A Study of Financial Markets with Heterogeneous Agents

A Numerical Approach

by

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Abstract

In this thesis, we present a model of a financial market with many heterogeneous agents in a continuous double auction market organization. We introduce the different concepts related to continuous double auction and electronic limit order book. We then construct a financial market model with heterogeneous agents that are using this electronic order book to trade. At each time, the next agent to trade is chosen independently of others according to a Poisson process. Each agent will decide to buy or to sell the stock according to the recommendation of a financial expert. These recommendations represent the expert expected price of the stock for the next period. There are three types of financial experts: the noise traders, the fundamentalist traders and the chartist traders. The recommendations from the noise traders are random, the recommendations from the fundamentalists are based on some fundamental value of the stock price while the recommendations of the chartists are formed using some extrapolatory or contrarian rules. The agents are choosing the financial expert by comparing each of them using a performance measure. In our case, they compare a discounted sum of past profits their recommendation would have generated. Once the financial expert is determined, the agent forms her excess demand and then executes her trade according to the sign of her excess demand. Finally, we present numerical results from the simulation of this model and compare them with previously found empirical results. Some interesting results include the relation between the volatility of the stock and the trading volume, the bid and ask spread and the stock returns, the analysis of the effects of the variation of some important parameters like the tick size, the market activity and the level of patience of the agents, the analysis of the effects of the chartist traders on the return and price distributions and the comparison of the tail of the return distribution with a power-law distribution. The most important result is the suggestion of the existence and uniqueness of a unique stationary distribution for the price process.
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À mes parents
Chapter 1

Introduction

The study of financial markets has interested many researchers since the very beginning of their existence. Many different approaches have been used on the road to a better understanding of the financial markets. Among them are the use of statistical techniques for both description and prediction of financial data and the creation of theoretical models with the hope to explain some of the phenomena found in these data. Other types of models try to link the financial market with general economic theory. More recently, some more sophisticated mathematical models have emerged from the field of mathematical finance. The goal of this thesis is the study of financial markets using multiple approaches. More precisely, we start with a mathematical theoretical model, make the link between this model and important concepts of economic theory, modify the model to take into account a particular form of financial market microstructure, use computer simulation to get intuition about the model and use statistical methods to find relations between different variables. Finally, we use statistical methods to compare our results with other results previously found in the empirical literature.

1.1 Review of Literature

Our study of financial markets or asset pricing models, as sometimes called in the literature, brings us back to the seminal paper of Merton (1973). Merton's model is included in the broader theory of general equilibrium under uncertainty. In these models, all agents solve an optimal investment problem and the equilibrium price of the asset is determined using a market clearing condition of zero excess demand. After the publication of Merton's paper, the influential model of Breeden (1979), known as the consumption based capital asset pricing model (CCAPM), appeared. This author develops the previous model further by also considering the consumption in the optimization problem. The importance of Breeden's model relies on the fact that his model can be empirically tested. This great characteristic of Breeden's model allowed many researchers to test the CCAPM empirically and concluded that it does not agree with the empirical financial data. In particular, these tests brought a considerable number of financial puzzles: equity premium puzzle, volatility

\footnote{For a complete presentation of the theory of general equilibrium we refer the reader to the book of Mas-Colell, Whinston and Green (1995).}
puzzle, risk free rate puzzle, temporal predictability puzzle, etc.  

This disagreement with the data has encouraged the fast development of extensions of the model within the last 28 years. An extensive listing of all the developments done during these years is too long and not the main goal of this thesis. We mention here only some of the different directions that the researchers took in extending the CCAPM. These extensions were made by considering: borrowing constraints, short selling constraints, incomplete markets, production economy, information, habits, transaction costs, taxes, etc.  

The general problem of all these models is that the equilibrium is found by considering a representative agent. As noted by many authors, models with representative agents have poor predictive power (in particular, see Meese and Rogoff (1983), Frankel and Rose (1995) and Cheung et al. (2002)). Moreover, an empirical study by Chinois and MacDonald (2002) suggests that there is considerable heterogeneity of expectations within the different agents acting on a financial market. Heterogeneous expectations are also necessary to explain other phenomena as pointed out by Bachetta and Wincoop (2005). For these different reasons, we need to introduce models that do not assume a representative agent, or more precisely, that consider agents with heterogeneous expectations. 

One of the difficulties of such models is that once we introduce heterogeneous expectations, we need to have another theory beside the usual rational expectation equilibrium. At this point, different options are available. Some authors have introduced agents with wrong expectations (see De Long et al. 1989 and 1990), other have use dispersed information models with higher order expectations (see Townsend (1983)) while finally some suggest that a unique stationary distribution for the price process can be considered as the proper equilibrium notion in such model (see Follmer, Horst and Kirman (2005)). Within these directions we have decided to retain the one by Follmer, Horst and Kirman (2005). The reason being that even when really sophisticated, the other models need to assume some irrationality from the agent to obtain interesting results. As an example, assuming irrationality is necessary to reproduce typical characteristics of financial time series such as bubbles and crashes. In particular, when bubbles happen in these models they usually explode, which is not the case in the model by Follmer, Horst and Kirman (2005). 

In their paper, Follmer, Horst and Kirman (2005) present a model where agents have heterogeneous expectations and choose the expected price for the next period by comparing a performance measure given by some financial experts. To be more precise, as mentioned by the authors: "The aim of their paper is to analyze a model in which:

- expectations can be heterogeneous, consistent with empirical observations,
• where agents learn from their experience of using different rules, how to form their expectations,

• where agents are not systematically wrong,

• and where significant departures from fundamentals can occur but where there is always a return to fundamentals.”

Our model shares a considerable part of the model of Föllmer, Horst and Kirman (2005) and for this reason, we explain in more details in the section 1.2 the results of their paper and give a similar presentation of their model in chapter 3. In fact, this thesis constitutes an extension of Föllmer, Horst and Kirman (2005) where we consider a different market organization.

The market organization constitutes a subdivision of a wider topic called market microstructure.

Definition 1.1. The market microstructure is the study of the process by which invertors demands are ultimately translated into prices and volumes.

A central idea of theory of market microstructure is that prices do not need to be equal full-information expectations of values. Important literature about market microstructure is surveyed in a paper by Madhavan (2000).

The particular market organization considered in this thesis is the continuous double auction. An introduction to such market organization is given by Luckock (2003). Indeed, one could resume the principal aim of this thesis as being the same as Föllmer, Horst and Kirman (2005) but considered in the double auction market organization instead of the Walrasian temporary equilibrium framework. Obviously the current work is less ambitious because we consider numerical simulations instead of rigorous mathematical analysis.

We have presented in this section a broader view of the position of this thesis with respect to some of the relevant literature. In the next section we will present in more detail some previous work that is considered particularly relevant for this thesis.

1.2 Details of the Most Relevant Literature

In this section we review in more detail a select portion of the literature that we consider as particularly relevant for the current work. To be more precise, we review the papers of Föllmer, Horst and Kirman (2005), Chiarella and Iori (2004) and Luckock (2003). It is an important section because it will allow the reader to understand clearly the next section where we explain the relevance of this thesis.
1.2.1 Models of Continuous Double Auction

In this section we present two particular models of continuous double auction. The model of Chiarella and Iori (2002 and 2004) and the model of Luckock (2003) because they both share similarity with our model.

The model introduced by Chiarella and Iori (2002 and 2004) is to some extent very similar to the model introduced here. They introduce an order based market model with heterogeneous agents that submit market or limit orders according to exogenously fixed rules. We give here a brief description of their model.

The orders arrive sequentially at random times and have a finite lifetime \( \tau \). The asset is assumed to have a fundamental value \( p^f \) which is constant and known by all the agents. The agents can submit limit orders at any price on a prespecified grid. The demand of each agent consists of three components, a fundamentalist component, a chartist component and a noise component. At time \( t \), the agent, makes an expectation about the spot return, \( \bar{r}^i_{t,t+\tau} \) that will prevail on the interval \( (t, t + \tau) \) during which her order will be active. Mathematically, \( \bar{r}^i_{t,t+\tau} \) is given by:

\[
\bar{r}^i_{t,t+\tau} = \frac{1}{g^1_i + |g^2_i| + n^i} \left[ g^1_i \frac{1}{\tau_f} \left( p^f_t - p_t \right) + g^2_i \bar{r}_t + n^i \epsilon_t \right]
\]

where \( \bar{r}_t \) is an average of past returns, \( g^1_i \) is the weight given to the fundamentalist, \( g^2_i \) is the weight given to the chartists and \( \epsilon_t \sim \mathcal{N}(0, \sigma^2) \) and \( \tau_f \).

The agent can choose between holding cash or stock. The authors assume the agents to have exponential utility function given by:

\[
U(W) = -e^{-\alpha W}
\]

where \( W \) is the wealth of the agent.

Then, the optimal composition of the portfolio is given by:

\[
\pi^f(p) = \frac{\log(p_{t+\tau}) - \log(p)}{\alpha \sigma_p^2}
\]

where \( \sigma_p^2 \) is the variance of returns. Restrictions are added on \( p(p) \) to avoid short selling.

Discussion 1.5. One of the differences with our model is that their model does not include a self-reinforcing term. The weight given to the fundamentalist \( g^1_i \) and the weight given to the chartist \( g^2_i \) vary randomly with time. This implies that the choice of the agent to use more a fundamentalist rule or a chartist rule is independent of the performance of the rules. As a result, an agent can decide randomly to use a more fundamentalist rule even if this rule performed poorly in the past. This assumption is relaxed in our model presented in section 3.

Another problem, as mentioned by the authors themselves, is that the model abstracts from important informational issues by assuming that agents have full knowledge of the
market fundamental. Not only do the agents have full knowledge of this value but it is also considered as constant. As noted by De Long et al. (1990) different types of risks are present in financial markets, in particular, there is risk coming from the uncertainty of the fundamental value and there is risk coming from noise trading. Our model relaxes these different assumptions of the model developed in Chiarella and Iori (2002 and 2004) by assuming that the agents do not know the fundamental value and that this value is not a constant.

The paper by Luckock (2003) is important for the presentation of the continuous double auction market organization and for some similar results than the one obtain in this thesis. In particular, the author found steady state distributions for the best ask, the best bid and the transaction prices and under fairly general assumptions that the prices are confined to a clearly defined window.

**Discussion 1.6.** Even if the results obtained in this paper are interesting, there are some problems with some of the assumptions used. In particular, the author assumes:

- All orders are for a single unit,
- the underlying supply and demand functions are time independent during the period of interest,
- the cancellation rate of unexecuted orders is negligible during the period of interest.

It is obvious that the first two assumptions are not true in reality. The third assumption is not good because as mentioned by several authors order cancellation is wide spread in financial markets. We have decided to relax these three assumptions in our model.

Finally, it is worth mentioning that while the model by Chiarella and Iori (2002 and 2004) may seems more complete than the one of Luckock (2003), their model is based on numerical simulation while the one of Luckock (2003) is based on rigorous mathematical analysis.

This completes the presentation of the two other models of continuous double auction. In the next section we present a more theoretical part of the literature.

### 1.2.2 Financial Market with Heterogeneous Agents

We present here a brief description of the model presented by Föllmer, Horst and Kirman (2005) and a part of the theory related to this model. These authors analyze a model with many heterogeneous agents acting on a financial market with one single asset. The model is considered in the Walrasian temporary equilibrium framework.

At time $t$, in reaction to a price $p$, the agents form their random excess demand, $e^t(p, \omega)$. The excess demand can be divided in two parts, the first part consists of a comparison, on

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*[Note: For the citations, please refer to the detailed content in the original text]*
a logarithmic level, of the difference between the suggested price \( p \) and a reference level price \( S_t(\omega) \) the second part consists of a random liquidity demand \( \eta_t(\omega) \). The reference level is given to the agent by a financial expert \( i \in I \). Each agent chooses this financial expert evaluating a discounted sum of past profits. Their recommendation would have produced in the past. The recommendation of the financial expert depends on the expert type. They distinguish two types of experts, fundamentalist and chartist. The fundamentalist believes that the price will eventually revert to a fundamental value while the chartists are using extrapolatory rules to make their recommendation. The actual asset price at time \( t \), denoted \( P_t \), is then determined via the market clearing condition of zero net excess demand:

\[
\sum_{a \in A} e^a_t(P_t(\omega), \omega) = 0 \quad (1.7)
\]

where \( A \) is a finite set of economic agents.

At this point, we will ignore some details and notation in order to arrive to the more theoretical results. Having this in mind, the dynamic sequence of temporary logarithmic price equilibria is governed by an equation of the following form:

\[
S_t = F(S_{t-1}, S_{t-2}, \tau_t) := [1 - \alpha_t + \beta_t]S_{t-1} - \beta_t S_{t-2} + F_t + \eta_t \quad (1.8)
\]

in a random environment described by the sequence

\[
\tau_t := (x_t, \gamma_t, \eta_t) \quad (1.9)
\]

where \( \gamma_t := ((c^t_a)_{a \in A}, (\alpha^i_t, \beta^i_t)_{i \in I}, (F^i_t)_{i \in I}) \). This environment summarizes the stochastic evolution \( \{x_t\}_{t \in \mathbb{N}} \) of the agents’ choices, the experts’ and the agents’ profiles as described by the process \( \{\gamma_t\}_{t \in \mathbb{N}} \) and the noise trading \( \{\eta_t\}_{t \in \mathbb{N}} \).

After introducing the definition of the performance process \( \{U_t\}_{t \in \mathbb{N}} \) we can view the logarithmic price process \( \{S_t\}_{t \in \mathbb{N}} \) as the first component of the Markov chain:

\[
\xi_t := (S_{t-1}, S_{t-2}, U_t) \quad (1.10)
\]

with state space \( E := \mathbb{R}^{m+2} \). Now the goal of the rest of their paper is to prove the existence and uniqueness of a stationary distribution of the Markov chain \( \{\xi_t\}_{t \in \mathbb{N}} \). As mentioned by the authors, the uniqueness of this distribution will imply that the Markov chain is ergodic.

To conclude this section, we recall the definition of stationary distribution and of ergodicity given by the authors and then we state the two main results from this paper.

**Definition 1.11.** A probability measure \( \mu \) is a stationary distribution for the price-performance process if \( \mu \prod = \mu \). In this case, the sequence \( \{\xi_t\}_{t \in \mathbb{N}} \) is stationary under the law

\[
\mathbb{P}_\mu(\cdot) := \int_E \mathbb{P}_\xi(\cdot) \mu(d\xi). \quad (1.12)
\]
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Definition 1.13. A Markov chain \( \{ \xi_t \}_{t \in \mathbb{N}} \) is ergodic if time averages converge to expectations under the unique stationary distribution \( \mu \), i.e.,

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\xi_t) = \int f(\xi) \mu(d\xi), \quad \mathbb{P}_\mu - \text{a.s.,}
\]

and hence \( \mathbb{P}_\xi - \text{a.s. for } \mu - \text{a.e. initial state } \xi \in \mathcal{E} \).

Theorem 1.15. Under some technical assumptions, \(^5\) the Markov chain \( \{ \xi_t \}_{t \in \mathbb{N}} \) has an invariant distribution.

While the detail assumption can be found in the paper directly, we still want to point out that the theorem 1.15 holds under the assumption that the proportion of chartists is bounded from above. It is this particular assumption that insured that the bubbles and crashes that occur do not explode.

Theorem 1.16. Under some technical assumptions, \(^6\) the Markov chain \( \{ \xi_t \}_{t \in \mathbb{N}} \) has a unique stationary distribution \( \mu \), and

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\xi_t) = \int f(\xi) \mu(d\xi), \quad \mathbb{P}_\xi - \text{a.s.,}
\]

for all bounded continuous functions \( f \) and each initial state \( \xi \in \mathcal{E} \).

The proof of the theorem 1.15 relies on a contraction property while the proof of theorem 1.16 is much more involved and requires the comparison of the price-performance process with a process whose convergence can be established using standard argument of the theory of iterated function systems. \(^7\)

This completes the short presentation of the results obtained by Föllmer, Horst and Kirman (2005). At this point, we want to bring to attention the fact that this presentation is far from complete and that the interested reader should consult the paper for more details. In the next section, we present the reasons why we believe that this thesis is relevant.

1.3 Relevance of this Thesis

We present here the reasons why we believe that this thesis constitutes a relevant work.

\(^5\)The precise assumptions are given in Föllmer, Horst and Kirman (2005) by assumption 2.9, 2.10 and 3.1 respectively. In particular, the assumption 3.1 imposes a quantitative bound on the impact of the chartist.

