so $g \notin S(Z)$, which is a contradiction. So Y is not an S-space, but by Corollary 4.9.3 Y is an S^*-space.

To extend some of the results above we look at a class of semi-groups referred to as Δ-semigroups.[6]

<u>Definition 4.11.1</u>. Let $\Delta(X)$ be a family of subsets of X such that $X \varepsilon \Delta(X)$; $\{x\} \varepsilon \Delta(X)$, $\forall x \varepsilon X$; and $\emptyset \notin \Delta(X)$. Then we say that

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[6] The following discussion of Δ-semigroups, hence 4.11.1 through to 4.12.3 are due to Magill in [3].

$T(X)$ is a Δ-semigroup if

i) $T(X) = \{f \varepsilon X^X : f(A) \varepsilon \Delta(X) \ , \forall A \varepsilon \Delta(X)\}$

and ii) $\forall A \varepsilon \Delta(X) \ , \exists f \varepsilon T(X)$ such that $f(X) = A$.

Note that $K(X) \subseteq T(X)$ because of property ii).

<u>Lemma 4.11.2</u>. Let $T(X)$ and $T(Y)$ be Δ-semigroups and $H : T(X) \to T(Y)$ be an isomorphism, then

i) $h(A) \varepsilon \Delta(Y) \ , \forall A \varepsilon \Delta(X)$,

and ii) $h^{-1}(B) \varepsilon \Delta(X) \ , \forall B \varepsilon \Delta(Y)$.

<u>Proof</u>. Let $A \varepsilon \Delta(X)$, then $\exists f \varepsilon T(X)$ such that $f(X) = A$, then $h(A) = h(f(X)) = h(f(h^{-1}(Y))) = H(f)(Y) \varepsilon \Delta(Y)$. Similarly for part ii).

<u>Theorem 4.11.3</u>. The class of T_1-spaces is C-admissible.

<u>Proof</u>. Let X and Y be T_1-spaces and $H : C(X) \to C(Y)$ be an isomorphism. Define $\Delta(X) = \{F \subseteq X : F$ is closed and $F \neq \emptyset\}$. Similarly for $\Delta(Y)$. Let $F \varepsilon \Delta(X)$, then choose any $x \varepsilon F$ and define $f \varepsilon X^X$ by $f(y) = \begin{cases} y \ , \ y \varepsilon F \\ x \ , \ y \varepsilon X \sim F \end{cases}$, then f is clearly closed and $f(X) = F$. So $C(X)$ is a Δ-semigroup. Similarly $C(Y)$ is a Δ-semigroup. But by Lemma 4.11.2 both h and h^{-1} are closed. So h is a homeomorphism.

Now we use somewhat the same approach to the space of connected functions. Unfortunately a weaker result must follow.

<u>Definition 4.12.1</u>. X is a U-space if it is connected and for every connected subset A of X , $\exists f \varepsilon U(X)$ such that $f(X) = A$.

One can easily see that not all U-spaces are U-admissible. We look at $X = \{(x,y) : y = \sin(1/x), \forall x \in (0,1]$ and $y = 0$ if $x = 0\}$. Then X , with the induced topology from E^2 , and I , with the usual topology, are U-spaces. But the map $h : X \to I$ defined by $h(x,y) = x , \forall (x,y) \in X$, is one-to-one, onto, and biconnected and so induces an isomorphism, H , from $U(X)$ to $U(I)$ given by $H(f) = h \circ f \circ h^{-1}, \forall f \in U(X)$. But h^{-1} is not continuous at 0 .

But, of course, from the same type of argument as in the proof of theorem 4.11.3 we get that if $H : U(X) \to U(Y)$ is an isomorphism and X and Y are U-spaces, then h is biconnected; i.e., both h and h^{-1} are connected. The question then is: when is such an h a homeomorphism? From Pervin and Levine in [7, Theorem 3.10, p. 495] we get that any biconnected map between two locally connected, compact, Hausdorff spaces is a homeomorphism. Hence:

<u>Theorem 4.12.2</u>. Locally connected, compact, Hausdorff U-spaces are a U-admissible class.

