Group quantum cohomology

by

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Abstract

Given a finite group $G$ acting on a smooth projective variety $X$, there exists a $G$-algebra $qA^*(X, G)$ whose structure constants are defined by integrals over moduli spaces of $G$-equivariant stable maps of Jarvis-Kaufmann-Kimura. It is a deformation of the Fantechi-Göttsche group cohomology, and its invariant part $qA^*(X, G)^G$ is canonically isomorphic to the Abramovich-Graber-Vistoli orbifold quantum cohomology of the quotient stack $[X/G]$. We provide the technology to study the associativity of the above algebra, and we prove it for some special cases.
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Chapter 1

Introduction

1.1 Overview

Gromov-Witten invariants are defined as integrals over moduli spaces of curves, and maps from these curves to a given target space. They give rise to an extremely rich theory, and in particular if one restricts to genus 0, one gets the so called quantum cohomology. One can extend the theory of Gromov-Witten invariants to study target spaces being orbifolds or stacks. In the symplectic category this study has been initiated by Chen and Ruan in [CR02] and [CR04], and in the algebraic category by Abramovich, Corti, Graber and Vistoli in [AV02], [ACV03] and [AGV02].

In this thesis, we will study a natural extension of the classical Gromov-Witten invariants to the category of $G$-objects where $G$ is a finite group. More precisely, one replaces curves by $G$-curves and maps by $G$-equivariant maps, and formally proceeds as in the classical theory (see [JKK03]). By analogy with the classical Gromov-Witten theory, one constructs the moduli spaces of marked $G$-curves and $G$-maps, the virtual fundamental classes and integrate cohomology classes. The genus 0 case is of particular interest for this paper. We construct a certain $G$-ring called the group quantum cohomology and show that its invariant part is the orbifold quantum cohomology of the quotient stack. This result is a generalization of a result in [FG03] on group cohomology and orbifold cohomology. We show that the degree 0 specialization of the group quantum cohomology is the group cohomology of [FG03]. We also show that the group quantum cohomology have functorial properties with respect to inclusions of groups.

We develop the theory necessary to prove the associativity of the group quantum cohomology (or in otherwords WDVV type equations). More precisely we relate it to equivalences of certain divisors on moduli spaces of marked $G$-curves. We prove these equivalences in certain cases. Proving them in general is still an open problem. The problem is that these moduli of marked $G$-curves are disconnected Deligne-Mumford curves with components of arbitrary genus. To prove the associativity, one probably needs a better understanding of the structure of those moduli.

1.2 Note on terminology

We use the names "quantum cohomology", "group cohomology" and "orbifold cohomology" for objects constructed based on the intersection theory (Chow rings). This seems like a bad choice, but the term "cohomology" is commonly used in the Gromov-Witten literature, even though the main references usually cited ([Ful98], [Kre99] and [Vis89]) are about the intersection theory. Also, we found it difficult to find good references about the cohomology of stacks. Therefore we will stick with intersection theory, and note that it seems that the same methods would work for cohomology.
1.3 Structure of the thesis

In the first few sections of Chapter 2, we make the standard constructions necessary for a Gromov-Witten type theory. We review the definitions of balanced and admissible marked $G$-curves, of their moduli stacks and of their basic properties. In Chapter 3, we describe the moduli stacks of $G$-equivariant maps from marked $G$-curves into a smooth projective $G$-variety. We relate the monodromies of the markings and the evaluation maps to the inertia variety. In Chapter 4, we describe natural obstruction theories on the above moduli stacks, construct their virtual fundamental classes using the approach of the intrinsic normal cone of Behrend and Fantechi ([BF97]), and compute their dimensions in terms of the ages, which is a set of numerical invariants associated to a variety with an action of a finite group.

In Chapter 5, we begin studying a purely equivariant phenomena, namely the behavior of the above objects under the inflation (sometimes called the induction), which is a functorial way of constructing a $G$-space from an $H$-space when $H$ is a subgroup of $G$. We show that there are inflation morphisms defined between the moduli stacks of equivariant maps, in such a way that the virtual fundamental classes are compatible.

We devote the Chapter 6 to the study of inertia varieties associated to $G$-varieties, of rigidified inertia stacks of global quotients and of their twisted sectors in a general context of groupoids. In particular we explain why the natural target spaces for the evaluation maps from the moduli stacks of twisted stable maps are rigidified inertia stacks (as opposed to the inertia stacks).

In Chapter 7, we specialize our study of admissible and balanced $G$-curves to the case where we have 3 markings and the quotient curve is $\mathbb{P}^1$. These are smooth $G$-covers of $\mathbb{P}^1$ with branching above 3 fixed points, and whose automorphisms are $G$-equivariant deck transformations. Denote their moduli stack by $\mathcal{M}_3^G$. It is a Deligne-Mumford stack of dimension 0, and we construct it in an explicit way. We consider the set $\mathcal{N}_3^G$ of triples in $G$ whose product is the identity. There exists a canonical $G$-cover of $\mathbb{P}^1$ whose monodromies at the 3 markings are given by any such triple. It can be constructed by taking a quotient of the product of $G$ and the upper half plane by an appropriate action of the principal congruence subgroup of level 2 (which can be identified with the fundamental group of 3-punctured sphere). This gives a morphism $\mathcal{N}_3^G \rightarrow \mathcal{M}_3^G$ which turns out to be a cross section for an action of $G^2$ by translation of any two of the three markings, hence providing a presentation of $\mathcal{M}_3^G$.

Let $\mathcal{N}_3^G(X)$ denote the stack of stable $G$-maps defined in Chapter 3, we will consider the fiber product $\mathcal{N}_3^G(X)$ of $\mathcal{M}_3^G(X)$ by $\mathcal{N}_3^G$ over $\mathcal{M}_3^G$. This is a compactification of the scheme of $G$-equivariant morphisms from $G$-covers of $\mathbb{P}^1$ into $X$. Note that we are compactifying an actual scheme, as we don't allow any $G$-equivariant deck transformations of the $G$-cover as isomorphisms of our moduli problem.

The Chapter 8 is dedicated to the construction of the group quantum cohomology, which is a $\mathbb{Q}$-graded and $G$-graded algebra whose structure constants are given by integrals over $\mathcal{N}_3^G(X)$. We prove that it behaves functorialy under the inclusion of groups, that degree 0 specialization is the group cohomology of Fantechi-Göttsche, and that the subalgebra of the $G$-invariants is the orbifold quantum cohomology of the quotient stack $[X/G]$. 

Finally in the Chapter 9, we verify two important properties of the virtual fundamental classes of the moduli stacks $\mathcal{M}^G_{g,n}(X)$, namely their compatibility under the cutting edges and the gluing morphisms. We link the associativity of the group quantum cohomology to equivalences of divisors on $\mathcal{M}^G_{0,4}$, obtained by gluing $G$-curves in two different ways. This is completely analogous to the classical case, except that $\mathcal{M}^G_{0,4}$ is far from being $\mathbb{P}^1$ in general.
Chapter 2

Moduli of $G$-curves

In this chapter we will define moduli stacks of marked $G$-curves with non-trivial isotropy at the nodes and markings. First we review the definitions of admissible and balanced $G$-curves (see [ACV03]). We then define the categories fibered in groupoids parameterizing stable admissible and balanced $G$-curves, and show that they form moduli stacks (with universal properties) by relating them to the moduli stacks of twisted stable curves. We also construct the Artin stack of (non-necessary stable) admissible and balanced $G$-curves.

2.1 Admissible and balanced $G$-curves

Let $G$ be a finite group. Given a $G$-scheme $X$ and a geometric point $Q$, denote by $G_Q$ the stabilizer of $Q$ in $G$. A morphism $f : Y \to X$ of $G$-schemes induces a canonical monomorphism of stabilizers $G_Q \to G_{f(Q)}$ for every $Q \in Y$.

**Definition 2.1.** A morphism $f : Y \to X$ of $G$-schemes is called fully faithful if for every point $Q \in Y$ the canonical monomorphism of stabilizers $G_Q \to G_{f(Q)}$ is an isomorphism.

A normalization of a $G$-scheme is a $G$-scheme by the universal property of normalization.

**Definition 2.2.** A $G$-action on a curve $E$ is called admissible if the normalization $\eta : \tilde{E} \to E$ is a fully faithful morphism.

Let $E$ be a possibly nodal $G$-curve over $\mathbb{C}$. Suppose the $G$-action on $E$ is admissible. Let $Q_1$ and $Q_2$ be the points on $\tilde{E}$ lying above a node $P \in E$. Then for any locally free $G-O_{\tilde{E}}$-module $F$ of finite rank, let $\chi_{Q_1,F}$ (resp. $\chi_{Q_2,F}$) be the character of the $G$-representation of $F_{Q_1}$ (resp. $F_{Q_2}$), the fiber at $Q_1$ (resp. $Q_2$) of the $G$-vector bundle $F$ corresponding to $\tilde{E}$. Let $\Omega_{\tilde{E}}$ be the $G-O_{\tilde{E}}$-sheaf of Kähler differentials on $\tilde{E}$.

**Definition 2.3.** An admissible $G$-action on a nodal curve $E$ is balanced at a node $P \in E$ if $\chi_{Q_1,\Omega_{\tilde{E}}}$ and $\chi_{Q_2,\Omega_{\tilde{E}}}$ are dual characters of $G_P$. A $G$-action is balanced if it is balanced at all nodes of $E$.

The admissibility of a $G$-action can be understood as a property of the action of the stabilizer of a node in an analytic or étale neighborhood, namely that it preserves each irreducible component of the curve. Balancing says that the action on the two components are compatible in the sense that they induce dual representations on the tangent spaces. In fact these two conditions imply that the nodal $G$-curve is a degeneration of a flat family of smooth $G$-curves, and conversely a flat degeneration of smooth $G$-curves is admissible and balanced (see [ACV03]).
2.2 Moduli of marked admissible G-curves

To define our moduli problems, we need the notions of category fibered in groupoids (in short CFG) and stack, whose definition can be found in [FanOl], [LMBOO] and [Beh97b]. Roughly a CFG is a pair consisting of a category and of a functor to the category of schemes of finite type over $\mathbb{C}$, having two properties: existence of pullbacks and unique factorization of pullback diagrams. Given two objects of a CFG over the same scheme, we can construct the presheaf of morphisms from one to the other (in the flat Grothendieck topology, see [Beh97b]). If this presheaf is a sheaf, then we call the CFG a prestack. A prestack is a stack if one can glue objects (every descent datum is effective). A stack $X$ is called Artin if there exists an affine scheme $U$ and a smooth epimorphism $p : U \rightarrow X$ (any such epimorphism is called a presentation of $X$) and if the diagonal morphism $X \rightarrow X \times X$ is representable and of finite type. An Artin stack is called Deligne-Mumford if it has an étale presentation.

Let $G$ be a finite group, $A$ be a finite set and $g$ be a non-negative integer. Let $\text{Sch}/\mathbb{C}$ be the category of schemes of finite type over $\mathbb{C}$.

**Definition 2.4.** Let $\mathfrak{M}^G_{g,A}$ be the CFG over $\text{Sch}/\mathbb{C}$ defined as follows. An object of $\mathfrak{M}^G_{g,A}$ over a scheme $T$ is a triple $\epsilon := (E_\epsilon, \Sigma_\epsilon, \sigma_\epsilon)$, where

1. $E_\epsilon$ is a flat and projective (in the sense of [Gro61a][5.5]) $G$-scheme over $T$, (that is it has a $G$-invariant structure morphism $p_{E_\epsilon} : E_\epsilon \rightarrow T$), whose fibers above geometric points are reduced nodal $G$-curves,

2. $\Sigma_\epsilon$ is a disjoint union of $G$-subschemes $\{\Sigma_{\epsilon,i}\}_{i \in A}$ of $E_\epsilon$ over $T$, such that for each $i \in A$ the structure morphism $\Sigma_{\epsilon,i} \rightarrow T$ is étale, and induces an isomorphism $\Sigma_{\epsilon,i}/G \simeq T$,

3. $\sigma_\epsilon$ is a set $\{\sigma_{\epsilon,i}\}_{i \in A}$ of sections of $\Sigma_{\epsilon,i} \rightarrow T$ indexed by $A$.

This data has to respect the following conditions:

(a) $\Sigma_\epsilon$ doesn’t intersect the nodes of the fibers of $p_\epsilon : E_\epsilon \rightarrow T$ (which is equivalent to requiring that its sheaf of ideals is invertible)

(b) for each $t : \text{Spec} \mathbb{C} \rightarrow T$, the $G$-action on $E_{\epsilon,t}$ is admissible, balanced and free away from the nodes and $\Sigma_{\epsilon,t}$, (where $E_{\epsilon,t}$ and $\Sigma_{\epsilon,t}$ denote the fibers of respectively $E_\epsilon$ and $\Sigma_\epsilon$ above $t$),

(c) for each $t : \text{Spec} \mathbb{C} \rightarrow T$ the dimension of $\text{H}^1(E_{\epsilon,t}, \mathcal{O}_{E_{\epsilon,t}})^G$ is $g$.

Let $\epsilon$ (resp. $f$) be an object of $\mathfrak{M}^G_{g,A}$ over $T$ (resp. $T'$). A morphism $\epsilon \rightarrow f$ of $\mathfrak{M}^G_{g,A}$ is a pair $(\alpha, t)$ consisting of a $G$-equivariant morphism $\alpha : E_\epsilon \rightarrow E_f$, and a morphism $t : T \rightarrow T'$ such that

$$
\begin{array}{ccc}
E_\epsilon & \xrightarrow{\alpha} & E_f \\
\downarrow p_\epsilon & & \downarrow p_f \\
T & \xrightarrow{t} & T'
\end{array}
$$

is cartesian and such that the sections of $\epsilon$ pullback from sections of $f$, that is

$$\alpha \circ \sigma_{\epsilon,i} = \sigma_{f,i} \circ t$$

for each $i \in A$. 
Definition 2.5. An object $e$ of $\mathcal{M}_{g,A}^G$ is called a marked admissible $G$-curve; $E_e$ is called the underlying $G$-curve of $e$; $\Sigma_{e,i}$ is called the $i^{th}$ twisted section of $e$; and $\sigma_{e,i}$ is called the $i^{th}$ section of $e$.

If we don't require the trivialization of the étale covers $\Sigma_{e,i} \to T$ (point 3 in the definition), we will obtain the moduli of twisted stable maps into the quotient stack (see [ACV03]).

Definition 2.6. Given an object $e$ of $\mathcal{M}_{g,A}^G$ over Spec $\mathbb{C}$, a point in the underlying $G$-curve of $e$ is called special if it is either a nodal point, or it belongs to a twisted section.

The CFG $\mathcal{M}_{g,A}^G$ is a prestack: given two objects $e$ and $f$ of $\mathcal{M}_{g,A}^G$ over $T$, the presheaf $\text{Isom}(e,f)$ is representable by a scheme over $T$ (which implies that it is a sheaf): it is a closed subscheme of the scheme of $G$-equivariant $T$-isomorphisms $\text{Isom}_T^G(E_e, E_f)$ cut out by the condition that the morphisms preserve the sections.

2.3 Forgetful morphism

Let $\mathcal{M}_{g,A}$ be the Artin stack of $A$-marked nodal curves (see [Beh97a]). When $G$ is the trivial group, then $\mathcal{M}_{g,A}^G$ is isomorphic to $\mathcal{M}_{g,A}$. For a general $G$, there is a morphism from $\mathcal{M}_{g,A}^G$ to $\mathcal{M}_{g,A}$, sending each object to its quotient by $G$, which is defined as follows.