\(^6\)The precise assumptions are given in Föllmer, Horst and Kirman (2005) in assumption 2.9, 3.1 and 3.5 respectively.

\(^7\)A part of the theory of iterated function systems is presented in the paper by Steinsaltz (1999).
First, we have presented in the section 1.2 some of the limitations of models considering a representative agent. We have also discussed the problems occurring in some models where agents have heterogeneous expectations. In particular, we recall that most of the models need to assume irrationality from the agents to obtain interesting results. The model of Föllmer, Horst and Kirman (2005) resolved most of these limitations and problems. Nevertheless, these authors consider a model in the Walrasian temporary equilibrium framework and as they mention themselves, it would be interesting to consider different market organizations.

Second, to some extent, this is done by the two models of Chiarella and Iori (2002 and 2004) and Luckock (2003) by considering heterogeneous agents in a continuous double auction market organization. On the other hand, these models use different assumptions than our model, as mentioned in discussions 1.5 and 1.6 respectively. This is where the present work becomes important. In here, we consider an application of the model of Föllmer, Horst and Kirman (2005) to the continuous double auction framework that use different assumptions that the one given in discussions 1.5 and 1.6.

Third, this thesis does not only apply the model of Föllmer, Horst and Kirman (2005) to the continuous double auction framework but also extends greatly the numerical simulation of such a model. In particular, we provide a wider range of numerical results than the one given by these authors. We also try to compare these results with some of the empirical results of the literature.

Finally, we believe that the presentation of the continuous double auction and order book dynamics given in the chapter 2 constitutes by itself a good reason for the relevance of this thesis. To our knowledge, the graphical approach used makes the explanations more clear than in the previous works on the subject.

1.4 Outline of the Thesis

The structure of the thesis is as follow. In chapter 2 we introduce the different concepts related to the continuous double auction and the order book dynamics. Several definitions are first presented in section 2.1 and then the different rules governing the order book are illustrated through numerous graphical examples in section 2.2. Chapter 3 is the core of the thesis and explains in details the financial market model considered. Section 3.2 presents how the different agents presented in this market take their decisions. Section 3.3 presents the three types of financial experts and shows how these experts choose their recommendation level. Section 3.4 presents the link between the agents and the financial experts through the performance measure process. This process is used by the agents to determine from which expert they will choose their reference level. Finally, section 3.5 presents a short algorithm to understand the different steps done by the market participants. In chapter 4 we present the different details related with the computer simulation. Section 4.1 presents the setting of the different parameters for the simulation. Section 4.2 presents the different
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results obtained by the computer simulations. Finally, chapter 5 presents the conclusion of the thesis. Section 5.1 presents a summary of our model, the different assumptions, the goals and the techniques used in this thesis. Section 5.2 presents the conclusions of the thesis. Finally, section 5.3 presents the different directions further research on this topic can take.
Chapter 2

Order Book Dynamics

In this chapter, we explain how the continuous double auction and the order book work. This is a really important chapter because it will explain how the prices are determined in these sort of market mechanisms. It is also relevant because all the concepts presented here are necessary for the creation of the computer simulation. In section 2.1 we introduce the concept of continuous double auction and other important definitions. In the section 2.2 we present the different rules governing the order book dynamics. In all the situations, we have decided to use graphical examples to illustrate clearly the different rules and mechanisms that occurs in these markets.

2.1 Continuous Double Auction and Definitions

In this section, we first introduce the concept of continuous double auction, referred as (CDA) and some definitions closely related to this concept. The next definition is from Luckock (2003).

Definition 2.1. A continuous double auction, denoted (CDA), is a continuously operating market for goods in which both buyers and sellers can announce offers to trade specified quantities at specified prices, and can also initiate trades by accepting such offers.

Here are the main differences between the CDA and a normal auction. First, in a CDA, both the buyers and sellers can announce offers and this is where the terminology double comes from. Second, the term continuous refers to the fact that this system allows trading at any point in time while the market is open. Some markets only allow trading at specific points in time and accordingly, they are not considered as being continuous.

As mentioned by Luckock (2003), the importance of the CDA mechanism relies on the fact that it is the basis of nearly all the automated trading systems that have been implemented in modern financial markets over the last 2 decades. For example, Paris, Tokyo, Toronto, Hong Kong, Sydney, Mumbai and Stockholm exchanges function as a continuous double auction. Another important point is that the CDA has been found in laboratory experiments to give a very rapid convergence to a competitive equilibrium and also to yield extremely efficient allocation as noted by Smith (1962), Smith et al. (1982) and Friedman (1993). 

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8For more details about the theory on competitive equilibrium and efficient allocations, we refer the reader to Mas-Colell, Whinston and Green (1995).
Chapter 2. Order Book Dynamics

Definition 2.2. Any buyers or sellers that operate in the CDA will be called an agent.

Definition 2.3. An order is a request given to someone or to an electronic system, to execute a certain task.

Definition 2.4. The order size is the number of units contained in the order request.

The most common example of a task to be executed is a request to buy or to sell a specified quantity of a specified good at a specified price. For example, we can think of a buy order of size four for an IBM stock. This is a request to buy four stocks of IBM. These definitions will become clear after the introduction of more precise examples.

There are many different versions of CDA and each of them is characterize by the types of orders that may be submitted: market order, limit order, stop-loss order, stop-limit order, market-if-touch order, etc. The interesting reader can consult the chapter 2 of Hull (2006) for a full description of all these types of order. In practice, most of the automated systems operate as limit order markets and accepts both market and limit orders. These are the two types of order that we consider in our model.

Definition 2.5. A market order is a request that a trade be executed at the best price available.

There are two types of market orders, a buy market order and a sell market order.

Definition 2.6. A buy market order of size n is a request to buy the specified quantity n of a specified good at the best sell price available.

Definition 2.7. A sell market order of size n is a request to sell the specified quantity n of a specified good at the best buy price available.

Definition 2.8. A limit order is a request that a trade be executed at a specified or a more favorable price.

The meaning of a more favorable price will become clear in the next two definitions. There are two types of limit orders, buy limit order and sell limit order.

Definition 2.9. A buy limit order of size n is a request to buy the specified quantity n of a specified good at a specified price b or at a lower price than b, in which case this price is considered as being a more favorable price.

Definition 2.10. A sell limit order of size n is a request to sell the specified quantity n of a specified good at a specified price a or at a higher price than a, in which case this is considered as being a more favorable price.

9The interested reader can consult the chapter 2 of Hull (2006) for a complete description of all these types of orders.
While the market orders are generally executed immediately, there is nothing that guarantees that it is the case for the limit orders. As an example, a buy limit order of price 4 cannot be executed if the best sell price available is 6. As a consequence, the limit orders need to be stored in an order book. For the market orders, any unexecuted part may be converted in a limit order at the same price (Paris bourse) or else executed at the next best available price (as on the Australian Stock Exchange). Indeed formally we have the following definition.

**Definition 2.11.** An order book is a system in which unexecuted or partially executed orders are stored and sometime displayed until their execution or their cancellation.

**Discussion 2.12.** We note that the question about which part of the order book or if the order book should be displayed is in itself of entire field of research.

We also want to point out the difference between the CDA and the order book. The CDA is a type of market organization or market microstructure while the order book is a system used to store the unexecuted or partially executed orders.

In the next few paragraphs we start using a more graphical approach that will help clarify the different concepts. In particular, we can use a graphical approach to represent and get some intuition about the order book. This is done in Figure 2.1.

**Discussion 2.13.** At this point, it is important to note that the different graphics presented in this section are only to make easier the exposition of the different concepts. The graphics presenting the simulation results will be more realistic in term of the different values of the parameters.

---

10 The interested reader is referred to Luckock (2003) for a more precise discussion about the difference between these two exchange platforms.
A less natural but more useful way to represent the order book is to place the buy limit orders on the negative part of the y axis. This technique allows us to see clearly the difference between the buy and sell limit orders as shown in Figure 2.2.
Order Book Dynamics

Chapter 2. Order Book Dynamics

Figure 2.2: Second order book representation

We are now ready for the introduction of other important concepts.

**Definition 2.14.** The best bid price at time $t$, denoted $b_t$, is the higher price for which there is buy limit orders currently stored in the order book.

**Definition 2.15.** The best ask price at time $t$, denoted $a_t$, is the lowest price for which there are sell limit orders currently stored in the order book.

**Definition 2.16.** The bid and ask spread at time $t$, denoted $q_t$, is the difference between the best ask price and the best bid price when they are both different from zero or is set to zero otherwise. Mathematically, it is given by the following formula:

$$q_t = \begin{cases} a_t - b_t & \text{if } a_t \neq 0 \text{ and } b_t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(2.17)

**Definition 2.18.** The middle point of the order book at time $t$, denoted $m_t$, is given by:

$$m_t = (a_t + b_t)/2.$$ 

(2.19)

These different concepts are shown in Figure 2.3.
Discussion 2.20. Here it is interesting to note that most of the mathematical finance theories assume that the bid and ask spread is actually zero, meaning that you can always buy and sell at the same price (see Shreve (2000a), Shreve (2000b), Duffie (2001) and Cvitanic and Zapatero (2004)). This assumption is never true in reality and can even be totally wrong for some particular stock. It is these types of frictions that are usually taken into account in the study of market microstructure.

Definition 2.21. Two orders of different sign, i.e. a sell order and a buy order, are executed against each other when they have the same price.

This simply means that one person wants to buy at the same price that another one wants to sell and so the two orders cancel each other and a transaction occurs. For this reason, we say that the orders are executed with orders of opposite signs. This complete the section about the CDA. In the next section we present the details of order book dynamics.

2.2 Order Book Dynamics

In this section we present and explain the different rules that governed the order book. At the same time, because in a limit order market, the price is determined by the interaction of the incoming orders with the orders already in the book we also explain the price
2.2.1 Case 1

The first situation is when a market order of size \( n \) is received and that the number of orders available at the best price \( p \) is \( m \geq n \). In this case, the number of orders available at the price \( p \) simply becomes \( m - n \) and the current price \( p_t \) is set to the price of the last transaction \( p \), i.e., \( p_t = p \). We present an example of this case in Figure 2.4.

![Figure 2.4: Receiving some market orders. The best ask price is \( a_{t-1} = 8 \). There are 6 sell limit orders available at this price. A buy market order of size 2 arrives. It is executed against sell limit orders at the best ask \( a_{t-1} = 8 \). The number of limit orders available at this price becomes \( 6 - 2 = 4 \). A transaction have occurred and the current price is modified accordingly, \( p_t = 8 \). The best bid price and best ask price remain unchanged.](image-url)
2.2.2 Case 2

A market order of size \( n \) is received and the number of orders available at the best price \( p \) is \( m < n \). In this case, we follow the Australian Stock Exchange mechanism, the unexecuted part will be executed at the next best available price. If there is not enough orders stored at this new best price, the reminder are executed at the next best available price and so on until the market order is completely executed. The current price will be the price of the last transaction. We present an example of this case in Figure 2.5.

![Order Book Before the Modification](image1)

**Order Book Before the Modification**
- Best bid price \( b_{t-1} = 6 \)
- Number of sell limit orders
- Number of buy limit orders
- Price

![Order Book After the Modification](image2)

**Order Book After the Modification**
- Best bid price \( b_t = 5 \)
- Number of sell limit orders
- Number of buy limit orders
- Price

**Figure 2.5:** Receiving some market orders. The best bid price is \( b_{t-1} = 6 \). There are 3 buy limit orders available at this price. A sell market order of size 4 arrives. Three orders are executed against buy limit orders at the best bid \( b_{t-1} = 6 \). The number of limit orders available at this price becomes \( 3 - 3 = 0 \). The rest of the order is executed at the next best price available 5. A transaction have occurred and the current price is modified to the price of the last transaction, \( p_t = 5 \). The best bid price is modified from \( b_{t-1} = 6 \) to \( b_t = 5 \). The best ask price remain unchanged.
2.2.3 Case 3

A buy limit order of size \( n \) and a price \( b < a_t \) (best ask price) is submitted. In this case, these buy orders are stored in the book at the price \( b \) and the price does not change because there is no transaction, i.e., \( p_t = p_{t-1} \). We present an example of this case in Figure 2.6.

![Order Book Dynamics](image)

Figure 2.6: Receiving some buy limit orders. A buy limit order of size 4 and price 6 arrives. The number of order at this price becomes \( 0 + 4 = 4 \). The best bid price is modified from \( b_{t-1} = 5 \) to \( b_t = 6 \).

**Discussion 2.22.** It is important to note that, as in the previous example, the best bid price or the best ask price can change even when no transaction occurs. Because the best bid and best ask are usually displayed, this conveys some information to the agents.
2.2.4 Case 4

A sell limit order of size $n$ and a price $a > b_t$ (best bid price) is received. In this case, these sell limit orders are stored in the book at the price $a$ and the price does not change, i.e., $p_t = p_{t-1}$. We present an example of this case in Figure 2.7.

![Order Book Before the Modification](image1)

![Order Book After the Modification](image2)

Figure 2.7: Receiving some sell limit orders. A sell limit order of size 2 and price 7 arrives. The number of orders at this price becomes $3 + 2 = 5$. The best bid price and best ask price remain unchanged.
2.2.5 Case 5

A buy limit order of size $n$ and price $b \geq a_t$ is received. If the number of sell limit order at price $a_t$, denoted $m$, is bigger or equal to $n$ (if $m \geq n$) then the buy limit orders are executed against the sell limit orders as if they were buy market orders and the price becomes $p_t = a_t$. We present an example of this case in Figure 2.8.

![Order Book Dynamics](image)

Figure 2.8: Matching buy limit orders. A buy limit order of size 4 and price $b = 9 \geq a_{t-1} = 7$ arrives. The number of orders at the price $a_{t-1} = 8$ becomes $5 - 4 = 1$. A transaction has occurred and the current price is modified accordingly, $p_t = 7$. The best bid price and best ask price remain unchanged.

**Discussion 2.23.** It is important to note that the buy limit order of price $b$ is not executed against sell limit order at this price. This is the meaning of best available price in the definition of limit order.

Now if we are in the other situation, i.e., $m < n$ then $m$ buy limit orders are executed against sell limit orders at price $a_t$ and the remaining $(n - m)$ buy limit orders are either executed at the next best available price $\bar{a}$ if $b \geq \bar{a}$ (case A) or they are stored in the book.
at the price $b$ if $b < a$ (case B). If $b \geq a$ the price becomes $p_t = a$. If $b < a_t$ the price is the price of the last transaction. Once again, all this is made more clear with a graphical representation. We present the case A in Figure 2.9 and the case B in Figure 2.10.

Figure 2.9: Matching buy limit orders, case A. The best price is $a_{t-1} = 7$. There are 2 sell limit orders available at this price. A buy limit order of size 5 and price $b = 8$ arrives. Two orders are executed against sell limit orders at the best ask $a_{t-1} = 7$. The number of limit orders available at this price becomes $2 - 2 = 0$. Because $b > a$ the next best available price, the rest of the order is executed at the next best price available 8. A transaction has occurred and the current price is modified to the price of the last transaction, $p_t = 8$. The best ask price is modified from $a_{t-1} = 7$ to $a_t = 8$. The best bid price remain unchanged.
Chapter 2. Order Book Dynamics

Figure 2.10: Matching buy limit orders, case B. The best ask price is $a_{t-1} = 8$. There are 3 sell limit orders available at this price. A buy limit order of size 5 and price $b = 8$ arrives. Three orders are executed against sell limit orders at the best ask $a_{t-1} = 8$. The number of limit orders available at this price becomes $3 - 3 = 0$. A transaction has occurred and the current price is modified to the price of the last transaction, $p_t = 8$. Because $b < a$, a buy limit order of size $5 - 3 = 2$ and price $b = 8$ is placed. The best ask price is modified from $a_{t-1} = 8$ to $a_t = 9$. The best bid price is modified from $b_{t-1} = 5$ to $b_t = 8$. 


By symmetry we obtain the same situation for the sell order. A sell limit order of size \( n \) and price \( a \leq b_t \) is received. If the number of buy limit orders at price \( b_t \), denoted \( m \), is bigger or equal to \( n \) (if \( m \geq n \)) then the sell limit orders are executed against the buy limit orders as if they were sell market orders and the price becomes \( p_t = b_t \). We present an example of this case in Figure 2.11.

