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H^* AND SOME RATHER NICE SPACES

by

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ABSTRACT

The Problem The integral cohomology algebra functor, H^*, was introduced to algebraic topology in hopes of deciding when spaces are homotopy-equivalent. With this in mind, let $T(A) = \{X \mid H^*(X) \simeq A\}$, the collection of all simply-connected finite complexes X, for which the cohomology algebra $H^*(X)$ is isomorphic to A. We ask: when are there only a finite number of homotopy equivalence classes in $T(A)$?

The Result Let A satisfy the condition:

$$A \otimes Q \simeq \bigotimes_{i=1}^{k} Q[x_i]/(x_i^{n_i}) \quad .$$

Then there are only a finite number of homotopy-equivalence classes in $T(A)$.

The Methods For a given A we construct a "model space" M and show that for any $X \in T(A)$ there exists a continuous map $X \overset{\phi}{\to} M$ "within N" . The concept of a map within N is less restrictive than that of a homotopy-equivalence, but more restrictive than the concept of a rational equivalence.

We then show that in the category T^N/M , whose objects are $\overset{\phi}{\to} M$, maps within N , having range M , there are only a finite number of equivalence classes. This is proved with the use of Postnikov Towers and algebraic arguments similar to Serre's mod C theory. The result then follows.

Applications The result applies to spaces having rational cohomology isomorphic to the rational cohomology of topological groups, H-spaces, Stiefel manifolds, the complex and quaternionic projective spaces and of some other homogeneous spaces.

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CHAPTER THE FIRST

Premises what is necessary to be known; concerns a few common-places of the cohomology Functor H^*, its early successes followed closely with a somewhat melancholic discourse on its later disappointments; finally a revelation of some bold aspirations for H^*, and a promise of a satisfactory conclusion, tho' the Reader perhaps will guard sundry doubts-

The early pioneers of algebraic topology Alexander, Eilenberg, Whitney and Zilber noticed that the integral cohomology functor H^* was equipt with a natural multiplicative structure, the cup product. It was hoped that H^* could thus distinguish homotopy equivalence classes of spaces, or homotopy types.

Let us agree that H^* is a functor of the topological-homotopy category T_h to the category of integral algebras. Thus we consider the category, T_h with objects spaces having the homotopy type of a <u>simply-connected</u>, <u>finite</u> CW complex, with basepoint, and with transformations which are homotopy classes of continuous maps. Let A_Z be the category with objects which are associative, graded commutative algebras over the integers, and with transformations which are graded algbra maps.

H^* gives a representation of the spaces of T_h in terms of the algebras of A_Z . The early pioneers noticed, that in some cases, it does this well:

Case 1: Moore spaces Y^n_G have a nontrivial reduced integral homology group G in only one dimension n . As always, Y^n_G is simply connected, $n > 1$.

1.1 <u>Lemma</u> If $H^*Y \simeq H^*Y^n_G$, then $Y \simeq Y^n_G$.

Pf. by the universal coefficient theorem and the Hurewicz isomorphism $\pi_n(Y;G) \simeq \mathrm{Hom}(G,G) + \mathrm{Ext}_G(G,\pi_{n+1}Y) \simeq_G [Y^n_G, Y]$ and id_G induces an isomorphism of integral homology. ¶

Case 11: Complex projective n-space $CP(n)$ is the 2n dimensional skeleton of $K(Z,2)$.

1.2. Lemma If $H^*Y \simeq H^*CP(n)$ then $Y \simeq CP(n)$.

Pf. The generator of $H^2(Y)$ is a map in $[Y;K(Z,2)]$. It induces a cohomology isomorphism on the 2n-skeleton. ¶

In such cases, we shall say that the integral cohomology H^* uniquely determines the homotopy type of the space.

Recently it was discovered that an associated functor $H^*(\ ;Q)$ gives quite a good representation of another class of spaces:

Case 111: Finite dimensional H-spaces. Let G be a space which supports a homotopy multiplication.

1.3. Theorem If $H^*Y \otimes Q \simeq H^*G \otimes Q$ and Y , an object of T_n , supports a homotopy multiplication then Y must belong to one of a finite number of homotopy types.

Pf. [Curjel-Douglas]. ¶

In this situation we may say that the underlying homotopy types of H-spaces are finitely determined by $H^*(\ ;Q)$.

Theorem 1.3, may be combined with another result of [Curjel] which states that a finite CW complex admits at most a finite number of mutually non-isomorphic structures as a group in T_h . The result is that on the category $\mathfrak{G}$ of group objects of T_h . $H^*(\ ;Q)$ finitely determines $\mathfrak{G}$-type.

1.4. <u>Definition</u> Let $F : C \to C'$ be a functor and X an object of C. Then F finitely determines C-type at X if the preimage of the C'-equivalence class of FX is a finite set of C-equivalence classes.

Despite these early success of H^*, as exemplified in Lemmas 1.1 and 1.2, pioneers of topology noticed that the functor did not give a complete picture of some tangibly evident spaces. Representative examples will demonstrate some of the weaknesses of H^*.

<u>Case IV</u>: Let $RP(n)$ denote n-dimensional real projective space and "v" the wedge product.

1.5 <u>Lemma</u> $H^*RP(3) \simeq H^*RP(2)vS^3$, but $RP(3) \not\simeq RP(2)vS^3$.

Pf. see [Hilton and Wylie]. This is a particularly galling example, since even $H^*(\ ;Z/2Z)$ gives a better showing. ¶

<u>Case V</u>: Denote a Whitehead product of the three inclusions of S^3 in $S^3vS^3vS^3$ by $k \in \pi_7(S^3vS^3vS^3)$. For $n \in Z^+$, denote the cofibre of nk by W_n (obtained by attaching e^8 with attaching map nk).

1.6 <u>Lemma</u>: $H^*W_n \simeq H^*W_m$ but $W_n \not\simeq W_m$ if $m \neq n$.

Pf. The cohomology groups of H^*W_n may be determined from the exact sequence for the cofibration $S^3vS^3vS^3 \xrightarrow{j} W_n \to S^8$. The gradation shows that <u>all multiplication</u> except by the identity is <u>trivial</u>.

k is of infinite order [Hilton 2]. Moreover $\pi_7 W_n \simeq \pi_7(VS^3)/(nk)$ where (nk) is the subgroup of $\pi_7 VS^3$ generated by nk. In $\pi_7 W_n$, $j_\#(k)$ is of order n. Thus $W_n \not\simeq W_m$ if $n \neq m$.

$\{W_n\}$ is an infinite family of spaces of mutually distinct homotopy types, but with isomorphic cohomology algebras. In this situation, H^* does not finitely determine homotopy type.

Nevertheless the main result of this thesis will be that for many of the spaces of T_h, H^* finitely determines homotopy type.

1.7. THEOREM Suppose that A is an algebra of A_Z which satisfies the condition $A \otimes Q \simeq \bigotimes_{i=1}^{k} \dfrac{Q[x_i]}{(x_i^{n_i})}$. Let $HT(A)$ be the set of homotopy types Y with $H^*Y \simeq A$. Then $HT(A)$ is a finite set.

Pf. see THEOREM 4.3.4.

The cohomology algebras of Cases 1, 11 and 111 satisfy the conditions of 1.7. For more examples see 5.3.0.

The proof of the THEOREM will progress in three stages. In Chapter Two we give a description of a "model space" M which has the desired rational cohomology. For any space X with $H^*X \simeq A$, a map $\phi_X : X \to M$ is constructed so that X is "near" to the model space. ϕ_X is "almost" an equivalence. With the help of the transformations ϕ_X, we show in Chapter Three that the homotopy groups of any space X with $H^*X \simeq A$ must also be "near" to those of M. Then in Chapter Four, again with the help of the maps ϕ_x , we show that the k-invariants of X must be "near" to those of M . Finally we remember that k-invariants provide a complete description of homotopy type, to affirm the THEOREM.

For a more complete discussion of the result, see Chapter Five.

