ALGEBRAS ARISING IN THEORETICAL GENETICS

by

JOHN T. P. KWEI

Hons B. Sc. University of British Columbia, 1969

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

MATHEMATICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April, 1971.
In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of MATHEMATICS

The University of British Columbia
Vancouver 8, Canada

Date 26 / APRIL / 1971
ABSTRACT

Certain non-associative algebras have important applications in theoretical Mendelian Genetics. In this thesis we will give definitions to these algebras and study their properties. Some examples will also be given.
TABLE OF CONTENTS

Chapter 1
Introduction p.1

Chapter 2
Multiplication in Genetics p.5

Chapter 3
Non-Associative Products and Powers p.8

Chapter 4
Baric Algebras p.10

Chapter 5
Train Algebras p.20

Chapter 6
Commutative Train Algebras of Ranks 2 and 3 p.22

Chapter 7
Special Train Algebras p.32

Chapter 8
A Stability Theorem for Special Train Algebra p.38
Chapter 9
Sequences of powers in Special Train Algebras  p.42

Chapter 10
Genetic Algebras " of Symmetrical Inheritance "  p.50

Chapter 11
Some Examples  p.57

Bibliography  p.62
ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. J. V. Whittaker for his suggestions and encouragement in writing this thesis. My indebtedness is also due to Dr. D. C. Murdoch for reading my thesis and providing valuable suggestions.

I am grateful to the University of B. C. and the National Research Council for their financial support.

Last, but not least, I wish to thank my wife Serena for her help and for typing the thesis.
CHAPTER 1

INTRODUCTION

Certain non-associative algebras have important applications in theoretical Mendelian Genetics. The study of such non-associative algebras was initiated in the 1930's by Dr. I.M. H. Etherington (2)-(6). Such studies were continued in the years to follow by many other authors such as R.D. Schafer (18) and H. Geiringer (7). It is carried further by current researchers in the field such as H. Gonshor (9),(10) and P. Holgate (13),(14), to name just a few.

The occurrence of the genetic algebras may be described in general terms as follows. (Etherington, Genetic Algebras 1939 (3) ). The mechanism of chromosome inheritance, in so far as it determines the probability distributions of genetic types in families and filial generations, and expresses itself through their frequency distributions, may be represented conveniently by algebraic symbols. Such a symbolism is described, for instance, by Jennings (16, Chap. ix); many applications are given by Geppert and Koller. The symbolism is equivalent to the use of a system of related linear algebra, in which multiplication is commutative (PQ=QP) but non-associative (PQ R = P QR). A population (i.e. a distribution of genetic types) is represented by
a normalized vector in one or other algebra, according to the point of view from which it is specified. If \( P, Q \) are populations, the filial generation \( P \times Q \) (i.e. the statistical population of offspring resulting from the random mating of individuals of \( P \) with individuals of \( Q \)) is obtained by multiplying two corresponding representations of \( P \) and \( Q \); and from this requirement of the symbolism it will be obvious why multiplication must be non-associative. It must be understood that a population may mean a single individual, or rather the information which we may have concerning him in the form of a probability distribution.

In this thesis, we endeaver to give a detailed account of many different algebras that arise in theoretical genetics. Definitions will be given of baric algebras, train algebras, special train algebras and genetic algebras in that order. Many properties and structural theorems will be developed. Some examples are given in chapter 11. Care was taken to blend the old and the new materials and to ensure the flow of the thesis.
Some useful definitions:

gamete: a reproductive cell that can unite with another similar one to form the cell that develops into a new individual.

zygote: any cell formed by the union of two gametes.

heterozygote: any cell formed by the union of two different types of gametes.

chromosome: any of the microscopic rod-shaped bodies which carry the genes that convey hereditary characteristics.

sex-chromosome: a sex-determining chromosome in the germ cells of most plants and animals: fertilized eggs containing two X-chromosomes (one from each parent germ cell) develop into females, those containing one X- and one Y-chromosome (male germ cells carry either one or the other) develop into males.

autosome: any chromosome other than the sex chromosome.

autosomal inheritance: inheritance of characteristics through autosomes.

progeny: descendants, offspring collectively.

genotype: an individual's genetic constitution (in terms of its hereditary factors).

phenotype: a type distinguished by visible characters rather
than by hereditary or genetic traits (i.e. referring to the appearance).

genomes: among organisms with chromosomes, each species has a characteristic set of genes, or genome.

autopolyploids: organisms having four similar genomes per nucleus.
CHAPTER 2

MULTIPLICATION IN GENETICS

DEFINITION: (5) The following definition is due to I.M.H. Etherington.

The multiplication of populations — individuals — gametes — means the calculation of progeny distribution resulting from their random mixing — mating — fusion. Defining a population as a probability distribution of genetic types, we may say in all cases that we are multiplying populations, (that is, a population may mean a single individual, or rather information which we may have concerning him in the form of a probability distribution).

Unlike multiplication in ordinary algebra, the associative law is not necessarily obeyed in genetical multiplication. As an example of non-associativity of genetical multiplication consider the following two populations (AB)C and A(BC). The progeny (AB)C, a mating between the offspring from A and B and a third one C is clearly different from the progeny from A and the hybrid population BC.

Genetical multiplication obeys the distributive law:

\[ A( B+C ) = AB + AC. \]

\[ D = \alpha B + \beta C \] has genetic interpretation only if \( \alpha + \beta = 1, \alpha, \beta \in F. \)
where \( F \) is a field. In that case it means that the individual \( D \) is composed of 100\( \alpha \)% of B and 100\( \beta \)% of C, (or produces 100\( \alpha \)% of B and 100\( \beta \)% of C as the case may be. c/f example 1.)
There are three cases to be considered regarding the commutative law:

CASE (1): For autosomal characters, the outcomes of reciprocal matings are generally identical and therefore multiplication is commutative, although in case (3) we shall see that there are exceptions.

CASE (2): For sex-linked characters the commutative law also holds (although one might be tempted to say that it is non-commutative). The reason is that for sex-linked characters we can only speak of reciprocal matings in connection with the phenotype classification of a population. On the other hand, the calculation of progeny distribution is only possible on the basis of the genotype classification. There are only the male genotype (which does not involve Y-chromosome) and the female genotype (which involves Y-chromosome). A reciprocal mating between genotypes is impossible. Therefore AB and BA means the same thing for a male genotype A and female genotype B, i.e. the multiplication is commutative.

CASE (3): It is possible for autosomal inheritance to be unsymmetrical in the sexes, though either crossing-over values or gametic selection being different in male and female. In such cases we can either treat the male and female genotypes as the same type with the same relevant gene content or as distinct types (since they produce different series of gametes). In the
former case, AB and BA are different, referring to reciprocal crosses which do not produce similar distribution of offspring, and multiplication is non-commutative. In the later case it is as case (2).

Therefore genetical multiplication is commutative and distributive with the exception of certain cases where we have the option of using a varied form of the symbolism in which the multiplication is non-commutative. Genetical multiplication is however, non-associative.

In view of the above remark, we shall assume that the multiplications to be encountered in this paper as both commutative and distributive.
CHAPTER 3

NON-ASSOCIATIVE PRODUCTS AND POWERS

Care must be taken when dealing with non-associative products and powers involving many factors. Brackets inserted in different ways would indicate different orders of association of the factors. To eliminate brackets, we shall use groups of dots to separate factors when necessary, fewness of dots implying precedence in multiplication. Thus \( A : . BC \cdot AD : E \) means \( A \{ [(BC)(AD)] E \} \) and represents the pedigree below:

\[
\begin{array}{c}
B \quad C \\
BC
\end{array}
\quad
\begin{array}{c}
A \quad D \\
AD
\end{array}
\quad
\begin{array}{c}
(BC)(AD) \\
(BC)(AD)
\end{array}

\quad
\begin{array}{c}
E \quad \text{E}
\end{array}

\quad
\begin{array}{c}
A \\
A
\end{array}

\quad
\begin{array}{c}
[(BC)(AD)] E \\
[(BC)(AD)] E
\end{array}

\quad
\begin{array}{c}
A \{ [(BC)(AD)] E \}
\end{array}
\]

We shall mainly be concerned with the two simple types of non-associative powers: the "principal" and "plenary" powers. It is important to distinguish between, e.g. \((A^2)^2\) and \(A^4 = A[A(A^2)]\). If mating takes place at random in an initial population \(A\), then the successive discrete generations are represented by
the sequence of plenary powers.

\[ A , A^2 , A^3 , \ldots , A^{n-1} , \ldots \]  \hspace{1cm} (3.1)

i.e.

\[ A^{[1]} = A \]
\[ A^{[n]} = A^{[n-1]} \cdot A \]  \hspace{1cm} (3.2)

While the sequence of principal powers:

\[ A , A^2 , A^3 , \ldots , A^n , \ldots \]  \hspace{1cm} (3.3)

i.e.

\[ A^n = A^{n-1} \cdot A \]  \hspace{1cm} (3.4)

refers to a mating system in which each generation is mated back to one original ancestor or ancestral population.

