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# IRREDUCIBLE REPRESENTATIONS OF ALGEBRAS

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# ABSTRACT

An element x of an associative algebra A is called diagonable provided A has a basis of characteristic vectors for the transformation ad x: $a \mapsto ax - xa$ of A. This notion immediately generalizes to that of a diagonable subspace L of A. The centralizer $A_o$ of L plays an important role in the representation theory of A, for there is a one-to-one correspondence between the "$\lambda$-weighted" irreducible modules of A and of $A_o$.

In Chapters Two and Three, we first explore various ring-theoretic properties of A and $A_o$, and then use the results obtained to classify the diagonable elements in different algebras. We also give conditions under which all A-modules are weighted.

The Cartan theory of Lie and Jordan algebras is linked in Chapter Four by the observation that Cartan subalgebras of simple finite dimensional Lie and Jordan algebras (over algebraically closed fields of characteristic 0) are diagonable subspaces of the respective universal enveloping algebras. Furthermore, in the Jordan case, the centralizer of a Cartan subalgebra is the centralizer of one of its elements and is a direct sum of complete matrix rings.

Finally, we are able to show that the universal enveloping algebra of any simple Jordan algebra which contains an idempotent whose Peirce one-space is one-dimensional, is generated by its idempotents.

# TABLE OF CONTENTS

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# INTRODUCTION

The universal enveloping algebra of a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0 has a basis relative to which the linear transformations ad h: $a \mapsto ah - ha$ for h in a Cartan subalgebra are simultaneously diagonizable. Thus any such Cartan subalgebra is an example of what we call a <u>diagonable</u> <u>subspace</u> of an associative algebra.

Let L be a diagonable subspace of an associative algebra A. In Chapter One, we define the concept of a <u>$\lambda$-weighted</u> <u>A-module</u>, where $\lambda$ is a linear functional on L, and then generalize a theorem of Lemire ([14]) by proving the existence of a one-to-one correspondence between the irreducible $\lambda$-weighted modules of A and $A_o$, the centralizer of L in A. This result leads us to believe there should be some close connections between the algebras A and $A_o$ and these we investigate in Chapter Two.

Several examples throughout the thesis indicate how frequently diagonable subspaces occur in algebras. In order to obtain many results, it is therefore necessary to impose some kind of restrictions on the algebra and the diagonable subspace. If we assume L has only a finite number of distinct <u>roots</u>, then the Jacobson radical of A is nilpotent whenever $A_o$ is semi-simple. If every A-module is weighted, $A_o$ is semi-simple whenever A is.

Several of the results in Chapter Two require that every irreducible A-module be weighted. In one of the major results of Chapter Three, we are able to prove that if L is a diagonable subspace spanned by a single algebraic element in a prime algebra A, and one irreducible A-module is weighted, then every A-module is weighted. In this chapter, we are able also to characterize the diagonable elements of a finite dimensional central simple algebra. They turn out to be exactly those elements which are expressible as linear combinations of primitive orthogonal idempotents with sum the identity of A. An interesting consequence of this is that the diagonable elements of a matrix ring are nothing but those matrices similar to diagonal matrices.

In Chapter Four, we give an illustration of the common Cartan theory of Lie and Jordan algebras established in Foster's doctoral dissertation ([3]). A Cartan subalgebra H of a finite dimensional simple Jordan algebra over an algebraically closed field of characteristic 0 is a diagonable subspace of its universal enveloping algebra A. Thus, exactly as in the Lie case, we obtain a one-to-one correspondence between the $\lambda$-weighted irreducible modules of A and of $C$, the centralizer in A of H. This result is particularly useful because $C$ can be realized as the centralizer of some element in H and is a direct sum of complete matrix rings over F.

As a final application of the theory of diagonable elements, we prove that the universal enveloping algebra of a simple Jordan algebra is generated as an algebra by its idempotents, provided the Jordan algebra contains an idempotent whose "Peirce one-space" is one-dimensional.

# PRELIMINARIES

All fields in this thesis are understood to have characteristic zero. An _algebra_ is a vector space A over a field F together with a bilinear map $A \times A \to A$ denoted $(a,b) \mapsto ab$ such that

$$a(b + c) = ab + ac \; ; \qquad (a + b)c = ac + bc$$

$$\text{and} \qquad \alpha(ab) = (\alpha a)b = a(\alpha b)$$

for any $\alpha \in F$, and $a, b, c \in A$. A _right ideal_ of A is a subspace I such that $ua \in I$ whenever $u \in I$ and $a \in A$; a _left ideal_ is defined in an analogous way. By an _ideal_ of A, we simply mean a subspace which is both a left and right ideal. A is a _simple algebra_ if it has no non-zero proper ideals. A is _nilpotent_ if there is a positive integer n such that every product of n elements of A is 0.

A _right A-module_ is an abelian group V together with a bilinear map $V \times A \to V$, denoted $(v,a) \mapsto va$, such that

$$v(a + b) = va + vb \; ; \qquad (v + w)a = va + wa \; ;$$

$$v(ab) = (va)b$$

for any $a,b \in A$ and $v,w \in V$. We also assume $v1 = v$ for every $v \in V$ if A has an identity 1. By an "A-module", we always mean "right A-module". A module V is _irreducible_ if it contains no non-zero proper submodules. _Schur's Lemma_ states that if V and W are irreducible A-modules, then any homomorphism from V to W is either zero or an isomorphism.

If A is an algebra over the field F and $x \in A$, we define the linear transformations $R_x$, $L_x$, and $\text{ad } x$ of A by

$$R_x: \quad a \mapsto ax$$
$$L_x: \quad a \mapsto xa$$
$$\text{ad } x: \quad a \mapsto ax - xa \qquad \text{for } a \in A$$

Note that ad $x = R_x - L_x$.

The <u>commutator</u> $(x,y)$ is defined by $(x,y) = xy - yx$, the <u>associator</u> $(x,y,z)$ by $(x,y,z) = xy.z - x.yz$. For any subsets X, Y, and Z of A, define $(X,Y) = \{(x,y)\colon x \in X,\ y \in Y\}$ and $(X,Y,Z) = \{(x,y,z)\colon x \in X,\ y \in Y,\ z \in Z\}$. The <u>centre</u> of A is $\{a \in A\colon (a,A) = 0\}$ and we always denote this by $Z(A)$. A is <u>commutative</u> if $(A,A) = 0$ and <u>associative</u> if $(A,A,A) = 0$.

A <u>Lie algebra</u> is an algebra $L$ such that $[x,x] = 0$ and $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for every x, y and z in $L$, where the product of elements x and y in $L$ is denoted $[x,y]$. An associative algebra A over a field F determines a Lie algebra $A_L$ by defining the product of elements $x,y \in A$ to be the commutator $(x,y)$.

A <u>Jordan algebra</u> is a commutative algebra J such that $(x^2,y,x) = 0$ for any $x,y \in J$. We always assume a Jordan algebra contains an identity 1. If $e^2 = e$ is an idempotent in J, then there exists a direct sum decomposition of the vector space J called the <u>Peirce decomposition</u>; namely,

$$J = J_0 + J_{\frac{1}{2}} + J_1 \ ; \qquad J_i = J_i(e) = \{x \in J\colon xe = ix\},\ i = 0,\tfrac{1}{2},1$$

More generally, if $e_1,\ldots,e_n$ are <u>pairwise orthogonal</u> idempotents ($e_ie_j = 0$ for $i \neq j$) with sum 1, the Peirce decomposition has the form

$$J = \bigoplus \sum_{\substack{i,j=1 \\ i<j}}^{n} J_{ij} \ ; \qquad \begin{array}{l} J_{ii} = J_1(e_i) = \{x \in J\colon xe_i = x\},\ i = 1,\ldots,n \\ J_{ij} = J_{\frac{1}{2}}(e_i) \cap J_{\frac{1}{2}}(e_j) = \{x \in J\colon xe_i = \tfrac{1}{2}x = xe_j\} \\ \hfill i<j \end{array}$$

The reader should consult Jacobson [11; pages 118-120] for details.

Finally, we point out that an element x in an algebra A over F is called <u>algebraic</u> if it is the solution to some polynomial in F[t] (i.e. a polynomial in t with coefficients in F). Among all such polynomials, there is one of least degree called the <u>minimal polynomial</u> of x which divides any other polynomial for which x is a solution. We assume that the minimal polynomial is monic, in which case it is unique.

We now make the convention that henceforth, "algebra" will always mean "associative algebra with identity".

# CHAPTER ONE

## REPRESENTATION THEORY

### 1.1 THE UNIVERSAL ENVELOPING ALGEBRA OF A LIE ALGEBRA

Let $L$ be a Lie algebra over a field F. A representation of $L$ is defined to be a linear map $S: L \to A$, where A is an associative algebra over F, such that for every a and b in $L$,

$$(1) \qquad S_{ab} = S_a S_b - S_b S_a \qquad \text{where } S: u \mapsto S_u \text{ for } u \in L$$

For example, $R: x \mapsto R_x$ defines a representation of $L$ in the associative algebra X generated over F by $\{R_x : x \in L\}$. An associative algebra $U(L)$ with identity is a universal enveloping algebra for $L$ if there is a canonical representation $S^*: L \to U(L)$ such that given any representation $S: L \to A$ of $L$ in an associative algebra A, there exists a unique homomorphism $\Psi: U(L) \to A$ such that $S = \Psi \circ S^*$; i.e. which makes the following diagram commutative:

$$\begin{tikzcd} & U(L) \arrow[d, "\psi"] \\ L \arrow[ur, "S^*"] \arrow[r, "S"'] & A \end{tikzcd}$$

We refer the reader to Chapter V of Jacobson ([9]) for a more comprehensive treatment of $U(L)$. What is important for us here is:

THEOREM 1.1.1: *$U(L)$ exists, is unique up to isomorphism, and is generated by* $\{S^*_x : x \in L\}$.

Suppose now that $L$ is finite dimensional and simple, and that F is algebraically closed. Then $L$ possesses a Cartan subalgebra $H$; i.e., a nilpotent subalgebra which is self-normalizing in the sense that $[x,H] \subseteq H$ implies $x \in H$. There exists a set $\Delta$ of linear functionals $\alpha: H \to F$ called roots with the property that the subspace $L_\alpha = \{x \in L: x(R_h - \alpha(h)1)^n = 0$, for all $h \in H\}$ is non-zero for any root $\alpha$. One introduces an order in $\Delta$ and is able to distinguish between positive and negative roots. A root is simple if it cannot be written as the sum of two positive roots. Denote by $\Delta_s$ and $\Delta_+$ the set of simple and positive roots respectively. Then there exists a basis B of $L$ called a Cartan basis, $B = \{e_\alpha, f_\alpha, h_\beta: \beta \in \Delta_s, \alpha \in \Delta_+\}$, where among the multiplicative relations for B, we have

$$\begin{aligned} &[h_\beta, h_{\beta'}] = 0 \\ (2) \qquad &[e_\alpha, h_\beta] = A_{\alpha,\beta} e_\alpha \\ &[f_\alpha, h_\beta] = B_{\alpha,\beta} f_\alpha \quad , \quad \alpha \in \Delta_+,\ \beta,\beta' \in \Delta_s \end{aligned}$$

where the $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ are integers.

By the Poincaré-Birkhoff-Witt Theorem ([ 9 ; §5.2 ]), the universal enveloping algebra $U(L)$ of $L$ has a linear basis consisting of all elements of the form

$$(3) \qquad \prod_{\alpha\in\Delta_+} f_\alpha^{n(\alpha)} \prod_{\beta\in\Delta_s} h_\beta^{r(\beta)} \prod_{\alpha\in\Delta_+} e_\alpha^{m(\alpha)}$$

where $n(\alpha)$, $r(\beta)$, $m(\alpha)$ are non-negative integers and the product respects the order in $\Delta$.

For a fixed $\beta_o \in \Delta_s$, observe that

$$\left(\prod_{\alpha\in\Delta_+} e_\alpha^{m(\alpha)}, h_{\beta_o}\right) = \left(\sum_{\alpha\in\Delta_+} m(\alpha) A_{\alpha,\beta_o}\right) \prod_{\alpha\in\Delta_+} e_\alpha^{m(\alpha)}$$

because of the identity

$$(4) \qquad (xy,z) = x(y,z) + (x,z)y$$

which holds in any associative algebra. Generally, it is true that $(u,h_{\beta_0})$ is an integral multiple of u for each basis element u of the form (3), and hence $U(L)$ decomposes as a vector space into a direct sum of spaces of the form $\{u \in U(L): (u,h_{\beta_0}) = nu\}$ for n an integer. With this motivation we make the following definition.

DEFINITION 1.1.2: *Let* A *be an associative algebra with unit element over a field* F. *Then an element* $x \in A$ *is* *diagonable* *if* $A = \bigoplus\sum_{\alpha \in F} A_\alpha(x)$, *a vector space direct sum, where* $A_\alpha(x) = \{a \in A: (a,x) = \alpha a\}$.

## 1.2 PROPERTIES OF ALGEBRAS POSSESSING DIAGONABLE ELEMENTS AND THEIR MODULES

We collect some important facts about diagonable elements in the next two propositions.

PROPOSITION 1.2.1: *Suppose* $A = \bigoplus \sum_{i \in I} R_i$ *is an algebra direct sum and* $x = \sum_{i \in I} x_i$ *is a diagonable element of* $A$. *Then each* $x_i$ *is a diagonable element in the algebra* $R_i$ *and* $(R_i)_\alpha(x_i) = A_\alpha(x) \cap R_i$.

PROOF: Let $u \in R_i$. Then $(u,x) = (u,x_i) = \sum_{0 \neq \alpha \in F} \alpha u_\alpha \in R_i$ where $u = \sum_{\alpha \in F} u_\alpha$, $u_\alpha \in A_\alpha(x)$. Next,

$$((u,x),x) = ((u,x_i),x_i) = \Big(\sum_{0 \neq \alpha \in F} \alpha u_\alpha, x\Big) = \sum_{0 \neq \alpha \in F} \alpha^2 u_\alpha \in R_i.$$

Continuing in this way, for every positive integer k, we have $\sum_{0 \neq \alpha \in F} \alpha^k u_\alpha \in R_i$. Now $u_\alpha = 0$ except for $\alpha$ in some finite set $\{\alpha_1, \ldots, \alpha_n\} \cup \{0\}$, no $\alpha_i = 0$. Thus

$$(1) \qquad \sum_{i=1}^{n} \alpha_i^{\,k} u_{\alpha_i} = v_k, \quad k = 1, \ldots, n$$

with $v_1, \ldots, v_n \in R_i$, is a system of n equations in n unknowns with coefficient matrix $(\alpha_i^{\,j})$, $i,j = 1, \ldots, n$ , a vandermonde matrix. Since $\alpha_1, \ldots, \alpha_n$ are distinct scalars, the system has a solution; i.e., each $u_{\alpha_i} \in R_i$. Also, $u_o = u - \sum_{i=1}^{n} u_{\alpha i} \in R_i$ and so $x_i$ is diagonable in $R_i$. Finally,

$$(R_i)_\alpha(x_i) = A_\alpha(x) \cap R_i$$

is an immediate consequence of $(u,x) = (u,x_i)$ for $u \in R_i$.

PROPOSITION 1.2.2: (i) *If* $x$ *is a diagonable element in an algebra* $A$ *, and* $B$ *is a subspace of* $A$ *invariant under ad* $x$*, then* $x$ *is diagonable on* $B$ *in the sense that* $B$ *decomposes as* $\oplus \sum_{\alpha \varepsilon F} B_\alpha(x)$ *with* $B_\alpha(x) = \{b \varepsilon B: (b,x) = \alpha b\}$. *(Note that we do not require that* $x \varepsilon B$*).*

(ii) *If* $x$ *is a diagonable element and* $t \varepsilon F$ *is non-zero, then* $tx$ *is diagonable, and* $A_\alpha(x) = A_{t\alpha}(tx)$.

(iii) *If* $x$ *and* $y$ *are commuting diagonable elements, then* $x + y$ *is diagonable and, in fact, if* $A_\alpha(x) = 0$ *except for* $\alpha \varepsilon \{\alpha_i: i \varepsilon I\}$ *and* $A_\beta(y) = 0$ *except for* $\beta \varepsilon \{\beta_j: j \varepsilon J\}$*, then* $A_\gamma(x+y) \neq 0$ *implies* $\gamma \varepsilon \{\alpha_i + \beta_j: i \varepsilon I, j \varepsilon J\}$. *If* $I$ *and* $J$ *are finite sets, the linear transformation ad(x+y) satisfies the polynomial* $\prod_{\substack{i\varepsilon I\\ j\varepsilon J}} (t - (\alpha_i+\beta_j)) \varepsilon F[t]$.