![Order Book Dynamics Diagram](image)

Figure 2.11: Matching sell limit orders. A sell limit order of size 3 and price \( a = 6 \leq b_{t-1} = 6 \) arrives. The number of orders at the price \( b_{t-1} = 6 \) becomes \( 4 - 3 = 1 \). A transaction has occurred and the current price is modified accordingly, \( p_t = 6 \). The best bid price and best ask price remain unchanged.
Chapter 2. Order Book Dynamics

The earlier discussion about the best available price is also applied in the present case. Now if the number of buy limit orders at price $b_t$ is smaller than $n$ (if $m < n$), then $m$ sell limit orders are executed against buy limit orders at price $b_t$ and the remaining $(n - m)$ sell limit orders are either executed at the next best available price $b$ if $a < b$ (case A) in which case the price is set to the price of the last transaction. Or they are stored in the book if $a > b$ (case B) and the price becomes $p_t = b$. We present the case A in Figure 2.12 and the case B in Figure 2.13.

Figure 2.12: Matching sell limit orders, case A. The best bid price is $b_{t-1} = 6$. There are 4 buy limit orders available at this price. A sell limit order of size 6 and price $a = 4$ arrives. Four orders are executed against buy limit orders at the best bid $b_{t-1} = 6$. The number of limit orders available at this price becomes $4 - 4 = 0$. Because $a < b$ the next best available price, the rest of the order is executed at the next best price available 5. A transaction has occurred and the current price is modified to the price of the last transaction, $p_t = 5$. The best bid price is modified from $b_{t-1} = 6$ to $b_t = 5$. The best ask price remain unchanged.
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Figure 2.13: Matching sell limit orders, case B. The bid ask price is $b_{t-1} = 6$. There are 4 buy limit orders available at this price. A sell limit order of size 6 and price $a = 6$ arrives. Four orders are executed against sell limit orders at the best bid $b_{t-1} = 6$. The number of limit orders available at this price becomes $4 - 4 = 0$. A transaction has occurred and the current price is modified to the price of the last transaction, $p_t = 6$. Because $a > b$, a sell limit order of size $5 - 4 = 2$ and price $a = 6$ is placed. The best bid price is modified from $b_{t-1} = 6$ to $b_t = 5$. The best ask price is modified from $a_{t-1} = 7$ to $a_t = 6$.

Discussion 2.24. The attentive reader will have noticed that we have avoided speaking about a particular case so far and it is the case when some market orders of size $n$ arrived and $n$ is bigger than the number of limit orders $\overline{m}$ of the opposite type in the book. This case is really important because it will require us to be really careful in the construction of the numerical simulation. Moreover, other authors usually avoid clear explanation of what happens in this case, (see Luckock (2003), Chiarella and Iori (2002) and Chiarella and Iori (2004)). We present this case in the next two subsections.
2.2.6 Case 6

A buy market order of size \( n \) is received and the total number of sell limit order in the book \( m \) is smaller \( (m < n) \). In this case, \( m \) buy market orders are executed against the \( m \) sell limit orders. The \( (n - m) \) remaining buy market orders are converted in a buy limit order of size \( (n - m) \) at the last price for which sell limit orders were available \( \bar{a} \). The price is set to the price of the last transaction, \( p_t = \bar{a} \). This case is shown in Figure 2.14.

Figure 2.14: Right side of the order book is empty. A buy market order of size 9 arrives. The total number of sell limit order available in the book \( m = 7 \). Seven buy market orders are executed against the sell limit orders in the book and the total number of sell limit orders in the book becomes \( m = 0 \). The price is modified to the price of the last transaction \( p_t = 9 \). The remaining market order are converted in a buy limit order of size \( 9 - 7 = 2 \) at the last transaction price \( \bar{a} = 9 \). The best ask price \( a_t = 0 \) and using equation 2.17 the spread becomes \( q_t = 0 \).
2.2.7 Case 7.

A sell market order of size $n$ is received and the total number of buy limit orders in the book $\overline{m}$ is smaller than $n$ ($\overline{m} < n$). In this case, $\overline{m}$ sell market orders are executed against the $\overline{m}$ buy limit orders. The $(n - m)$ remaining sell market orders are converted in a sell limit order of size $(n - m)$ at the last price for which buy limit orders were available $\tilde{b}$. The price is set to the price of the last transaction, $p_t = \tilde{b}$. This case is shown in Figure 2.15.

![Diagram](image-url)

Figure 2.15: Left side of the order book is empty. A sell market order of size 9 arrives. The total number of buy limit order available in the book $\overline{m} = 6$. Six sell market orders are executed against the buy limit orders in the book and the total number of buy limit orders in the book becomes $\overline{m} = 0$. The price is modified to the price of the last transaction $p_t = 4$. The remaining market order are converted in a sell limit order of size $9 - 6 = 3$ at the last transaction price $\tilde{b} = 4$. The best bid price becomes $b_t = 0$ and using equation 2.17 the spread becomes $q_t = 0$.

This concludes the chapter on the continuous double auction and the order book dynamics. In this chapter we have introduced the different important concepts related to the CDA microstructure and explained through numerous graphics and examples the different
rules governing the order book dynamics. In the next chapter, we present the detailed description of our model.
Chapter 3

The Model

In this chapter we give a presentation of our model with the different market participants. While in the chapter 2 we gave the different rules that governed the order flow, we have omitted any explanations about who were submitting these orders and why. This is done in this chapter. The section 3.1 gives a concise description of our model without all the details. The section 3.2 explains how the agents take their decisions in our model and how we need to adjust the rules of the order book presented in the chapter 2 to fit our model. In section 3.3 we introduce the three types of financial experts and we explain how they choose their reference level at each period. Section 3.4 makes the link between the agents and the financial experts through the concept of performance measure. Finally, in section 3.5 we present a short algorithm to make the understanding of the whole model easier.

3.1 The Model in Brief

In this section we present a concise description of our model without all the details.

We consider a financial market with a single risky asset. There is a finite set $\mathcal{A}$ of economic agents trading on this market. The agents submit orders according to Poisson dynamics with constant rate $\mu$. The choice of the agent to submit buy or sell orders will be determined using her excess demand. The excess demand of each agent is based on two components, the comparison of the last price with a reference level and a exogenous random liquidity demand. There is a finite set $\mathcal{I}$ of financial expert each proposing a reference level for the next period. The agent’s choice of a particular reference level will depend on a measure of the performance of the expert recommendation. Finally, the price is not a equilibrium price determined using a market clearing condition like in Föllmer, Horst and Kirman (2005). The price is determined using the different rules that governed the order book as explained in chapter 2.

This complete the brief and concise presentation of our model. In the next section we will explain in more details the different components of the agents excess demand and how the agent chooses between placing market or limit orders.
3.2 The Agents

This section follows closely the theoretical model given in Föllmer, Horst and Kirman (2005). We consider the usual probability space framework, \((\Omega, \mathcal{F}, \mathbb{P})\). We consider a finite set of economic agents \(A = \{1, 2, \ldots, N\}\). The financial market consist of a single risky asset with price at time \(t\) given by \(p_t\). At time \(t\), each agent \(a \in A\) forms an excess demand for the risky asset, denoted \(e_a^t(p_{t-}, \omega)\), where \(\omega\) represents a particular element of \(\Omega\) and where \(t-\) represents the time of the last transaction. 11 This excess demand will depend on an endogenous component which compares, on a logarithmic basis, the last price \(p_{t-}\) and some reference level \(\hat{S}_t\) and an exogenous component which consists of a random liquidity demand \(\eta_t^a\).

The agents submit orders according to Poisson dynamics with constant rate \(\mu\). The excess demand of agent \(a \in A\) is then determined via the following formula:

\[
e_a^t(p_{t-}, \omega) := c_a^t(\omega) [\hat{S}_t^a(\omega) - \log(p_{t-})] + \eta_t^a(\omega)
\]  

(3.1)

where:

- \(c_a^t(\omega)\) is non-negative, i.e., \(c_a^t(\omega) \geq 0\). This parameter represents the speed perceived by agent \(a \in A\) at which the price will go back to the reference level,

- \(\hat{S}_t^a(\omega)\) is the logarithmic reference level chosen for the time interval \([t, t+]\). This reference level is based on the recommendation \(\varphi_i^t\) of an expert \(i \in I = \{1, 2, \ldots, M\}\)

\[
\hat{S}_t^a(\omega) \in \{\varphi_1^t(\omega), \varphi_2^t(\omega), \ldots, \varphi_M^t(\omega)\}
\]  

(3.2)

and

- \(\eta_t^a(\omega)\) is a exogenously determined random liquidity demand of agent \(a\).

Discussion 3.3. For the moment, we do not explain where the logarithmic reference level \(\hat{S}_t^a(\omega)\) comes from nor how it is chosen by the agent. This will be done in Sections 3.3 with the description of the experts and 3.4 with the introduction of the concept of performance measure.

Now after an agent is chosen and this agent \(a \in A\) has determined the reference level she will follow, she needs to determine the sign of order she wants to place, a buy or a sell order, and the order size. These two decisions are determined directly using the excess demand \(e_a^t(p_{t-}, \omega)\). The type of order depends on the sign of the excess demand, if \(e_a^t(p_{t-}, \omega) > 0\) the agent submits a buy order while if \(e_a^t(p_{t-}, \omega) < 0\) the agent submits a sell order. In the

11Following this notation, \(t-\) will represent the time of the second last transaction and \(t+\) will represent the time of the next transaction.
Chapter 3. The Model

case \( e^a_t(p_{t-}, \omega) = 0 \) the agent does not submit any order. The size of the order is simply given by the absolute value of the excess demand, which we denote \( \zeta \), i.e.,

\[
\zeta = |e^a_t(p_{t-}, \omega)|.
\]

**Discussion 3.5.** There are two reasons why the random liquidity demand \( \eta^a_t(\omega) \) is important. First, it serves as a sort of budget constraint for the agent when she is taking her investment decisions. Suppose for example that the term \( \hat{S}^a_t(\omega) - \log(p_{t-}) \) is positive which implies that the agent expects a price increase. Following this logic, the agent wants to buy some stock to profit from this expected price increase. Now suppose that the random liquidity demand is negative and that:

\[
|\eta^a_t(\omega)| > c^a_t(\omega)[\hat{S}^a_t(\omega) - \log(p_{t-})]
\]

then the excess demand will be negative and the agent will sell some stock. Indeed, we are in a situation where the agent expects a price increase but still wants to sell the stock. It is for this reason that the random liquidity demand constitutes a sort of budget constraint.

Second, as we will see later the random liquidity demand is also useful to represent the fact that an agent can follow a noise trader advice. This will happen when the reference level \( \hat{S}^a_t(\omega) \) and the price of the last time interval \( p_{t-}(\omega) \) are the same. This last statement will become clear in the next section where we describe the experts.

**Discussion 3.7.** It is important to note that our model assumes implicitly that the agents are allowed to short sell the stock and that the agent does not have a borrowing constraint (other than the random liquidity demand). By short selling the stock, we mean selling the stock even if they do not hold it. This is another difference between our model and the model by Chiarella and Iori (2004) where the authors consider a model where short selling is not allowed. Which one of these two assumptions is the best is not clear, both assumptions can be considered as being unrealistic.

At this point, the agent needs to decide if she will submit a market or a limit order. This is accomplished by comparing the best ask \( a_t \) or the best bid \( b_t \) with the reserve price of the agent. This implicitly assumes that all the agents of our model always know the best bid price and the best ask price. In order to define the concept of reserve price properly we first need to introduce the following definition.

**Definition 3.8.** The time horizon of an agent \( a \in A \), denoted \( h^a \), is the amount of time an agent will wait until she decides to cancel her unexecuted limit orders still in the order book. It is given by the following formula:

\[
h^a := \nu^a + \frac{1}{\gamma^a}
\]

where \( \nu^a > 0 \) is a constant and \( \gamma^a \) is the discount factor of agent \( a \).
This means that at each time $t$, every agent that is active on this market will verify if she should cancel her limit orders in the order book or keep waiting. For this reason, the time horizon of an agent can be interpreted as her level of patience. We also note that if the discount factor of an agent $\gamma^a$ is increasing then the time horizon of this agent $h^a$ is decreasing which is consistent from an economic point of view.

For simplicity of notation, we introduce the following variable:

$$S^a_t := e^{\hat{S}^a_t}$$

(3.10)

We recall that $\hat{S}^a_t$ is the logarithmic reference level chosen by the agent $a \in A$ for the time interval $[t, t+]$ then, $e^{\hat{S}^a_t}$ represent the actual price reference level for this time interval.

**Definition 3.11.** The reserve price of an agent $a \in A$ at time $t$, denoted $R^a_t$, is the price at which the agent is ready to sell or to buy the stock directly for the time interval $[t, t+]$. In other words, at this price, the agent submits a market order. It is mathematically given by the following formula:

$$R^a_{t, \text{sell}} := S^a_t + \xi h^a \quad \text{for a sell order and}$$

$$R^a_{t, \text{buy}} := S^a_t - \xi h^a \quad \text{for a buy order}$$

(3.12) (3.13)

where $\xi > 0$ is constant.

**Discussion 3.14.** The reserve price also influences how far from the middle of the order book the agent will place her limit order. This parameter needs to be related to the time horizon because as noted by Bouchaud and Potters (2003) the more patient agent will place her limit order further from the middle of the book because she is willing to wait a long time for her order to be executed. Indeed, if you are impatient it will not make sense to place a limit order far from the middle unless you are expecting a large price movement. Consequently, the position where the agents are placing their limit orders gives important information about the level of patience of the agents and their expected level of variability of the price process.

It is important to note another difference with Chiarella and Iori (2004). In our model the reserve price $R^a_t$ depends on the discount factor of agent $a$ via the time horizon parameter $h^a$. This is not the case in the model of Chiarella and Iori (2004). In fact, these authors do not consider any discount factor in their model.

Now we divide this section in two symmetrical sections where we take time to explain the mechanism of decision between market and limit orders for both the case of buy order and sell order.

\[\text{12} A \text{ relation between order placement and patience is also discussed in Farmer and Zovko (2002).}\]
Chapter 3. The Model

3.2.1 The Case of Buy Orders

In this case, to decide which type of order the agent will submit we need to compare the value of the best ask at with the reserve price of the agent $R_t^{a,\text{buy}}$. If $R_t^{a,\text{buy}} \geq a_t$ the agent is willing to pay more than $a_t$ to obtain the stock and so she is submitting a buy market order. This order will be executed against the sell limit order at the price $a_t$ as long as the size of the order $\zeta \neq 0$ or that $R_t^{a,\text{buy}} \geq a_t$. If it happens that in the process of executing the order from the agent, $R_t^{a,\text{buy}} < a_t$ then because the best ask become bigger than the agent reserve price, the agent will submit the rest of her order as limit buy order at the price $R_t^{a,\text{buy}}$. We represent these two situations in Figure 3.1 and Figure 3.2 respectively.

![Order Book Before the Modification](image1.png)

![Order Book After the Modification](image2.png)

Figure 3.1: Submitting some buy market orders. The best ask price is $a_{t-1} = 7$. There are 2 sell limit orders available at this price. The reference level given by the expert is $S_t^a = 10$. A buy limit order of size $\zeta = 5$ and price $R_t^{a,\text{buy}} = 9$ arrives. Two orders are executed at the price 7 and then because $R_t^{a,\text{buy}} = 9 \geq a_t = 8$, the rest of the order is executed at the price 8.
3.2.2 The Case of Sell Orders

In this case, to decide which type of order the agent will submit we need to compare the value of the best bid $b_t$ with the reserve price of the agent $R_{t}^{a, sell}$. If $R_{t}^{a, sell} \leq b_t$ someone is willing to pay more than the reserve price of the agent to obtain the stock and so she is submitting a sell market order. This order will be executed against the buy limit order at the price $b_t$ as long as the size of the order $\zeta \neq 0$ or that $R_{t}^{a, sell} \leq b_t$. If it happens that in the process of executing the order from the agent, $R_{t}^{a, sell} > b_t$ then because the best bid has become smaller than the agent reserve price, the agent will submit the rest of her order as limit sell order at the price $R_{t}^{a, sell}$. We present these two situations in Figure 3.3 and Figure 3.4 respectively.
Chapter 3. The Model

Order Book Before the Modification

Order Book After the Modification

Figure 3.3: Submitting some sell market orders. The best bid price is \( b_{t-1} = 6 \). There are 4 buy limit orders available at this price. The reference level given by the expert is \( S_t^b = 3 \).