CHAPTER THE SECOND

In which the quest commences H^**'s rationalizations give rise to an abstract model to which it can give the appropriate ring structure (at least rationally!). However such is the vexatious estate of the topological universe that an abstract model space can seldom satisfy our functor's integral requirements. Providently, and at wondrous length, the Author recites how* H^* *begins to take stock of integrally satisfactory prospects, to discern the Similarities and Particularities between them and its rational model. The recitation includes an account of calculations of so BASE a character, that some may not think it worthy of their notice. The Reader is encouraged to approach the chapter with Charity and Eclectic Diligence.*

2.1.0 MODEL SPACES

Let us agree on some nomenclature:

2.1.1 <u>Definition</u> An algebra of A_Z over Q is <u>rather nice</u> if

$$A \simeq \bigotimes_{i=1}^{k} \frac{Q[x_i]}{(x_i^{n_i})}$$. Here $Q[x]$ is the (graded-commutative) polynomial ring on one generator x, and (x^n) is the ideal therein generated by x^n.

Colloquially, we shall say an algebra over the integers A is rather nice if $A \otimes Q$ is; a space, or homotopy type, or even rational homotopy type is rather nice if $H^*(\ ;Q)$ is.

We shall now construct a space M for which $H^*(M;Q) \simeq A$. This particular space shall be called the model space for A.

2.2.0 LOOP SPACES OF SPHERES AND THEIR SKELETA

The loop space of a space Y in T_n is the function space of all continuous basepoint preserving maps $\alpha : S^1 \to Y$, with the compact open topology. It is denoted $\Omega(Y)$, Ω is a functor. There exists a natural isomorphism $\pi_n(Y) \simeq \pi_{n-1}(\Omega Y)$.

The loop space of an odd dimensional sphere S^{m+1} (m even) is infinite dimensional, with rational cohomology $H^* \Omega S^{m+1} \otimes Q \simeq Q[x]$ the free polynomial ring of one generator of dimension m, over Q. The integral cohomology is a divided polymomial algebra on one generator (see 2.5.6 Example). We shall denote the loop space of S^{m+1} as $(S^m)_\infty$ for m even, cf [James]. For m odd, let $(S^m)_\infty$ denote S^m. In either case, $(S^m)_\infty$ has rational homotopy groups trivial in all dimensions except m .

[James] gives a canonical skeletal decomposition of $(S^m)_\infty$. Denote the mn dimensional skeleton of $(S^m)_\infty$ by $(S^m)_n$. Then its rational cohomology is $H^*(S^m)_n \otimes Q = Q[x]/(x^{n+1})$, where x is an m dimensional generator. More recondite is the fact that $(S^m)_n$ (m even) has exactly two non-trivial rational homotopy groups: $\pi_m(S^m)_n \otimes Q \simeq Q$ and $\pi(n+1)m-1(S^m)_n \otimes Q \simeq Q$ [Toda].

2.2.1 Definition Let A be a rational, rather nice algebra. i.e.

$$A \simeq \bigotimes_{i=1}^{k} \frac{Q[x_i]}{(x_i^{n_i+1})}$$ and the degree of x_i is called m_i . Then the model space for A is $M = \displaystyle\mathop{X}_{i=1}^{k} (S^{m_i})_{n_i}$, where X indicates topological direct product.

M may also be called the rational model space for any Z algebra A' with $A' \otimes Q \simeq A$.

A simple calculation involving the Kuenneth Theorem shows that $H^*(M;Q) \simeq A$.

Another fact of prime importance for later theorems of this chapter is a result of [Mimura-Toda; Lemma 2.4]: If $i \neq (n+1)m$, there exists an endomorphism of the relative complex $((S^m)_\infty , (S^m)_n)$ which induces a trivial map on the i^{th} relative homotopy group $\pi_i((S^m)_\infty , (S^m)_n)$, while maintaining an induced automorphism on all rational relative homotopy groups $\pi_k((S^m)_\infty\ , (S^m)_n)\otimes Q$. This is why rational model spaces are useful. They have internal transformations which preserve their rational structure, yet which ignore torsion in their cohomology and homotopy groups.

2.3.0 HOW H^* COMPARES SPACES or IN SEARCH OF A WHITEHEAD THEOREM WITHIN N .

2.3.1 Definition Let $\phi : A \to B$ be a homomorphism of finitely-generated abelian groups. ϕ is a (group)map within N if Ker(ϕ) and Coker(ϕ) are finite groups of order a factor of N.

A map ϕ^n: $A^n \to B^n$ of graded abelian groups is within N if the order of the kernel of ϕ^n = o(ker ϕ^n) raised to the n^{th} power is a factor of N, for all n.

An algebra map of A_Z is within N if it induces a map within N on the underlying graded abelian groups.

A map of T_h is within N if $H^*(\phi)$ is within N .

Finally, two spaces Y and Y' are within N if there exists a T_h morphism within N between them.

Group maps within N enjoy some elementary properties:

2.3.2 Lemma Let $A \xrightarrow{\phi} B$, $B \xrightarrow{\psi} C$ be group maps within N and N' resp. Then $\psi \circ \phi$ is within N'N . ¶

2.3.3 Lemma Let $\phi : A \to B$ be an abelian group map within N . Then there exists a map $\psi : B \to A$ (within N^2) such that $\phi \circ \psi = N^2$ and $\psi \circ \phi = N^2$.

i.e.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "N^2"'] & B \arrow[d, "N^2"] \arrow[dl, "\psi"'] \\ A \arrow[r, "\phi"'] & B \end{tikzcd}$$

is a commutative diagram.

2.3.4 <u>Lemma</u> Let $\phi : C_* \to C'_*$ be a map of chains of abelian groups within N. Then the maps induced on the homology of the chain complexes, $H_n(\phi)$ are maps within N^2.

Pf. Diagram hassling. ¶

Suppose that A is a given, integral, rather nice algebra. The main result of this chapter will be to show that all spaces Y with $H^*Y \simeq A$ are within a uniform N of the model space M.

2.4.0 <u>OBSTRUCTIONS</u>

To construct maps within N between rather nice spaces and the model, we recall some of the results of obstruction theory. See [Hu; Chapter VI, in particular E].

Let $* \xrightarrow{i} Y^1 \xrightarrow{i} Y^2 \hookrightarrow \ldots \hookrightarrow Y$ be a skeletal CW decomposition for Y. Suppose a map has been defined $f_t : (Y,Y^t) \to (Z,Z')$ has been defined and we wish to extend this map to a map $f_{t+1} : (Y,Y^{t+1}) \to (Z,Z')$ so that

$$\begin{tikzcd}
(y,y_{t+1}) \arrow[r, "f_{t+1}"] & (Z,Z') \\
(y,y_t) \arrow[u, "(1,i_t)"] \arrow[ur, dashed, "f_t"'] &
\end{tikzcd}$$

is a homotopy commutative.

It is a standard result of obstruction theory [Hu] that an obstruction class $\gamma(f_t) \in H^{t+1}(Y;\pi_{t+1}(Z,Z')$ measures the obstruction to this extension. The map f_{t-1} can be extended to f_{t+1} iff $\gamma(f_t) = 0$.

The obstruction class is defined naturally in the following two senses.

2.4.1 Lemma Consider the following diagram in the category of excisive pairs of T_h

$$\begin{tikzcd}
(W,W^{n+1}) \arrow[r, "f"] & (X,X^{n+1}) & & \\
(W,W^n) \arrow[u, dash] \arrow[r, "f"] & (X,X^n) \arrow[u, dash] \arrow[r, "g"] & (Y,Y') \arrow[r, "h"] & (Z,Z')
\end{tikzcd}$$

Then a) h induces a coefficient map : $H^{n+1}(X;\pi_{n+1}(Y,Y')) \xrightarrow{h_\#} H^{n+1}(X;\pi_{n+1}(Z,Z'))$ and $h_\#(\gamma(g)) = \gamma(h\circ g)$.

b) f induces a cohomology map $H^{n+1}(X,\pi_{n+1}(Y,Y')) \xrightarrow{f^*} H^{n+1}(W,\pi_{n+1}(Y,Y'))$ and $f^*\gamma(g) = \gamma(g\circ f)$.

Pf. from definitions. ¶

2.5.0 <u>MEDITATIONS ON THE INTERNAL STRUCTURE OF A Z-ALGEBRA</u>

Let A be an integral rather nice algebra in A_Z . Then

$$A \otimes Q \simeq \bigotimes_{i=1}^{k} \frac{Q[\overline{x}_i]}{(\overline{x}_i^{\,n_i+1})} \quad \text{with degree } \overline{x}_i = m_i \ .$$

Let $i_Q : A \to A \otimes Q$ be the rationalization of A . For each $i = 1,2, \ ,k$ we have the inclusions $Q[\overline{x}_i]/(\overline{x}_i^{\,n_i+1}) \longrightarrow A \otimes Q$.

