UNIFYING THE BAIRE CATEGORY THEOREM

by

GEORGE MICHAEL HUBER

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in the Department
of
MATHEMATICS

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August, 1970

Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date December 18, 1970

ABSTRACT

The formulation of the Baire category theorem found in most elementary topology texts deals with two distinct classes of spaces: locally compact spaces, and complete metric spaces. This "dual theorem" status of Baire's theorem suggests the problem of finding one class of topological spaces for which the Baire category theorem can be proved and which includes both the locally compact spaces and the complete metric spaces. This thesis surveys and compares the three approaches to this problem taken by three methamticians.

The classical results of E. Čech achieve a unified Baire theorem by a definition of completeness different from that in current common usage. Johannes de Groot introduced a notion of subcompactness, generalizing compactness. K. Kunugi worked in the setting of complete ranked spaces which generalize uniform spaces and eliminate the need to assume regular separation in the space. This last point is the basis for the construction of a complete ranked space which is neither subcompact nor complete in the sense of Čech. It is also shown in the paper that there exist spaces subcompact but not complete in the sense of Čech, and that in certain special cases completeness in the sense of Čech implies subcompactness.

Table of Contents

UNIFYING THE BAIRE CATEGORY THEOREM

Formulations of the Baire category theorem found in most elementary topology texts are actually two somewhat distinct theorems: A locally compact Hausdorff space cannot be written as the countable union of nowhere dense subsets (i.e. is of second Baire category); a complete metric space is of second Baire category. This peculiar situation suggests the problem of finding one property of topological spaces for which the following three assertions are true: Every locally compact Hausdorff space enjoys this property; every complete metric space enjoys this property; every space enjoying this property is of second Baire category. This thesis is a survey of three papers by three mathematicians, each with a different solution for this problem.

Section 1 deals with the classical results of E. Čech [1] who proved the Baire category theorem using a definition of completeness which is more general than that currently in use. For metric spaces completeness in the sense of Cech is equivalent to topological completeness. It being the case that every locally compact Hausdorff space is complete in the sense of Čech, the desired unity is achieved.

In 1963 Johannes de Groot - apparently without reference to the work of Cech - approached the unifying problem with his generalization of compactness: <u>subcompactness</u>.

De Groot also had the intention of achieving a formulation of Baire's theorem which allowed the countability conditions to be changed to an arbitrary cardinality. In our presentation of his work in section 2 we have supressed these generalizations somewhat. Specializing to the standard classical definitions has facilitated the comparison of de Groot's subcompactness with completeness in the sense of Cech. This comparison is the content of the third section.

Kinjiro Kunugi proves the Baire category theorem in the settings of ranked spaces which is a generalization of uniform spaces [5]. This work is of particular interest in virtue of being the only formulation of the Baire theorem known to us which eliminates all separability requirements. This important weakening of the hypotheses allows us to close section 4 with an example of a "complete ranked" topological space which is neither complete in the sense of Cech nor subcompact as defined by de Groot.

ACKNOWLEDGEMENT

The author wishes to acknowledge his indebtedness to Dr. J.V. Whittaker for his patient guidance in formulating this thesis.

The financial support of the National Research Council of Canada during the writing of this thesis is also gratefully acknowledged.

Throughout this paper the definitions used will follow the usage of R. Engelking where his text [2] contains the corresponding terms. With important exception of paragraph 4 the topological spaces under consideration will be regular (T_3 separation) unless a stronger separation axiom is specified. It may be helpful to recall (or make, as the case may be) these few definitions: Given two subsets U and V of some set X the set theoretic difference of U and V will be written as $U \setminus V$; $U \setminus V$ is the intersection of the subset U with the complement (in X) of the subset V. We will tend to use capital Roman letters from the first part of the alphabet to denote families of subsets, and to use capital Roman letters from the last part of the alphabet to denote single subsets. In keeping with this tendency one will see the notation

$$A = \{U_s\}_{s \in S}$$

where A will be a family of subsets U_s indexed by some suitably large index set S.

A family of subsets is said to have the <u>finite intersection property</u> if each finite subcollection of subsets from the family has nonvoid intersection. A family of subsets is said to have the descending chain condition if every strictly decreasing sequence of subsets from the family is finite. A topological space is a <u>Baire space</u> if it satisfies either of the two equivalent conditions (a) The intersection of every countable collection of open everywhere dense subsets is everywhere dense, (b) The union of every countable collection of nowhere dense subsets is a boundary set. A boundary set is a subset having void interior; a set is nowhere dense iff (if and only if) its closure is a boundary set.

(1) COMPLETENESS IN THE SENSE OF CECH

This section is devoted to a summary of the classical treatment of the Baire Theorem of E. Cech who formulated the definition of completeness in the sense of Cech in 1937 [1]. Further bibliographic notes can be found in [2] whose text we are summarizing, pages 142 to 146.

Given a Tychonoff topological space X, we shall use a symbol of the form cX to denote a compactification of X; cX denotes a compact topological space and c denotes a homeomorphic embedding of X into cX. $c(X)$ shall denote the image of X under the mapping c, hence $\overline{c(X)} = cX$. It is well known that every topological space has a compactification iff it is a Tychonoff space. The symbol βX shall be reserved to denote the Cech-Stone compactification.

The following theorem is fundamental to the definition of completeness in the sense of Cech.

THEOREM (1.1): Let X be a Tychonoff space ($T_{3\frac{1}{2}}$ separation); then the following are equivalent.

(i) For every compactification cX of the space X the remainder $cX\setminus c(X)$ is an F_σ-set in cX.

(ii) The remainder $\beta X\setminus \beta(X)$ is an F_σ-set in βX.

(iii) For some compactification cX of X the remainder $cX\setminus c(X)$ is an F_σ-set in cX .

A Tychonoff topological space is said to be complete in the sense of Cech if it satisfies one of the equivalent conditions of Theorem (1.1).