Definition 2.7. Let $e$ be an object of $\mathcal{M}_{g,A}^G$. Let $\bar{E}_e := E_e/G$ be the geometric quotient of $E_e$ by $G$ over $T$. For each $i \in A$, taking the quotient of $\Sigma_{e,i}$ induces a morphism $\bar{\sigma}_{e,i} : T \simeq \Sigma_{e,i}/G \to \bar{E}_e$ into the smooth locus of $\bar{E}_e$. Let $\bar{e} := (\bar{E}_e, \bar{\sigma}_{e,i})$, which is an object of $\mathcal{M}_{g,A}$.

This construction is functorial and commutes with base change. The following proposition follows.

Proposition 2.8. There is a morphism $\mathcal{M}_{g,A}^G \to \mathcal{M}_{g,A}$ that sends an object $e$ to $\bar{e}$.

2.4 Properness

To avoid repetitions, we use the following terminology.

Definition 2.9. Let $\mathcal{M}$ be a prestack with a representable diagonal. Given an object $e$ of $\mathcal{M}$ over $T$, let $\text{Aut}_T(e)$ be the $T$-group of automorphisms of $e$. Then the object $e$ is called stable if $\text{Aut}_T(e)$ is of relative dimension 0 over $T$.

Note that if $\mathcal{M}$ is a prestack with a representable diagonal, then $\text{Aut}_T(e) = \text{Isom}_T(e,e)$ is representable, and we can talk of relative dimension of $\text{Aut}_T(e)$ over $T$.

Let $\text{Aut}_{\mathcal{M}}$ be the universal automorphism group over $\mathcal{M}$. Then the subprestack $\overline{\mathcal{M}}$ of stable objects of $\mathcal{M}$ is the locus where the morphism $\text{Aut}_{\mathcal{M}} \to \mathcal{M}$ is of relative dimension 0. It follows that $\overline{\mathcal{M}}$ is open in $\mathcal{M}$.

Definition 2.10. Let $\overline{\mathcal{M}}_{g,A}^G$ be the prestack of stable objects of $\mathcal{M}_{g,A}^G$. 
Chapter 2. Moduli of G-curves

Next result follows from [ACV03] by interpreting $\overline{\mathcal{M}}_{g,A}^G$ as a fiber product of the universal twisted sections over $\overline{\mathcal{M}}_{g,A}^{\text{ab}}(BG)$ (see [JKK03]).

**Theorem 2.11.** Suppose that $2g - 3 + o(A) \geq 0$, then the prestack $\overline{\mathcal{M}}_{g,A}^G$ is a proper and smooth Deligne-Mumford stack of pure dimension $3g - 3 + o(A)$.

**Corollary 2.12.** The prestack $\mathcal{M}_{g,A}^G$ is a smooth Artin stack, and if $2g - 3 + o(A) \geq 0$, the substack $\overline{\mathcal{M}}_{g,A}^G$ is open and dense in $\mathcal{M}_{g,A}^G$.

**Proof.** That $\overline{\mathcal{M}}_{g,A}^G$ is open follows from the fact that the stability is an open condition. That it is dense follows from Theorem 2.11 and the observation that the complement of $\overline{\mathcal{M}}_{g,A}^G$ is of proper codimension in $\mathcal{M}_{g,A}^G$.

The first statement is proven by an argument analogous to the one proving the same statement about $\overline{\mathcal{M}}_{g,A}$ and $\mathcal{M}_{g,A}$, and which can be found in [Beh97a]. The key to constructing a presentation of $\mathcal{M}_{g,A}^G$ is to consider the morphism $\pi : \mathcal{M}_{g,A}^{\mu_* = \text{id}} \to \mathcal{M}_{g,A}^G$ that forgets the section $*$, where $\mathcal{M}_{g,A}^{\mu_* = \text{id}}$ is the locus in $\mathcal{M}_{g,A}^G$ where the section $\sigma_*$ has no non-trivial stabilizer. Then $\mathcal{M}_{g,A}^{\mu_* = \text{id}}$ is a smooth $G$-curve over $\mathcal{M}_{g,A}^G$, namely it is the complement of the nodes and twisted sections in the universal $G$-curve. In particular $\pi$ is a smooth morphism. Restricting to the stable locus, we get a morphism from a Deligne-Mumford stack $\overline{\mathcal{M}}_{g,A}^{\mu_* = \text{id}}$ to $\mathcal{M}_{g,A}^G$. Notice that this morphism is not necessarily surjective (and it is possible that $\overline{\mathcal{M}}_{g,A}^{\mu_* = \text{id}}$ is empty), but we can remedy to this problem by adding more non-stabilized sections. By adding enough such sections we can cover a neighborhood of any point in $\mathcal{M}_{g,A}^G$. It follows that $\mathcal{M}_{g,A}^G$ is an Artin stack. 

2.5 Universal families

The construction of universal structures over a stack is a simple exercise, and reflects the fact that stacks have been created precisely for the purpose of serving as moduli spaces with the right universal properties. Therefore we leave out the details of the constructions involved in the following definitions.

**Definition 2.13.** There is a stack $\mathcal{E}_{g,A}^G$ over $\mathcal{M}_{g,A}^G$ such that the fiber over a marked admissible $G$-curve $\epsilon$ is isomorphic to the underlying $G$-curves of $\epsilon$. It is called the **universal $G$-curve over $\mathcal{M}_{g,A}^G$**. For any $i \in A$, there is a substack $\Sigma_{g,A,i}^G \subseteq \mathcal{E}_{g,A}^G$ such that the fiber over $\epsilon$ is identified under the above natural isomorphism with the $i$th twisted section of $\epsilon$. This is called the **$i$th universal twisted section over $\mathcal{M}_{g,A}^G$**. Each universal twisted section has a universal section denoted $\sigma_{g,A,i}^G$ with the obvious universal property. Denote by $\mathcal{E}_{g,A}^G$ the restriction of $\mathcal{E}_{g,A}^G$ to $\overline{\mathcal{M}}_{g,A}^G$.

2.6 Stabilization morphism

Denote $\overline{\mathcal{M}}_{g,A}^G$, $\mathcal{M}_{g,A}^G$, $\mathcal{E}_{g,A}^G$ and $\mathcal{E}_{g,A}^G$ by $\mathcal{M}$, $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{E}$ respectively. The stabilization morphism is a retraction $s : \mathcal{M} \to \mathcal{M}$, together with a $G$-morphism $c : \mathcal{E} \to s^*\mathcal{E}$ over $\mathcal{M}$.
compatible with the universal sections, such that given an object $c$ of $\mathcal{M}$ over $T$, then $c$ induces $E_c \to E_{s(c)}$ which is universal in the following sense: given an object $f$ of $\overline{\mathcal{M}}$ over $T$, and a $G$-morphism $E_c \to E_f$ compatible with the markings, there exists a unique arrow $\alpha : s(c) \to f$ in $\overline{\mathcal{M}}$ such that

\[
\begin{array}{ccc}
E_c & \xrightarrow{\alpha} & E_f \\
\downarrow & & \downarrow \\
E_{s(c)} & \xrightarrow{\alpha} & E_f
\end{array}
\]

commutes.

One can prove the following Proposition using the existence of stabilization morphism for nodal curves and its universal property.

**Proposition 2.14.** *Stabilization morphism exists for $\mathcal{M} = \mathcal{M}_g$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}}_g*. 

\[
\begin{array}{ccc}
E_c & \xrightarrow{\alpha} & E_f \\
\downarrow & & \downarrow \\
E_{s(c)} & \xrightarrow{\alpha} & E_f
\end{array}
\]
Chapter 3

Admissible G-maps

We define the moduli stacks of G-equivariant maps from a (admissible and balanced) G-curve into a fixed G-variety X. We describe different structures found on these moduli stacks, namely the universal G-curves and maps, the evaluation maps into the inertia variety, and the G-actions by translation of the sections.

3.1 Spaces of morphisms

Given two G-schemes E and X over a base scheme T, let \( \text{Mor}^G_{T}(E, X) \) be the scheme over \( T \) of G-equivariant T-morphisms from E to X. Its functor of points is defined as follows: a morphism \( S \to \text{Mor}^G_{T}(E, X) \) is a pair \((s, \phi)\) where \( s : S \to T \) and \( \phi \) is a G-equivariant \( S \)-morphism \( s^*E \to s^*X \). This definition generalizes to the case were \( E \to T \) and \( X \to T \) are representable morphisms of algebraic stacks. Then \( \text{Mor}^G_{T}(E, X) \) is an algebraic stack over \( T \) with the property that the structure morphism \( \text{Mor}^G_{T}(E, X) \to T \) is representable.

Let \( X \) be a G-variety over \( \mathbb{C} \). We want to construct a space whose points are G-equivariant morphisms from G-curves into \( X \). Consider the diagram

\[
\begin{array}{ccc}
\mathcal{E}^G_{g,A} & \to & X \times \mathcal{M}^G_{g,A} \\
\downarrow & & \downarrow \downarrow \\
\mathcal{M}^G_{g,A} & \to & \mathcal{M}^G_{g,A}
\end{array}
\]

where \( \mathcal{E}^G_{g,A} \) is the universal G-curve over \( \mathcal{M}^G_{g,A} \) and \( X \times \mathcal{M}^G_{g,A} \) is the product of \( X \) and \( \mathcal{M}^G_{g,A} \) (with the G-action on the first factor). These are both G-stacks representable over \( \mathcal{M}^G_{g,A} \).

**Definition 3.1.** Let \( X \) be a G-variety over \( \mathbb{C} \). Let

\[
\mathcal{M}^G_{g,A}(X) := \text{Mor}^G_{\mathcal{M}^G_{g,A}}(\mathcal{E}^G_{g,A}, X \times \mathcal{M}^G_{g,A}),
\]

and let \( \overline{\mathcal{M}}^G_{g,A}(X) \) be the open substack of \( \mathcal{M}^G_{g,A}(X) \) consisting of stable objects (see Definition 2.9).

The Artin stack \( \mathcal{M}^G_{g,A}(X) \) comes with a representable morphism to \( \mathcal{M}^G_{g,A} \). Given a geometric point \( \epsilon : \text{Spec} \mathbb{C} \to \mathcal{M}^G_{g,A} \), the pull back of \( \mathcal{M}^G_{g,A}(X) \) by \( \epsilon \) is the space of G-equivariant morphisms \( \text{Mor}^G_{T}(E, X) \):

\[
\begin{array}{ccc}
\text{Mor}^G_{T}(E, X) & \to & \mathcal{M}^G_{g,A}(X) \\
\downarrow & & \downarrow \downarrow \\
\text{Spec} \mathbb{C} & \to & \mathcal{M}^G_{g,A}
\end{array}
\]
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In general an object \( e \) of \( \mathcal{M}_{g,A}^G(X) \) is a quadruple \( e := (E_e, \Sigma_e, \sigma_e, f_e) \) where \( (E_e, \Sigma_e, \sigma_e) \) is an object of \( \mathcal{M}_{g,A}^G \) and \( f_e : E_e \to X \) is a \( G \)-equivariant morphism over \( C \). A morphism \( e \to f \) of \( \mathcal{M}_{g,A}^G(X) \) is a morphism \( (\alpha, \tau) : (E_e, \Sigma_e, \sigma_e) \to (E_f, \Sigma_f, \sigma_f) \) of \( \mathcal{M}_{g,A}^G \) compatible with \( f_e \) and \( f_f \), that is such that \( f_f \circ \alpha = f_e \).

**Definition 3.2.** Let \( \mathcal{E}_{g,A}^G(X) \) (resp. \( \mathcal{E}_{g,A}^+(X) \)) be the universal \( G \)-curve over \( \mathcal{M}_{g,A}^G(X) \) (resp. \( \mathcal{M}_{g,A}^+(X) \)) that is the pullback of the universal \( G \)-curve over \( \mathcal{M}_{g,A}^G \); let \( \Sigma_{g,A}^G(\tau) \) be the \( i^{th} \) universal twisted section over \( \mathcal{M}_{g,A}^G(X) \); let \( \sigma_{g,A}^G(\tau) \) be the \( i^{th} \) universal section over \( \mathcal{M}_{g,A}^G(X) \); and let \( f_{g,A}^G(X) : \mathcal{E}_{g,A}^G(X) \to X \) be the universal morphism over \( \mathcal{M}_{g,A}^G(X) \).

### 3.2 Morphisms of specified degree

Let \( A_1(X) := \text{Hom}(\mathbb{A}^1(X, \mathbb{Z}), \mathbb{Z}) \), the dual of the Picard’s group Pic(\( X \)) = \( \mathbb{A}^1(X, \mathbb{Z}) \), and let \( A_1^+(X) \subseteq A_1(X) \) be the semi-group of linear functions whose values on positive line bundles are non-negative. There is a natural \( G \)-module structure on Pic(\( X \)), and thus on \( A_1(X) \), and since any automorphism of \( X \) induces an isomorphism of Pic(\( X \)) that preserves the property of being a positive line bundle, the \( G \)-action preserves \( A_1^+(X) \).

Given a curve \( E \) and a morphism \( f : E \to X \), let \( \text{deg} f \in A_1^+(X) \) be the linear function sending the class of a line bundle to the degree of its pullback under \( f \). If \( E \) is a \( G \)-curve, and \( f \) is \( G \)-equivariant, then \( \text{deg} f \in A_1^+(X)^G \).

**Definition 3.3.** Given \( \beta \in A_1^+(X)^G \), let \( \mathcal{M}_{g,A}^G(X, \beta) \) be the substack of \( \mathcal{M}_{g,A}^G(X) \) whose geometric points are morphisms of degree \( \beta \).

### 3.3 Properness

Next result follows from [AV02] by interpreting \( \mathcal{M}_{g,A}^G(X, \beta) \) as a fiber product of the universal twisted sections of \( \mathcal{M}_{g,A}^\text{bal}(X/G, \beta) \) (see [JKK03]) where \( \beta \) is the pushforward of \( \beta \) to \( A_1^+(X/G) \).

**Theorem 3.4.** Given \( \beta \in A_1^+(X)^G \), the stack \( \mathcal{M}_{g,A}^G(X, \beta) \) is a proper Deligne-Mumford stack, and has a projective coarse moduli space.

### 3.4 Group actions

For each \( i \in A \), there is a \( G \)-action on \( \mathcal{M}_{g,A}^G(X, \beta) \) (and on \( \mathcal{M}_{g,A}^G(X, \beta) \)) obtained by translating the \( i^{th} \) section: given an object \( e \) over \( T \), and \( g \in G \), let \( g \cdot e := (E_e, \Sigma_e, g \cdot \sigma_e, f_e) \) where

\[
(g \cdot \sigma_e)_j = \begin{cases} 
\sigma_{e,j} & \text{if } i \neq j, \\
g \cdot \sigma_{e,i} & \text{if } i = j.
\end{cases}
\]

This is called the \( G \)-action by translation of the \( i^{th} \) marking.
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3.5 Monodromies

Given a generically free action of a finite group on a Riemann surface, the stabilizer of a point is cyclic and acts faithfully on the tangent space. The monodromy at this point is an element of the stabilizer whose action on the tangent space is by rotation by a smallest possible angle, in counterclockwise direction. The monodromies classify faithful one dimensional representations of a given cyclic group.