PROOF: (i) If $b \varepsilon B$, we can write $b = \sum_{\alpha\varepsilon F} b_\alpha$ relative to $x$, and since $(B,x) \subseteq B$, $(b,x^{(k)}) = \sum_{0\neq\alpha\varepsilon F} \alpha^k b_\alpha \varepsilon B$ for every integer $k > 0$. Here we write

$$(b,x^{(k)}) = (\ldots(b,x),x),\ldots,x), \quad x \text{ repeated } k \text{ times.}$$

A vandermonde matrix argument identical to that used in the proof of 1.2.1 shows that each $b_\alpha \varepsilon B$. (ii) is trivial, and (iii) follows from (i) because $(x,y) = 0$ implies each $A_{\alpha_i}(x)$ is invariant under ad $y$, and hence decomposes as $\oplus \sum_{j\varepsilon J} (A_{\alpha_i}(x))_{\beta_j}(y)$ relative to $y$. Clearly $(A_{\alpha_i}(x))_{\beta_j}(y) \subseteq A_{\alpha_i+\beta_j}(x+y)$. The last part of (iii) follows from the observation that if $x$ is diagonable and $A_\alpha(x) = 0$ except for $\alpha$ in a finite set $\{\alpha_1,\ldots,\alpha_n\}$, then the linear transformation ad $x$ of $A$ is algebraic with minimum polynomial $\prod_{i=1}^{n}(t - \alpha_i) \varepsilon F[t]$.

Now let $L$ be a linear subspace of an algebra $A$ over $F$ which is spanned by commuting diagonable elements. The preceding proposition implies that every element of L is necessarily diagonable.

DEFINITION 1.2.3: *A map* $\alpha$*: L → F such that*

$$A_\alpha(L) = \{a \in A: (a,x) = \alpha(x)a, \text{ for every } x \in L\}$$

*is non-zero is called a root of L in A, and* $A_\alpha(L)$ *is the corresponding root space.* With L fixed, we will write $A_\alpha$ instead of $A_\alpha(L)$ and refer to "roots of A". *If V is a right A-module which is also a vector space over F, a map* $\lambda$*:L → F is a weight of L in V if,*

$$V_\lambda = \{v \in V: v(x - \lambda(x)1)^n = 0, \text{ for every } x \in L \text{ and } n = n(x) > 0\}$$

*is non-zero.* $V_\lambda$ *is the corresponding weight space and V is said to be L-weighted or sometimes* $\lambda$*-weighted if we wish to emphasize that* $\lambda$ *is a weight of* L.

We remind the reader that if B is any vector space over a field F, B*, the dual space of B, is defined as the space of all linear functionals on B; i.e., those maps $\psi$: B → F satisfying

$$\text{(i) } \psi(b_1 + b_2) = \psi(b_1) + \psi(b_2) \quad \text{and}$$
$$\text{(ii) } \psi(\alpha b_1) = \alpha\psi(b_1) \quad \text{for every } b_1, b_2 \in B,\ \alpha \in F.$$

The * notation is standard and will be used freely in the rest of this thesis.

PROPOSITION 1.2.4: *If* $\lambda$ *is any weight of L in an A-module V, and* $\alpha$ *is a root of L, then both* $\lambda$ *and* $\alpha$ *are in L*.*

PROOF: We show first that for any $u \in L$, $\lambda(u)$ is the only characteristic root of u on $V_\lambda$. For this, let $0 \neq v \in V_\lambda$. Then

$v(u - \alpha 1)^s = 0 = v(u - \lambda(u)1)^t$ implies $v((u - \lambda(u)) - (u - \alpha 1))^{t+s} = 0$, where we can expand using the binomial theorem because $u - \lambda(u)1$ and $u - \alpha 1$ commute. Thus $v((\alpha - \lambda(u))1)^{t+s} = 0$ and $\lambda(u) = \alpha$. That $\lambda$ is linear now follows from

(a) $x, y \in L$ implies $\lambda(x) + \lambda(y)$ is a characteristic root of $x + y$ on $V_\lambda$

and (b) $x \in L$ and $\alpha \in F$ implies $\alpha\lambda(x)$ is a characteristic root of $\alpha x$ on $V_\lambda$

To see (a), we have positive integers n and m with $0 = v(x - \lambda(x)1)^n = v(y - \lambda(y)1)^m$ for any $0 \neq v \in V_\lambda$. Since $x - \lambda(x)1$ and $y - \lambda(y)1$ commute, $v((x+y) - (\lambda(x)+\lambda(y))1)^{n+m} = v((x - \lambda(x)1) + (y - \lambda(y)1))^{n+m} = 0$. (b) follows in the same way. Finally, that a root $\alpha$ is also in $L^*$ is proven exactly as above, using the linearity of the map ad x.

<u>DEFINITION 1.2.5</u>: *A linear subspace* L *of an algebra* A *over* F, *spanned by commuting diagonable elements, is called a <u>diagonable subspace</u>, if* A *decomposes as a vector space relative to the collection* $\Delta$ *of roots of* L *i.e.,* $A = \bigoplus \sum_{\alpha \in \Delta} A_\alpha$ .

Suppose that L is any two-dimensional subspace of A with basis consisting of the commuting diagonable elements x and y. In the course of proving Proposition 1.2.2, we saw

$$(1) \qquad A = \sum_{\alpha,\beta} (A_\alpha(x))_\beta(y), \qquad (A_\alpha(x))_\beta(y) = A_\alpha(x) \cap A_\beta(y)$$

the sum taken over all characteristic roots $\alpha$ of ad x and characteristic roots $\beta$ of ad y. The linearity of the roots of L implies immediately that if $(A_\alpha(x))_\beta(y) \neq 0$, then this space is exactly $A_\lambda(L)$, where $\lambda$ is the root of L uniquely determined by the conditions $\lambda(x) = \alpha$, $\lambda(y) = \beta$. Thus (1) is a decomposition of A relative to L; i.e., L is a diagonable subspace. By induction, we see that any finite-dimensional subspace L spanned by commuting diagonable elements is in fact a diagonable subspace. For example, a Cartan subalgebra of a finite dimensional simple Lie algebra over an algebraically closed field is abelian ([9 ; page 110]) and hence its image under the canonical embedding in the universal enveloping algebra is a diagonable subspace.

Now fix a diagonable subspace L of A, and let $A = \oplus \sum_{\alpha\varepsilon\Delta} A_\alpha$ be the decomposition of A relative to the collection $\Delta$ of roots of L. We come now to an investigation of L-weighted A-modules.

<u>LEMMA 1.2.6</u>: *If* V *is an* L*-weighted* A*-module,* $V_\lambda A_\alpha \subseteq V_{\lambda+\alpha}$ *for any weight* $\lambda$, *where we define* $V_{\lambda+\alpha}$ *to be zero if* $\lambda + \alpha$ *is not a weight of* L *in* V.

<u>PROOF</u>: An easy induction reveals that $x^k a = a(x - \alpha(x)1)^k$ for all $a \varepsilon A_\alpha$ and $x \varepsilon L$. Thus if $v(x - \lambda(x)1)^n = 0$, we have

$$\begin{aligned} 0 = v(x - \lambda(x)1)^n a &= v \sum_{k=0}^{n} \binom{n}{k} x^k (-\lambda(x)1)^{n-k} a \\ &= v \sum_{k=0}^{n} \binom{n}{k} (-\lambda(x))^{n-k} a(x - \alpha(x)1)^k \\ &= va(x - (\lambda(x) + \alpha(x))1)^n \end{aligned}$$

and so $va \; \varepsilon \, V_{\lambda+\alpha}$.

<u>COROLLARY 1.2.7</u>: *Let* V *be an irreducible* L*-weighted* A*-module. Then* V *decomposes as* $\oplus \sum_{\lambda\varepsilon\Lambda} V_\lambda$, *relative to the set* $\Lambda$ *of all weights of* L *in* V.

PROOF: We simply note that $V_\lambda \neq 0$ for some $\lambda \in \Lambda$ and so by the lemma, $\oplus \sum_{\lambda \in \Lambda} V_\lambda$ is a non-zero A-submodule of V.

Now the identity (4) in 1.1 implies

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta} \qquad (2)$$

where we again define $A_{\alpha+\beta}$ to be zero if $\alpha + \beta$ is not a root. In particular, $A_o = \{a \in A\colon ax = xa, \text{ for every } x \in L\}$ is a subalgebra containing 1 and x; namely, the centralizer of L in A. Also (2) implies that each $A_\alpha$ is an $A_o$-module. By 1.2.6 so is any weight space $V_\lambda$ of an A-module, and we can further show:

LEMMA 1.2.8: *If* V *is an irreducible weighted* A*-module,* $V_\lambda$ *is an irreducible* $A_o$*-module for any weight* $\lambda$ .

PROOF: If $W_{\lambda_o}$ is a proper $A_o$-submodule of $V_{\lambda_o}$, then $W = W_{\lambda_o}A = W_{\lambda_o} \oplus \sum_{0 \neq \alpha \in \Delta} W_{\lambda_o}A_\alpha$ is a proper A-submodule of V because of 1.2.6.

LEMMA 1.2.9: *If* K *is any maximal right ideal of* $A_o$ *and* $u \in Z(A_o)$*, the centre of* $A_o$*, then* $au \in K$ *with* $a \in A_o$ *implies* $a \in K$ *or* $u \in K$.

PROOF: If $a \notin K$, $K + aA_o = A_o$, and so $k + ab = 1$ for some $k \in K$ and $b \in A_o$. Hence $ku + abu = u$ with $ku$ and $abu$ ($=aub$) both in K. So $u \in K$.

Now suppose V is an irreducible L-weighted A-module and $V_\lambda$ is a non-zero weight space. Let $0 \neq v \in V_\lambda$. Then $\tau\colon a \mapsto va$ is an $A_o$-module homomorphism $A_o \to V$ , necessarily surjective by 1.2.8, and so $V_\lambda \cong A_o/T_o$ where $T_o$ is the kernel of $\tau$. Since $T_o$ is a maximal right ideal of $A_o$ which contains $(x - \lambda(x)1)^n$, and $x - \lambda(x)1$ is in $Z(A_o)$, we have that $T_o$ actually contains $x - \lambda(x)1$ for every $x \in L$ by 1.2.9. But now, letting T be the kernel of the A-module homomorphism $A \to V$ defined by $a \mapsto va$, similar reasoning gives $V \approx A/T$. Noting that $T \cap A_o = T_o$, we have obtained:

THEOREM 1.2.10: *If* V *is an irreducible* L-*weighted* A-*module and* $\lambda$ *is a weight of* L *in* V, *then* $V \approx A/T$, *where* $x - \lambda(x)1 \in T$ *for every* $x \in L$.

We close this section with an easy result which nevertheless will prove quite useful.

PROPOSITION 1.2.11: *Suppose* $A = \bigoplus \sum_{i\in I} R_i$ *is an algebra direct sum. Let* L *be a diagonable subspace of* A *and write* $L = \bigoplus \sum_{i\in I} L_i$. *Then if every irreducible* A-*module is* L-*weighted, every irreducible* $R_{i_o}$-*module is* $L_{i_o}$-*weighted, for any* $i_o \in I$.

PROOF: Any irreducible $R_{i_o}$-module V is also an irreducible A-module under the definition $va = va_{i_o}$ for $v \in V$ and $a = \sum_{i\in I} a_i \in A$. Thus V is L-weighted and $V_\lambda \neq 0$ for some weight $\lambda$. Let $x \in L$. Then writing $1 = \sum_{i\in I} e_i$ and $x = \sum_{i\in I} x_i$, $v(x - \lambda(x)1)^n = 0$ implies $v(x_{i_o} - \lambda(x)e_{i_o})^n = 0$. Since $e_{i_o}$ is the identity in $R_{i_o}$, upon defining $\lambda_{i_o}(x_{i_o}) = \lambda(x)$, we see that V is $L_{i_o}$-weighted with weight $\lambda_{i_o}$.

## 1.3 A CORRESPONDENCE BETWEEN A- AND $A_o$-MODULES

Let $H$ be a Cartan subalgebra of a simple finite-dimensional Lie algebra $L$ over an algebraically closed field of characteristic 0, and let $C$ be the centralizer of $H$ in the universal enveloping algebra $U(L)$. Lemire ([14]) has shown that there is a one-to-one correspondence between the set of equivalence classes of irreducible representations of $L$ possessing a weight $\lambda \in H^*$ and $\lambda$-weighted representations of $C$; i.e., representations whose associated $C$-modules are $\lambda$-weighted. In this section, it is shown that this result follows from the fact that $H$ is a diagonable subspace of $U(L)$.

We denote by $W_\lambda$ the collection of all isomorphism classes $[V]$ of irreducible L-weighted A-modules V for which $V_\lambda \neq 0$, $\lambda \in L^*$. $W_\lambda{}^o$ denotes the collection of all isomorphism classes $[V]$ of non-zero irreducible $A_o$-modules V for which $V(x - \lambda(x)1) = 0$ for all $x \in L$. For convenience, we will write V when we strictly mean the isomorphism class $[V]$.

The discussion prior to 1.2.10 establishes a map $\Phi: W_\lambda \to W_\lambda{}^o$ given by $\Phi V = V_\lambda$. $\Phi$ is well-defined because of the next result:

LEMMA 1.3.1: *If W is an L-weighted irreducible A-module and V is any A-module isomorphic to W, then V is L-weighted and irreducible, and $W_\lambda$ is isomorphic to $V_\lambda$ as $A_o$-modules for every $\lambda \in L^*$.*

PROOF: We simply observe that if $\psi: W \to V$ is the A-module isomorphism, then by restricting $\psi$ to $W_\lambda$ and the ring of scalars to $A_o$, we obtain an isomorphism $W_\lambda \to V_\lambda$ because each module is irreducible and $\psi(W_\lambda) \subseteq V_\lambda$.

A one-to-one correspondence between $W_\lambda$ and $W_\lambda{}^o$ will be established by showing that the map $\Phi$ has an inverse. First, a result in linear algebra:

LEMMA 1.3.2: *Let* V *be any vector space over a field* F *and* $T_1,\ldots,T_n$ *distinct elements of* $V^*$. *Then there is a* $v \in V$ *such that* $\{T_1(v),\ldots,T_n(v)\}$ *is a set of* n *distinct scalars.*

PROOF: Consider the finitely many linear functionals $\{T_i - T_j : i \neq j\}$ of $V^*$. Since $T_1,\ldots T_n$ are distinct, $\{\ker(T_i - T_j): i \neq j\}$ is a finite set of proper subspaces of V. We are required to find a $v \in V$ such that $v \notin \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \ker(T_i - T_j)$. But this follows from the fact that V cannot be expressed as a finite union of proper subspaces because F is infinite (char F = 0). Indeed, if $V = \bigcup_{i=1}^{t} V_i$ , we may assume inductively that no $v_i$ is contained in $\bigcup_{j\neq i} V_j$ and hence choose $v_1 \in V_1 - \bigcup_{j=2}^{t} V_j$ and $v_2 \notin V_1$. There must be distinct scalars $\alpha_1$ and $\alpha_2$ such that $\alpha_1 v_1 + v_2$ and $\alpha_2 v_1 + v_2$ belong to the same $V_i$. Then $i \neq 1$ because $v_2 \notin V_1$ and so for some $i > 1$, $(\alpha_1 v_1 + v_2) - (\alpha_2 v_1 + v_2) \in V_i$. But then $(\alpha_1 - \alpha_2)v_1 \in V_i$ which is impossible.

DEFINITION 1.3.3: *An ideal* I *(right, left, or two-sided) of* $A = \oplus \sum_{\alpha\in\Delta} A_\alpha$ *is called* <u>*homogeneous*</u> *if* $I = \oplus \sum_{\alpha\in\Delta} I \cap A_\alpha$.

PROPOSITION 1.3.4: *Any two-sided ideal of* A *is homogeneous. Any right (or left) ideal containing* $x - \lambda(x)1$ *for every* $x \in L$, *and some* $\lambda \in L^*$ *is homogeneous.*

PROOF: Let I be a two-sided ideal of A and suppose $a = \sum_{\alpha\in\Delta} a_\alpha \in I$ with $a_\alpha \in A_\alpha$. Then for any $x \in L$, $(a,x) = \sum_{0\neq\alpha\in\Delta} \alpha(x)a_\alpha \in I$. Also $((a,x),x) = \sum_{0\neq\alpha\in\Delta} \alpha(x)^2 a_\alpha$ is in I, and continuing in this way, we get that for any integer $k > 0$,

$$(1) \qquad \sum_{0\neq\alpha\in\Delta} \alpha(x)^k a_\alpha = i_k \in I$$

Now $a_\alpha = 0$ for all $\alpha$ except $\alpha \in \{\alpha_1,\ldots,\alpha_n\}$. The $\alpha_i$ are in $L^*$ by 1.2.4, and so by 1.3.2, we can find an $x \in L$ for which $\{\alpha_1(x),\ldots,\alpha_n(x)\}$ is a set of n distinct scalars. Then letting k run from 1 to n, (1) is a system of linear equations over F whose matrix of coefficients is a vandermonde matrix with non-zero determinant. Solving, it is clear that each $a_{\alpha_i}$ is a linear combination of $i_1,\ldots,i_n$ and so in I. Of course $a_o = a - \sum_{i=1}^{n} a_{\alpha_i}$ is then in I too.