An intrinsic characterization of completeness in the sense of Cech is given by

THEOREM (1.2): A Tychonoff space is complete in the sense of Cech iff there exists a countable family $\{A_i\}_{i=1}^{\infty}$ of open coverings of the space X satisfying the condition: If $\{V_s\}_{s\in S}$ is a family of closed subsets of X with (a) the finite intersection property and (b) for each $i = 1,2,\ldots$ there exist a $V_{s(i)}$ and an open subset $U_{(i)} \in A_i$ with $V_{s(i)} \subset U_{(i)}$; then the inequality $\bigcap_{s\in S} V_s \neq \emptyset$ holds. To illustrate the role of the family of open coverings, $\{A_i\}_{i=1}^{\infty}$ we interpolate the following example.

Let X be the half open interval $(0,1]$ with its usual topology. Then X is complete in the sense of Cech where for the family $\{A_i\}_{i=1}^{\infty}$ we may choose the single open covering $A_1 = \{(1/n,1] : n = 2,3,\ldots\}$. Indeed if $\{V_s\}_{s\in S}$ is a family of closed subsets satisfying (a) and (b) of Theorem (1.2), let $V_{s(1)}$ be the closed set and n_1 the integer with $V_{s(1)} \subset (1/n_1, 1]$. Then $\{V_s \cap V_{s(1)}\}_{s\in S}$ is a family of subsets of the compact topological space $[1/(n_1 + 1), 1]$. But this new family of closed subsets retains property (a) of Theorem (1.2), hence

$$\bigcap_{s\in S} (V_s \cap V_{s(i)}) \neq \emptyset \quad .$$

We conclude that

$$\emptyset \neq \bigcap_{s\in S} V_s \supset \bigcap_{s\in S} (V_s \cap V_{s(1)}) \ ,$$

and also that X is indeed complete in the sense of Cech.

The following theorem summarizes some of the properties of completeness in the sense of Cech.

THEOREM (1.3): (i) Completeness in the sense of Cech is a hereditary property with respect to closed subsets.

(ii) Completeness in the sense of Cech is a hereditary property with respect to subspaces which are G_δ-sets.

(iii) The topological sum of a family of disjoint topological spaces is complete in the sense of Cech iff each component of the sum is.

(iv) The Cartesian product of a countable number of topological spaces complete in the sense of Cech is complete in the sense of Cech.

Before explicitely recording Čech's form of the Baire theorem, two remarks are in order. A locally compact Hausdorff space is complete in the sense of Cech. To apply the definition we shall use condition (iii) of theorem (1.1). Indeed, the remainder of the space in its Alexandroff one point compactification is a single point; hence this remainder is closed, so in particular it is an F_σ-set. In the paper previously cited [1] E. Cech showed that in a metric topological space the notions of topological completeness and completeness in the sense of Cech are equivalent.

THEOREM (1.4): (Baire) Every space complete in the sense of Cech is a Baire space.

(2) SUBCOMPACTNESS OF DE GROOT

The work of Cech gives a unified Baire category theorem in the sense that it treats one class of topological spaces which includes simultaneously locally compact Hausdorff spaces as well as complete metric spaces. J. de Groot [3] deals with another class of topological spaces-subcompact ones-which also has this unity. The work of de Groot also explores the possibility of a formulation of Baire's theorem which replaces the countability conditions with m-conditions, m being any cardinal. All spaces are assumed to be regular (T_3 separation).

Let A denote a base of nonvoid open subsets of the space X. A nonvoid subset F of A is a regular filter base relative to A if

(i) Each set $U \in F$ contains some set $U' \in F$ with $\overline{U'} \subset U$,

(ii) F has the finite intersection property.

Let $F = \{U_s\}_{s \in S}$ be a regular filter base in X relative to some open base. F is preconvergent if $\bigcap_{s \in S} U_s \neq \emptyset$. A topological space is subcompact if there exists an open base A such that every regular filter base relative to A is preconvergent. Specializing to the countable case, a topological space is countably subcompact if there exists some open base A such that every countable regular filter base with respect to A is preconvergent. Completeness in the sense of Cech is in general only preserved by the formation of countable Cartesian products. Subcompact spaces enjoy the stronger property:

THEOREM (2.1): (i) The Cartesian product of subcompact spaces is subcompact.

(ii) The topological sum of disjoint subcompact spaces is subcompact.

Proof of (i): Let $\{X_s\}_{s \in S}$ be a collection of topological spaces with X_s subcompact with respect to open base A_s for each $s \in S$. If the whole space X_s is not an element of A_s for any s, then add X_s to the open base A_s so that $\{A_s\}_{s \in S}$ may be used to construct a basis for the Tychonoff topology of $\underset{s \in S}{P} X_s$. It remains to show that $\underset{s \in S}{P} X_s$ is subcompact with respect to this open basis.

If $\{F_r\}_{r \in R}$ is a regular filter base in $\underset{s \in S}{P} X_s$, then for each s, the projections $\{\pi_s F_r\}_{r \in R}$ will be a subset of A_s and a regular filter base. But each X_s is assumed subcompact so $\bigcap_{r \in R} (\pi_s F_r) \neq \emptyset$ for each $s \in S$. Hence $\bigcap_{r \in R} F_r \neq \emptyset$ and $\underset{s \in S}{P} X_s$ is subcompact with respect to the base generated by the bases A_s.

Proof of (ii): Suppose again that $\{X_s\}_{s \in S}$, is a collection of disjoint topological spaces with X_s subcompact with respect to open base A_s for each $s \in S$. Let $\{F_r\}_{r \in R}$ be a regular filter base in the topological sum of the X_s and select any r' from R . $F_{r'}$ is then a union of basis elements selected from among the bases A_s , $s \in S$. Some of the base sets of this union may be the void set, but not all of them; let $X_{s'}$ denote one of the spaces such that $F_{r'} \cap X_{s'} \neq \emptyset$. Relativizing now to the subcompact space $X_{s'}$ we can see that $\{F_r \cap X_{s'}\}_{r \in R}$ will be a regular filter base in $X_{s'}$. So invoking the definition of subcompactness, $\bigcap_{r \in R} (F_r \cap X_{s'}) \neq \emptyset$. A fortiori $\bigcap_{r \in R} F_r \neq \emptyset$, and the topological sum of the spaces X_s is subcompact.

