INTERNAL WAVES IN A RANDOMLY STRATIFIED OCEAN

by

ROBERT ERNEST McGORMAN

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ABSTRACT

In this thesis we consider the propagation of internal waves in a rotating stratified unbounded ocean with randomly varying Brunt-Väisälä frequency, N. Keller's method is used to obtain the dispersion relation for the mean wave field correct to second order in ϵ when N is of the form $N^2 = N_0^2(1+\epsilon\mu)$ where N_0 = constant, $0 < \epsilon^2 << 1$ and μ is a centered stationary random function of either depth or time separately. From the dispersion relation there are derived general formulas for the change in phase speed and growth or attenuation rates due to the random fluctuations μ. These formulas are dependent on the statistics of μ only through the autocovariance function.

The phase speed change and growth rate formulas for depth dependent μ, which constitutes a model of the temperature and salinity fine-structure in the ocean, are presented for various special cases including the limiting cases of correlation lengths of μ that are long or short with respect to the wavelength. Observations at station P (50°N, 145°W) indicate that, to a good approximation, the μ are "white noise" and a close examination is made of the theoretical results for this case. With the aid of the station P data it is estimated that, although the phase speed changes are generally small, the amplitude of a wave increases (decreases) significantly in propagating upward (downward) through a depth of a few kilometers. In addition it is found that the mean effect of the depth dependent fluctuations μ is to increase the

effective Brunt-Väisälä frequency, or "stiffen" the fluid. This may explain why some recently observed frequency spectra of internal waves do not exhibit a sharp cut-off at N_o, the deterministic theoretical upper bound for the wave frequency. Finally an attempt is made to asses the range of validity of Keller's method in the context of the present problem.

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It has long been known that internal gravity wave motion exists in the ocean (see Chapter 5 of Phillips[14]). In order to propagate, these waves require a stratified medium such as the ocean where less dense surface layers rest on more dense deeper layers. A measure of the strength of the stratification is the Brunt-Väisälä frequency, N, defined by the equation

$$N^2 = \frac{-g\,\frac{d\rho_0(z)}{dz}}{\rho_0(z)}$$

where g is the acceleration due to gravity and $\rho_0(z)$ is the rest density of the water as a function of depth z, which is positive upwards.

Observations[18] indicate that internal waves frequently do not propagate with constant amplitude but grow to the point of breaking or become damped out. Possible mechanisms have been advanced to account for this behaviour including diffusive processes[12] and interaction with currents.[16] In this thesis we shall examine the effect on the waves of random variations in the quantity N^2 by studying the dispersion relation.

Many studies of internal waves have assumed a stratification such that $N^2(z) = N_0^2$, a constant. This is done for two reasons: first, it implies a smoothly varying dependence of density on depth, which may be a reasonable approximation to the actual dependence at least sectionally; second, it reduces a z-dependent coefficient in the internal wave equation to a constant, i.e., a purely mathematical convenience. The properties of free wave solutions for this case are well known and are summarized in Appendix A. Here we shall assume N^2 is subject to small random fluctuations about a constant mean N_0^2. In Chapter I the random fluctuations will be depth dependent and in Chapter II time dependent fluctuations will be considered. In Chapter III we discuss graphs of the relative phase speed change and growth rate plotted using real oceanic data in the corresponding formulas for "white noise" fluctuations derived in Chapter I.

I.1 Formal Dispersion Relation

In Appendix B we begin with the equations of mass and momentum conservation of an incompressible, inviscid fluid in a right-handed system of Cartesian coordinates uniformly rotating with angular frequency $f/2$ about the z-axis, which is vertical and positive upward. A basic state of hydrostatic equilibrium is assumed and upon this small perturbations in the fluid velocity, pressure and density are imposed. If the resulting equations are linearized and attention is restricted to two dimensional motion then the stream function is shown to satisfy the equation

$$\Phi_{xxtt} + \Phi_{zztt} + N^2\Phi_{xx} + f^2\Phi_{zz} - \frac{N^2}{g}\Phi_{ztt} - \frac{f^2N^2}{g}\Phi_z = 0 \qquad (I\text{-}1)$$

where x is the horizontal coordinate and t is the time.

We assume depth dependent random fluctuations in N^2 of the form

$$N^2 = N_0^2(1+\epsilon\mu(z))$$

where $\mu(z)$ is a zero-mean, wide-sense stationary random process" and ϵ is a size parameter such that $0<\epsilon^2<<1$. Restricting attention to the case of harmonic time dependence we set

$$\Phi(x,z,t) = e^{-i\sigma t}\,\Psi(x,z), \quad \sigma > 0$$

This gives equation I-1 the form

$$(\mathcal{M} + \mathcal{N})\Psi = 0 \qquad (I\text{-}2)$$

where

$$\mathcal{M} = \partial_z^2 - \frac{N_0^2-\sigma^2}{\sigma^2-f^2}\partial_x^2 - \frac{N_0^2}{g}\partial_z$$

a deterministic operator, and

$$\mathcal{N} = -\frac{\epsilon N_0^2\mu(z)}{\sigma^2-f^2}\partial_x^2 - \frac{\epsilon N_0^2\mu(z)}{g}\partial_z$$

a "small" random operator.

Equation I-2 is in a form suitable for the application of Keller's method[8,9,10,11] of which we give an account in Appendix C. There we show that, correct to second order in ϵ, the dispersion relation of the mean wave $\langle\Psi\rangle$ (where $\langle\ \rangle$ indicates ensemble average) is

$$e^{-i(kx+\ell z)}\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\}e^{i(kx+\ell z)} = 0 \qquad \text{(I-3)}$$

where $\mathcal{M}^{-1}$ is the integral operator defined by

$$\mathcal{M}^{-1}f(x,z) = \iint_{-\infty}^{\infty} G(x,x',z,z')f(x',z')\,dx'dz'$$

with G, the Green's function, being the solution of

$$\mathcal{M}G = \delta(x-x')\delta(z-z')$$

Since $\mathcal{M}$ is translationally invariant in x and z, the Green's function is a displacement kernel in $\mathcal{M}^{-1}$, that is,

$$G(x,x',z,z') = G(x-x',z-z')$$

so it is necessary to consider only

$$\mathcal{M}G = \delta(x)\delta(z) \qquad \text{(I-4)}$$

Equation I-4 can be solved by taking a double Fourier transform of both sides, i.e. multiply both sides by $e^{-i(kx+\ell z)}$ and integrate from $-\infty$ to $+\infty$ successively with respect to x and z. The inversion in ℓ can be readily performed to yield the Fourier transform in x of G, which we denote by $\hat{G}(k,z)$, so that

$$\hat{G}(k,z) = \frac{-H(-z)\,e^{dz/2}\sin\left(\sqrt{c^2k^2-d^2/4}\;z\right)}{\sqrt{c^2k^2-d^2/4}}$$

where H is the Heaviside function and we have set

$$d = N_o^2/g$$

and

$$c^2 = \frac{N_o^2-\sigma^2}{\sigma^2-f^2}$$

for convenience. In the deterministic theory there are two passbands of σ^2 defined by inequalities:

$$f^2 < \sigma^2 < N_o^2$$

represents passband I waves and

$$N_o^2 < \sigma^2 < f^2$$

represents passband II waves. In either case $c^2 > 0$.

I-2 Simplifying the Dispersion Relation

It is now possible to substitute $\mathcal{M}$, $\mathcal{M}^{-1}$ and $\mathcal{N}$ into I-3 to find the dispersion relation explicitly.

Set

$$T_1 = e^{-i(kx+\ell z)} \mathcal{M} e^{i(kx+\ell z)}$$

and

$$T_2 = -e^{-i(kx+\ell z)} \langle \mathcal{N}\mathcal{M}^{-1}\mathcal{N} \rangle e^{i(kx+\ell z)}$$

Then

$$T_1 = c^2k^2 - \ell^2 - i\ell d$$

and

$$T_2 = -e^{-i(kx+\ell z)} \langle \epsilon\mu(z)\left[-\frac{N_0^2}{\sigma^2-f^2}\partial_x^2 - d\partial_z\right] \iint_{-\infty}^{\infty} G(x-x', z-z') \cdot$$

$$\cdot\, \epsilon\mu(z')\left[-\frac{N_0^2}{\sigma^2-f^2}\partial_{x'}^2 - d\partial_{z'}\right] \rangle e^{i(kx'+\ell z')} dx'dz'$$

Simplifying

$$T_2 = -\epsilon^2 e^{-i(kx+\ell z)} \langle \mu(z)\left[\frac{N_0^2}{\sigma^2-f^2}\partial_x^2 + d\partial_z\right] \iint_{-\infty}^{\infty} G(x-x'; z-z') \cdot$$

$$\cdot\, \mu(z')\left[\frac{-N_0^2k^2}{\sigma^2-f^2} + i\ell d\right] e^{i(kx'+\ell z')} dx'dz'$$

This can be written

$$T_2 = -\epsilon^2\left[\frac{-N_0^2k^2}{\sigma^2-f^2} + i\ell d\right] e^{-i(kx+\ell z)} \langle \mu(z)\left[\frac{N_0^2}{\sigma^2-f^2}\partial_x^2 + d\partial_z\right] \cdot$$

$$\cdot \iint_{-\infty}^{\infty} G(x-x'; z-z')\mu(z') \rangle e^{i(kx+\ell z)} e^{-i\{k(x-x')+\ell(z-z')\}} dx'dz'$$

Make the change of variables

$$x'' = x - x'$$

$$z'' = z - z'$$

Then this gives

$$T_2 = -\epsilon^2\left[\frac{-N_0^2 k^2}{\sigma^2 - f^2} + il d\right] e^{-i(kx+lz)} \left\langle \mu(z)\left[\frac{N_0^2}{\sigma^2 - f^2}\partial_x^2 + d\partial_z\right]\iint_{-\infty}^{\infty} G(x'', z') \cdot\right.$$

$$\left. \cdot \mu(z - z'')\right\rangle e^{i(kx+lz)} e^{-i(kx''+lz'')} dx'' dz''$$

Now taking into account that the integral is a function of Z, the differentiation can be performed to give

$$T_2 = -\epsilon^2\left[\frac{-N_0^2 k^2}{\sigma^2 - f^2} + il d\right] e^{-i(kx+lz)} \left\langle \mu(z)\left[\frac{-N_0^2 k^2}{\sigma^2 - f'^2} + il d\right] e^{i(kx+lz)} \cdot\right.$$

$$\left. \cdot \iint_{-\infty}^{\infty} G(x'', z'')\mu(z - z'')\right\rangle e^{-i(kx''+lz'')} dx'' dz''$$

$$-\epsilon^2\left[\frac{-N_0^2 k^2}{\sigma^2 - f^2} + il d\right] e^{-i(kx+lz)} e^{i(kx+lz)} \left\langle \mu(z) d \cdot\right.$$

$$\left. \cdot \iint_{-\infty}^{\infty} G(x'', z'')\mu_z(z - z'')\right\rangle e^{-i(kx''+lz'')} dx'' dz''$$

Using the stationarity of $\mu(z)$ we define the autocovariance function, $\Gamma(z'')$, by

$$\Gamma(z'') = \langle \mu(z)\mu(z - z'')\rangle$$

Now

$$\mu_z(z-z'') = -\mu_{z''}(z-z'')$$

and

$$-\langle\mu(z)\mu_{z''}(z-z'')\rangle = -\partial_{z''}\langle\mu(z)\mu(z-z'')\rangle$$

Thus

$$T_2 = -\epsilon^2\left[\frac{-N_0^2k^2}{\sigma^2-f^2}+ild\right]^2\iint_{-\infty}^{\infty} G(x'',z'')\,\Gamma(z'')\,e^{-i(kx''+lz'')}\,dx''dz''$$

$$+\epsilon^2 d\left[\frac{-N_0^2k^2}{\sigma^2-f^2}+ild\right]\iint_{-\infty}^{\infty} G(x'',z'')\,\Gamma'(z'')\,e^{-i(kx''+lz'')}\,dx''dz''$$

and in each case the integration over x'' produces $\hat{G}(k,z'')$ in the above integrals.