**Definition 3.5.** Let $E$ be a $G$-curve with a generically free $G$-action, and let $p \in E$ be a smooth point. Define the **monodromy of $E$ at $p$** as the unique element $m_p \in G_p$ such that the induced action of $m_p$ on the tangent space $T_p E$ is by multiplication by $\exp\left(\frac{2\pi i}{o(G_p)}\right)$, where $o(G_p)$ is the order of the stabilizer at $p$.

Note that a monodromy of a point is always a generator of the stabilizer of this point.

We decompose the stack $\mathcal{M}^G_{g,A}$ into a finite disjoint union of substacks, by specifying the monodromies at each section. This can be done as follows. Let $\omega$ be the relative dualizing sheaf of the universal $G$-curve over $\mathcal{M}^G_{g,A}$, and let $\tau_i$ be the dual of the restriction of $\omega$ to the $i^{th}$ universal twisted section $\Sigma_i$. Then $\tau_i$ is a line bundle, and its fiber over a point $p$ is the tangent space to the $G$-curve containing $p$. Given $m \in G$, let $\Sigma_i^{(m)}$ be the locus of points $p$ in $\Sigma_i$ fixed by $m$ and where $m$ acts on $\tau_i$ by multiplication by $\exp\left(\frac{2\pi i}{o(G_p)}\right)$.

Let $\mathcal{M}^G_{g,A} = m$ be the inverse image of $\Sigma_i^{(m)}$ under the $i^{th}$ universal section.

In general, given $m \in G$ let $\mathcal{M}^G_{g,A}$ be the intersection of the substacks $\mathcal{M}^G_{g,A} = m_i$ where $i$ runs over $A$.

The definitions of this decomposition for $\overline{\mathcal{M}}_{g,A}$, $\mathcal{M}^G_{g,A}(X)$ and $\overline{\mathcal{M}}^G_{g,A}(X)$ are similar.

3.6 Evaluation maps

Given an object $e$ of $\overline{\mathcal{M}}^G_{g,A}(X)$, the composition of the $G$-equivariant map $f_e$ with a section $\sigma_{i,e}$ defines a morphism from the base scheme into $X$, called the evaluation at the $i^{th}$ section. We can be more precise about the target space of the evaluation: since the section is invariant under the action of its monodromy and the morphism $f_e$ is $G$-equivariant, the evaluation maps into the subvariety of $X$ whose points are fixed by this monodromy. It is therefore useful to define a space which is the union of the fixed points sets of group elements.

**Definition 3.6.** Define the **inertia variety** of the $G$-variety $X$ as

$$X^{(G)} := \bigcup_{g \in G} \{g\} \times X^g \subseteq G \times X.$$ 

The diagonal $G$-action on $G \times X$ where $G$ acts on itself by left conjugation induces a $G$-action on $X^{(G)}$.

**Definition 3.7.** For each $i \in A$, let

$$ev_i : \mathcal{M}^G_{g,A}(X) \to X^{(G)}$$
be the morphism whose projection on $G$ is the monodromy $\mu_i$ of the $i^{th}$ section and whose projection on $X$ is the composition of the $i^{th}$ universal section $\sigma_{g,A_i}^G(X)$ with the universal morphism $\eta_{g,A}^G(X)$. It is a $G$-equivariant morphism where $G$ acts on $\mathcal{M}_{g,A}^G(X)$ by translation of the $i^{th}$ section.
Chapter 4

Fundamental class

As in the classical Gromov-Witten theory, the moduli stacks $\overline{\mathcal{M}}_{g,A}(X, (\beta))$ are expected to have dimensions given by a Riemann-Roch formula, but their actual dimension is in general different. Therefore one needs to construct a homology class of the expected dimension and having properties of a fundamental class. We will use the intrinsic normal cone construction of [BF97], with an appropriate obstruction theory.

4.1 Obstruction theory

In this section we generalize the ideas of [Beh97a] to the $G$-equivariant setup. To simplify, we will use the following notation for the moduli spaces

\[
\mathcal{M} := \mathcal{M}_{g,A}^G, \quad \overline{\mathcal{M}}(X) := \overline{\mathcal{M}}_{g,A}(X, \beta).
\]

Let $\pi : \mathcal{E}(X) \to \overline{\mathcal{M}}(X)$ (resp. $\mathcal{E} \to \mathcal{M}$) be the universal $G$-cover over $\overline{\mathcal{M}}(X)$ (resp. $\mathcal{M}$), and let $f : \mathcal{E}(X) \to X$ the universal $G$-morphism over $\overline{\mathcal{M}}(X)$.

The fibers of the forgetful morphism $p : \overline{\mathcal{M}}(X) \to \mathcal{M}$ are spaces of $G$-equivariant morphisms from a fixed $G$-cover to $X$. Therefore $p$ is representable, and one can construct the relative cotangent complex $L_{\mathcal{E}(X)/\mathcal{M}}$ using [Ill71] as an object of $D(O_{\overline{\mathcal{M}}(X)_{et}})$, the derived category of sheaves of modules on the étale site of $\overline{\mathcal{M}}(X)$. A perfect relative obstruction theory is an element $E$ of $D(O_{\overline{\mathcal{M}}(X)_{et}})$ concentrated in degrees 0 and $-1$, together with a morphism $\phi_E : E \to L_{\mathcal{E}(X)/\mathcal{M}}$ which induces an isomorphism on $H^0$ and a surjection on $H^{-1}$ (see [BF97]).

As $X$ is smooth over $\mathbb{C}$, the complex consisting of $\Omega_X$ in degree 0 is quasi-isomorphic to the cotangent complex of $X$ over $\mathbb{C}$. Define the complex

\[
E := R\pi^*_G(\omega \otimes f^*\Omega_X)[1]
\]

where $\omega$ is the relative dualizing sheaf of $\mathcal{E}(X)$ over $\overline{\mathcal{M}}(X)$, which has a natural structure of a $G$-sheaf, and where $\pi^*_G$ is the invariant pushforward (see [Gro57]). We construct $\phi_E : E \to L_{\mathcal{E}(X)/\mathcal{M}}$ as follows. By the functoriality of the cotangent complex we have

\[
f^*\Omega_X \to L_{\mathcal{E}(X)/\mathcal{M}} \to L_{\mathcal{E}(X)/\mathcal{M}} \simeq \pi^*L_{\overline{\mathcal{M}}(X)/\mathcal{M}}.
\]

Tensoring with $\omega$, we get a morphism of $D(O_{\mathcal{M}(X)_{et}})$

\[
\omega \otimes f^*\Omega_X \to \omega \otimes \pi^*L_{\overline{\mathcal{M}}(X)/\mathcal{M}} \simeq \pi^!L_{\overline{\mathcal{M}}(X)/\mathcal{M}}[-1].
\]

And finally as $\pi^!$ is the right adjoint of $R\pi^*_G$, we get

\[
\phi_E : E = R\pi^*_G(\omega \otimes f^*\Omega_X)[1] \to R\pi^*_G\pi^!L_{\overline{\mathcal{M}}(X)/\mathcal{M}} \to L_{\mathcal{E}(X)/\mathcal{M}}.
\]
Chapter 4. Fundamental class

Proposition 4.1. The morphism $\phi_E : E \rightarrow L_{\overline{M}(X)/\mathfrak{m}}$ is a perfect relative obstruction theory.

Proof. One adapts [BF97, Section 6] to the $G$-equivariant setup. First we show that $\phi_E$ is a relative obstruction theory. Denote by $p : \overline{M}(X) \rightarrow \mathfrak{m}$ the forgetful morphism. Let $t : T \rightarrow T'$ be an infinitesimal deformation of $T$ by an ideal $\mathcal{J}$, let $e$ be an object of $\overline{M}(X)$ over $T$ and let $f$ be a deformation of $f := p(e)$ over $t$:

$$
\begin{array}{ccc}
T & \rightarrow & T' \\
| & | \\
\downarrow t & & \downarrow t' \\
T & \rightarrow & T'
\end{array}
$$

We calculate:

$$
\text{Ext}^1_T(t^* E, \mathcal{J}) = \text{Ext}^1_T(R\pi_*^G(\omega_e \otimes f_*^* \Omega_X)[1], \mathcal{J})
= \text{Ext}^G_{\mathcal{E}}(f_*^* \Omega_X, \pi_*^* \mathcal{J}).
$$

By [Ill71], $\text{Ext}^G(\pi_*^* \Omega_X, \pi_*^* \mathcal{J})$ contains a canonical element $\alpha_t$ which is 0 if and only if the deformation $f'$ of $f$ lifts to a deformation of $e$, in which case the set of all such deformations is a torsor under $\text{Ext}^G(f_*^* \Omega_X, \pi_*^* \mathcal{J})$. By a relative version of [BF97, Theorem 4.5], $E \rightarrow L_{\overline{M}(X)/\mathfrak{m}}$ is a relative obstruction theory. That the complex $E$ is of perfect amplitude contained in $[-1, 0]$ follows from the fact that $\pi$ is of codimension 1. \(\square\)

The following is a consequence of Proposition 4.1, and of [Kre99, Theorem 5.2.1] which allows avoiding a technical hypothesis in [BF97, Section 5] about global resolutions.

Definition 4.2. We have a class $[\overline{M}(X), E] \in A_*(\overline{M}(X))$

as constructed in [BF97, Section 5], called the virtual fundamental class of $\overline{M}(X)$ with obstruction theory $E$.

When there is no confusion about the obstruction theory used, we will denote the virtual fundamental class by $[\overline{M}(X)]^{vir}$.

4.2 Expected dimension

We recall the definition of age (see [FG03]). Given a complex vector space $V$, and an idempotent automorphism $\nu$ of $V$, the age is a numerical invariant associated to this pair, defined as follows. Let $C_\nu$ the one dimensional representation of the cyclic group $\langle \nu \rangle$ with monodormy $\nu$ at 0 (recall from Section 3.5 that it means that $\nu$ acts by multiplication by $\exp \left( \frac{2\pi i}{\sigma(\nu)} \right)$). Let $m(\nu, j, V)$ be the multiplicity of $(C_\nu)^{\otimes j}$ in $V$. The age of $\nu$ on $V$ is the following weighted average of the above multiplicities

$$
a(\nu, V) := \frac{1}{\sigma(\nu)} \sum_{j=0}^{\sigma(\nu)-1} j m(\nu, j, V).
$$
Equivalently, if $V = V_1 \oplus \cdots \oplus V_n$ is the decomposition of $V$ into irreducible representations of $\langle \nu \rangle$ and $k_1, \ldots, k_n \in \{0, \ldots, \o(\nu) - 1\}$ are such that $\nu$ acts by multiplication by $\exp \left( \frac{2\pi i k_j}{\o(\nu)} \right)$ on $V_j$, then
\[
a(\nu, x, V) = \frac{1}{\o(\nu)} \sum_{j=1}^{n} k_j.
\]

Let $\Omega X$ be the $G$-vector bundle of Kähler differentials on $X$. The number $m(\nu, f, \Omega_{t,x} X)$ depends only on the connected component $Z$ of $X^\nu$ containing $x$, hence we can write $a(\nu, Z, X)$ for $a(\nu, \Omega_{t,x} X)$. Call $a(\nu, Z, X)$ the age of $\nu$ in $Z$. The following is proven in Appendix A.

**Theorem 4.3.** Let $\epsilon$ be an object of $\overline{M}(X)$ over $\text{Spec} \, \mathbb{C}$. Let $\nu_{\epsilon,i}$ be the monodromy of $E_\epsilon$ at $\sigma_{\epsilon,i}$. For each $i \in A$, let $Z_{\epsilon,i}$ be the connected component of $X^{\nu_{\epsilon,i}}$ containing the point $f_{\epsilon} \circ \sigma_{\epsilon,i} : \text{Spec} \, \mathbb{C} \to X$. Then the rank of $E$ at $\epsilon$ is
\[
rk_{\epsilon} E = (1 - g) \dim X + \frac{1}{\o(G)} \deg f_{\epsilon}^* TX - \sum_{i \in A} a(\nu_{\epsilon,i}, Z_{\epsilon,i}, X)
\]
where $\deg f_{\epsilon}^* TX = \beta(\det TX)$, the value of $\beta$ on the class of the highest wedge power of $TX$. The rank $\rk E$ of a bounded complex of vector bundles $E$, is the alternating sum of the ranks of each degree $\rk E := \sum_{i} (-1)^i \rk E^i$.

**Definition 4.4.** The number $\rk_{\epsilon} E$ is called the relative expected dimension of $\overline{M}(X)$ over $\mathcal{M}$ at $\epsilon$. By [BF97], the rank of the class $[\overline{M}(X), E]$ at the component containing $\epsilon$ is
\[
\rk_{\epsilon} [\overline{M}(X), E] := \rk_{\epsilon} E + \dim \mathfrak{M} = 3g - 3 + \o(A) + (1 - g) \dim X + \frac{1}{\o(G)} \deg f_{\epsilon}^* TX - \sum_{i \in A} a(\nu_{\epsilon,i}, Z_{\epsilon,i}, X)
\]
and it is called the expected dimension of $\overline{M}(X)$ at $\epsilon$. 

Chapter 5

Inflation

5.1 Inflation of spaces

Suppose that $H$ is a subgroup of $G$. There is a functor $\text{Inf}_H^G = G \times_H -$ from the category of $H$-schemes to the category of $G$-schemes which takes an $H$-scheme $X$ and sends it to the quotient $G \times_H X$ of the product $G \times X$ by the right $H$-action $(g, x) \cdot h \rightarrow (g \cdot h, h^{-1}x)$. The scheme $\text{Inf}_H^G X = G \times_H X$ has a natural left $G$-action induced by the multiplication on $G$. If $f : X \rightarrow Y$ is a morphism of $H$-schemes, then the induced morphism $G \times X \rightarrow G \times Y$ is $H$-equivariant under the above right $H$-action, hence taking the quotient induces a morphism $\text{Inf}_H^G f : \text{Inf}_H^G X \rightarrow \text{Inf}_H^G Y$ which is $G$-equivariant.

Consider the functor $\text{Res}_G^H$ from the category of $G$-schemes to the category of $H$-schemes, which restricts the action of $G$ to $H$. If $X$ is an $H$-scheme, there is a natural $H$-morphism $\iota^G_H X : X \rightarrow \text{Res}_G^H \text{Inf}_H^G X$ induced by the $H$-morphism $H \times X \rightarrow G \times X$ and the natural identification $X \simeq H \times_H X$.

If $X$ is an $H$-scheme over $T$ (that is the $H$-action preserves the fibers of $X \rightarrow T$), then $\text{Inf}_H^G X$ has a unique structure of a $G$-scheme over $T$ such that $\iota^G_H : X \rightarrow \text{Res}_G^H \text{Inf}_H^G X$ is a morphism of $H$-schemes over $T$.

Lemma 5.1. The functor $\text{Res}_G^H$ is a right adjoint of $\text{Inf}_H^G$.

Proof. The natural transformation $\text{id} \rightarrow \text{Res}_G^H \circ \text{Inf}_H^G$ is given by $\iota^G_H$ and the natural transformation $\text{Inf}_H^G \circ \text{Res}_G^H \rightarrow \text{id}$ is given by the $G$-action $G \times_H F \rightarrow F$ for a $G$-space $F$. •

Lemma 5.2. The functor $\text{Inf}_H^G$ commutes with base change, that is if $X$ is an $H$-scheme over a scheme $T$ and $t : S \rightarrow T$ is a morphism of schemes, then there is a natural $G$-isomorphism $G \times_H t^{-1}X \simeq t^{-1}(G \times_H X)$ over $S$. More precisely it means that

\[
\begin{array}{ccc}
\text{Sch}^H/T & \overset{\text{Inf}_H^G}{\longrightarrow} & \text{Sch}^G/T \\
\downarrow t^{-1} & & \downarrow t^{-1} \\
\text{Sch}^H/S & \overset{\text{Inf}_H^G}{\longrightarrow} & \text{Sch}^G/S
\end{array}
\]

is a commutative diagram of categories (where two functors are said to commute if they are naturally isomorphic).