There is a third kind of power which occurs frequently in genetics and will be discussed here at times. This is the primary product of the form:

\[ X \{ Y^{n-1} \} = \left\{ [(XY) \cdot Y] \right\} \cdot Y \ldots \]
\[ = XY \cdot Y : Y \ldots \]  \hspace{1cm} (3.5)

The primary product refers to the descendants of a single individual or a subpopulation \( X \) mating at random within a population \( Y \). A sequence of primary products is called the operational sequence:

\[ X , XY , XY \cdot Y , XY \cdot Y : Y \ldots \]  \hspace{1cm} (3.6)
BARIC ALGEBRAS

DEFINITION: A baric algebra is a linear algebra $A$, associative or not, that possesses a non-trivial representation onto its coefficient field $F$,

\[ W : A \rightarrow F \]
\[ a \mapsto W(a) \]

such that

\[ W(a + b) = W(a) + W(b) \]
\[ W(\alpha a) = \alpha W(a) \] \hspace{1cm} (4.1)
\[ W(ab) = W(a) \cdot W(b) \]

$\alpha \in F, a, b \in A$

$W(a)$ is called the weight of $a$, and $W$ is a weight function of $A$.

If $W(a)$ is $\neq 0$, $a$ can be normalized as:

\[ \overline{a} = \frac{a}{W(a)} \] \hspace{1cm} (4.2)

of unit weight.

The set of elements of unit weight is closed with respect to multiplication. Elements of zero weight are called
nil elements. The set $N$ of all nil elements is an invariant subalgebra

i.e.

$$AN \subseteq N,$$

the nil subalgebra.

By suitable linear transformations, the basis of a baric algebra $(c_0, c_1, \cdots, c_{n-1})$ may be so chosen that one of its elements, say $c_0$, has weight unity, and the remainder, $c_1, \cdots, c_{n-1}$ have weight zero.

For, suppose the basis of a baric algebra is $(b_0, b_1, \cdots, b_{n-1})$ and it has a weight function $W$. Without loss of generality assume $W(b_0) \neq 0$.

Now, let

$$c_0 = \frac{b_0}{W(b_0)}, \quad c_i = b_i - W(b_i) c_0$$

for $i = 1, \cdots, n-1$

then

$$W(c_0) = \frac{W(b_0)}{W(b_0)} = 1$$

and

$$W(c_i) = W(b_i) - W(b_i) W(c_0) = W(b_i) - W(b_i) = 0$$

$(c_0, c_1, \cdots, c_{n-1})$ is the required basis.

Let the multiplication table of a linear algebra $A$ with
basis \( (c_0, c_1, \ldots, c_{n-1}) \) be:
\[
c_i c_j = \sum_{k=0}^{n-1} r_{ij}^k c_k
\]
(\(i, j, k = 0, 1, \ldots, n-1\))
(4.3)

and let the arbitrary element \( a \in A \) be denoted by
\[
a = \sum_{i=0}^{n-1} \alpha_i c_i
\]
where \( \alpha_i \in F \)
(4.4)

**THEOREM (4-A):**

A linear algebra \( A \) is a baric algebra if and only if the equation
\[
x_i x_j = \sum_{k=0}^{n-1} r_{ij}^k x_k
\]
(\(i, j, k = 0, 1, \ldots, n-1\))

regarded as ordinary simultaneous equations in \( F \) for the unknowns \( x_i \), should possess a non-null solution \( x_i = w_i \).

**Proof: Necessary:**

Let \( W_i = W(x_i) = x_i \)

(\( x_i \) regarded as unknowns in \( F \))
(4.5)

and since \( W \) is a non-trivial representation there exists a non-
null solution $x_i = w_i$.

Sufficiency:

Let $W(a) = \sum \alpha_i w_i$ \hspace{1cm} (4.6)

$\exists x_i \in A \exists W(x_i) = w_i \neq 0$ and (4.1) are at once deducible from the definition.

The $w_i$'s are called the basic weights and they form the weight vector of $A$.

Let us denote the kernel of the homomorphism $W$ by $R$.

Then a necessary and sufficient condition that a linear algebra $A$ be a baric algebra is that $A$ contains an ideal $R$ such that $A/R \cong F$ \hspace{1cm} (4.7)

This is clear from the definition of a baric algebra.

Let the general element $x \in A$ be denoted by

$x = \sum_{l=0}^{n-1} x_l c_l$ \hspace{1cm} (4.8)

Let the rank equation or equation of lowest degree connecting the principal powers

$x, x^1, x^3, \ldots$ \hspace{1cm} (4.9)

be

$g(x) = x^r + s_1 x^{r-1} + s_2 x^{r-2} + \cdots + s_{r-1} x = 0$ \hspace{1cm} (4.10)
THEOREM (4-B):

\( s_m \) in (4.10) is a homogeneous polynomial of degree \( m \) in the coefficients \( x_i \) of \( x \).

Remark: The rank equation is the same as the minimal polynomial for a linear transformation or matrix and for each \( x \in A \), the linear transformation in question is left multiplication by \( x \).

Proof: Suppose \( A \) has a unit element. For each \( x \in A \) there is a one-to-one correspondence (isomorphism) \( h \) from \( A \) into the algebra \( E \) of all left multiplication of \( x \), \( x \in A \),

that is,

\[
h : A \longrightarrow E
\]

\[
x \longmapsto h(x)
\]

where

\[
h(x) : A \longrightarrow A \text{ is the left multiplication by } x
\]

\[
a \longmapsto xa \text{ for each } a \in A
\]

(4.11)

(4.12)

Let \( \{ c_i \} \) be a basis of \( A \) with multiplication table:

\[
c_i c_j = \sum_{k=1}^{n} r_{ijk} c_k
\]

(4.13)

\[
( i, j, k = 1, 2, \ldots, n )
\]

Let \( x = \sum_{i=1}^{n} x_i c_i \)
Then

\[ xc_j = \sum_{k=1}^{n} y_{jk} c_k \]  \hspace{1cm} (4.14)

where

\[ y_{jk} = \sum_{i=1}^{n} x_i r_{ij} \]  \hspace{1cm} (4.15)

\[ h(x)(c_j) = \sum_{k=1}^{n} y_{jk} c_k \]  \hspace{1cm} (4.16)

i.e. there exists

an n x n matrix \( H(x) \)

such that

\[ h(x)(a) = H(x) [a] \]  \hspace{1cm} (4.17)

where

\[
H(x) = \begin{bmatrix}
    y_{11} & y_{21} & \cdots & y_{n1} \\
    y_{12} & y_{22} & \cdots & y_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    y_{1n} & y_{2n} & \cdots & y_{nn}
\end{bmatrix}
\]  \hspace{1cm} (4.18)

Let \( F \) be extended to its algebraic closure, so that the characteristic polynomial \( G(x) \) of \( H(x) \) factors completely

\[ G(x) = (x - \alpha_1)^{d_1} \cdots (x - \alpha_r)^{d_r} \]  \hspace{1cm} (4.19)

Then there is an ordered basis \( (c_1', c_2', \cdots, c_n') \)

for \( A \) in which \( H(x) \) is represented by a matrix which is in
Jordan form:

\[
H(x) = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
1 & \lambda_1 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & \lambda_r \\
0 & \cdots & \cdots & 0 \\
\end{bmatrix}
\]

(4.20)

where some of the entries in the second diagonal may be zero and

\[
c_i'c_j' = \sum_{k=1}^{r} r_{ij} c_k' \quad (i', j', k = 1, 2, \cdots, n)
\]

(4.20 a)

\[
xc_j = \sum_{k=1}^{r} y_{jk} c_k
\]

(4.20 b)

\[
y_{j}^k' = \sum_{k=1}^{r} x_i r_{ij} c_k'
\]

(4.20 c)

\[
\lambda_j' = y_{j}^k' = \sum_{k=1}^{r} x_i r_{ij} c_k'
\]

(4.21)

and the minimal polynomial of \( H(x) \) is

\[
\psi(x) = (x - \lambda_1') \cdots (x - \lambda_r')
\]

(4.22)

where the \( \lambda_1', \ldots, \lambda_r' \) are not necessarily distinct.

Since the minimal polynomial is unchanged by linear transforma-
tion it is clear that $s_m$ is homogeneous of degree $m$ in the coefficients $x$.