2.5.1 Definition The i^{th} faculty of A, $Fac_i A$ is the subalgebra

$$i_Q^{-1}(Q[\overline{x}_i]/(\overline{x}_i^{\,n_i+1}))$$

In dimension m_i the i^{th} faculty, considered as an abelian group, has free rank 1. Choose a non-torsion generator of this group, called x_i.

2.5.2 Definition The power subalgebra of x, $Pow(x)$ is the subalgebra of a generated as a Z-module by the powers of x ($x^0 = 1, x, x^2$ etc.) For the next definition, we consider only the torsion-free graded quotient algebras of the two subalgebras. Call them $FreePow(x_i)$ and $FreeFac_i A$.

2.5.3 Definition The depth of $Fac_i A$ is the order of the cokernel of the inclusion $FreePow(x_i) \longrightarrow FreeFac_i A$, considered as a map of graded abelian groups.

It can easily be shown that definition 2.5.3 is independent of the choice of x_i : any two choices have $\pm$ the same representative in $FreePow(x)$. Similarly for more than one generator,

2.5.4 Definition The power algebra of $(x_1, x_2, , x_k)$, denoted $Pow(x_1, , x_k)$ is the subalgebra of A generated as a Z-module by all products of the x_i.

The algebra $Pow(x_1, , x_k)$ modulo the ideal of torsion elements will be denoted $FreePow(x_1, , x_k)$. Similarly for Free A.

2.5.5 Definition The depth of A, dA is the order of the quotient of the inclusion $\mathrm{FreePow}(x_1,,x_k) \longrightarrow \mathrm{Free}\ A$ considered as a map of graded abelian groups.

An example will surely not further mystify the situation. Perhaps it will clear up most of the fog:

2.5.6 Example The free divided polynomial algebra on one generator of height $n+1$ may be denoted $\Gamma_n(x)$. It is a free Z-module of rank $n+1$ with generators $(x_{(0)}, x_{(1)},, x_{(n)})$. It has a multiplication defined by the relations $x_{(m)} \cdot x_{(n)} = \binom{n+m}{n} x_{(m+n)}$. $\binom{n}{k}$ means "n choose k". In particular, let $x = x_{(1)}$. Then $x^k = k!x_{(k)}$. The power algebra of x is the free module generated by the powers of x. There is exactly one faculty and its depth in the k^{th} dimension is $k!$. The depth of $\Gamma_n(x)$ is $1!2!3!..n!$

One more definition and we can start demonstrating something.

2.5.7 Definition The torsion number of A in dimension k is the order, $t_n(A)$ of the subgroup of torsion elements of degree k.

2.6.0 HOW THE OBSTRUCTIONS ARE OVERCOME; A COMPARISON IS POSSIBLE

We consider the class of all spaces Y such that $H^*Y \simeq A$, a given particular, rather nice algebra. For each such space, we wish to produce a map $\phi_Y : Y \to M$. While we shall be doing this, we shall take care that all maps ϕ_Y are maps within some (uniform) N_A, an integer defined solely in terms of A. A sometimes delicate quantification puzzle presents itself at this time. Think it through! The result of this chapter will be

2.6.1 Theorem Pick a rather nice algebra A . Then there is an (explicitly stated) integer N_A so that all spaces Y with $H^*Y \simeq A$ are within N_A of A's rational model space M .

Pf. The discussion will largely follow [Mimura and Toda; Lemma 2.5] Since A is an integral rather nice algebra in A_Z

$$A \otimes Q \simeq \bigotimes_{i=1}^{k} Q[\bar{x}_i]/(\bar{x}_i^{\,n_i+1}) \; ; \; M = \mathop{X}_{i=1}^{k} (S^{m_i})_{n_i}$$

where m_i is the degree of $\bar{x}_i$

a) We can consider the model space one factor at a time:

To obtain a map within N_A $\phi_Y : Y \to M$, it is sufficient to construct a map within (a uniform) N_i $\phi_Y^i : Y \to (S^{m_i})_{n_i}$. Indeed if $\phi_Y^{i*} : H^*(S^{m_i})_{n_i} \to \mathrm{FreePow}(x_i)$ is within N_i for all Y , set $N_A = (d_A) \prod_{i=1}^{k} (N_i^{n_i} \cdot t_i(A))$ .. Then set $\phi_Y = (\phi^1, \phi^2, , \phi^k)$ and the result will follow.

Let us now agree to suppress the index i from $\bar{x}_i, m_i, n_i, N_i, \mathrm{Fac}_i A, \phi^i$

b) If m is odd: Then $n(=n_i) = 1$ and [Serre] demonstrates there exists $\phi : Y \to S^m$. One observes that, upon the required restriction, this induces a map within $N = 2^{\dim A - m} \prod_{j=m+1}^{\dim A} ((\text{order of } \pi_j(S^m))^{\dim A}$

Since the order of $\pi_j S^m$ is finite for $j > m$, N is a well defined integer, an invariant of A .

c) If m is even: the demonstration is more complex. [Berstein] is credited with early work in this direction. The idea is to produce a map $\theta : Y \to (S^m)_\infty$. By cellular approximation, θ may be viewed as a map $(Y,Y^{mn}) \to ((S^m)_\infty , (S^m)_n)$. We then try to extend this to a map $(Y,Y) \to ((S^m)_\infty , (S^m)_n)$. This almost works, but not quite; we must modify θ by an endomorphism of $((S^m)_\infty , (S^m)_n)$ to get a map which does extend, and this map will be the required ϕ_Y .

Choose a torsion-free generator x of dimension m in FacA . Without loss of generality we may assume $i_Q(x) = \bar{x}$ (see 2.5.1) x is an $m+1$ dimensional cohomology generator of Y . Let us call the generator of $H^{m+1}S^{m+1}$, $\bar{u}$, and let u be a generator of $H^m(S^m)$. Again invoking the work of [Serre], we know there exists a map $\bar{\theta} : \sum Y \to S$ S^{m+1} such that $\bar{\theta}^*(\bar{u}) = N'\sum x$ where N' is given by the formula in b) substituting $m+1$ for m , and $\dim A + 1$ for $\dim A$.

The adjoint map $\theta : Y \to \Omega S^{m+1} = (S^m)_\infty$ has the similar property $\theta^*(u) = N'x$. We thus have a map $(Y,Y^{mn}) \to ((S^m)_\infty , (S^m)_n)$. A moment's reflection on the cell structure of $(S^m)_\infty$ indicates that this may automatically be extended to a map $(Y,Y^{m(n+1)-1}) \to ((S^m)_\infty , (S^m)_n)$.

Thus the first (and most difficult) obstruction to extending θ to a map $(Y,Y) \to ((S^m)_\infty , (S^m)_n)$ occurs in $H^{m(n+1)}(Y;\pi_{m(n+1)}((S^m)_\infty , (S^m)_n))$. Now $\pi_{m(n+1)}$ is easily calculable: $\pi_{m(n+1)}((S^m)_\infty , (S^m)_n) \simeq H^{m(n+1)}((S^m)_\infty , (S^m)_n) \simeq Z$. The obstruction cocycle $c(\theta)$ is also readily calculated: Let $w = m(n+1)$. $c(\theta)$ is equivalent to $\theta_* : C_w(Y) = H_w(Y^w, Y^{w-1}) \to H_w((S^m)_\infty , (S^m)_n) \simeq H_w(S^m)_\infty \simeq Z$. $\gamma(\theta) = \overset{*}{\theta}(u_{(n+1)})$, where $u_{(n+1)}$ is the generator of the w^{th} degree of $\Gamma(u) = H^*(S^m)_\infty$.

We now note that $(n+1)!\gamma(\theta) = \theta^*(u^{n+1})$ which must be a torsion element: $\theta^*(u^{n+1}) = (N'_x)^{n+1}$ and x^{n+1} is torsion for $\bar{x}^{n+1} = 0$. Thus $\gamma(\theta)$ is a torsion obstruction.

In all other dimensions, the obstruction to extension of any map is a torsion cohomology class since all the coefficient groups $\pi_k((S^m)_\infty , (S^m)_n)$ are torsion groups. Let us denote the order of the group $\pi_k((S^m)_\infty , (S^m)_n)$ by p_k.

[Mimura Toda; Lemma 2.4] show the existence of endomorphisms of $((S^m)_\infty , (S^m)_n)$ which trivialize torsion homotopy groups. We compose theta with a sequence of these endomorphisms, and call the composite ϕ. From Lemma 2.4.1 we confirm that the obstructions to extending to (Y,Y) all vanish and the theorem 2.6.1 shall be proven.