So, correct to second order in ϵ, the dispersion relation reduces to

$$T_1+T_2 = c^2k^2-l^2-ild+\epsilon^2\left[\frac{-N_0^2k^2}{\sigma^2-f^2}+ild\right]^2\int_{-\infty}^{0}\frac{e^{dz/2}\sin(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma(z)\,e^{-ilz}}{\sqrt{c^2k^2-d^2/4}}\,dz$$

$$-\epsilon^2 d\left[\frac{-N_0^2k^2}{\sigma^2-f^2}+ild\right]\int_{-\infty}^{0}\frac{e^{dz/2}\sin(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma'(z)\,e^{-ilz}}{\sqrt{c^2k^2-d^2/4}}\,dz = 0$$

Using integration by parts we can write

$$\int_{-\infty}^{0}\frac{e^{dz/2}\sin(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma'(z)\,e^{-ilz}}{\sqrt{c^2k^2-d^2/4}}\,dz = -\int_{-\infty}^{0} e^{dz/2}\cos(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma(z)\,e^{-ilz}\,dz$$

$$-(d/_2 - il)\int_{-\infty}^{0} \frac{e^{dz/2}\sin(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma(z)\,e^{-ilz}}{\sqrt{c^2k^2-d^2/4}}\,dz$$

as the integrated term vanishes at 0 and ∞ . Collecting terms we obtain

$$c^2k^2-l^2-ild+\epsilon^2\left[\frac{N_0^2k^2}{\sigma^2-f^2}-d^2/_2\right]\left[\frac{N_0^2k^2}{\sigma^2-f^2}-ild\right]\int_{-\infty}^{0}\frac{e^{dz/2}\sin(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma(z)\,e^{-ilz}}{\sqrt{c^2k^2-d^2/4}}\,dz$$

$$-\epsilon^2 d\left[\frac{N_0^2k^2}{\sigma^2-f^2}-ild\right]\int_{-\infty}^{0} e^{dz/2}\cos(\sqrt{c^2k^2-d^2/4}\,z)\,\Gamma(z)\,e^{-ilz}\,dz = 0 \quad (I\text{-}5)$$

as the dispersion relation for the mean field $\langle\Psi\rangle$, correct to $O(\epsilon^2)$.

We note that the $O(\epsilon^2)$ terms, which are the corrections to the dispersion relation for internal waves in the non random fluid (i.e. the above with $\epsilon = 0$), are dependent on the passband of σ^2.

I-3 Polar Representation of the Dispersion Relation

In discussing the 0th order dispersion relation in Appendix A we employ a polar representation of the wavenumbers, i.e.

$$k = \mathcal{H}_0 \cos\theta$$

and

$$l = \mathcal{H}_0 \sin\theta - i d/_2$$

where $\mathcal{H}_0$ is real.

We want to extend this representation to the $O(\epsilon^2)$ dispersion relation, i.e. set

$$k = \mathcal{H}\cos\theta \qquad (I\text{-}6)$$

and

$$\ell = \varkappa \sin\theta - i d/2 \qquad (I-7)$$

where θ is a real angle and $\varkappa$ may be complex. Substituting I-6 and I-7 into I-5 and simplifying we obtain

$$c^2\varkappa^2\cos^2\theta - \varkappa^2\sin^2\theta - d^2/4 + \epsilon^2\left[\frac{N_o^2}{\sigma^2-f^2}\varkappa^2\cos^2\theta - \frac{d^2}{2}\right]\cdot$$

$$\cdot\left[\frac{N_o^2}{\sigma^2-f^2}\varkappa^2\cos^2\theta - id\varkappa\sin\theta - \frac{d^2}{2}\right]\int_{-\infty}^{0}\Gamma(z)\frac{\sin\left(\sqrt{c^2\varkappa^2\cos^2\theta - d^2/4}\,z\right)}{\sqrt{c^2\varkappa^2\cos^2\theta - d^2/4}}e^{-i\varkappa\sin\theta\, z}\,dz$$

$$-\epsilon^2 d\left[\frac{N_o^2}{\sigma^2-f^2}\varkappa^2\cos^2\theta - id\varkappa\sin\theta - \frac{d^2}{2}\right]\int_{-\infty}^{0}\Gamma(z)\cos\left(\sqrt{c^2\varkappa^2\cos^2\theta - d^2/4}\,z\right)e^{-i\varkappa\sin\theta\, z}\,dz = 0 \qquad (I-8)$$

Now if we fix an angle of propagation, θ, and specify the quantities σ, f, N_o, d and $\Gamma(z)$ then equation I-8 is an equation for $\varkappa$ which can be solved by iteration (see Keller and Veronis[11]). Setting $\epsilon = 0$ gives the 0th order solution of Appendix A

$$\varkappa_o = \frac{d}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} \qquad (I-9)$$

Substituting $\varkappa_o$ for $\varkappa$ in the $O(\epsilon^2)$ terms gives the first iteration solution (which is valid if the $O(\epsilon^2)$ terms are small, see discussion Appendix C)

$$\frac{\varkappa^2}{\varkappa_o^2} = 1 - \frac{4\epsilon^2}{d^2}\left[\frac{N_o^2}{\sigma^2-f^2}\varkappa_o^2\cos^2\theta - \frac{d^2}{2}\right]\left[\frac{N_o^2}{\sigma^2-f^2}\varkappa_o^2\cos^2\theta - id\varkappa_o\sin\theta - \frac{d^2}{2}\right]\cdot$$

$$\cdot\int_{-\infty}^{0}\Gamma(z)\,\frac{\sin(\mathcal{H}_0|\sin\theta|\,z)}{\mathcal{H}_0|\sin\theta|}\,e^{-i\mathcal{H}_0\sin\theta\, z}\,dz+\frac{4\epsilon^2}{d}\left[\frac{N_0^2}{\sigma^2 f^2}\mathcal{H}_0^2\cos^2\theta-id\mathcal{H}_0\sin\theta-d^2/2\right]\cdot$$

$$\cdot\int_{-\infty}^{0}\Gamma(z)\cos(\mathcal{H}_0|\sin\theta|\,z)\,e^{-i\mathcal{H}_0\sin\theta\, z}\,dz \qquad (I-10)$$

Since $\sin(\alpha z)/\alpha$ and $\cos(\alpha z)$ are even functions of α, the absolute value signs can be dropped. For convenience set $\alpha=\mathcal{H}_0\sin\theta$ and

$$A=\frac{N_0^2}{\sigma^2 f^2}\mathcal{H}_0^2\cos^2\theta-d^2/2$$

Taking the square root of both sides of I-10 and using the binomial expansion on the R.H.S. (assuming the $O(\epsilon^2)$ terms are small relative to 1), we obtain

$$\mathcal{H}/\mathcal{H}_0=1-\frac{2\epsilon^2}{d^2}A(A-id\alpha)\int_{-\infty}^{0}\Gamma(z)\,\frac{\sin\alpha z}{\alpha}\,e^{-i\alpha z}\,dz$$

$$+\frac{2\epsilon^2}{d}(A-id\alpha)\int_{-\infty}^{0}\Gamma(z)\cos\alpha z\,e^{-i\alpha z}\,dz$$

Hence it readily follows that

$$Re\{\mathcal{H}/\mathcal{H}_0\}=1-\frac{\epsilon^2A^2}{d^2\alpha}\int_{-\infty}^{0}\Gamma(z)\sin 2\alpha z\,dz+\frac{2\epsilon^2A}{d}\int_{-\infty}^{0}\Gamma(z)\,dz$$

$$-\epsilon^2\alpha\int_{-\infty}^{0}\Gamma(z)\sin 2\alpha z\,dz \qquad (I-11)$$

and

$$\mathrm{Im}\{\mathcal{H}/\mathcal{H}_0\} = \frac{\epsilon^2 A^2}{d^2\alpha}\int_{-\infty}^{0}\Gamma(z)(1-\cos 2\alpha z)\,dz \;-\epsilon^2\alpha\int_{-\infty}^{0}\Gamma(z)(1+\cos 2\alpha z)\,dz \qquad (I\text{-}12)$$

I-4 Special Cases

First of all we shall assume that the fluctuations in $\mu(z)$ are "white noise" so that

$$\Gamma(z) = \Gamma\delta(z)$$

Observations taken on February 23, 1971 and July 29, 1970 at weather ship station P (50°N, 145°W) and analyzed by Richard Thomson of Environment Canada indicate that the assumption is well justified over certain depth ranges. The quantity $N^2(z)$ was computed at 10 meter intervals from the ocean surface to a depth of 1500 meters. The results indicate that $N^2(z)$ exhibits a strong trend with depth in the upper 700 meters. However, between the depths of 700 m and 1500 m $N^2(z)$ appears to have little trend with mean $N_0^2 \simeq 3\times10^{-6}\,\mathrm{sec}^{-2}$ and stationary fluctuations, $\epsilon\mu(z)$, about N_0^2 with $\epsilon\mu \lesssim O(\frac{1}{2})$. Furthermore, the autocovariance functions for both the summer and winter data computed using the mean, N_0^2 , over this range indicate the $\epsilon\mu(z)$'s are "white noise" to a good approximation. We include for illustration in table I the autocovariance function computed from the winter data.

TABLE I, SAMPLE COMPUTED AUTOCOVARIANCE FUNCTION

DEPTH LAG, z m	$\epsilon^2 \Gamma(z)$
0	0.19580
10	0.09219
20	0.01881
30	0.06659
40	0.02786
50	0.00000
60	0.02475
70	0.06593
80	0.02659
90	0.03095
100	0.06968
110	0.04162
120	0.01218
130	0.0597[illegible]
140	0.07506
150	0.01380

Substituting $\Gamma\delta(z)$ for $\Gamma(z)$ in formulas I-11 and I-12 yields

$$Re\{\mathcal{H}/\mathcal{H}_0\} = 1 + \frac{\epsilon^2 A \Gamma}{d}$$

and

$$Im\{\mathcal{H}/\mathcal{H}_0\} = -\epsilon^2 \alpha \Gamma$$

Using equation I-9 and recalling the values of A and α we can write

$$Re\{\mathcal{H}/\mathcal{H}_0\} = 1 + \epsilon^2 d \Gamma \left[\frac{N_0^2}{\sigma^2 f^2} \cdot \frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2}\right] \qquad (I\text{-}13)$$

and

$$Im\{\mathcal{H}/\mathcal{H}_0\} = -\epsilon^2 d \Gamma \frac{\sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} \qquad (I\text{-}14)$$

We denote the dimensionless quantity $d\Gamma$ by $\hat{d}$ where

$$\hat{d} \equiv d\int_{-\infty}^{\infty}\Gamma(z)\,dz$$

If S_0 is the phase speed of the 0th order wave and S is the phase speed of the $O(\epsilon^2)$ wave then

$$Re\{\mathscr{H}/\mathscr{H}_0\} = S_0/S$$

and when $Im\{\mathscr{H}/\mathscr{H}_0\}$ is negative the wave grows, thus we can rewrite I-13 and I-14 as

$$\frac{S_0/S - 1}{\epsilon^2\hat{d}} = \frac{N_0^2}{\sigma^2 - f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2} \qquad (I\text{-}15)$$

and

$$\frac{-Im\{\mathscr{H}/\mathscr{H}_0\}}{\epsilon^2\hat{d}} = \frac{\sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} \qquad (I\text{-}16)$$

Equations I-15 and I-16 are discussed at length in Chapter III where they appear as equations III-1 and III-2. At this point we just mention that from I-16 it is clear that, independent of the σ^2 passband, waves propagating with an upward component of direction grow and waves propagating with a downward component decay with the magnitude of the growth or decay dependent on the passband of σ^2 only through the dimensionless quantity $\epsilon^2\hat{d}$.