For the second statement of the proposition, if I is a right ideal of A containing $x - \lambda(x)1$ for all $x \in L$ and some $\lambda \in L^*$, and if $a = \sum_{\alpha\in\Delta} a_\alpha \in I$, we note that $(a,x) = (a,x - \lambda(x)1)$ is in I too, and hence the conclusion follows just as above.

<u>LEMMA 1.3.5</u>: *Any (proper) right ideal* I *of* $A_o$ *generates a right ideal of* A *contained in* $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$ *and hence is contained in a maximal right ideal* $I^*$ *of* A. *If* I *is maximal,* $I^* \cap A_o = I$, *and if, in addition, there is a* $\lambda \in L^*$ *such that* $x - \lambda(x)1 \in I$ *for every* $x \in L$, *then* $I^* \subseteq I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$ *and* $I^*$ *is unique.*

<u>PROOF</u>: The right ideal IA of A which I generates is contained in $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$ because $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. The existence of $I^*$ then follows from a standard argument involving an application of Zorn's Lemma. Now $I^* \cap A_o \neq A_o$ because $1 \notin I^*$, and since $I^* \cap A_o \supseteq I$ and is a right ideal of $A_o$, we must have $I^* \cap A_o = I$ if I is maximal. Next, if there is a $\lambda \in L^*$ with $x - \lambda(x)1 \in I$ for every $x \in L$, then

$$(2) \qquad J \subseteq I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$$

for any (proper) right ideal J of A which contains I. To obtain (2), we observe that such a J is homogeneous by 1.3.4 and so is contained in $(J \cap A_0) + \sum_{0 \neq \alpha \in \Delta} A_\alpha$. As $I \subseteq J \cap A_0 \subsetneq A_0$, $J \cap A_0 = I$ by the maximality of I. Thus the sum of all proper right ideals of A containing I is contained in $I \oplus \sum_{0 \neq \alpha \in \Delta} A_\alpha$ and so must again be proper. Clearly this is the unique maximal right ideal I*.

Now define a map $\Psi: W_\lambda^{\,o} \to W_\lambda$ as follows: if $V \in W_\lambda^{\,o}$, then $V \simeq A_0/I$ where I is a maximal right ideal of $A_0$ containing $x - \lambda(x)1$ for all $x \in L$ and some $\lambda \in L^*$. This is so because $V(x - \lambda(x)1) = 0$ implies $(I + 1)(x - \lambda(x)1) = 0$. By Lemma 1.3.5, I extends uniquely to a maximal right ideal I* of A. $A/I^* \in W_\lambda$ because $(I^* + 1)(x - \lambda(x)1) = 0$ for every $x \in L$. Define $V = A/I^*$. $\Psi$ is well-defined because of:

<u>LEMMA 1.3.6</u>: *Suppose* $A_0/I_1$ *and* $A_0/I_2$ *are irreducible and isomorphic* $A_0$*-modules, and for some* $\lambda \in L^*$, $x - \lambda(x)1 \in I_1 \cap I_2$ *for every* $x \in L$. *Then* $A/I_1^* \simeq A/I_2^*$ *as* A*-modules, where* $I_1^*$ *and* $I_2^*$ *are the right ideals of* A *given by* 1.3.5.

<u>PROOF</u>: Suppose that $\sigma(I_1 + 1) = I_2 + a_0$, where $\sigma: A_0/I_1 \to A_0/I_2$ is the given isomorphism. Lift $\sigma$ to $\sigma^*: A/I_1^* \to A/I_2^*$ by $\sigma^*: I_1^* + a \mapsto I_2^* + a_0 a$. $\sigma^*$ is well-defined for if $a \in I_1^*$ but $a_0 a \notin I_2^*$, then $I_2^* + a_0 aA = A$, and so there exists $u \in A$ such that

$$(3) \qquad a_0 au - 1 \in I_2^* \subseteq I_2 \oplus \sum_{0 \neq \alpha \in \Delta} A_\alpha .$$

Now because $au \in I_1^* \subseteq I_1 \oplus \sum_{0 \neq \alpha \in \Delta} A_\alpha$, we can write $au = b_0 + \sum b_\alpha$ where $b_0 \in I_1$, and so from (3) we see that $a_0 b_0 - 1 \in I_2$. This is impossible because $\sigma(I_1 + b_0) = 0 = I_2 + a_0 b_0$. Finally, since $I_2^* \subseteq I_2 \oplus \sum_{0 \neq \alpha \in \Delta} A_\alpha$, $a_0 \notin I_2^*$.

Hence $\sigma^*$ is not zero and therefore must be an isomorphism by Schur's Lemma.

We are now able to prove the main result of this section.

<u>THEOREM 1.3.7</u>: *$\Phi$ defines a one-to-one correspondence between $W_\lambda$ and $W_\lambda{}^o$.*

<u>PROOF</u>: We prove that $\Phi$ and $\Psi$ are inverse maps. Given $V \in W_\lambda$, we recall that $\Phi V = V_\lambda \simeq A_o/I$ for some maximal right ideal $I$ of $A_o$ containing $x - \lambda(x)1$ for every $x \in L$. $\Psi(\Phi V)$ is then $A/I^*$ where $I^*$ is that ideal of $A$ given by Lemma 1.3.5. That $\Psi(\Phi V) \simeq V$ is now apparent from the discussion preceding Theorem 1.2.10 upon observing that the ideal $T$ defined there is the ideal $I^*$ by uniqueness. Conversely, given $V \in W_\lambda{}^o$, $V \simeq A_o/J$ where $J$ is a maximal right ideal of $A_o$ containing $x - \lambda(x)1$ for every $x \in L$, for some $\lambda \in L^*$. Letting $J^*$ be that ideal of $A$ given by 1.3.5, $\Phi(\Psi V)$ is then $(A/J^*)_\lambda$. To see that this is isomorphic to $V$, define a map $\sigma: A_o \to (A/J^*)_\lambda$ by $a_o \mapsto J^* + a_o$. For any $a_o \in A_o$, $a_o(x - \lambda(x)1) = (x - \lambda(x)1)a_o \in J \subsetneq J^*$ and so $\sigma(A_o) \subsetneq (A/J^*)_\lambda$. $\sigma$ is surjective because it is non-zero and $(A/J^*)_\lambda$ is irreducible. The kernel of $\sigma$ is $\{a_o \in A_o: a_o \in J^*\} = J^* \cap A_o = J$ by 1.3.5. Thus, $(A/J^*)_\lambda \simeq A_o/J \simeq V$ as required; i.e., $\Phi(\Psi V) \simeq V$.

## 1.4 FURTHER RESULTS CONCERNING $W_\lambda$ AND $W_\lambda{}^o$

Let V be an L-weighted irreducible A-module with $\Lambda$ the set of weights of L in V. We have seen how the associated family $A(V) = \{V_\lambda : \lambda \in \Lambda\}$ of irreducible $A_o$-modules determines via the map $\Psi$ the same irreducible A-module V, up to isomorphism (1.2.10). A natural question to ask at this point is the following: given some subset $\Lambda$ of L*, the dual space of L, and a family $F = \{V_\lambda \in W_\lambda{}^o : \lambda \in \Lambda\}$ of $A_o$-modules, does there exist an A-module V such that $F = A(V)$?

Suppose $I_\lambda$ and $I_\mu$ are maximal right ideals of $A_o$ containing $x - \lambda(x)1$ and $x - \mu(x)1$ for every $x \in L$ respectively, where $\lambda, \mu \in L^*$. We know that there exist (unique) maximal right ideals $I_\lambda{}^*$ and $I_\mu{}^*$ of A with $A/I_\lambda{}^* \in W_\lambda$, $A/I_\mu{}^* \in W_\mu$, and such that

$$(1) \qquad (A/I_\lambda{}^*)_\lambda \cong A_o/I_\lambda \quad ; \quad (A/I_\mu{}^*)_\mu \cong A_o/I_\mu$$

This was established in the proof of Theorem 1.3.7. Suppose now that $A/I_\lambda{}^* \cong A/I_\mu{}^*$. Then (1) certainly implies $(A/I_\lambda{}^*)_\mu \neq 0$ and thus from the discussion preceding Theorem 1.2.10, $I_\mu{}^*$ is the kernel of the homomorphism $\tau: A \to A/I_\lambda{}^*$ given by $\tau(a) = I\,^* + b_\mu a$, where $I_\lambda{}^* + b_\mu$ is any non-zero element of $(A/I_\lambda{}^*)_\mu$. By the irreducibility of $A/I_\lambda{}^*$, $(I_\lambda{}^* + b_\mu)A = A/I_\lambda{}^*$ and there is a $y \in A$ such that $b_\mu y - 1 \in I_\lambda{}^*$. Clearly $y \notin I_\mu{}^*$ because then $b_\mu y$ and hence 1 would belong to $I_\lambda{}^*$; but $yI_\lambda{}^* \subseteq I_\mu{}^*$ because if $a \in I_\lambda{}^*$, $b_\mu ya = (b_\mu y - 1)a + a \in I_\lambda{}^*$ and so $ya \in \ker \tau = I_\mu{}^*$. We have established the necessity part of the next theorem.

<u>THEOREM 1.4.1</u>: *A family $\{A_o/I_\lambda \in W_\lambda{}^o : \lambda \in \Lambda \subseteq L^*\}$ of $A_o$-modules is the family corresponding to an A-module V if and only if there exist elements*

$y_{\mu,\lambda}$ *in* A, *for each* $\lambda$ *and* $\mu$ *in* $\Lambda$, *such that* $y_{\mu,\lambda} \notin I_\mu{}^*$ *but* $y_{\mu,\lambda} I_\lambda{}^* \subseteq I_\mu{}^*$.

PROOF: We need only prove the sufficiency. If $\lambda, \mu \in \Lambda$ and there is a y in A, $y \notin I_\mu{}^*$, $yI_\lambda{}^* \subseteq I_\mu{}^*$, then the map $A/I_\lambda{}^* \to A/I_\mu{}^*$ defined by $I_\lambda{}^* + a \mapsto I_\mu{}^* + ya$ is well-defined and non-zero, hence an isomorphism by the irreducibility of the two modules. Let $V = A/I_\lambda{}^*$. By (I) and Lemma 1.3.1, we see that V has the desired property.

We now consider the extent to which the classes $W_\lambda$ and $W_\lambda{}^o$ determined by the weight $\lambda$. The situation is ideal for $A_o$-modules, somewhat less than ideal for A-modules as the following theorem shows.

THEOREM 1.4.2: (i) *If* $W_\lambda{}^o \cap W_\mu{}^o \neq \phi$, *then* $\lambda$ *is identically equal to* $\mu$ $(\lambda \equiv \mu)$.

(ii) *If* $W_\lambda \cap W_\mu \neq \phi$, *then for some* $\alpha \in \Delta$, $\mu = \lambda + \alpha$ *and* $-\alpha \in \Delta$.

PROOF: (i) If $V \in W_\lambda{}^o \cap W_\mu{}^o$, then there exist maximal right ideals $I_\lambda$ and $I_\mu$ of $A_o$ containing $x - \lambda(x)1$ and $x - \mu(x)1$ respectively, for every $x \in L$, and such that $V \simeq A_o/I_\lambda \simeq A_o/I_\mu$. Suppose $\sigma\colon A_o/I_\lambda \to A_o/I_\mu$ is the isomorphism and $\sigma(I_\lambda + 1) = I_\mu + a_o$, $a_o \in A_o$. Then $\sigma(I_\lambda + (x - \lambda(x))1) = 0$ implies $a_o(x - \lambda(x)1) \in I_\mu$ and since $a_o$ cannot belong to $I_\mu$, by Lemma 1.2.9 we must have $x - \lambda(x)1 \in I_\mu$ for every $x \in L$. Thus $I_\mu$ contains

$$(x - \lambda(x)1) - (x - \mu(x)1) = (\mu - \lambda)(x)1 \quad \text{for every } x \in L.$$

Since $I_\mu$ is proper, $\mu \equiv \lambda$.

(ii) If $V \in W_\lambda \cap W_\mu$, then there are maximal right ideals $I_\lambda$ and $I_\mu$ of A containing $x - \lambda(x)1$ and $x - \mu(x)1$ respectively, for every $x \in L$, and such that $V \simeq A/I_\lambda \simeq A/I_\mu$. We now notice that the decomposition of A relative to L induces decompositions of V;

$$(2) \qquad V \simeq \bigoplus \sum_{\alpha\varepsilon\Delta} A_\alpha + I_\lambda/I_\lambda \;\simeq\; \bigoplus \sum_{\alpha\varepsilon\Delta} A_\alpha + I_\mu/I_\mu$$

Moreover, for $a_\alpha \varepsilon A_\alpha$ and $x \varepsilon L$, $a_\alpha(x - \lambda(x)1) = (x - \lambda(x)1)a_\alpha + \alpha(x)a_\alpha$ so that $a_\alpha(x - (\lambda+\alpha)(x)1) \varepsilon I_\lambda$. By Lemma 1.3.1, $V_{\lambda+\alpha} \simeq A_\alpha + I_\lambda/I_\lambda$ and the weights of L in V are contained in the set $\{\lambda + \alpha: \alpha \varepsilon \Delta\}$. Since $\mu$ is a weight, there is an $\alpha$ in $\Delta$ with $\mu = \lambda + \alpha$. By symmetry, there is an $\alpha' \varepsilon \Delta$ with $\lambda = \mu + \alpha'$. Clearly $\alpha' = -\alpha$.

We single out the following two results, now obvious.

COROLLARY 1.4.3: *If* V *is an irreducible* L-*weighted* A-*module, then the weights of* L *in* V *are of the form* $\lambda_o + \alpha$, $\alpha \varepsilon \Delta$, *where* $\lambda_o$ *is any fixed weight. In particular if* A *has only finitely many non-zero roots,* V *has only finitely many weights.*

COROLLARY 1.4.4: *If* V *is an irreducible* L-*weighted* A-*module, then any two weights of* L *in* V *differ by a root of* A.

In the sequel, we will frequently have cause to suppose that A has only finitely many non-zero roots. We therefore make the

DEFINITION 1.4.5: L *is a finitely diagonable subspace of* A *if it is diagonable, and* $A_\alpha = 0$ *for all but a finite number of* $\alpha$ *in* $L^*$.

# CHAPTER TWO

## RING-THEORETIC CONNECTIONS BETWEEN $A_o$ AND $A$

### 2.1 CHAIN CONDITIONS

If R is a ring (associative) and M an R-module, then M has the descending chain condition on submodules if every decreasing sequence $M_1 \supseteq M_2 \supseteq \ldots$ of submodules of M terminates; i.e., for some $k > 0$, $M_t = M_k$ for all $t \geq k$. This is equivalent to saying that M has the minimum condition on submodules: every collection of submodules of M has a smallest member (with respect to inclusion). Definitions of the ascending chain condition and maximum condition are made in the obvious way. If M = R considered as a right R-module, submodules of M are right ideals of R and if M has the descending (respectively ascending) chain condition on ideals, R is said to be artinian (respectively noetherian).

We employ the same notation as that of Chapter 1. L is a diagonable subspace of an algebra A over a field F. A decomposes as a vector space, $A = \sum_{\alpha \in \Delta} A_\alpha$, relative to the set $\Delta$ of roots of L in A.