Proof. We construct the natural isomorphism $G \times_H t^{-1}X \simeq t^{-1}(G \times_H X)$ as follows. By applying the base change to the $H$-morphism $\iota^G_H : X \rightarrow \text{Res}_G^H \text{Inf}_H^G X$ we get an $H$-morphism $t^{-1}X \rightarrow t^{-1}\text{Res}_G^H \text{Inf}_H^G X$ over $S$. It is clear that $\text{Res}_G^H$ commutes with base change, hence we get $\text{Inf}_H^G t^{-1}X \rightarrow \text{Inf}_H^G \text{Res}_G^H t^{-1}\text{Inf}_H^G X \rightarrow t^{-1}\text{Inf}_H^G X$ as $\text{Res}_G^H$ is a right adjoint of $\text{Inf}_H^G$. •
Lemma 5.3. The functor $\text{Inf}^G_H$ is faithful.

Proof. Given a morphism $f : X \to Y$ of $H$-schemes, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\iota^G_X} & \text{Res}_G^H \text{Inf}^G_H X \\
\downarrow & & \downarrow \text{Res}_G^H f \\
Y & \xrightarrow{\iota^G_Y} & \text{Res}_G^H \text{Inf}^G_H Y
\end{array}
$$

The result follows from the observation that the morphism $\iota^G_X$ is injective. \qed

Lemma 5.4. Given an $H$-scheme $X$, the projection morphism $G \times X \to G$, induces a morphism $\text{Inf}^G_H X \to G/H$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{\iota^G_X} & \text{Res}_G^H \text{Inf}^G_H X \\
\downarrow & & \downarrow \\
\{\text{id}_H\} & \xrightarrow{} & \text{Res}_G^H G/H
\end{array}
$$

is a cartesian diagram in the category of $H$-schemes, where the bottom arrow is induced by the inclusion of the identity element in $G$. Moreover the $G$-scheme $\text{Inf}^G_H X$ together with the projection onto $G/H$ is uniquely defined by the above property, up to a unique isomorphism commuting with the projection onto $G/H$.

Proposition 5.5. The functor $\text{Inf}^G_H$ induces an equivalence of categories between $\text{Sch}^H$ and $\text{Sch}^G/(G/H)$ whose objects are $G$-schemes with a $G$-morphism to $G/H$ and arrows are morphisms of $G$-schemes commuting with the projection onto $G/H$.

Proof. Construct the inverse functor using Lemma 5.4. \qed

Corollary 5.6. The functor $\text{Inf}^G_H$ preserves cartesian diagrams.

Proof. Follows from Proposition 5.5 and the fact that the forgetful functor $\text{Sch}^G/(G/H) \to \text{Sch}^G$ preserves cartesian diagrams. \qed

5.2 Inflation of admissible curves

Suppose that $H$ is a subgroup of $G$. Given $\beta \in \Lambda^+_1(X)^H$, we will define morphisms

- $\text{Inf}^G_H : \mathfrak{m}^H_{g,n} \to \mathfrak{m}^G_{g,n}$
- $\text{Inf}^G_H : \mathcal{M}^H_{g,n} \to \mathcal{M}^G_{g,n}$
- $\text{Inf}^G_H : \mathcal{M}^H_{g,n}(X, \beta) \to \mathcal{M}^G_{g,n}(X, \alpha)$

where $\alpha = \text{Inf}^G_H(\beta) := \sum_{g \in G/H} g \cdot \beta$, that inflate the underlying $H$-curves.
Definition 5.7. Given an object $e$ of $\mathfrak{M}_{g,n}^H$, let $f = \text{Inf}_H^G(e)$ be an object of $\mathfrak{M}_{g,n}^G$ defined by

$$
E_i := \text{Inf}_H^G E_e \\
\Sigma_{t,i} = \text{Inf}_H^G \Sigma_{e,i} \\
\sigma_{t,i} = \iota_{H}^G(E_e) \circ \sigma_{e,i}
$$

for each $i$. This construction is functorial and commutes with base change. Therefore it defines a morphism $\text{Inf}_H^G : \mathfrak{M}_{g,n}^H \to \mathfrak{M}_{g,n}^G$ called the inflation morphism.

Proposition 5.8. The inflation $\text{Inf}_H^G : \mathfrak{M}_{g,n}^H \to \mathfrak{M}_{g,n}^G$ is a representable étale morphism, and if in addition $n \geq 1$, then it is an embedding (that is fully faithful).

Proof. First, that it is representable is a direct consequence of Lemma 5.3.

Next we show that the inflation is étale. We need to study the $H$-equivariant infinitesimal deformations of $E_e$ over a fixed $G$-equivariant infinitesimal deformation of $G \times H E_e$. But $\iota_{H}^G(E_e) : E_e \to G \times H E_e$ is a closed and open $H$-equivariant embedding, and the $H$-equivariant relative cotangent complex $L_{E_e/G \times H E_e}$ is 0. This implies that there is no obstructions to lifting the deformations and the deformation space is trivial, which implies the étaleness.

Finally we show the full faithfulness. Since it is étale, we only need to show that it is fully faithful when restricted to the open dense substack $\mathcal{M}_{g,n}^H$ where the underlying $H$-curves are smooth. Moreover since it commutes with the projections onto $\mathcal{M}_{g,n}$, we only need to show it is fully faithful on the fibers, which is a result in Appendix B.

The inflation being étale preserves stability, and therefore induces a morphism $\text{Inf}_H^G : \overline{\mathfrak{M}}_{g,n}^H \to \overline{\mathfrak{M}}_{g,n}^G$.

Corollary 5.9. Suppose $n \geq 1$. Then the following diagram

$$
\begin{array}{ccc}
\mathfrak{M}_{g,n}^H & \xrightarrow{\text{Inf}_H^G} & \mathfrak{M}_{g,n}^G \\
\downarrow s & & \downarrow s \\
\overline{\mathfrak{M}}_{g,n}^H & \xrightarrow{\text{Inf}_H^G} & \overline{\mathfrak{M}}_{g,n}^G
\end{array}
$$

is cartesian, where the vertical arrows are the stabilization morphisms.

Proof. Since $n \geq 1$, using the inflation morphism, we identify $\mathfrak{M}^H$ (resp. $\overline{\mathfrak{M}}^H$) with an open substack $\mathfrak{M}^{H'}$ (resp. $\overline{\mathfrak{M}}^{H'}$) of $\mathfrak{M}^G$ (resp. $\overline{\mathfrak{M}}^G$), that is characterized by the property: an object $e$ of $\mathfrak{M}^G$ (resp. $\overline{\mathfrak{M}}^G$) is in $\mathfrak{M}^{H'}$ (resp. $\overline{\mathfrak{M}}^{H'}$) if and only if there exists a $G$-equivariant morphism $h : E_e \to G/H$ such that all the markings lie in the fiber above the identity $H \in G/H$. Given an object $e$ of $\mathfrak{M}^G$ such that the stabilization $e^s$ is an object of $\overline{\mathfrak{M}}^{H'}$, we have a $G$-equivariant morphism $h^s : E_{e^s} \to G/H$ such that the markings of $e^s$ lie in the fiber of the identity $H$. Composing the $G$-equivariant stabilization morphism $E_e \to E_{e^s}$ with $h^s$, we get a $G$-equivariant morphism $h : E_e \to G/H$ such that the markings of $e$ lie in the fiber of the identity $H$, which implies that $e$ is an object of $\mathfrak{M}^{H'}$. 

$\square$
5.3 Inflation of morphisms

**Definition 5.10.** Continuing with the notation of Definition 5.7, if we are given a $G$-scheme $X$ and an $H$-equivariant morphism $f_e : E_e \to \text{Res}_G^H X$, define $f_1$ as the composition of the morphism $\text{Inf}_G^G f_e : \text{Inf}_G^G E_e \to \text{Inf}_G^G \text{Res}_G^H X$ with the action $\text{Inf}_G^G \text{Res}_G^H X \to X$. This defines $\text{Inf}_H^G : \mathcal{M}_{g,n}^H(\text{Res}_G^H X) \to \mathcal{M}_{g,n}^G(X)$, and again it induces $\text{Inf}_H^G : \overline{\mathcal{M}}_{g,n}^H(\text{Res}_G^H X) \to \overline{\mathcal{M}}_{g,n}^G(X)$ as it preserves stability. We will write $\overline{\mathcal{M}}_{g,n}^H(X)$ (resp. $\mathcal{M}_{g,n}^H(X)$) for $\mathcal{M}_{g,n}^H(\text{Res}_G^H X)$ (resp. $\mathcal{M}_{g,n}^H(\text{Res}_G^H X)$).

**Lemma 5.11.** The diagram

$$
\begin{array}{ccc}
\mathcal{M}_{g,n}^H(X) & \xrightarrow{\text{Inf}_H^G} & \mathcal{M}_{g,n}^G(X) \\
\downarrow \pi_H & & \downarrow \pi_G \\
\mathcal{M}_{g,n}^H & \xrightarrow{\text{Inf}_H^G} & \mathcal{M}_{g,n}^G
\end{array}
$$

is cartesian.

**Proof.** For simplicity write $\mathcal{M}_H$ (resp. $\mathcal{M}_G$) for $\mathcal{M}_{g,n}^H$ (resp. $\mathcal{M}_{g,n}^G$). Let $\mathcal{E}_G \to \mathcal{M}_G$ be the universal curve over $\mathcal{M}_G$, and let $\mathcal{E}'_G \to \mathcal{M}_H$ be the pullback of $\mathcal{E}_G$ under $\text{Inf}_H^G : \mathcal{M}_H \to \mathcal{M}_G$. By definition of spaces of morphisms, the natural diagram

$$
\begin{array}{ccc}
\text{Mor}_{\mathcal{M}_H}(\mathcal{E}_G, X \times \mathcal{M}_H) & \xrightarrow{\text{Inf}_H^G} & \text{Mor}_{\mathcal{M}_G}(\mathcal{E}_G, X \times \mathcal{M}_G) \\
\downarrow & & \downarrow \\
\mathcal{M}_H & \xrightarrow{\text{Inf}_H^G} & \mathcal{M}_G
\end{array}
$$

is cartesian. By definition of inflation, we have a cartesian diagram

$$
\begin{array}{ccc}
\text{Inf}_H^G \mathcal{E}_H & \xrightarrow{} & \mathcal{E}_G \\
\downarrow & & \downarrow \\
\mathcal{M}_H & \xrightarrow{\text{Inf}_H^G} & \mathcal{M}_G
\end{array}
$$

which implies that $\mathcal{E}'_G \simeq \text{Inf}_H^G \mathcal{E}_H$. Moreover, the morphism

$$
\text{Mor}_{\mathcal{M}_H}(\text{Inf}_H^G \mathcal{E}_H, X \times \mathcal{M}_H) \to \text{Mor}_{\mathcal{M}_H}(\mathcal{E}_H, X \times \mathcal{M}_H)
$$

induced by the inclusion $\mathcal{E}_H \to \text{Res}_G^H \text{Inf}_H^G \mathcal{E}_H$ is an isomorphism by Lemma 5.1, which shows the result.

**Proposition 5.12.** Suppose that $n \geq 1$. The following diagram

$$
\begin{array}{ccc}
\mathcal{M}_{g,n}^H & \xrightarrow{\text{Inf}_H^G} & \mathcal{M}_{g,n}^G \\
\downarrow \pi & & \downarrow \\
\mathcal{M}_{g,n}^H & \xrightarrow{\pi} & \mathcal{M}_{g,n}^G
\end{array}
$$

where the disjoint union is over all $\beta \in \Lambda_1^+(X)^H$ such that $\text{Inf}_H^G \beta = \alpha$, is cartesian.
Chapter 5. Inflation

Proof. This follows from Lemma 5.11, Lemma 5.9 and the observation that if \( \beta \in A^+(X)^H \) is such that \( \text{Inf}^G_H \) maps \( \mathcal{M}^H_{g,n}(X, \beta) \) into \( \mathcal{M}^G_{g,n}(X, \alpha) \) then \( \text{Inf}^G_H \beta = \alpha \). \( \square \)

Corollary 5.13. Suppose that \( n \geq 1 \). The following diagram

\[
\begin{array}{ccc}
\bigsqcup_{\beta} \mathcal{M}^H_{g,n}(X, \beta) & \xrightarrow{\text{Inf}^G_H} & \mathcal{M}^G_{g,n}(X, \alpha) \\
\downarrow & & \downarrow \pi \\
\mathcal{M}^H_{g,n} & \xrightarrow{\pi} & \mathcal{M}^G_{g,n}
\end{array}
\]

where the disjoint union is over all \( \beta \in A^+(X)^H \) such that \( \text{Inf}^G_H \beta = \alpha \), is cartesian. In particular the inflation morphism \( \text{Inf}^G_H : \mathcal{M}^H_{g,n}(X, \beta) \rightarrow \mathcal{M}^G_{g,n}(X, \alpha) \) is an open and closed embedding of Deligne-Mumford stacks.

Proof. By Corollary 5.12 and the observation that an object of \( \mathcal{M}^H_{g,n}(X, \beta) \) is stable if and only if its image in \( \mathcal{M}^G_{g,n}(X, \alpha) \) is stable. \( \square \)

5.4 Compatibility of virtual fundamental classes under inflation

Let \( H \) be a subgroup of \( G \). To prove the compatibility of the virtual fundamental classes under the inflation \( \text{Inf}^G_H \) morphism, we only need to show that the obstruction theories are compatible (see [BF97]). More precisely if \( E^G \) (resp. \( E^H \)) is an obstruction theory for \( \mathcal{M}^G_{g,A}(X, \beta) \) (resp. \( \mathcal{M}^H_{g,A}(X, \alpha) \)), then we want to show that \( (\text{Inf}^G_H)^* E^G = E^H \) in \( D^G(\mathcal{O}_{\mathcal{M}^G_{g,A}(X, \alpha)}) \) (where \( \beta = \text{Inf}^G_H \alpha \)). The following Lemma is straightforward.

Lemma 5.14. Let \( E \) be an \( H \)-scheme over \( T \), with structure morphism \( p : E \rightarrow T \). Let \( q : \text{Inf}^G_H E \rightarrow T \) be the structure morphism of \( \text{Inf}^G_H E \) over \( T \), and let \( \iota = \iota^G_H(E) \). Then the diagram

\[
\begin{array}{ccc}
\text{Mod}^G(\mathcal{O}_{\text{Inf}^G_H E}) & \xrightarrow{\iota^*} & \text{Mod}^H(\mathcal{O}_E) \\
\downarrow q^G \downarrow & & \downarrow p^H \downarrow \\
\text{Mod}(\mathcal{O}_T) & & 
\end{array}
\]

commutes, that is there exists a natural isomorphism \( p^H_* \circ \iota^* \cong q^G_* \).

Proof. The result follows from the observation that if \( U \) is an \( H \)-invariant open in \( E \), and \( F \) is an object of \( \text{Mod}^G(\mathcal{O}_{\text{Inf}^G_H E}) \), then \( F(\text{Inf}^G_H U)^G = F(U)^H \). \( \square \)

The following shows that the obstruction theories of \( \mathcal{M}^H_{g,A}(X) \) and \( \mathcal{M}^G_{g,A}(X) \) are compatible under the inflation.