If one imposes the restriction that the cardinality of the index set R be that of the natural numbers, the proofs just given become applicable to the

COROLLARY (2.2): (i) The Cartesian product of countably subcompact spaces is countably subcompact.

(ii) The topological sum of countably subcompact spaces is countably subcompact.

Before proceeding to the Baire theorem for subcompact spaces, it is in order to present the following two theorems.

THEOREM (2.3): A locally compact space is subcompact.

Proof: For one having in mind the Bourbaki definition of compactness, it is clear that a compact topological space is subcompact relative to any open basis. The locally compact case is reduced to the compact case as follows.

For each point x in the locally compact space X, select some open neighborhood of x with compact closure, V_x. Let A_x be the family of all open neighborhoods of x which are contained in this V_x. $B = \{A_x : x \in X\}$ is an open basis for the space X.

Suppose that F is a regular filter base in B and select one subset U from the collection F. Since F is a subcollection of B and each basis element in B has a compact closure, $\overline{U}$ is compact. As was noted at the onset the compact space $\overline{U}$ will be subcompact. But the family $E = \{U \cap W : W \in F\}$ is a regular filter base in the subcompact space $\overline{U}$. Hence

$$\emptyset \neq \bigcap_{W \in F} (U \cap W) \subset \bigcap_{W \in F} W$$

which shows the locally compact space X to be subcompact.

The countable case requires no further proof.

COROLLARY (2.4): A locally countably compact space is countably subcompact.

THEOREM (2.5): A metrizable space is complete iff it is subcompact.

Proof: We shall first assume that the metrizable space X is complete and construct a base for the space with respect to which X is subcompact. We will select a base from among the open sets of diameters $1/n$, $n = 1,2,\ldots$, by means of the following

LEMMA: Every cover of a set by a collection of subsets has a subcover satisfying the descending chain condition.

Proof of lemma: The lemma hinges on the fact that every decreasing sequence of ordinal numbers is eventually constant. Let any cover be indexed by well ordered set S . From the cover $\{C_s\}_{s \in S}$ select a subset B by the rule

$$B = \{C_s : C_s \not\subset C_t \text{ for all } t < s \text{ in } S\} .$$

B is a cover for X , and every sequence of decreasing subsets of B will correspond to a decreasing sequence of ordinal numbers. So B possesses the descending chain condition.

For each positive integer n the open sets of diameter $1/n$ constitute a cover for the space, hence we may apply the lemma to obtain K_n , a cover of the space which satisfies the descending chain condition and consists of open sets of diameter $1/n$. Denote by K the collection of all of the elements of all of these covers K_n . Since K is a collection of open covers of X by sets of diameter $1/n$, K is a base for X . We complete the first half of the proof by showing contradiction upon the assumption that X is not subcompact with respect to K .

To this end assume that there is a regular filter base in K which is not preconvergent. It is clear that each member of this regular filter base must properly contain some other member; in particular, let $\{V_i\}_{i=1}^{\infty}$ be a properly decreasing sequence of filter base elements. Because each of the families K_n satisfies the descending chain condition, only finitely many of the sets V_i can be selected from K_n for each $n = 1,2,\ldots$. Thus $\{V_i\}_{i=1}^{\infty}$ is not only a decreasing sequence of subsets in complete space X but the diameters of the sets V_i must tend to zero. We conclude that the intersection of the closures of the V_i' is nonvoid. Let x be some point in this intersection and suppose that U is a member of the given regular filter base such that $x \notin U$. Since the filter base is regular we may select a U' in the filter base with $\overline{U'} \subset U$. Clearly $x \notin \overline{U'}$, so by the separation in the space X we may find an integer j sufficiently large such that $\overline{V}_j \cap \overline{U'} = \emptyset$. This gives the contradiction in our filter base: $V_j \cap U' = \emptyset$. From this we conclude that x is an element of every member of the given regular filter base; hence the filter base is preconvergent and X is subcompact with respect to K .

Before proceeding with the details, we will outline our procedure for showing a subcompact metrizable space to be complete. Given metrizable space X , the metric completion of X is a metric space $\tilde{X}$ such that X can be mapped onto a dense subset of $\tilde{X}$ by a metric preserving homeomorphism. Considering X as a subspace of $\tilde{X}$, for each subset U open in X there is a subset $\tilde{U}$ open in $\tilde{X}$ such that $U = \tilde{U} \cap X$, and the closure of U in $\tilde{X}$ contains $\tilde{U}$. It is well known that a G_δ-set in a complete metric space is a complete metric subspace [2, p. 189]. In particular given a subcompact metric space X, we shall demonstrate that X can be written as a G_δ-set in $\tilde{X}$.

Suppose that the metric space X is subcompact relative to some open base $A = \{U_s\}_{s \in S}$. In the metric completion $\tilde{X}$ of X we can find a $\tilde{U}_s$ open in $\tilde{X}$ such that $U_s = \tilde{U}_s \cap X$, for each $s \in S$. Let this be done for each $s \in S$ and define $\tilde{A} = \{\tilde{U}_s\}_{s \in S}$. Each $\tilde{U}_s$ has a non-negative diameter, and since U_s is dense in $\tilde{U}_s$, diameter of U_s equals diameter of $\tilde{U}_s$ for each s in S. For each positive integer i let $\tilde{O}_i$ be the union of all the sets $\tilde{U}_i$ in $\tilde{A}$ whose diameters are less than $1/i$. Then the sets $\tilde{O}_i$ are open in $\tilde{X}$ and $\bigcap_{i=1}^{\infty} \tilde{O}_i$ is a G_δ set in $\tilde{X}$.