Suppose $A$ does not have a unit, and $A$ have a basis \{ $c_1, c_2, \ldots, c_n$ \}. Consider the algebra $A^*$ with unit $c_0$ and basis \{ $c_0, c_1, \ldots, c_n$ \}

where

$$c_i \cdot c_j = c_{i+j}, \quad 0 \leq j < i \leq n$$

Thus $c_0$ is a unit of $A^*$. Set

$$x = \sum_{i=0}^{n} x_i c_i$$

$$x^* = x_0 c_0 + x$$

where $x_i \in F \quad i = 0, 1, 2, \ldots, n$

1-1 correspondence (isomorphism) $h$ from $A^*$ into the algebra $E$ of all left multiplication by $x^*$, $x^* \in A^*$

i.e.

$$h : A^* \rightarrow E$$

$$x^* \rightarrow h(x^*)$$

such that

$$h(x^*) : A^* \rightarrow A^*$$

is the left multiplication by $x^*$

$$a \rightarrow x^* a$$

Then

$$x^* c_0 = (x_0 c_0 + x) c_0 = x^*$$
\[ x^* c_j = (x_0 c_0 + x) c_j = x_0 c_j + \sum_{k=1}^{n} y_{jk} c_k \]
\[ (j = 1, 2, \ldots, n) \]

where

\[ y_{jk} = \sum_{i=1}^{n} x_i r_{ij} \]  

\[ h(x^*) (c_j) = x_0 c_j + \sum_{k=1}^{n} y_{jk} c_k \]

i.e. there exists

\[ n + 1 \times n + 1 \] matrix \( H(x^*) \)

\[
H(x^*) =
\begin{bmatrix}
  x_0 & 0 & 0 & \ldots & \ldots & 0 \\
  x_1 & x_0 + y_{11} & y_{12} & \ldots & \ldots & y_{1n} \\
  x_2 & y_{21} & x_0 + y_{22} & \ldots & \ldots & y_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  x_n & y_{n1} & y_{n2} & \ldots & x_0 + y_{nn} \\
\end{bmatrix}
\]

When we set \( x_0 = 0 \), the characteristic polynomial takes the form

\[ G(x) = x(x - \lambda_1)^{d_1} \ldots (x - \lambda_r)^{d_r} \]

and the minimal polynomial (or equivalently the rank equation):
\[ \Psi(x) = x(x - \alpha_1) \cdots (x - \alpha_r) \]

(4.32)

where the \( \alpha_i \)'s may not be distinct. This completes the proof.

Since

\[ g(x) = x^r + s_1 x^{r-1} + s_2 x^{r-2} + \cdots + s_{r-1} x = 0, \]

\( g(x) \) is of zero weight. Hence the equation is satisfied when we substitute \( W(x) \) for \( x \): consequently

\[ x - W(x) \mid g(x) \]

i.e. \( W(x) \) is a root of the rank equation.
CHAPTER 5

TRAIN ALGEBRAS

Consider the rank equation (4.10). In general the \(s\) will depend on \(x\), but if in so far as they depend on \(x\), they depend only on \(W(x)\), then the baric algebra is called a train algebra of principal rank \(k\); (or rank \(k\)).

Since \(s^m\) is homogeneous of degree \(m\) in the coefficients \(x_i\) of \(x\), it must in a train algebra be a numerical multiple of \(W(x)\). Therefore, the rank equation can be factored:

\[
g(x) = x(x - w)(x - \lambda_1 w)(x - \lambda_2 w) \cdots = 0
\]  

(5.1)
in an extended field if necessary.

The numbers

\(1, \lambda_1, \lambda_2, \ldots\)

are called the principal train roots of the algebra.

For each element \(\vec{x}\) of unit weight (4.10) becomes

\[
g(\vec{x}) = \vec{x}^r + s_1 \vec{x}^{r-1} + \cdots + s_{r-1} \vec{x} = 0
\]  

(5.2)

where now the \(s_k\)'s are constant and (5.1) becomes
Since (5.2) can be multiplied by \( \bar{x} \) any number of times, it can be regarded as a linear recurrence equation with constant coefficient connecting the principal powers of the general normalized element \( \bar{x} \).
I. Nilproduct and Triple Nilproduct

Let $X$ be a commutative baric algebra of order $n$ with basis $(c_0, c_1, \ldots, c_{n-1})$. Let $N$ denote the nil ideal. $N$ consists of all elements of zero weight, the nil elements. Then $N = \langle c_1, \ldots, c_{n-1} \rangle$ the subalgebra generated by $c_1, \ldots, c_{n-1}$ and hence is of order $n - 1$.

Let $x, y \in X$ be of weight $j, \ell$. The nilproduct is defined as the nil element

$$ x \cdot y = xy - \frac{1}{2}jy - \frac{1}{2}\ell x \quad (6.1) $$

Clearly

$$ x \cdot (y + z) = x \cdot y = x \cdot z $$

$$(\alpha x) \cdot (\beta y) = \alpha \beta (x \cdot y) \quad (6.2) $$

The nilsquare $x \cdot x = x^2 - \frac{3}{2}x$ is written $x''$

DEFINITION: A p-element is an element of the form $\sum_{i=1}^{n} \alpha_i x_i^{\cdot \cdot}$

i.e. a linear combination of nilsquares. \quad (6.3)

Let $P$ be the set of all p-elements.
THEOREM (6-A)

All nilproducts are p-elements.

Proof: If \( x = \frac{1}{2} \bar{x}, y = \frac{1}{2} \bar{y}, \bar{x} \neq 0, \bar{y} \neq 0 \), then
\[
x \cdot y = \frac{1}{2} \{ 4 (\frac{1}{4} \bar{x} + \frac{1}{4} \bar{y}) - x - y \} \in P
\]
(6.4)

If \( \bar{x} = 0 \) or \( \bar{y} = 0 \), then \( x \) or \( y \) can be expressed as the difference of two elements of equal non-zero weight; therefore, with (6.2) the theorem follows.

Corollary: \( N^2 \subseteq P \subseteq N \)

THEOREM (6-B)

(i) \( P \) is an ideal.

(ii) \( X/P \) is a train algebra with the train equation
\[
\bar{x}(\bar{x} - 1) = 0
\]

Proof: (i) Let \( x \in X \) and \( p \in P \)

then
\[
\bar{x} \cdot \bar{p} = x \cdot p + \frac{1}{2} \{ 4 (\frac{1}{4} \bar{x} + \frac{1}{4} \bar{y}) - x - y \} \in P
\]
(6.4)

\( \bar{x}(\bar{x} - 1) = \bar{x} \cdot \bar{x} \in P \)

therefore \( \bar{x}(\bar{x} - 1) = 0 \in X/P \). This completes the proof.
Let $\lambda \in \mathbb{F}$ be fixed, and let $x, y, z \in X$ be of weights $\lambda, \gamma, \zeta$. Their triple nilproduct is defined as

$$\lambda x \cdot y \cdot z = \lambda x \cdot y \cdot z = \lambda (xy)z + y(\lambda x)z + x(\gamma y)z - (1 + \lambda)(\lambda x + \gamma y + \zeta z)$$

$$= \lambda (x - \lambda^2)(y - \lambda^2)(z - \lambda^2) + \lambda (y - \lambda^2)(x - \lambda^2)(z - \lambda^2) + \lambda (x - \lambda^2)(y - \lambda^2)(z - \lambda^2)$$

(6.5)

The triple nilproduct is commutative, associative, and distributive, i.e.

$$\lambda x \cdot y \cdot z = \lambda x \cdot y \cdot z = \lambda (xy)z + y(\lambda x)z + x(\gamma y)z = \lambda x \cdot y \cdot z$$

$$= \lambda (x - \lambda^2)(y - \lambda^2)(z - \lambda^2) + \lambda (y - \lambda^2)(x - \lambda^2)(z - \lambda^2) + \lambda (x - \lambda^2)(y - \lambda^2)(z - \lambda^2)$$

(6.6)

The nilcube of $X$ is denoted $\lambda^3 x$ and for a normalized element

$$\lambda^3 x = \lambda^3 x = \lambda^3 x = \lambda^3 x = \lambda^3 x = \lambda^3 x = \lambda^3 x = \lambda^3 x$$

(6.7)

DEFINITION: A q-element is an element of the form $\sum_{i=1}^{\lambda} \lambda x$ i.e. a linear combination of nilcubes.

Let $Q_\lambda$ be the set of all q-elements.
THEOREM (6-C)

All triple nilproducts are q-elements.

Proof: For $\bar{x}, \bar{y}, \bar{z}$ of unit weight in $\mathcal{X}$, we have

$$6^{\bar{x}} \cdot \bar{y} \cdot \bar{z} = (\bar{x} + \bar{y} + \bar{z}) \cdots - (\bar{y} + \bar{z}) \cdots$$

$$- (\bar{z} + \bar{x}) \cdots - (\bar{x} + \bar{y}) \cdots$$

$$4^{\bar{x}} \cdots + 4^{\bar{y}} \cdots + 4^{\bar{z}} \cdots$$

(6.9)

and together with (6.6) the theorem follows.

THEOREM (6-D)

$Q_\lambda \subseteq P$

Proof: This follows from (6.7) and the fact that $P$ is an ideal.