[Mimura Toda; Lemma 2.4] supplies us with an endomorphism h_q which induces a map of degree q^{n+1} on $H^{m(n+1)}((S^m)_\infty , (S^m)_n)$. For this reason if we let $q = t_w(A)$
$\gamma(h_q \circ \theta) = \theta^* h_q^*(u_{(m(n+1)}) = q^{n+1}\gamma(\theta) = 0$. $h_q \circ \gamma$ extends to the next dimension.

[Mimura Toda; Lemma 2.4] also shows the existence of $h_{(p_k)}$ an endomorphism which induces a map of degree p_k on $\pi_k((S^m)_\infty , (S^m)_n)$ when $k \neq m(n+1)$. Hence

$$h_{(p_{\dim A})} \circ h_{(p_{\dim A-1})} \circ \cdots\cdots \circ h_{(p_{m(n+1)+1})} \circ h_q \circ \;\; = \phi$$

has no obstructions and extends to a map $\phi : (Y,Y) \to ((S^m)_\infty , (S^m)_n)$.

A laborious calculation shows that $h_{(p_k)}$ induces a map of degree $p_k^{k!}$ in $H^m((S^m)_n)$. So finally $\phi^*(u) = Nx$ where

$$N = N' \cdot t^A_{m(n+1)} \cdot \prod_{j=m(n+1)+1}^{\dim A} p_j^{\,j!} \quad . \; ¶$$

The main result of Chapter Two has been proven: All spaces Y with $H^*Y \cong A$, a given rather nice algebra, are within some uniform N of the model space M . Recall that two spaces are within N if there exists a map with induced <u>cohomology</u> map within N between them. We shall see that this kind of comparison "H^* - within N" is important in discovering how π_* compares spaces.

CHAPTER THE THIRD

Containing such serious matter that the Reader cannot laugh once through the whole chapter, unless peradventure he should laugh at the author.

The Reader is cautioned that without Familiarity of the Riggings of the language of the following introduction, or without Sensibility of the direction of the demonstration, he will soon welter in a desolate Sargasso of minutiae and conundra.

HERE BE SERPENTS !!

To the Intrepid Reader, the Author now pledges Solemn Promise of Prodigious Treat upon completion of his Endeavours.

To the Prodigal Reader, the Author imparts begrudgingly that heretofore bespoken Treat might perhaps be ferreted out at 4.3.5.

Of the Disciplined Reader, the Author commends Zeal and Patience, and invites him to prove his Mettle forthwith.

3.1.0 A NOVEL CATEGORY AND A QUESTION OF INFLUENCE.

Let us agree that A denotes a rather nice algebra in A_Z, M its rational model space as defined in 2.2.1 and N the integer defined in Theorem 2.6.1. If Y is a space of T_h and $H^*Y \simeq A$, we have already constructed a map $\phi : Y \to M$ which is within N. Let us affirm the importance of the role of ϕ with a change of viewpoint (and a dextrous modification of our categories).

3.1.1 Definition The category of spaces within N over M, T^N/M has objects which are homotopy classes of continuous maps : $Y \xrightarrow{\phi} M$ within N. Maps are (homotopy) commutative triangles of T_h.

On T^N/M let us introduce auxiliary functors D and Fb which distinguish respectively the domain for ϕ, and its fibre. There is a fibration $Fb(\phi) \to D(\phi) \to M$.

We shall also require a notion of when a functor ϕ influences a functor ψ.

3.1.2 Definition Let $\phi : C_1 \to C_2$ and $\Psi : C_1 \to C_3$ be two functors. Denote the class of equivalence classes of a category C by $\bar{\bar{C}}$. Then the functors ϕ and Ψ induce maps $\bar{\bar{\phi}} : \bar{\bar{C}}_1 \to \bar{\bar{C}}_2$ and $\bar{\bar{\Psi}} : \bar{\bar{C}}_1 \to \bar{\bar{C}}_3$. ϕ finitely determines ψ if $\bar{\bar{\Psi}}\,\bar{\bar{\phi}}^{-1}\,\bar{\bar{x}}$ is a finite set, for all $\bar{\bar{x}}$ in $\bar{\bar{C}}_2$.

3.1.3 Example On T_h, H^* finitely determines $H^*(\ ;Z/pZ)$. Similarly $H^*(\ ;Z/pZ)$ finitely determines $H^*(\ ;Z/p^2Z)$. Thus H^* finitely determines $H^*(\ ;Z/p^2Z)$. H^* uniquely determines $H^*(\ ;Q)$ 1.5.

Similarly for several functors:

3.1.4 Definition Let $\Psi : C \to C_o$ be a functor and $\{\phi_i : C \to C_i\}$ be a set of functors. Then $\{\phi_i\}$ finitely determine Ψ if $\bar{\bar{\Psi}}(\bigcap_i \bar{\bar{\phi}}_i^{-1}((\bar{\bar{X}}_i))$ is a finite set, for all $\bar{\bar{X}}_i$ in $\bar{\bar{C}}_i$.

3.1.5 Examples $H_{n-1}(\ ;Z)$ and $H_n(\ ;Z)$ finitely determine $H^n(\ ;Z)$. Indeed by the universal coefficient theorem, they uniquely determine $H^n(\ ;Z)$. More difficult: see Theorem 3.4.1 $\{\pi_n(Fb)$, $\pi_{n-1}(Fb)$ and $\pi_n M\}$ finitely determine $\pi_n(D)$ on T^N/M .

3.1.6 More Examples Consider the category Z^+ whose objects are the non-negative natural numbers. If $m \neq n$ $\mathrm{Map}_{Z^+}(m,n)$ has only one element, $*$. If $m = n$, $\mathrm{Map}_{Z^+}(m,m)$ has exactly two elements $*$ and id_m . Composition with the id_m does not change any map. All other compositions $= *$. This might be called the discrete category on the set Z^+ . The functors $C \to Z^+$ are exactly the integer invariants of C .

Let Ab be the category of finitely generated abelian groups, $Rk : Ab \to Z^+$ be such that $Rk(G)$ is the minimal number of cyclic direct summands of G . It is an invariant of Ab-isomorphism class, and hence a functor. Similarly $Tn : Ab \to Z^+$ is the order of the torsion subgroup. Then Rk and Tn finitely determine the identity functor on Ab , or in the language of 1.4, Rk and Tn finitely determine Ab-type.

3.1.6 Exposition The first aim of Chapter Three shall be to demonstrate that on the category of rather nice spaces, H^* finitely determines π_n, for all n. In other words if A is rather nice, we shall deduce that the class $\{\pi_n(Y) \mid H^* Y \approx A\}$ resolves into a finite number of isomorphism classes of groups. We shall approach this result (Theorem 3.4.2) carefully, in three stages (3.2, 3.3, 3.4).

Task the First: Let $Fb(\phi)$ be the fibre of the map $\phi : Y \to M$ given in Theorem 2.5.1. With the help of the Serre spectral sequence, $H^*(D)$ finitely determines $H_n(Fb)$ for all n. Moreover $\pi_1(Fb)$ is finitely determined by $H^*(D)$ (Hurewicz).

Task the Second: Let $CC^n(Fb)$ be the n^{th} connective covering of Fb. There is a fibration $K(\pi_n Fb, n) \to CC^n(Fb) \to CC^{n-1}(Fb)$ Applying the Serre spectral sequence again $\{\pi_n(Fb), H_i(CC^{n-1})$ $H_i(CC^{n-1}(Fb)) | i \leq k\}$ finitely determine $H_k(CC^n(Fb))$.

Task the Third: $\pi_n(Fb) \approx H_n(CC^{n-1}(Fb))$ from (Hurewicz).
We can calculate the homotopy groups of D from those of Fb and M.

3.2.0 THE FIBRE SPACE OF A MAP WITHIN N

Given a map $\phi : Y \to M$ of T_h , within N we may construct the fibre of ϕ , $Fb(\phi)$, so that $Fb(\phi) \to Y \to M$ is a fibration. There are two things to notice initially. $Fb(\phi)$ is not in general finite dimensional, though for our purposes this will not matter. $\pi_n(Fb(\phi))$ and $H_n(Fb(\phi))$ are finite groups for $n > 0$, by the use of "mod C" theory [Serre].