A sell limit order of size \( \zeta = 3 \) and price \( R_t^a = 5 \) arrives. It is executed at the price 6.
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Figure 3.4: Submitting some sell market and limit orders. The best bid price is \( b_{t-1} = 8 \). There are 4 buy limit orders available at this price. The reference level given by the expert is \( S^t = 5 \). A sell limit order of size \( \zeta = 6 \) and price \( R^s_{t, \text{sell}} = 6 \) arrives. Four orders are executed at the price 6 and then because \( R^s_{t, \text{sell}} = 6 > b_t = 5 \), the rest of the order is converted to a sell limit order of size 2 and price 6.

**Discussion 3.15.** It is important to note that in fact in our model there is no pure market order. The reason is that the agents always have a reserve price and will not be willing to buy at a price higher or sell at a price lower than this reserve price. This means that in our model all the market orders can be viewed as a limit order for which \( R^a_{t, \text{buy}} > a_t \) in the case of a buy order and \( R^a_{t, \text{sell}} < b_t \) in the case of sell order.

This completes the section describing the decision mechanism of the agent. In the next section, we will explain where the reference level \( \varphi \) comes from by describing the financial experts.
3.3 The Experts

In this section we describe the role of the financial experts present in our model. The agent of our model takes the expected price for the next time interval \([t, t+1]\), called the reference level, from a financial expert. Indeed, we need to describe how these experts choose this reference level.

We consider a finite set of financial experts \(I = \{1, 2, \ldots, M\}\). Each expert \(i \in I\) of our model has her own personal perception of the fundamental value of the stock price. To be more precise:

**Definition 3.16.** The fundamental value or benchmark of expert \(i \in I\), denoted \(F_i^t(\omega)\), is the value, on a logarithmic scale, at which this expert expects the price to return in the long run.

**Discussion 3.17.** We note that the fundamental value of our model \(F_i^t(\omega)\) is different from expert to expert and it is not constant because of the dependence on \(\omega\). This is one of the differences with the model of Chiarella and Iori (2002 and 2004) mentioned in the introduction.

In practice, the fundamental value is usually a discounted sum of future earnings. As mentioned by Lux and Marchesi (1999) the most commonly used value represents a discounted sum of future dividends payments.

Each expert \(i \in I\) determines their reference level for the next time interval using the following equation:

\[
\phi_i^t(\omega) := \phi_{t-} + \alpha_i^t(\omega) F_i^t(\omega) - \phi_{t-} + \beta_i^t F_i^t(\omega) \phi_{t-} - \phi_{t-}(\omega) \tag{3.18}
\]

where \(\phi_{t-}(\omega), \phi_{t-}\) represent the price in logarithmic scale at time \(t-\) and \(t-\) respectively, i.e., \(\phi_t(\omega) := \log(p_t(\omega))\).

**Discussion 3.19.** The coefficients \(\alpha_i^t\) and \(\beta_i^t\) represent the expert estimate of the speed of price adjustment. These quantities can be negative, for example \(\beta_i^t < 0\) means that the expert follows a contrarian strategy. This as been noted by Brock and Hommes (1997) and Föllmer, Horst and Kirman (2005).

The determination of \(\phi_i^t(\omega)\) through the formula 3.18 is general enough to allow us to consider the three usual types of traders: noise traders, fundamentalist traders and chartist traders. Their precise definition is given below.

**Definition 3.20.** A fundamentalist is an expert for which \(\beta_i^t(\omega) \equiv 0\).

**Definition 3.21.** A chartist is an expert for which \(\alpha_i^t(\omega) \equiv 0\).

**Definition 3.22.** A noise trader is an expert for which both \(\alpha_i^t(\omega) \equiv 0\) and \(\beta_i^t(\omega) \equiv 0\).
Discussion 3.23. As mentioned by Bouchaud et al. (2003) and De Long et al. (1990) it is well known in the literature that these three types of traders need to be present in a model if we want to reproduce realistic financial market data.

It is important to understand where the term noise trader comes from. For this purpose, we recall the discussion 3.5 where we pointed out the importance of the random liquidity demand $\eta^a_\ell(\omega)$. If agent $a \in A$ follows the advice from a noise trader expert $i \in I$ then using equation 3.18 she obtains the following reference level:

$$\varphi^i_\ell(\omega) = \phi_{t-}(\omega)$$  \hfill (3.24)

Now using this reference level with the fact that $\hat{S}^i_\ell(\omega) = \phi_{t-}(\omega) = \log(p_{t-})$ and the equation 3.1 we obtain the agent excess demand which is simply given by:

$$e^i_\ell(p_{t-}, \omega) = \eta^a_\ell(\omega)$$  \hfill (3.25)

In other words, when an agent follows the advice from a noise trader her excess demand becomes simply her random liquidity demand.

This complete the section on the description of the financial experts. In the next section we explain the process by which the agents decide from which expert they should follow advice.

3.4 The performance measure

The performance measure defined in this section links the agents and the experts. Each agent $a \in A$ associates a performance measure to each expert $i \in I$. The probability to follow the reference level of expert $i \in I$ will depend on this performance measure. We first introduce the notion of conditional profit realized by expert $i \in I$ and than we define the performance measure.

Definition 3.26. We define the immediate conditional profit associated with the expert $i \in I$, noted $\pi^i_t$, as the profit that an agent will have realized between the time interval $[t-, t]$ if she had followed expert $i$ recommendation. More precisely,

$$\pi^i_t := (\varphi^i_\ell - S_{t-})(e^{\phi_{t-}} - e^{\phi_{t-}}).$$  \hfill (3.27)

We note that using equation 3.18 $\pi^i_t$ can be rewritten as follow:

$$\pi^i_t = \alpha^i_\ell(\omega)(F^i_\ell(\omega) - \phi_{t-}(\omega)) + \beta^i_\ell(\omega)(\phi_{t-}(\omega) - \phi_{t-}(\omega))(p_{t} - p_{t-})$$  \hfill (3.28)

Now we can define the performance measure.
Definition 3.29. The performance measure an agent $a \in A$ associates to the expert $i \in I$ at time $t$, denoted $U_{t}^{a,i}$, is the discounted sum of the past profit the expert $i \in I$ recommendation would have generate. Mathematically, it is given by:

$$U_{t}^{a,i} = \sum_{j=0}^{t} \frac{\pi_{t}^{i}}{(1 + \gamma^{a})^{t-j}}$$  \hspace{1cm} (3.30)

Discussion 3.31. We note that if an agent has a smaller discount factor $\gamma$, and then a longer time horizon $h$ (using equation 3.9), she will give more weight to past profit than an agent with a higher discount factor $\bar{\gamma}$. This is totally consistent. The agent with a shorter time horizon is less interested in the profits realized long ago because they will not wait for very long to cancel their orders from the book and so for them these profits will never be realized.

Discussion 3.32. At this time, we want to bring the reader's attention to the importance of the discount factor $\gamma$. Each agent $a \in A$ has a discount factor, denoted $\gamma^{a}$. This discount factor enters in three different equations:

- First, it enters in the determination of the agent time horizon $h^{a}$ through equation 3.9.
- Second, it enters in the calculation of the reserve prices $R_{t}^{a,\text{buy}}$ and $R_{t}^{a,\text{sell}}$ through equations 3.12 and 3.13 respectively.
- Finally, it is used to discount the past profit in the calculation of the performance measure of the experts through equation 3.28.

Finally, we use the performance measure of each expert as the parameter of an exponential distribution. We then draw a number from these distributions for each expert and the agent will choose the expert with the highest number.

This completes the section linking the agents and the financial experts. In the next section we present a short algorithm to help understanding the complete model.

3.5 Order Submission Mechanism

In this section we present an easy algorithm to explain the complete model step by step. This section is particularly relevant for understanding how the computer simulation is created. The first thing that need to consider a discrete version of our model. We then take $t = 0$ as the first time and the subsequent times are simply $t = 1$, $t = 2$, etc. In particular, this will change the notation of $t-$ for $t - 1$, $t -$ for $t - 2$ and $t+$ for $t + 1$.  

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1. For each agent \( a \in A \) we draw a number from a Poisson distribution with constant parameter \( \mu \). If we let, \( \theta^a(\mu) \) be the result of such a drawing for each agent \( a \in A \), then we chose the agent \( \hat{a} \) with the highest \( \theta^a \).

\[
\hat{a} = \arg\max_{n \in A} \{\theta^n(\mu)\}. \tag{3.33}
\]

2. Now that we have determined which agent will be trading for the next period we need to determine from which financial expert this agent will pick up the reference level for this period. This is done by calculating the performance measure \( U_{t,i}^{a,i} \) of every expert \( i \in I \) given by the equation \( 3.30 \):

\[
U_{t,i}^{a,i} = \sum_{j=0}^{t} \frac{\pi^i_t}{(1 + \gamma^a)^{t-j}}
\]

Then, we draw a number for each expert from an exponential distribution with parameter \( U_{t,i}^{a,i} \) and choose the expert with the highest draw.

3. Now that we have determined our expert \( \hat{i} \) we can calculate her recommendation \( \varphi_{t}^{\hat{i}} \) for this period using equation \( 3.18 \):

\[
\varphi_{t}^{\hat{i}}(\omega) := \phi_{t-1}(\omega) + \alpha_{t}^{\hat{i}}(\omega) [F_{t}^{\hat{i}}(\omega) - \phi_{t-1}(\omega)] + \beta_{t}^{\hat{i}}(\omega) [\phi_{t-1}(\omega) - \phi_{t-2}(\omega)]
\]

4. At this point, we let \( \tilde{S}_{t}^{\hat{a}}(\omega) = \varphi_{t}^{\hat{i}}(\omega) \) and we are able determine the agent's excess demand using the equation \( 3.1 \):

\[
e_{t}^{\hat{a}}(p_{t-1}, \omega) = \tilde{c}_{t}^{\hat{a}}(\omega) [\tilde{S}_{t}^{\hat{a}}(\omega) - \log(p_{t-1})] + \eta_{t}^{\hat{a}}(\omega)
\]

5. As explained in the previous sections, the choice of buy or sell order is then determined by considering the sign of the excess demand \( e_{t}^{\hat{a}}(p_{t-1}, \omega) \):

- if \( e_{t}^{\hat{a}}(p_{t-1}, \omega) > 0 \) the agent submits a buy order and
- if \( e_{t}^{\hat{a}}(p_{t-1}, \omega) < 0 \) the agent submits a sell order.

6. Then the agent submits her orders using the absolute value of her excess demand \( \zeta \) and her reserve price \( R_{t}^{a} \):

- In case of a buy order, the agent submits a buy limit order of size \( \zeta \) and price \( R_{t}^{a,\text{buy}} \).
- In case of a sell order, the agent submits a sell limit order of size \( \zeta \) and price \( R_{t}^{a,\text{sell}} \).
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7. Finally the process of order submission is completed. The modification of the order book, the best bid price $b_t$, the best ask price $a_t$, the spread $q_t$ and the price $p_t$ will is accomplished according to the different rules presented in chapter 2. Then, at time $t + 1$ we redo step 1 to 7.

This completes the chapter presenting our complete model. In this chapter, we have presented in detail a description of the participants present in our model; the agents and the experts, how these participants are linked to each other via the performance measure and finally, an easy algorithm to clarify the model. In the next chapter we will present the results from the computer simulation of our model.
Chapter 4

Simulation and Results

In this chapter we present the different results obtained by a computer simulation of the model presented in the previous chapters. Section 4.1 give a description of the different parameters that we use for the simulation and section 4.2 presents the results as a series of graphics and tables.

4.1 Setting the parameters

In this section we present the choice of the different parameters for the simulation. In all of the simulations, the parameters mentioned here keep the same value unless it is otherwise mentioned. The only reasons why we do change these parameters are:

- We want to analyze the effect of this particular parameter on the model or
- one parameter is modified to allow a faster execution of the program after we already know what influence this parameter can have on the results.

This remark is important because this allows us to claim that even if the model seems to have an enormous amount of parameters it is still relevant. The different parameters are given in table 4.1 as a quick reference. If needed, these parameters will be explained throughout the chapter after they have been introduced.

---

The main function of our program is given in appendix A. In appendix B, we give a list and a short description of the other functions of the program.
### Table 4.1: Setting the parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Parameters Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of agent</td>
<td>( N )</td>
<td>( N = 100 )</td>
</tr>
<tr>
<td>Number of expert</td>
<td>( M )</td>
<td>( M = 20 )</td>
</tr>
<tr>
<td>Excess demand coefficient</td>
<td>( c_f^i(\omega) )</td>
<td>( c_f^i(\omega) = c^\alpha ) a constant on ( \mathcal{U}(110,120) )</td>
</tr>
<tr>
<td>Random liquidity demand</td>
<td>( \eta_f^i(\omega) )</td>
<td>follows a normal law ( \mathcal{N}(0,c^\alpha) )</td>
</tr>
<tr>
<td>Discount factor</td>
<td>( \gamma^\alpha )</td>
<td>a constant on ( \mathcal{U}(0.05,0.1) )</td>
</tr>
<tr>
<td>Time horizon parameter</td>
<td>( \nu^\alpha )</td>
<td>( \nu^\alpha = 30 ) ( \forall \alpha \in \mathbb{A} )</td>
</tr>
<tr>
<td>Reserve price parameter</td>
<td>( \xi )</td>
<td>( \xi = 0.1 )</td>
</tr>
<tr>
<td>Expert fundamentalist coefficient</td>
<td>( \alpha_f^i(\omega) )</td>
<td>( \alpha_f^i(\omega) = \alpha^\beta ) a constant on ( \mathcal{U}(1,2) )</td>
</tr>
<tr>
<td>Expert chartist coefficient</td>
<td>( \beta_f^i(\omega) )</td>
<td>( \beta_f^i(\omega) = \beta^\beta ) a constant on ( \mathcal{N}(0,1) )</td>
</tr>
<tr>
<td>Tick size</td>
<td>( \Delta )</td>
<td>( \Delta = 0.1 )</td>
</tr>
<tr>
<td>lowest price</td>
<td>( p_{\min} )</td>
<td>( p_{\min} = 500 )</td>
</tr>
<tr>
<td>highest price</td>
<td>( p_{\max} )</td>
<td>( p_{\max} = 1500 )</td>
</tr>
<tr>
<td>Initial price</td>
<td>( p_0 )</td>
<td>( p_0 = 1000 )</td>
</tr>
<tr>
<td>Initial fundamental value</td>
<td>( F_0^i )</td>
<td>( F_0^i = 1000 ) ( \forall i \in \mathbb{I} )</td>
</tr>
<tr>
<td>fundamental value volatility</td>
<td>( \delta^i )</td>
<td>a constant on ( \mathcal{U}(10,18) )</td>
</tr>
<tr>
<td>Interval to compute trading volume</td>
<td>( \tau )</td>
<td>( \tau = 10 )</td>
</tr>
<tr>
<td>Market liquidity</td>
<td>( \lambda )</td>
<td>( \lambda = 1 )</td>
</tr>
</tbody>
</table>

In all of the cases where we say a constant on \( \mathcal{U}(a,b) \) we mean that at the beginning of the program we chose a number uniformly on the interval \([a,b]\) for the concerned participants. For example, \( c^\alpha \) a constant on \( \mathcal{U}(110,120) \) means that for each \( \alpha \in \mathbb{A} \) we choose a value \( c^\alpha \) uniformly on the interval \([110,120]\) and this value stays the same for this entire simulation. The same comment applies for the constant on \( \mathcal{N}(a,b) \) but we choose the number from a normal law with parameters \((a,b)\).

The parameters \( N, M, c_f^i(\omega), \eta_f^i(\omega), \gamma^\alpha, \nu^\alpha, \xi, \alpha_f^i(\omega) \) and \( \beta_f^i(\omega) \) were introduced in the previous chapters. The parameters, \( \Delta, p_{\min}, p_{\max}, p_0, F_0^i \) and \( \delta^i \) are needed for the computer simulation and are explained below. The parameter \( \tau \) is related to the calculation of certain results and is explained in the subsection 4.2.2. Finally, the parameter \( \lambda \) is an artificial parameter introduced and explained in the subsection 4.2.3.

The parameters \( \Delta, p_{\min} \) and \( p_{\max} \) are used to create a price grid on which the agents will submit their orders. This grid is simply a vector taking values between 500 and 1500 by increment of 0.10.

The parameter \( p_0 \) simply gives the initial value of the stock price and the parameter \( F_0^i \) gives the initial fundamental value of expert \( i \in \mathbb{I} \).