Proposition 5.15. Let \( \omega_p \) (resp. \( \omega_q \)) be the relative dualizing complex of \( p \) (resp. \( q \)). Suppose \( f : E \rightarrow \text{Res}_{X}^G \) is an \( H \)-equivariant morphism where \( X \) is a \( G \)-variety. Then we have

\[
\text{Rp}^H(\omega_p \otimes f^* \Omega_X) = \text{Rq}^G(\omega_q \otimes f^* \Omega_X)
\]

in the derived category of \( \mathcal{O}_T \)-modules.
Proof. First notice that $\omega_p = \iota^* \omega_q$ as $\iota$ is a closed and open embedding over $T$. Thus by Lemma 5.14,

\[ R_p^H (\omega_p \otimes f^* \Omega_X) = R_p^H \iota^* (\omega_q \otimes f^* \Omega_X) = R_q^G (\omega_q \otimes f^* \Omega_X). \]

\[ \Box \]
Chapter 6

Groupoid actions, inertia varieties and stacks

Suppose that $G$ acts on a variety $X$. We've seen in Section 3.6 that there is a variety $X^{(G)}$ called the inertia variety that is a natural target for the evaluation maps. In this section we will construct the inertia variety using the action of $(G)$, and describe the natural subvarieties corresponding to conjugacy classes of $G$. Also we describe the rigidified inertia stack as the quotient of the inertia variety by the groupoid $G_{(G)}$, as opposed to the inertia stack which is the quotient by $G$. The rigidified inertia stack is the natural target space for the evaluation maps from moduli spaces of twisted stable maps of [AV02], and hence appears in the construction of the orbifold (quantum) cohomology.

6.1 Quotient by a product of cyclic subgroups

Let $G$ be a finite group, and $\gamma \subseteq G$ be a subset invariant under conjugation (for example a conjugacy class, or the whole group $G$). Let $(\gamma)$ be the subset of $G \times \gamma$ given by

$$(\gamma) = \{(g, m) \in G \times \gamma \mid g \in \langle m \rangle\} = \{(g, m) \in \langle m \rangle \times \{m\} \mid m \in G\}.$$ 

It is a subgroup of $G \times \gamma$ (as groups over $\gamma$ or $\gamma$-groups). The fiber of the projection $(\gamma) \rightarrow \gamma$ above $m \in \gamma$ is the subgroup generated by $m$. Let $G_{(\gamma)}$ be the quotient of $G \times \gamma$ by the right action of $(\gamma)$ over $\gamma$. More concretely $G_{(\gamma)}$ is the quotient of $G \times \gamma$ by the relation

$$(g, m) \sim (h, n) \iff m = n, h^{-1}g \in \langle m \rangle$$

and denote by $[g, m]$ the equivalence class of $(g, m)$. Given $m \in \gamma$, we have a cartesian diagram

$$G/\langle m \rangle \longrightarrow G_{(\gamma)} \quad \downarrow$$

$$\{m\} \longrightarrow \gamma$$

The space $G_{(\gamma)}$ has a natural structure of a groupoid over $\gamma$:

1. the source map is $[g, m] \mapsto m$,
2. the target map is $[g, m] \mapsto gmg^{-1}$,
3. the inverse is $[g, m] \mapsto [g^{-1}, gmg^{-1}]$,
4. the unit is $m \mapsto [id, m]$. 

5. and the product is $([g, m], [h, n]) \mapsto [gh, m]$ for $m = hn^{-1}$.

Denote this groupoid by $BG(\gamma)$, where it is understood that the set of objects is $\gamma$. The following result shows that if $\gamma$ is a conjugacy class, then $BG(\gamma)$ is isomorphic as a groupoid (that is equivalent as a category) to a group. Recall that any groupoid has a canonical functor from the set of object (which is a groupoid with only trivial arrows) to itself. Hence we have a functor $\gamma \to BG(\gamma)$.

**Lemma 6.1.** Suppose that $\gamma$ is a conjugacy class in $G$. Choose an element $m \in \gamma$, and let $Z(m)$ be the quotient group $Z_G(m)/\langle m \rangle$ where $Z_G(m)$ is the centralizer of $m$ in $G$. We have a canonical equivalence of categories $BZ(m) \cong BG(\gamma)$. More precisely the diagram

$$
\begin{array}{ccc}
Z(m) & \longrightarrow & \{m\} \\
\downarrow & & \downarrow \\
\{m\} & \longrightarrow & BG(\gamma)
\end{array}
$$

is cartesian, where the morphism $\{m\} \to BG(\gamma)$ is the composition of the inclusion $\{m\} \to \gamma$ with $\gamma \to BG(\gamma)$.

**Proof.** The groupoid $G(\gamma)$ acts transitively on $\gamma$ (as $\gamma$ is a conjugacy class), and it follows that $\{m\} \to BG(\gamma)$ is essentially surjective. The result follows from the observation that the arrows of $BG(\gamma)$ fixing $m$ form a group isomorphic to $Z(m)$. 

**6.2 Actions of $G(\gamma)$**

Let $\gamma$ be an invariant subset of $G$ (under conjugation). We present how natural actions of $G(\gamma)$ arise from $G$-varieties.

**Lemma 6.2.** Suppose $Y$ is a $G$-variety, and $a : Y \to \gamma$ is a $G$-equivariant morphism (where $G$ acts on $\gamma$ by conjugation) such that for all $m \in \gamma$, $m$ acts trivially on $Y_m := a^{-1}(m)$. Then the groupoid $G(\gamma)$ acts on $Y$ with anchor $a$.

**Proof.** The group $(\gamma)$ acts trivially on $Y$ over $\gamma$, hence the canonical action of the groupoid $G \times \gamma$ on $Y$ induces an action of $G(\gamma)$. 

In particular we have a cartesian diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & \gamma \\
\downarrow & & \downarrow \\
[Y/G(\gamma)] & \longrightarrow & BG(\gamma)
\end{array}
$$

where $[Y/G(\gamma)]$ denotes the quotient stack of the $G(\gamma)$-action on $Y$.

**Lemma 6.3.** Suppose $a_1 : Y_1 \to \gamma$ and $a_2 : Y_2 \to \gamma$ are like in the Lemma 6.2, and suppose $f : Y_1 \to Y_2$ is a $G$-equivariant morphism commuting with the morphisms to $\gamma$. Then $f$ is $G(\gamma)$-equivariant with the $G(\gamma)$-action defined in Lemma 6.2.

**Proof.** The action of $[g,m] \in G(\gamma)$ defines a morphism $Y^m \to Y^{gmg^{-1}}$, which is the same as the morphism defined by the action of a representative $g$ of $g \langle m \rangle$, and the result follows since $f$ is $G$-equivariant.
6.3 Inertia variety and rigidified inertia stack

Suppose that $G$ acts on a variety $X$.

**Definition 6.4.** The group $(\gamma)$ acts on $\gamma \times X$ over $\gamma$, and denote by $X^{(\gamma)}$ its fixed point set, that is the fiber product

$$
\begin{array}{c}
X^{(\gamma)} \longrightarrow \gamma \times X \\
\downarrow \\
X \longrightarrow X \times X
\end{array}
$$

where the rightmost vertical arrow is the product of the action with the projection, and the bottom arrow is the diagonal. When $\gamma$ is the whole group $G$, the variety $X^{(G)}$ is called the **inertia variety**. When $\gamma$ is a conjugacy class, the variety $X^{(\gamma)}$ is called the $\gamma$-twisted sector of the inertia variety $X^{(G)}$. Given a group homomorphism $h : H \to G$, there is a natural closed and open morphism $X^{(H)} \to X^{(G)}$, which is injective when $h$ is injective, and which sends twisted sectors into twisted sectors.

Since the image of $X^m$ under the action of an element $g \in G$ is $X^{gm^{-1}}$, the diagonal $G$-action on $\gamma \times X$ induces a $G$-action on $X^{(\gamma)}$ such that the projections $X^{(\gamma)} \to \gamma$ and $X^{(\gamma)} \to X$ are $G$-equivariant. By Lemma 6.2, $G(\gamma)$ acts on $X^{(\gamma)}$ over $\gamma$.

**Lemma 6.5.** Suppose $f : Z \to X$ is a $G$-equivariant morphism. Then there exists a $G^{(\gamma)}$-equivariant morphism

$$
Z^{(\gamma)} \to X^{(\gamma)}
$$

such that

$$
\begin{array}{c}
\{m\} \times Z^m \longrightarrow Z^{(\gamma)} \\
\downarrow \\
\{m\} \times X^m \longrightarrow X^{(\gamma)}
\end{array}
$$

is cartesian for all $m \in \gamma$.

**Proof.** The morphism $f$ being $G$-equivariant, it sends the fixed point set $Z^m$ onto $X^m$ for all $m \in \gamma$. It follows that above cartesian diagrams define the desired morphism, which is $G^{(\gamma)}$-equivariant. \hfill $\Box$

**Lemma 6.6.** Suppose $a : Y \to \gamma$ is as in Lemma 6.2. Then the projection $Y^{(\gamma)} \to Y$ is $G^{(\gamma)}$-equivariant and has a canonical $G^{(\gamma)}$-equivariant section.

**Proof.** Given $m \in \gamma$, the fiber $a^{-1}(m)$ is fixed by $m$ and hence a subspace of $Y^m$. Compose this inclusion with the natural identification of $Y^m$ with the fiber of $Y^{(\gamma)}$ above $m$. This defines a $G$-equivariant section of $Y^{(\gamma)} \to Y$, which is $G^{(\gamma)}$-equivariant by Lemma 6.3. \hfill $\Box$

**Proposition 6.7.** Suppose $a : Y \to \gamma$ is as in Lemma 6.2, and suppose $f : Y \to X$ is a $G$-equivariant morphism. Then we have a $G^{(\gamma)}$-equivariant morphism

$$
Y \to X^{(\gamma)}
$$
given by composing the morphism \( Y(\gamma) \to X(\gamma) \) constructed in Lemma 6.5 with the section of \( Y(\gamma) \to Y \) constructed in Lemma 6.6. In particular we have an induced morphism of stacks

\[ [G_{(\gamma)} \backslash Y] \to [G_{(\gamma)} \backslash X(\gamma)] \]

**Definition 6.8.** The stack \([G_{(\gamma)} \backslash X^{(\gamma)}]\) is called the **rigidified inertia stack** of \([G \backslash X]\).

Given a conjugacy class \( \gamma \), the closed and open substack \([G_{(\gamma)} \backslash X(\gamma)]\) of \([G_{(\gamma)} \backslash X^{(\gamma)}]\) is called the \( \gamma \)-**twisted sector** of \([G \backslash X]\).

**Lemma 6.9.** The quotient \( X(\gamma) \to [G_{(\gamma)} \backslash X(\gamma)] \) is a morphism of degree \( o(G) o(m) / o(m) \) for any \( m \in \gamma \).

**Proof.** The diagram

\[
\begin{array}{ccc}
X(\gamma) & \longrightarrow & \gamma \\
\downarrow & & \downarrow \\
[G_{(\gamma)} \backslash X(\gamma)] & \longrightarrow & [G_{(\gamma)} \backslash \gamma]
\end{array}
\]

is a cartesian, therefore, the degree of the morphism is equal to the degree of the quotient \( \gamma \to [G_{(\gamma)} \backslash \gamma] \) which is

\[ o(\gamma) o(Z_G(m)) o(G) / o(m) = o(G) / o(m). \]

\[ \square \]

### 6.4 A note on evaluation morphisms

Suppose that \( E \to T \) is a \( G \)-curve over \( T \) with an admissible and balanced action, and suppose that \( \Sigma \subseteq E \) is a twisted section, which means that the morphism \( \Sigma \to T \) is étale and \( \Sigma/G \to T \). Notice that the monodromy morphism \( \Sigma \to G \) is an anchor map (see Lemma 6.2) for the action of the groupoid \( G_{(\Sigma)} \) on \( \Sigma \). This action has no fixed points, and the stack quotient \([G_{(\Sigma)} \backslash \Sigma]\) is isomorphic to \( T \). Suppose that \( f : E \to X \) is a \( G \)-equivariant morphism. By Proposition 6.7, there is natural morphism

\[ \Sigma \to X^{(G)} \]

into the inertia variety, and a morphism

\[ T \simeq [G_{(\Sigma)} \backslash \Sigma] \to [G_{(\Sigma)} \backslash X^{(G)}] \]

into the rigidified inertia stack.
Chapter 7

Presentation of the moduli space of admissible 3-marked genus 0 curves

The GW type integrals over $\overline{M}_3^{G,m}(X,\beta)$ do not have good properties analogous to the ordinary 3-pointed genus 0 GW-invariants. To get the GW type invariants that are structure constants of a ring, we will construct a compactification of the moduli space of maps from a canonical marked $G$-cover of $\mathbb{P}^1$ ramified over 0, 1 and $\infty$ associated to any triple of elements of $G$ whose product is the identity.

In what follows we will use the notation: $\overline{M}_3^G := \overline{M}_{0,\{0,1,\infty\}}$ and $\overline{M}_3^G(X,\beta) = \overline{M}_{0,\{0,1,\infty\}}(X,\beta)$.

7.1 Canonical element of $\overline{M}_3^G$

To any triple $m = (m_0, m_1, m_\infty) \in G^3$ such that $m_0 m_1 m_\infty = \text{id}$, we construct a canonical object of $\overline{M}_3^{G,m}$ over Spec $\mathbb{C}$, as the quotient of the cartesian product of $G$ with the upper half plane by an appropriate action of a free group on 2 generators (the fundamental group of $\mathbb{P}^1 \setminus \{0,1,\infty\}$).

Let $\mathbb{H} := \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$ be the upper half plane in $\mathbb{C}$, endowed with the hyperbolic metric. It is a Riemann surface isomorphic to the unit disk. The holomorphic automorphism group of $\mathbb{H}$ is $\text{PSL}(2, \mathbb{R})$, and acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Let $\mathbb{H} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. The action of the subgroup $\text{PSL}(2, \mathbb{Q})$ of $\text{PSL}(2, \mathbb{R})$ extends naturally to an action on $\mathbb{H}$ given by the same formula (with the obvious convention about dividing by 0, and acting on $\infty$). The space $\mathbb{H}$ has the coarsest topology such that the inclusion $\mathbb{H} \to \mathbb{H}$ is continuous, the sets $\mathbb{H}_r := \{ z \in \mathbb{H} \mid \Im(z) > r \} \cup \{\infty\}$ for $r > 0$ are open, and $\text{PSL}(2, \mathbb{Q})$ acts by continuous morphisms. Note that the action of the subgroup $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{Q} \cup \{\infty\}$ is transitive.