THEOREM (6-E)

If $\lambda \notin \mathcal{U}$, then $Q_\lambda + Q_\mu = P$

Proof: $(\mathcal{U} - \lambda) \bar{x} = \frac{\lambda}{\bar{x}} \cdots - \frac{\lambda}{\bar{x}} \cdots$ implies

$$P \subseteq Q_\lambda + Q_\mu$$

but

$$Q_\lambda + Q_\mu \subseteq P \quad \text{by theorem (6-A)}$$

therefore
THEOREM (6–F)

(i) If $Q\lambda$ is an ideal of $X$, then $X/Q\lambda$ is a train algebra with train equation

$$\bar{x}(\bar{x} - 1)(\bar{x} - \lambda) = 0$$

(ii) If further $X$ is a train algebra, then $\lambda$ must be one of its train roots.

Proof: (i) $\bar{x}(\bar{x} - 1)(\bar{x} - \lambda) = \lambda \bar{x} \cdots \in Q\lambda$

therefore

$$\bar{x}(\bar{x} - 1)(\bar{x} - \lambda) = 0 \in X/Q$$

(ii) If $X$ is a train algebra, then the rank equation of $X$ is of the form $x( x - \lambda ) ( x - \lambda ) f( x ) = 0$, for some $f(x)$,

i.e. $\lambda$ is one of its train roots.

II. (Commutative) train algebra of rank 2

Let $X$ be a train algebra of rank 2. The train equation is

$$\bar{x}(\bar{x} - 1) = 0$$

But

$$\bar{x}'' = \bar{x}(\bar{x} - 1) = 0$$
therefore

\[ P = N^3 = 0 \quad (6.11) \]

Hence for \( x, y \in X \) with weights,

\[ x'y = xy - \frac{1}{2}y'x + \frac{1}{2}y = 0 \]

implies

\[ xy = \frac{1}{2}y'x + \frac{1}{2}y \quad (6.12) \]

In particular, taking the base elements to be all of unit weight, say \( c_0, c_1, \cdots, c_{n-1} \), then

\[ c_i c_j = \frac{1}{2}c_i + \frac{1}{2}c_j \quad (6.13) \]

Or taking one base element \( c_0 \) of unit weight and \( c_1, \cdots, c_{n-1} \) of zero weight, we have

\[ c_0^2 = c_0 \]
\[ c_0 c_i = \frac{1}{2}c_i \quad i = 1, \cdots, n - 1 \]
\[ c_i c_j = 0 \quad i, j = 1, 2, \cdots, n-1 \]

\[ (6.14) \]

**THEOREM (6-G)**

In a commutative train algebra of rank 2.

(i) Multiplication is associative for powers.

(ii) Any sequence of powers forms a train† with the

† For definition of trains see next chapter.
same train equation \( \bar{x}( \bar{x} - 1 ) = 0 \).

(iii) The operational sequence forms a train with the train equation \( \bar{x} \{ \bar{y} - 1 \} \{ \bar{y} - \frac{1}{2} \} = 0 \).

Proof: (i) From (6.12)
\[ x^2 = \frac{1}{3} x \]
therefore
\[ x^n = \frac{1}{3} x \quad \text{for all } n \quad (6.15) \]

(ii) Since \( \bar{x}^2 = \bar{x} \), all powers of \( x \) are equal.

(iii) \( \bar{x} \bar{y} = \frac{1}{2} \bar{x} + \frac{1}{2} \bar{y} \)
or
\[ \bar{x}( \bar{y} - \frac{1}{2} ) = \frac{1}{2} \bar{y} \]
multiplying on both sides by \( ( \bar{y} - 1 ) \),
\[ \bar{x}( \bar{y} - \frac{1}{2} )( \bar{y} - 1 ) = \frac{1}{2} \bar{y}( \bar{y} - 1 ) = 0 \]

(6.16)

III. (Commutative) train algebra of rank 3

Let the train equation be
\[ \bar{x}( \bar{x} - 1 )( \bar{x} - \lambda ) = 0 \]
\[ = \bar{x}^3 - ( 1 + \lambda ) \bar{x}^2 + \lambda \bar{x} \]

(6.17)
The rank equation is then
\[ x( x - \frac{1}{2} )( x - \lambda \frac{1}{2} ) = 0 \]
Take \( x^{(1)} \) as the train root, then from (6.17)

\[
Q_{\lambda} = 0
\]

Therefore

\[
\lambda \bar{x} \cdot \bar{x} \cdot \bar{x}^2 = \bar{x}( \bar{x} \bar{x}^2 + \bar{x}^2 \bar{x} + \bar{x}^2 ) - (1 + \lambda)
\]

\[
( \bar{x} \bar{x}^2 + \bar{x}^2 \bar{x} + \bar{x}^2 ) + \lambda( \bar{x} + \bar{x}^2 + \bar{x}^2 )
\]

\[
= 2\bar{x}^4 + \bar{x}^{1+2} - (1 + \lambda)(2\bar{x}^3 + \bar{x}^2) + \lambda(2\bar{x} + \bar{x}^2)
\]

But from (6.17)

\[
2\bar{x}^4 - 2(1 + \lambda)\bar{x}^3 + 2\lambda\bar{x}^2 = 0
\]

Hence

\[
\bar{x}^{2+1} - (1 + 2\lambda)\bar{x}^2 + 2\lambda\bar{x} = 0
\]

We have just proved the following theorem.

**THEOREM (6-H)**

A train algebra with the principal train equation (6.17) possesses the plenary train equation

\[
\bar{x}(\bar{x} - 1)(\bar{x} - 2\lambda) = 0
\]

\[
= \bar{x}^{2+1} - (1 + 2\lambda)\bar{x}^2 + 2\lambda\bar{x}
\]

(6.19)

The plenary rank equation is obtained by multiplying by \( \sqrt[3]{4} \) and is
\[ x^{2,2} - (1 + 2\lambda)\bar{y}^2 + 2\lambda j^3 x = 0 \quad (6.20) \]

Substituting \(\lambda \bar{x} + \beta \bar{y} + \gamma \bar{z} + \delta \bar{w} \) for \(x\) in equation (6.20), and \(j\) by \(\lambda + \beta + \gamma + \delta\), for some \(\lambda, \beta, \gamma, \delta \in F\), simplying and collecting terms one has

\[
\bar{x}\bar{y} (\bar{z}\bar{w}) + \bar{x}\bar{z} (\bar{y}\bar{w}) + \bar{x}\bar{w} (\bar{y}\bar{z})
- (1 + \lambda) (\bar{y}\bar{z} + \bar{z}\bar{x} + \bar{x}\bar{y} + \bar{w}\bar{y} + \bar{y}\bar{w} + \bar{z}\bar{w})
+ \frac{1}{2} \lambda (\bar{x} + \bar{y} + \bar{z} + \bar{w})
= 0 \quad (6.21)
\]

If we let \(\bar{w} = \bar{z} = \bar{y}\) in (6.21), we obtain

\[
\begin{align*}
\bar{x}\bar{y} \bar{y}^2 &- (1 + \lambda) (\bar{y}^2 + 3\bar{x}\bar{y} + 2\bar{y}^2) \\
&+ \frac{3}{2} \lambda (\bar{x} + 3\bar{y}) \\
&= 0 \quad (6.22)
\end{align*}
\]

Since

\[
\begin{align*}
\bar{x}\bar{y} \bar{y}^2 &- (1 + \lambda) (\bar{y}^2 + 3\bar{x}\bar{y} + 2\bar{y}^2) \\
&+ \frac{3}{2} \lambda (\bar{x} + 3\bar{y}) \\
&= 0 \quad (6.23)
\end{align*}
\]

Equating equations (6.22) and (6.23) and eliminating the term \((xy) y^2\) we obtain

\[
[\bar{x}\bar{y}] \bar{y} - (1 + \lambda)(\bar{x}\bar{y})\bar{y} + (\frac{3}{2} + \lambda)\bar{x}\bar{y} - \frac{3}{2} \lambda \bar{x}
\]
\begin{align*}
  &= \frac{1}{4} \bar{y}^2 - \frac{1}{4} \lambda \bar{y} \\
  \text{or} \quad \bar{x} \left[ \bar{y} - \frac{1}{2} \right]^2 (\bar{y} - \lambda) = \frac{1}{4} \bar{y} (\bar{y} - \lambda) \\
  \text{(6.24)}
\end{align*}

Hence multiplying both sides of (6.24) by (\bar{y} - 1) we see that the operational sequence

\[
\bar{x} \left\{ \bar{y}^{n-1} \right\} \quad \text{forms a train satisfying the equation}
\]

\[
\bar{x} \left\{ \bar{y} - 1 \right\} \left\{ \bar{y} - \frac{1}{2} \right\} \left\{ \bar{y} - \lambda \right\} = 0
\]
CHAPTER 7

SPECIAL TRAIN ALGEBRAS

DEFINITION:

A special train algebra is a commutative linear algebra for which there exists a basis

\[ \{ a_0, a_1, \ldots, a_n \} \]

such that

\[ a_i a_j = \sum_{k=0}^{n} x_{i,j,k} a_k \]

\[ x_{0,0} = 1 \] \hspace{1cm} (7.1)

\[ x_{i,j,k} = 0 \] for \( k < j \) \hspace{1cm} (7.2)

\[ x_{i,j,k} = 0 \] for \( i,j > 0 \) and \( k \leq \max(i,j) \) \hspace{1cm} (7.3)

and furthermore all powers of the ideal \( \{ a_0, a_1, \ldots, a_n \} \) are ideals.