We shall limit the cohomology groups of the fibre by applying the Serre spectral sequence for cohomology with integral coefficients.[Serre 2]. The E_2 level of the spectral sequence has the property

$$E_2^{p,q} \simeq H^p(M;H^q(Fb)) \Longrightarrow \bigoplus_{p+q=n} G^{p,q}$$

where $G_\phi^{*,q}$ is the graded group associated with $H^*D(\phi)$ with respect to the filtration $(F_\phi^{*,q-1} \longrightarrow F_\phi^{*,q})$ of $H^*D(\phi)$.

3.2.1 Lemma $E_\infty^{0,q}$ is finitely determined by H^*D (on the category T^N/M)

Pf. Consider

$$\begin{tikzcd}
E_\infty^{0,q} = G^{0,q} & F^{0,q} \arrow[l, two heads] & F^{1,q-1} \arrow[l, hook'] & F^{q,0} = E_\infty^{q,0} \arrow[l, hook'] \\
\mathrm{cok}H^q(\phi) \arrow[u, "\theta"] & H^qD \arrow[l] \arrow[u, equal] & & H^qM \arrow[ll, "H^q(\phi)"] \arrow[u, two heads]
\end{tikzcd}$$

Show that θ exists and is epic. Thus the number of direct irreducible summands $Rk(E_\infty^{0,q}) \leq RkH^qD$, and the torsion order $Tn(E_\infty^{0,q}) \leq N$. ¶

3.2.2 <u>Lemma</u> $\{E_2^{0,j}\}_{j<q}$ and H^*D finitely determine $E_2^{0,q}$.

Pf. The number of isomorphism classes of subgroups of a finitely generated abelian group is finite. The number of isomorphism classes of quotient groups of a finite abelian group is finite.

$$E_2^{i,j} \simeq H^i(M;H^jFb) \simeq H^iM \otimes H^jFb \oplus Tor(H^{i+1}M,H^jFb)$$
$$\simeq H^iM \otimes E_2^{0,j} + Tor(H^{i+1}M,E_2^{0,j})$$

is uniquely determined by $E_2^{0,j}$. It is also a finite group (-valued functor) when $j \neq 0$, because $E_2^{0,j}$ is, when $j \neq 0$. $E_s^{i,j}$, a sub-quotient of $E_2^{i,j}$ is finitely determined by $E_2^{0,j}$ if $j \neq 0$.

Let $d_s : E_s^{0,q} \longrightarrow E_s^{s,q-s+1}$ be an s^{th} differential. When $s \neq q+1$ $Im(d_s)$ is finitely determined by $E_2^{0,q-s+1}$.

When $s = q+1$ consider

$$\begin{tikzcd}
\operatorname{Im} d_s \arrow[r] & E_s^{q+1,0} \arrow[r, two heads] & E_{s+1}^{q+1,0} = E_\infty^{q+1,0} \arrow[d, hook] \\
\ker H^{q+1} \arrow[u, "\theta"] \arrow[r] & H^{q+1}M \arrow[u, two heads] \arrow[r] & H^{q+1}D
\end{tikzcd}$$

Show that θ exists and is epic. Then $Tn(Imd_s) \leq Tn(\ker H^{q+1}) \leq N$. Hence $Im(d_s)$, a finite group (-v.f.) is finitely determined by $E_2^{0,q-s+1}$ for all s. We have a short exact sequence

$$E_{s+1}^{0,q} \hookrightarrow E_s^{0,q} \twoheadrightarrow Im(d_s) \quad \text{for all } s$$

Since the isomorphism class of $E_s^{0,q}$ is uniquely determined by the extension class of d_s which is in $Ext(Im(d_s), E_{s+1}^{0,q})$, a finite group (v.f) it might not be a bad idea to notice that Ext( ) is finitely determined by $E_{s+1}^{0,q}$ and $\{E_2^{0,j}\}_{j<q}$. $E_s^{0,q}$ is finitely determined by $E_{s+1}^{0,q}$ and $\{E_2^{0,j}\}_{j<q}$. By the convergence properties of the Serre spectral sequence $E_{q+2}^{0,q} \simeq E_\infty^{0,q}$, and by Lemma 3.2.1, Lemma 3.2.2 is proved. ¶

3.2.3 Corollary H^*D finitely determines $E_2^{0,q} = H^q Fb$, for all q .

Pf. Induction on q of Lemma 3.2.3. ¶

For the next task, we shall find it somewhat more convenient to restate this as

3.2.4 Corollary H^*D finitely determines $H_q Fb$.

Pf. Universal Coefficient Theorem. ¶

3.3.0 THE n^{th} CONNECTIVE COVERING OF THE FIBRE

The homotopy exact sequence includes the sequence

$$\pi_2 D \longrightarrow \pi_2 M \twoheadrightarrow \pi_1 Fb \longrightarrow \pi_1 D = 0$$

This implies that $\pi_1 Fb$ is abelian and there is a natural isomorphism $\pi_1 Fb \simeq H_1 Fb$. Comparing this with Corollary 3.2.4 we get

3.3.1 Lemma $\pi_1 Fb$ is finitely determined by $H^* D$ (on T^N/M) .

We now perform the Cartan-Serre-Whitehead method of killing higher homotopy groups [Mosher-Tangora] with a map $i_W : Fb \longrightarrow K(\pi_1 Fb,1)$. The fibre of this map will be denoted $CC^1(Fb)$ and is called the simply-connected covering of Fb .

3.3.2 Recursive Definition The n^{th} connective covering of a space X , denoted $CC^n X$ is the fibre space of the Whitehead-Serre-Cartan map $i_W : CC^{n-1}X \to K(\pi_n X,n)$.

As its name implies, it is n-connected, Furthermore $\pi_j CC^n X \simeq \pi_j X$ if $j > n$.

3.3.3 Lemma Let $\mathcal{F}_q$ be the set of functors $\{\pi_n Fb, H_q CC^{n-1} Fb, H_j CC^n Fb \mid j<q\}$. Then $\mathcal{F}_q$ finitely determines $H_q CC^n Fb$.

Pf. Note that all are finite abelian (group valued functors), from "mod C theory".

Consider the Serre spectral sequence of $CC^nFb \to CC^{n-1}Fb \to K(\pi_n Fb,n)$. The E^2 level: $E^2_{s,t} \simeq H_s(\, K(\pi_n Fb,n)\ ;\ H_t(CC^nFb))$

$$\simeq H_s K(\pi_n Fb,n) \otimes H_t CC^nFb \oplus \mathrm{Tor}(H_{s-1}K(\pi_n Fb,n)\ ,\ H_t CC^nFb)$$

is uniquely determined by $\mathcal{F}_q$ if $t < q$.

On the E^r level $E^r_{s,t}$ is a sub-quotient of $E^2_{s,t}$ and hence is finitely determined by it. If $t < q$ $\mathcal{F}_q$ finitely determines $E^r_{s,t}$.

Consider the r^{th} differential $d^r : E^r_{r,q-r+1} \to E^r_{0,q}$. The image of d^r is a quotient of $E^r_{r,q-r+1}$ and so is finitely determined by $\mathcal{F}_q$.

We have an exact sequence $\mathrm{Im}(d^r) \hookrightarrow E^r_{0,q} \twoheadrightarrow E^{r+1}_{0,q}$. From the properties of Ext, we have that $E^r_{0,q}$ is finitely determined by $\mathcal{F}_q \cup \{E^{r+1}_{0,q}\}$.

By the convergence of the Serre spectral sequence, and by the use of the edge homomorphism we have $E^{q+2}_{0,q} = E^{\infty}_{0,q} \hookrightarrow F_{o,q} = H_q CC^{n-1}Fb$. Hence $E^{q+2}_{0,q}$ is finitely determined by $H_q CC^{n-1}Fb$, and hence by $\mathcal{F}_q$. Lemma 3.3.3 follows by downward induction on r , of the previous paragraph. ¶

3.3.4 <u>Corollary</u> $\pi_n(Fb)$ is finitely determined by H^*D .