For usefulness we need some knowledge of the extent of U-spaces.

<u>Theorem 4.12.3</u>. If X is a connected, completely regular, Hausdorff space with cardinality c (the cardinality of the continuum), then X is a U-space.

<u>Proof</u>. Let x,y be two distinct points in X . Then complete regularity implies that $\exists f : X \to I$ such that f is continuous and $f(x) = 0$ and $f(y) = 1$. Now since $f(X)$ is connected and contains both 0 and 1 , then $f(X) = I$. If $a \in I$, let

$$0\cdot a_1 a_2 \ldots a_n \ldots \quad (\text{where } a_i = 0 \text{ or } 1)$$

denote the non-terminating binary expansion of a. Define g on I by
$$g(a) = \begin{cases} 0, & a = 0 \\ \limsup \left(\frac{1}{n} \sum_{i=1}^{n} a_i\right), & a \neq 0 \end{cases}$$
Then from Kurotowski [2, p. 82] we see that g has the property that if A is any non-degenerate half-open, open, or closed interval in I, then $g(A) = I$. If $B \subseteq I$ is connected, then by assumption the cardinality of B is less than or equal to c, hence there exists some onto map $h : I \to B$. Thus $h \circ g \circ f \in X^X$. Let D be any connected subset of X, then $f(D)$ is either a singleton or a non-degenerate interval. In the former case $(h \circ g \circ f)(D)$ is a singleton, hence, connected. In the latter case $g(f(D)) = I$, so $(h \circ g \circ f)(D) = B$, which is assumed connected. Hence $h \circ g \circ f \in U(X)$ and $(h \circ g \circ f)(X) = B$.

As an end to this chapter, and as an introduction to the neat, we note:

<u>Corollary 4.13</u>. Let X be a topological space, then every automorphism of $T(X)$ is inner if any of the following hold:

i) X is an S-space and $T(X) = S(X)$.

ii) X is either a T^*-or a T_*-space and $\{h \in X^X : h$ is a homeomorphism$\} \subseteq T(X) \subseteq \Gamma(X)$;

iii) $T(X)$ is a Δ-semigroup;

or iv) X is T_1 and $T(X) = C(X)$.

Proof. i) Theorem 4.3;

ii) Theorem 4.8;

iii) Lemma 4.11.2;

and iv) Theorem 4.11.3.

5. Automorphisms of $T(X)$. Before we begin we introduce a few more notations to facilitate the discussion. Suppose $\mathcal{D}$ is some subsemigroup of X^X, then:

Definition 5.1.1. $\mathrm{Aut}(\mathcal{D})$ = the set of all automorphisms of $\mathcal{D}$.

Definition 5.1.2. $\mathrm{Inn}(\mathcal{D}) = \{H \in \mathrm{Aut}(\mathcal{D}) : H \text{ is inner}\}$.

Definition 5.1.3. $Z(\mathcal{D})$ = center of $\mathcal{D} = \{f \in \mathcal{D} : f \circ g = g \circ f, \forall g \in \mathcal{D}\}$.

Definition 5.1.4. $B(\mathcal{D}) = \{\text{bijections } h \in \mathcal{D} : h^{-1} \in \mathcal{D}\}$.

Definition 5.1.5. For any set A, i_A denotes the identity map on A; i.e., i_A is that map in A^A such that $i_A(x) = x$, $\forall x \in A$.

We always here assume that $K(X) \subseteq T(X)$.

Lemma 5.1.6. If $i_X \in T(X)$, then $Z(T(X)) = \{i_X\}$.

Proof. Certainly we always have $i_X \in Z(T(X))$. Now let $f \in Z(T(X))$, then $f \circ g = g \circ f$, $\forall g \in T(X) \Rightarrow [f(x)] = f \circ [x] = [x] \circ f = [x]$, $\forall x \in X \Leftrightarrow f = i_X$.