Next we consider the special case of I-11 and I-12 for which

$$\Gamma(z) = \Gamma e^{-|z|/L}$$

which is the autocovariance function of a Uhlenbeck-Ornstein process.[6] The resulting integrals are easily evaluated as they have the form of Laplace transforms. Equations I-11 and I-12 become

$$Re\{\mathcal{H}/\mathcal{H}_0\} = 1 + \frac{2\epsilon^2\Gamma d^2L^2}{1+4\alpha^2L^2}\left[\frac{N_0^2}{\sigma^2-f^2}\cdot\frac{\cos\theta}{4(c^2\cos^2\theta-\sin^2\theta)}-\frac{1}{2}\right]^2$$

$$+\frac{2\epsilon^2\Gamma\alpha^2L^2}{1+4\alpha^2L^2}+2\epsilon^2\Gamma d L\left[\frac{N_0^2}{\sigma^2-f'^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta-\sin^2\theta)}-\frac{1}{2}\right] \quad (I\text{-}17)$$

and

$$Im\{\mathcal{H}/\mathcal{H}_0\} = \frac{2\epsilon^2\Gamma d L\sqrt{c^2\cos^2\theta-\sin^2\theta}}{\sin\theta}\left[\frac{N_0^2}{\sigma^2-f'^2}\cdot\frac{\cos\theta}{4(c^2\cos^2\theta-\sin^2\theta)}-\frac{1}{2}\right]\left\{\frac{4\alpha^2L^2}{1+4\alpha^2L^2}\right\}$$

$$-\frac{\epsilon^2\Gamma d L\sin\theta}{2\sqrt{c^2\cos^2\theta-\sin^2\theta}}\left\{\frac{2+4\alpha^2L^2}{1+4\alpha^2L^2}\right\} \quad (I\text{-}18)$$

If we set

$$\Gamma = \Gamma'/2L$$

and

$$\Gamma' d = \hat{d}$$

then as $L \to 0$ we have

$$\lim_{L\to 0}\Gamma'\frac{e^{-|z|/L}}{2L} = \Gamma'\delta(z)$$

Letting the dimensionless quantities $dL, \alpha L \to 0$ we have the limiting case of I-17 and I-18

$$\frac{S_0/S - 1}{\epsilon^2 \hat{d}} = \frac{N_0^2}{\sigma^2 f^2} \cdot \frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2}$$

and

$$\frac{-\operatorname{Im}\{\varkappa/\varkappa_0\}}{\epsilon^2 \hat{d}} = \frac{\sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}}$$

which is a recapturing of equations I-15 and I-16 of the "white noise" case.

Finally, we define a quantity, L, called the correlation length of $\mu(z)$; it is the smallest length such that

$$|z| > L \implies \Gamma(z) \simeq 0$$

It is now possible to deduce the form of the dispersion relation in the two limiting cases of short and long correlation length with respect to the wavelength.

First we examine the case of short correlation length, i.e. $L/\lambda_0 << 1$ where $\varkappa_0 = 2\pi/\lambda_0$. Non-dimensionalizing the depth coordinate z with L, i.e. setting $z = Lz^*$ it is now possible to rewrite I-11 and I-12 as

$$\operatorname{Re}\{\varkappa/\varkappa_0\} = 1 - \frac{\epsilon^2 A^2 L^2}{d^2}\int_{-\infty}^{0}\Gamma(Lz^*)\,\frac{\sin(2\alpha L z^*)}{\alpha L}\,dz^* + \frac{2\epsilon^2 A}{d}\int_{-\infty}^{0}\Gamma(z)\,dz$$

$$-\epsilon^2\alpha L\int_{-\infty}^{0}\Gamma(Lz^*)\,\sin(2\alpha L z^*)\,dz^* \qquad \text{(I-19)}$$

and

$$\mathcal{I}m\{\mathcal{H}/\mathcal{H}_0\} = \frac{\epsilon^2 A^2 L^2}{d^2}\int_{-\infty}^{0}\Gamma(Lz^*)\left\{\frac{1-\cos(2\alpha L z^*)}{\alpha L}\right\}dz^* - \epsilon^2\alpha\int_{-\infty}^{0}\Gamma(z)\,dz$$

$$-\epsilon^2\alpha L\int_{-\infty}^{0}\Gamma(Lz^*)\cos(2\alpha L z^*)\,dz^* \qquad (I-20)$$

By Appendix D we have that in the limiting case $L << \lambda_0$

$$\mathcal{R}e\{\mathcal{H}/\mathcal{H}_0\} = 1 - \frac{2\epsilon^2 A^2}{d^2}\int_{-\infty}^{0} z\,\Gamma(z)\,dz + \frac{2\epsilon^2 A}{d}\int_{-\infty}^{0}\Gamma(z)\,dz + O(\beta) \qquad (I-21)$$

and

$$\mathcal{I}m\{\mathcal{H}/\mathcal{H}_0\} = -2\epsilon^2\alpha\int_{-\infty}^{0}\Gamma(z)\,dz + O(\beta) \qquad (I-22)$$

where

$$\beta = 4\pi \sin\theta\; L/\lambda_0$$

If the term in I-21 involving the integral

$$\int_{-\infty}^{0} z\,\Gamma(z)\,dz$$

is small (which seems plausible for short correlation lengths) then I-21 and I-22 are qualitatively the same as the "white noise" formulas I-15 and I-16 since the integral

$$\int_{-\infty}^{0}\Gamma(z)\,dz > 0$$

as it is one half of the power spectrum of $\mu(z)$ evaluated at the origin.

Now in the case of long correlation length, i.e. $L/\lambda_0 >> 1$

we can obtain the asymptotic expansions[4] for the integrals in I-11 and I-12 involving the sine and cosine functions by repeated integration by parts. By Appendix E we have that in the limiting case $L >> \lambda_0$, with $\hat{\varkappa}_0 = \varkappa_0/d$ and $\hat{d} = d\int_{-\infty}^{\infty}\Gamma(z)\,dz$

$$\mathcal{R}e\{\varkappa/\varkappa_0\} = 1 + \frac{\epsilon^2\Gamma(0)}{2\sin^2\theta\,\hat{\varkappa}_0^2}\left[\frac{N_0^2}{\sigma^2-f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta-\sin^2\theta)} - \frac{1}{2}\right]^2 + \frac{\epsilon^2\Gamma(0)}{2}$$

$$+\epsilon^2\hat{d}\left[\frac{N_0^2}{\sigma^2-f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta-\sin^2\theta)} - \frac{1}{2}\right] + O(1/\beta) \qquad \text{(I-23)}$$

and

$$\mathcal{I}m\{\varkappa/\varkappa_0\} = \frac{\epsilon^2\hat{d}}{2\hat{\varkappa}_0\sin\theta}\left[\frac{N_0^2}{\sigma^2-f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta-\sin^2\theta)} - \frac{1}{2}\right]^2 - \frac{\epsilon^2\hat{d}\hat{\varkappa}_0\sin\theta}{2} + O(1/\beta) \qquad \text{(I-24)}$$

valid for θ not near $\theta = 0$ or $\theta = \pi$.

In view of the experimental data mentioned earlier in this chapter and the remarks of Appendix C we cannot expect to treat the case $L >> \lambda_0$ in the oceans. However, formulas I-23 and I-24 may have some relevance to internal wave motion in stellar atmospheres.

II.1 Formal Dispersion Relation

As in Chapter I the equation satisfied by the stream function $\Phi(x,z,t)$ is

$$\Phi_{xxtt} + \Phi_{zztt} + N^2\Phi_{xx} + f^2\Phi_{zz} - \frac{N^2}{g}\Phi_{ztt} - \frac{f^2N^2}{g}\Phi_z = 0 \qquad (II-1)$$

This time we assume $N^2 = N_o^2(1+\epsilon\mu(t))$ where $\mu(t)$ is again a zero-mean, wide-sense stationary random process. This gives equation II-1 the form

$$(\mathcal{M} + \mathcal{N})\Phi = 0 \qquad (II-2)$$

where

$$\mathcal{M} = \partial_x^2\partial_t^2 + \partial_z^2\partial_t^2 + N_o^2\partial_x^2 + f^2\partial_z^2 - \frac{N_o^2}{g}\partial_z\partial_t^2 - \frac{N_o^2 f^2}{g}\partial_z$$

and

$$\mathcal{N} = \epsilon\mu(t)N_o^2\left[\partial_x^2 - \frac{1}{g}\partial_z\partial_t^2 - \frac{f^2}{g}\partial_z\right]$$

In this case the dispersion relation to second order in ϵ becomes

$$e^{-i(kx+\ell z-\sigma t)}\left\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\right\}e^{i(kx+\ell z-\sigma t)} = 0 \qquad (II-3)$$

where $\mathcal{M}^{-1}$ is the integral operator defined by

$$\mathcal{M}^{-1}f(x,z,t) = \iiint_{-\infty}^{\infty} G(x,x';z,z';t,t')\,f(x';z';t')\,dx'dz'dt'$$

with G, the Green's function, being the solution of

$$\mathcal{M}G = \delta(x-x')\,\delta(z-z')\,\delta(t-t')$$

As in Chapter I, the Green's function is a displacement kernel

$$G(x,x',z,z',t,t') = G(x-x',z-z',t-t')$$

so it is necessary to consider only

$$\mathcal{M}G = \delta(x)\,\delta(z)\,\delta(t) \qquad (\text{II}-4)$$

We solve by taking a triple Fourier transform of both sides of II-4 and define

$$\hat{\hat{G}}(k,\ell,\sigma) = \iiint_{-\infty}^{\infty} G(x,z,t)\, e^{-i(kx+\ell z-\sigma t)}\,dx\,dz\,dt$$

Using the causality property, $G \equiv 0$ for $t<0$, the inversion in σ can be performed to yield the Fourier transform in x and z of G:

$$\hat{\hat{G}}(k,\ell,t) = \frac{-H(t)\sin\omega t}{\omega(k^2+\ell^2+i\ell d)}$$

with $d = N_0^2/g$ and

$$\omega = \sqrt{\frac{N_0^2k^2+\ell^2f^2+i\ell f^2 d}{k^2+\ell^2+i\ell d}}$$

for convenience.

<u>II.2 Simplifying the Dispersion Relation</u>

It is now possible to substitute $\mathcal{M}$, $\mathcal{M}^{-1}$ and $\mathcal{N}$ intc II-3 to find the dispersion relation explicitly.

Set

$$T_1 = e^{-i(kx+\ell z-\sigma t)} \mathcal{M}\, e^{i(kx+\ell z-\sigma t)}$$

and

$$T_2 = -e^{-i(kx+\ell z-\sigma t)} \langle \mathcal{N}\mathcal{M}^{-1}\mathcal{N} \rangle\, e^{i(kx+\ell z-\sigma t)}$$

Then

$$T_1 = \sigma^2(k^2+\ell^2) - N_o^2k^2 - \ell^2 f^2 + i\sigma^2\ell d - if^2\ell d$$

and

$$T_2 = -e^{-i(kx+\ell z-\sigma t)} \langle \epsilon\mu(t) N_o^2 [\,\partial_x^2 - \tfrac{1}{g}\partial_z\partial_t^2 - \tfrac{f^2}{g}\partial_z] \cdot$$

$$\cdot \iiint_{-\infty}^{\infty} G(x-x', z-z', t-t')\, \epsilon\mu(t')\, N_o^2[\partial_{x'}^2 - \tfrac{1}{g}\partial_{z'}\partial_{t'}^2 - \tfrac{f^2}{g}\partial_{z'}] \rangle \cdot$$

$$\cdot\, e^{i(kx'+\ell z'-\sigma t')} dx'dz'dt'$$

Simplifying,

$$T_2 = -\epsilon^2 N_o^4[-k^2 + i\ell\sigma^2/g - i\ell f^2/g]\, e^{-i(kx+\ell z-\sigma t)} \langle \mu(t)[\partial_x^2 - \tfrac{1}{g}\partial_z\partial_t^2 - \tfrac{f^2}{g}\partial_z] \cdot$$

$$\cdot \iiint_{-\infty}^{\infty} G(x-x', z-z', t-t')\, \mu(t') \rangle\, e^{i(kx'+\ell z'-\sigma t')} dx'dz'dt'$$

This can be written

$$T_2 = -\epsilon^2 N_o^4[-k^2 + i\ell\sigma^2/g - i\ell f^2/g]\, e^{-i(kx+\ell z-\sigma t)} \langle \mu(t)[\partial_x^2 - \tfrac{1}{g}\partial_z\partial_t^2 - \tfrac{f^2}{g}\partial_z] \cdot$$

$$\cdot\, e^{i(kx+\ell z-\sigma t)} \iiint_{-\infty}^{\infty} G(x-x', z-z', t-t')\, \mu(t') \rangle\, e^{-i[k(x-x')+\ell(z-z')-\sigma(t-t')]} dx'dz'dt'$$