Pf. Remark that $\pi_n Fb \simeq H_n CC^{n-1}Fb$. Apply Lemma 3.3.3 inductively first on q , then on n , to show that $H_p CC^m Fb$ is finitely determined by H^*D for any p,m . ¶

3.4.0 THE HOMOTOPY GROUPS OF A RATHER NICE SPACE

3.4.1 Theorem On T^N/M $\pi_n D$ is finitely determined by $H^* D$.

Pf. The homotopy exact sequence includes the sequence

$$\pi_n Fb \xrightarrow{i_n} \pi_n D \xrightarrow{\phi} \pi_n M \longrightarrow \pi_{n-1} Fb \quad \text{and so}$$

$$Im(i_n) \longrightarrow \pi_n D \longrightarrow Im(\phi) \quad \text{is short exact.}$$

Considered as a quotient of $\pi_n Fb$, $Im(i_n)$ is finitely determined by $H^* D$ (Lemma 3.3.4). $Im(\phi)$ is a subgroup of $\pi_n M$, and there are only a finite number of extensions, up to isomorphism. ¶

3.4.2 Corollary Let A be an integral, rather nice algebra. Then $\{\pi_n Y \mid H^* Y \simeq A\}$ resolves into only a finite number of group isomorphism classes.

3.4.3 Lemma There exists an integer N_n^π such that $\pi_n(\phi)$ is within N_n^π for all objects ϕ of T^N/M .

Pf. Let N_n^π be the least common multiple of the finite set $\{Tn(\pi_n Fb(\phi))\ ,\ Tn(\pi_{n-1} Fb(\phi)) \mid \phi \in obj\ T^N/M\}$. ¶

CHAPTER THE FOURTH

A wonderful long chapter concerning the marvellous; containing much clearer matters, but which flow from the same fountain with those in the preceding Chapter. The memorable Transactions which occur within may encourage the Reader felicitously to adapt some of his Categories. In any case, these Transactions are indeed integral in a happy resolution of our Recitation.

4.1.0 <u>n-TYPE AND POSTNIKOV TOWERS</u>

4.1.1 <u>Definition</u>: Two CW complexes K,L have the same <u>n-type</u> if there exist maps $f : K^n \to L^n$ and $g : L^n \to K^n$ where L^n, K^n are the n^{th} dimensional skeleta of K and L . The maps satisfy:

a) For every map from any (n-1) dimensional CW complex $h : M \to K^n$ $g \circ f \circ h \simeq h$.

b) For every map from any (n-1) dimensional CW complex $h : M \to L^n$ $f \circ g \circ h \simeq h$.

One of the most important observations about n-type is

4.1.2 <u>Lemma</u> Let K and L be n-dimensional complexes. Then they have the same (n+1) type iff they have the same homotopy type. ¶

Given a CW complex Y there is a canonical space P_nY with the same (n+1) type as Y : The Cartan-Serre-Whitehead method of killing higher homotopy groups constructs a map $i_W : Y \to P_nY$, with the property that $i_{W\#} : \pi_iY \simeq \pi_iP_nY$ for $i \leq n$, and $\pi_iP_nY = 0$ for $i > n$.

4.1.3 <u>Lemma</u> X and Y have the same (n+1) type iff $P_nX \simeq P_nY$.

Pf. Whitehead theorem. ¶

If we now apply the Cartan-Serre-Whithead method of killing the (n+1)st homotopy group of $P_{n+1}Y$, we obtain a map $P_{n+1}Y \to P_nY$. Such maps form a Postnikov tower for Y in the sense of [Mosher-Tangora].

4.1.4 <u>Definition</u>: If X is a simply connected complex, the diagram

$$\begin{tikzcd}
 & \vdots \\
 & X_{n+2} \arrow[d] \\
X \arrow[ur] \arrow[r] \arrow[dr, "\rho_n"'] & X_{n+1} \arrow[d, "\rho_n^{n+1}"] \\
 & X_n \\
 & \vdots
\end{tikzcd}$$

is called a Postnikov Tower for X if it satisfies four conditions:

a) the diagram is homotopy commutative

b) $\pi_i X_m = 0$ for $i \geq m$

c) ρ_m induces isomorphisms $\pi_i X \cong \pi_i X_m$ for $i < m$

d) ρ^m_{m-1} is a principal $K(\pi_m X, m)$ fibration.

Happily, there is a classification theory for princiapl fibrations [Meyer] and ρ^m_{m-1} is classified by a homotopy class $k_{(m-1)} : PX_{m-1} \to K(\pi_m X, m+1)$. This homotopy class is often called the $(m-1)$st k-invariant of the Postnikov tower. Postnikov Towers enjoy some functorial properties:

4.1.5 <u>Lemma</u> Let X and Y be simply-connected CW complexes, and let $f : X \to Y$ be a continuous map. Suppose P_*X and P_*Y are Postnikov Towers for X and Y . Then there exists a family of maps $P_n f : P_n X \to P_n Y$ with the following properties:

a)
$$\begin{tikzcd} P_{n-1}X \arrow[r, "k_n^X"] \arrow[d, "P_{n-1}f"'] & K(\pi_n X, n+1) \arrow[d, "f\#"] \\ P_{n-1}Y \arrow[r, "k_n^Y"] & K(\pi_n Y, n+1) \end{tikzcd}$$
is homotopy commutative

b)
$$\begin{tikzcd} P_n X \arrow[r, "\rho_n f"] \arrow[d, "\rho^n_{n-1}"'] & P_n Y \arrow[d, "\rho^n_{n-1}"] \\ P_{n-1}X \arrow[r, "\rho_{n-1}f"'] & P_{n-1}Y \end{tikzcd}$$
is homotopy commutative

c)
$$\begin{tikzcd} X \arrow[r, "\rho_n"'] \arrow[d, "f"'] & P_n X \arrow[d, "P_n f"] \\ Y \arrow[r, "\rho_n"'] & P_n Y \end{tikzcd}$$
is homotopy commutative

Pf. [Kahn, Mosher-Tangora]. ¶

It will be important to know when two spaces having the same n-type have the same $(n+1)$ type.

4.1.6 Lemma Consider the following diagram:

$$\begin{tikzcd} P_n X \arrow[r, "\rho^n_{n-1}"] & P_{n-1}X \arrow[r, "k_n^X"] \arrow[d, "\wr"] & K(\pi_n X, n+1) \\ P_n Y \arrow[r, "\rho^n_{n-1}"] & P_{n-1}Y \arrow[r, "k_n^Y"] & K(\pi_n Y, n+1) \end{tikzcd}$$

There exists a homotopy equivalence, allowing the diagram to commute, on the left, iff there exists one on the right.

Pf. If there exists a homotopy equivalence on the right (commutative) the induced fibre map is the required commutative homotopy equivalence on the left.

If there exists a homotopy equivalence on the left (commutative) say $P_n f : P_n X \xrightarrow{\sim} P_n Y$, note that $\{P_m X\}$, $\{P_m Y\}$ for $m \leq n$ form Postnikov towers for $P_n X$ and $P_n Y$. Let the map on the right be $P_n f_{\#}$. Property a) of Lemma 4.1.5 shows that the diagram commutes on the right, and the homotopy exact sequence for fibrations shows that it is a homotopy equivalence. ¶

We shall now begin to prove the main result of this thesis:

THEOREM: $H^* D$ finitely determines homotopy type on T^N/M .

Recall that an object of T^N/M is a homotopy class $\phi : Y \to M$ which is within N , and that a map of T^N/M is a (homotopy) commutative triangle.

4.1.7 <u>Definition</u> If ϕ and ϕ' are objects of T^N/M , they have the same $(n+1)$ type (over M) if there exists a homotopy commutative diagram

$$\begin{tikzcd} P_n(D\phi) \arrow[rr, "\simeq"] \arrow[dr, "P_n\phi"'] & & P_n(D\phi') \arrow[dl, "P_n\phi'"] \\ & P_n M & \end{tikzcd}$$

Since everything is simply connected, all objects in T^N/M have the same 1-type.

4.1.8 <u>Explication</u> Suppose we have demonstrated that H^*D finitely determines n-type. We wish to show that {n-type, $\pi_n D$ and H^*D} finitely determines (n+1) type. Consider the following diagram

$$\begin{tikzcd}
 & P_{n-1}D\phi \arrow[rr, "k\phi"] \arrow[dl, "\sim"'] \arrow[dd] & & K(\pi_n D\phi, n+1) \arrow[dl, dotted, "\alpha"'] \arrow[dd] \\
P_{n-1}D\phi \arrow[rr, "k\phi'"] \arrow[dr] & & K(\pi_n D\phi', n+1) \arrow[dr] & \\
 & P_{n-1}M \arrow[rr, "k_M"] & & K(\pi_n M, n+1)
\end{tikzcd}$$

ϕ and ϕ' have the same (n+1)-type iff there exists a homotopy equivalence α, indicated as a dotted line, which allows the entire diagram to commute.