Remarks: It follows from the definition that

(i) The map which sends each element into its coefficient \( a \) is the only non-trivial homomorphism of the algebra into its coefficient field. Therefore, a special train algebra is a baric algebra.

(ii) \( \{ a_0, a_1, \ldots, a_n \} = N \), the nil-ideal, is nilpotent.
In such algebras, there are many other sequences whose properties resemble those of the sequence of principal power, i.e.

(1) sequences of elements derived from a general element, which satisfy linear recurrence equations.

(2) coefficients of the equations depend only on the weight of the element.

(3) coefficients become constants for each element of unit weight.

Such sequences will be called trains.

For example: the sequence of plenary powers:

\[ x, x^2, x^{2-2}, x^{2^3}, \]  \( (7.4) \)

forms a train in a special train algebra and so does the operational sequence of primary products:

\[ x, xy, xy.y, \ldots \]  \( (7.5) \)

We will denote the \( r \)th element of a train as \( x^r \) and regard it as a symbolic \( r \)th power of \( x \), and we shall use this symbolism throughout this paper in algebraic operations.

Let the normalized recurrence equation, or train equation be

\[ f[\bar{x}] = \bar{x}^{(s)} + t_1 \bar{x}^{(s-1)} + \cdots + t_{s-1} \bar{x} = 0 \]  \( (7.6) \)

and symbolically

\[ f[\bar{x}] = \bar{x} [\bar{x} - 1] [\bar{x} - \mu_s] \cdots [\bar{x} - \mu_{s-1}] = 0 \]  \( (7.7) \)
The square brackets indicate that after expansion, powers of $\bar{x}$ are to be interpreted as symbolic powers. Assuming no repeated factors present, $s$ is the rank of the train, and the number 1, $\mu_1, \mu_2, \cdots$ are the train roots.

By the train property we shall mean that the determination of the $n$th term (or generation in genetics) depends ultimately on a linear recurrence equation with constant coefficients.

The significance of special train algebra is seen in the application to genetics. It will be seen that all the fundamental symmetrical genetic algebras are special train algebras. Various trains have genetical significance; the $\bar{x}^{(r)}$ represent successive discrete generations of an evolving population, and the train equation is the recurrence equation which connects them.

For example: plenary powers (7.4) refer to populations with random mating; principal powers (3.3) refer to a mating system in which each generation is mated back to one original ancestor or ancestral population, and the primary products refer to the descendants of a single individual or subpopulation $x$ mating at random within a population $y$. Other mating system are described by other sequences, and in various well-known cases these have the train property.
It usually happens that the train roots are real, distinct and between zero and one. Therefore it may be shown that \( x^{(r)} \rightarrow \) equilibrium as \( r \rightarrow \infty \); the rate is ultimately that of a geometrical progression with common ratio equal to the largest train root excluding 1, but it may be some generations (depending on the number of train roots) before the rate of approach is manifest.

Train roots may be described as the eigen values of the operation of symbolic multiplication by \( x \) or in genetic language, train root may be described as the operation of passing from one generation to the next.

**THEOREM (7-A)**

A special train algebra is a train algebra.

Proof: Let \( B \) be a special train algebra, then there exists a basis

\[
\{ a_0, a_1, \ldots, a_n \}
\]

such that

\[
a_i a_j = \sum_{k=0}^{n} x_{ij}^k a_k (7.8)
\]

\[
x_{00} = 1 (7.9)
\]

\[
x_{ij}^k = 0 \quad \text{for} \quad k < j (7.10)
\]

\[
x_{ij}^k = 0 \quad \text{for} \quad i, j > 0 \text{ and} \quad k \leq \text{mn}(i, j) (7.11)
\]

Let \( x \in B \)
\[ x = wc_0 + \sum_{i=1}^{N} \alpha_i c_i \]  \hspace{1cm} (7.12)

where

\[ w = W(x) \text{ is its weight} \]

Then using (7.8), (7.9), (7.10), (7.11)

\[ xc_0 = xc_0 + \cdots \]
\[ xc_i = x\lambda_i c_i + \cdots \]  \hspace{1cm} (7.13)

which implies

\[ 0 = (w - x)c_0 + \cdots \]
\[ 0 = (w\lambda_i - x)c_i + \cdots \]
\[ 0 = (w\lambda_i - x)c_2 + \cdots \]  \hspace{1cm} etc.  \hspace{1cm} (7.14)

The characteristic equation of the algebra is obtained by equation to zero \( x \) times the determinant of this set of equations which is

\[ x(x-w)(x-\lambda_1w) \cdots (x-\lambda_kw) = 0 \]  \hspace{1cm} (7.15)

Powers of \( x \) in the expanded form of this equation are to be interpreted as principal powers.

If the rank equation of \( B \) is \( g(x) = 0 \), then \( g(x) \) must be a factor of the left-hand side of the characteristic equation (7.15).
i.e.

\[ x( x-w ) ( x-\mu_1 w ) ( x-\mu_2 w ) \cdots ( x-\mu_r w ) = 0 \]

where

\[(\mu_1, \ldots, \mu_r) \subset (\lambda_1, \ldots, \lambda_n)\]

B has thus the essential (defining) property of a train algebra. This completes the proof.
CHAPTER 8

A STABILITY THEOREM FOR SPECIAL TRAIN ALGEBRA

The \( x_{ij} \) (abbreviated \( \lambda_j \)) that appeared in (7.1), (7.2) and (7.3) will be called the train roots of the algebra. Clearly \( \lambda_0 = 1 \).

**THEOREM (8-A)**

Every special train algebra which has no train root \( \lambda \) satisfying \( 2\lambda = 1 \) has a unique non-zero idempotent.

Proof: Construct \( x_i \) inductively such that

\[
\left( \sum_{i=0}^{m} x_i a_i \right)^2 = \sum_{i=0}^{m} x_i a_i + \sum_{i=m+1}^{n} y_i a_i
\]  

(8.1)

Since \( x_{...} = 1 \), we have \( x_0 = 1 \) for a non-zero idempotent. Suppose \( x_0, x_1, \ldots, x_m \) has been chosen.

Then

\[
\left( \sum_{i=0}^{m} x_i a_i \right)^2 = \left( \sum_{i=0}^{m} x_i a_i \right)^2 + 2z a_{m+1} \left( \sum_{i=0}^{m} x_i a_i \right) + z^2 a_{m+1}^2
\]

\[
= \left( \sum_{i=0}^{m} x_i a_i \right)^2 + 2z \sum_{i=0}^{m} x_i a_i a_{m+1} + z^2 a_{m+1}^2
\]

\[
= \left( \sum_{i=0}^{m} x_i a_i \right)^2 + \sum_{i=0}^{m} y_i a_i + 2z x_{m+1} a_{m+1} + 2z x_{m+1} a_{m+1} a_{m+1} + 2z \sum_{k=m+2}^{n} y_k a_k
\]

[by (6.2) and (6.3)]
\[ y_{m+1} = f(x_1, x_2, \ldots, x_m) \text{ is a function of } x_1, \ldots, x_m \]

Since \(2 \lambda_{m+1} \neq 1\), the equation in \(Z\)
\[ 2\lambda_{m+1} Z + f(x_1, x_2, \ldots, x_m) = Z \]
(8.2)

has a unique solution which is taken as \(x_{m+1}\). This proves the existence and uniqueness simultaneously.

In the following the algebra is taken over the reals. We may then define
\[
\sum_{m=0}^{n} x_m a_m \rightarrow \sum_{m=0}^{n} x_m a_m \iff x_m \rightarrow x_m
\]

THEOREM (8-B)

The sequence of plenary powers of an element of weight 1 in a special train algebra whose train roots other than \(\lambda = 1\) all have absolute value less than \(\frac{1}{2}\) tends to an idempotent.

Proof: Suppose \(a_i = \sum_{m=0}^{n} x a_m\). We want to show \(x_m \rightarrow x_m\) for \(x\) defined as in theorem (8-A).

Clearly
\[ x_{\omega i} = 1 \text{ for all } i \quad (\text{weight} = 1) \]
suppose
\[ x_m \rightarrow x_{m+1} \] for \( m = 1, \ldots, k \).

consider
\[ x_{k+1, i+1} = 2\lambda_{k+1} x_{k+1, i} + f( x_{i', i}, \ldots, x_{m', i} ) \quad (8.3) \]

which is derived in the same way as in theorem (8-A) and where
\[ f( x_{i', i}, \ldots, x_{m', i} ) \]
is a quadratic function of \( x_{i', i}, \ldots, x_{m', i} \) and depends on the multiplication table of the algebra.