PROPOSITION 2.1.1: *If A is artinian (respectively noetherian), then each $A_o$-module $A_\alpha$ has the descending (respectively ascending) chain condition on its submodules. In particular, $A_o$ is artinian (respectively noetherian). Conversely, if each $A_\alpha$ has the descending (respectively ascending) chain condition on $A_o$-submodules and L is a finitely diagonable subspace of A, then A is artinian (respectively noetherian).*

PROOF: Suppose A is artinian, and $M_1 \supseteq M_2 \supseteq \ldots$ is a descending chain of $A_o$- submodules of $A_\alpha$. For each i, $M_i$ generates the right ideal $M_i^* = M_iA$ of A, and this is contained in $M_i + \sum_{\alpha \neq \beta \varepsilon \Delta} A_\beta$ because $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ for any $\alpha$, $\beta \varepsilon \Delta$ (§1.2). Since $M_1^* \supseteq M_2^* \supseteq \ldots$, there is an integer $k > 0$ for which $M_t^* = M_k^*$ for every $t \geqslant k$. Now if $t \geqslant k$ and $m \varepsilon M_k$, $m \varepsilon M_k^* = M_t^*$ and $M_t^* \subseteq M_t + \sum_{\alpha \neq \beta \varepsilon \Delta} A_\beta$, so considering the components of m relative to L, $m \varepsilon M_t$. Thus $M_k \subseteq M_t$ and the original sequence of $A_o$-modules terminates at $M_k$. On the other hand, if L is finitely diagonable and each $A_\alpha$ has the descending chain condition on $A_o$-submodules, let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of right ideals of A. Notice that each $I_j$ can be written as $\oplus \sum_{\alpha \varepsilon \Delta} (I_j)_\alpha$, where $(I_j)_\alpha$ is the $A_o$-module of $\alpha$-components of elements in $I_j$. Certainly $k \leqslant \ell$ implies $(I_k)_\alpha \supseteq (I_\ell)_\alpha$ for each $\alpha$. Thus we obtain finitely many descending chains $\{(I_j)_\alpha : j = 1,2,\ldots\}$ of $A_o$-submodules of $A_\alpha$, one for each $\alpha \varepsilon \Delta$. Each chain terminates after a finite number of steps, so they have all terminated by the $n^{th}$ step, say. This implies that $I_m = I_n$ for any $m \geq n$, and so A is artinian. The proof for the noetherian case simply mimics the above.

EXAMPLES 2.1.2: In the converse of the previous proposition, the hypothesis that L be a finitely diagonable subspace of A is essential as the following examples show.

(i) Let $A = R[y]$ be the ring of polynomials in y over the algebra $R = F(x)$ of rational functions in x over a field F. Here x and y are indeterminates and $yx = (x+1)y$. It is not difficult to show that for any $r(x) \varepsilon R$, and integer $k \geqslant 0$, $y^k r(x) = r(x+k)y^k$. $L = Fx$ is a diagonable subspace of A, and relative to L, A decomposes as $A_o + \sum_{n=1}^{\infty} A_n$, where $A_n = Ry^n$ for all integers $n \geqslant 0$. Clearly $A_o$ is a field and each $A_o$-module $A_n$ is one dimensional. However, A is not artinian ($yA \supset y^2A \supset \ldots$ is a decreasing infinite chain of right ideals) and L is not finitely diagonable.

(ii) Here we let $A = R[y_1, y_2, \ldots]$ be the algebra of polynomials in commuting, algebraically independent indeterminates $y_1, y_2, \ldots$ over the usual (commutative) ring $R = F[x]$ of polynomials in an indeterminate x over the field F. Define $y_i x = (x + i)y_i$. Then it is straightforward to show that

$$y_{i_1}{}^{n_1} \ldots y_{i_k}{}^{n_k} f(x) = f(x + n_1 i_1 + \ldots + n_k i_k) y_{i_1}{}^{n_1} \ldots y_{i_k}{}^{n_k}$$

for any $f(x) \in R$. Moreover, it follows that $L = Fx$ is a diagonable subspace of A, $A = A_o + \sum_{n=1}^{\infty} A_n$, and $A_n = \sum_{\sum_j i_j s_j = n} R y_{i_1}{}^{s_1} \ldots y_{i_r}{}^{s_r}$ for $n \geq 0$. $A_n$ is finitely generated as an $A_o$-module and $A_o = R$ is a principal ideal domain (hence noetherian), so $A_n$ has the ascending chain condition on $A_o$-submodules (see, for example, [2; §11.14-11.16]). However, A is not noetherian ($y_1 A \subsetneq y_1 A + y_2 A$ ... is a strictly increasing sequence of right ideals of A) and L is not finitely diagonable.

## 2.2 NILPOTENT IDEALS AND SEMI-PRIMENESS

An ideal I (right, left, or two-sided) of a ring R is nilpotent if for some integer $n > 0$, $I^n = 0$, where $I^n$ denotes the ideal of R consisting of sums of monomials $x_1x_2...x_n$, each $x_i \in I$. R is called semi-prime if (0) is the only nilpotent ideal. If V is an R-module, $(0:V) = \{ x \in R: Vx = 0\}$ is the annihilator of V in R. The Jacobson radical of R, J(R), is then defined to be $\bigcap\{(0:V):$ V an irreducible R-module$\}$. It is a two-sided ideal of R containing every nilpotent left, right and two-sided ideal of R. R is said to be semi-simple if $J(R) = 0$.

Now define a new associative binary operation on R by defining $x \circ y = x + y - xy$ for any x and y in R. The pair $(R,\circ)$ is then a monoid with identity the 0 element of R. If $x \circ y = 0$, we say y is a right quasi-inverse of x, and x is right quasi-regular; x is a left quasi-inverse of y and y is left quasi-regular. If every element of a right ideal is right quasi-regular, then every element x of the ideal is quasi-regular in the sense that x is both right and left quasi-regular with unique quasi-inverse. Assuming that R has an identity 1, this is equivalent to saying that $1 - x$ is invertible in R, because on writing this inverse in the form $1 - y$, the identity $(1 - x)(1 - y) = 1 + x \circ y$ shows that y is the quasi-inverse of x. It turns out that the Jacobson radical can be characterized as a quasi-regular ideal (every element in it is quasi-regular) which contains every quasi-regular right ideal. This characterization will prove useful in the sequel. A good discussion of all these ideas can be found in [5].

Now throughout this section, assume that L is a finitely diagonable subspace of the algebra A, so that $A_\alpha = 0$ except for $\alpha$ in the set

$\Delta = \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. The reader is reminded that for any $\alpha, \beta$ in $\Delta$, $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$, where $A_{\alpha+\beta}$ is 0 if $\alpha + \beta \notin \Delta$. We first prove the following key lemma:

<u>LEMMA 2.2.1</u>: *Given elements* $x_1, x_2, \ldots, x_{k+1}$ *of* A *with* $x_i \in A_{\beta_i}$ *and* $\beta_i \in \Delta$, $i = 1, \ldots, k+1$, *then there exist integers* m *and* n *with* $1 \leq m \leq n \leq k+1$ *so that* $x_m \ldots x_n \in A_o$.

<u>PROOF</u>: Let $b_1 = \beta_1$, $b_2 = \beta_1 + \beta_2$, $\ldots$, $b_{k+1} = \beta_1 + \beta_2 + \ldots + \beta_{k+1}$. If some $b_i \notin \{\alpha_1, \ldots, \alpha_k\}$ either $b_i \notin \Delta$ in which event $x_{b_i} = 0$, or $b_i = 0$. In either case $x_{b_i} \in A_o$ and the assertion of the lemma is true. Otherwise, $\{b_1, \ldots b_{k+1}\}$ is a set of $k + 1$ elements in a set of cardinality k. So there exist distinct integers r and n with $b_r = b_n$. Assuming $r < n$, we have $0 = b_n - b_r = \beta_{r+1} + \ldots + \beta_n$. Letting $m = r + 1$, it follows that $x_m \ldots x_n \in A_o$.

This enables us to prove

<u>LEMMA 2.2.2</u>: *Suppose* I *is a right ideal of* A *such that* $(I \cap A_o)^2 = 0$, *then* I *is nilpotent.*

<u>PROOF</u>: We show that $I^{(k+1)(k+2)} = 0$. For this, let $x_i$, $i = 1, \ldots, (k+1)(k+2)$ be $(k+1)(k+2)$ elements of I, and define $y_i$ for $i = 1, \ldots, k+2$ by $y_i = x_{(i-1)(k+1)+1} \ldots x_{i(k+1)}$. Each $y_i \in I$, and so by the previous result, we may write

$$\prod_{i=1}^{k+2} y_i = a u_1 a_{\beta_1} u_2 a_{\beta_2} \ldots u_{k+1} a_{\beta_{k+1}} u_{k+2} b$$

where a and b are either 1 or in (possibly different) root spaces $A_\gamma$, $\gamma \neq 0$, each $u_i \in I \cap A_o$, $i = 1, \ldots k+2$, and each $\beta_i \in I \cap A_{\beta_i}$ where for $i = 1, \ldots, k+1$,

$\beta_1 \in \{\alpha_1,\ldots,\alpha_k\}$. Now $a_{\beta_1}u_2\ldots a_{\beta_{k+1}}$ belongs to $A_{\beta_1}\ldots A_{\beta_{k+1}}$ and so again using 2.2.1, there exist integers m and n with $1 \leq m \leq n \leq k+1$ such that $a_{\beta_m}u_{m+1}\ldots a_{\beta_n}$ belongs to $I \cap A_o$. Thus $u_m a_{\beta_m}u_{m+1}\ldots a_{\beta_n} \in (I \cap A_o)^2 = 0$ and

$$\prod_{j=1}^{(k+1)(k+2)} x_j = \prod_{i=1}^{k+2} y_i = 0.$$

Since $I^{(k+1)(k+2)}$ consists of sums of products of $(k+1)(k+2)$ elements from I, the lemma is established.

The main result of this section now follows.

THEOREM 2.2.3: *If* A *is semi-prime, so is* $A_o$. *Conversely if* $A_o$ *is semi-prime, then the nilpotent right ideals of* A *are exactly those contained in* $\sum_{0\neq\alpha\in\Delta} A_\alpha$.

PROOF: Suppose A is semi-prime and I is a right ideal of $A_o$ with $I^n = 0$. By an easy induction argument, we can assume $n = 2$. Now IA is a right ideal of A contained in $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$. Also $IA \cap A_o = I$ has square 0, so IA is nilpotent by the previous lemma. Hence $IA = 0$ and $I = 0$. Conversely, if $A_o$ is semi-prime and I is a nilpotent right ideal of A, then $I_1 = I + AI$ is a two-sided ideal of A which is nilpotent because $I_1{}^k \subseteq I^k + AI^k$ for any positive integer k. If $I_1{}^t = 0$, $(I_1 \cap A_o)^t = 0$, so $I_1 \cap A_o = 0$ and $I_1 \subseteq \sum_{0\neq\alpha\in\Delta} A_\alpha$ by homogeneity (1.3.4). $I \subseteq I_1$ so I is contained in $\sum_{0\neq\alpha\in\Delta} A_\alpha$ too. Finally, we note that any ideal contained in $\sum_{0\neq\alpha\in\Delta} A_\alpha$ is nilpotent by 2.2.2 since $I \cap A_o = 0$.

COROLLARY 2.2.4: *If* $A_o$ *is semi-simple, then* J(A) *is the sum of all right ideals* I *of* A *contained in* $\sum_{0\neq\alpha\in\Delta} A_\alpha$ *and hence is nilpotent.*

PROOF: Let T denote the sum defined here. Certainly $T \subseteq J(A)$ because

$J(A)$ contains all nilpotent ideals of $A$. We prove that the reverse inclusion holds by establishing

$$(1) \qquad J(A) \cap A_0 \subseteq J(A_0)$$

To see this, we use the fact that the Jacobson radical of a ring with 1 can also be characterized as the intersection of all maximal right ideals ([4; page 11]). But every maximal right ideal of $A_0$ is contained in a maximal right ideal of $A$ by 1.3.5.

Before continuing, we remark that once again the underlying constraint that $L$ be finitely diagonable is essential, for otherwise, we obtain a counter-example by taking $A = F[x]\langle y\rangle$, the ring of formal power series in $y$ over the pölynomial ring $F[x]$, where $x$ and $y$ are indeterminates, and $yx = (x+1)y$. $A_0 = F[x]$ is semi-simple, but $J(A)$ is not nilpotent; indeed, it contains the quasi-regular non-nilpotent ideal $B = \{ \sum_{i=1}^{\infty} f_i(x)y^i : f_i(x) \in F[x]\}$. That $B$ is quasi-regular can be seen by noting that for a given element $\sum_{i=1}^{\infty} f_i(x)y^i$ of $B$, the equation

$$(1 - \sum_{i=1}^{\infty} f_i(x)y^i)(1 + \sum_{i=1}^{\infty} g_i(x)y^i) = 1$$

can be solved inductively for the $g_i(x) \in F[x]$, $i = 1,2,\ldots$

We close this section with

<u>THEOREM 2.2.5</u>: *If every* $A$*-module is* $L$*-weighted, then* $J(A_0) = A_0 \cap J(A)$ *and hence* $A_0$ *is semi-simple if* $A$ *is.*

<u>PROOF</u>: The inclusion (1) is always valid, so we need prove only that $J(A_0) \subseteq J(A) \cap A_0$. Let $V$ be an irreducible $A$-module. Then $V = \oplus \sum_{\lambda \in \Lambda} V_\lambda$

relative to the set $\Lambda$ of weights of L in V. Now suppose $va_o = 0$ with $a_o \in A_o$ and $v = \sum_{\lambda\varepsilon\Lambda} v_\lambda \in V$. Then each $v_\lambda a_o = 0$ because $V_\lambda A_\alpha \subseteq V_{\lambda+\alpha}$ (1.2.6). Thus, we obtain

$$(2) \qquad (0:V) \cap A_o = \bigcap_{\lambda\varepsilon\Lambda} (0:V_\lambda)_o$$

where $(0:M)_o$ indicates the annihilator in $A_o$ of an $A_o$-module M. Let $I$ (respetively $I_o$) denote the class of all irreducible right A- (respectively $A_o$-) modules. Then (2) implies

$$\left(\bigcap_{V\varepsilon I} (0:V)\right) \cap A_o = \bigcap_{V\varepsilon I} ((0:V) \cap A_o) = \bigcap_{V\varepsilon I} \bigcap_{\lambda\varepsilon\Lambda} (0:V_\lambda)_o \supseteq \bigcap_{V\varepsilon I_o} (0:V)_o$$

the last inclusion following because each $V_\lambda$ is an irreducible $A_o$-module by Lemma 1.2.8. Thus $J(A) \cap A_o \supseteq J(A_o)$ as required.

## 2.3 PRIMITIVITY

An ideal P of a ring R is primitive if it is the largest (two-sided) ideal of R contained in a maximal right ideal of R. Primitive ideals are exactly those which are annihilators of irreducible R-modules. A primitive ring is one in which (0) is a primitive ideal; i.e., there is a maximal right ideal containing no non-zero two-sided ideal, or, alternatively, there exists an irreducible module V which is faithful in the sense that its annihilator is 0.

Suppose now that the algebra A is primitive with faithful irreducible module V. Taking a closer look at (2) of the previous section, we see that we have a collection of irreducible $A_o$-modules $\{V_\lambda : \lambda \in \Lambda\}$ with $\bigcap_{\lambda\in\Lambda} (0:V_\lambda)_o = 0$. Now $P_\lambda = (0:V_\lambda)_o$ is a primitive ideal of $A_o$ containing $x - \lambda(x)1$ for every $x \in L$ as the discussion after Lemma 1.2.9 showed. Moreover, for $\lambda,\mu \in \Lambda$ and $\lambda \neq \mu$, there is an $x \in L$ for which $\lambda(x) \neq \mu(x)$ and so $(x - \mu(x)1) - (x - \lambda(x)1) = (\lambda - \mu)(x)1$ is in $P_\lambda + P_\mu$. This implies $P_\lambda + P_\mu = A_o$. Assuming L is finitely diagonable, we know by 1.4.3 that $\Lambda$ is finite, and so we can apply the Chinese Remainder Theorem to obtain $A_o \approx \bigoplus\sum_{\lambda\in\Lambda} A_o/P_\lambda$. The quotient of any ring by a primitive ideal is a primitive ring, so here we have realized $A_o$ as a direct sum of primitive algebras. Finally we notice that Propositions 1.2.1 and 1.2.11 enable us to extend our result to direct sums, thus giving

THEOREM 2.3.1: *Suppose* L *is a finitely diagonable subspace of an algebra* A *which is a direct sum of primitive algebras and further suppose that every irreducible* A*-module is weighted. Then* $A_o$ *is a direct sum of primitive algebras.*

As a partial converse, we obtain:

THEOREM 2.3.2: *If* L *is a finitely diagonable subspace of* A, *and* $A_o$ *is a direct sum of primitive algebras, then* A/J(A) *can be embedded in a direct sum of primitive algebras. Assuming also that* A *is semi-prime,* A *itself is so embeddable.*

PROOF: $A_o$ is necessarily a finite direct sum of primitive algebras $R_j$, $j = 1,\ldots,n$ because it contains 1. Let $I_j$ be a maximal right ideal of $R_j$ containing no non-zero two-sided ideal. Then $\sum_{j=1}^{n} I_j$ can contain no ideal of $A_o$ except (0), because such an ideal T would decompose as $\bigoplus\sum_{j=1}^{n} T \cap R_j$ (for $1 \in A_o$), and $T \cap R_j \subsetneq I_j$. For each j, let

$$T_j = I_j \oplus \sum_{\substack{i=1\\ i\neq j}}^{n} R_i$$

Then $T_j$ is a maximal right ideal of $A_o$, and so contained in a maximal right ideal $T_j^*$ of A such that $T_j^* \cap A_o = T_j$ by Lemma 1.3.5. Then $P_j = \{a \in A\colon Aa \subseteq T_j^*\}$ is contained in $T_j^*$ (since $1 \in A$) and is a primitive ideal because it is the annihilator of the irreducible A-module $A/T_j^*$. Notice that $(\bigcap_{j=1}^{n} P_j) \cap A_o \subseteq \bigcap_{j=1}^{n} (T_j^* \cap A_o) = \bigcap_{j=1}^{n} T_j = \bigoplus \sum_{j=1}^{n} I_j$, and so, because $(\bigcap_{j=1}^{n} P_j) \cap A_o$ is a two-sided ideal, it must be (0) by what was stated above. Thus $\bigcap_{j=1}^{n} P_j \subseteq \sum_{0\neq\alpha\in\Delta} A_\alpha$ by homogeneity and hence contained in J(A) by 2.2.4. (Note that a primitive ring is semi-simple). The reverse inclusion always holds (see for example [5; page 40]) so we have that $J(A) = \bigcap_{j=1}^{n} P_j$. Clearly A/J(A) can be embedded in $\bigoplus \sum_{j=1}^{n} A/P_j$. Since 2.2.4 implies J(A) is nilpotent, the last statement of the theorem is obvious.