Let $\Gamma(2)$ be the principal congruence subgroup of level 2, which is defined as the kernel

$$1 \to \Gamma(2) \to \text{PSL}(2, \mathbb{Z}) \to \text{PSL}(2, \mathbb{Z}/2\mathbb{Z}) \to 1$$

of the morphism reducing the matrix coefficient modulo 2. The reason we consider this group is that the quotient $\Gamma(2) \backslash \mathbb{H}$ is conformally isomorphic to $\mathbb{P}^1 \setminus \{0,1,\infty\}$, and the quo-
Chapter 7. Presentation of the moduli space of admissible 3-marked genus 0 curves

Consider the 3 orbits of $\Gamma(2)$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$:

\[ \mathcal{O}_0 := \left\{ \begin{array}{ll} \text{even} & \text{odd} \in 2\mathbb{Z} + 1, \text{even} \in 2\mathbb{Z} \\ \text{odd} & \text{odd} \in 2\mathbb{Z} + 1 \end{array} \right\} \]

\[ \mathcal{O}_1 := \left\{ \begin{array}{ll} \text{odd}_1 & \text{odd}_2 \in 2\mathbb{Z} + 1 \end{array} \right\} \]

\[ \mathcal{O}_\infty := \left\{ \begin{array}{ll} \text{odd}_2 & \text{even} \in 2\mathbb{Z} + 1, \text{even} \in 2\mathbb{Z} \setminus \{0\} \cup \{\infty\} \end{array} \right\} \]

Hence after composing with an appropriate automorphism of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ we can assume that

\[ u : \mathbb{H} \to \mathbb{P}^1 \]

\[ 0 \mapsto 0 \]

\[ 1 \mapsto 1 \]

\[ \infty \mapsto \infty. \]

For each $i \in \{0, 1, \infty\}$ denote by $\Gamma_i(2)$ the stabilizer of $i$ in $\Gamma(2)$. These are cyclic groups generated by the parabolic transformations

\[ \alpha_0 := \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \in \Gamma_0(2), \quad \alpha_1 := \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \in \Gamma_1(2), \quad \alpha_\infty := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \in \Gamma_\infty(2). \]

These are chosen so that $\alpha_0 \alpha_1 \alpha_\infty = \text{id}$. The group $\Gamma(2)$ is a free group generated by any two elements of $\{\alpha_0, \alpha_1, \alpha_\infty\}$. The automorphisms $\alpha_0, \alpha_1$ and $\alpha_\infty \in \Gamma(2)$ are called the monodromies of respectively 0, 1 and $\infty$. This terminology is justified by the following construction.

Let $G$ be a finite group, and let $m_i \in G$ for $i \in \{0, 1, \infty\}$ be such that $m_0 m_1 m_\infty = \text{id}$. We will construct a canonical object $\varepsilon_m = (\mathcal{E}_m, \Sigma_m, \sigma_m)$ of $\mathcal{M}_3^{G,m}$. Let $\theta_m : \Gamma(2) \to G$ be the homomorphism given by $\theta_m(\alpha_i) = m_i$ for $i \in \{0, 1, \infty\}$. Denote by $G^m$ the right $\Gamma(2)$-space $G$ with structure given by $\theta_m$. The group $\Gamma(2)$ acts on $G^m \times \mathbb{H}$ by $\alpha \cdot (g, x) = (g \theta_m(\alpha)^{-1}, \alpha \cdot x)$, and let $E_m = G^m \times_{\Gamma(2)} \mathbb{H}$ be the quotient by this action. Denote by $[g, \tau]$ the image of the pair $(g, \tau) \in G^m \times \mathbb{H}$ in $E_m$, and let $u_m : \mathbb{H} \to E_m$ be the composition of the inclusion $\mathbb{H} \to G^m \times \mathbb{H}$, $x \mapsto (id, x)$ with this quotient. For $i \in \{0, 1, \infty\}$, let $\sigma_{m,i} := u_m(i) = [id, i] \in E_m$, and let $\Sigma_m := G^m \times_{\Gamma(2)} \mathcal{O}_i$.

**Lemma 7.1.** The triple $\varepsilon_m := (E_m, \Sigma_m, \sigma_m)$ is an object of $\mathcal{M}_3^{G,m}$.

**Proof.** We need to verify the conditions in the Definition 2.4 in Chapter 2. Condition (a) is trivial as $E_m$ is smooth. Condition (b) follows from the fact that $\Gamma(2)$ acts freely on $\mathbb{H}$ and that the union of the three orbits $\mathcal{O}_0, \mathcal{O}_1$ and $\mathcal{O}_\infty$ is the complement of $\mathbb{H}$ in $\mathbb{H}$. For (c) we need the quotient of $E_m$ by $G$ to be $\mathbb{P}^1$ which follows from the fact that the quotient of $\mathbb{H}$ by the action of $\Gamma(2)$ is $\mathbb{P}^1$. That $\Sigma_m$ is a $G$-orbit of $\sigma_m$ follows from the fact that $\mathcal{O}_i$ is a $\Gamma(2)$-orbit. Hence we only need to show that the monodromy of $\sigma_{m,i}$ in $E_m$ is $m_i = \theta_m(\alpha_i)$ for each $i \in \{0, 1, \infty\}$. 

Chapter 7. Presentation of the moduli space of admissible 3-marked genus 0 curves

We start with \( i = \infty \). Consider the open set \( \mathbb{H}_1 \) (see Section 7.1). It is an \( \alpha_\infty \)-invariant neighborhood of \( \infty \). Moreover \( \alpha_\infty \cdot \mathbb{H}_1 \cap \mathbb{H}_1 = \emptyset \). It follows that the image of \( \mathbb{H}_1 \) in \( E_m \) is isomorphic to the quotient \( \left( \frac{\alpha_\infty(m \infty)}{\mathbb{H}_1} \right) \). But \( \left( \frac{\alpha_\infty(m \infty)}{\mathbb{H}_1} \right) \) is \( \Gamma(2) \)-equivariently isomorphic to an open disk on which \( \alpha_\infty \) acts by multiplication by \( \exp \left( \frac{2\pi i}{\alpha_\infty(m \infty)} \right) \). This isomorphism is given by \( \tau \mapsto \exp \left( \frac{2\pi i}{\alpha_\infty(m \infty)} \right) \) for \( \tau \neq \infty \) and \( \infty \mapsto 0 \). The result follows.

The case \( i = 0 \) (resp. \( i = 1 \)) follows from the case \( i = \infty \) by applying the automorphism \( z \mapsto \frac{z}{\tau} \) (resp. \( z \mapsto \frac{-z}{\tau} \)) to \( \mathbb{H} \). It sends 0 (resp. 1) to \( \infty \) and conjugates \( \alpha_0 \) (resp. \( \alpha_1 \)) into \( \alpha_\infty \).

\[ \square \]

7.2 Canonical presentation of \( \overline{M}_3^G \)

Recall that the monodromies at the markings give rise to 3 maps \( \mu_i : \overline{M}_3^G \rightarrow G \), for \( i \in \{0, 1, \infty\} \). Translation of the markings gives rise to 3 actions of \( G \) on \( \overline{M}_3^G \), and since the monodromy of a marking stabilizes it, by Lemma 6.2 we have 3 actions of the groupoid \( G(\mathbb{G}) \) on \( \overline{M}_3^G \) over \( \mathbb{G} \) (where the anchor \( \mathbb{G} \rightarrow \mathbb{G} \) for each action is the monodromy morphism \( \mu_i \) corresponding to the marking being translated). Let \( G^2(\mathbb{G}) \) act on \( \overline{M}_3 \) over \( \mathbb{G} \) by translation of the markings indexed by 0 and 1.

Consider the morphism \( G^2 \rightarrow \overline{M}_3^G \), given by \( (m_0, m_1) \mapsto e_m \) where \( m_\infty = (m_0m_1)^{-1} \).

For simplicity, denote the element \( ([g_0, m_0], [g_1, m_1]) \) of \( G^2(\mathbb{G}) \) by \([g, m] \). We have a morphism

\[ e_- : G^2(\mathbb{G}) \rightarrow \overline{M}_3^G \]

\[ [g, m] \mapsto e_{[g, m]} := g \cdot e_m \]

where \( g \cdot e_m \) is the translation of the marking 0 by \( g_0 \) and the marking 1 by \( g_1 \).

We will show that \( e_- : G^2(\mathbb{G}) \rightarrow \overline{M}_3^G \) is a stack quotient by a right action of the group \( \mathbb{G} \) over \( \mathbb{G} \). The anchor for this action is given by \( G^2(\mathbb{G}) \rightarrow \mathbb{G}, [g, m] \mapsto m_0m_1 \), and the action is given by

\[ (([g_0, m_0], [g_1, m_1]) \cdot a = ([g_0 a, a^{-1}m_0 a], [g_1 a, a^{-1}m_1 a]) \]

for \( a \in (m_\infty) \). This action preserves the anchor map, and hence is an action of a group over \( \mathbb{G} \). Now let \( \epsilon_a : e_{[g, m]} \rightarrow e_{[g, m] \cdot a} \) be the arrow of \( \overline{M}_3^G \) given by the \( G \)-equivariant isomorphism of \( G \)-curves

\[ E_{e_{[g, m] \cdot a}} = G^m \times \Gamma(2) \mathbb{H} \rightarrow G^{a^{-1}ma} \times \Gamma(2) \mathbb{H} = E_{e_{[g, m]} \cdot a} \]

\[ [h, \tau] \mapsto [ha, \tau] \]

Note that this is a well defined isomorphism as \( \theta_{a^{-1}ma} = a^{-1}\theta_{ma} \).

**Proposition 7.2.** The functor \( e_- \) is an isomorphism of groupoids

\[ \left[ G^2(\mathbb{G})/\mathbb{G} \right] \simeq \overline{M}_3^G \]
Chapter 7. Presentation of the moduli space of admissible 3-marked genus 0 curves

Proof. First we prove the essential surjectivity. Let \( e \) be an object of \( \mathcal{M}_3^G \). Let \( E_\infty \) be the connected component of \( \sigma_{e,\infty} \) in \( E_\infty \). We can assume that \( E_\infty \) is a branched cover of \( \mathbb{P}^1 \) with ramification points over \( 0, 1 \) and \( \infty \). Let \( u_\infty : \mathbb{H} \to E_\infty \) be the completion of the lift of the universal cover \( u : \mathcal{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \). After acting by an automorphism of \( \mathbb{H} \) we can assume that \( u_\infty(\infty) = \sigma_{e,\infty} \). Let \( m_i \) be the monodromy of \( u_\infty(i) \) for \( i \in \{0, 1, \infty\} \). For \( i \in \{0, 1\} \), there is a unique \( g_i \in G \) such that \( g_i \cdot u_\infty(i) = \sigma_{e,i} \).

We have an element \( ([g_0, m_0], [g_1, m_1]) \) of \( G^2 \). I claim that \( e \cdot [g, m] = e \). The morphism \( u_\infty : \mathbb{H} \to E_\infty \) induces \( v_\infty : E_{g^{-1}m} := G^m \times \Gamma(2) \mathbb{H} \to E_\infty \) given by \( v_\infty([g, \tau]) = g \cdot u_\infty(\tau) \). This is injective as \( \Gamma(2) \) acts transitively on the intersection of any \( G \)-orbit on \( E_\infty \) with the connected component \( E_\infty^{\text{loc}} \). It is surjective as any point of \( E_\infty \) can be translated by an element of \( G \) into \( E_\infty^{\text{loc}} \). Moreover \( v_\infty([g, m], i) = v_\infty([g, m], i) = g \cdot u_\infty(i) = \sigma_{e,i} \) for \( i \in \{0, 1, \infty\} \). This shows that \( v_\infty \) is the required isomorphism.

We now prove the full faithfulness. Let \( \psi \) be an isomorphism between \( \epsilon_1 := \epsilon_{[g, m]} \) and \( \epsilon_2 := \epsilon_{[h, n]} \). By definition we have maps \( u_i : \mathbb{H} \to E_i \) such that \( \psi \circ u_1 = \psi \circ u_2 \) where \( p_{\epsilon,\infty} \circ u_1 = p_{\epsilon,\infty} \circ u_2 \) where \( p_{\epsilon,\infty} \) is the quotient morphism \( E_\infty \to \mathbb{P}^1 \). As \( \psi \) is a \( G \)-isomorphism \( E_\infty \to \mathbb{P}^1 \), there exits a unique (and hence we have the faithfulness) \( a \in G \) such that \( \psi(u_1(\tau)) = a \cdot u_2(\tau) \) for all \( \tau \in \mathbb{H} \). The morphism \( \psi \) sends markings onto markings: \( \psi(\sigma_{e,i}) = \sigma_{e_2,i} \). For \( i = \infty \) we have \( \psi(u_1(\infty)) = a \cdot u_2(\infty) = u_2(\infty) \), which implies that \( m_\infty = n_\infty \) and that \( a = (n_\infty) \).

For \( i \in \{0, 1\} \) we have
\[
\psi(g \cdot u(i)) = g \cdot \psi(u(i)) = g \cdot a \cdot u_2(i) = h \cdot u_2(i)
\]
and it follows that \( g \cdot m \cdot g^{-1} = h \cdot n \cdot h^{-1} \) and \( h^{-1} g \cdot a = (n_i) \), which shows \( [g, m] \cdot a = [h, n] \).

Hence the full faithfulness.

7.3 Compactification of the moduli space of \( G \)-equivariant morphisms from a \( G \)-cover of \( \mathbb{P}^1 \) with 3 ramifications

Denote by \( \mathcal{N}_3^G \) the subset of \( G^3 \) of triples \((m_0, m_1, m_\infty)\) such that \( m_0 m_1 m_\infty = \text{id} \). Given \( m \in \mathcal{N}_3^G \), we consider the scheme of \( G \)-equivariant morphism \( \mathcal{N}_3^{G, m}(X) := \text{Mor}^G_{\mathcal{M}_3^G}(X, X) \) from \( E_m \) to \( X \), and the subschemes \( \mathcal{N}_3^{G, m}(X, \beta) \) of morphism of a given degree \( \beta \in \text{A}^+ G \). We are looking for a compactification of \( \mathcal{N}_3^{G, m}(X, \beta) \) such that the 3 evaluation maps extends. Notice that \( \mathcal{N}_3^{G, m}(X, \beta) \) is the fiber of the forgetful morphism \( \mathcal{M}_3^{G, m}(X, \beta) \to \mathcal{M}_3^{G, m} \) above \( \epsilon_m \). Hence the fiber above \( \epsilon_m \) of the composition \( \mathcal{M}_3^{G, m}(X, \beta) \to \mathcal{M}_3^{G, m} \) (where the second morphism is the stabilization morphism) is proper and contains \( \mathcal{N}_3^{G, m}(X, \beta) \) as a dense open substack. The evaluation morphisms extend to the morphisms induced by the evaluation morphisms of \( \mathcal{M}_3^{G, m}(X, \beta) \), and hence it is the desired compactification.
Definition 7.3. Define $\mathcal{N}_3^G(X, \beta)$ (resp. $\mathcal{M}_3^{3,G}(X, \beta)$), by the cartesian diagram

\[
\begin{array}{ccccccccc}
\mathcal{N}_3^G(X, \beta) & \longrightarrow & \mathcal{M}_3^{3,G}(X, \beta) & \longrightarrow & \mathcal{M}_3^G(X, \beta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_3^G & \longrightarrow & G^2_G & \longrightarrow & \mathcal{M}_3^G
\end{array}
\]

where $\mathcal{N}_3^G \to G^2_G$ is given by $(m_0, m_1, m_\infty) \to ([\text{id}, m_0], [\text{id}, m_1])$.

Define the virtual fundamental classes $[\mathcal{N}_3^G(X, \beta)]_{\text{vir}}$ (resp. $[\mathcal{M}_3^{3,G}(X, \beta)]_{\text{vir}}$) of $\mathcal{N}_3^G(X, \beta)$ (resp. $\mathcal{M}_3^{3,G}(X, \beta)$) by pulling back the virtual fundamental class of $\mathcal{M}_3^G(X, \beta)$.