The parameter \( \delta^i \) is used to model the evolution of the fundamental value of the expert \( i \in \mathbb{I} \). For the simulation, we model the evolution of the fundamental value as follow:

\[
F_t^i = F_{t-1}^i + 0.05(m_t - F_{t-1}^i) + \delta \mathcal{N}(0,1)
\]  

(4.1)
where we recall that $m_t$ is the middle of the order book defined by equation 2.19. The initial fundamental value is given by $F_0^i = 1000 \ \forall i \in I$. We present in Figure 4.1 a typical path for the fundamental value using equation 4.1.

![Fundamental Value Path](image)

Figure 4.1: Typical fundamental value path

This completes the section 4.1. In the next section, we present the different results.

4.2 Results

In this section we present the different results obtained by the simulation of our model. When this is possible, we compare some of them with empirical studies or with other simulation models.

4.2.1 Shape of the Order Book

In the first series of graphics we are interested in the shape of the order book.

The Figure 4.2 represents a typical order book at the end of the simulation. At this point, it is important to note that the book attains two maxima away from the best bid and best ask price which is just below a thousand and just over a thousand respectively. Moreover, we also note that there are orders far away from the best bid and the best ask,
some as low as 800 and some as high as 1200. Finally, we note a certain symmetry of the book. These three facts have been found also for real stock market empirically by Bouchaud, Mezard and Potters (2002) and by Bouchaud and Potters (2003). As noted by these authors, the long tail of the book suggest that some agents expect a large variation in the price or have a very long time horizon. Contrarily to what Luckock (2003) suggests, we do not see a price window at the end of which orders accumulate in the order book. This is normal and it is because in our model the agents have a finite time horizon at the end of which they cancel the unexecuted part of their orders. All the orders that are placed too far from the best bid and ask prices are eventually cancelled by the agent that place them.

![Order Book at the End](image)

Figure 4.2: A typical representation of the order book at the end of the simulation. We see from this figure that the order book attains two maxima away from the best bid and best ask price. We also note that there are some orders far away from the best bid and best ask price.

Because the Figure 4.2 is only a representation of the order book at the end of the simulation, we cannot confirm that the shape of the order book is similar during the simulation itself. In order to speak about the shape of the order book for the entire simulation we introduce the concept of average order book.

**Definition 4.2.** The average order book is simply an average of all the order books represented during the entire simulation.
Chapter 4. Simulation and Results

We show in Figure 4.3 that the average order book has in fact the same shape than the order book at the end of the simulation and indeed our conclusions about the shape of the book still hold. We have not yet verified but we claim in fact that the shape of the average order book converges to a unique order book configuration for a specific set of parameters.

![Figure 4.3: Average order book](image)

Figure 4.3: We see from this figure that the order book attains two maxima away from the best bid and best ask price. We also note that there are some orders far away from the best bid and best ask price.

This completes the section presenting the results related to the shape of the order book.

4.2.2 The Volatility in the Asset Prices

The second series of graphics and tables explain a part of the asset price volatility by using other parameters of the model. Several authors have argued that the high volatility in the price is associated with large trading volume (see Gabaix et al. (2003)) or large gap in the book, i.e., large bid and ask spread (see Farmer et al. (2004)) and small number of orders in the book (see Weber and Rosenow (2006)). In order to verify these three affirmations we need to introduce the following definitions.

**Definition 4.3.** The trading volume between time $t$ and $t - \tau$, denoted $V_{t-\tau}^t$, is given by:

$$V_{t-\tau}^t = \sum_{i=t-\tau}^t V^i$$  \hspace{1cm} (4.4)
where \( V^i \) is the trading volume at time \( i \).

**Definition 4.5.** The volatility between time \( t \) and \( t - \tau \), denoted \( \sigma_{t-\tau}^t \), is simply the price variance between period \( t - \tau \) and \( t \).

**Definition 4.6.** The cumulative bid and ask spread between time \( t \) and \( t - \tau \), denoted \( q_{t-\tau}^t \), is given by:

\[
q_{t-\tau}^t = \sum_{i=t-\tau}^{t} q^i
\]

(4.7)

where \( q^i \) is the bid and ask spread at time \( i \).

**Definition 4.8.** The returns between time \( t - \tau \) and \( t \), denoted \( \chi_{t-\tau}^t \), are given by:

\[
\chi_{t-\tau}^t = \frac{p_t - p_{t-\tau}}{p_{t-\tau}}
\]

(4.9)

The Figure 4.4 illustrates the relation between the trading volume and the price volatility.

---

Figure 4.4: Volatility and trading volume. We note a positive correlation between the price volatility and the trading volume. Each big spike from the volatility figure seems to be accompanied by spike from the trading volume figure.

As it is easy to see from the figure, there is a clear positive correlation between the trading volume and the price volatility. More precisely, we found a correlation of 0.4375
between these two variables. We present this result differently in Figure 4.5 by showing the price evolution and the trading volume on the same graphic.

Figure 4.5: The relation between the price and the volume. We can see from this figure that large price movements are usually accompanied with high trading volume.

The two last figures and the correlation found suggest that the periods of high price volatility are usually accompanied by periods of large trading volume.

The figure 4.6 illustrates the relation between the spread and the price volatility.
Figure 4.6: Volatility and cumulative bid and ask spread. We note a positive correlation between the price volatility and the cumulative bid and ask spread.

The figure suggests a positive correlation between the spread and the price volatility. More precisely, we found a correlation of 0.4099 between these two variables. We present this result differently in Figure 4.7 by showing the price evolution and cumulative bid and ask spread on the same graphic.
Figure 4.7: The relation between the price and the bid and ask spread. The figure illustrates that the large price movements and large values of the bid and ask spread usually happen at the same time.

The two last figures and the correlation found suggest that the periods of high price volatility are usually accompanied by periods with a large cumulative bid and ask spread. This is absolutely coherent. When the bid and ask spread is large, the price will vary significantly every time the orders change sign. If the order is a buy order the given price will be the best ask price and if the next order is a sell order than the given price will be the best bid price. If the bid and ask spread is large it means that the difference between the best bid and best ask price is large and consequently there are large price variations each time the order change sign. This means that we could use the bid and ask spread to partially predict the price volatility.

In Figure 4.8 we present the relation between the returns and the price volatility. The figure suggests a positive correlation between these two variables. More precisely, we found a correlation of 0.4876 between the absolute value of the returns and the price volatility. This result is nothing surprising because large absolute returns arrive only when there are large price fluctuations. Indeed, an investor who is seeking large returns over a short period of time should choose to invest in a stocks with large volatility. One has to be careful with this statement because large returns in absolute value also means large losses.
Figure 4.8: Volatility and the return. We note a positive correlation between the variability of the returns and the volatility of the price.

Finally, we present in Figure 4.9 the relation between the number of orders in the order book and the asset price volatility.
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Figure 4.9: Volatility and Number of Orders. We note a clear negative correlation between the number of orders in the book and the price volatility.

The figure suggests a negative correlation between these two variables. More precisely, we found a correlation of $-0.4613$. The negative sign is coherent with the study of Weber and Rosenow (2006). In this paper, the authors suggest that we can not explain large price movements only with the trading volume but we also need to consider the number of orders in the book. When the number of orders in the book is small than large orders can make the price vary greatly because we need to move further in the book to execute these orders entirely. This result is important because it contributes to the debate of how much information about the order book should be displayed to the different economic agents. An agent who knows in addition to the best bid and best ask price the number of orders in the book has an obvious advantage than the agent who only knows the best bid and best ask price. This is particularly relevant in the case of large order size. Knowing that the number of orders in the book is low when someone wants to place a very large order is relevant because this agent can anticipate that her order might creates large price movements. At this point, the agent can decide to submit her order anyway or to divide her order in several small orders to minimize her impact on the price.

This completes the results section related with the different possible explanations about the stock price volatility. In summary, we can say that periods of high price volatility are usually accompanied by periods of large trading volume, large bid and ask spread and relatively small number of orders in the book. In the next section we are interested in the effects of some important parameters.
4.2.3 Some Relevant Parameters

The third series of graphics tries to verify the importance of three specific parameters on the trading volume, the volatility, the spread and the number of bids and asks in the book. The three parameters of interest are the market liquidity \( \lambda \), the tick size \( \Delta \) and the time horizon \( h \) by varying the value of \( v^a \).

For each of these parameters we are considering four different dependent variables: the trading volume given as the average trading volume for the entire simulation, the volatility given as the price variance for the entire simulation, the spread given as an average for the entire simulation and the number of bids and asks also as an average for the entire simulation. We present and comment on the three different graphics obtained and we give the correlation between the different variables in table 4.2.

The market liquidity is an artificial parameter that represents the level of market activity. It represents the probability that the chosen agent will trade at time \( t \). This parameter was also considered in other models (in particular see Chiarella and Iori (2002)). For all other simulation its value is set to one, which means that once an agent is chosen this agent will always trade. For the first graphic of this section, figure 4.10, we vary the value of \( \lambda \) between 0 and 1.

In Figure 4.10 we show the dependence on the market activity \( \lambda \). For the market activity \( \lambda \), it seems completely natural to obtain a quasi perfect correlation between \( \lambda \) and the trading volume. It is the same thing for the number of bids and asks. The correlations with the number of bids and asks in the book are strong and positive, 0.8749 and 0.8574 respectively. This is also perfectly coherent, when the market is more active, the agents submit more limit orders and so the number of bids and asks will increase. The correlation with the volatility is not terribly strong but still positive 0.3967 which is also coherent. If once chosen, the probability that an agent submits an order is higher then we can expect the price volatility to be higher as well. The number that needs some more explanation is the negative correlation with the spread. The important concept to understand here is that when the market becomes more active, the agents are submitting more orders of both types. Because the limit orders will generally reduces the spread it is not incoherent to obtain a negative correlation between the market activity \( \lambda \) and the spread \( q \). The results about the dependence on the market activity are important in the applications of our model to reality. In particular, the relation between \( \lambda \) and the number of bids and asks can be use as an indication of the number of orders in the book even if this information is not directly displayed to the agents.
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Figure 4.10: Dependence on the market activity $\lambda$. We note a strong positive correlation between $\lambda$ and the trading volume and $\lambda$ and the number of bids and asks in the book. We also note a positive correlation between $\lambda$ and the price volatility. Finally, we note a negative correlation between $\lambda$ and the bid and ask spread which we explain further above.
In figure 4.11 we show the dependence on the tick size $\Delta$. For the tick size, in this case, most of the correlation are almost irrelevant (see table 4.2). A priori, this seems to be wrong but it is coherent with other studies for these values of the tick size (see Chiarella and Iori (2002)). These results are really important in terms of market design. It suggests that when the market designers decide the tick size for a particular stock, they can choose a value between 0.01% and 0.5% of the initial value of the stock without significantly affecting the trading volume, the price volatility, the bid and ask spread and the number of bids and asks in the book.

<table>
<thead>
<tr>
<th>Volume as a Function of the Tick Size</th>
<th>Volatility as a Function of the Tick Size</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Volume</strong></td>
<td><strong>Volatility</strong></td>
</tr>
<tr>
<td>11.5</td>
<td>2000</td>
</tr>
<tr>
<td>11</td>
<td>1500</td>
</tr>
<tr>
<td>10.5</td>
<td>1000</td>
</tr>
<tr>
<td>10</td>
<td>500</td>
</tr>
<tr>
<td>9.5</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
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<td></td>
<td>3</td>
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<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spread as a Function of the Tick Size</th>
<th>Bid and Ask as a Function of the Tick Size</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spread</strong></td>
<td><strong>Bids and Asks</strong></td>
</tr>
<tr>
<td>35</td>
<td>Bids</td>
</tr>
<tr>
<td>30</td>
<td>*</td>
</tr>
<tr>
<td>25</td>
<td>*</td>
</tr>
<tr>
<td>20</td>
<td>*</td>
</tr>
<tr>
<td>15</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>Asks</td>
</tr>
<tr>
<td></td>
<td>*</td>
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<td>*</td>
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</tbody>
</table>

Figure 4.11: Dependence on the tick size $\Delta$

<table>
<thead>
<tr>
<th>Table 4.2: The Different Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\Delta$</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
</tbody>
</table>
Chapter 4. Simulation and Results

Finally, in figure 4.12 we show the dependence on the time horizon $h$. The modification of $h$ is made through an increase of the constant parameter $\nu$ (see equation 3.9). For the time horizon, we ignore the correlation with the trading volume and the spread because they are too weak. Nevertheless, one can note that at least the correlation with the spread has the correct sign, i.e., the spread is decreasing in $h$. This is coherent if we consider that if all agents have a longer horizon then the number of limit orders in the book should increase. The correlation with the volatility -0.3141 is not really strong but is also in the right direction. When the agents have a longer time horizon more limit orders are placed by these agents and the price volatility is reduced. For the correlation with the number of bids and asks, it is natural to obtain a positive correlation because if the agents are waiting longer before canceling their limit orders than there are more orders in the order book.

Figure 4.12: Dependence on the time horizon $h$. We note a strong positive correlation between the time horizon $h$ and the number of bids and asks in the book. There is also a negative correlation between the time horizon and the volatility. The other correlations are explained in more details above.

This completes the section about the influence of some relevant parameters. We recall
here the most important findings of this section. First, the results about the dependence on the market activity are similar to results from other studies (see Chiarella and Iori (2002 and 2004)) and have important applications in reality. In particular, the relation between the market activity \( \lambda \) and the number of bids and asks in the book can be used as a proxy of the number of orders in the order book. Second, the results about the tick size are also consistent with other studies (see Chiarella and Iori (2002 and 2004)) and are relevant for market designer. Finally, the results about the time horizon are different from other studies. Once again the relation between the time horizon and the number of bids and asks in the book can be used to estimate the number of orders in the order book. In the next section we study the importance of chartists in our model.

4.2.4 The Effects of the Chartists

This series of graphics attempts to demonstrate the important effects of the presence of chartists in the model. The results found in this section are similar to the one found by other studies (see Chiarella and Iori (2002), Chiarella and Iori (2004), De Long et al. (1990) and Föllmer, Horst and Kirman (2005)).

In Figure 4.13 we show the differences in the evolution of the returns and the evolution of the price in a model with fundamentalists and noise traders only and in a model with fundamentalists, noise traders and chartists. It is easy to see from the figure that the chartists create an increase in the variability of the returns and of the asset price.

\footnote{We need to warn the reader that DeLong et al. (1990) used the term noise trader for what we consider as chartist trader.}
Figure 4.13: Evolution of the price and the returns. The top figure shows a model where there is no chartist present and the bottom figure shows a model where the chartists are present. For the returns we choose $\tau = 50$. We note that the presence of chartists increases the variability of the returns and the price processes.
In Figure 4.14 and Figure 4.15 we look at the returns more closely. Figure 4.14 shows the distribution of the returns of the model without chartists, the model with chartists and a normal distribution with the same mean and variance than the model without chartists. We see that the presence of chartists definitively fattens the tail of the returns distribution. This result was also found in the more theoretical model of Föllmer, Horst and Kirman (2005).

Figure 4.14: Return distributions. On the same graphic we represent the model without chartist, the model with chartists as well as a normal distribution with the same mean and variance than the model without chartist. For the returns we choose \( \tau = 10 \). We note that the presence of chartists fattens the tails of the returns distribution.
Chapter 4. Simulation and Results

In Figure 4.15 we compare the returns distribution of the model with chartist and a normal distribution with the same mean and variance. Once again, we see that the distribution of the model has a fatter tail than the normal distribution.

Figure 4.15: Return distribution. On the same graphic we represent a normal distribution with the same mean and variance than the return distribution of our complete model. For the returns we choose $r = 10$. Once again, it is easy to note that the return distribution of our model has a fatter tail than a normal distribution.

The two last figures suggest that in the presence of chartists, the returns are not normally distributed. Empirical studies have demonstrated that the returns are not normally distributed in reality. This is why the presence of chartists is essential in the modelization of financial market. To confirm the fact that our return distribution is not a normal distribution we present a normal plot of our returns in the figure 4.16.
Chapter 4. Simulation and Results

Figure 4.16: Normal plot of the returns distribution for $\tau = 10$. The dashed line represents the theoretical distribution while the + signs represent our data. It is easy to see from this figure that our returns are not normally distributed. If it were the case, the + signs would all be on the dashed straight line.