For each i, the space X is covered by base elements of diameter less than $1/i$. Since $U_s \subset \tilde{U}_s$ for all s we have $X \subset \tilde{O}_i$ for each positive integer i, and in particular $X \subset \bigcap_{i=1}^{\infty} \tilde{O}_i$. To show that $X = \bigcap_{i=1}^{\infty} \tilde{O}_i$ we will show the reverse inclusion.

Take $x \in \bigcap_{i=1}^{\infty} \tilde{O}_i$ and let $\tilde{V}_1$ be any element of $\tilde{A}$ of diameter less than one with $x \in \tilde{V}_1$. Such a $\tilde{V}_1$ exists because $x \in \tilde{O}_1$. Let n be an integer greater than one and assume that the $\tilde{V}_i$ have all been selected for i less than n, subject to the conditions

(a) $x \in \tilde{V}_i$,

(b) $\tilde{V}_i \in \tilde{A}$ with diameter of $\tilde{V}_i$ less than $1/i$,

(c) $\overline{\tilde{V}_i} \subset \tilde{V}_{i-1}$ (closure taken in $\tilde{X}$).

Now x is an interior point of $\tilde{V}_{n-1}$ so the distance from x to the complement of $\tilde{V}_{n-1}$ is some nonzero number, d. Let

$\tilde{V}_n$ be any element of $\tilde{A}$ which contains x and has diameter less than minimum $\{1/n, d/3\}$. Such a $\tilde{V}_n$ exists because $x \in \bigcap_{i=1}^{\infty} \tilde{O}_i$ assures us that we can find elements of $\tilde{A}$ containing x and having diameter less than $1/i$ for each positive i . The condition that the diameter of $\tilde{V}_n$ is less than $d/3$ assures that $x \in \overline{\tilde{V}_n} \subset \tilde{V}_{n-1}$. Inductively we have constructed a sequence of elements of $\tilde{A}$ satisfying (a), (b), and (c) above for all i greater than 1 .

Note first that the sequence $\{\tilde{V}_i\}_{i=1}^{\infty}$ is a decreasing sequence of nonvoid subsets of $\tilde{X}$ with diameters tending to zero. Since $\tilde{X}$ is complete Hausdorff space, the intersection of the $\overline{\tilde{V}_i}$ must be exactly one point. A fortiori, $\{x\} = \bigcap_{i=1}^{\infty} \overline{\tilde{V}_i}$.

Recall now that for every element of A , and in particular for the sets $\tilde{V}_i$ we have $\tilde{V}_i \cap X = V_i$, where V_i is an element of the given basis A . Condition (c) can be relativized to the space X :

$$\overline{V_i} \subset V_{i-1} \text{ for each } i \text{ greater than one.}$$

Thus $\{V_i\}_{i=1}^{\infty}$ is a regular filter base in A and $\bigcap_{i=1}^{\infty} V_i \neq \emptyset$. But this intersection is contained in $\bigcap_{i=1}^{\infty} \overline{\tilde{V}_i}$ because $V_i \subset \tilde{V}_i$ for each i . We conclude immediately: $\{x\} = \bigcap_{i=1}^{\infty} V_i$; and therefore $x \in X$. So $\bigcap_{i=1}^{\infty} \tilde{O}_i \subset X$. This concludes the proof that $X = \bigcap_{i=1}^{\infty} \tilde{O}_i$, a G_δ-set in complete metric space $\tilde{X}$.

Thus X itself is metrizable in a complete manner.

In the above theorem the proof that every subcompact metric space is complete only required the assumption that every countable regular filter base in A was preconvergent. Since a subcompact space is trivially countably subcompact, the following theorem is seen to be true.

THEOREM (2.6): In a metrizable space the following are equivalent:

(i) subcompactness,

(ii) countable subcompactness,

(iii) completeness in a suitably chosen metric.

The metric selected according to part (iii) will of course give rise to an equivalent topology in the space.

De Groot states a form of the Baire category theorem in a setting where the countability conditions are replaced by conditions with an arbitrary infinite cardinal, $\mathcal{M}$ as a parameter. Specifically:

> Definition. Let $\mathcal{M}$ be an infinite cardinal. A (e.g. closed) subset S of a topological space T is called $\mathcal{M}$-thin, if the intersection of any family of less than $\mathcal{M}$ open subsets of T is not (fully) contained in S, unless this intersection is empty.
> Complementarily, a (open) set O in T is called $\mathcal{M}$-puffed, if the intersection of any family of less than $\mathcal{M}$ open subsets of T meets O, unless this intersection is empty. So the complement of an $\mathcal{M}$-puffed set is $\mathcal{M}$-thin and conversely.

For the countable case we have

THEOREM (2.7): A subset is $\aleph_0$-thin iff it is a boundary set.

<u>Proof</u>: Let U be an $\aleph_0$-thin subset of topological space X. We must show that U has void interior. But the intersection of less than $\aleph_0$ open sets is a finite intersection of open sets,

which intersection is an open set. So U contains no open subset and hence has void interior.

Conversely if U is a boundary set in X then it contains no nonvoid open set; hence it does not contain a finite intersection of open sets unless that intersection is void. Hence U is $\aleph_0$-thin.

COROLLARY (2.8): (i) A closed subset is $\aleph_0$-thin iff it is nowhere dense.

(ii) An open set is $\aleph_0$-puffed iff it is everywhere dense.

Proof: We recall that a set is nowhere dense iff its closure is a boundary set. Also, the complement of an everywhere dense set is nowhere dense and conversely.