Make the change of variables

$$x'' = x - x'$$

$$z'' = z - z'$$

$$t'' = t - t'$$

Then this gives

$$T_2 = -\epsilon^2 N_0^4 [-k^2 + i\ell\sigma^2/g - i\ell f^2/g] e^{-i(kx+\ell z-\sigma t)} \langle \mu(t) [\partial_x^2 - \tfrac{1}{g}\partial_z \partial_t^2 - \tfrac{f^2}{g}\partial_z] \cdot$$

$$\cdot e^{i(kx+\ell z-\sigma t)} \int_{-\infty}^{\infty} \hat{\hat{G}}(k,\ell,t'') \mu(t-t'') \rangle e^{i\sigma t''} dt''$$

Now taking account of the integral being a function of t, the differentiation can be performed to give

$$T_2 = -\epsilon^2 N_0^4 [-k^2 + i\ell\sigma^2/g - i\ell f^2/g]^2 \langle \mu(t) \int_{-\infty}^{\infty} \hat{\hat{G}}(k,\ell,t'') \mu(t-t'') \rangle e^{i\sigma t''} dt''$$

$$+ \frac{\epsilon^2 N_0^4}{g} [-k^2 + i\ell\sigma^2/g - i\ell f^2/g] \Big\{ 2\sigma\ell \langle \mu(t) \int_{-\infty}^{\infty} \hat{\hat{G}}(k,\ell,t'') \mu_t(t-t'') \rangle e^{i\sigma t''} dt''$$

$$+ i\ell \langle \mu(t) \int_{-\infty}^{\infty} \hat{\hat{G}}(k,\ell,t'') \mu_{tt}(t-t'') \rangle e^{i\sigma t''} dt'' \Big\}$$

Using the stationarity of $\mu(t)$ we define the autocovariance function $\Gamma(t'')$ by

$$\Gamma(t'') = \langle \mu(t) \mu(t-t'') \rangle$$

Then

$$T_2 = -\epsilon^2 N_0^4 [-k^2 + i\ell\sigma^2/g - i\ell f^2/g]^2 \int_{-\infty}^{\infty} \hat{\hat{G}}(k,\ell,t'') \Gamma(t'') e^{i\sigma t''} dt''$$

$$+\frac{\epsilon^2 N_o^4}{g}\left[-k^2+\frac{i\ell\sigma^2}{g}-\frac{i\ell f^2}{g}\right]\int_{-\infty}^{\infty}\hat{G}(k,\ell,t'')\left[-2\sigma\ell\,\Gamma'(t'')+i\ell\,\Gamma''(t'')\right]dt''$$

The dispersion relation, correct to second order in ϵ, is thus

$$T_1+T_2=\sigma^2(k^2+\ell^2)-N_o^2k^2-\ell^2f^2+i\sigma^2\ell d-if^2\ell d$$

$$+\epsilon^2N_o^4\left[k^2-\frac{i\ell\sigma^2}{g}+\frac{i\ell f^2}{g}\right]^2\int_0^{\infty}\frac{\sin\omega t}{\omega(k^2+\ell^2+i\ell d)}\,\Gamma(t)\,e^{i\sigma t}\,dt$$

$$+\frac{\epsilon^2N_o^4}{g}\left[k^2-\frac{i\ell\sigma^2}{g}+\frac{i\ell f^2}{g}\right]\int_0^{\infty}\frac{\sin\omega t}{\omega(k^2+\ell^2+i\ell d)}\left[-2\sigma\ell\,\Gamma'(t)+i\ell\,\Gamma''(t)\right]e^{i\sigma t}dt=0 \quad (\text{II-5})$$

Setting $\epsilon=0$ and rearranging the terms of equation II-5 gives the 0th order dispersion relation as in Appendix A, where we had σ, k, ℓ_R real and

$$\ell=\ell_R-id/2$$

It is again physically meaningful to substitute the above form for ℓ into equation II-5, the general dispersion relation to $O(\epsilon^2)$, and then solve the equation for σ. Making the substitution we obtain

$$\sigma^2(k^2+\ell_R^2-d^2/4-i\ell_R d)-N_o^2k^2-f^2(\ell_R^2-d^2/4-i\ell_R d)+i\sigma^2(\ell_R-id/2)d$$

$$-if^2(\ell_R-id/2)d+\epsilon^2N_o^4\left[k^2-\frac{i\sigma^2}{g}(\ell_R-id/2)+\frac{if^2}{g}(\ell_R-id/2)\right]^2.$$

$$\cdot\int_0^\infty \frac{\sin\omega t}{\omega(k^2+\ell_R^2+d^2/4)}\Gamma(t)e^{i\sigma t}dt + \frac{\epsilon^2 N_o^4}{g}\left[k^2 - \frac{i\sigma^2}{g}(\ell_R - id/2) + \frac{if^2}{g}(\ell_R - id/2)\right]\cdot$$

$$\cdot\int_0^\infty \frac{\sin\omega t}{\omega(k^2+\ell_R^2+d^2/4)}\left[-2\sigma(\ell_R - id/2)\Gamma'(t) + i(\ell_R - id/2)\Gamma''(t)\right]e^{i\sigma t}dt = 0$$

This reduces to

$$\sigma^2(k^2+\ell_R^2+d^2/4) - N_o^2k^2 - f^2(\ell_R^2+d^2/4) + \epsilon^2 N_o^4\left[k^2 - \frac{i\ell_R}{g}(\sigma^2-f^2) - \frac{d}{2g}(\sigma^2-f^2)\right]^2\cdot$$

$$\cdot\int_0^\infty \frac{\sin\omega t}{\omega(k^2+\ell_R^2+d^2/4)}\Gamma(t)e^{i\sigma t}dt + \frac{\epsilon^2 N_o^4}{g}\left[k^2 - \frac{i\ell_R}{g}(\sigma^2-f^2) - \frac{d}{2g}(\sigma^2-f^2)\right]\cdot$$

$$\cdot\int_0^\infty \frac{\sin\omega t}{\omega(k^2+\ell_R^2+d^2/4)}\left[-2\sigma\ell_R\Gamma'(t) + i\sigma d\Gamma'(t) + i\ell_R\Gamma''(t) + \frac{d}{2}\Gamma''(t)\right]e^{i\sigma t}dt = 0 \quad (\mathrm{II}\text{-}6)$$

Equation II-6 is an equation for σ, and can be solved by iteration. Setting $\epsilon = 0$ gives the 0th order solution

$$\sigma_o = \sqrt{\frac{N_o^2k^2+f^2\ell_R^2+f^2d^2/4}{k^2+\ell_R^2+d^2/4}}$$

where we choose the principal branch of the square root function for definiteness. Replacing σ by σ_o in the $O(\epsilon^2)$ terms and rearranging yields the approximate solution

$$\sigma^2 = \sigma_o^2 - \frac{\epsilon^2 N_o^4}{\sigma_o}\cdot\frac{\left[k^2 - \frac{i\ell_R}{g}(\sigma_o^2-f^2) - \frac{d}{2g}(\sigma_o^2-f^2)\right]^2}{(k^2+\ell_R^2+d^2/4)^2}\int_0^\infty \sin\sigma_o t\,\Gamma(t)\,e^{i\sigma_o t}dt$$

$$- \frac{\epsilon^2 N_o^4}{g\sigma_o} \cdot \frac{\left[k^2 - i\,{}^{\ell_R}\!/\!_g(\sigma_o^2 - f^2) - {}^{d}\!/\!_{2g}(\sigma_o^2 - f^2)\right]}{(k^2 + \ell_R^2 + d^2/4)^2} \cdot \int_0^\infty \sin\sigma_o t \left[-2\sigma_o \ell_R \Gamma'(t)\right.$$

$$\left. + i\sigma_o d\,\Gamma'(t) + i\ell_R \Gamma''(t) + d/_2\,\Gamma''(t)\right] e^{i\sigma_o t}\, dt$$

Dividing both sides by σ_o^2 and using the binomial approximation for the square root we can write

$$\sigma/_{\sigma_o} - 1 = - \frac{\epsilon^2 N_o^4}{2\sigma_o^3} \cdot \frac{\left[k^2 - i\,{}^{\ell_R}\!/\!_g(\sigma_o^2 - f^2) - {}^{d}\!/\!_{2g}(\sigma_o^2 - f^2)\right]^2}{(k^2 + \ell_R^2 + d^2/4)^2} \cdot \int_0^\infty \sin\sigma_o t\, \Gamma(t)\, e^{i\sigma_o t}\, dt$$

$$- \frac{\epsilon^2 N_o^4}{2g\sigma_o^3} \cdot \frac{\left[k^2 - i\,{}^{\ell_R}\!/\!_g(\sigma_o^2 - f^2) - {}^{d}\!/\!_{2g}(\sigma_o^2 - f^2)\right]}{(k^2 + \ell_R^2 + d^2/4)^2} \cdot \int_0^\infty \sin\sigma_o t \cdot \left[-2\sigma_o \ell_R \Gamma'(t)\right.$$

$$\left. + i\sigma_o d\,\Gamma'(t) + i\ell_R \Gamma''(t) + d/_2\,\Gamma''(t)\right] e^{i\sigma_o t}\, dt \qquad (\text{II-}7)$$

For a given choice of autocovariance function $\Gamma(t)$, $Re\{\sigma/_{\sigma_o} - 1\}$ gives the relative change in the phase speed of the mean wave, and $Im\{\sigma/_{\sigma_o} - 1\}$ represents the growth or attenuation factor. If both are positive then the mean wave travels faster and grows. It is also possible from equation II-7 to derive asymptotic results for the limiting cases of long and short (with respect to the wave period) correlation times similar to equations I-21, I-22, I-23 and I-24 of Chapter I.

CHAPTER III: FURTHER EXAMINATION OF THE CASE OF DEPTH DEPENDENT "WHITE NOISE" FLUCTUATIONS

III.1 Introduction

In this chapter we shall be concerned primarily with the case of "white noise" $\mu(z)$ variations and the equations

$$\frac{S_0/S - 1}{\epsilon^2 \hat{\mathcal{L}}} = \frac{N_0^2}{\sigma^2 f^2} \cdot \frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2} \qquad (\text{III}-1)$$

and

$$\frac{-\mathrm{Im}\{\mathcal{H}/\mathcal{H}_0\}}{\epsilon^2 \hat{\mathcal{L}}} = \frac{\sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} \qquad (\text{III}-2)$$

which are formulas I-15 and I-16 respectively of Chapter I.

In order to obtain a graphical representation corresponding to these formulas we employed the data from station P (50°N, 145°W) discussed in section I.4. We put $f^2 = 1.22 \times 10^{-8}\,\sec^{-2}$ throughout corresponding to 50°N latitude, $N_0^2 = 3 \times 10^{-6}\,\sec^{-2}$ for the passband I case, and in the absence of any data we chose $N_0^2 = 10^{-10}\,\sec^{-2}$ for the passband II case. Representative values of σ^2 were selected for both passbands to give the values of asymptote angle, $\theta_A = \tan^{-1}\sqrt{c^2}$ indicated on the curves.

We then let θ run over the range $0, 0.1\theta_A, \ldots 0.9\theta_A$ as III-1 (III-2) is even (odd) in θ, and computed by machine the R.H.S. of III-1 and III-2 for these radian angles of propagation. The print-out was hand plotted on graph paper, smooth curves were drawn through the points and the curves were inked onto tracing paper to give figures 1, 2 and 3.