At this point, maybe one can think that this is true for the particular value of $\tau = 10$. We prove that it is not the case in figure 4.17 showing another normal plot but this time with $\tau = 1$. In fact, we have verified that our returns distributions are not normal for $\tau \in [1, 1000]$. This result is important because empirical studies also found that the returns are not normally distributed for a wide range of $\tau$ (see Plerou et al. (2002)).
This completes the subsection about the effects of chartists. In the next subsection we want to make more precise comparison of the tail of our returns distributions to some theoretical distribution.

4.2.5 Power-law Distribution

In this subsection, we are interested to compare the tail of our returns distributions to the tail of a power-law distribution.

Some empirical studies have found precise expressions for the tail of the returns distribution. In particular, Gabaix et al. (2003) have found empirically that the tail of the return distributions follows a power-law with an exponent \( \zeta = 3 \). Another study by Plerou et al. (2002) found similar results with an exponent between \( \zeta = 2 \) and \( \zeta = 4 \). The precise statement is as follow. We first introduce the following definition.

**Definition 4.10.** The log returns between period \( t \) and \( t - \tau \), denoted \( r_{t-\tau} \), are given by

\[
    r_{\tau} = \ln(p_t) - \ln(p_{t-\tau})
\]  

(4.11)
Then these authors found that the probability that a return has an absolute value larger than \( x \) is
\[
P(|r_{t-1}| > x) \sim c x^{-\zeta}
\] (4.12)
where \( c \) is a constant.

This result is found to be quite robust as it is holding for up to 80 standard deviations for some stocks in the market and it is holding for values of \( \tau \) that range between one minute up to one month. The actual reason of such phenomenon is a subject of disagreement between the different researchers. Our goal for the moment is not to explain such phenomenon but verify if our model can reproduce it.

We present in Figure 4.18 a representation of the tail of our distribution and the tail of a distribution given by equation 4.12 with \( \zeta = 3 \).

---

\(^{15}\)See Plerou et al. (1999) for more details.

Figure 4.18: Tail of the return distribution compared to a power law with $\zeta^* = 3$ and $\tau = 50$. The solid line represents the power tail with $c = 1$, the dashed line represents a translation of the power tail by modifying the value of $c$ and the curve represents the tail of our model. The axis are in a logarithmic scale and as a consequence the power law is represented by a straight line with slope -3. We note from this figure that there is a non trivial set of parameters for which the tail of our distribution agree with the tail of a power law distribution.

We present on Figure 4.19 the set of parameters for which the power law and our distribution agrees.
Chapter 4. Simulation and Results

Figure 4.19: Tail of the return distribution compared to a power law with $\zeta^r = 3$ and $\tau = 50$. The x axis values are between 5 and 15. The thick gray line represents the power tail with $c = 1$, the black dashed line represents a translation of the power tail by modifying the value of $c$ and the thin black solid line represent the tail of our model. The axis are in a logarithmic scale and as a consequence the power law is represented by a straight line with slope -3. We see easily with this figure that our distribution agree almost perfectly, after a translation, with a power law for these values of returns.

We are now interested to verify if the tail of our return distribution agree with a power-law for a wider range of values of $\tau$. We verify the robustness of our result in Figure 4.20 where we show the tail of the returns distributions for different choice of $\tau$. 
Figure 4.20: Tail of the return distribution for different values of $\tau \in [1, 1000]$. The solid line represents the power tail with $\zeta' = 3$ and $c = 1$, the dashed line represent a translation of the power tail by modifying the value of $c$ and the curve represent the tails of our model for different values of $\tau$. The axis are in a logarithmic scale and as a consequence the power law is represented by a straight line with slope $-3$. Once again, we note from this figure that there is a non trivial set of parameters for which the tail of our distribution agree with the tail of a power law distribution. Moreover, we note this result seems to be robust to the variation of $\tau$.

Once again, this confirm that the tail of our return distributions agree with a power law with exponent $\zeta' = 3$ for a non trivial set of parameters (see figure 4.21). Finally, this figure also suggest that the result is independent of $\tau$ for $\tau \in [1, 1000]$. This is consistent with the empirical literature, as noted by Plerou et al. (1999).
Figure 4.21: Tail of the return distribution for different values of $\tau \in [1, 1000]$. The solid gray line represents the power tail with $\zeta^* = 3$ and $c = 1$, the dashed gray line represents a translation of the power tail by modifying the value of $c$ and the black curves represent the tails of our model for different values of $\tau$. The axis are in a logarithmic scale and as a consequence the power law is represented by a straight line with slope -3. Once again, we note from this figure that there is a non trivial set of parameters for which the tail of our distribution agree with the tail of a power law distribution. Moreover, we note this result seems to be robust to the variation of $\tau$.

Discussion 4.13. It is important to note that even if the tails of the returns distributions are similar for different values of $\tau$, the actual returns distributions can look quite different. We illustrate this idea in figure 4.22 where we present two returns distributions for two different values of $\tau$. 
Chapter 4. Simulation and Results

Figure 4.22: Returns distribution with different values of \( \tau \). The solid line represents the returns distribution for \( \tau = 1 \) and the dashed line represents the returns distribution for \( \tau = 10 \). The strange shape of the distribution with \( \tau = 10 \) is due to the adjustment made by putting the two distributions on the same figure.

This completes the subsection related to the comparison with a power-law distribution. In summary, we can say that the tails of our returns distributions agree with the tail of a power law for a non trivial set of returns values. Moreover, this result is independent of the value of \( \tau \) at least for \( \tau \in [1,1000] \). In the next subsection we present the results related with the stationary distributions for the price and the returns processes.

4.2.6 In Search of the Stationary Distribution

In this section we present the main result of this project. We are interested in finding the unique and stationary distribution for the price process. As mentioned in the introduction, this result has been found analytically in a Walrasian market framework by Föllmer, Horst and Kirman (2005). The goal is to verify numerically if this result can also be found in the continuous double auction framework.

One of the problems of such verification is that we cannot let the time go to infinity in our simulation because the computation will never finish. In our context, letting the time
Chapter 4. Simulation and Results

go to infinity means repeating the steps 1 to 7 given in the section 3.5 forever. Instead, to
test the stationarity of our price and returns time series we use a statistical test.\(^\text{17}\)

The usual method used for testing the stationarity of a time series data is the Kwiatkowski,
Phillips, Schmidt and Shin test, short to KPSS test, as explained in Kwiatkowski, Phillips,

Discussion 4.14. *It is important to note that the KPSS test is better than the Dickey-
Fuller (DF) or the Phillips and Perron (PP) tests. These two tests use the series follows a unit root process as their null hypothesis \(H_0\). The KPSS test uses the series is stationary as the null hypothesis \(H_0\) and it then constitutes a better method to test stationarity.* \(^\text{18}\)

We have performed the KPSS test for the price and the returns using the free statistical
software \(R\). In both cases, we cannot reject the null hypothesis that our time series is
stationary. \(^\text{19}\) This confirms our intuition that changing the market organization from the Walrasian framework to the continuous double auction does not change the stationarity of
the price process.

Unfortunately, verifying the uniqueness of the two distributions causes some problems.
We present here an idea of how this could be done correctly. First, we need to choose the
number of time we want to repeat the steps 1 to 7 presented in section 3.5. We have decided
to choose 50000 which already require approximately 4 hours of computation. There is no a
priori indication of how big this number should be. In fact, this particular choice is related
to the computation time more than any other reasons. Second, we need to repeat this
computation several times in order to create a sample of stationary distributions. Third,
one this sample is complete, we could build a statistical test to verify the true mean,
variance and higher order moments of the potential stationary distribution. Unfortunately,
this is not necessarily feasible. In order to obtain a good statistical test we would need a
relatively large sample because the size of the sample is inversely proportional to the square
of the margin error of the test (see McCabe and Moore 1999, chapter 6). This causes a
problem with the computation time of our model. As an example, obtaining a sample size
of 1000 would require approximately 166 days of computation. This does not seem to be
easily feasible.

Instead, we decide to present a graphic on which we plot the price distribution every
1000 steps of our 50 000 iteration to show that the distribution of the price become closer
and closer to the distribution at the very end of the simulation. This is done in the Figure
4.23.

\(^{17}\)For a review about statistical test in general we refer the reader to McCabe and Moore (1999) and

\(^{18}\)See Dickey and Fuller (1979), Phillips and Perron (1988) and Kwiatkowski, Phillips, Schmidt and Shin

\(^{19}\) The complete details of our tests is given in appendix C.
Figure 4.23: The Price Stationary Distribution. The solid line represents the price distribution at the end of the simulation while the dotted lines represent the price distribution every 1000 steps. We see that the distributions gradually approaches the distribution at the end of the simulation with less significant differences each time.

We present the same type of graphic in Figure 4.24 for the price returns.
Figure 4.24: Returns stationary distribution. The circles represent the price distribution at the end of the simulation while the dotted lines represent the price distribution every 1000 steps. We see that the distributions gradually approach the distribution at the end of the simulation with smaller differences each time. We choose $\tau = 1$. One would obtain another distribution with a different $\tau$ but it will still be unique for each choice of $\tau$. For this reason the price distribution is more interesting because it does not require us to precise the choice of $\tau$.

These two graphics obviously do not constitute a rigorous statistical test nor than a rigorous mathematical proof. On the other hand, they confirm the intuition that even in a continuous double auction market model there seems to exist a unique stationary distribution for the price process which justifies further analytical research in this direction.

This completes the chapter on the computer simulation and the numerical results. In the next chapter, we present the conclusion of this thesis.
Chapter 5

Discussion and Conclusions

5.1 Discussion

In this section we recall the model developed in this thesis, some of the main important assumptions made, the main goals and the different techniques used to achieved these goals.

We have presented in the previous chapters a model of a financial market with many heterogeneous agents in a continuous double auction market organization. We have introduced the different concepts related to continuous double auction and electronic limit order book. We have constructed a financial market model with heterogeneous agents that are using this electronic order book to trade. Agents submit orders according to a Poisson dynamics with constant parameter \( \mu \). Each agent decides to buy or to sell the stock according to the recommendation of a financial expert. These recommendations represent the expert expected price of the stock for the next period. There are three types of financial experts: the noise traders, the fundamentalist traders and the chartist traders. The recommendation from the noise traders are random, the recommendation from the fundamentalists are based on some fundamental value of the stock price while the recommendation of the chartists are based on some extrapolatory or contrarian rules. The agents are choosing the financial expert by comparing each of them using a performance measure. In our case, they compare a discounted sum of past profits their recommendations would have generated. Once the financial expert is determined, the agent forms her excess demand and then executes her trade according to the sign of her excess demand. The price is then determined using the different rules governing the order book dynamics.

One of the most important assumption is related with the information available to the agents. Each agent has a full knowledge of the past history of the price process and all the past recommendation of every experts present in the market. Moreover, our model makes the assumption that the agents are able to use this information properly to calculate the performance measure of each expert. This represents a limitation of our model.

Our principal goal was to verify numerically the existence and uniqueness of a stationary distribution for the price process. We also verify that our model can reproduce some of the phenomena found in financial time series.

Finally, to attain these goals we have used computer simulation, statistical techniques (correlation, tests, KPSS, etc.) and lots of graphical representation of our results. In the next section, we recall the results obtained.
Chapter 5. Discussion and Conclusions

5.2 Conclusion

We present in this section the main results obtained by the computer simulation.

First, we recall the interesting results about the shape of the order book. The order book has two local maximums away from the best bid and best ask price which was also found in empirical studies (see Bouchaud and Potters (2003)). Second, we have found a positive correlation between the stock price volatility and trading volume and cumulative bid and ask spread (0.4375 and 0.4099 respectively). We also found a negative correlation (-0.4613) between the price volatility and the number of order in the book. This negative correlation is consistent with the explanation of Weber and Rosenow (2006) that the price volatility cannot be entirely explained by the trading volume but that the number of orders in the book is also important. Third, we have found interesting relationships between the market activity, the tick size and the time horizon of the agents and other important variables. In particular, we recall the strong positive relations between the market activity and the trading volume and the number of bids and asks in the book. We also recall the positive relation between the agent time horizon and the number of bids and asks in the book. Fourth, we have illustrated using graphics the important effects of the presence of chartists in our model. In particular, the presence of chartists increases the variability of the price and the returns processes and as a consequence fattens the tail of the returns distribution compare to a model without chartists. Fifth, we compare our return distribution with a power law and found that the tail of our distribution agree with a power law for a non trivial set of parameters. This is an important result because empirical studies have shown that the tail of returns distribution usually follow a power law (see Plerou et al. (2002) and Gabaix et al. (2003)). Finally, we have verified that the price distribution is stationary using the KPSS statistical test. We claim that the this distribution is unique even if the numerical verification was not make here because of the computing time it requires.

In conclusion, we want to point out the importance of heterogeneity, the effects of the presence of chartists and the existence of a stationary distribution for the price process. This last result is essential because it justifies that further efforts should be done to prove analytically the existence of such distribution for the continuous double auction market organization. We present in the next section other directions that further research on this topics can take.

5.3 Further Research

Even if at the first glance the model presented in this thesis seems complete and the results numerous, there are many ways one can do further research related to this thesis. We separate them in two categories: statistical and simulation researches and theoretical research.

For the statistical and simulation research, we suggest the following extensions.
Chapter 5. Discussion and Conclusions

- Try to find predictive power of certain variables by verifying correlation with variables at different times.

- Try to compute the market impact function (found in the model of Gabaix et al. (2003) and also mentioned in Farmer and Lillo (2003)).

- We need further comparison of the model and the results found in the empirical literature.

- There is also a need for a calibration exercise to be able to use the model for predictions.

- One could consider more types of orders: stop loss order, stop limit orders, etc.

- There are many relations found in financial data that have not been analyzed in the current project and indeed many other graphic results could be created.

- The excess demand could come from an optimization problem as in Chiarella and Iori (2004).

- The dependence of the results on other parameters should also be verified.

For the theoretical research, we suggest the following extensions.

- Analyze an optimal order type choice (between market and limit orders) as it is done in Luckock (2003).

- Do the mathematical analysis for the existence and the uniqueness of the stationary distribution for this market organization as it is done by Föllmer, Horst and Kirman (2005) for the temporary Walrasian equilibrium framework.

This completes this thesis. We hope that the reader enjoyed the presentation of "A Study of Financial Markets with Heterogeneous Agents: A Numerical Approach."
Bibliography


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Appendix A

Main Function of the Program

In this chapter, we present the main function of the our program made with Matlab®. This does not constitute the entire program but it does give a good idea to the reader of how the program works. We remind the reader that text following the % is a comment in the Matlab® environment.

function order_book_new_graph1(repetition);

% This function is the main function of the project.

tic
% This function is used to calculate the computation time.

% Throughout the entire code, the matrices have name in capital letters and the vectors name always start with a "v_".