<u>COROLLARY 2.3.3</u>: *Under the conditions of the theorem, if* $A_o$ *is primitive, then* $A/J(A)$ *is primitive.*

<u>PROOF:</u> If I is a maximal right ideal of $A_o$ containing no non-zero two-sided ideal of $A_o$, and $I^*$ is the maximal right ideal of A generated by I, then $P \cap A_o = 0$, where P is the largest ideal of A contained in $I^*$. To see this, we simply note that $P \cap A_o \subseteq I^* \cap A_o = I$. Thus $P = J(A)$ as in the theorem. Since P is a primitive ideal, the conclusion follows.

# CHAPTER THREE

# DIAGONABLE ELEMENTS AND WEIGHTED MODULES

## 3.1 IDEMPOTENTS ARE DIAGONABLE

In this chapter we give examples of diagonable elements and look for conditions on an algebra which allow something to be said about the nature of its diagonable elements. They often turn out to be algebraic. The results of Chapter Two will be of great help in giving information both about a diagonable element x and its centralizer because of course $A_0(x) = A_0(Fx)$. When employing results from Chapter Two, the diagonable subspace L is always understood to be Fx. A module is said to be x-weighted if it is Fx-weighted and we will refer to "weights of x" and "roots of x" rather than weights and roots of Fx. Also, since any weight or root is a linear functional on L (1.2.4) it is completely determined by its value at x. We will identify a root $\alpha$ with the scalar $\alpha(x)$ so that the "roots of x" will actually be the roots of the minimum polynomial of ad x in the event all but finitely many root spaces are zero. Similarly, the weights $\lambda$ will be identified with the scalars $\lambda(x)$.

Any algebra containing idempotents abounds in diagonable elements, for we have:

THEOREM 3.1.1: *An element* $x = \sum_{i=1}^{n} \alpha_i e_i$ *with* $\alpha_1, \ldots, \alpha_n \in F$ *is a linear combination of orthogonal idempotents* $e_1, \ldots, e_n$ *with sum* 1 *if and only if* x *is algebraic with minimal polynomial having distinct roots in* F. *Such an element is diagonable, and* $A_\alpha(x) = \sum_{\alpha_j - \alpha_i = \alpha} e_i A e_j$.

PROOF: First assume $x = \sum_{i=1}^{n} \alpha_i e_i$ as above. Let $S$ be any subset of $\{1,\ldots,n\}$ with the property that $\{\alpha_i: i \in S\}$ is the set of distinct coefficients of $e_1,\ldots,e_n$ appearing in $\sum_{i=1}^{n} \alpha_i e_i$. We prove that x has the minimal polynomial

$$(1) \qquad \prod_{i \in S} (t - \alpha_i) \in F[t].$$

For this it is sufficient to assume that $S = \{1,\ldots,n\}$ for if $\alpha_1,\ldots,\alpha_n$ are not all distinct, define for each $i \in S$, $f_i = \sum_{\alpha_j=\alpha_i} e_j$. Then $x = \sum_{i \in S} \alpha_i f_i$, $\{f_i: i \in S\}$ are orthogonal idempotents with sum 1, and $\{\alpha_i: i \in S\}$ is a distinct set of scalars. Now let $f(t) = \prod_{i=1}^{n}(t - \alpha_i)$. Since

$$x - \alpha_j 1 = x - \alpha_j\left(\sum_{i=1}^{n} e_i\right) = \sum_{\substack{i=1 \\ i \neq j}}^{n} (\alpha_i - \alpha_j)e_i$$

it is clear that $f(x) = 0$. To see that f is actually the minimal polynomial, note that if $g(t) = \sum_{i=1}^{n} a_i t^i \in F[t]$, then

$$g(x) = \sum_{i=1}^{n} a_i\left(\sum_{j=1}^{n} \alpha_j{}^i e_j\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_i \alpha_j{}^i\right)e_i$$

and so $g(x) = 0$ if and only if $\alpha_1,\ldots,\alpha_n$ are roots of g (because $e_1,\ldots,e_n$ are orthogonal idempotents). Since $\alpha_1,\ldots,\alpha_n$ are distinct, degree $g = m$ is at least as big as $n =$ degree f. Conversely, if x is algebraic with minimal polynomial $f(t) = \prod_{i=1}^{n}(t - \alpha_i) \in F[t]$ where $\alpha_1,\ldots,\alpha_n$ are distinct scalars, define for each $i = 1,\ldots,n$, $h_i(t) = \prod_{j \neq i}(t - \alpha_j)$. Then $h_1(t),\ldots,h_n(t)$ are relatively prime elements of the Euclidean domain $F[t]$, so there exist $a_1(t),\ldots,a_n(t)$ in $F[t]$ with $\sum_{i=1}^{n} a_i(t)h_i(t) = 1$. Let $e_i = a_i(x)h_i(x)$, for $i = 1,\ldots,n$. Then $e_1,\ldots,e_n$ are orthogonal idempotents with sum 1, $x = \sum_{i=1}^{n} xe_i$ and $xe_i = (x - \alpha_i 1)e_i + \alpha_i e_i = \alpha_i e_i$. Finally, to see that an element $x = \sum_{i=1}^{n} \alpha_i e_i$ is diagonable, we note that if $a \in A$,

$$a = \left(\sum_{i=1}^{n} e_i\right)a\left(\sum_{j=1}^{n} e_j\right) = \sum_{i,j=1}^{n} e_i a e_j$$

and $(e_i a e_j, x) = (\alpha_j - \alpha_i) e_i a e_j$. The theorem now follows.

In the above proof, we notice that if x is an algebraic diagonable element whose minimal polynomial has distinct roots in F, then $L = Fx$ is a finitely diagonable subspace, because the root spaces are of the form $A_{\alpha-\beta}$, where $\alpha$ and $\beta$ are roots of the minimal polynomial. This result generalizes considerably.

THEOREM 3.1.2: *Any algebraic diagonable element* x *in an algebra* A *over a field* F *generates a finitely diagonable subspace of* A. *Root spaces are of the form* $A_{\alpha-\beta}$ *where* $\alpha$ *and* $\beta$ *are roots (not necessarily in* F*) of* $p(t)$*, the minimal polynomial of* x.

PROOF: If x is in the centre of A there is nothing to prove, so assume some root space $A_\gamma(x)$ is non-zero for $\gamma \neq 0$. Considering for a moment ~~A as an algebra over $\overline{F}$, the algebraic closure of F, the subspace $A_\gamma(x)$~~ is invariant under the linear transformation $R_x$. Now $R_x$ is algebraic, also with minimal polynomial $p(t)$ since $1 \in A$. Thus, since $A_\gamma(x)p(R_x) = 0$ and p splits into linear factors over $\overline{F}$, there exists some $\beta \in \overline{F}$ and $a_\gamma \neq 0$ in $A_\gamma(x)$ such that $a_\gamma(x - \beta 1)^n = 0$, for some integer $n > 0$. Now $a_\gamma x = (x + \gamma 1)a_\gamma$ implies $a_\gamma f(x) = f(x+\gamma)a_\gamma$ for any $f(t) \in \overline{F}[t]$ and so $(x - (\beta-\gamma)1)^n a_\gamma = 0$. But this says that $\alpha = \beta - \gamma$ is a root of the minimal polynomial of $L_x$, which is also $p(t)$. Hence $\gamma = \beta - \alpha$ is a difference of roots of $p(t)$ and we have the theorem.

Over a field of characteristic 0, it is well-known that any irreducible polynomial has distinct roots. This result extends as we now see.

<u>LEMMA 3.1.3</u>: *If* $p(t) \in F[t]$ *is irreducible and of degree at least* 2, *then* $\alpha - \beta \notin F$ *for any distinct roots* $\alpha$ *and* $\beta$ *of* $p(t)$.

<u>PROOF</u>: Let K be the splitting field of $p(t)$ over F and $G$ the Galois group of K over F. Then there exists $\sigma \in G$ such that $\sigma\alpha = \beta$ ([4; page 204]). Suppose $\alpha - \beta \in F$. Then $\tau(\alpha - \sigma\alpha) = \alpha - \sigma\alpha$ for every $\tau \in G$. In particular, with $\tau = \sigma^{-1}$, we obtain $\sigma^{-1}\alpha + \sigma\alpha = 2\alpha$. Applying $\sigma$ to both sides of this relation gives $\alpha + \sigma^2\alpha = 2\sigma\alpha$. Another application of $\sigma$ gives $\sigma\alpha + \sigma^3\alpha = 2\sigma^2\alpha = 2(2\sigma\alpha - \alpha)$, and so $2\alpha + \sigma^3\alpha = 3\sigma\alpha$. An easy induction argument reveals that for any integer $t > 0$, $t\alpha + \sigma^{t+1}\alpha = (t+1)\alpha$ . Now $\sigma^k$ is the identity of $G$ for some k. Hence $(k-1)\alpha + \alpha = k\sigma\alpha$; i.e., $\alpha = \sigma\alpha = \beta$. This contradiction implies the lemma.

With the aid of 3.1.2, we obtain immediately

<u>THEOREM 3.1.4</u>: *Let* x *be an algebraic diagonable element whose minimal polynomial is irreducible. Then* x *is in the centre of* A.

We next give:

<u>PROPOSITION 3.1.5</u>: *Let* x *be an algebraic diagonable element with minimal polynomial* $p(t) = \prod_{i=1}^{n}(t - \alpha_i) \in F[t]$. *Then every non-zero A-module is* x*-weighted, every non-zero* $A_0$*-module is* x*-weighted, and weights of* x *are in the set* $\{\alpha_1,\ldots,\alpha_n\}$. *(Note: we do not assume here that* $\alpha_1,\ldots,\alpha_n$ *are all distinct.)*

<u>PROOF</u>: Let V be an A-module different from zero, and let $0 \neq v \in V$. Then either $v(x - \alpha_1 1) = 0$ or, since $\prod_{i=1}^{n}(x - \alpha_i 1) = 0$, there is some integer k, $1 < k \leq n$ with $v\prod_{i=1}^{k-1}(x - \alpha_i 1) \neq 0$ but $(v\prod_{i=1}^{k-1}(x - \alpha_i 1))(x - \alpha_k 1) = 0$, and

so $V$ is $x$-weighted. The proof that $A_o$-modules are $x$-weighted is identical. Next let $\gamma$ be any weight of $x$ in an $A$- (or $A_o$-) module $V$. Thus there is some $v \neq 0$ in $V$ with $v(x - \gamma 1) = 0$. Now if $\gamma \notin \{\alpha_1, \ldots, \alpha_n\}$, then the polynomials $t - \gamma$ and $p(t)$ in $F[t]$ are relatively prime, and so there are polynomials $a(t)$ and $b(t)$ in $F[t]$ with $a(t)p(t) + b(t)(t - \gamma) = 1$. Setting $t = x$ gives $b(x)(x - \gamma 1) = 1$ and hence $v = v(x - \gamma 1)b(x) = 0$, which is untrue.

The converse of Theorem 3.1.2 is generally false; for example, if $A$ is the algebra generated over $F$ by two elements $x$ and $y$ with $x$ transcendental, $y^2 = 0$, and $yx = (x+1)y$, then $Fx$ is a finitely diagonable subspace of $A$ (the only roots of $x$ are 0 and 1) but of course $x$ is not algebraic. The best result we could obtain in this direction now follows:

<u>THEOREM 3.1.6:</u> *Suppose* $L = Fx$ *is a finitely diagonable subspace of an algebra* $A$ *which is a direct sum of primitive algebras. Suppose further that every irreducible* $A$*-module and every irreducible* $A_o$*-module is* $x$*-weighted. Then* $x$ *is algebraic.*

<u>PROOF</u>: By Theorem 2.3.1, $A_o$ itself is a direct sum of primitive algebras $R_i$, $i = 1, \ldots, n$, only finitely many because $1 \in A_o$. If $I$ is any maximal right ideal of $A_o$, then $A_o/I$ is weighted, so there is some $a_o \notin I$ and $\lambda \in F$ with $a_o(x - \lambda 1)^n \in I$, and this implies $x - \lambda 1 \in I$ by 1.2.9. Hence, letting $J_i$ be a maximal right ideal of $R_i$ containing no non-zero ideal of $R_i$, $i = 1, \ldots, n$, for each $i$, $J_i \oplus \sum_{j \neq i} R_j$ is a maximal right ideal of $A_o$ and so contains $x - \lambda_i 1$ for some $\lambda_i \in F$. Writing $x = \sum_{i=1}^{n} x_i$, $x_i \in R_i$, and $1 = \sum_{i=1}^{n} e_i$, $e_i$ the identity of $R_i$, we have $x_i - \lambda_i e_i \in J_i$ by looking at the $i^{th}$ component of $x - \lambda_i 1$. Now $x_i - \lambda_i e_i$ is in the centre of $R_i$ because $x - \lambda 1$ is in the centre of $A_o$ and so $(x_i - \lambda_i e_i)R_i$ is an ideal of $R_i$ contained in $J_i$. Therefore, $x_i = \lambda_i e_i$ and since $e_1, \ldots, e_n$ are orthogonal idempotents, $x$ is algebraic by Theorem 3.1.1.

## 3.2 PRIME ALGEBRAS

A ring R is <u>prime</u> if $aRb \neq 0$ whenever a and b are non-zero elements of R; or, equivalently, if the product of non-zero right ideals of R is non-zero. In particular a prime ring is semi-prime and it is also a fact that any primitive ring is prime ([5; page 95]).

In general one can not expect the irreducible factors of the minimal polynomial of a diagonable algebraic element to be distinct; for example, let R by the quotient of the usual polynomial ring F[x] by the principal ideal generated by $x^3 - x^2$ and let $A = R[y]$ be the polynomial ring over R in the indeterminate y, where $yx = (x+1)y$. A straight-forward computation reveals that A is just the four-dimensional algebra over F with basis 1, x, $x^2$, y and multiplication table:

| | 1 | x | $x^2$ | y |
|---|---|---|---|---|
| 1 | 1 | x | $x^2$ | y |
| x | x | $x^2$ | $x^2$ | 0 |
| $x^2$ | $x^2$ | $x^2$ | $x^2$ | 0 |
| y | y | y | y | 0 |

x is a diagonable algebraic element whose minimal polynomial $p(t) = t^3 - t^2$ has t as a repeated factor.

If A is semi-prime, this cannot happen, for we have:

<u>PROPOSITION 3.2.1</u>: *If* x *is a diagonable algebraic element in a semi-prime algebra* A *over the field* F, *then the minimal polynomial of* x *has distinct irreducible factors.*

<u>PROOF</u>: $L = Fx$ is a finitely diagonable subspace of A and so by Theorem 2.2.3 $A_0(x)$ is semi-prime. Suppose the minimal polynomial of x is $p(t) = \prod_{i=1}^{r} p_i(t)^{n_i}$

where $p_1(t),\ldots,p_r(t)$ are the distinct irreducible factors of $p(t)$ in $F[t]$. Then if any $n_i > 1$, $u = \prod_{i=1}^{r} p_i(x)$ is a non-zero nilpotent element in the centre of $A_o(x)$ and hence generates a non-zero nilpotent ideal, and this is impossible.