A topological space is called an $\mathcal{M}$-Baire space if it is not the union of at most $\mathcal{M}$ closed $\mathcal{M}$-thin subsets. With this definition an $\aleph_0$-Baire space corresponds with Baire space as defined previously. De Groot states both an $\mathcal{M}$-Baire theorem and its specialization to the countable case. We will not repeat his proof of the $\mathcal{M}$-Baire theorem but will supply the proof of the countable case which he omits.

THEOREM (2.9): (Baire-de Groot): A subcompact regular space is an $\mathcal{M}$-Baire space for every infinite cardinal $\mathcal{M}$.

THEOREM (2.10): (Baire- deGroot): A countably subcompact regular space is a Baire space.

Proof: Let there be given a topological space X which is subcompact with respect to base A, and let $\{U_i\}_{i=1}^{\infty}$ be a sequence of nowhere dense subsets of X. If O is any open subset of the space we must show that $0 \not\subset \bigcup_{i-1}^{\infty} \overline{U}_i$.

Since U_1 is nowhere dense there exists some point $x \notin 0\setminus\overline{U}_1$. Since X is regular we can further find some open neighborhood of x whose closure does not meet $\overline{U}_1$. Now A is a base for the space, so select some non void base element $V_1 \in A$ such that $x \in V_1 \subset 0$ and $\overline{V}_1 \cap \overline{U}_1 = \emptyset$.

Take some integer n greater than one and suppose that for each $j = 1,2,\ldots n-1$ we have determined base elements $V_j \in A$ subject to the conditions

(a) $\overline{V}_j \subset V_{j-1}$,

(b) $\bigcap_{i=1}^{j} V_i \neq \emptyset$,

(c) $\overline{V}_j \cap U_j = \emptyset$.

We would select a V_n satisfying (a), (b), and (c) .

Now $\bigcap_{i=1}^{n-1} V_i$ is an open set which by virtue of the induction hypothesis is nonvoid. U_n is a nowhere dense set so there is a point $x \in (\bigcap_{i=1}^{n-1} V_i)\setminus\overline{U}_n$. Since X is regular we can also find some open neighborhood of x whose closure does not meet $\overline{U}_n$. Select a base element $V_n \in A$ with $\overline{V}_n \cap \overline{U}_n = \emptyset$ and $x \in V_n \subset \overline{V}_n \subset \bigcap_{i=1}^{n-1} V_i$. This V_n satisfies conditions (a), (b), and (c), a fortiori.

The sequence $\{V_i\}_{i=1}^{\infty}$ is a countable regular filter base in A ; since $V_1 \subset 0$ and X is subcompact with respect to A we have

$$0 \supset \bigcap_{i=1}^{\infty} V_i \neq \emptyset .$$

But from condition (c) we see that $(\bigcap_{i=1}^{\infty} V_i) \cap (\bigcup_{j=1}^{\infty} \overline{U}_j) = \emptyset$.

We have thus produced a nonvoid subset of the arbitrary open set O which is disjoint from the union of the closures of the nowhere dense sets U_i . Hence $\bigcup_{i=1}^{\infty} U_i$ is a boundary set and the space X is a Baire space.

(3) TOWARD A COMPARISON

The classic results of E. Cech achieve a Baire category theorem by means of a definition of completeness which is a bit more general than completeness as the term is currently used. The work of J. de Groot has accomplished a similar end with a generalization of compactness. The task of comparing these two approaches to the Baire category theorem is, unfortunately, only partially accomplished in this thesis.

Subcompactness is an hereditary property under the formation of arbitrary Cartesian products (Theorem (2.1), i). Completeness in the sense of Cech is in general only preserved by the formation of countable Cartesian products (Theorem (1.3), iv). These general considerations give birth to the following example which demonstrates a regular topological space which is subcompact but not complete in the sense of Cech.

EXAMPLE (3.1): Let R be the real numbers and for each $r \in R$ let X_r be the half open interval $(0,1]$ with the usual topology. Let $X = \underset{r \in R}{P} X_r$ with the (usual) Tychonoff topology.

Each space X_r is a locally compact Hausdorff space and, as a result, subcompact (Theorem (2.3)). As was recalled immediately above, X, being the Cartesian product of subcompact spaces, is subcompact.

To show that X is not complete in the sense of Cech we will show that X cannot be a G_δ-set in some compactification. For each $r \in R$ let $\tilde{X}_r$ be the closed interval [0,1] with the usual topology. Then $\tilde{X} = \underset{r \in R}{P} \tilde{X}_r$ is a compactification of X.

For an open set in the Tychonoff product topology it is the case that the component projections map <u>onto</u> the components with at most a finite number of exceptions. So for a G_δ-set it will be the case that the component projections will map onto components, with at most countably many exceptions. In particular if X could be represented as a G_δ-set in $\tilde{X}$ it would be necessary that the component projections of X into the spaces $\tilde{X}_r$ would map onto all of the $\tilde{X}_r$ with at most countably many exceptions. This is not the case and so X is not a G_δ-set in $\tilde{X}$ and, by definition, X is not complete in the sense of Cech.

This example establishes that some subcompact spaces are not complete in the sense of Cech. To the converse question we can only answer that in certain special cases spaces complete in the sense of Cech are subcompact. In the case of metrizable topological spaces completeness in the sense of Cech and subcompactness are mutually equivalent to topological completeness, hence equivalent to each other. Another partial result is

THEOREM (3.2): Let X be a regular topological space. Then X is subcompact if there is an open cover A_1 of X such that for every family of closed subsets $\{E_s\}_{s \in S}$ with

(a) the finite intersection property,

(b) some element of $\{E_s\}$ is contained in some element of A_1,

we have $\bigcap_{s \in S} E_s \neq \emptyset$.

According to the Theorem (1.2) X is complete in the sense of Cech iff it has $T_{3\frac{1}{2}}$ separation and has a countable family of open coverings $\{A_i\}_{i=1}^{\infty}$ such that families of closed subsets will have nonvoid intersection if they satisfy conditions (a) and (b) for all $i = 1,2,\ldots$.