Let
\[
\begin{align*}
x_{k+1, i+1} &= a_{i+1} \\
x_{k+1, i} &= a_i \\
2\lambda_{k+1} &= \lambda \\
f( x_{i', i}, \ldots, x_{m', i} ) &= b_i
\end{align*}
\]
we have
\[ a_{i+1} = \lambda a_i + b_i \quad (8.5) \]
and
\[ b_i \rightarrow b = f( x_i, \ldots, x_m ) \]

Let \( a \) be such that
\[ a = \lambda a + b \quad (8.6) \]
write
\[
\begin{align*}
a_i &= a + u_i \\
b_i &= b + y_i
\end{align*}
\]
we have
\[ u_{i+1} = \lambda u_i + y_i \quad \text{where } |\lambda| < 1 \]
and $y_i \to 0$, and if $u_i \to 0$, then $a_i \to a$ and we are done.

Since

$$u_{i+1} = \lambda^i u_i + \lambda^{i-1} y_i + \cdots + y_i$$

if we define

$$Z_n = \lambda^{i-1} y_r + \lambda^{i-2} y_2 + \cdots + y_n$$

then

$$Z_n \to 0$$

also

$$\lambda^i u_i \to 0$$

implies

$$u_i \to 0$$

This completes the proof.
CHAPTER 9

SEQUENCES OF POWERS IN SPECIAL TRAIN ALGEBRAS

Let $A$ denote the special train algebra over a space of dimension $n + 1$ whose canonical basis is $\{c_0, c, \cdots, c_n\}$ in which a typical element of unit weight is

$$x = c_0 + u_1c + \cdots + u_n c_n$$

$$= (1, u_1, \cdots, u_n) \quad (9.1)$$

We shall assume that the algebra contains an idempotent element and it can be taken as $c_0$, in which case $\lambda_{0k} = 0$, $k > 0$.

Let $A_{n-1}$ be a subspace of $A_n$ whose basis elements are $\{c_0, c_1, \cdots, c_{n-1}\}$. Then $A_i$ has basis elements $\{c_0, c_i\}$ with multiplication table

$$\begin{array}{c|cc}
  & c_0 & c_i \\
\hline
c_0 & c & \lambda_{0i}, c_i \\
c_i & 0 & \\
\end{array} \quad (9.2)$$
and \( A \), has basis elements \( \{ c_0, c_1, c_2 \} \) with multiplication table

<table>
<thead>
<tr>
<th></th>
<th>( c_0 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 )</td>
<td>( c_0 )</td>
<td>( \lambda_{011} c_1^{-1} \lambda_{012} c_2 )</td>
<td>( \lambda_{022} c_2 )</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>( \lambda_{112} c_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( E_n \) be the operator on \( A_n \) which transform \( \bar{x} \) into \( \bar{x}^2 \). For example

\[
E_2 \bar{x} = \bar{x}^2 = c_0 + 2 \lambda_{011} u_1 c_1 + (2 \lambda_{012} u_2 + \lambda_{112} u_1^2 + 2 \lambda_{022} u_2) c_2
\]

We think of \( E_n \) as operating on the coefficient \( u_i \) (which is regarded as a variable). For example,

\[
\begin{align*}
E_2 1 &= 1 \\
E_2 u_1 &= 2 \lambda_{011} u_1 \\
E_2 u_2 &= 2 \lambda_{012} u_1 + \lambda_{112} u_1^2 + 2 \lambda_{022} u_2
\end{align*}
\]

THEOREM (9-A)

The plane of unit weight in \( A_n \) may be mapped into a variety \( V_n \) lying in a space \( B_n \) of minimal dim \( B_n \geq n + 1 \) by a

\[\text{†In biology, a variety is a group having characteristics of its own species; subdivision of a species, subspecies. Here, a variety means a subset consisting of elements of the form \( (l, v_1, \ldots, v_m) \).}\]
function $R$

$$R_n(1, u, \ldots, u_n) = (1, v_1, \ldots, v_n)$$

where

$$v_i = u_1 u_2 \ldots u_n$$

(9.4)

in such a way that a linear transformation of $B_n$ with matrix $E$
can be found having the following properties:

(i) $E_n(R_n x) = R_n(E_n x)$

(ii) $E_n$ is lower triangular

(iii) The characteristic value $\lambda_i$ of $E_n$ such that $(E_n - \lambda_i I)(R_n x)$ has no component in $v_i$ given by (9.4) is

$$\left(2\lambda_{i1}\right)^{p_{i1}} \left(2\lambda_{i2}\right)^{p_{i2}} \cdots \left(2\lambda_{in}\right)^{p_{in}}$$

Informally, $E_n u_n$ involves $u_{n-1}^2$. Hence if $B_n^{-1}$ has been found corresponding to $A_{n-1}$, and a mapping $R$ given by (9.4), $B_n$ will need to contain a dimension corresponding to each distinct product of powers given by multiplying pairs of expressions on the right of (9.4) including squares and one corresponding to $u_n$.

Proof: For the case $A$;

Since

$$(1, u_1) \cdot (1, u_1) = (1, 2\lambda_n u_1)$$

we have

$$R_1 = I$$

the identity mapping
and

\[ \hat{E}_1 = E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2\lambda_{011} \end{pmatrix} \]  

(9.5)

For the case \( A \):

Since

\[ (1, u_1, u_2)(1, u_1, u_2) = (1, 2\lambda_{011} u_1, 2\lambda_{012} u_1 + \lambda_{112} u_1^2 + 2\lambda_{022} u_2) \]

we have

\[ R_2(1, u_1, u_2) = (1, v_1, v_2, v_3) = (1, u_1, u_2, u_3) \]

The induced transformation \( \hat{E}_2 \) is

\[ \hat{E}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\lambda_{011} & 0 & 0 \\ 0 & 0 & 4\lambda_{011}^2 & 0 \\ 0 & 2\lambda_{012} & \lambda_{112} & 2\lambda_{022} \end{pmatrix} \]  

(9.6)

The theorem is thus seen to be true for \( A_1 \) and \( A_2 \).

Suppose that it is true for \( A_{n-1} \). Let a general element of \( B_{n-1} \) be \((v_0, v_1, \cdots, v_{m'})\)

(9.7)

and the elements of \( \hat{E}_{n-1} \) denoted by \( e_{i,j} \) \( i, j = 0, \cdots, m' \).

The required space \( B_n \) and matrix \( \hat{E}_n \) will be constructed in two
stages and let the intermediate construction be \( \tilde{B} \) and \( \tilde{E} \).

For \( \tilde{B} \) we take a space of dimension

\[
m + 1 = \frac{1}{2}(m' + 1)(m' + 2) + 1
\]

Define:

\[
\tilde{R} : \mathbb{A}_n \longrightarrow \tilde{B}
\]

\[
( l, u_1, u_2, \cdots, u_n ) \longmapsto ( b_0, \cdots, b_{m-1}, b_m )
\]

(9.8)
as follows: The first \( m \) coordinates \( b_0, b_1, \cdots, b_{m-1} \) are the products \( v_i v_j, \) \( i = 0, \cdots, m' \) and \( j \geq i \), ordered so that \( v_r v_s \) precedes \( v_k v_{\ell} \) if either \( r < k \) or \( r = k, s < \ell \). Then \( b_m = u_n \).

Define the matrix \( \tilde{E} \) as follows: for the first \( m \) rows and columns of \( \tilde{E} \) we take the kroneeker square\(^\dagger\), and delete the row and column corresponding to \( v_i v_j \) for which \( i < j \). The \( (m + 1) \)th row and column of \( \tilde{E} \) are defined by

\[
\begin{align*}
e^i_{jm} &= 0, \quad j = 0, \cdots, m - 1 \\
e^m_m &= 2\lambda_{m,m} \\
e^m_k &= 2\lambda_{i,j} \quad \text{if the kth column of } \tilde{E} \text{ expressed in terms of (9.4) corresponds to } u_i u_j, \quad i \neq j. \\
e &\equiv 0 \quad \text{if the kth column of } \tilde{E} \text{ corresponds to } u_i^2 \quad \text{ otherwise (9.11)}
\end{align*}
\]

E satisfies (ii) and (iii) of theorem (9-A), which is clear from the construction.

It is possible that \( v_k v_l = v_r v_s \) for some \( k, l, r, s \).