There are occasions when all the irreducible modules of an algebra are weighted; for example, when x is a diagonable algebraic element whose minimal polynomial has all its roots in the ground field (Proposition 3.1.5). In [15] Lemire gives an example to show that this need not always be the case. He considers the universal enveloping algebra $U(A_1)$ of the three-dimensional simple Lie algebra $A_1$ ([ 9 ; page 137]) where the diagonable element is the generator H of a Cartan subalgebra of $A_1$. Relative to H, there are infinitely many root spaces, for otherwise, $U(A_1)$ would contain nilpotent elements (because $(A_\alpha(H))^n \subseteq A_{n\alpha}(H)$), and this is known to be false ([ 9 ; page 166]). Lemire constructs an irreducible $U(A_1)$-module which is not H-weighted. We further pursue in this section the interrelations between an algebra A, a diagonable element x in A, and the property that the irreducible modules of A should be x-weighted.

EXAMPLE 3.2.2: We give here an example similar to Lemire's. Let A be the universal enveloping algebra of the two-dimensional non-abelian Lie algebra over a field F (of characteristic 0). This has generators x and y with the relation $[y,x] = y$. A is in fact the ring of non-commuting polynomials over F in the two indeterminates x and y, where $yx = (x+1)y$. Now x is a diagonable element in A, and there are infinitely many roots, because $A = \bigoplus \sum_{n=0}^{\infty} A_n(x)$, $A_n(x) = F[x]y^n$. Since x is not invertible, there exists a maximal right ideal $I$ of $A$ which contains $x$, and so the $A$-module $V = A/I$ is $x$-weighted (for $(I + 1)x = 0$). We prove however that not all irreducible $A$-modules are weighted.

$y + 1$ is not invertible in $A$ and so is contained in a maximal right ideal $J$ which has the property

$$(1) \qquad f(x)y^k \in J \text{ for } k > 0 \text{ and } f(x) \in F[x] \text{ implies } f \equiv 0.$$

To see this, we use induction on $n$ = degree $f$ and $k$. For $n = 0$, if $\alpha y^k$ and $y + 1$ are both in $J$, then $J$ contains $1$ because these two polynomials are relatively prime in the Euclidean domain $F[y]$, hence certainly in $A$. Assuming the validity of (1) for polynomials $f$ with degree less than $n$, suppose $f(x) \in F[x]$ has degree $n$ and $f(x)y^k$ and $y + 1$ are both in $J$. Then $J$ must contain

$$(f(x)y^k,\ y+1) = f(x)y^{k+1} - yf(x)y^k = (f(x) - f(x+1))y^{k+1}.$$

Since $f(x) - f(x+1)$ has degree less than the degree of $f$, by induction we have $f(x) = f(x+1)$. This implies $\xi + n$ is a root of $f$ for every integer $n > 1$ and any root $\xi$ of $f$ and this is impossible unless degree $f = 0$. This possibility has already been eliminated above and so we have obtained (1). It follows immediately that $J$ can contain no two-sided ideal of $A$ except $(0)$ because any such ideal is homogeneous by 1.3.4. Therefore $A/J$ is a simple, faithful $A$-module (its annihilator is a two-sided ideal of $J$) and so $A$ is a primitive algebra. $A/J$ is not weighted however, for we can prove

$$(2) \qquad \left(\sum_{i=0}^{n} a_i(x)y^i\right)(x - \alpha 1) \in J \text{ implies } \sum_{i=0}^{n} a_i(x)y^i \in J.$$

For $n = 0$, $a_o(x)(x - \alpha 1) \in J$ implies $a_o(x) = 0$ or $x = \alpha 1$ by (1), and the latter possibility is impossible. Now if $J$ contains

$$\left(\sum_{i=0}^{n} a_i(x)y^i\right)(x - \alpha 1) = \sum_{i=0}^{n} y^i(x-i1)(x - \alpha 1)$$

then it also contains

$$(y+1)^n a_n(x)(x - \alpha 1) = \sum_{i=0}^{n} \binom{n}{i} y^i a_n(x)(x - \alpha 1)$$

because $y + 1 \in J$, and hence $J$ contains the difference of these two elements; namely,

$$\sum_{i=0}^{n-1} y^i\left(a_i(x-i1) - \binom{n}{i}a_n(x)\right)(x - \alpha 1) = \sum_{i=0}^{n-1}\left(a_i(x) - \binom{n}{i}a_n(x+i1)\right)y^i(x - \alpha 1).$$

By induction, we may assume that $a_i(x) = \binom{n}{i}\ a_n(x+i1)$; i.e., that $a_i(x) = \binom{n}{i}\ a_o(x+i1)$, for $i = 0,\ldots,n-1$. Thus

$$\sum_{i=0}^{n} a_i(x)y^i = \sum_{i=0}^{n}\binom{n}{i}\ a_o(x+i1)y^i = \sum_{i=0}^{n}\binom{n}{i}y^i a_o(x) = (1+y)^n a_o(x)$$

is in $J$. Hence we have (2), and $A/J$ is not weighted.

Thus we see that even for a primitive algebra, one is able to say little about which modules are weighted. We can however prove:

THEOREM 3.2.3: *Let* $A$ *be a prime algebra over the field* $F$ *and* $x \in A$ *be a diagonable algebraic element with minimum polynomial* $p(t) \in F[t]$. *Suppose some irreducible* $A$*-module is* $x$*-weighted. Then all* $A$*-modules are* $x$*-weighted.*

PROOF: The key step in the proof is to show that $p(t)$ has the form

$$(3) \qquad p(t) = \prod_{\alpha\in S} q(t+\alpha)$$

where $q(t) \in F[t]$ is irreducible and $S \subseteq \Delta$, the set of roots of $x$.
By Proposition 3.2.1, $p(t)$ is of the form $p_1(t)\ldots p_s(t)$ where $p_1(t),\ldots,p_s(t)$ are the distinct monic irreducible factors of $p(t)$ in $F[t]$. Let $A_{ij}$ denote the subspace $\{a \in A\colon p_i(x)a = 0 = ap_j(x)\}$ for $1 \leq i,j \leq s$. Then we have:

$$(4) \qquad A = \oplus \sum_{i,j=1}^{s} A_{ij}$$

To prove this, we note that the linear transformation $L_x$ of A is algebraic with minimal polynomial also $p(t)$ because $1 \in A$. Thus $A = \oplus\sum_{i=1}^{s} A_i$ where $A_i = \{a \in A\colon p_i(x)a = 0\}$. Herstein proves this in [4; page 256] for A finite-dimensional, but the proof uses only the fact that the linear transformation in question is algebraic. Now each subspace $A_i$ is invariant under $R_x$ which is also an algebraic linear transformation of A with minimal polynomial $p(t)$. Thus the restriction of $R_x$ to $A_i$ has minimal polynomial dividing $p(t)$ and so each $A_i$ decomposes into a direct sum of the subspaces $A_{ij}$ for some integers j, $1 \leq j \leq s$.

Now for each k, $1 \leq k \leq s$, define $a_k = \prod_{i \neq k} p_i(x)$. Then $a_k \neq 0$ and for any $i,j \in \{1,\ldots,s\}$, $a_i A a_j \neq 0$ (because A is prime). But $a_i A a_j$ is contained in $A_{ij}$ and so each of the spaces $A_{ij} \neq 0$. Suppose $0 \neq a \in A_{j1}$ and $a = \sum_{\alpha\in\Delta} a_\alpha$ relative to x. Then $0 = ap_1(x) = \sum_{\alpha\in\Delta} a_\alpha p_1(x)$. For some $\alpha$, $a_\alpha \neq 0$ and because $A_\alpha(x)A_o(x) \subseteq A_\alpha(x)$ and the sum $\sum_{\alpha\in\Delta} A_\alpha(x)$ is direct, $a_\alpha p_1(x) = 0$. As we have seen before, this implies $p_1(x+\alpha 1)a_\alpha = 0$. We also have $p_j(x)a = 0$ and hence, just as above, $p_j(x)a_\alpha = 0$. The polynomials $p_1(t+\alpha)$ and $p_j(t)$ cannot be relatively prime, and so because they are irreducible and monic, $p_j(t) = p_1(t+\alpha)$. This establishes (3), where $q(t) = p_1(t)$.

Some irreducible A-module is x-weighted by hypothesis, and so by Theorem 1.2.10 (with L = Fx) there is a scalar $\lambda \in F$ with $x - \lambda 1$ not invertible. This means that the polynomials $p(t)$ and $t - \lambda$ are not relatively prime and so $t - \lambda$ divides $p(t)$. It follows that for some $\alpha \in S$, $q(t+\alpha) = t - \lambda$ and hence $q(t) = t - (\lambda+\alpha)$ and $p(t)$ has all its roots in F. The theorem now follows directly from Proposition 3.1.5.

Looking back at the form of the minimal polynomial p(t) in (3) we see that in particular, if x is in the centre of A, $S = \Delta = \{0\}$. Together with 3.1.4, we obtain immediately

<u>COROLLARY 3.2.4</u>: *Let* x *be an algebraic diagonable element in a prime algebra* A. *Then* x *is in the centre of* A *if and only if its minimal polynomial is irreducible.*

## 3.3 CENTRAL SIMPLE ALGEBRAS

As might be expected, an element in an algebra A over a field F which is not diagonable, can become diagonable when A is considered to be an algebra over the algebraic closure $\overline{F}$ of F. For example, in the algebra of 2 × 2 matrices over the real numbers, $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonable because $A_0(x) = \{ a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: a,b \text{ real}\}$, while $A_\alpha(x) = 0$ for $\alpha \neq 0$. Over the complex numbers, however, the minimum polynomial of x has the distinct roots $\pm i$ and so x is diagonable by Theorem 3.1.1. More generally, if x is a diagonable element in the complete ring A of matrices over any algebraically closed field F, then since A is semi-prime, the minimal polynomial of x, whose roots of course lie in F, has distinct roots by 3.2.1 and x is similar to a diagonal matrix as is well-known. This curious observation that diagonability in our sense is equivalent to the usual meaning of the word in a matrix ring does not depend on the algebraic closure of the underlying base field. The proof of this is the principal aim of this section.

LEMMA 3.3.1: *Let A be any associative ring with 1 and $e \in A$ an idempotent. Then if A is simple, so is the subring $A_1(e) = \{a \in A: ae = ea = a\}$.*

PROOF: Relative to e, we have the Pierce decomposition of A,

$$A = A_{00} + A_{10} + A_{01} + A_{11}, \quad A_{ij} = \{a \in A: ae = ja, \ ea = ia\}.$$

([8 ; page 48]). Suppose A is simple and I is a non-zero ideal of $A_{11} = A_1(e)$. Then $I + A_{01}I + IA_{10} + A_{01}IA_{10}$ is a non-zero ideal of A and hence equals A. Writing 1 as a sum of elements from I, $A_{01}I$, $IA_{10}$, and $A_{01}IA_{10}$ and multiplying both sides of the equation so obtained by e first on the left and then on the right, we deduce that $e \in I$. But then, if

$a \in A_1(e)$, $a = ae \in I$ and so $I = A_1(e)$.

An idempotent in any ring (associative or not) is primitive if it cannot be written as the sum of two orthogonal idempotents. An algebra over F is said to be central over F if its centre is F. We require the following result, a proof for which can be found in sections 3.6 and 3.7 of [12].

LEMMA 3.3.2: *Let* R *be an artinian ring with* 1. *Then any idempotent can be written as a sum of primitive orthogonal idempotents. If* 1 *can be represented in two ways as a sum of primitive orthogonal idempotents,* $1 = \sum_{i=1}^{n} e_i = \sum_{j=1}^{m} f_j$, *then* $m = n$ *and there is a unit* v *of* R *and a permutation* $\pi$ *of* $\{1,\ldots,n\}$ *such that* $v^{-1}e_i v = f_{\pi(i)}$.

We use this to establish

LEMMA 3.3.3: *Let* A *be a finite dimensional central simple algebra over a field* F *and suppose* $e \in A$ *is an idempotent. Then* $A_1(e)$ *is also central simple over* F.

PROOF: By Lemma 3.3.1, we need only show that $A_1(e)$ is central over F. Using the previous lemma, it is easy to show that we can find pairwise orthogonal primitive idempotents $e_1,\ldots,e_n$ in A such that $\sum_{i=1}^{n} e_i = 1$ and $e = \sum_{i=1}^{t} e_i$ for some $t < n$. Now $A \cong D_m$, the ring of $m \times m$ matrices over a division algebra D (necessarily central over F) by the Wedderburn Theorem ([6 ; page 98]). Let $\{f_{ij}: i,j = 1,\ldots,m\}$ be the usual matrix units in A. Then $1 = \sum_{j=1}^{m} f_{jj}$ and $f_{11},\ldots,f_{mm}$ are orthogonal primitive idempotents. By 3.3.2, $m = n$ and there is a unit $v \in A$ and a permutation $\pi$ of $\{1,\ldots,n\}$

such that $v^{-1}e_i v = f_{\pi(i)\pi(i)}$. Let $B = v^{-1}A_1(e)v$ and $f = \sum_{i=1}^{t} f_{\pi(i)\pi(i)}$. Then one can check easily that $B = A_1(f) \simeq D_t$ which is central over F because D is. Since $B \simeq A_1(e)$, the lemma follows.

Our main result is

THEOREM 3.3.4: *Suppose* A *is a finite dimensional central simple algebra over the field* F. *Then* $x \in A$ *is diagonable if and only if* x *is a linear combination of orthogonal idempotents. In particular, if* A *is the complete ring of matrices over* F, *then any diagonable element is similar to a diagonal matrix.*

PROOF: Since A is semi-prime, the minimal polynomial of x has the form $q_1(t)\ldots q_n(t)$, where $q_1,\ldots,q_n$ are distinct irreducible polynomials in F[t] (by 3.2.1). We can write $1 = \sum_{i=1}^{n} e_i$ and $x = \sum_{i=1}^{n} xe_i$, where $e_1,\ldots,e_n$ are pairwise orthogonal idempotents which are polynomials in x just as in Herstein, [4; page 256]. Moreover, $x_i = xe_i$ has the minimum polynomial $q_i(t)$, $i = 1,\ldots,n$ because $f(x_i) = f(x)e_i$ for any $f(t) \in F[t]$. Let $B_i = A_1(e_i)$ for $i = 1,\ldots,n$. Notice that $x_i \in B_i$ and $B_i$ is a subalgebra, so $B_i$ is invariant under ad x. The restriction of ad x to $B_i$ is ad $x_i$ and by 1.2.2(i) this restriction satisfies a polynomial with distinct roots in F; i.e., $x_i$ is diagonable in $B_i$, which is central simple over F by 3.3.3. By 3.1.4, $x_i$ is in the centre of $B_i$; hence, for some $\alpha_i \in F$, $x_i = \alpha_i e_i$. Therefore $x = \sum_{i=1}^{n} \alpha_i e_i$. The last statement of the theorem now follows from Theorem 3.1.1 and linear algebra.

COROLLARY 3.3.5: *If* A *is a direct sum of finite dimensional central simple algebras over a field* F, *then any diagonable element in* A *is a linear* *combination of pairwise orthogonal idempotents with sum* 1*, and its centralizer is semi-simple.*

PROOF: That a diagonable element x is a linear combination of orthogonal idempotents follows directly from the Theorem together with Proposition 1.2.1. x is therefore algebraic and so Fx is a finitely diagonable subspace of A. Proposition 3.1.5 implies every A-module is x-weighted and so $A_0(x)$ is semi-simple by Theorem 2.2.5.

In conclusion, we remark that the assumption of finite-dimensionality in Theorem 3.3.4 is crucial. If F(x) denotes the ring of rational functions over F in an indeterminate x, and we adjoin an indeterminate y so that yx = (x+1)y, then the resulting algebra of non-commuting power series in y over F(x) is central simple (in fact it is a division algebra as Jacobson proves in [8 ; pages 187-188]), x is diagonable, but certainly not a linear combination of idempotents.