%XXXXXXXXXXXXXXXXXX INITIALIZATION OF THE MOST IMPORTANT PARAMETERS %XXXXXXXXXXXXXXXXXX

N = 100;
% - Place: In the main
% - Explanation: This is the number of agents that will be trading in the
% market.
% - Effects: see explanation
% - Suggested values: [10,10000]

M = 20;
% - Place: In the main
% - Explanation: This is the number of experts for which the agents will be
% able find a suggested value for the price the next period.
% - Effects: see explanation
% - Suggested values: [10;N]

nu = 30;
% - Place: In the function initialization

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Appendix A. Main Function of the Program

% - Explanation: This parameter is related to the horizon $h$ of each agent.
% - Effects: An increase of $\nu$ will increase the time horizon of every
% agent in the model.
% - Suggested values: [1,10]

$\lambda = 1$;
% - Place: In the main
% - Explanation: This parameter is related to the rate at which orders are
% submitted.
% - Effects: see explanation
% - Suggested values: [0,1]

$\text{low}_c = 10$;
% - Place: In the function initialization
% - Explanation: This parameter is the lower bound for the excess demand
% - parameter $c$.
% - Effects: An increase of $\text{low}_c$ will increase the excess demand for every
% - agents and will probably create more activities in the market.
% - Suggested values: [10,50]

$\text{high}_c = 15$;
% - Place: In the function initialization
% - Explanation: This parameter is the higher bound for the excess demand
% - parameter $c$.
% - Effects: An increase of $\text{high}_c$ will increase the excess demand for every
% - agents and will probably create more activities in the market.
% - Suggested values: [$\text{low}_c+5,100$]

$\text{low}_\gamma = 0.05$;
% - Place: In the function initialization
% - Explanation: This parameter is the lower bound for the discount factor
% of each agent.
% - Effects: An increase of $\text{low}_\gamma$ will increase the time horizon of the
% agents.
% - Suggested values: [0.01,0.5]

$\text{high}_\gamma = 0.1$;
% - Place: In the function initialization
% - Explanation: This parameter is the higher bound for the discount factor
% of each agent.
% - Effects: An increase of $\text{high}_\gamma$ will increase the time horizon of the

80
% agents.
% - Suggested values: [low_gamma+0.0.99]

% NOTE: The 2 parameters low_gamma and high_gamma have to be strictly less
% than 1.

low_delta = 10;
% - Place: In the function initialization
% - Explanation: This parameter is the lower bound for the volatility of
% - the benchmark value of every experts.
% - Effects: An increase of low_delta will increase the volatility of the
% - benchmark value for each experts and will probably create more
% - noise in the model.
% - Suggested values: [10,20]

high_delta = 18;
% - Place: In the function initialization
% - Explanation: This parameter is the higher bound for the volatility of
% - the benchmark value of every experts.
% - Effects: An increase of high_delta will increase the volatility of the
% - benchmark value for each experts and will probably create more
% - noise in the model.
% - Suggested values: [low_delta+0.5,30]

low_alpha = 1;
% - Place: In the function initialization
% - Explanation: This parameter is the lower bound for the strength or the
% - speed at which the expert expect the price to go back to
% - his fundamental value.
% - Effects: An increase of low_alpha will create higher variation from the
% - variation between the price and the fundamental value of the
% - expert in the calculation of the expert's recommendation.
% - Suggested values: [1,5]

high_alpha = 2;
% - Place: In the function initialization
% - Explanation: This parameter is the higher bound for the strength or the
% - speed at which the expert expect the price to go back to
% - his fundamental value.
% - Effects: An increase of high_alpha will create higher variation from the
Appendix A. Main Function of the Program

\begin{verbatim}
\% variation between the price and the fundamental value of the
\% expert in the calculation of the expert's recommendation.
\% - Suggested values: [low_alpha+1,10]

beta_parameter = 1;
\% - Place: In the function initialization
\% - Explanation: This parameter is the strength of the trend chasing in the
\% calculation of the expert performance measure.
\% - Effects: An increase of parameter_beta will create higher variation from
\% the price difference in the expert's recommendation.
\% - Suggested values: [1,5]

expert_horizon1 = 10;
\% - Place: In the function initialization
\% - Explanation: This parameter enter in the calculation of the expert's
\% horizon and his not related to the type of the expert.
\% - Effects: An increase of expert_horizon1 will make all the experts to
\% change their fundamental values less often.
\% - Suggested values: [5,25]

expert_horizon2 = 10;
\% - Place: In the function initialization
\% - Explanation: This parameter enter in the calculation of the expert's
\% horizon and his related with fundamentalist experts.
\% - Effects: An increase of expert_horizon2 will make experts with a
\% fundamentalist part to increase their time horizon, and make
\% them change their fundamental value less often.
\% - Suggested values: [1,15]

expert_horizon3 = 10;
\% - Place: In the function initialization
\% - Explanation: This parameter enter in the calculation of the expert's
\% horizon and his related with chartist experts.
\% - Effects: An increase of expert_horizon3 will make experts with a
\% chartist part to increase their time horizon, and make
\% them change their fundamental value less often.
\% - Suggested values: [1,15]

prop_fund = 0.5;
\% - Place: In the function initialization
\% - Explanation: This parameter represent the proportion of pure
\end{verbatim}
Appendix A. Main Function of the Program

7, fundamentalist (beta=0) within the experts.
7, - Effects: An increase of prop_fund will increase the number of experts
7, that follow only fundamentalist rules. This should probably
7, reduce the appearance of bubbles and crashes.
7, - Suggested values: [0.1,0.8]

prop_fund = 0.2;
7, - Place: In the function initialization
7, - Explanation: This parameter represent the proportion of pure chartist
7, (alpha=0) within the experts.
7, - Effects: An increase of prop_fund will increase the number of experts
7, that follow only chartist rules. This should probably
7, increase the appearance of bubbles and crashes.
7, - Suggested values: [0.0,0.5]

prop_chart = 0.2;
7, - Place: In the function initialization
7, - Explanation: This parameter represent the proportion of pure chartist
7, (alpha=0) within the experts.
7, - Effects: An increase of prop_chart will increase the number of experts
7, that follow only chartist rules. This should probably
7, increase the appearance of bubbles and crashes.
7, - Suggested values: [0.0,0.5]

prop_noise = 0.3;
7, - Place: In the function initialization
7, - Explanation: This parameter represent the proportion of pure chartist
7, (alpha=0 and beta=0) within the experts.
7, - Effects: An increase of prop_noise will increase the noise in the
7, model. Most likely that will get the return distribution
7, closer to a normal distribution.
7, - Suggested values: [0.0,0.5]

NOTE: The last 3 parameters are not independent of each other. In
7, particular their sum must never be higher than 1 but it can be strickly
7, less than one without problem.

low_performance = 0.01;
7, - Place: In the function performance_measure2
7, - Explanation: This parameter is the lowest bound at which we stop adding
7, the profit when we are calculating the performance measure
7, of each expert. The goal of this parameter is to make the
7, code faster.
7, - Effects: An increase of low_performance will eventually make the code
7, faster.
7, - Suggested values: [0.0001,1]

lowest_price = 500;
7, - Place: In the function initialization
7, - Explanation: This parameter represents the lowest bound for the price.
Appendix A. Main Function of the Program

highest_price = 1500;
% - Place: In the function initialization
% - Explanation: This parameter represents the highest bound for the price.
% - Effects: see explanation
% - Suggested values: [lowest_price + 1000, ??]

tick_size = 0.5;
% - Place: In the function initialization
% - Explanation: This parameter represents the tick size.
% - Effects: Will create a bigger variation in the price and a bigger value
% for the gap in the order book.
% - Suggested values: [0.5, 2]

step_size1 = 1000;
% - Place: In the main
% - Explanation: This parameter is related with the creation of the
% vector_return. In a certain way these returns represent the
% returns of agent with a long time horizon
% - Effects: Raising it will certainly change the first returns graphic at
% the end of the code.
% - Suggested values: [50, 1000]

step_size2 = 1;
% - Place: In the main
% - Explanation: This parameter is relation with the creation of the
% vector_return. In a certain way these returns represent the
% returns of agent with a short time horizon.
% - Effects: Raising it will certainly change the first returns graphics at
% the end of the code.
% - Suggested values: [1, 200]

interval = 10;
% - Place: In the main
% - Explanation: This parameter determines on which interval the volatility and
% the volume will be computed.
% - Effects: Unknown
% - Suggested values: [5, 100]
Appendix A. Main Function of the Program

```matlab
lag = max(step_size1, step_size2) + 1;

[AGENTS, CANCEL, EXPERTS, v_x, middle] = initialization(N, M, lowest_price, ...
  highest_price, tick_size, nu, low_c, high_c, low_delta, high_delta, ...
  prop_fund, prop_chart, prop_noise, low_alpha, high_alpha, beta_parameter, ...
  expert_horizon1, expert_horizon2, expert_horizon3, low_gamma, ...
  high_gamma);

% The function initialization initializes the matrices AGENTS, CANCEL and ...
% EXPERTS as well as the value of c and the vector v_x. For more ...
% information about these 3 matrices refer to the respective legends below.

%%%%%%%%%%%%%%%%%%%%% LEGEND MATRIX AGENTS %%%%%%%%%%%%%%%%%%%%
% % %
% AGENTS(a, 1) = c excess demand parameter of agent a % %
% AGENTS(a, 2) = gamma discount factor of agent a % %
% AGENTS(a, 3) = h time horizon of agent a % %
% AGENTS(a, 4) = nu excess demand random parameter % %
% AGENTS(a, 5) = teta rate of market participation % %
% AGENTS(a, 6) = tau timer for the cancelation in case the agent a % %
% has some limit orders in the book % %
% NOTE: The three parameters nu, teta and tau are time dependent and they %
% will need to be change at every step of the main loop. % %

%%%%%%%%%%%%%%%%%%%%% LEGEND MATRIX CANCEL %%%%%%%%%%%%%%%%%%%%
% % %
% CANCEL(i, 1) = i give the position in the matrix CANCEL % %
% CANCEL(i, 2) = n give the position in the order book % %
% CANCEL(i, 3) = n give the number of order at that position % %
% CANCEL(i, 4) = 1 type of limit order: buy % %
% CANCEL(i, 4) = 2 type of limit order: sell % %
```

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Appendix A. Main Function of the Program

\%
\% CANCEL(i,5) = a give which agent submit the order
\%
\% CANCEL(i,6) = tau give the time left before cancelation
\%
\% NOTE: The last parameter tau is time dependent and need to be modified
\% at every step of the main loop.
\%
\%-------------------------------------------------------------------------------------

\% LEGEND MATRIX EXPERTS
\%
\% EXPERTS(i,1) = delta benchmark "volatility" parameter
\% EXPERTS(i,2) = alpha fundamentalist strength parameter
\% EXPERTS(i,3) = beta chartist strength parameter
\% EXPERTS(i,4) = g time horizon for the expert i
\% EXPERTS(i,5) = t timer for the modification of the benchmark
\% NOTE: The last parameter t is time dependent and need to be modified at
\% every step of the main loop.
\%
\%-------------------------------------------------------------------------------------

l_x = length(v_x);
\% Keep in trac the length of the vector x

v_y = translation(CANCEL,l_x,middle);
\% This function make the conversion between the matrix CANCEL and the
\% vector y. The vector y is the vector that gives you the number of orders
\% in the order book for each prices given by the vector x.

\% FIGURE OF THE ORDER BOOK AT THE BEGINNING
figure; bar(v_x,v_y,0.5);
\% Create a bar plot to represent the order book with the price on the x axis
\% and the number or order on the y axis
xlabel('Price','FontAngle','italic'); ylabel('Number of order at
\% each price','FontAngle','italic'); title('Order book at the
\% beginning','FontWeight','bold');

p_mean = v_x(middle); p_var = 6*tick_size; v_price = p_mean +
p_var*randn(1,lag);
Appendix A. Main Function of the Program

% Create the initial condition for the vector price
l_price = length(v_price);
% Keep in trac the length of the vector price that comes from the initial % conditions.

[BENCHMARK,EXPERTS] = initialization2(EXPERTS,M,lag,v_x(middle));
% Initialisation of the matrix BENCHMARK to allow a lag in the calcul of the % expert profit measure. For more information about the matrix BENCHMARK % see the legend below.

v_volume = zeros(1,repetition);
% Vector to compute the volume of trading at each time t

v_best_ask = zeros(1,repetition);
% Vector to keep in track the best ask at each time t

v_best_bid = zeros(1,repetition);
% Vector to keep in track the best bid at each time t

v_total_order = zeros(1,repetition);
% Vector to keep in track the total number of order in the book at each % time t.

average_book = zeros(1,length(v_y));
% Vector to keep in track the average order book.

AGENT EXPERTS
%Show the matrices AGENTS and EXPERTS at the beginning of the code.

l_x = length(v_x);
% Keep the length of the vector x.

for i = 1 : 1 : repetition
Appendix A. Main Function of the Program

% This is the main loop of the code

v_total_order(i) = sum(abs(v_y));
% Compute the number of orders in the book.

average_book = average_book + v_y;
% This is used for computing the average order book.

v_best_ask(i) = best_ask(v_x,v_y);
% Put the value of the best ask in the vector best_ask

v_best_bid(i) = best_bid(v_x,v_y);
% Put the value of the best bid in the vector best_bid

time = lag + i;
% Update the time value

[AGENTS,CANCEL,v_y] = update_matrices(AGENTS,CANCEL,v_y,...
   best_bid(v_x,v_y));
% Update time dependent information for the matrices AGENTS and CANCEL.
% In particular, this function will cancel the order that are in the
% order book since too long.

[max_value,agent] = max(AGENTS(:,5));
% Determine who will be the next agent to submit orders

[BENCHMARK,EXPERTS] = update_matrices2(BENCHMARK,EXPERTS,v_x(middle));
% Update the time dependent information for the matrices BENCHMARK and
% EXPERTS.

[expert,reference] = performance_measure2(AGENTS(agent,2),EXPERTS,...
   BENCHMARK,v_price,low_performance);
% Return the expert with the highest performance measure and her
% recommendation.

l_p = length(v_price);
% Give the length of vector price

order_type = AGENTS(agent,1)*(reference - log(v_price(l_p))) +...
   sqrt(AGENTS(agent,1))*randn;
% Determine the order sign.
% Creation of the excess demand for the chosen agent using the
% reference level given by the chosen expert.
if(order_type == 0)
    excess_demand = round(AGENTS(agent,1)*(reference -
        log(v_price(l_p)) + ...
        sqrt(AGENTS(agent,1))*randn);
else
    excess_demand = round(sqrt(AGENTS(agent,1))*randn);
end

v_volume(i) = abs(excess_demand);
% Put the information in the vector volume.

if(excess_demand>0)
    reference = exp(reference);
    % Transform the reference value in a price value
    % Following the advice from the chosen expert, the agent expects the
    % price to goes up so she wants to submit some buy orders.
    [AGENTS,CANCEL,v_y,v_price] = submit_buy_order(AGENTS,CANCEL,...
        agent,v_x,v_y,time,excess_demand,reference,v_price,tick_size);
    % Modify AGENTS, EXPERTS, CANCEL, v_y and v_price when buy orders
    % arrive.
elseif(excess_demand<0)
    reference = exp(reference);
    % Transform the reference value in a price value
    % Following the advice from the chosen expert, the agent expects the
    % price to goes down so she wants to submit some sell orders.
    [AGENTS,CANCEL,v_y,v_price] = submit_sell_order(AGENTS,CANCEL,...
        agent,v_x,v_y,time,excess_demand,reference,v_price,tick_size);
    % Modify AGENTS, EXPERTS, CANCEL, v_y and v_price when sell orders
    % arrive.
else

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Appendix A. Main Function of the Program

% In this case the agent random liquidity demand most likely cancel % his demand for selling or buying the security. It is a bit like % the agent changes idea and finally do not do anything. We still % need to modify the price vector.

if(best_ask(v_x,v_y) ~=0 & best_bid(v_x,v_y) ~= 0)
    v_price = [v_price,...
        (v_x(best_ask(v_x,v_y)) + v_x(best_bid(v_x,v_y)))/2];
else
    v_price = [v_price,v_price(length(v_price))];
end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% END OF THE MAIN LOOP %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% THE NEXT SECTION IS THE CREATION OF ALL THE INTERESTING GRAPHICS %%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% ORDER BOOK AT THE END %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure; bar(v_x,v_y,0.5); % Create a bar plot to represent the order book with the price on the x axis % and the number or order on the y axisxlabel('Price','FontAngle','italic'); ylabel('Number of order at each price','FontAngle','italic'); title('Order book at the end','FontWeight','bold');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% FIRST LOG RETURN %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price - step_size1)); for i = l_price + step_size1 : step_size1 : length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size1));
    retur = retur*100;
Appendix A. Main Function of the Program

```matlab
v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end

step_size1_str = num2str(step_size1);

mu = mean(v_log_return); va = var(v_log_return); k =
kurtosis(v_log_return);

mu_str = num2str(mu); va_str = num2str(va); k_str = num2str(k);

v_log_return = abs(v_log_return);
figure; [f, xi] = ecdf(v_logReturn); loglog(xi, 1-f); hold on; coccin =
max(xi); xi2 = 1:0.01:coccin; loglog(xi2, power_law(xi2, 3));
xlabel('Returns in percentage','FontAngle','italic');
ylabel('Proportion','FontAngle','italic'); title('log return
distribution','FontWeight','bold'); hold off;

figure; normplot(v_log_return);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

- Appendix A. Main Function of the Program

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end

step_size1_str = num2str(step_size2);

mu = mean(v_log_return); va = var(v_log_return); k =
kurtosis(v_log_return);

mu_str = num2str(mu); va_str = num2str(va); k_str = num2str(k);
```

SECOND LOG RETURN

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end
```

- Appendix A. Main Function of the Program

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end
```

SECOND LOG RETURN

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end
```

SECOND LOG RETURN

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end
```