The existence of a homotopy equivalence allowing the right-hand triangle to commute is a purely group theoretic question about the isomorphisms of $\pi_n D\phi \rightarrow \pi_n D\phi'$. It shall be dealt with in 4.2.

The question of the homotopy commutativity of the whole diagram shall be considered in 4.3, with our knowledge of $\pi_n\phi$. Thus shall we perform the induction step.

4.2.0 THE CATEGORY Ab/C

4.2.1 Definition The category of abelian groups over C, Ab/C has objects which are group homomorphisms $B \xrightarrow{\phi} C$, where B and C are finitely generated abelian groups. Morphisms in Ab/C are commutative triangles.

The full subcategory of groups over C and within N , denoted Ab^N/C has objects which are group maps within N .

Of course, two objects ϕ and ϕ' will be equivalent in Ab^N/C iff

$$\begin{tikzcd} B \arrow[rr, "\simeq"] \arrow[dr, "\phi"'] & & B \arrow[dl, "\phi'"] \\ & C & \end{tikzcd}$$ is commutative.

4.2.2 Theorem There are only a finite number of equivalence classes in Ab^N/C .

Sketch of proof: Let $D(\phi)$ denote the domain of ϕ . Then the free rank of $D(\phi)$ = free rank of C and the order of the torsion subgroup of $D(\phi)$ must divide N times the order of the torsion subgroup of C . Thus there are only a finite number of group isomorphism classes for D .

Since $Hom(B,C) \cong Hom(FB,FC) \oplus Hom(FB,TC) \oplus Hom(TB,TC)$ where $B \cong TB \oplus FB$, the torsion and free summands of B , we may restrict our attention to the case when B and C are free groups. The result then follows from elementary observations. ¶

4.2.3 Definition The category $Ab_N \backslash C$, abelian groups under C and within N has objects which are group maps $C \xrightarrow{\phi} B$ within N .

4.2.4 Theorem There are only a finite number of equivalence classes in $Ab_N \backslash C$.

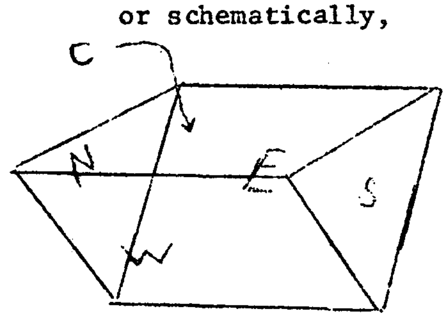
4.3.0 THE MAIN RESULT ; THE WHITEHEAD THEOREM WITHIN N

Let us review: with the assumption that $H^{*}D$ finitely determines the n-type of ϕ , we consider the following diagram for two objects ϕ and ϕ_o of T^N/M .

4.3.1 Diagram

$$\begin{tikzcd}
 & P_{n-1}D(\phi) \arrow[rrr, "k_{n-1}D\phi"] \arrow[dl, "\alpha"', "\sim"] \arrow[dd, "P_{n-1}\phi"] & & & K(\pi_n D(\phi),n+1) \arrow[dl, "\alpha_\#", "\sim"'] \arrow[dd, "\pi_n(\phi)"] \\
P_{n-1}D(\phi_o) \arrow[rrr, "k_{n-1}D\phi_o"] \arrow[dr, "P_{n-1}\phi_o"'] & & & K(\pi_n D(\phi_o),n+1) \arrow[dr, "\pi_n\phi_o"] & \\
 & P_{n-1}M \arrow[rrr, "k_{n-1}M"] & & & K(\pi_n M,n+1)
\end{tikzcd}$$

or schematically,



That we can consider side N to be (homotopy) commutative is a consequence of the induction hypothesis. (Definition 4.1.7).

Sides W and E are homotopy commutative, from the properties of Postnikov towers. (Lemma 4.1.5 a)).

That we can consider side S to be commutative is the result of 4.2.2 .

To decide when the cover C is homotopy commutative, is to decide when ϕ and ϕ_o have the same $(n+1)$-type. (Take fibres "to the north" and compare Definition 4.1.7).

The direction of the proof should now be evident.

4.3.2 Lemma: Under the assumption that N,E,W,S are commutative, the torsion elements of $H^{n+1}(P_{n-1}D(\phi_o), \pi_n D(\phi_o))$ characterize commutativity of C.

Pf. For any ϕ, an object of T^N/M for which there exists a Diagram 4.3.1, let $t(\phi) = \alpha_{\#}\circ(k_{n-1}D\phi)\circ\alpha^{-1} - k_{n-1}D\phi_o$, where each of the terms is considered as an element of $H^{n+1}(P_{n-1}D\phi_o, \pi_n D\phi_o)$. If $t(\phi) = t(\phi')$, then

$$\begin{array}{ccc} P_{n-1}(D(\phi)) & \xrightarrow{k_{n-1}D\phi} & K(\pi_n D(\phi), n+1) \\ \wr\downarrow \alpha'^{-1}\circ\alpha & & \wr\downarrow \alpha_{\#}'^{-1}\circ\alpha_{\#} \\ P_{n-1}D(\phi') & \xrightarrow{k_{n-1}D\phi'} & K(\pi_n D(\phi'), n+1) \end{array} \text{ is commutative.}$$

To see that $t(\phi)$ is a torsion element, note that $\phi_{o\#}$ is a rational equivalence. From the commutativity of N,E,W,S, $t(\phi)\circ\phi_{o\#} = 0$. ¶

4.3.3 THEOREM ("Whitehead Theorem within N"): There are a finite number of homotopy types of objects in T^N/M.

Pf. Lemma 4.3.2 shows that H^*D finitely determines n-type for all n . Apply Lemma 4.1.2 to show that H^*D finitely determines homotopy-(equivalence) type. Then note there are only a finite number of isomorphism classes of $\{H^*D(\phi) \mid \phi \in \mathrm{obj}T^N/M\}$. ¶

4.3.4 <u>COROLLARY</u> Let A be a finitely-generated, associative, graded-commutative simply-connected algebra such that

$$A \times Q \simeq \bigotimes_{i=1}^{k} \frac{Q[x_i]}{(x_i^{n_i})} .$$

Then $\{Y \mid H^*Y \simeq A, Y$ has the homotopy type of a simply connected finite CW complex$\}$ resolves into only a finite set of homotopy types.

4.3.5 <u>TREAT</u> Now define two finite CW complexes X and Y to be within N if there exists a CW complex Z such that there exist maps f $f : Z \to X$ within N and $g : Z \to Y$ within N . Dualize the arguments of Chapter Three and Four to show that there are a finite number of homotopy types of $T_N \backslash M$. Choose an object X of a rational homotopy type (5.1.1). Define W_X^N to be the class of all spaces within N of X . Show there are only a finite number of homotopy types of W_X^N . Finally, $\varinjlim W_X^N = W_X^\infty$ = the rational homotopy type of X .

For rather nice spaces, Theorem 2.6.1 shows there is an easily calculable invariant, namely H^*Y which guarantees that all such spaces are in some W^N filtration of the rational homotopy type. Can the Reader think of a larger class of spaces for which the same result is true?

CHAPTER THE FIFTH

Showing what kind of History this is; what it is like and what it is not like. Lastly a heartfelt farewell to the Reader.

5.0.0 DISCUSSION

5.1.0 THE HISTORICAL SETTING

5.1.1) In 1969 [Quillen] introduced the concept of rational homotopy theory. With apologies to that author, a rational homotopy type may be considered a subcategory of T_h , where the morphisms are 0-equivalences. (i.e. within N for some N). Moreover, for any two objects X,Y there exists another object Z and morphisms $Z \to X$ and $Z \to Y$.

5.1.2) In 1970 [Mimura and Toda] announced that for rather nice spaces, the rational cohomology functor uniquely determines rational homotopy type. In particular, given a rational, rather nice algebra A , one can construct a space Z_o such that 1) $H^*(Z_o;Q) \cong A$ and 2) for any other space X with $H^*(X;Q) \cong A$ there exists a 0-equivalence $X \to Z_o$.

5.1.3) In the same year [Curjel and Douglas] succeeded in showing that there are a finite number of homotopy types of dimension N which support an H-space multiplication. In other words, the dimension finitely determines the underlying homotopy type.