# CHAPTER FOUR

# APPLICATIONS TO THE UNIVERSAL ENVELOPING ALGEBRA OF A JORDAN ALGEBRA

## 4.1 THE UNIVERSAL ENVELOPING ALGEBRA OF A JORDAN ALGEBRA

Let J be a Jordan algebra over a field F of characteristic 0. Then a representation of J is defined to be a linear map $S: J \to A$ where A is an associative algebra over F, such that for every a, b, and c in J, we have

(1) $$S_aS_{bc} + S_bS_{ca} + S_cS_{ab} = S_{bc}S_a + S_{ca}S_b + S_{ab}S_c \quad \text{and}$$

(2) $$S_aS_bS_c + S_cS_bS_a + S_{ac.b} = S_aS_{bc} + S_bS_{ca} + S_cS_{ab}$$

where $S: u \to S_u$ for $u \in J$. As in the Lie case, the map $x \to R_x$ of J into the associative algebra generated by right multiplications is an example of a representation of J. An associative algebra $\mathcal{U}(J)$ with identity is a universal enveloping algebra for J if there is a canonical representation $S^*: J \to \mathcal{U}(J)$ such that for any representation $S: J \to A$ of J in an associative algebra A, there exists a unique homomorphism $\Psi: \mathcal{U}(J) \to A$ such that $S = \Psi o S^*$; i.e., which makes the following diagram commutative:

$$\begin{tikzcd} & \mathcal{U}(J) \arrow[d, "\psi"] \\ J \arrow[ur, "S^*"] \arrow[r, "S"'] & A \end{tikzcd}$$

For any positive integer n, let $J^n$ denote the linear span of all products of n elements of J. Define $J^{(n)}$ inductively by $J^{(1)} = J$, and $J^{(n)} = (J^{(n-1)})^2$ for $n > 1$. Then J is called solvable if $J^{(n)} = 0$ for some n. If J is finite-dimensional, it contains a maximal solvable ideal called the <u>radical</u> of J. J is then <u>semi-simple</u> if its radical is 0. By a theorem of Dieudonné, any finite-dimensional semi-simple Jordan algebra is a direct sum of simple Jordan algebras.

We will require several facts about the universal enveloping algebra $U(J)$ of a Jordan algebra J. These can be found in Jacobson [7 ], and are summarized in

<u>THEOREM 4.1.1</u>: *$U(J)$ exists, is generated by $\{S^*_a : a \in J\}$, and is unique up to isomorphism. If J is finite-dimensional then so is $U(J)$; if in addition J is semi-simple, then so also is $U(J)$.*

We will also make extensive use of the following rather technical result.

<u>PROPOSITION 4.1.2</u>: *Let e be an idempotent in a Jordan algebra J over a field F. Then letting $a \mapsto \bar{a}$ denote the canonical embedding of J in $U(J)$, $\bar{e}$ is a linear combination of orthogonal idempotents and is diagonable with roots in the set $\{0, \frac{1}{2}, -\frac{1}{2}, 1, -1\}$. If $J = J_0 + J_{\frac{1}{2}} + J_1$, with $J_i = \{a \in J: ae = ia\}$ for $i = 0, \frac{1}{2}, 1$ is the Peirce decomposition of J relative to $\bar{e}$, then*

$$\bar{J}_0 + \bar{J}_1 \subseteq A_0(\bar{e}) \qquad \text{and} \qquad \bar{J}_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}(\bar{e}) + A_{-\frac{1}{2}}(\bar{e}), \qquad A = U(J)$$

<u>PROOF</u>: Setting $a = b = c = e$ in (2) gives immediately that $2\bar{e}^3 - 3\bar{e}^2 + \bar{e} = 0$. Thus $\bar{e}$ is algebraic with minimal polynomial dividing $f(t) = 2t^3 - 3t^2 + t$, a polynomial with the distinct roots 0, $\frac{1}{2}$, and 1. Thus $\bar{e}$ is a linear combination of orthogonal idempotents and diagonable by Theorem 3.1.1. Its roots are in $\{0, \frac{1}{2}, -\frac{1}{2}, 1, -1\}$ by 3.1.2. Next, if $u \in J_0$, put $a = u$, $b = c = e$

in (1). Then $\overline{u}\overline{e} = \overline{e}\overline{u}$ so $\overline{u} \in A_o(\overline{e})$. Similarly if $u \in J_1$, putting $a = u$, $b = c = e$ in (1) gives $\overline{u}\overline{e} + 2\overline{e}\overline{u} = \overline{e}\overline{u} + 2\overline{u}\overline{e}$ and again $\overline{u} \in A_o(\overline{e})$. Finally, assuming $u \in J_{\frac{1}{2}}$, equation (2) with $a = c = e$ and $b = u$ implies $2\overline{e}\overline{u}\overline{e} + \frac{1}{2}\overline{u} = \overline{e}\overline{u} + \overline{u}\overline{e}$ and with $a = u$, $b = c = e$ implies $\overline{u}\overline{e}^2 + \overline{e}^2\overline{u} + \frac{1}{4}\overline{u} = \overline{u}\overline{e} + \overline{e}\overline{u}$. Therefore, $\overline{u}\overline{e}^2 + \overline{e}^2\overline{u} + \frac{1}{4}\overline{u} = 2\overline{e}\overline{u}\overline{e} + \frac{1}{2}\overline{u}$; i.e., $((\overline{u},\overline{e}),\overline{e}) = \frac{1}{4}\overline{u}$. Now write $\overline{u} = u_o + u_{\frac{1}{2}} + u_{-\frac{1}{2}} + u_1 + u_{-1}$ with $u_\alpha \in A_\alpha(\overline{e})$. Then we see that $\frac{1}{4}u_{\frac{1}{2}} + \frac{1}{4}u_{-\frac{1}{2}} + u_1 + u_{-1} = \frac{1}{4}\overline{u}$ and hence $u_o = u_1 = u_{-1} = 0$ and $\overline{u} \in A_{\frac{1}{2}}(\overline{e}) + A_{-\frac{1}{2}}(\overline{e})$.

Now suppose a and b are two elements in a Jordan algebra J. We define the linear transformation $R_{a,b}$ of J by $uR_{a,b} = (u,a,b)$ for any $u \in J$. J is associator nilpotent if there is a positive integer k such that $R_{a_1,b_1} \cdots R_{a_k,b_k} = 0$ for all $a_i$, $b_i \in J$, $i = 1,\ldots,k$.

A Cartan subalgebra of J is an associator nilpotent subalgebra H containing 1 which has the property that if $(x,H,H) \subseteq H$ for any $x \in J$ then $x \in H$. In [10; page 601], Jacobson shows that if J is finite-dimensional over an algebraically closed field of characteristic 0, then J possesses a Cartan subalgebra $H = \sum_{i=1}^{t} J_{ii}$, where $J = \sum_{i,j} J_{ij}$ is the Peirce decomposition of J relative to a set of pairwise orthogonal idempotents with sum 1 which are also primitive. If J is simple, Albert ([1; page 561]) has shown that $J_{ii} = Fe_i$ and so $H = \sum_{i=1}^{t} Fe_i$. In this case, let $L = \overline{H}$, the image of H in $U(J)$ under the canonical embedding. By Proposition 4.1.2, L is spanned by diagonable elements which commute, because setting $a = e_i$, $b = c = e_j$ in (1), $i \neq j$, gives $(\overline{e}_i,\overline{e}_j) = 0$. As was pointed out in the discussion after 1.2.5, L is therefore a diagonable subspace of $U(J)$.

In his doctoral dissertation, Foster showed that the Cartan theory of Lie and Jordan algebras is essentially the same ([3]). It is therefore not surprising to discover that the Cartan subalgebras of simple Lie and Jordan algebras (over algebraically closed fields of characteristic 0) share the common property that they are both diagonable subspaces of the corresponding universal enveloping algebras. In particular, using Theorem 1.3.7, we obtain the analogue of Lemire's result ([14]) for Jordan algebras.

THEOREM 4.1.3: *For a fixed linear functional* $\lambda \in H^*$, *there is a one-to-one correspondence between the set of isomorphism classes of* $\lambda$*-weighted irreducible* $U(J)$*-modules and the set of isomorphism classes of* $\lambda$*-weighted irreducible* $C$*-modules, where* $C$ *is the centralizer of* $\overline{H}$ *in* $U(J)$.

This theorem is particularly useful because the algebra $C$ is generally much "smaller" than A; moreover, we shall see in §4.2 that $C$ can be characterized as the centralizer of a single element of H.

We close this section with the following example:

EXAMPLE 4.1.4: Let J be a simple Jordan algebra over an algebraically closed field F of characteristic 0 which is if degree two; i.e., $1 = e_1 + e_2$ where $e_1$ and $e_2$ are orthogonal primitive idempotents. Then as in Albert ([1]), $J = Fe_1 + Fe_2 + J_{12}$ where $J_{12}$ is the subspace $\{a \in J: ae_1 = ae_2 = \frac{1}{2}a\}$, and $J_{12}$ has a basis $u_1$, $u_2$ such that $u_1u_2 = u_2u_1 = 0$ and $u_1^2 = u_2^2 = 1$. Let $C$ be the centralizer in $U(J)$ of the embedding of the Cartan subalgebra $H = Fe_1 + Fe_2$. Then $C$ is spanned over F by the following seven elements:

$$
\begin{aligned}
f_1 &= \overline{e}_1 + \overline{e}_2 - (\overline{u}_1^{\,2} + \overline{u}_2^{\,2})\\
f_2 &= \overline{u}_1^{\,2} - \tfrac{1}{2}(\overline{e}_1 + \overline{e}_2) + 2\,\overline{e}_1\overline{e}_2\\
f_3 &= \overline{u}_2^{\,2} - \tfrac{1}{2}(\overline{e}_1 + \overline{e}_2) + 2\overline{e}_1\overline{e}_2\\
f_4 &= 2\overline{e}_1 - 4\overline{e}_1\overline{e}_2\\
f_5 &= 2\overline{e}_2 - 4\overline{e}_1\overline{e}_2\\
u &= 8\overline{e}_1\overline{u}_1\overline{u}_2\\
v &= 8\overline{e}_2\overline{u}_1\overline{u}_2
\end{aligned}
$$

$f_1,\ldots,f_5$ are pairwise orthogonal idempotents which satisfy

$$f_i u = uf_i = f_i v = vf_i = 0, \quad i = 1,2,3$$
$$f_4 u = uf_4 = u; \qquad f_4 v = vf_4 = 0$$
$$f_5 u = uf_5 = 0; \qquad f_5 v = vf_5 = v$$

Thus $C = (Ff_1 \oplus Ff_2 \oplus Ff_3) \oplus B$, where B is the subalgebra of $C$ spanned by $f_4$, $f_5$, u and v, and $\oplus$ denotes an algebra direct sum. The multiplicative relations among these elements are given by the next table:

| | $f_4$ | $f_5$ | u | v |
|---|---|---|---|---|
| $f_4$ | $f_4$ | 0 | u | 0 |
| $f_5$ | 0 | $f_5$ | 0 | v |
| u | u | 0 | $-f_4$ | 0 |
| v | 0 | v | 0 | $-f_5$ |

Thus, denoting the square root of -1 by j, we have $B = Fg_1 \oplus Fg_2 \oplus Fg_3 \oplus Fg_4$, where

$$g_1 = \tfrac{1}{2}(f_4 + ju)$$
$$g_2 = \tfrac{1}{2}(f_4 - ju)$$
$$g_3 = \tfrac{1}{2}(f_5 + jv)$$
$$g_4 = \tfrac{1}{2}(f_5 - jv)$$

are pairwise orthogonal idempotents. So we see that $C$ is the direct sum of seven copies of F.

## 4.2 $\Lambda_o(x)$ AS A CARTAN SUBALGEBRA

Let $L$ be an n-dimensional Lie algebra with basis $u_1,\ldots,u_n$ over a field F. Then any $a \in L$ determines the so-called Engel subalgebra $L_o(a) = \{x \in L: xR_a^{\,n} = 0 \text{ for some } n\}$. If $a = \sum_{i=1}^{n} \xi_i u_i \in L$, let $f(t,a)$ be the characteristic polynomial of $R_a$. Then

$$f(t,a) = t^n + p_1(\xi_1,\ldots,\xi_n)t^{n-1} + \ldots + p_n(\xi_1,\ldots,\xi_n)$$

where the coefficients $p_1,\ldots,p_n$ are polynomials in n-variables over F. Since $aR_a = 0$ for any $a \in L$, $p_n \equiv 0$, and so there is a well-defined integer s, $1 \leq s < n$ such that $p_s \not\equiv 0$, but $p_r \equiv 0$ for $s < r \leq n$. Jacobson in [13; page 60] calls an element $a \in L$ regular if $p_s(a) \neq 0$. He then shows that $H$ is a Cartan subalgebra of $L$ if and only if for some $a \in L$, $H = L_o(a)$ is a minimal Engel subalgebra (with respect to inclusion).

We now distinguish a certain class of diagonable elements in an associative algebra and show that they behave very much like regular elements of a Lie algebra.

DEFINITION 4.2.1: *A diagonable element* x *in an associative algebra* A *over a field* F *is called finitely diagonable if* Fx *is a finitely diagonable subspace of* A*; i.e., if* ad x *is an algebraic linear transformation of whose minimal polynomial has distinct roots.* x *is regular if* $A_o(x)$ *is minimal (always with respect to inclusion) among centralizers of all finitely diagonable elements in* A.

The key result of this section now follows.

THEOREM 4.2.2: *If $x$ is a regular element in the algebra $A = \sum_{\alpha \varepsilon F} A_\alpha(x)$, then $A_o(x) \subseteq A_o(y)$ for every finitely diagonable element $y$ in $A_o(x)$; in other words, every finitely diagonable element of $A_o(x)$ is in the centre of $A_o(x)$.*

PROOF: Suppose $y$ is a finitely diagonable element in $A_o(x)$. For each $t \varepsilon F$, define $y_t = x + t(y - x)$. Now $y - x$ is finitely diagonable by Proposition 1.2.2. Also $A_\alpha(x)$ is invariant under ad $(y-x)$ for any $\alpha \varepsilon F$ and so by the same proposition $y - x$ is finitely diagonable on $A_\alpha(x)$. Now assuming $t \neq 0$, $\beta$ is a root of $y - x$ on $A_\alpha(x)$

if and only if $A_\alpha(x) \cap A_\beta(y-x) \neq 0$

if and only if $A_\alpha(x) \cap A_{t\beta}(t(y-x)) \neq 0$ (by Proposition 1.2.2)

if and only if $\alpha + t\beta$ is a root of $y_t$ on $A_\alpha(x)$.

$y_t$ is finitely diagonable on $A_\alpha(x)$ and we now see that for $t \neq 0$, the minimal polynomial of the restriction of ad $y_t$ to $A_\alpha(x)$ is

$$(1) \qquad f_\alpha(\lambda,t) = \prod_\beta(\lambda - (\alpha+t\beta)) = \lambda^{m_\alpha} + \beta_1{}^\alpha(t)\lambda^{m_\alpha-1} + \ldots + \beta_{m_\alpha}{}^\alpha(t)$$

the product taken over the roots $\beta$ of $y - x$ on $A_\alpha(x)$. Here the $\beta_i{}^\alpha(t)$, $i = 1,\ldots,m_\alpha$ are polynomials in $t$, and $m_\alpha$ depends only on $\alpha$, not on $t$; in fact $m_\alpha$ is just the number of roots of $y - x$ on $A_\alpha(x)$. Now if $\alpha \neq 0$, $\beta_{m_\alpha}{}^\alpha(0) = (-\alpha)^{m_\alpha} \neq 0$, and so letting $\alpha$ range over the non-zero roots of $x$, we have finitely many polynomials $\beta_{m_\alpha}{}^\alpha(t)$ none of which is identically 0. Since the characteristic of $F$ is 0, $F$ is infinite, so there is an infinite subset $D \subseteq F$ such that $\beta_{m_\alpha}{}^\alpha(t) \neq 0$ for any $t \varepsilon D$ and non-zero $\alpha$. But for $t \varepsilon D$, $A_o(y_t) \subseteq A_o(x)$ because if $a \varepsilon A_o(y_t)$ and we write $a = \sum_{\alpha\varepsilon F} a_\alpha$ relative to $x$, then $(a,y_t) = 0$ implies $(a_\alpha,y_t) = 0$. But ad $y_t$ is non-singular on $A_\alpha(x)$ for $\alpha \neq 0$ and so $a_\alpha = 0$. Hence $a = a_o \varepsilon A_o(x)$. By the minimality of $A_o(x)$, we have $A_o(y_t) = A_o(x)$. Therefore the minimal polynomial of ad $y_t$ on $A_o(x)$ is $\lambda$ for $t \varepsilon D$; i.e., $\beta_i{}^o(t) = 0$ for infinitely many $t$,

$i = 1,\ldots,m_o$. Therefore the polynomials $\beta_i^{\,o}(t)$ are identically 0 and $\operatorname{ad} y_1 = \operatorname{ad} y$ has the minimal polynomial $\lambda$ on $A_o(x)$. This says that $A_o(x) \subseteq A_o(y)$ as we wanted to show.

We obtain immediately the important corollaries:

<u>COROLLARY 4.2.3</u>: *Let* L *be a finitely diagonable subspace of* A *and assume that any collection of centralizers of finitely diagonable elements from* A *has a minimal member. Then the centralizer of* L *is* $A_o(x)$ *for some* $x \in L$.