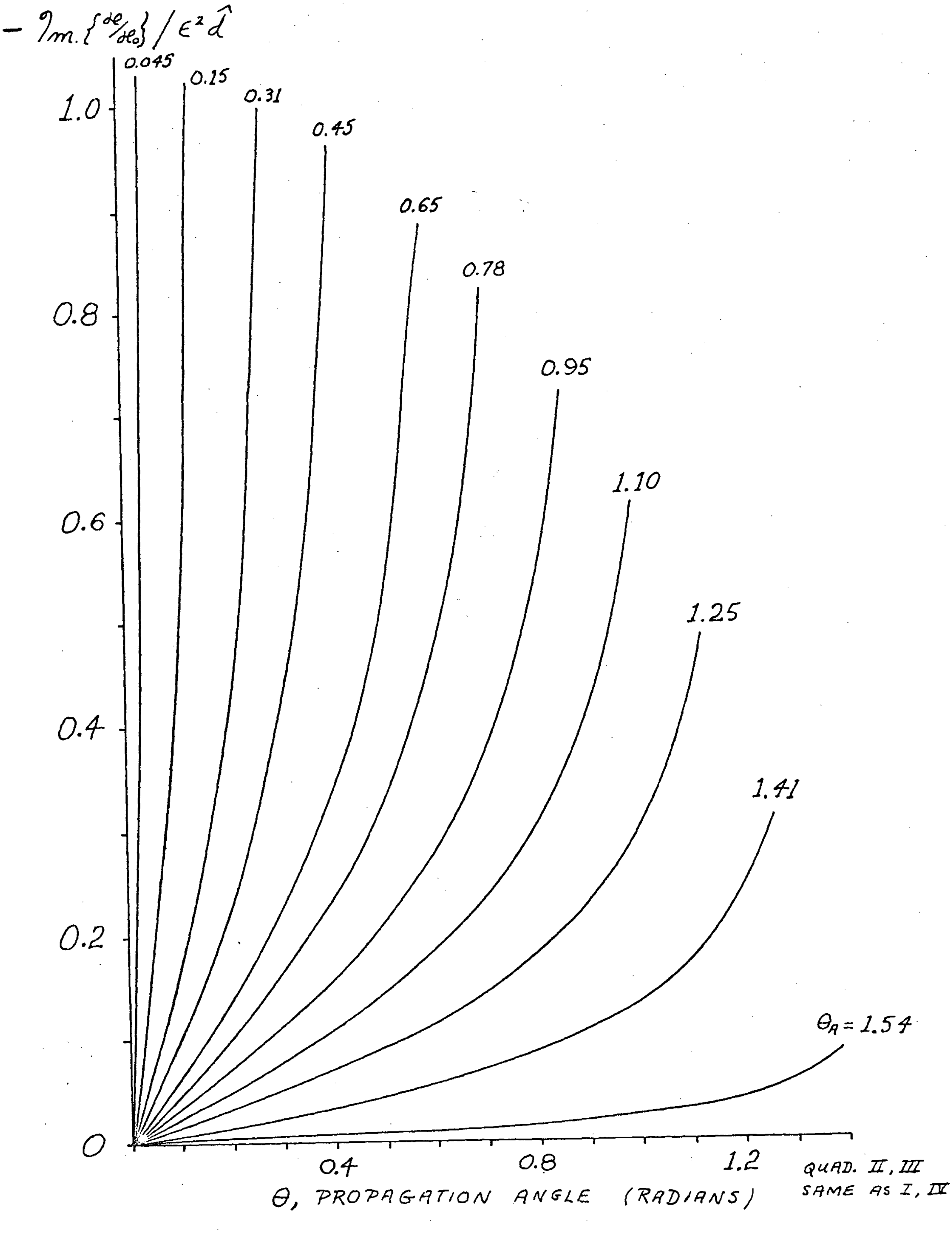

FIG. 1 GROWTH ($\theta > 0$) AND ATTENUATION ($\theta < 0$) RATES

III.2 Discussion of the "White Noise" Growth Curves

In order to interpret the graph of $-\mathcal{I}m\{\partial\kappa/\partial\kappa_0\}/\epsilon^2\hat{d}$ versus the radian angle of propagation for both passbands of σ^2, it should be understood that the dimensionless quantity $\epsilon^2\hat{d}$ will have a different value for each of the two passbands. We recall

$$\hat{d} = d\int_{-\infty}^{\infty}\Gamma(z)\,dz$$

$$= d\Gamma$$

for $\Gamma(z)=\Gamma\delta(z)$ and for the approximate value

$$g = 10^3\ cm\ sec^{-2}$$

we have

$$d = 3\times 10^{-9}\ cm^{-1}$$

for the passband I case, and

$$d = 10^{-13}\ cm^{-1}$$

for the passband II case, with the above values of N_o^2.

Examining figure 1 it is clear that waves propagating with an upward component of direction grow but that waves with a downward component of direction are attenuated, since the growth rate is an odd function of the propagation angle. This behaviour is also found in the 0th order case where an explanation for it is given in Appendix A.

There we saw that to 0th order a plane wave has the form

$$e^{dz/2}\cdot e^{i(kx+\ell_R z-\sigma t)}$$

For a polar representation we set

$$(k, \ell_R) = \mathcal{K}_0 (\cos\theta, \sin\theta)$$

and

$$(x, z) = \rho (\cos\varphi, \sin\varphi)$$

which gives the form $e^{dz/2} \cdot e^{i[\mathcal{K}_0 \rho \cos(\theta-\varphi) - \sigma t]}$

Now in the $O(\epsilon^2)$ dispersion relation we have allowed $\mathcal{K}$ to be complex, i.e.

$$\mathcal{K} = \mathcal{K}_R + i\mathcal{K}_I$$

and the mean wave has the form

$$e^{dz/2} \cdot e^{i[\mathcal{K}_R \rho \cos(\theta-\varphi) - \sigma t]} \cdot e^{-\mathcal{K}_I \rho \cos(\theta-\varphi)}$$

But, since in the "white noise" case

$$\frac{-\operatorname{Im}\{\mathcal{K}/\mathcal{K}_0\}}{\epsilon^2 \hat{\Gamma}} = \sin\theta \, \hat{\mathcal{K}}_0$$

or

$$-\mathcal{K}_I = \epsilon^2 \Gamma \mathcal{K}_0^2 \sin\theta$$

then the mean wave form can be written as

$$e^{(d/2 + \epsilon^2 \Gamma \mathcal{K}_0^2 \sin^2\theta) z + \epsilon^2 \Gamma \mathcal{K}_0^2 \cdot \frac{\sin 2\theta}{2} \cdot x} \cdot e^{i[\mathcal{K}_R \rho \cos(\theta-\varphi) - \sigma t]}$$

which implies that growth or attenuation occurs only if the wave has a vertical component of propagation (i.e. $\theta \neq 0, \pi$) and that the growth rate depends on the square of the corresponding 0th order wave number. In figure 1 wave number is a "hidden variable"; as the angle of propagation approaches the asymptote angle we find that the growth or attenuation rate becomes large because $\mathcal{H}_0$ becomes large.

So for a given distance h shorter waves will grow faster than longer waves, possibly to the point of breaking, in travelling upward from the plane $z = 0$ to the plane $z = h$. It is reasonable that shorter waves should be more affected by the presence of the "white noise" fluctuations. The analysis is not uniformly valid in the wavelength, as discussed in Appendix C; very short waves are excluded from consideration.

Since $d = -\rho_{0z}/\rho_0$ we define ℓ_ρ, a length scale determined by the variation in ρ, by

$$d = 1/\ell_\rho$$

Also let us define

$$d_{eff.} = d + 2\epsilon^2 \Gamma \mathcal{H}_0^2 \sin^2\theta$$

and by analogy with the above we define $\ell_{\rho eff.}$ by

$$d_{eff.} = 1/\ell_{\rho eff.}$$

Now $d_{eff.} > d$ so $\ell_{\rho eff.} < \ell_\rho$; in words, the mean effect of the random inhomogeneities is to shorten the length scale of the ρ variations. Consequently by analogy with the corresponding result of Appendix A growth or attenuation of the mean wave is necessary to preserve kinetic energy in the presence of the decreased mean density variation sacle $\ell_{\rho eff}$. Similarly defining $N_{o\,eff.}^2$ by

$$d_{eff.} = \frac{N_{o\,eff.}^2}{g}$$

we see this can be interpreted as an increase in the Brunt-Väisälä frequency or a "stiffening" of the fluid.

Using the station P data discussed in I.4 it is possible to estimate the magnitude of the change in growth or attenuation rates and the quantities $d_{eff.}$ and $N_{o\,eff}^2$. For both summer and winter in the 700 - 1500 m range we find

$$\epsilon^2 \int_{-\infty}^{\infty} \Gamma(z)\,dz = \epsilon^2 \Gamma \simeq 2 \times 10^3\,cm$$

Since $d \simeq 3 \times 10^{-9}\,cm^{-1}$ we have

$$d_{eff.}/d - 1 = 2\epsilon^2 \Gamma \varkappa_o^2 \sin^2\theta / d \simeq 10^{12} \varkappa_o^2 \sin^2\theta$$

Zalkan[18] observed for internal waves in the Pacific that $\varkappa_o \gtrsim 2\pi \times 10^{-5}\,cm^{-1}$ or $\lambda_o \lesssim 10^5\,cm = 1\,km$. Thus

$$d_{eff.}/d - 1 \gtrsim 4 \times 10^3 \sin^2\theta >> 1$$

unless $\theta \lesssim 10^{-2}$ radian. Thus the stochastic effects are dominant.

Since the vertical growth rate is essentially $\epsilon^2 \Gamma \varkappa_o^2 \sin^2\theta$ and the horizontal growth rate is $\epsilon^2 \Gamma \varkappa_o^2 \frac{\sin 2\theta}{2}$ then with $\epsilon^2 \Gamma \simeq 2 \times 10^3\,cm$ we find vertical and horizontal growth rates respectively of approximately $10^{-5} \sin^2\theta\,cm^{-1}$ and $4 \times 10^{-6} \sin 2\theta\,cm^{-1}$ for a 1 km wave, and $10^{-3} \sin^2\theta\,cm^{-1}$ and $4 \times 10^{-4} \sin 2\theta\,cm^{-1}$ for a 100 m wave.

If the waves are essentially horizontally propagating, as Zalkan[18] finds, i.e. $\theta \ll 1$, e.g. $\theta = 0.1$ then $\sin^2\theta \simeq \theta^2$ and $\sin 2\theta \simeq 2\theta$ and the rates for a 100 m wave are approximately $10^{-5}\,cm^{-1}$ and $10^{-4}\,cm^{-1}$. This represents an e-folding length of 10 wavelengths in the vertical and 1 wavelength in the horizontal.

Since $N_{o\,eff.}^2 = d_{eff.}\, g$ then

$$N_{o\,eff}^2 / N_o^2 = d_{eff.}/d \simeq 10^{12} \mathcal{H}_o^2 \sin^2\theta$$

with our data. Thus $N_{o\,eff.}/N_o \simeq 10^6 \mathcal{H}_o |\sin\theta| \simeq 2\pi$ for $\theta \simeq 0.1$ and a wavelength of 1 km. Hence the mean effect of the random fluctuations is to substantially "stiffen" the fluid rendering the Brunt-Väisälä frequency effectively greater. This suggests internal waves in passband I might exist with frequency $\sigma > N_o$, the deterministic cut-off. In fact some recently observed internal wave frequency spectra do not exhibit a sharp cut-off at N_o.[5,7,17]

Looking at the curves of figure 1 we can account for their overall appearance as follows:

1) Independence of $-\mathrm{Im}\{\mathcal{H}/\mathcal{H}_o\}/\epsilon^2\hat{d}$ on the passband of σ^2 follows as it is only $c^2 = (N_o^2 - \sigma^2)/(\sigma^2 - f^2)$ which determines the asymptotes, not the parameters individually.

2) Growth or attenuation requires vertical motion of the wave; hence for $\theta = 0, \pi$ we have no growth or attenuation.