That is

\[
\begin{pmatrix}
  u_1^{t_k} & \ldots & u_n^{t_k} \\
  \vdots & \ddots & \vdots \\
  u_1^{t_r} & \ldots & u_n^{t_r}
\end{pmatrix}
\begin{pmatrix}
  u_1^{s_k} & \ldots & u_n^{s_k} \\
  \vdots & \ddots & \vdots \\
  u_1^{s_r} & \ldots & u_n^{s_r}
\end{pmatrix}
\]

Therefore the corresponding characteristic values would also be equal. For each occurrence of this type, add the column of \( \tilde{E} \) corresponding to \( v_r v_s \) to that corresponding to \( v_k v_l \), and delete the row and column corresponding to \( v_r v_s \). Also delete the coordinate of \( \tilde{B} \) corresponding to \( v_r v_s \). Further, since the last row of \( \tilde{E} \) has non-zero entries only for coordinate corresponding to \( u_i u_d \), the elements of this row are not affected by the reduction procedure except for relabelling of the column number.

The results of the reduction procedure are the required space \( \mathcal{B} \) and matrix \( \tilde{E} \) and the associated map \( R^\omega \).

To prove (i) is true for \( A_\omega, B_\omega, E_\omega \) and \( \tilde{E}_\omega \), and \( R_\omega \) so constructed, consider \( x \in A_\omega, \overline{x} = (1, u_1, \ldots, u_{n-1}, u_n) \) and let \( \mathcal{P} \) be the projection of \( \overline{x} \) into its first \((n-1)\) coordinates, i.e.

\[
\mathcal{P}\overline{x} = \mathcal{P}(1, u_1, \ldots, u_{n-1}, u_n)
= (1, u_1, \ldots, u_{n-1}, 0)
\]

(9.12)
Then
\[ E_n \mathbf{x} = E_n( P \mathbf{x} + u_n \mathbf{c} ) \]
\[ = (P \mathbf{x})^2 + \left\{ 2 \sum_{i}^{n-1} \sum_{j>i}^{n-1} \lambda_{ij} n u_i u_j + \sum_{i=1}^{n-1} \lambda_i \ln u_i x + 2 \lambda_{\text{orn}} u_n \right\} \]
\[ (9.13) \]

From the definition of \( R_n \), one sees that the first term of (9.13) shows that the appropriate transform of the first \( m \) coordinates of \( R_n \) is the Kronecker square of \( E_{n-1} \) reduced to allow for identities among the \( v_i v_j \), and the remainder of (9.13) shows that (9.10) and (9.11) gives the required last row of \( \hat{E}_n \).

To prove that \( B_n \) so constructed is minimal, consider
\[ E_n u_{\mathbf{n}} = \lambda_{n-1}, n-1, n \quad u_{n-1} + \cdots \]
\[ (9.14) \]
By hypothesis, \( E_{n-1} u_{n-1} \) generates all the distinct products of powers of the \( u_i \) involved in the right-hand side of (9.4). Hence after the reductions of the above, no further reduction of \( B_n \) is possible. This completes the proof.

For particular sets of \( \lambda_{ij} \), spaces of smaller dimension than the \( B_n \) just constructed may satisfy the conditions of the theorem.

THEOREM (9-B)

Plenary powers in \( A_n \) form a train. The plenary train roots of \( A_n \) are included in the following set: the product taken
in pairs of those plenary train roots of the $A_{n-1}$, including $2\lambda_{nn}$ and squares of those plenary train root of $A_{n-1}$.

Proof: From theorem (9-A),

$$\hat{E}_n( RX ) = R( E_nX )$$

Hence, if $f( \hat{E}_n )$ is a polynomial operator which annihilates $B$, $f( \hat{E}_n )$ will annihilate $A_n$. Hence the minimal polynomial of $\hat{E}_n$ contains as a factor a polynomial which corresponds to the plenary rank equation of $A_n$. In view of its lower triangular form, it can be seen that the proper values of $\hat{E}_n$, and hence the plenary train roots of $A_n$, are included in the set stated in the theorem.
CHAPTER 10

GENETIC ALGEBRAS "OF SYMMETRICAL INHERITANCE"

Let $U$ be a commutative non-associative algebra of order $n$ over a field $F$. Let $R_x$ for $x \in U$ be the homomorphism defined as right multiplication by $x$ on $U$,

that is,

$$R_x : U \rightarrow U$$

$$a \mapsto ax \quad a \in U$$

(10.1)

Let $m$ be a subset of $\text{Hom}_F(U,U)$

DEFINITION: The enveloping algebra of $M$ is the smallest sub-algebra of $\text{Hom}_F(U,U)$ containing the homomorphisms in $M$. We shall denote it by $\text{env}(M)$. Alternatively, $\text{env}(M)$ is the algebra of all polynomials in the transformations (homomorphisms) in $M$ with coefficients in $F$.

DEFINITION: The enveloping algebra of the set which consists of the identity $I$ in $\text{Hom}_F(U,U)$ together with the right multiplications of $U$, is the transformation algebra $T(U)$ of $U$. Clearly any $T$ in $T(U)$ may be written in the form

$$T = \alpha I + f(R_{x_1}, R_{x_2}, \cdots)$$

(10.2)

where $\alpha \in F$ and $f$ is a polynomial.
If \( U \) is a baric algebra, then \( T( U ) \) is also a baric algebra. For if \( U \) has weight function \( W \), a weight function \( Z \) for \( T( U ) \) is defined by

\[
Z(T) = W(T\bar{u})
\]

(10.3)

for \( \bar{u} \) is an arbitrary element in \( U \) of unit weight. By (10.2) we have

\[
Z(T) = \lambda + f( W(x_1), W(x_2), \cdots)
\]

(10.4)

and \( Z \) nontrivial, since \( Z( I ) = 1 \).

Let

\[
T = \lambda I + f( R_{x_1}, R_{x_2}, \cdots)
\]

(10.5)

\( \lambda \in F, x_i \in U, T \in T( U ) \).

The characteristic function or determinant \( |\lambda I - T| \) of \( T \) in (10.5) has coefficients which are polynomials in \( \lambda \) and the coordinates of the \( x_i \), polynomials which depend on the function \( f \).

**DEFINITION:** A commutative baric algebra \( U \) over \( F \) with weight function \( W \) is called a genetic algebra in case the coefficients of the characteristic function of \( T \) in (10.5), insofar as they depend on the \( x_i \), depend only on the weight \( W(x) \).

**THEOREM:** (10-A)

A genetic algebra \( U \) over \( F \) is a train algebra.
Proof: Let $T = R_x$ in (10.5) and write $w(x) = \frac{1}{2}$. Then by the definition of a genetic algebra, the characteristic function

$$\phi(\lambda) = |\lambda I - R_x|$$

has the form

$$\phi(\lambda) = \lambda^n + r_1\lambda^{n-1} + \cdots + r_n\lambda$$

where $r_i \in F$ (10.6)

Now (10.6) factors in an algebraic closure $R$ of $F$ as

$$\phi(\lambda) = (\lambda - \lambda_1^1)(\lambda - \lambda_2^2) \cdots (\lambda - \lambda_r^r)$$

(10.7)

If

$$\lambda^r + \psi_1\lambda^{r-1} + \cdots + \psi_{r-1}\lambda$$

(10.8)

is the rank function of $U$, where $\psi_j$ a homogeneous polynomial of degree $j$ in the coordinates of $x$ then (10.8) divides $\lambda\phi(\lambda)$.

The $\lambda_i$ in (10.7) may then be ordered so that (10.8) equals

$$\lambda(\lambda - \lambda_1^1) \cdots (\lambda - \lambda_r^r)$$

from which it follows that

$$\psi_j = (-1)^j \sum_{i=1}^{j} \lambda_i \cdot \cdots \lambda_{i+j}$$

(10.9)

The rank equation is

$$x^r + \rho_1 w(x)x^{r-1} + \cdots + \rho_{r-1}[w(x)]^{r-1} x = 0$$
with \( \beta_j = ( -1 )^j \sum \lambda_i \cdots \lambda_{ij} \) in \( F \). Therefore \( U \) is a train algebra.

THEOREM (10-B)

A special train algebra \( U \) over \( F \) is a genetic algebra.