Lemma 7.4. Consider the left $G^2_G$ action on $\mathcal{M}_3^{3,G}(X, \beta)$ by translation of the two first sections. Then the composition $\mathcal{N}_3^G(X, \beta) \to \mathcal{M}_3^{3,G}(X, \beta) \to [G^2_G \backslash \mathcal{M}_3^{3,G}(X, \beta)]$ is an isomorphism.

Proof. The map $G^2 \to G^2_G$ is a cross section of the groupoid $G^2_G$ over $G^2$, that is it induces an isomorphism $G^2 \times_{G^2} G^2 \to G^2_G \times_{G^2} G^2_G \to G^2_G$ where the second map is the product map of the groupoid. It follows that the composition $G^2 \to G^2_G \to [G^2_G \backslash G^2_G]$ is an isomorphism, and the result follows from the definitions. \[\square\]

7.4 Inflation

Suppose $H$ is a subgroup of $G$. To distinguish between the groups $H$ and $G$, let $\epsilon^H$ and $\epsilon^G$ denote the morphism $\epsilon$ of the last section corresponding to the given group. For $m \in \mathcal{N}_3^H$, the inflation of $\epsilon^H_m$ is naturally isomorphic to $\epsilon^G_m$. It follows that the inclusion of $H$ into $G$ induces inclusions $\mathcal{M}_3^{3,H} \to \mathcal{M}_3^{3,G}$ and $\mathcal{N}_3^H \to \mathcal{N}_3^G$ such that the diagram

\[
\begin{array}{ccccccccc}
\mathcal{N}_3^H & \longrightarrow & \mathcal{N}_3^G \\
\downarrow & & \downarrow \\
\mathcal{M}_3^{3,H} & \longrightarrow & \mathcal{M}_3^{3,G} \\
\downarrow & & \downarrow \\
\mathcal{N}_3^H & \underset{\text{Inf}_H G}{\longrightarrow} & \mathcal{M}_3^G
\end{array}
\]

is cartesian. We pullback the definition of inflation to stack of $H$-maps.
Lemma 7.5. We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{N}_3^G(X) & \rightarrow & \mathcal{M}_3^G(X) \\
\mathcal{N}_3^H(X) & \rightarrow & \mathcal{M}_3^H(X) \\
\mathcal{N}_3 & \rightarrow & \mathcal{M}_3 \\
\end{array}
\]

where all faces are cartesian.

Let \( ev_G^N : \mathcal{N}_3^G(X) \rightarrow (X^{(G)})^3 \) be the composition of \( \mathcal{N}_3^G(X) \rightarrow \mathcal{M}_3^G(X) \) with the product of the three evaluation maps \( ev_i : \mathcal{M}_3^G(X) \rightarrow X^{(G)} \).

Proposition 7.6. Given \( \alpha \in A_3^+(X)^G \), we have a cartesian diagram

\[
\begin{array}{ccc}
\coprod_{\beta} \mathcal{N}_3^H(X,\beta) & \rightarrow & \mathcal{N}_3^G(X,\alpha) \\
\downarrow ev_H^N & & \downarrow ev_G^N \\
(X^{(H)})^3 & \rightarrow & (X^{(G)})^3
\end{array}
\]

where the disjoint union is over all \( \beta \in A_3^+(X)^H \) such that \( \alpha = \sum_{g \in G/H} g \cdot \beta \), is cartesian.

\textbf{Proof.} The result follows by cartesian diagram chasing from Lemma 7.5, Corollary 5.13 and the following two cartesian diagrams

\[
\begin{array}{ccc}
(X^{(H)})^3 & \rightarrow & (X^{(G)})^3 \\
\downarrow & & \downarrow \\
H^3 & \rightarrow & G^3
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{N}_3^H & \rightarrow & \mathcal{N}_3^G \\
\downarrow & & \downarrow \\
H^3 & \rightarrow & G^3
\end{array}
\]
Chapter 8
Group quantum cohomology

In this section we first review the definition of group cohomology of [FG03]. Then we define the algebra of invariant quantum parameters, and build the group quantum cohomology as the tensor product of this algebra with the group cohomology. We define a product on the group quantum cohomology using GW type integrals over the moduli stack constructed in Chapter 7. We then prove a functoriality property of the group quantum cohomology under the inclusion of groups with compatible actions on the variety $X$.

8.1 Group cohomology

The following is an overview of [FG03]. Let $G$ be a finite group acting on a smooth projective variety $X$. We first define the “cohomology” group.

**Definition 8.1.** Using the intersection theory with coefficients in $\mathbb{Q}$, define the $\mathbb{Q}$-module


It is a left $G$-module under the action given by

$$ g \cdot \sigma := (g^{-1})^* \sigma $$

for $g \in G$ and $\sigma \in A^* (X^G)$. Let $A^* (X, G)_m$ be the direct summand of $A^* (X, G)$ corresponding to the closed and open subvariety $\{ m \} \times X^m \subseteq X^G$. Give $A^* (X, G)_m$ a $\mathbb{Q}$-grading as follows: given $c \in \mathbb{Q}$, let

$$ A^c (X, G)_m := \bigoplus_{Z \subseteq X^m} A^{a(m, Z, X)} (Z) \subseteq A^* (X, G)_m $$

where $Z$ runs over the connected components of $X^m$, and where $a(m, Z, X)$ was defined in Section 4.2.

**Lemma 8.2.** The $\mathbb{Q}$-grading on $A^* (X, G)$ is invariant under the $G$-action.

**Proof.** To show that the $\mathbb{Q}$-grading is $G$-invariant, it suffices to verify that $m(m, j, T_p X) = m(gmg^{-1}, j, T_{g p} X)$ for all $g \in G$, which is a consequence of the fact that $T X$ is a $G$-vector bundle.

Suppose $\phi : H \to G$ is a homomorphism of groups. Then $\phi$ induces an action of $H$ on $X$. Define

$$ A^* (X, \phi) : A^* (X, H) \to A^* (X, G) $$

as the morphism of $H$-modules induced by pushing forward under the open and closed $H$-equivariant morphism $X^H \to X^G$.

**Proposition 8.3.** This defines a functor $A^* (X, -)$ from the category of groups acting on $X$ and homomorphisms compatible with the actions to the category of $\mathbb{Q}$-graded $\mathbb{Q}$-modules.
8.2 Quantum parameters

Let $G$ be a finite group acting on a smooth projective variety $X$.

**Definition 8.4.** Denote by $Q^*(X, G)$ the semi-group algebra of the semi-group of $G$-invariant effective curve classes $A^+_+(X)^G$, that is the algebra on the symbols $q^G_\beta$ with $\beta \in A^+_+(X)^G$, and relations $q^G_\beta q^G_\alpha = q^G_{\beta + \alpha}$ for $\alpha, \beta \in A^+_+(X)^G$. Denote $Q^*(X, G)$ by $Q^*(X)$ when $G$ is trivial, and its generators by $q^\beta$ with $\beta \in A^+_+(X)$. The algebra $Q^*(X, G)$ is $Q$-graded by

$$\deg q^G_\beta := \frac{1}{|\Gamma|} \beta(c_1 T_X).$$

where $T_X$ is the tangent sheaf of $X$.

Suppose $H$ is another finite group, and $\phi : H \to G$ is a homomorphism of groups. Then $\phi$ induces an action of $H$ on $X$. Define

$$Q^*(X, \phi)(q^H_\gamma) = q^{1/\ker \phi} \sum_{\gamma' \in \phi(H)} \epsilon^{\gamma,\beta}$$

which induces a homomorphism of algebras

$$Q^*(X, \phi) : Q^*(X, H) \to Q^*(X, G).$$

This is a graded homomorphism as $c_1(T_X)$ is $G$-invariant and $T_X$ is a $G$-sheaf.

**Proposition 8.5.** This defines a functor $Q^*(X, \phi)$ from the category of groups acting on $X$ and homomorphisms compatible with the actions to the category of $Q$-graded algebras.

8.3 Construction of the group quantum cohomology

**Definition 8.6.** The **group quantum cohomology** is the $Q$-graded $Q[\Gamma]$-module defined as the tensor product

$$qA^*(X, G) := Q^*(X, G) \otimes_Q A^*(X, G)$$

where $G$ acts trivially on $Q^*(X, G)$.

Given a homomorphism of groups $\phi : H \to G$, the homomorphisms $Q^*(X, \phi)$ and $A^*(X, \phi)$ induce a homomorphism of $Q$-graded $Q[H]$-modules

$$qA^*(X, \phi) : qA^*(X, H) \to qA^*(X, G).$$

Let $\iota$ be the involution of $X^G$ sending the component $\{g\} \times X^g$ isomorphically onto $\{g^{-1}\} \times X^{g^{-1}}$, using the identification $X^g = X^{g^{-1}}$.

Consider the diagram

$$
\begin{array}{ccc}
N^G_2(X, \beta) & \xrightarrow{\text{ev}} & (X^G)^{\{0,1,\infty\}} \\
\downarrow \pi_{G,i} & & \\
X^G & & 
\end{array}
$$
where \( \pi_G, i \) is the projection on the \( i \)th factor for \( i \in \{0,1,\infty\} \). Given \( \rho_i \in A^*(X,G) \) for \( i \in \{0,1\} \), define
\[
\rho_0 \ast_G \rho_1 := \sum_{\beta \in \Lambda^+_G(X)^G} q^\beta_G \left( \pi_G,0 \rho_0 \cdot \pi_G,1 \rho_1 \cdot (ev_0^G) \ast \left[ N_3^G(X,\beta) \right]^{vir} \right).
\]

**Lemma 8.7.** If \( \beta \neq 0 \), then the morphism \( f : \overline{M}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \to \overline{M}_2^{G,\text{id},G,\beta^{-1}}(X,\beta) \) forgetting the first marking is the universal \( G \)-curve over \( \overline{M}_2^{G,\text{id},G,\beta^{-1}}(X,\beta) \).

**Proof.** This result follows from a widely accepted result of Abramovich and Vistoli (it appears without proof in [AV02]): the stack of twisted stable maps \( \overline{M}_2([X/G],\beta) \) (where \([X/G]\) is the quotient stack) has the universal curve identified with a stack that they denote by \( \overline{M}_3([X/G],\beta) \). It turns out that this stack pulls back to \( A^*(X,\beta) \) under the morphism \( \overline{M}_3([X/G],\beta) \to \overline{M}_2([X/G],\beta) \) that we will construct in Section 8.7. The result follows from the fact that the universal \( G \)-curve over \( \overline{M}_2([X/G],\beta) \) pulls back to the universal \( G \)-curve over \( \overline{M}_2^{G,\text{id},G,\beta^{-1}}(X,\beta) \).

**Proposition 8.8.** This makes \( qA^*(X,G) \) into a \( \mathbb{Q} \)-graded algebra with identity.

**Proof.** First of all we prove that the product preserves the \( \mathbb{Q} \)-grading. This follows from the computation of the dimension of the virtual fundamental class at \( f \):
\[
\dim X + \frac{1}{o(G)} \deg f^*_T X - a(m_0,f_1(\sigma_{1,0}),X) - a(m_1,f_1(\sigma_{1,1}),X) - a(m_\infty,f_1(\sigma_{1,\infty}),X)
= \frac{1}{o(G)} \deg f^*_T X - a(m_0,f_1(\sigma_{1,0}),X) - a(m_1,f_1(\sigma_{1,1}),X)
+ a(m_\infty,f_1(\sigma_{1,\infty}),X) + \dim f_1(\sigma_{1,\infty}) X^{m_\infty}
\]
by using the formula \( a(g,x,X) + a(g^{-1},x,X) = \dim X - \dim_x X^g \) (see [FG03]).

We prove now that the identity class on the non-twisted sector \( \text{id} \in A^*(X) \) is the identity of \( qA^*(X,G) \). Let \( \sigma \in qA^*(X,G) \) be a class supported on the sector \( X^g \) for some \( g \in G \). The coefficient of \( q^\beta_G \) in the product \( \text{id} \ast \sigma \) is by definition
\[
(\iota \circ ev_3)_* (ev_2^* \sigma \cdot \left[ N_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \right]^{vir})
= (\iota \circ \pi_2)_* (\pi_1^* \sigma \cdot \left[ N_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \right]^{vir})
\]
where \( ev = ev_2 \times ev_3 : N_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \to X^g \times X^{g^{-1}} \) and \( \pi_1 : X^g \times X^{g^{-1}} \to X^g \) and \( \pi_2 : X^g \times X^{g^{-1}} \to X^{g^{-1}} \) are the projections. First notice that \( \overline{N}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) = \overline{M}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \). We will show that if \( \beta \neq 0 \), \( ev_* \left[ \overline{N}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \right] = 0 \). First we note that the morphism \( ev \) factors through the morphism \( f : \overline{M}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \to \overline{M}_2^{G,\text{id},G,\beta^{-1}}(X,\beta) \) forgetting the first marking. As this morphism is proper of pure codimension 1 by Lemma 8.7, we have \( f_* \left[ \overline{M}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \right] = 0 \) which implies that \( \overline{ev}_* \left[ \overline{N}_3^{G,\text{id},G,\beta^{-1}}(X,\beta) \right] = 0 \).

If \( \beta = 0 \), then \( \overline{N}_3^{G,\text{id},G,\beta^{-1}}(X,0) = X^g \) and the above product is just the product by the identity on \( A^*(X^g) \). The proposition follows.
8.4 G-graded G-algebra and the braid group action

Definition 8.9. A G-algebra $A$ is called G-graded if there is a decomposition $A = \bigoplus_{g \in G} A^g$ compatible with the G-action in the sense that for all $g$ and $h \in G$

1. $h \cdot A^g \subseteq A^{gh^{-1}}$ and
2. $A^g A^h \subseteq A^{gh}$.

It is G-graded commutative if

$$
\sigma_g \sigma_h = (g \cdot \sigma_h) \sigma_g = \sigma_h h^{-1} \cdot \sigma_g.
$$

for all $\sigma_g \in A^g$ and $\sigma_h \in A^h$.

We will show that the group quantum cohomology is a G-graded G-algebra. That it is G-graded commutative is related to the fact that we can lift the action of the symmetric group on $\mathcal{M}_3$ by permutation of the markings to an action of the braid group on $G^2$. We first lift the $S_3$-action to a braid group action on $\mathcal{M}_3^G$.

Let the symmetric group $S_3$ act on $\mathcal{M}_3^G$ by permuting the markings. In general, this action doesn't lift to an action on $G^2$, but a modification lifts to an action of the braid group $B_3$ on two generators $b_{01}$ and $b_{1\infty}$. This $B_3$-action on $\mathcal{M}_3^G$ and more generally on $\mathcal{M}_3^G(X)$ is defined by

$$
b_{01} := p_{01} \circ t_1(\mu_0)
$$

$$
b_{1\infty} := p_{1\infty} \circ t_\infty(\mu_1)
$$

where $p_{ij} \in S_3$ is the permutation of the markings $i$ and $j$, and $t_i(\mu_j)$ is the translation of the $i^{th}$ marking by the monodromy of the $j^{th}$ marking. It is straightforward to check the braid relation.

This action lifts to $G^2$ as follows. Let $b_{01}$ and $b_{1\infty}$ act on $G^3$ by

$$
b_{01} : (m_0, m_1, m_\infty) \mapsto (m_0 m_1 m_0^{-1}, m_0, m_\infty)
$$

$$
b_{1\infty} : (m_0, m_1, m_\infty) \mapsto (m_0, m_1 m_\infty m_1^{-1}, m_1)
$$

These two maps preserve $G^2 \subseteq G^3$ and respect the braid relation, inducing an action of $B_3$ on $G^2$.