Lemma 5.1.7. Suppose $i_X \in T(X)$ and $\phi : B(T(X)) \to \mathrm{Inn}(T(X))$ is defined by $\phi(h) = H$ where $H(f) = h \circ f \circ h^{-1}$, $\forall f \in T(X)$. Then ϕ is an isomorphism and $Z(B(T(X))) = \{i_X\}$.

Proof. That ϕ is a homomorphism onto is trivial. And the uniqueness part of Lemma 3.3 tells us that if $\phi(h_1) = \phi(h_2)$, then $h_1 = h_2$; i.e., ϕ is one-to-one.

Now suppose $h \in Z(B(T(X)))$, then $\phi(h)(f) = h \circ f \circ h^{-1}$, $\forall f \in T(X)$. But $h \in Z(B(T(X))) \Rightarrow \phi(h)(f) = h \circ f \circ h^{-1} = f = i_X \circ f \circ i_X^{-1}$, so $h = i_X$.

So for any $T(X)$ we can describe the inner automorphisms of $T(X)$. We are here interested in the case where every automorphism is inner. Corollary 4.13 yields many results in this regard.

We first concern ourselves with a specific problem.

Theorem 5.2.[7] Let X be the reals and $\mathcal{D} = D(X) = \{f \in X^X : f$ has a (finite) derivative everywhere$\}$, then $\mathrm{Aut}(\mathcal{D}) = \mathrm{Inn}(\mathcal{D})$.

Proof. Let $H \in \mathrm{Aut}(\mathcal{D})$. It is clear that X is a D_*-space. For example, suppose F is a closed subset of X, then $X \sim F = \bigcup_{n=0}^{\infty} (a_n, b_n)$ as a disjoint union of non-degenerate open intervals. Define $k \in X^X$ by

$$k(x) = \begin{cases} 0 & , x \in F \\ (b_n - a_n)\left[\cos\left(\dfrac{2\pi x - \pi(a_n + b_n)}{(b_n - a_n)}\right) + 1\right] & , x \in (a_n, b_n) \end{cases}$$

then $k \in \mathcal{D}$ and $k^{-1}(\{0\}) = F$. So by Theorem 4.8 we have that h is a homeomorphism. But being a homeomorphism of the reals we know that h is strictly monotone, hence is differentiable somewhere (see Royden [8, p. 96]), say at x_1. Let $x_o \in X$ and define

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7 This is Theorem 2.1 of [4].

$$\left.\begin{aligned} f(x) &= x + x_o - x_1 \\ t(x) &= h(x+x_1) - h(x_1) \end{aligned}\right\} \forall x \varepsilon X \ .$$

Then $f \varepsilon \mathcal{D}$ and t is a homeomorphism such that $t(0) = 0$. So we get that if $x \neq 0$ and $g = H(f) = h \circ f \circ h^{-1}$, then

$$\frac{g(h(x_1) + t(x)) - g(h(x_1))}{t(x)} \cdot \frac{h(x_1+x) - h(x_1)}{x} = \frac{h(x_o+x) - h(x_o)}{x}$$

This equation is seen to be valid if we merely substitute for $f(x)$ and $t(x)$ and note that $g \circ h = h \circ f$. Now since the limit as $x \to 0$ on the left side of this equation exists and equals $g'(h(x_1)h'(x_1)$, then the limit on the right side exists and must equal, by definition, $h'(x_o)$. But x_o is arbitrary, hence $h \varepsilon \mathcal{D}$. Similarly (by considering H^{-1}) $h^{-1} \varepsilon \mathcal{D}$.

From Lemma 5.1.7 we immediately now get:

<u>Corollary 5.2.1</u>. Under the assumptions of Theorem 5.2 $\mathrm{Aut}(\mathcal{D})$ is isomorphic to $B(\mathcal{D})$ = the set of strictly monotonic functions in X^X which have a finite derivative everywhere.