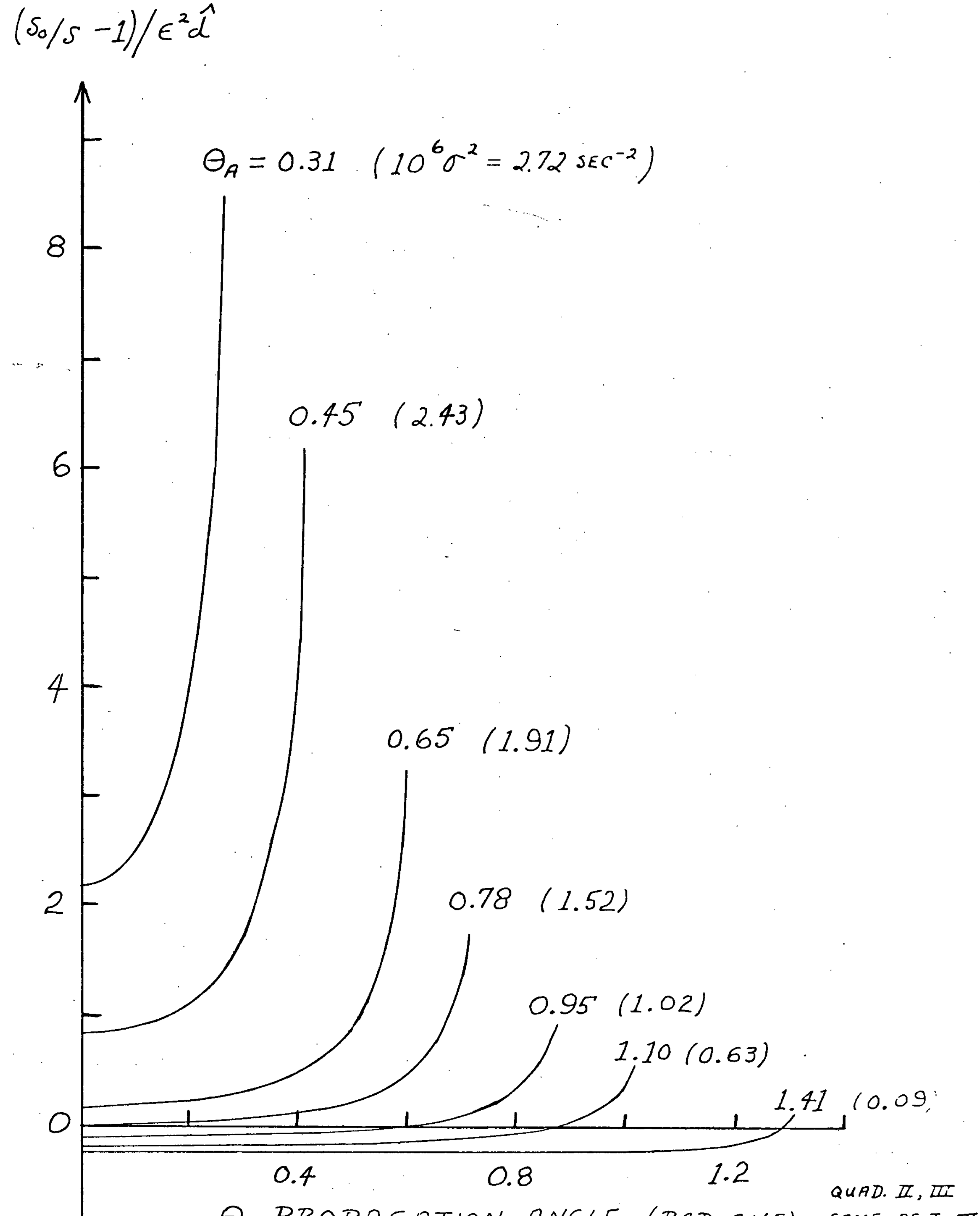

FIG. 2 RELATIVE PHASE SPEED CHANGE FOR PASSBAND I: $f^2 < \sigma^2 < N_0^2$

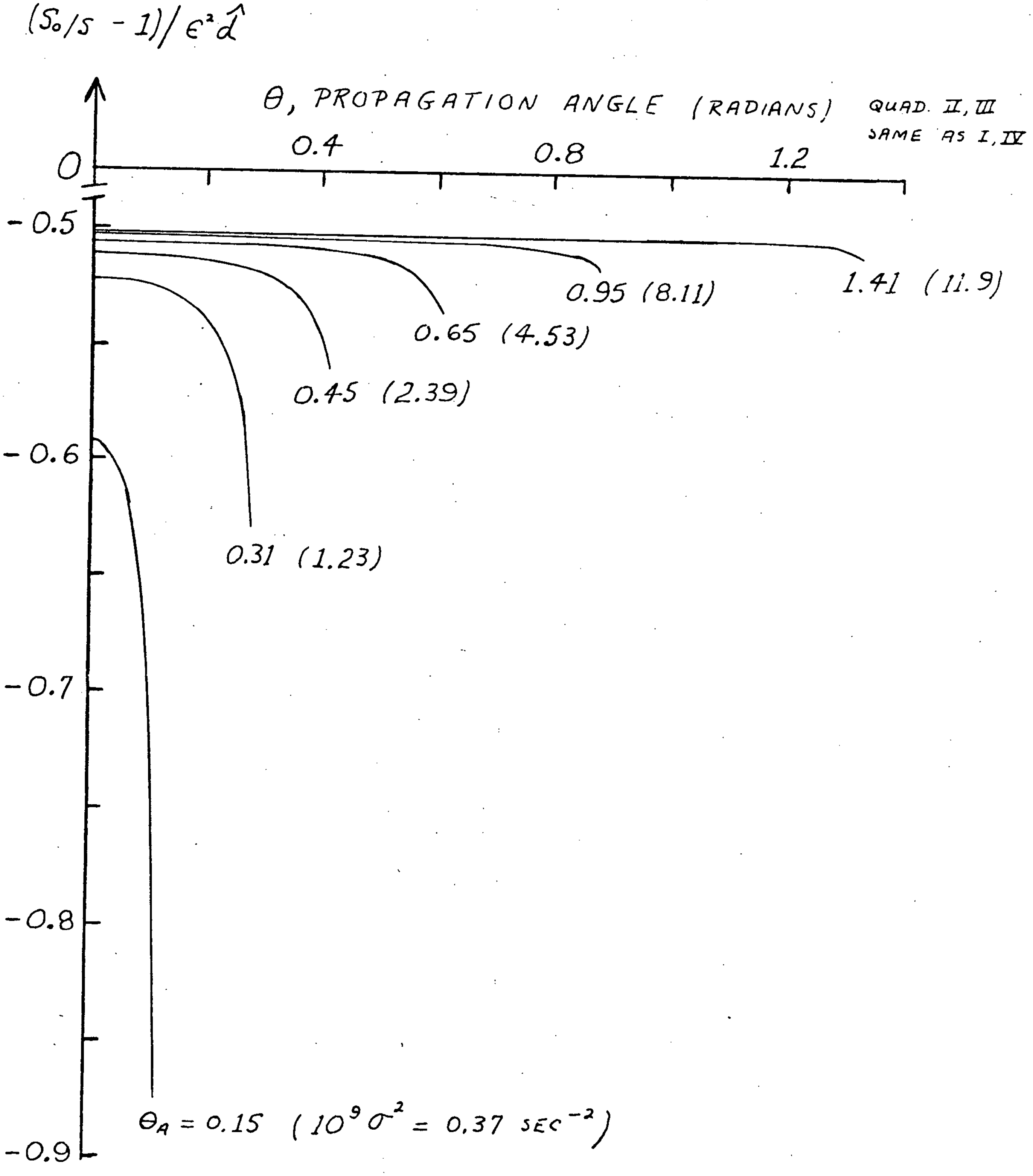

FIG. 3 RELATIVE PHASE SPEED CHANGE FOR PASSBAND II: $N_0^2 < \sigma^2 < f^2$

3) On a given curve as $|\theta|$ increases the waves are shorter and thus have a higher growth or attenuation rate.

4) As θ_R decreases from curve to curve the waves travelling in a fixed direction θ become shorter and thus have a higher growth or attenuation rate.

III.3 Discussion of the "White Noise" Phase Speed Curves

In this section we shall attempt to interpret the graphs in figures 2 and 3, which correspond to formula III-1 for passbands I and II respectively. Since the passband I case, $f^2 < \sigma^2 < N_o^2$, is the usual circumstance we shall give more attention to it.

We note that when the ordinate $(S_o/S - 1)/\epsilon^2\hat{\mathcal{L}} > 0$ this means S_o or the mean phase speed to $O(\epsilon^2)$ is less than the phase speed in the non random case. For our data $\epsilon^2\hat{\mathcal{L}} \simeq 6\times10^{-6}$ or $\simeq 10^{-5}$ and so $S_o/S - 1 \simeq 10^{-5}(O(1) - O(10))$ on examining figure 2. Thus phase speed changes are small, generally speaking.

Let us define $F(\theta)$ = R.H.S. of equation III-1, then $F(\theta)$ is monotonic increasing in passband I and monotonic decreasing in passband II. Also

$$F(0) = \frac{N_o^2}{4(N_o^2 - \sigma^2)} - \frac{1}{2}$$

in the limiting case $\theta \rightarrow \pi/2$, we have $\sigma^2 \downarrow f^2$ with $f^2 << N_o^2$ so $F(0) \simeq -1/4$ in passband I, and $\sigma^2 \uparrow f^2$ with $f^2 >> N_o^2$ so $F(0) \simeq -1/2$ in passband II.

In passband I the transition case for which the phase speed to $O(\epsilon^2)$ is nowhere greater than the 0th order phase speed occurs when $F(0)=0$ i.e. $\sigma^2 = N_0^2/2$. For $f^2 << N_0^2$ this means $\sigma^2 \simeq (N_0^2+f^2)/2$ and $(N_0^2-\sigma^2)/(\sigma^2-f^2) \simeq 1$ or $c^2 \simeq 1$; thus $\theta_A \simeq \pi/4$.

On examining figure 2 we see that when $\theta_A = 1.41$ corresponding to $10^6\sigma^2 = 0.09$, the phase speed to $O(\epsilon^2)$ is greater than the 0th order phase speed except for a small range of θ near θ_A. Since for large θ_A we have $\sigma^2 \downarrow f^2$ it appears the increase in phase speed is a result of the rotation. However, as $\theta \longrightarrow \theta_A$ the waves become shorter and are relatively slowed down, refraction effects apparently becoming dominant. The shorter the wave the more it is scattered by the random inhomogeneities in the medium and hence the farther it must travel to get from one place to another. This appears as a decrease in the mean phase speed.

As θ_A decreases from curve to curve in figure 2, C^2 decreases and, from Appendix A, this means $\mathcal{H}_0$ increases for fixed θ, i.e. the wavelength is decreased. Thus the refraction effects become more important resulting in the upper curves of figure 2 lying wholly above the θ axis. On any given curve for θ nearing θ_A the ends of the curve turn up sharply due to refraction of the rapidly shortening waves.

As was mentioned in the discussion of figure 1, the size of the dimensionless quantity $\epsilon^2\hat{\alpha}$ will depend on the passband of σ^2. To indicate which passband we are considering we shall use the subscript I or II. We have

$$\hat{d}_I = 3 \times 10^{-9} cm^{-1}$$

and

$$\hat{d}_{II} = 10^{-13} cm^{-1}$$

thus

$$\frac{\epsilon_I^2 \hat{d}_I \int_{-\infty}^{\infty} \Gamma_I(z) dz}{\epsilon_{II}^2 \hat{d}_{II} \int_{-\infty}^{\infty} \Gamma_{II}(z) dz} = 3 \times 10^4 \cdot \frac{\epsilon_I^2 \Gamma_I}{\epsilon_{II}^2 \Gamma_{II}}$$

It seems unlikely that the ratio $\epsilon_I^2 \Gamma_I / \epsilon_{II}^2 \Gamma_{II}$ could be small enough to yield a value of $\epsilon_{II}^2 \hat{d}_{II}$ larger than $\epsilon_I^2 \hat{d}_I$. However, in the absence of experimental data for the passband II case we shall consider the 0th order dispersion relation in order to obtain some idea of the importance of the fluctuations in N^2 for passband II relative to passband I.

Now

$$C_I^2 = \frac{N_{oI}^2 - \sigma_I^2}{\sigma_I^2 - f^2}$$

or

$$C_I^2 = \frac{3 \times 10^{-6} - \sigma_I^2}{\sigma_I^2 - 1.22 \times 10^{-8}} \qquad (III-3)$$

and

$$C_{II}^2 = \frac{\sigma_{II}^2 - N_{oII}^2}{f^2 - \sigma_{II}^2}$$

or

$$C_{II}^2 = \frac{\sigma_{II}^2 - 10^{-10}}{1.22 \times 10^{-8} - \sigma_{II}^2} \qquad (III-4)$$

Let $C_I^2 = C_{II}^2$ with C^2 their common value. Now let us increase N_o^2 by 10% in each case and let Δc^2 be the increment in C^2. Then

$$\Delta c_I^2 = \frac{3\times10^{-7}}{\sigma_I^2 - 1.22\times10^{-8}}$$

and

$$\Delta c_{II}^2 = \frac{-10^{-11}}{1.22\times10^{-8} - \sigma_{II}^2}$$

Now III-3 and III-4 can be solved for σ_I^2 and σ_{II}^2 and the results substituted into the above to yield

$$\Delta c_I^2 = \frac{3\times10^{-7}(1+c^2)}{3\times10^{-6} - 1.22\times10^{-8}} \backsimeq 10^{-1}(1+c^2)$$

and

$$\Delta c_{II}^2 = \frac{-10^{-11}(1+c^2)}{1.22\times10^{-8} - 10^{-10}} \backsimeq -10^{-3}(1+c^2)$$

Hence

$$\left|\frac{\Delta c_I^2}{\Delta c_{II}^2}\right| \backsimeq 10^2$$

Thus the asymptotes of the 0th order dispersion relation are much more affected by changes in the Brunt-Väisälä frequency in passband I than in passband II. This analysis suggests that the change in phase speed of the mean wave for passband II will be negligible, so we shall not concern ourselves with interpreting figure 3, which is included for completeness.