Proof: Let X be a regular topological space and A_1 an open cover of X as described in the hypotheses of the theorem. For each point $x \in X$ suppose W_x is an open set in A_1 containing x . Define A to be the family of all subsets of X which can be written in the form $W_x \cap O$ where $x \in X$ and O is open in X . We shall prove that X is subcompact with respect to this open basis, A .

Suppose that $F = \{U_s\}_{s \in S}$ is a regular filter base in A . For each $U_s \in F$ select some $U'_s \in F$ with $U'_s \subset \overline{U'_s} \subset U_s$. Define $V_s = \overline{U'_s}$.

$\{V_s\}_{s \in S}$ is a family of closed subsets of X . To see that it has the finite intersection property, we note that if $\bigcap_{i=1}^{n} V_{s_i} = \emptyset$, then since each V_{s_i} is the closure of an element of F (namely U'_{s_i}) we have

$$\bigcap_{i=1}^{n} U'_{s_i} \subset \bigcap_{i=1}^{n} \overline{U'_{s_i}} = \bigcap_{i=1}^{n} V_{s_i} = \emptyset ,$$

a contradiction.

By the way the base A was constructed, condition (b) is satisfied a fortiori. By the hypotheses of the theorem concerning the open cover A_1 we conclude that $\bigcap_{s\in S} V_s \neq \emptyset$. But then since $V_s \subset U_s$ for each $s \in S$,

$$\bigcap_{s\in S} U_s \supset \bigcap_{s\in S} V_s \neq \emptyset$$

showing X to be subcompact with respect to the open basis A.

COROLLARY (3.3): Let X be a regular topological space. Then X is subcompact if there is a finite family of open covers $\{A_i\}_{i=1}^{n}$ of X such that for every family of closed subsets $\{E_s\}_{s\in S}$ with

(a) the finite intersection property,

(b) for each $i = 1,2,\ldots, n$ there is some $U_i \in \{E_s\}$ and some open subset $W_i \in A_i$ with $U_i \subset W_i$,

we have $\bigcap_{s\in S} E_s \neq \emptyset$.

<u>Proof</u>: The extension to the case of finitely many open coverings is not essentially different from the main theorem. Set

$$B = \{V_1 \cap V_2 \cap \ldots \cap V_n | V_i \in A_i \text{ for each } i = 1,2,\ldots, n\}$$

Then B is a single covering of the space and everything reduces to the previous theorem with B playing the role of A_1. Indeed suppose we have a finite family of open coverings of X and a family of closed subsets satisfying conditions (a) and (b). So we have $\{U_i\}_{i=1}^n \subset \{E_s\}$ and $W_i \in A_i$ with $U_i \subset W_i$ for each $i = 1,2,\ldots, n$. Having assumed the finite intersection property we see that

$$\emptyset \neq \bigcap_{i=1}^{n} U_i \subset \bigcap_{i=1}^{n} W_i \in B .$$

Substituting for $\{E_s\}$ the family of all finite intersections of elements in $\{E_s\}$, theorem (3.2) applies.

We state explicitely the application of these results to spaces complete in the sense of Cech.

COROLLARY (3.4): If X is a topological space complete in the sense of Cech and the equivalent condition stated in theorem (1.2) can be achieved with only a finite family of open coverings, then X is subcompact.

Cases to which corollary (3.4) apply exist.

EXAMPLE (3.5): Let Q denote the rational numbers and let $X = (0,1)\setminus Q$ with the topology of the Euclidian metric. For each rational number q define $V_q = (0,1)\setminus\{q\}$. $X = \bigcap_{q\in Q} V_q$ is a representation of the metric space X as a G_δ-set in the complete metric space $[0,1]$. Then it is well known that this is sufficient for X to be topologically complete [2, p. 189]; so X is both subcompact and complete in the sense of Cech. We assert that corollary (3.4) does not apply to X. To demonstrate this we will show that every open subset of X contains a decreasing sequence of closed subsets with void intersection. To this end we consider X as a subspace of the unit interval $(0,1)$ with the induced topology. Any open set in X contains the intersection of X with some nondegenerate interval. If q is a rational point interior in this interval (speaking of q has meaning in the space $(0,1)$), then eventually the subsets V_i defined to be $X \cap [q, q + 1/i]$ will be contained in this interval; if N is the first integer for which this is true, then $\{V_i \cap X\}_{i=N}^{\infty}$ will be a decreasing sequence of subsets closed in X and having void intersection.

(4) RANKED SPACES OF KUNUGI

The Japanese mathematician K. Kunugi has proved a form of the Baire category theorem which does not assume an axiom of separation. In addition the theorem of Kunugi allows in some cases the conclusion that more than countably many open everywhere dense subsets have dense intersection. This generalization is obtained without strengthening the definition of everywhere dense set as was the case with the M-puffed sets defined by J. de Groot. The work exposited here is to be found in [5].

Before presenting the definitions of the structures used by Kunugi we recall that F. Hausdorff formulated these three neighborhood axioms in his classic text [4, p. 259]:

(a) Every point x has at least one neighborhood U_x; and U_x always contains x.

(b) For any two neighborhoods U_x and V_x of the same point, there exists a third, $W_x \subset U_x \cap V_x$.

(c) Every point $y \in U_x$ has a neighborhood $U_y \subset U_x$.

Professor Kunugi deals with a point set X together with a system of (open) neighborhoods satisfying axioms (A) and (C) of Hausdorff. A decreasing sequence (possibly transfinite) of neighborhoods of a point $x \in X$ is <u>maximal</u> if the intersection of all neighborhoods in the sequence does not contain a neighborhood of x. Denoting a sequence of neighborhoods by $\{V_s(x)\}\ 0 \leq s < \beta$, the ordinal number β is the <u>type</u> of the sequence.