In [4, Corollary 2.3] Magill proves that if X is the reals and $D(X)$ is the differentiable maps, then every automorphism of $D(X)$ has a unique extension to an inner automorphism of $S(X)$. We can generalize this to:

<u>Theorem 5.3</u>. Suppose $\mathcal{B}$ and $\mathcal{C}$ are subsemigroups of X^X such that $K(X) \subseteq \mathcal{B} \subseteq \mathcal{C}$ and $\mathrm{Aut}(\mathcal{B}) = \mathrm{Inn}(\mathcal{B})$, then every automorphism of $\mathcal{B}$ has a unique extension to an inner automorphism of $\mathcal{C}$.

Proof. Let $H \in Aut(\mathcal{B})$. Define $H^* : \mathcal{C} \to \mathcal{C}$ by $H^*(f) = h \circ f \circ h^{-1}$, $\forall f \in \mathcal{C}$. Then H^* is an extension of H , because $Aut(\mathcal{B}) = Inn(\mathcal{B}) \Rightarrow h \in B(\mathcal{B}) \Rightarrow h \in B(\mathcal{C}) \Rightarrow H^* \in Inn(\mathcal{C})$. Now, if K is some other extension of H , then, using Lemma 3.3, we have, $\forall x \in X$

$$K([x]) = [k(x)] = H([x]) = [h(x)] \Rightarrow h = k \Rightarrow H^* = K .$$

Now to try to extend Theorem 5.2 to say the entire (everywhere analytic) maps on the complex numbers, we immediately run into difficulty: if $\mathcal{D}$ is this semigroup of entire maps and h is the complex conjugate function (i.e., if $z = x + iy$, x,y real, then $h(z) = x - iy = \overline{z}$) then $H \in Aut(\mathcal{D})$ defined by $H(f) = h \circ f \circ h^{-1}$, $\forall f \in \mathcal{D}$, is an automorphism but is not inner since $h \notin \mathcal{D}$.

Another method of extension has been attempted, and this is to the semigroup of Fréchet-differentiable maps on a real Banach space, X . To say a function $f \in X^X$ is Fréchet-differentiable we mean that there exists a map $\delta f : X \times X \to X$ which is continuous and linear in the second variable and is such that $\forall a \in X$ we have

$$\lim_{x \to 0} \frac{\|f(a+x) - f(a) - \delta f(a,x)\|}{\|x\|} = 0 .$$

Then $\delta f(a,x)$ is called the first derivative of f at a with increment x . The problem, stated by Yamanuro in [9], is: if $\mathcal{D}$ is the set of all Fréchet-differentiable maps on X , then is $Aut(\mathcal{D}) = Inn(\mathcal{D})$? The answer is not yet known. The best result so far is that of Yamanuro in [10]: an automorphism $H \in Aut(\mathcal{D})$ is inner iff it is uniform; where uniform means that $\forall \varepsilon > 0$ and for every sequence of real numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n \neq 0$, $\forall n$ but $\alpha_n \to 0$ as $n \to \infty$, there exists a $\delta > 0$ such that if $\|x\| < \delta$, then

$$\sup_{n\geq 1} \left\| \frac{h(\alpha_n h^{-1}(x+h(0))) - h(0)}{\alpha_n} - x \right\| < \varepsilon\|x\| .$$

The proof of this is long and complicated.

If we try and look at other semigroups, we find there is usually very much difficulty in showing whether or not all automorphisms are inner. An example which is not so difficult is the following.

<u>Theorem 5.4</u>. Let X be a set and $\mathcal{D}$ a subsemigroup of X^X such that $K(X) \subseteq \mathcal{D}$. If there exists a collection of subsets of X , $\underline{S}$, containing all finite subsets of X , such that

i) $\mathcal{D} = \{f \in X^X : f^{-1}(A) \in \underline{S}, \forall A \in \underline{S}\}$

and ii) $\forall A \in \underline{S}, \exists f \in \mathcal{D}$ and a finite subset F of X such that $f^{-1}(F) = A$;

then $\mathrm{Aut}(\mathcal{D}) = \mathrm{Inn}(\mathcal{D})$.