A difference in the behaviour of waves of passbands I and II is not surprising when it is recalled that the two classes of internal waves have essentially different dependencies on the media through which they propagate. Passband I waves cannot exist in an unstratified ocean and so are likely to be more sensitive to fluctuations in N^2 than passband II waves, which are inertial and can exist in an unstratified ocean.

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A- The 0th Order Dispersion Relation

For a plane wave $\psi = e^{i(kx+\ell z-\sigma t)}$ propagating in a stratified fluid with constant Brunt-Väisälä frequency N_0 that is uniformly rotating about the vertical (z) axis, the dispersion relation is

$$c^2k^2 - \ell^2 - i\ell d = 0$$

where $c^2 = (N_0^2-\sigma^2)/(\sigma^2-f^2)$, $d = N_0^2/g$, f = Coriolis parameter, x = horizontal distance, t = time.

Putting $\ell = \ell_R + i\ell_I$ with ℓ_R, ℓ_I real and requiring k real (as there is no physical reason for growth or attenuation of the wave along the horizontal direction) gives

$$\ell_I = -d/2$$

and

$$c^2k^2 - \ell_R^2 = d^2/4 \tag{A-1}$$

A plane wave now has the form $e^{dz/2}\,e^{i(kx+\ell_R z-\sigma t)}$. Waves propagating with an upward component of direction grow, and waves with a downward component of direction are attenuated. Now having $N^2 = N_0^2$, a constant, implies $\rho_0(z) \propto e^{-dz}$. The kinetic energy of a wave is proportional to ρA^2 where A is the wave amplitude. Hence the factor $e^{dz/2}$ is seen to be necessary to preserve the kinetic energy of the wave.

The equation $c^2k^2-\ell_R^2 = d^2/4$ corresponds to a dispersion diagram consisting of a rectangular hyperbola opening toward large $|k|$ values, for fixed σ. We introduce a polar coordinate representation of the real wave numbers. Set

$$(k, \ell_R) = \mathcal{H}_0(\cos\theta, \sin\theta)$$

Substituting into A-1, we find

$$\mathcal{H}_0 = \frac{d}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} > 0 \qquad (A-2)$$

for $\theta \in (-\theta_A, \theta_A) \cup (\pi - \theta_A, \pi + \theta_A)$ where θ_A, the angle at which the asymptotes to the hyperbola are inclined to the k -axis, is given by

$$\theta_A = \tan^{-1}\sqrt{c^2}$$

Thus it is clear that an increase in c^2 corresponds to an increase in θ_A or a spreading of the asymptotes. In addition equation A-2 implies that for fixed θ an increase in c^2 results in a decrease in $\mathcal{H}_0$.

B- Derivation of the Stream Function Equation

Let (x, y, z) be a point in a right-handed system of Cartesian coordinates rotating uniformly about the z -axis, which is vertical and positive upward. Let

$$f = 2\Omega \sin\varphi$$

be the Coriolis parameter, where Ω is the magnitude of the earth's rot. on vector and φ is the latitude. Then the system rotates with angular frequency $f/2$. If g is the magnitude of the effective gravitational acceleration, anti-parallel to the z -axis, (u, v, w) are the fluid velocity components, ρ is the fluid density and p is the pressure, then the momentum conservation equations are

$$u_t + uu_x + vu_y + wu_z - fv + 1/\rho\, p_x = 0$$

$$v_t + uv_x + vv_y + wv_z + fu + 1/\rho\, p_y = 0$$

$$w_t + uw_x + vw_y + ww_z + g + 1/\rho\, p_z = 0$$

for an inviscid fluid.

The equation of mass conservation is

$$\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z = 0$$

and the incompressibility condition is

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0$$

Initally setting

$$(u, v, w) = (0, 0, 0)$$

gives with $\rho = \rho_0(z)$, $p = p_0(z)$ the basic state of hydrostatic equilibrium

$$g\rho_0(z) + p_0(z) = 0$$

Introducing perturbations such that

$$\rho(x, y, z, t) = \rho_0(z) + \rho_1(x, y, t)$$

$$(u, v, w) = (u_1, v_1, w_1)$$

$$p = p_0(z) + p_1(x, y, t)$$

and linearizing the equations in the quantities subscripted with 1 we obtain

$$\rho_0(u_{1t} - fv_1) + p_{1x} = 0 \qquad (B-1)$$

$$\rho_0(v_{1t} + fu_1) + p_{1y} = 0 \qquad (B-2)$$

$$\rho_o w_{1t} + \rho_1 g + p_{1z} = 0 \qquad (B-3)$$

$$u_{1x} + v_{1y} + w_{1z} = 0 \qquad (B-4)$$

$$\rho_{1t} + w_1 \rho_{oz} = 0 \qquad (B-5)$$

Dropping the subscript 1, B-1, B-2 yield

$$\rho_o L u = -p_{xt} - f p_y$$

$$\rho_o L v = -p_{yt} + f p_x$$

where

$$L = \partial_t^2 + f^2$$

Thus

$$\rho_o L u_x = -p_{xxt} - f p_{xy} \qquad (B-6)$$

and

$$\rho_o L v_y = -p_{yyt} + f p_{xy} \qquad (B-7)$$

Applying $\rho_o L$ to B-4 yields

$$\rho_o L u_x + \rho_o L v_y + \rho_o L w_z = 0$$

Substituting B-6, B-7 gives

$$-p_{xxt} - p_{yyt} + \rho_o L w_z = 0$$

or

$$p_{xxzt} + p_{yyzt} - (\rho_o L w_z)_z = 0 \qquad (B-8)$$

Equation B-3 gives

$$\rho_o w_{tt} + \rho_t g + p_{zt} = 0$$

Using equation B-5

$$\rho_o w_{tt} + p_{zt} - g \rho_{oz} w = 0$$

or

$$p_{zt} = g \rho_{oz} w - \rho_o w_{tt}$$

Thus

$$p_{xxzt} = g\rho_{oz} w_{xx} - \rho_o w_{xxtt}$$

and

$$p_{yyzt} = g\rho_{oz} w_{yy} - \rho_o w_{yytt}$$

Substituting this into B-8 produces

$$-g\rho_{oz} w_{xx} + \rho_o w_{xxtt} - g\rho_{oz} w_{yy} + \rho_o w_{yytt} + (\rho_o L w_z)_z = 0$$

Alternatively,

$$w_{xxtt} + w_{yytt} + w_{zztt} - g\rho_{oz}/\rho_o\, w_{xx} - g\rho_{oz}/\rho_o\, w_{yy}$$

$$+ \rho_{oz}/\rho_o\, w_{ztt} + f^2 \rho_{oz}/\rho_o\, w_z + f^2 w_{zz} = 0 \qquad (B-9)$$

Put $N^2 = -g\rho_{oz}/\rho_o$ and $\partial_y \equiv 0$, then B-9 reduces to

$$w_{xxtt} + w_{zztt} + N^2 w_{xx} - N^2/g\, w_{ztt} - f^2N^2/g\, w_z + f^2 w_{zz} = 0$$

Now

$$u_x + w_z = 0$$

implies

$$w = -\Phi_x$$

where Φ is a stream function.[1] Thus the equation for Φ is seen to be

$$\Phi_{xxtt} + \Phi_{zztt} + N^2\Phi_{xx} + f^2\Phi_{zz} - N^2/g\, \Phi_{ztt} - \frac{f^2N^2}{g}\Phi_z = 0$$

the same as equation I-1 of the text.

C- Keller's Method

We now give a brief account of Keller's method[8,9,10,11] for deriving the dispersion relation for the mean wave in a random medium. A more general treatment will be found in Keller's paper in Electromagnetic Scattering[10].

If $\mathcal{L}$ is an invertible random linear operator and χ is a known function, then the equation

$$\mathcal{L}\Phi = \chi \tag{C-1}$$

implies Φ is a random process[2]. Applying $\mathcal{L}^{-1}$, the inverse of $\mathcal{L}$, to both sides of equation C-1 we obtain

$$\Phi = \mathcal{L}^{-1}\chi$$

Taking ensemble averages, denoted by $\langle \quad \rangle$, this becomes

$$\langle \Phi \rangle = \langle \mathcal{L}^{-1} \rangle \chi$$

Inverting $\langle \mathcal{L}^{-1} \rangle$ gives

$$\langle \mathcal{L}^{-1} \rangle^{-1} \langle \Phi \rangle = \chi \tag{C-2}$$

Now if $\mathcal{L}$ is statistically homogeneous and $\chi = 0$ equation C-2 has as eigendifferentials the plane wave solutions given by

$$\langle \Phi(x,z,t) \rangle = e^{i(kx + \ell z - \sigma t)}$$

which obey the dispersion relation

$$e^{-i(kx+\ell z-\sigma t)} \langle \mathcal{L}^{-1} \rangle^{-1} e^{i(kx+\ell z-\sigma t)} = 0 \tag{C-3}$$

Since $\mathcal{L} = \mathcal{M} + \mathcal{N}$ and $\mathcal{N}$ is "small" compared to $\mathcal{M}$ the binomial expansion is used to get

$$\mathcal{L}^{-1} = \mathcal{M}^{-1} - \mathcal{M}^{-1}\mathcal{N}\mathcal{M}^{-1} + \mathcal{M}^{-1}\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\mathcal{M}^{-1} - \ldots.$$

which is valid if $\|\mathcal{M}^{-1}\mathcal{N}\| < 1$. Averaging this equation with $\langle\mathcal{N}\rangle = 0$, one obtains

$$\langle\mathcal{L}^{-1}\rangle = \mathcal{M}^{-1} + \mathcal{M}^{-1}\langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\mathcal{M}^{-1} - \ldots.$$

This is inverted to yield

$$\langle\mathcal{L}^{-1}\rangle^{-1} = \mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle \qquad (C-4)$$

which is correct to second order in $\mathcal{N}$, and hence ϵ .

Substituting C-4 into C-3 gives

$$e^{-i(kx+\ell z-\sigma t)}\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\}e^{i(kx+\ell z-\sigma t)} = 0 \qquad (C-5)$$

as the dispersion relation of the infinitesimal amplitude mean wave $\langle\Phi\rangle$ correct to $O(\epsilon^2)$. Equation C-5 is identical with II-3.

In Chapter I we have

$$\Phi(x,z,t) = e^{-i\sigma t}\psi(x,z)$$

Thus in this case C-5 reduces to

$$e^{-i(kx+\ell z)}\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\}e^{i(kx+\ell z)} = 0$$

which coincides with equation I-3.

The foregoing analysis is dependent for its validity on the not easily applicable condition that $\|\mathcal{M}^{-1}\mathcal{N}\|$ be small. In order to achieve another perspective on the validity of equation C-5 we present an alternative derivation of it.

Proceeding directly from

$$(\mathcal{M} + \mathcal{N})\Phi = 0 \tag{C-6}$$

we have

$$\mathcal{M}\Phi = -\mathcal{N}\Phi \tag{C-7}$$

Applying $\mathcal{M}^{-1}$ to both sides of C-7 we obtain

$$\Phi = -\mathcal{M}^{-1}\mathcal{N}\Phi \tag{C-8}$$

Now C-8 is an integral equation and the first substitution yields

$$\Phi = \mathcal{M}^{-1}\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\Phi$$

Taking ensemble averages

$$\langle\Phi\rangle = \mathcal{M}^{-1}\langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\Phi\rangle$$

Thus

$$\mathcal{M}\langle\Phi\rangle = \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\Phi\rangle \tag{C-9}$$

We now make the so-called closure assumption[3,6]

$$\langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\Phi\rangle = \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\langle\Phi\rangle \tag{C-10}$$

Using C-10 in C-9 gives

$$\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\}\langle\Phi\rangle = 0$$

and again $\langle\Phi\rangle$ has the plane wave solutions $e^{i(kx+\ell z-\sigma t)}$ so the approximate dispersion relation becomes

$$e^{-i(kx+\ell z-\sigma t)}\{\mathcal{M} - \langle\mathcal{N}\mathcal{M}^{-1}\mathcal{N}\rangle\}e^{i(kx+\ell z-\sigma t)} = 0$$

identical with equation C-5.