5.1.4) Then in August of 1971 [Curjel and Douglas 2] asked:

Let A be a candidate for $H^*(X;Z)$ i.e. an associative, graded simply connected algebra over the integers Z . Let $T(A)$ be the set of homotopy types of simply connected finite complexes having integral cohomology ring isomorphic to A .

<u>*Question*</u> *For which A is $T(A)$ a finite set?*

[Curjel and Douglas] then announced a proof of a positive result when $A \otimes Q$ is an exterior algebra on odd dimensional generators. In some sense, the result is the easy case of Theorem 4.3.3: the obstructions to getting a map from Y to $\underset{i}{X} S^{m_i}$ are torsion due to the work of [Serre]. Moreover all k-invariants must be torsion. Thus this thesis is a generalization of the announced result of [Curjel and Douglas 2], and in the same spirit.

5.2.0 COGNOMINA AND METHODS

5.2.1 [Larry Smith] discusses "nice" and "super-nice" algebras in a paper which introduced the author to the conept of a space over a space. Since the algebras we are concerned with in this thesis are all nice, though somewhat less free than super-nice algebras, they merit the appellation "rather nice". Moreover this thesis demonstrates that they permit a rather nice distinction of homotopy types.

Some of the methods of demonstration have been commonplace and sometimes perhaps elephantine. The Serre spectral sequence, Postnikov Towers and the higher connective coverings are heavyduty methods of producing results; the author is certain the Reader is well aware of their manifest and wide-ranging applicability.

Three novel concepts are central to this thesis and deserve comment.

5.2.2) Categories over a base. The question of when two spaces are homotopy equivalent has long puzzled topologists. Yet the problem we consider is apparently much more difficult: When are two spaces over a model space homotopy equivalent? i.e. when does there exist a homotopy commutative triangle

$$\begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr] & & Y \arrow[dl] \\ & M & \end{tikzcd}$$

That this apparently more difficult problem sometimes admits a solution rests perhaps on the fact the category of abelian groups over a group has more structure than the category of abelian groups considered in isolation. Hence when a classification in Ab/C is possible, there may be more chance that it will carry over to the topological situation.

5.2.3) Maps within N Consider a (Serre) congruence class of groups "modulo finite groups". There is a free group in the class. Call it C . Then Ab^N/C may be considered to be a filtration of the congruence class Ab^∞/C . Moreover there are a finite number of isomorphism classes in each Ab^N/C .

The result of this thesis is that we can perform a similar procedure on rather nice rational homotopy types with the added feature that such a filtration is compatible with integral cohomology. When this happens we may conclude as in Theorem 4.3.3 that HT(A) is a finite set.

5.2.4) Category theory seems sadly lacking in language to describe how well a functor functions. Think of a homotopy functor as drawing a likeness of a space. Can we recognize the original from its likeness? If we can, let us say that the functor uniquely determines the (equivalence class of the) space. It is with this in mind that the author introduces the notion of "<u>finitely determines</u>". The name however seems fraught with unwonted ambiguities, and the Author will welcome emendations.

5.3.0 APPLICATIONS AND EXAMPLES

5.3.1 By an early theorem of Hopf, finite dimensional rational Hopf algebras are multiplicatively exterior algebras on odd dimensional spheres generators. Thus Lie groups, topological groups, H-spaces and H-spaces mod F are all rather nice.

5.3.2 From the work of [Borel] and [Cartan], real and complex Stiefel manifolds also have rational cohomology algebras which are exterior algebras. In general, homogeneous spaces are often rather nice. See [Atiyah-Hirzebruch] and [Baum] for some interesting examples.

5.3.3 By the Kuenneth Theorem, a product of spaces is rather nice iff each factor is. More generally

5.3.4 Lemma If $X, Z \in \operatorname{obj} T_h$, and X is a retract of Z a rather nice space, then X is also rather nice.

Sketch of proof: Let $q : A \twoheadrightarrow QA$ be the quotient map onto the (Q-vector space) of indecomposables. i.e. $QA \simeq \tilde{A}/\tilde{A}^2$. An endomorphism α of A is an automorphism iff the induced map $\tilde{\alpha} : QA \to QA$ is also. Characterize rational rather nice algebras as follows:

*) For any set of elements $x_1, x_2, , , x_k \in A$ such that $q(x_1), q(x_2), , q(x_k)$ are linearly independent, then $\prod_{i=1}^{k} x_i^{m_i} \neq 0$, where m_i is the largest integer such that $x_i^{m_i} \neq 0$.

*) is equivalent to the statement that A is rather nice. If X is a retract of Z $Q(H^* X \otimes Q) \hookrightarrow Q(H^* Z \otimes Q)$ and $H^* X \otimes Q \longrightarrow H^* Z \otimes Q$ are both monomorphisms. Apply *) to show that $H^* X \otimes Q$ is rather nice. ¶

CITATION OF RELEVANT LITERATURE

[Atiyah-Hirzebruch] "Vector Bundles and Homogeneous Spaces", Proc. Sympos. Pure Math. Vol. III (Providence R.I. A.M.S. 1961), pp. 7-38.

[Baum], Paul "On the Cohomology of Homogeneous Spaces" (thesis), Princeton University 1961.

[Berstein], Israel Commentarii Mathematici Helvetici, V. 35:1 (1961), pp. 9-14.

[Cartan], Elie "Notions d'algèbre différentielle; applications aux variétés où opère un groupe de Lie", Colloque de Topologie (Espaces fibrés) (Liège: Centre Belge de Recherches Mathématiques 1950), p. 25.

[Curjel], Caspar "On the H-space structures of finite complexes", Commentarii Mathematici Helvetici, 43:1 (1969), pp. 1-17.

[Curjel-Douglas], "On H-spaces of finite domension", Topology 10 (1971), pp. 385-389.

[Curjel-Douglas 2] "On Stasheff's Fifth Problem", Notices of the A.M.S. 18:5, (August, 1971), p. 687.

[Hilton 2], Peter "On the homotopy groups of the union of spheres", Journal of the London Mathematical Society, 30 (1955), p. 154-172.

[Hu], Sze-Tsen Homotopy Theory, (New York: Academic Press 1959), in particular Chapter V : Obstruction Theory, and Exercise E.

[James], Ioan "Reduced Product Spaces", Annals of Mathematics 62 (1955), pp. 170-192.

[James 2], "Over Homotopy Type", Symposia Mathematica, Vol. IV, Rome: INDAM 1969/69. (London: Academic Press 1970), pp. 219-229.

[Kahn], Donald W. "Induced Maps for Postnikov Systems", Transactions of the A.M.S. 197 (1963), pp. 432-450.

[Meyer], J.P. "Principal Fibrations", Transactions of the A.M.S. 107, (1963), pp. 177-185.

[Mimura-Toda] Mamoru Mimura and Hirosi Toda "On p-equivalences and p-universal spaces", Commentarii Mathematici Helvetici, 46:1 (1971), pp. 87-97.

[Mimura-O'Neill-Toda] "On p-equivalence in the sense of Serre", Japanese Journal of Mathematics, V. XL (1971), pp. 1-10.

[Mosher-Tangora] Robert Mosher and Martin Tangora, Cohomology Operations and Applications in Homotopy Theory, (New York: Harper & Row, 1968).

[Quillen] Dan, "Rational Homotopy Type", Annals of Mathematics 90, (Sept., 1969), p. 205.

[Serre] Jean-Paul "Groupes d'homotopie et classes de groupes abéliens", Annals of Mathematics 58 (1953), pp. 258-294, in particular Chapitre V, p. 288, "Applications d'un polyhèdre dans une sphère". Also p. 259, "I. La notion de classe".

[Serre2] "Homologie Singulière des espaces fibrés", Annals of Mathematics 54 (1951), pp. 425-505.

[Smith], Larry "On the Eilenberg-Moore Spectral Sequence", A.M.S. Lecture Notes, Summer Institute on Algebraic Topology 1970, Lecture #12.

[Toda], Hirosi Composition Methods in Homotopy Groups of Spheres, (Princeton, J.J.: Princeton University Press 1962), Chapter II, in particular Theorem 2.4, p. 19.

[Whitehead] J.H.C. "Combinatorial Homotopy I-II", Bulletin of the A.M.S. 55 (1949), pp. 213-245 and pp. 453-496.

[Hilton and Wylie] Homology Theory, An Introduction to Algebraic Topology (Cambridge: 1960), p. 151, Eg. 4.3.4.