We define the depth of the space X at the point x to be the least ordinal number $W(X,x)$ for which x has a maximal sequence, subject to the two conventions: (i) if axiom (B) of Hausdorff is not satisfied at some point x_o, $W(X,x_o)$ is equal to zero; (ii) if some point has no maximal sequence (i.e. the point has a smallest neighborhood) then the depth at that point is equal to the first ordinal of potency $2^{\overline{\overline{X}}}$. $W(X) = \inf \{W(X,x) : x \in X\}$ is the <u>rank</u> of the space.

The space X is <u>ranked</u> if for some limit ordinal number $b \leq w(W)$, there exists a family of open coverings $B = \{A_a\} a < b$ indexed by the ordinals less than b such that:

(a) Let there be given x an arbitrary point of X and $V(x)$ its neighborhood, and an ordinal less than b. Then there exists some ordinal a' with $a < a' < b$ and some subset $U \in A_a{}' \in B$ with $U \subset V(x)$.

The ordinal number b of this definition is called the <u>indicator</u> of ranked space X. Note that if the space X is not a topological space (i.e. axiom (B) fails at some point) then $b \leq w(X) = 0$; if X is a topological space then $w(X) \geq \omega$, the ordinal of the natural numbers. An open set has rank a if it is a member of the cover $A_a \in B$, and if this is not the case for any $A_a{}'$, $a' > a$.

omega

Let $V_a(x_a)$ $(a < c)$ be a decreasing sequence of neighborhoods where $c < b$, the indicator of X. The sequence is <u>fundamental</u> if the rank of each of the neighborhoods $V_a(x_a)$ is increasing and for each $a < c$ there is an a', $a \leq a' < c$ such that $x_{a'} = x_{a'+1}$ and the rank of $V_{a'}(x_{a'})$ is strictly less than the rank of $V_{a'+1}(x_{a'+1})$. A ranked space is complete if every <u>fundamental</u> sequence has nonvoid intersection.

THEOREM (4.1): Every complete metric space can be given the structure of a <u>complete ranked</u> space.

<u>Proof</u>: This theorem can be approached via the theory of uniform spaces. Take some metric d for which the space X is complete and construct a uniform structure for the space in the usual manner. A base for this uniformity is given by the family

$$\{(x,y) \in X \times X : d(x,y) < \tfrac{1}{n}\}_{n=1}^{\infty} .$$

X thus takes on the structure of a complete uniform space [2, p. 335], which can also be looked upon as a rank structure with indicator ω.

In a complete uniform space every Cauchy filter is convergent [2, p. 341]. Recall that a filter in X is Cauchy if for each entourage of the diagonal some member of the filter is contained in a neighborhood generated by the entourage of the diagonal. Let $\{V_i(x_i)\}_{i=1}^{\infty}$ be a fundamental sequence of neighborhoods of the ranked space. The conditions that the ranks of the neighborhoods $V_i(x_i)$ must eventually grow arbitrarily large imply that for each positive integer n some neighborhood $V_{i(n)}(X_{i(n)})$ is contained in an open ball of diameter i/n:

$$V_{i(n)}(X_{i(n)}) \subset \{y : d(X_{i(n)}, y) < 1/n\} .$$

Let F denote the (Cauchy) filter generated by $\{V_i(x_i)\}$. Then by the completeness of X as a uniform space we see that $\bigcap_{i=1}^{\infty} V_i(x_i) \supset \bigcap_{W \in F} W \neq \emptyset$. Hence the fundamental sequence $\{V_i(x_i)\}$ has nonvoid intersection and so X is complete as a ranked space.

THEOREM (4.2): (Baire-Kunugi): Let X be a complete ranked space with indicator ordinal b. Then the intersection of any nonvoid family of open everywhere dense subsets, indexed by the ordinals less than the ordinal $d \leq b$, is everywhere dense.

Proof: Given a sequence of open everywhere dense subsets

$\{B_a\}$ $(o \leq a < d)$ we can, for technical reasons, compress it by defining for each even $a < d$,

$$A_a = B_a \cap B_{a+1} .$$

The sets A_a will each be open everywhere dense subsets.

Suppose U is some nonvoid open subset of X . To show that $U \cap (\bigcap_{a<d} B_a) \neq \emptyset$ it will now suffice to show that $U \cap \{\bigcap_a A_a : a< d , a \text{ even}\} \neq \emptyset$.

We begin the (transfinite) induction by selecting some $x_o \in U$. By the definition of ranked space there is a rank c_o and an open neighborhood $V_o(x_o)$ of rank c_o with $V_o(x_o) \subset (U \cap A_o)$. Let $a < d$ be given and suppose that for each $e < a$ there has been defined a point x_e , a rank c_e , and a neighborhood $V_e(x_e)$ of rank c_e with the neighborhoods decreasing and the ranks increasing. Suppose further that for each even ordinal $e < a$, $x_e = x_{e+1}$, $c_e < c_{e+1}$, and $\underline{V_e (x_e) \subset A_e}$.

Suppose first that a is an even non-limit ordinal. Set $U_a = V_{a-1}(x_{a-1})$ and let x_a' be any point in the open nonvoid set $A_a \cap U_a$. This set is nonvoid because A_a is everywhere dense. In the case that a is a limit ordinal set $U_a = \bigcap_{e<a} V_e(x_e)$. U_a is nonvoid in virtue of the fact that $\{V_e(x_e)\}$ $(e < a)$ is a fundamental sequence in the complete ranked space X . We assert further that U_a is open. Indeed if $y \in \bigcap_{e<a} V_e(x_e)$ for each e we can find a neighborhood $V_{e'}(y)$ with ranks increasing, the

neighborhoods $V_{e'}(y)$ decreasing, and $V_{e'}(y) \subset V_e(x_e)$. Then $\{V_{e'}(y)\}$ $(e' < a)$ will be a decreasing sequence of neighborhoods of type less than than the depth of the space $w(X)$; in particular this sequence cannot be maximal. We conclude that there is some neighborhood $V'(y)$ with

$$y \in V'(y) \subset \bigcap_{e<a} V_{e'}(y) \subset U_a .$$

So U_a is open and we may therefore designate by x_a some point in the nonvoid open set $A_a \cap U_a$.