The crucial point in the analysis is the approximation C-10.

On p. 45 W.E.Boyce[3] gives an account of R.C.Bourret's attempt to justify this approximation by an argument somewhat similar to the following.

In the term $\langle \mathcal{N}\mathcal{M}^{-1}\mathcal{N}\Phi \rangle$ it is quite clear that the statistics of Φ cannot be independent of those of $\mathcal{N}$ in view of equation C-6. However, if the scales of variation of $\mathcal{N}$ and Φ are greatly different then some justification for C-10 can be given. If the random processes μ and Φ are assumed to have the ergodic property, i.e. space and time averages are equivalent to ensemble averages, then it seems clear that if the scales of variation of $\mathcal{N}$ and Φ are greatly different it would be a good approximation to regard the more slowly varying of $\mathcal{N}\mathcal{M}^{-1}\mathcal{N}$ and Φ as constant while a small scale space or time average is made of the other. This average could be regarded as an ensemble average. Then a large scale average could be taken of the remaining process with the previously averaged quantity being regarded as a constant. This second average could also be identified as an ensemble average in view of the assumption of ergodicity.

When we used the 0th order solution, equation I-9, in the $O(\epsilon^2)$ terms and expected the first iteration of equation I-8 to give a good approximation for $\mathcal{H}$ it was necessary to assume that the $O(\epsilon^2)$ terms be small, or equivalently that $\mathcal{H}$ be not greatly different from $\mathcal{H}_0$. This assumption enabled us to use the binomial approximation to obtain equations I-11 and I-12. Similar remarks apply to the derivation of equation II-7 from II-6.

Now if we resubstitute

$$\alpha = \frac{d \sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}}$$

into equations I-11 and I-12, it will be clear that the assumption that the $O(\epsilon^2)$ terms must be small implies that the quantity $c^2\cos^2\theta - \sin^2\theta$ cannot approach 0 arbitrarily closely, i.e. θ cannot approach θ_R , defined in Appendix A, arbitrarily closely, and in other words the analysis cannot be expected to hold for very short waves. That the $O(\epsilon^2)$ perturbations cease to remain small as the waves become shorter and shorter is indicated in figures 1, 2 and 3 for "white noise" $\mu(z)$.

Excluding very short waves from the analysis is consistent with keeping only second order terms in ϵ in equation C-4. This is so because higher powers of $\mathcal{N}$ would add higher powers of the wave number to the dispersion relation and these could not be expected to be negligible for very short waves, as has been assumed in using C-5.

As a final note we point out the fact that the analyzed experimental data considered in Chapter I indicated $\mu(z)$ was "white noise" to a good approximation; and the analysis cannot be expected to be valid for very short waves, i.e. waves whose scale of variation approaches the scale of variation of $\mu(z)$. This is in agreement with the discussion of the alternative derivation of equation C-5 involving the closure assumption C-10.

D- The Limiting Case $L << \lambda_o$

With reference to formulas I-19 and I-20 we define

$$I_1 = L^2 \int_{-\infty}^{0} \Gamma(Lz^*) \frac{\sin(2\alpha L z^*)}{\alpha L} dz^*$$

$$I_2 = L \int_{-\infty}^{0} \Gamma(Lz^*) \sin(2\alpha L z^*) dz^*$$

$$I_3 = L \int_{-\infty}^{0} \Gamma(Lz^*) \cos(2\alpha L z^*) dz^*$$

and

$$I_4 = L^2 \int_{-\infty}^{0} \Gamma(Lz^*) \left\{ \frac{1-\cos(2\alpha L z^*)}{\alpha L} \right\} dz^*$$

Now $\alpha = \mathcal{H}_o \sin\theta$ and we define

$$\beta = 4\pi \sin\theta \, L/\lambda_o$$

then

$$I_i = I_i(\beta) \qquad (i = 1, 2, 3, 4)$$

and we want to take

$$\lim_{\beta \to 0} I_i(\beta) \qquad (i = 1, 2, 3, 4)$$

We have

$$I_1(\beta) = 2L^2 \int_{-\infty}^{0} z^* \Gamma(Lz^*) \frac{\sin(\beta z^*)}{\beta z^*} dz^*$$

$$I_2(\beta) = L \int_{-\infty}^{0} \Gamma(Lz^*) \sin(\beta z^*) dz^*$$

$$I_3(\beta) = L \int_{-\infty}^{0} \Gamma(Lz^*) \cos(\beta z^*) dz^*$$

and

$$I_4(\beta) = 2L^2 \int_{-\infty}^{0} \Gamma(Lz^*) \left\{ \frac{1-\cos(\beta z^*)}{\beta z^*} \right\} dz^*$$

Let

$$g(z^*) = |z^* \Gamma(Lz^*)|$$

and

$$h(z^*) = |\Gamma(Lz^*)|$$

Now the integrals

$$\int_{-\infty}^{0} h(z^*)\,dz^*$$

and

$$\int_{-\infty}^{0} g(z^*)\,dz^*$$

converge as Γ is an autocovariance function. It is easy to show that the functions $\sin(\beta z^*)/\beta z^*$, $\sin(\beta z^*)$, $\cos(\beta z^*)$ and $[1-\cos(\beta z^*)]/\beta z^*$ are absolutely bounded by 1 independent of β. Hence

$$\left| z^* \Gamma(Lz^*) \sin(\beta z^*)/\beta z^* \right| \leq g(z^*)$$

$$\left| \Gamma(Lz^*) \cos(\beta z^*) \right| \leq h(z^*)$$

$$\left| \Gamma(Lz^*) \sin(\beta z^*) \right| \leq h(z^*)$$

and

$$\left| z^* \Gamma(Lz^*) \left\{ \frac{1-\cos(\beta z^*)}{\beta z^*} \right\} \right| \leq g(z^*)$$

Then by Theorem VIII p. 667 of Taylor[15] the improper integrals of the first kind $I_i(\beta)$ $(i = 1, 2, 3, 4)$ above are uniformly convergent for β in $[-\infty, \infty]$. Hence we may take $\lim_{\beta \to 0}$ under the integral signs to obtain

$$\lim_{\beta \to 0} I_1(\beta) = 2L^2 \int_{-\infty}^{0} z^* \Gamma(Lz^*)\,dz^*$$

$$\lim_{\beta \to 0} I_2(\beta) = 0$$

$$\lim_{\beta \to 0} I_3(\beta) = L\int_{-\infty}^{0} \Gamma(Lz^*)\,dz^*$$

and

$$\lim_{\beta \to 0} I_4(\beta) = 0$$

Changing variables

$$2L^2\int_{-\infty}^{0} z^*\Gamma(Lz^*)\,dz^* = 2\int_{-\infty}^{0} z\,\Gamma(z)\,dz$$

and

$$L\int_{-\infty}^{0} \Gamma(Lz^*)\,dz^* = \int_{-\infty}^{0} \Gamma(z)\,dz$$

Hence equations I-21 and I-22 follow.

E- The Limiting Case $\lambda_o << L$

Here we derive formulas I-23 and I-24 for the limiting case $\lambda_o << L$ and prove we have obtained a true asymptotic result.

We have

$$\alpha = \mathcal{H}_o \sin\theta = \frac{d\sin\theta}{2\sqrt{c^2\cos^2\theta - \sin^2\theta}} = \frac{2\pi\sin\theta}{\lambda_o}$$

Define

$$I = \int_{-\infty}^{0} \frac{d\Gamma(z)}{dz}\cos(2\alpha z)\,dz + i\int_{-\infty}^{0} \frac{d\Gamma(z)}{dz}\sin(2\alpha z)\,dz$$

Then

$$I = \int_{-\infty}^{0} \frac{d\Gamma(z)}{dz}\, e^{i2\alpha z}\,dz$$

With $z = Lz^*$

$$I = \int_{-\infty}^{0} \frac{d\Gamma(Lz^*)}{dz^*}\, e^{i\frac{4\pi}{\lambda_o}\sin\theta\, Lz^*}\,dz^*$$

Letting $\beta = 4\pi \sin\theta/(\lambda_0/L)$ we can obtain the asymptotic expansion[4] of I as $\beta \to \infty$ along the real axis by repeated integration by parts. It is necessary to assume Γ is sufficiently differentiable and to restrict θ so it is not near $\theta = 0$ or $\theta = \pi$.

Letting

$$\Gamma^{(k)} = \frac{d^k \Gamma(Lz^*)}{dz^{*k}}$$

$$J_k = \int_{-\infty}^{0} \Gamma^{(k)} e^{i\beta z^*} dz^*$$

and

$$I_k = e^{i\beta z^*} \Gamma^{(k)} \Big|_{-\infty}^{0} = \Gamma^{(k)}_{(0)}$$

as Γ is an autocovariance function. We obtain after N integrations by parts

$$I = \sum_{m=1}^{N} \frac{(-1)^{m+1}}{(i\beta)^m} \Gamma^{(m)}_{(0)} + R_N$$

where

$$R_N = \frac{(-1)^N}{(i\beta)^N} \int_{-\infty}^{0} \frac{d^{(N+1)}\Gamma(Lz^*)}{dz^{*(N+1)}} e^{i\beta z^*} dz^*$$

To prove that this is the valid asymptotic expansion of I as $\beta \to \infty$ along the real axis it is necessary to prove that for fixed N.

$$\lim_{\beta \to \infty} \left[\frac{I - \sum_{m=1}^{N} \frac{(-1)^{m+1}}{(i\beta)^m} \Gamma^{(m)}_{(0)}}{1/\beta^N} \right] = 0 \qquad (E-1)$$

i.e.

$$\lim_{\beta \to \infty} \frac{(-1)^N}{i^N} \int_{-\infty}^{0} \frac{d^{(N+1)}\Gamma(Lz^*)}{dz^{*\,(N+1)}} e^{i\beta z^*} dz^* = 0 \qquad (E-2)$$

Now the integral in E-2 is just the sum of two expressions proportional to the Fourier sine and cosine transforms of the (N+1)th derivative of an autocovariance function and hence must die out at infinity in any physical system. Thus equation E-1 is proved and we have obtained the true asymptotic expansion of I as $\beta \to \infty$.

Keeping only the 0th order in $1/\beta$ gives

$$\lim_{\beta \to \infty} \int_{-\infty}^{0} \frac{d\Gamma(z)}{dz} \cos(2\alpha z)\, dz = 0$$

and

$$\lim_{\beta \to \infty} \int_{-\infty}^{0} \frac{d\Gamma(z)}{dz} \sin(2\alpha z)\, dz = 0$$

Hence in this limiting case the formulas I-11 and I-12 reduce to

$$Re\{\mathcal{H}/\mathcal{H}_0\} = 1 + \frac{\epsilon^2\Gamma(0)}{2\hat{\mathcal{H}}_0^2 \sin^2\theta}\left[\frac{N_0^2}{\sigma^2 f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2}\right]^2 + \frac{\epsilon^2\Gamma(0)}{2}$$

$$+ \epsilon^2 \hat{d}\left[\frac{N_0^2}{\sigma^2 f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2}\right] + O(1/\beta)$$

and

$$Im\{\mathcal{H}/\mathcal{H}_0\} = \frac{\epsilon^2\hat{d}}{2\hat{\mathcal{H}}_0 \sin\theta}\left[\frac{N_0^2}{\sigma^2 f^2}\cdot\frac{\cos^2\theta}{4(c^2\cos^2\theta - \sin^2\theta)} - \frac{1}{2}\right]^2 - \frac{\epsilon^2\hat{d}\,\hat{\mathcal{H}}_0 \sin\theta}{2} + O(1/\beta)$$

where $\hat{\mathcal{H}}_0 = \mathcal{H}_0/d$ and $\hat{d} = d\int_{-\infty}^{\infty}\Gamma(z)\,dz$ valid for θ not near $\theta = 0$ or $\theta = \pi$. These are formulas I-23 and I-24 respectively of the text.