SECOND LOG RETURN

```matlab
v_log_return(l) = log(v_price(l_price)) - log(v_price(l_price -
step_size2)); for i = l_price + step_size2 : step_size2 :
length(v_price)
    retur = log(v_price(i)) - log(v_price(i - step_size2));
    retur = retur*100;
    v_log_return = [v_log_return, retur];
% Put the value retur in the vector return
end
```
Appendix A. Main Function of the Program

\[ v_{\log\text{\_return}} = \text{abs}(v_{\log\text{\_return}}); \]

\[ \text{figure; } [f,xi] = \text{ecdf}(v_{\log\text{\_return}}); \text{loglog(xi,1-f); } \text{hold on; } \text{coccin} = \text{max(xi); } \text{xi2 = 1:0.01:coccin; loglog(xi2,power\_law(xi2,3)); xlabel('Returns in percentage','FontAngle','italic'); ylabel('Proportion','FontAngle','italic'); title('log return distribution','FontWeight','bold'); hold off; } \]

\[ \text{figure; normplot(v_{\log\text{\_return}}); } \]

\text{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%}

\text{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%} LOG RETURNS DIFFERENT STEP %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

\[ \text{number\_of\_step = 1000; step = 50; figure; hold on; } \]

\[ \text{for j = 1 : step : number\_of\_step } \]
\[ \text{v_{log\_return}(1) = log(v\_price(l\_price)) - log(v\_price(l\_price - j)); } \]
\[ \text{for i = l\_price + j : j : length(v\_price) } \]
\[ \text{retur = log(v\_price(i)) - log(v\_price(i - j)); } \]
\[ \text{retur = retur*100; } \]
\[ \text{v_{log\_return} = [v_{log\_return},retur]; } \]
\[ \text{end} \]
\[ \text{v_{log\_return} = \text{abs}(v_{log\_return}); } \]
\[ \text{[f,xi] = ecdf(v_{log\_return}); loglog(xi,1-f); coccin = max(xi); } \]
\[ \text{xi2 = 1:0.01:coccin; loglog(xi2,power\_law(xi2,3),'.r'); xlabel('Returns in percentage','FontAngle','italic'); ylabel('Proportion','FontAngle','italic'); title('log return distribution','FontWeight','bold'); end} \]

\text{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%}

\text{%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%} FIRST RETURN GRAPHICS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

\[ v\_return(l) = ((v\_price(l\_price) - v\_price(l\_price - \text{step\_size1})) /... \]

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Appendix A. Main Function of the Program

v_price(l_price - step_size1)*100;
% Initialisation of the vector return

for i = l_price + step_size1 : step_size1 : length(v_price)
    % Creation of the vector return
    retur = (v_price(i) - v_price(i - step_size1)) / v_price(1,i - step_size1);
    retur = retur*100;
    v_return = [v_return,retur];
% Put the value retur in the vector return
end

figure; plot(v_return); xlabel('Time','FontAngle','italic');
ylabel('Returns in percentage','FontAngle','italic'); step_size1_str = num2str(step_size1); txt = ['Evolution of the returns' blanks(2) 'Step size = ' step_size1_str]; title(txt,'FontWeight','bold');

mu = mean(v_return); va = var(v_return); k = kurtosis(v_return);
mu_str = num2str(mu); va_str = num2str(va); k_str = num2str(k);

figure; [f,xi] = ksdensity(v_return); % Create a density for the vector return
plot(xi,f,:r); hold on; max_re = max(xi); min_re = min(xi); x = min_re-1:0.01:max_re+1;
plot(x,1/sqrt((2*pi*va)).*exp((-x-mu).^2./(2*va))); legend('Returns distribution','Normal distribution',2);
xlabel('Returns in percentage','FontAngle','italic');
ylabel('Proportion','FontAngle','italic'); txt = ['Returns distribution' blanks(2) 'Step size = ' step_size1_str blanks(2) 'mean = ' mu_str blanks(2) 'variance = ' va_str blanks(2) 'kurtosis = ' k_str]; title(txt,'FontWeight','bold'); hold off;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% SECOND RETURNS GRAPHICS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

v_return2(l) = ((v_price(l_price) - v_price(l_price - step_size2))
end

v_price(l_price - step_size2)*100;
% Initialisation of the vector return
for i = l_price + step_size2 : step_size2 : length(v_price)


% Creation of the vector return
retur = (v_price(i)-v_price(i - step_size2)) / v_price(i - step_size2);
retur = retur*100;
v_return2 = [v_return2,retur];
% Put the value retur in the vector return
end

figure; plot(v_return2); xlabel('Time','FontAngle','italic');
ylabel('Returns in percentage','FontAngle','italic');
step_size2_str = num2str(step_size2);
txt = ['Evolution of the returns' blanks(2) 'Step size = ' step_size2_str];
title(txt,'FontWeight','bold');
mu = mean(v_return2); va = var(v_return2); k = kurtosis(v_return2);
mu_str = num2str(mu); va_str = num2str(va); k_str = num2str(k);
figure; [f,xi] = ksdensity(v_return2);
% Create a density for the vector return
plot(xi,f,:r'); hold on; max_re = max(xi); min_re = min(xi);
x = min_re-1:0.01:max_re+1;
plot(x,1/sqrt((2*pi*va)).*exp((-x-mu).^2./(2*va)));
legend('Returns distribution','Normal distribution',2);
xlabel('Returns in percentage','FontAngle','italic');
ylabel('Proportion','FontAngle','italic');
txt = ['Returns distribution' blanks(2) 'Step size = ' step_size2_str blanks(2) 'mean = ' mu_str blanks(2) 'variance = ' va_str blanks(2) 'kurtosis = ' k_str];
title(txt,'FontWeight','bold'); hold off;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Returns of different steps %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure; [f,xi] = ksdensity(v_return2);
% Create a density for the vector return
plot(xi,f); hold on; [f,xi] = ksdensity(v_return);
% Create a density for the vector return
plot(xi,f,:r'); legend('Step Size = 1', 'Step Size = 20');
xlabel('Returns in percentage','FontAngle','italic');
ylabel('Proportion','FontAngle','italic');
title('Returns in Percentage','FontWeight','bold');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Appendix A. Main Function of the Program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

PRICE GRAPHIC %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

v_price_graph = v_price(l_price:length(v_price));

figure;  % Open a new graphic window
plot(v_price_graph);  % Make the plot of the price process
title('Evolution of the price','FontWeight','bold');  % Put a title on a graphic
xlabel('"Time"','FontAngle','italic');
ylabel('Price','FontAngle','italic');
axis([-inf inf 500 1500]);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

BENCHMARK PATH %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure; plot(BENCHMARK(1,:));
title('Typical fundamental value path for a fundamentalist','FontWeight','bold');
xlabel('Time','FontAngle','italic');
ylabel('Fundamental value','FontAngle','italic');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

VOLATILITY AND VOLUME %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

var1 = repetition/interval; v_volume2 = zeros(1,var1); v_volatility = zeros(1,var1);

for i = 1 : 1 : var1-1
v_volume2(i) = sum(v_volume(i*interval:i*interval+interval));
v_volatility(i) = var(v_price(l_price+i*interval:l_price+i*interval+... interval));
end

figure; subplot(2,1,1); plot(v_volume2);
title('Trading volume','FontWeight','bold');
xlabel('Time','FontAngle','italic');
ylabel('Trading volume','FontAngle','italic');
subplot(2,1,2);
plot(v_volatility); title('Volatility','FontWeight','bold');
xlabel('Time','FontAngle','italic');
ylabel('Volatility','FontAngle','italic');
test1 =
Appendix A. Main Function of the Program

corrcoef(v_volume2',v_volatility')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

BID AND ASK SPREAD

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

v_spread = v_best_ask - v_best_bid; v_spread_graph = v_spread(1:interval:length(v_spread)); figure; subplot(2,1,1); plot(v_spread_graph); title('Bid and Ask Spread', 'FontWeight', 'bold'); xlabel('Time', 'FontAngle', 'italic'); ylabel('Bid and Ask Spread', 'FontAngle', 'italic'); subplot(2,1,2); plot(v_volatility); title('Volatility', 'FontWeight', 'bold'); xlabel('Time', 'FontAngle', 'italic'); ylabel('Volatility', 'FontAngle', 'italic'); test2 = corrcoef(v_spread_graph',v_volatility')

v_spread_graph2 = zeros(1,var1); for i = 1 : 1 : var1-1
v_spread_graph2(i) = sum(v_spread(i*interval:i*interval+interval));
end

figure; subplot(2,1,1); plot(v_spread_graph2); title('Cummulative Bid and Ask Spread', 'FontWeight', 'bold'); xlabel('Time', 'FontAngle', 'italic'); ylabel('Bid and Ask Spread', 'FontAngle', 'italic'); subplot(2,1,2); plot(v_volatility); title('Volatility', 'FontWeight', 'bold'); xlabel('Time', 'FontAngle', 'italic'); ylabel('Volatility', 'FontAngle', 'italic'); test3 = corrcoef(v_spread_graph2',v_volatility') test4 = corrcoef(v_spread_graph2(1:length(v_spread_graph2)-1)',v_volatility(2:length(v_volatility))')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

RETURN AND VOLATILITY

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

figure; subplot(2,1,1); plot(v_return); xlabel('Time', 'FontAngle', 'italic'); ylabel('Returns in percentage', 'FontAngle', 'italic'); step_size2_str = num2str(step_size2); txt = ['Evolution of the returns' blanks(2) 'Step size = ' step_size1_str]; title(txt, 'FontWeight', 'bold'); subplot(2,1,2); plot(v_volatility); title('Volatility', 'FontWeight', 'bold');
Appendix A. Main Function of the Program

```matlab
xlabel('Time','FontAngle','italic');
ylabel('Volatility','FontAngle','italic');
v_return_spec = v_return(1:length(v_return)-1);
return_volatility = corrcoef(abs(v_return_spec),v_volatility)

%%%%%%%%%%%%%%%%%%%%%%%%%%%% VOLUME AND PRICE %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure; subplot(2,1,1); plot(v_volume2); title('Trading volume','FontWeight','bold'); xlabel('Time','FontAngle','italic');
ylabel('Trading volume','FontAngle','italic');
subplot(2,1,2); plot(v_price_graph);
title('Evolution of the price','FontWeight','bold'); xlabel('Time','FontAngle','italic');
ylabel('Price','FontAngle','italic');
axis([-inf inf 500 1500]);

%%%%%%%%%%%%%%%%%%%%%%%%%%%% SPREAD AND PRICE %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure; subplot(2,1,1);
plot(v_price_graph); ylabel('Price','FontAngle','italic');
axis([-inf inf 500 1500]);

subplot(2,1,2); plot(v_spread); title('Bid and Ask Spread','FontWeight','bold'); xlabel('Time','FontAngle','italic');
ylabel('Bid and Ask Spread','FontAngle','italic');
v_spread_spec = v_spread(1:length(v_spread)-10);
price_spread = corrcoef(v_price_graph,v_spread_spec)
```
Appendix A. Main Function of the Program

% AVERAGE ORDER BOOK

average_book = average_book./repetition; figure;
plot(v_x,average_book); title('Average order book','FontWeight','bold'); xlabel('Price','FontAngle','italic');
ylabel('Order book','FontAngle','italic');

% VOLATILITY AND NUMBER OF ORDERS

figure; hold on; subplot(2,1,1);
plot(v_total_order(100:length(v_total_order))); subplot(2,1,2);
plot(v_volatility(100:length(v_volatility)));

v_total_order2 = zeros(1,var1);
for i = 1 : 1 : var1-1
    v_total_order2(i) = sum(v_total_order(i*interval:i*interval+interval));
end
v_total_order2_special = v_total_order2(100:length(v_total_order2));
v_volatility_special = v_volatility(100:length(v_volatility));
number_volatility =
corrcoef(v_total_order2_special,v_volatility_special)

toc
% End of the time counter

% RESULTS FROM THE CORRELATION

% test1 =
%  
%  1.0000  0.4375
%  0.4375  1.0000
%
% test2 =
%  
%  1.0000  0.2151
%  0.2151  1.0000

Appendix A. Main Function of the Program

\%
\%
\% test3 =
\%
\% 1.0000  0.4099
\% 0.4099  1.0000
\%
\%
\% test4 =
\%
\% 1.0000  0.3261
\% 0.3261  1.0000
\%
\%
\% return_volatility =
\%
\% 1.0000  0.4876
\% 0.4876  1.0000
\%
\%
\% volume_price =
\%
\% 1.0000  -0.0043
\% -0.0043  1.0000
\%
\%
\% price_spread =
\%
\% 1.0000  -0.0061
\% -0.0061  1.0000
\%
\%
\% number_volatility =
\%
\% 1.0000  -0.4613
\% -0.4613  1.0000
Appendix B

List of Other Functions

In this appendix, we present a list and a brief description of the different functions. In section B.1 we present the functions that are used to produce the different figures of this thesis. In the section B.2 we present the other functions used by the main function presented in A.

B.1 How to Produce the Figures

- Figures 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 3.1, 3.2, 3.3 and 3.4 were created using the function graphique.
- Figures 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.16, 4.17, 4.18, ??, 4.20 and 4.22 were created using the function order book new graph1.
- Figure 4.10 was created using the function dependence lambda.
- Figure 4.11 was created using the function dependence tick size.
- Figure 4.12 was created using the function dependence time horizon.
- Figures 4.13, 4.14 and 4.15 were created using the function stationary distribution1.
- Figures 4.23 and 4.24 were created using the function stationary distribution final.

B.2 Other Functions

- Function best ask: This function uses the vector x and y to return the indices of the best ask price.
- Function best bid: This function uses the vector x and y to return the indices of the best bid price.
- Function initialization: This function initialize the matrices AGENTS,CANCEL and EXPERTS as well as the value of middle and the vector x.
- Function initialization2: This function create some initial condition for the vector benchmark.
Appendix B. List of Other Functions

- Function matrix search: This function search for which line in the matrix CANCEL there are some orders that correspond to the number position in book.

- Function order book new1: This function is a faster version of the main function order book new graph1 that do not reproduce the figures of the main function. It is used by the functions dependence lambda, dependence tick size, dependence time horizon and stationary distribution 1.

- Function order book new graph1: This is the main function of the program.

- Function performance measure2: This function calculate the performance measure for each expert using the discount factor of the agent that will trade in the next period.

- Function submit buy order: This function is the modification of the order book when the agent submit some buy orders. It can be either some market buy orders or some limit buy orders.

- Function submit sell order: This function is the modification of the order book when the agent submit some sell orders. It can be either some market sell orders or some limit sell orders.

- Function translation: This function find back the vector y when you give it the matrix CANCEL.

- Function update matrices: This is the function that update all the time dependent information for the matrix AGENTS and for the matrix CANCEL.

- Function update matrices2: This is the function that update all the time dependent information for the matrices BENCHMARK and EXPERTS.
Appendix C

Statistical Tests

In this appendix, we present in more details the results of our statistical tests.

C.1 Converting the Data

Unfortunately, *Matlab®* does not provide any function to perform the KPSS test. The first thing we need to do is then to save the our results in *Matlab®* and then load them in *R* to perform the KPSS test. Here are the necessary commands to execute these tasks.

**IN MATLAB:**
```matlab
save filename -ascii v_price
% This command save the vector v_price in a filename in ASCII format
```

**IN R:**
```r
price <- scan("filename")
% Load the data from filename
price <- as.ts(price)
% Transform the price data for the application of the KPSS test
test <- KPSS.test(price)
% Perform the KPSS test
test
% Show the results of the KPSS test
```

C.2 Results

Here are the presentation of the results we obtain from the tests.

For the price:

---

KPSS test
---

Null hypotheses: Level stationarity and stationarity.
Appendix C. Statistical Tests

Alternative hypothesis: Unit root.

Statistic for the null hypothesis of
level stationarity: 0.301

$p$-value: 0.10 0.05 0.025 0.01
Critical values: 0.347 0.463 0.574 0.739

For the returns:

KPSS test

Null hypotheses: Level stationarity and stationarity.

Alternative hypothesis: Unit root.

Statistic for the null hypothesis of
level stationarity: 0.097

$p$-value: 0.10 0.05 0.025 0.01
Critical values: 0.347 0.463 0.574 0.739

In both cases, we cannot reject the null hypothesis $H_0$ because the test statistic is smaller than all the critical values. We do note a significant difference between the test statistic for the price 0.301 and the test statistic for the returns 0.097. This means that it is harder to not reject $H_0$ for the price than for the returns. In fact, while the returns do not fail the test of stationarity for a very small number of iterations of our program (around 5000), we need to do at least 50 000 iterations to obtain a valid test for the price.