Proof: It was shown in theorem (7-A) that \( \exists \) scalars \( \lambda = \lambda_1, \lambda_2, \ldots, \lambda_n \) such that the matrix of \( R_\lambda \) has the form

\[
R_\lambda = \begin{bmatrix}
\lambda_1 & * & \cdots & * \\
0 & \lambda_2 & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n \\
\end{bmatrix}
\]  

(10.10)

Then the characteristic function of \( R \) has the form

\[
x( x - \lambda_1 ) ( x - \lambda_2 ) \cdots ( x - \lambda_n ) = 0
\]

with \( \lambda_i \) as in (10.10)

(10.11)

Hence

\[
( -1 )^t r_t = \sum \lambda_i \cdots \lambda_{it}
\]

(\( t = 1, 2, \ldots, n \))

(10.12)
From (10.10) we obtain

\[
\begin{bmatrix}
  f(\lambda_1, \lambda_2, \lambda_3, \cdots) & \ast & \cdots & \ast \\
  0 & f(\lambda_1, \lambda_2, \lambda_3, \cdots) & \cdots & \ast \\
  \vdots & \vdots & \ddots & \ddots \\
  0 & 0 & \cdots & f(\lambda_1, \lambda_2, \lambda_3, \cdots)
\end{bmatrix}
\]

where \( \lambda_k = w(x_k) \) (10.13)

Then \( T \) in (10.5) has characteristic equation

\[
|\lambda I - T| = |(\lambda - \lambda)I - f(R_{x_1}, R_{x_2}, \cdots)|
\]

\[
= [(\lambda - \lambda) - \xi_1] [(\lambda - \lambda) - \xi_2] \cdots [(\lambda - \lambda) - \xi_n]
\]

\[
= 0
\]

where

\[
\xi_i = f(\lambda_i, \lambda_i, \lambda_i, \cdots)
\]

then

\[
|\lambda I - T| = (\lambda - \lambda)^n + \mu_1(\lambda - \lambda)^{n-1} + \cdots + \mu_n
\]

(10.15)

where \((-1)^i \mu_i\) is a linear combination of \(\xi_i\) which is in turn a linear combination of \(\lambda_i\) with coefficients which are symmetric function of the \(\lambda_i\). Hence the coefficients of the characteristic function are independent of \(x_i\), that is \(U\) is a genetic algebra. This completes the proof.
We have thus far established the following relation:
a special train algebra is a genetic algebra and a genetic algebra is a train algebra and all are baric algebras.

THEOREM (10-C) Structure of genetic algebra.

Let $R$ be the kernel of the weight function $W$ of a genetic algebra $U$ over $F$. Then $R$ is the radical rod $U$ of $U$, and is nilpotent.

Proof: Let $T$ in $T(U)$ have the form (10.5), and write $w(x_i) = j_i$. Then the characteristic function

$$|\lambda I - T| = \chi^\alpha + \psi_1 \chi^\beta + \cdots + \psi_k. \quad (10.16)$$

of $T$ has coefficients $\psi_k = 0$ for any $T$ in $T(U)$ which may be written in the form (10.5) with

$$\lambda = w(x_i) = 0 \quad i = 1, 2, \cdots \quad (10.17)$$

For by definition of a genetic algebra, $\psi_k$ depends only on $w(x_i)$ and $T = 0$ is such a function with $\lambda = x_i = 0$. In this case

$$|\lambda I - T| = \chi^n = 0$$

for any $T \in T(U)$ which has the form (10.15) and satisfies (10.17), that is $T^n = 0$, $T$ is nilpotent. Now let $T$ be in the enveloping algebra $R^*$ of the nilsubalgebra $R$. Then (10.17) is satisfied, $T$ is nilpotent. Since $\text{env} \ (R)$ is an associative
algebra consisting of nilpotent elements, env ( R ) is nilpotent. Thus R is a nilpotent ideal of U, and is contained in the radical rad U of U. On the other hand:

\[ U/R \cong F \]

which implies rad U \( \subseteq \) R, or rad U = R.
EXAMPLE 1: An example of a train algebra.

Mendelian Geometric Algebra (D, R)

Consider a single autosomal gene difference (D, R) and the corresponding genotypes

\[ A = DD, \quad B = DR, \quad C = RR \]  \hspace{1cm} (11.1)

According to the Mendelian principles:

\[ DD = D, \quad DR = \frac{1}{2}D + \frac{1}{2}R, \quad R^2 = R \]  \hspace{1cm} (11.2)

These give the series of gametes produced by each type of zygote. For example the second of equations (11.2) means that a heterozygote produces D and R gametes in equal numbers, (that is 50 percent D and 50 percent R).

A population P can be described by the frequencies of the gametes which it produces:

Gametic representation

\[ P = sD + tR \]  \hspace{1cm} (11.3)

with the normalizing condition:
The algebra of the symbols $D, R$ defined by the multiplication table (11.2) will be called the \textit{gametic algebra for single mendelian inheritance}, and referred to as $G_2$.

If

\[ x \in G_2, \]

then

\[ x = sD + tR \quad \text{some } s, t \in \text{Reals} \quad (11.5) \]

$x \in G_2$ is interpreted as a population only if the coefficients satisfy (11.4).

The principal rank equation is

\[ x^2 - (s + t)x = 0 \quad (11.6) \]

A population $P$ is represented by an element of unit weight in the algebra $G$. The ratio $s:t$ gives the relative frequencies of the gametic types which it produces. In the case (11.6) becomes the train equation

\[ P^2 = P \text{ or } P (P - 1) = 0 \quad (11.7) \]

and $1$ is the unique principal train root of the algebra $G_2$. 
EXAMPLE 2: An example of a special train algebra.

The gametic algebra $G_3$ with multiplication table

\[
\begin{align*}
  x^2 &= x, \quad y^2 = xz = \frac{1}{6} x + \frac{3}{2} y + \frac{1}{2} z \\
  z^2 &= z, \quad yz = \frac{1}{2} y + \frac{1}{2} z \\
  xy &= \frac{1}{2} x + \frac{1}{2} y
\end{align*}
\]

(11.8, 11.9)

(11.10, 11.11)

(11.12)

refers to the inheritance of a single autosomal gene difference (x, y, z) in autotetraploids (c/f Haldane, 1930 chapter 11) the case m=2, with x, y, z written for $x^2 = x$, $x\alpha$, $a^2$.

Let

\[
a_0 = x, \quad a_1 = x - y, \quad a_2 = x - 2y + z
\]

(11.13)

Then

\[
\begin{align*}
  a_0 a_0 &= x = a_0 \\
  a_0 a_1 &= \frac{1}{2} x - \frac{1}{2} y = a_1 \\
  a_0 a_2 &= \frac{1}{6} x - \frac{1}{2} y + \frac{1}{6} z = \frac{1}{6} a_2 \\
  a_1 a_1 &= \frac{1}{2} a_2 \\
  a_1 a_2 &= 0 \\
  a_2^2 &= 0
\end{align*}
\]

(11.14)

(11.15)

(11.16)

(11.17)

(11.18)

(11.19)

Since

\[
a_1^2 = \frac{1}{6} a_2, \quad a_1 a_2 = a_2 a_2 = 0
\]

all powers of the ideal (a_1, a_2) are ideals. Therefore, $G_3$ is
a special train algebra. It has the principal train equation

\[ P ( P - 1 ) ( P - \frac{1}{6} ) = 0 \]  
(11.20)

and plenary train equations

\[ P [ P - 1 ] [ P - \frac{1}{3} ] = 0 \]  
(11.21)

where

\[ P = a_0 + sa_1 + ta_2 \quad s, t \in \text{Reals} \]  
(11.22)

**EXAMPLE 3:** An example of a genetic algebra.

Recall that the gametic algebra \( G_2 \) of simple mendelian inheritance is a train algebra. Its multiplication table is

\[ D^2 = D, \; DR = \frac{1}{2}D + \frac{1}{2}R, \; R^2 = R \]  
(11.23)

and \( (D, R) \) is its basis.

Now let

\[ u = \frac{1}{2} D + \frac{1}{2} R \]  
(11.24)

\[ z = \frac{1}{2} D - \frac{1}{2} R \]  
(11.25)

Then \( (u, z) \) is a new basis of \( G_2 \) such that

\[ u^2 = u, \; uz = \frac{1}{2} z, \; z^2 = 0 \]  
(11.26, 11.27, 11.28)

Writing \( x = \frac{3}{2} u + \sqrt{3} z \), we have weight function

\[ w : x \rightarrow W(x) = \frac{3}{2}. \]
For the nil-subalgebra $R = (z)$, we have $R^2 = 0$, and $G_2$ is a special train algebra.

The transformation algebra $T( G_2 )$ has order 3 over $F$, and any element $T$ of $T( G_2 )$ may be written in the form

$$T = \lambda I + \frac{3}{2} R_u + \gamma R_z$$

$$= \lambda I + R_x$$

where $\lambda, \gamma, \chi \in R$

$$x = \frac{1}{3} u + \gamma z \in G_2$$

(11.29)

The characteristic function of $T$ in (11.29) is

$$|I - T| = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} \alpha + \frac{3}{2} & 0 \\ \frac{1}{2} & \alpha + \frac{3}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - (\alpha + \frac{3}{2}) & 0 \\ -\frac{1}{2} & \lambda - (\alpha + \frac{3}{2}) \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - (\alpha + \frac{3}{2}) \end{vmatrix} \begin{vmatrix} \lambda - (\alpha + \frac{3}{2}) \end{vmatrix}$$

Therefore the coefficients are independent of $x$, and are dependent only on its weight $W(x) = \frac{3}{2}$. Thus $G_2$ is a genetic algebra.
BIBLIOGRAPHY

Dickson, L. E. 1914 (reprinted 1930), Linear Algebra, Cambridge Tract, no. 16.


Gonshor, H., "Special Train Algebras Arising in Genetics I ", 


Knoop, K., Theory and Application of Infinite Series, Blackie, 1928.