In both cases the construction performed immediately above can be accomplished for the case $y = x_a$ to find a neighborhood $V_a(x_a)$ of rank c_a with $c_a > c_e$ and $V_a(x_a) \subset V_e(x_e)$ for all $e < a$. In particular, $V_a \subset A_a$.

Finally if a is some odd ordinal set $x_a = x_{a-1}$ and select according to the definition of ranked space a rank $c_a > c_{a-1}$ and a neighborhood $V_a(x_a)$ of rank c_a with $V_a(x_a) \subset V_{a-1}(x_{a-1})$.

Thus we have a fundamental sequence $\{V_a(x_a)\}$ such that

$$\bigcap_{a<d} V_a(x_a) \subset U \cap \{\bigcap_a A_a : a < d , \ a \ \text{even}\} .$$

Since X is complete this fundamental sequence has nonvoid ~~intersection and so~~ $\bigcap_{a<d} B_a \cap U$ is nonvoid, the desired result.

It will be recalled that ranked spaces need not be topological spaces. But in the case that axiom (B) of Hausdorff fails for any point of the space the above theorem reduces to the assertion: In a complete ranked space every open everywhere dense set is everwhere dense. This because the indicator of a non-topological ranked space is defined to be zero.

COROLLARY (4.3): A ranked topological space is a Baire space.

Proof: A fortiori.

We close this section with the following example demonstrating a topological space which is a complete ranked space but is neither complete in the sense of Cech nor subcompact.

EXAMPLE (4.4): Let X consist of the real numbers with the cofinite topology. That is, every open set is a subset of X which is the complement of some finite point set in X. X is a T_1 separated space but is not T_2; so X is neither complete in the sense of Cech nor subcompact.

Define $U(x_1, x_2, \ldots, x_n, x, n) = X \setminus \{x_1, \ldots, x_n\}$; where n is a positive integer, $x \in X$, and x_i is a point of X different from x for each $i = 1, 2, \ldots, n$. Further defining

$$A_n = \{U(x_1, \ldots, x_n, x, n) : x_1 \ldots, x_n \in X \text{ ; } n \text{ fixed}\},$$

$$B = \{A_n\}_{n=o}^{\infty} .$$

X is seen to be a ranked space with ranking structure B and indicator ω. Indeed B contains every open set and given

$U(x_1, \ldots, x_n, x, n)$ an open neighborhood of x and $m > 0$, set $p = \text{maximum } \{n + 1, m\}$. Setting $x_n = x_{n+1} = \cdots = x_p$ we have $x \in U(x_1, \ldots, x_n, \ldots, x_p, x, p) \subset U(x_1, \ldots, x_n, x, n)$.

For the completeness note that the intersection of any countable sequence of subsets in B will have the cardinality of the continuum; a fortiori such an intersection is nonvoid and X is a complete ranked space.

(5) CONCLUDING REMARKS

A fourth approach to the problem of unifying the Baire category theorem is given by E. Elias Zakon in [8]. There is presented a Baire-like theorem in the context of uniform topological spaces [8, theorem 5.1, p. 383]. Like the work of de Groot, this paper attempts to formulate a theorem which allows for uncountable unions. While the work of Zakon is interesting in the cases of higher cardinality, it does not specialize to the usual Baire theorem. Applied to the real line with the usual topology, Zakon's theorem gives conclusions applicable only to the coarser indiscrete topology (the whole space and the null set being the only open sets).

Completeness in the sense of Cech is defined in the setting of Tychonoff spaces; but the proof of the Baire theorem only requires regularity. If in the setting of regular spaces one uses the equivalent intrinsic condition of Theorem (1.2) as the definition, then one achieves a modified definition of completeness for which all of the results presented in this paper remain true.

Using the fact that a complete ranked space need not have any separation properties it was not difficult to construct a space which was a complete ranked space but neither subcompact nor complete in the sense of Cech. It seems that a very pertinent question in this regard would be the relationship among spaces complete in the sense of Cech, spaces subcompact, and regular spaces with a completely ranked structure.

Conspicuously lacking from section 4 is the theorem which assures that every locally compact Hausdorff space can be given the structure of a complete ranked space. This theorem is announced in [5] and a proof is sketched in [6]; the present writer has been unable to verify that proof. Y. Yoshida, a student of Kunugi, has proved this theorem only after strengthening the hypotheses [7].

All of the papers encountered in the research for this thesis dealt with the task of finding sufficient conditions that a space be a Baire space. A final fully unified Baire theorem would be achieved by defining some property-property B-which would allow the theorem: Let X be a regular topological space; then X is a Baire space iff X has property B. The results of Kunugi give hope that the assumption of regular separation could be eliminated.

REFERENCES

[1] Cech, E., On bicompact spaces, Annals of Mathematics 38 (1937), 823-844.

[2] Engelking, R., Outline of general topology, North-Holland, Amsterdam (1968).

[3] deGroot, J., Subcompactness and the Baire category theorem, Koninklijke Nederlands Akademie Van Wetenschappen: Proceedings, Series A, Mathematical Sciences 66 (1963), 761-767.

[4] Hausdorff, F., (J.R. Anman, et al., Tr.) Set theory, Chelsea, New York (1957).

[5] Kunugi, K., Sur les espaces complets et regulierement complets I, Proceedings of the Japan Academy 30 (1954), 553-556.

[6] ______, Sur les espaces completes et regulierement complets II, Ibid., 912-916.

[7] Yoshida, Y., Compactness and completeness in ranked spaces, Proceedings of the Japan Academy 44 (1968), 251-254.

[8] Zakon, E., On uniform spaces with quasi-nested base, Transactions of the American Mathematical Society 133 (1968), 373-385.