<u>Proof</u>. Let $H \in \mathrm{Aut}(\mathcal{D})$ and $A \in \underline{S}$. Then by assumption there exists an $f \in \mathcal{D}$ and a finite subset F of X such that $f^{-1}(F) = A$. Let $g = H(f) = h\circ f\circ h^{-1}$. Then $h(A) = h(f^{-1}(F)) = g^{-1}(h(F))$. And since $h(F)$ is finite, property i) says that $h(A) \in \underline{S}$. Hence $h^{-1} \in \mathcal{D}$. Similarly $h \in \mathcal{D}$.

Examples of where Theorem 5.4 would apply are any measure space where finite sets are measurable and $\underline{S}$ is the collection of measurable sets, or where $\underline{S}$ is the collection of closed sets of some S_*-space.

6. Conclusions. Some of the results given above give criteria for two spaces being homeomorphic. For example if X and Y are T-admissable and there exists an $H : T(X) \to T(Y)$ which is an isomorphism, then X and Y are homeomorphic.

If $Aut(T(X)) = Inn(T(X))$ we can sometimes say something too. For instance if $D(X)$ = the near-ring of Fréchet differentiable maps, where sum and composition are the near-ring operations, then it is easily seen that all ring automorphisms of $D(X)$ are uniform, hence inner. So $D(X)$ and $D(Y)$ are isomorphic iff X and Y are diffeomorphic.

It is of general mathematical interest to know when all the automorphisms of a semigroup are inner. In this regard there is much room for research. For instance, more examples of familiar semigroups with this property could be searched for. Also open for investigation is the question, if $\mathcal{D}$ is a semigroup such that $Aut(\mathcal{D}) = Inn(\mathcal{D})$, what then can we conclude about the semigroup $\mathcal{D}$ itself?

These questions for research are interesting in that they can be simply stated and understood, but not so simply can they be answered. For example Theorem 5.4 concerns semigroups having the property that there is some collection $\underline{S}$ of subsets such that the property $f^{-1}(A) \in \underline{S}, \forall A \in \underline{S}$, completely characterizes the elements of the semigroup. If for a certain semigroup such a collection of subsets is found, then much about the semigroup may be deducible from study of the collection of subsets. If such a collection were found for say the Fréchet-differentiable maps, then perhaps the question of whether all automorphisms are inner or not could be answered.

References

1. Howard Cook, "A continuum which admits only the identity mapping onto a non-degenerate subcontinuum", Abstract 625-3, Amer. Math. Soc. Notices 12(1965), p. 545.

2. C. Kuratowski, Topologie II (Warsaw, 1950).

3. K.D. Magill, Jr., "Semigroups of functions on a topological space", Proc. London Math. Soc. (3) 16(1966), p. 507-518.

4. ______________, "Automorphisms of the semigroup of all differentiable functions", Glasgow Math. Journal 8(1967), pp. 63-66.

5. ______________, "Semigroups of continuous functions", Amer. Math. Monthly 71(1964), pp. 984-988.

6. ______________, "Another S-admissible class of spaces", Proc. Amer. Math. Soc. 18(1967), pp. 295-298.

7. W.J. Pervin and N. Levine, "Connected mappings of Hausdorff Spaces", Proc. Amer. Math. Soc. 9(1958), pp. 488-496.

8. H.L. Royden, Real Analysis, 2nd edition, The MacMillan Co., New York (1968).

9. S. Yamamuro, "A note on semigroups of mappings on Banach spaces", Journal Australian Math. Soc. 9(1969), pp. 455-464.

10. ___________, "On the semigroup of differentiable mappings", Journal Australian Math. Soc. 10(1969), pp. 503-510.