<u>PROOF</u>: Let $x \in L$ be such that $A_o(x)$ is minimal in the set $\{A_o(y)\colon y \in L\}$. Then for any $y \in L$ and $t \in F$, $y_t = x + t(y - x)$ is in L, and exactly as in the proof of the theorem, we can show that $A_o(y) \supseteq A_o(x)$. Thus the centralizer of L, which is $\bigcap_{y \in L} A_o(y)$ must equal $A_o(x)$.

<u>COROLLARY 4.2.4</u>: *If* x *is a regular element in an algebra* A *and* $A_o(x)$ *is spanned by finitely diagonable elements, then* $A_o(x)$ *is commutative.*

Now let $L = \sum_{i=1}^{t} F\overline{e}_i$ denote the embedding of a Cartan subalgebra $H = \sum_{i=1}^{t} Fe_i$ of a finite dimensional simple Jordan algebra J in its universal enveloping algebra A, as in §4.1. We saw in that section that L is a diagonable subspace of A, and of course it is finitely diagonable, because A is finite dimensional. Applying 4.2.3, we deduce that the centralizer $C$ of L is $A_o(x)$ for some $x = \sum_{i=1}^{t} \alpha_i\overline{e}_i \in L$, $\alpha_1,\ldots,\alpha_t \in F$. A is also semi-simple, and so $C$ is semi-prime by Theorem 2.2.3. But the Jacobson radical of any artinian ring is nilpotent, and hence $C$ is actually semi-simple. By the Wedderburn Theorem and the fact that F is algebraically closed, we see that

$C$ is a direct sum of matrix rings over F ([5; page 51]). Now any $n \times n$ matrix ring over F is spanned by idempotents (hence by finitely diagonable elements) for it has the basis of matrix units $\{e_{ij}: i,j = 1,\ldots,n\}$ and $e_{ij} = (e_{ii} + e_{ij}) - e_{ii}$ is a difference of idempotents for $i,j = 1,\ldots,n$. In particular, $C = A_o(x)$ is spanned by finitely diagonable elements. If x could be chosen regular, then $C$ would be commutative by Corollary 4.2.4, and a direct sum of matrix rings over F can only be commutative if each matrix ring is $1 \times 1$. We conjecture that this is indeed the case: that the centralizer of a Cartan subalgebra in the universal enveloping algebra has the same algebra structure as that of the Cartan subalgebra; namely, it is a direct sum of fields. This was indeed the case in the example 4.1.4. Because of the important role the centralizer plays in the representation theory of Jordan algebras (Theorem 4.1.3), the validity of our conjecture would simplify this theory considerably. The structure theorem we do have is summarized in:

THEOREM 4.2.5: *The centralizer of a Cartan subalgebra of a finite dimensional simple Jordan algebra over an algebraically closed field* F *of characteristic* 0 *in its universal enveloping algebra is the centralizer of a single element of the Cartan subalgebra, and is a direct sum of complete matrix rings over* F.

We finish this section with an attempt to characterize regular elements in an algebra. Suppose D is a set of finitely diagonable elements in an algebra A. Then for x and y in D, $A_o(y) \subseteq A_o(x)$ if and only if x is in $Z(A_o(y))$, the centre of $A_o(y)$, and so $A_o(x)$ is minimal in the set $\{A_o(y): y \in D\}$ if and only if for each y in D, $x \in Z(A_o(y))$ implies $y \in Z(A_o(x))$. We use this observation to establish

PROPOSITION 4.2.6: *If* $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_1,\ldots,\alpha_n \in F$ *and* $e_1,\ldots,e_n$ *orthogonal idempotents in a prime algebra* $A$, *then if* $A_o(x)$ *is minimal over the set of all centralizers of linear combinations of orthogonal idempotents, the* $e_1,\ldots,e_n$ *are primitive and* $\prod_{i \neq j}(\alpha_i - \alpha_j) \neq 0$.

PROOF: Suppose $\alpha_1 = \alpha_2$. Then let $y = \sum_{i=1}^{n} \beta_i e_i$ where $\beta_1,\ldots,\beta_n$ are arbitrary distinct scalars. By Theorem 3.1.1, y is diagonable and $A_o(y) = \sum_{i=1}^{n} e_i A e_i$, and so $x \in Z(A_o(y))$. However, $y \notin Z(A_o(x))$ because $0 \neq e_1 A e_2 \subseteq A_o(x)$ but $(e_1 A e_2, y) \neq 0$. It is immediate that $e_1,\ldots,e_n$ are primitive, because if $e_1 = f_1 + f_2$ with $f_1$, $f_2$ orthogonal idempotents, $x = \alpha_1 f_1 + \alpha_1 f_2 + \sum_{i=2}^{n} \alpha_i e_i$ and we just showed that $A_o(x)$ could not be minimal.

LEMMA 4.2.7: *Suppose* $x = \sum_{i=1}^{n} \alpha_i e_i = \sum_{j=1}^{m} \beta_j f_j$ *where* $\{e_1,\ldots,e_n\}$ *and* $\{f_1,\ldots,f_m\}$ *are two sets of orthogonal idempotents with sum* 1 *and* $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m$ *are in a field* F. *Then* $\{\alpha_1,\ldots,\alpha_n\} = \{\beta_1,\ldots,\beta_m\}$; *hence if* $\alpha_1,\ldots,\alpha_n$ *are distinct, either* $m > n$ *or* $m = n$ *and* $\beta_1,\ldots,\beta_m$ *are also distinct.*

PROOF: The first statement is just a statement of the fact that the coefficients of x when expressed as a linear combination of orthogonal idempotents with sum 1, are uniquely determined as the roots of the minimal polynomial of x (Theorem 3.1.1). The rest is obvious.

This lemma enables us to prove

THEOREM 4.2.8: *Let* $A$ *be a finite dimensional central simple algebra over a field* F. *Then* $x \in A$ *is regular if and only if* $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_1,\ldots,\alpha_n$ *distinct scalars and* $e_1,\ldots,e_n$ *primitive orthogonal idempotents with sum* 1.

PROOF: We first note that the only diagonable elements in A are linear combinations of orthogonal idempotents by Theorem 3.3.4. Then since a simple algebra with 1 is prime, 4.2.6 gives the result in the "only if" direction. Conversely, suppose $x = \sum_{i=1}^{n} \alpha_i e_i$ with $\alpha_1,\ldots,\alpha_n \in F$ distinct, and $e_1,\ldots,e_n$ primitive orthogonal idempotents with sum 1. By the remarks before Proposition 4.2.6, it is sufficient to establish

(2) If $y \in A$ is diagonable and $x \in Z(A_o(y))$, then $y \in Z(A_o(x))$.

Thus suppose y is a diagonable element in A. By 3.3.5, $A_o(y)$ is semisimple and so the Wedderburn Theorem implies

(3) $A_o(y) = \bigoplus \sum_{i=1}^{t} D_i$, $D_1,\ldots,D_t$ matrix rings over division algebras $\Delta_i$ which are finite-dimensional over F.

Let $f_i$ be the identity of $D_i$ for $i = 1,\ldots,t$. Then $f_1,\ldots,f_t$ are orthogonal idempotents with sum 1. Writing each of these (if necessary) as a sum of primitive orthogonal idempotents, we see from Lemma 3.3.2 that $t \leq n$. The key step in the proof is to show

(4) u diagonable in $Z(A_o(y))$ implies $u = \sum_{i=1}^{t} \xi_i f_i$, $\xi_1,\ldots,\xi_t \in F$.

To see this, we note first that such a u can be written $u = \sum_{i=1}^{t} u_i$ with $u_i \in Z(D_i)$ for $i = 1,\ldots,t$. But $Z(D_i) = Z(\Delta_i)f_i$ and so $u_i = \delta_i f_i$, $\delta_i \in Z(\Delta_i)$. Since u is diagonable, we know $u = \sum_{i=1}^{m} \xi_i g_i$ where $g_1,\ldots,g_m$ are orthogonal idempotents with sum 1 and $\xi_1,\ldots,\xi_m \in F$, and by re-ordering if necessary, we may suppose that $\{\xi_1,\ldots,\xi_m\} = \{\xi_1,\ldots,\xi_s\}$, $s \leq m$ and $\xi_1,\ldots,\xi_s$ distinct. By Theorem 3.1.1, the minimal polynomial of u is $p(t) = \prod_{i=1}^{s}(t - \xi_i)$. Since $u_i u_j = 0$ for $i,j \in \{1,\ldots,t\}$ and $i \neq j$, $p(u) = \sum_{i=1}^{t} p(u_i)$ and because (3) is a direct sum, $p(u_i) = 0$, $i = 1,\ldots,t$. But $p(u_i) = p(\delta_i)f_i$ implies $p(\delta_i) = 0$. Now over the field $F[\delta_i]$, p(t) which is of degree s, has the s + 1 roots

$\delta_i, \xi_1, \ldots, \xi_s$ and so $\delta_i \in \{\xi_1, \ldots, \xi_s\}$ because $\xi_1, \ldots, \xi_s$ are distinct. This establishes (4).

In particular, $y \in Z(A_o(y))$ implies $y = \sum_{i=1}^{t} \beta_i f_i$, $\beta_1, \ldots, \beta_t \in F$, and if $x \in Z(A_o(y))$, $x = \sum_{i=1}^{t} \gamma_i f_i$, $\gamma_1, \ldots, \gamma_t \in F$. But $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_1, \ldots, \alpha_n$ distinct elements of F; therefore because $t \leq n$, by the Lemma 4.2.7, $t = n$ and $\gamma_1, \ldots, \gamma_n$ are distinct too. Hence $A_o(x) = \sum_{i=1}^{n} f_i A f_i$ (3.1.1) and so $y \in Z(A_o(x))$, verifying (2) and giving the theorem.

<u>REMARK 4.2.9</u>: If A is the ring of $n \times n$ matrices over a field F and $A_L$ denotes the associated Lie algebra, then it is known ([ 9 ; page 66]) that $x \in A_L$ is regular in the Lie sense if and only if x has n distinct characteristic roots in the algebraic closure of F. Theorem 4.2.8 shows that in this case, the two concepts of regularity coincide if F is algebraically closed, although the example at the beginning of §3.3 shows that if F is not algebraically closed, there may exist regular elements in the Lie sense which are not even diagonable.

## 4.3 SIMPLE JORDAN ALGEBRAS

We saw in the discussion after Corollary 4.2.4 that the universal enveloping algebra $\mathcal{U}(J)$ of a finite-dimensional simple Jordan algebra J over an algebraically closed field F is spanned over F by its idempotents. This followed directly because $\mathcal{U}(J)$ is semi-simple and finite-dimensional. In this section we prove that the universal enveloping algebra of any simple Jordan algebra over any field of characteristic 0 is generated as an algebra by its idempotents, provided J contains an idempotent e whose Peirce one-space $J_1(e) = Fe$. We remark that any reduced simple Jordan algebra has this property (see for example, Jacobson [11; page 202]).

By a derivation of a (not necessarily associative) algebra A over a field F, we mean a linear map $D: A \to A$ such that for any a and b in A,

$$D(ab) = D(a)b + aD(b) \tag{1}$$

If $D_1$ and $D_2$ are derivations of A and $\alpha \in F$, then $\alpha D_1 + D_2$ as well as the commutator $(D_1, D_2)$ are also derivations and so the derivations of any algebra form a Lie algebra under the commutator product. If D is a nilpotent derivation, then $\exp(D) = 1 + D + \frac{D^2}{2!} + \ldots$ is always an automorphism of A (see §1.2 of[9]).

DEFINITION 4.3.1: *Diagonable elements* x *and* y *in an algebra* A *over* F *are said to be of the same type if there is an automorphism* $\psi$ *of* A *such that* $\psi(x) = y$, *and in this case, we write* $x \sim y$. It is immediate that $\sim$ is an equivalence relation on the class of diagonable elements in A; i.e., it is a symmetric, reflexive, and transitive relation.

PROPOSITION 4.3.2: *If $x_o$ is a finitely diagonable element in an algebra* A *over* F, *then the linear space* S *spanned by* $\{x \in A: x \sim x_o\}$ *is a Lie ideal of* A *containing* $\sum_{\alpha\neq 0} A_\alpha(x)$; *i.e., a subspace such that* $x \in S$ *and* $a \in A$ *implies* $(a,x) \in S$.

PROOF: We note that if $\psi$ is an automorphism of A, then $\psi(A_\alpha(x_o)) = A_\alpha(\psi(x_o))$ and thus if $x \sim x_o$, the roots of x and $x_o$ are the same. In particular x has only finitely many roots, and so if $0 \neq a_\alpha \in A_\alpha(x)$ and $\alpha \neq 0$, then ad $a_\alpha$ is a derivation of A which is nilpotent because $A_\beta(x)(\text{ad } a_\alpha)^k \subseteq A_{\beta+k\alpha}(x)$ for any root $\beta$ of x. Thus $\exp(-\frac{1}{\alpha}\text{ad } a_\alpha)$ is an automorphism of A which sends x to $x + a_\alpha$; i.e., $x + a_\alpha \sim x$. Since $\sim$ is a transitive relation, $x + a_\alpha \sim x_o$ and $a_\alpha = (x + a_\alpha) - x \in S$ for any $\alpha \neq 0$. If $a \in A$, and $a = \sum_{\alpha\in F} a_\alpha$ relative to x, then $(a,x) = \sum_{\alpha\neq 0} \alpha a_\alpha \in S$. Thus S is a Lie ideal.

We can now prove our main result.

THEOREM 4.3.3: *Let* J *be a simple Jordan algebra over a field* F *which contains an idempotent* e *such that* $J_1(e) = Fe$. *Then* $U(J)$ *is generated by its idempotents.*

PROOF: We have seen in 4.1.2 that $\bar{e}$ is an algebraic diagonable element in $U(J)$ with roots in the set $\{0,\frac{1}{2},-\frac{1}{2},1,-1\}$. By the previous proposition the linear span S of $\{x \in U(J): x \sim \bar{e}\}$ is a Lie ideal of $U(J)$. Moreover, S is spanned by idempotents since $x \sim \bar{e}$ clearly implies that x satisfies the same minimal polynomial as $\bar{e}$ and hence is also a linear combination of idempotents. It is therefore sufficient to prove S generates $U(J)$.

For this, define $S' = \{a \in J: \bar{a} \in S\}$. Then S' is an associator ideal of J in the sense that

(2) $\qquad (S',J,J) + (J,S',J) + (J,J,S') \subseteq S'.$

To establish this, we first note that by the identity (2) of 4.1, we have

(3) $\qquad \overline{a}\overline{b}\overline{c} - \overline{a}\overline{bc} + \overline{c}\overline{b}\overline{a} - \overline{c}\overline{ba} + \overline{ca.b} - \overline{b}\overline{ca} = 0$ for any $a,b,c \in J$

Interchanging a and b in (3) and then subtracting from (2), we obtain

(4) $\qquad \overline{a}\overline{bc} + \overline{c}\overline{b}\overline{a} - \overline{b}\overline{a}\overline{c} - \overline{c}\overline{a}\overline{b} + \overline{ca.b} - \overline{cb.a} = 0$

and this can be re-written as

(5) $\qquad \overline{(a,c,b)} = (\overline{c},(\overline{a},\overline{b}))$

Suppose $a \in S'$, $b,c \in J$. Then we see from (5) that $\overline{(a,c,b)} \in S$ because S is a Lie ideal; hence $(S',J,J) \subseteq S'$. By similar arguments we also see that $(J,S',J) \subseteq S'$ and $(J,J,S') \subseteq S$ and so we have (2). But then $S'' = \{a \in S' : aJ \subseteq S'\}$ is an ideal of J because if $a \in S''$ and $b,c \in J$, $ab.c = a.bc + (a,b,c) \in S'$ implies $ab \in S''$. Let $J = J_0 + J_{\frac{1}{2}} + J_1$ be the Peirce decomposition of J relative to e. Then we see that $e \in S''$, for

$\overline{eJ_0} = 0 \in S;$

$\overline{eJ_{\frac{1}{2}}} \subseteq \overline{J_{\frac{1}{2}}} \subseteq A_{\frac{1}{2}}(\overline{e}) + A_{-\frac{1}{2}}(\overline{e}) \subseteq S$ by Propositions 4.1.2 and 4.3.4;

and $\quad \overline{eJ_1} \subseteq \overline{eFe} \subseteq \overline{F}\overline{e} \subseteq S.$

Thus $S''$ is a non-zero ideal of J and so equals J by simplicity. But $S'' \subseteq S'$ implies $S' = J$; i.e., $\overline{J} \subseteq S$. Since $\overline{J}$ generates $U(J)$, so does S.

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