Lemma 8.10. The morphism $\epsilon_- : \mathcal{N}_3^G \rightarrow \mathcal{M}_3^G$ is $B_3$-equivariant.

Proof. We will prove equivariance for the action of $b_{01}$, the case of $b_{1\infty}$ being similar. Let $m \in \mathcal{N}_3^G$ and $n = b_{01} \cdot m$. We need a $G$-equivariant isomorphism $\phi : E_m \cong E_n$ such that $\phi(\sigma_{m,0}) = \sigma_{n,1}$, $\phi(\sigma_{m,1}) = \sigma_{n,0}$ and $\phi(\sigma_{m,\infty}) = \sigma_{n,\infty}$. Recall that $E_m := G^m \times_{\Gamma(2)} \mathbb{H}$ and $E_n := G^n \times_{\Gamma(2)} \mathbb{H}$ (see Section 7.1). The above conditions are equivalent to $\phi([id,0]) = [id,1]$, $\phi([m_0,1]) = \phi([id,0 \cdot 1]) = [id,0]$ and $\phi([id,\infty]) = [id,\infty]$. Let

$$
\phi_0 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{PSL}(2, \mathbb{Z})
$$
and let $\phi : G^m \times \Gamma(2) \tilde{H} \to G^n \times \Gamma(2) \tilde{H}$ be the unique $G$-equivariant isomorphism such that $\phi([id, \tau]) = [id, \phi_0(\tau)]$. That $\phi$ is well defined is a consequence of the relations $\phi_0\alpha_0\phi_0^{-1} = \alpha_1$ and $\phi_0\alpha_1\phi_0^{-1} = \alpha_1^{-1}\alpha_0\alpha_1$. That it has the above properties is a consequence of the following calculation: $\phi_0(0) = 1$, $\phi_0\alpha_0(1) = \phi_0(-1) = 0$ and $\phi_0(\infty) = \infty$. 

This induces a unique $B_3$-action on $\mathcal{N}_3^G(X)$ such that the morphisms in the cartesian diagram

$$
\mathcal{N}_3^G(X) \xrightarrow{\varepsilon} \mathcal{M}_3^G(X)
$$

are $B_3$-equivariant.

**Lemma 8.11.** The class $[\mathcal{M}_3^G(X, \beta)]^{vir}$ is invariant under the action of $S_3$ by permutation of the markings.

**Proof.** Let $p$ be a permutation of the markings. The pull back of the universal curve over $\mathcal{M}_3^G(X, \beta)$ under $p$ is naturally isomorphic to itself, and this isomorphism commutes with the universal morphisms to $X$. It follows that the obstruction theory is compatible under $p$, and hence the virtual fundamental class.

The class $[\mathcal{N}_3^G(X, \beta)]^{vir}$ is invariant under the $B_3$-action as it is invariant under the action of $G$ by translation of any marking and under the action of $S_3$ by permutation of the markings. By $B_3$-equivariance of $\mathcal{N}_3^G(X, \beta) \to \mathcal{M}_3^G(X, \beta)$ it follows that $[\mathcal{N}_3^G(X, \beta)]^{vir}$ is $B_3$-invariant too.

The braid group $B_3$ has an action on $(X^G)^{(0,1,\infty)}$ given by

$$
b_{01} : ((m_0, x_0), (m_1, x_1), (m_\infty, x_\infty)) \mapsto ((m_0 m_1 m_0^{-1}, m_0 \cdot x_1), (m_0, x_0), (m_\infty, x_\infty))
$$

$$
b_{1\infty} : ((m_0, x_0), (m_1, x_1), (m_\infty, x_\infty)) \mapsto ((m_0, x_0), (m_1 m_\infty m_1^{-1}, m_1 \cdot x_\infty), (m_1, x_1))
$$

making the evaluation $\text{ev} : \mathcal{N}_3^G(X) \to (X^G)^{(0,1,\infty)}$ into a $B_3$-equivariant morphism.

Consider the following $G$-actions on $\mathcal{N}_3^G$, $\mathcal{M}_3^G$ and $\mathcal{M}_3^G(X)$. Given $g \in G$, and $m \in \mathcal{N}_3^G$ let $\delta_g \cdot m = (gm_0 g^{-1}, gm_1 g^{-1}, gm_\infty g^{-1})$. On $\mathcal{M}_3^G$ and more generally on $\mathcal{M}_3^G(X)$ the action is $\delta_g \cdot f = t_0(g) \circ t_1(g) \circ t_\infty(g) \cdot f$ for any object $f$ of $\mathcal{M}_3^G$ or $\mathcal{M}_3^G(X)$.

**Lemma 8.12.** The morphism $\varepsilon : \mathcal{N}_3^G \to \mathcal{M}_3^G$ is $G$-equivariant under the above actions.

**Proof.** Given $g \in G$ and $m \in \mathcal{N}_3^G$, we need a natural isomorphism $\delta_g \cdot \varepsilon_m \cong \varepsilon_{\delta_g \cdot m}$. This is given by the morphism $G^m \times \Gamma(2) \tilde{H} \to G^{\delta_g \cdot m} \times \Gamma(2) \tilde{H}$, $[h, \tau] \mapsto [hg^{-1}, \tau]$. 

As before, this induces a unique $G$-action on $\mathcal{N}_3^G(X)$ such that the morphisms to $\mathcal{N}_3^G$ and $\mathcal{M}_3^G(X)$ are $G$-equivariant. The virtual fundamental class of $\mathcal{N}_3^G(X, \beta)$ is invariant under this $G$-action because the virtual fundamental class of $\mathcal{M}_3^G(X, \beta)$ is invariant under any translation of the markings and because of the $G$-equivariance of the morphism $\mathcal{N}_3^G(X, \beta) \to \mathcal{M}_3^G(X, \beta)$. 

---

Chapter 8. Group quantum cohomology
Theorem 8.13. The group quantum cohomology $\text{qA}^*(X, G)$ is a $G$-graded commutative $G$-algebra.

Proof. To show that it is a $G$-algebra we need to show that the $G$-action is compatible with the product structure. Suppose $\sigma_0$ and $\sigma_1$ are two classes of $\text{A}^*(X, G)$. Then

$$g \cdot (\pi_{G,\infty})_* \left( \pi_{G,0}^* \sigma_0 \pi_{G,1}^* \sigma_1 \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

$$= (\pi_{G,\infty})_* \left( \delta_g \cdot \pi_{G,0}^* \sigma_0 \pi_{G,1}^* \sigma_1 \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

$$= (\pi_{G,\infty})_* \left( \pi_{G,0}^* (g \cdot \sigma_0) \pi_{G,1}^* (g \cdot \sigma_1) \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

by the invariance of the virtual fundamental class of $\mathcal{N}_3^G(X, \beta)$, where $\delta_g$ denotes the diagonal action of $g$ on $(X(G))^{0,1,\infty}$.

Suppose $\sigma_0$ (resp. $\sigma_1$) is supported on $\{m_0\} \times X_{m_0}^m$ (resp. $\{m_1\} \times X_{m_1}^m$), then the product $\sigma_0 \ast_G \sigma_1$ is supported on $\{m_0 m_1\} \times X_{m_0 m_1}^m$ because $(\text{ev}_N^*)^* \sigma_0 (\text{ev}_N^*)^* \sigma_1$ is supported on $\mathcal{N}_3^{G, m_0 m_1}(X, \beta)$ which maps into $\{m_0 m_1\} \times X_{m_0 m_1}^m$ under $i \circ \text{ev}_N^*$.

That this product is $G$-graded commutative is a consequence of the $B_3$-invariance of the virtual fundamental class of $\mathcal{N}_3^G(X, \beta)$ and the $B_3$-equivariance of $\text{ev}_N$: suppose $\sigma_i$ is supported on the twisted sector corresponding to $m_i$ for $i \in \{0, 1\}$, then

$$i \cdot (\pi_{G,\infty})_* \left( \pi_{G,0}^* \sigma_0 \pi_{G,1}^* \sigma_1 \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

$$= (\pi_{G,\infty})_* \left( \pi_{G,0}^* \sigma_0 \pi_{G,1}^* \sigma_1 \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

$$= (\pi_{G,\infty})_* \left( \pi_{G,0}^* \sigma_0 \pi_{G,1}^* \sigma_1 \text{ev}_N^* \left[ \mathcal{N}_3^G(X, \beta) \right] \right)$$

which shows that $\sigma_0 \ast_G \sigma_1 = \sigma_1 \ast_G m_1^{-1} \cdot \sigma_0$. The relation $\sigma_0 \ast_G \sigma_1 = (m_0 \cdot \sigma_1) \ast_G \sigma_0$ follows similarly by applying $b_{01}^*$ instead of $b_{01}$.

8.5 Inflation is a homomorphism

Theorem 8.14. For any injective homomorphism $\phi : H \to G$ of groups, the morphism $\text{qA}^*(X, \phi)$ is a homomorphism of Q-graded algebras.

Proof. We have to show that for $\rho_1 \in \text{A}^*(X, H)$, $\text{qA}^*(X, \phi)(\rho_1 \ast_H \rho_2) = \text{qA}^*(X, \phi)(\rho_1) \ast_G \text{qA}^*(X, \phi)(\rho_2)$. Fix $\alpha \in \text{A}^+_H(X)^G$. By Proposition 7.6

$$i! \left[ \mathcal{N}_3^G(X, \alpha) \right]^{\text{vir}} = \sum_{\beta \in A^+_H(X)^G \mid \text{Inf}_G^H \beta = \alpha} \left[ \mathcal{N}_3^G(X, \beta) \right]^{\text{vir}}.$$

where $i : (X(H))^3 \to (X(G))^3$ is the product of the inclusion $j : X(H) \to X(G)$. Since $\text{ev}_H$ is proper and $i$ is a regular embedding $(\text{ev}_H)_*i! = i!(\text{ev}_G)_*$ (see [Ful98] and [Vis89]). Now
we calculate:

\[ qA^*(X, \phi)(\rho_1 \ast_H \rho_2) = \]

\[ = \sum_{\beta \in A_1^+(X)^H} qA^*(X, \phi) \left\{ q_H^\beta (\ell \circ \pi_{H,3})_* \left( \pi_{H,1}^* \rho_1 \cdot \pi_{H,2}^* \rho_2 \cdot (ev_H)_* \left[ N_3^H (X, \beta) \right]^{vir} \right) \right\} \]

\[ = \sum_{\beta \in A_1^+(X)^H} q_H^{inf_{\beta}^G} A^*(X, \phi) \left\{ (\ell \circ \pi_{H,3})_* \left( \pi_{H,1}^* \rho_1 \cdot \pi_{H,2}^* \rho_2 \cdot (ev_H)_* \left[ N_3^H (X, \beta) \right]^{vir} \right) \right\} \]

\[ = \sum_{\alpha \in A_1^+(X)^G} q_G^0 A^*(X, \phi) \left\{ (\ell \circ \pi_{H,3})_* \left( \pi_{H,1}^* \rho_1 \cdot \pi_{H,2}^* \rho_2 \cdot (ev_H)_* \sum_{\beta \in A_1^+(X)^H} \left[ N_3^H (X, \beta) \right]^{vir} \right) \right\} \]

\[ = \sum_{\alpha \in A_1^+(X)^G} q_G^0 A^*(X, \phi) \left\{ (\ell \circ \pi_{H,3})_* \left( \pi_{H,1}^* \rho_1 \cdot \pi_{H,2}^* \rho_2 \cdot (ev_H)_* \left[ N_3^G (X, \alpha) \right]^{vir} \right) \right\} \]

\[ = \sum_{\alpha \in A_1^+(X)^G} q_G^0 A^*(X, \phi) \left\{ (\ell \circ \pi_{G,3})_* \left( \pi_{G,1}^* \rho_1 \cdot \pi_{G,2}^* \rho_2 \cdot (ev_G)_* \left[ N_3^G (X, \alpha) \right]^{vir} \right) \right\} \]

\[ = \sum_{\alpha \in A_1^+(X)^G} q_G^0 (\ell \circ \pi_{G,3})_* (\ell \circ \pi_{G,3})^* \left( \pi_{G,1}^* \rho_1 \cdot \pi_{G,2}^* \rho_2 \cdot (ev_G)_* \left[ N_3^G (X, \alpha) \right]^{vir} \right) \]

\[ = qA^*(X, \phi)(\rho_1) \ast_G qA^*(X, \phi)(\rho_2). \]

\[ \square \]

### 8.6 Degree 0 specialization and Fantechi-Göttsche group cohomology

In this section we prove that the degree 0 specialization of the group quantum cohomology is isomorphic to the group cohomology of Fantechi-Göttsche (see [FG03]).

Let \( m \in \mathcal{N}_3^G \), and suppose we have \( \sigma_i \in A^*(X, G) \) supported on \( \{ m_i \} \times X^{m_i} \) for \( i \in \{ 0, 1 \} \). We consider the coefficient of \( q_G^0 \) in the product \( q_0 \ast_G \sigma_1 \). Notice that

\[ N_3^{G, m}(X, 0) \simeq X^{m_0} \cap X^{m_1}, \]

and denote this intersection by \( X^m \). Let \( \pi : E_m \times X^m \to X^m \) be the universal curve over \( X^m \), and let \( f : E_m \times X^m \to X \) be the universal \( G \)-morphism. Then \( f \) is the unique \( G \)-equivariant morphism whose restriction \( f^* \) to \( E_m \times X^m \) (where \( E_m \) is the connected component of \( E_m \) containing all the markings \( \sigma_{m_i} \)) factors through the projection \( \pi^* : E_m \times X^m \to X^m \) and the natural inclusion \( \iota_m : X^m \to X \).

The obstruction theory on \( N_3^{G, m}(X, 0) \simeq X^m \) is \( R\pi^*_G (f^* \Omega_X \otimes \omega_f)[1] \) and its dual is \( R\pi^*_G f^* T_X \). The 0th cohomology of the above complex is just the tangent sheaf \( T_X \) of
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$X^m$ which is locally free as $X^m$ is smooth. It follows that $R^1\pi^G_* f^* T_X$ is locally free, and by [BF97][Proposition 5.6], the virtual fundamental class is given by

$$[\overline{\mathcal{M}}^G_{3, n} (X, 0)]^{vir} = c_r R^1\pi^G_* f^* T_X \cdot [X^m]$$

$$= (-1)^r c_r \pi^G_*(f^*\Omega_X \otimes \omega_{\pi^G}) \cdot [X^m]$$

where $r$ is the rank of $R^1\pi^G_* f^* T_X$. This is precisely the class defined in [FG03], and this proves the following result.

**Proposition 8.15.** Consider the $\mathbb{Q}$-graded ideal $qA^*(X, G)^+$ in $qA^*(X, G)$ generated by all $q G$ with $0 G A \sim l(X)$ such that $0 \neq 0$. Then there is an isomorphism

$$qA^*(X, G)/qA^*(X, G)^+ \simeq A^*(X, G)$$

of $\mathbb{Q}$-graded $G$-rings, where $A^*(X, G)$ has a ring structure defined in [FG03].

Note that the fiber of $R^1\pi^G_2 f^* T_X$ over a point $x \in X^m$ is $B. E^m G$, where $f^x : E^m G \rightarrow X$ is the unique $G$-equivariant morphism which is constant on $E^m G$ with value $x$.

The rank $r$ of $R^1\pi^G_* f^* T_X$ at $x$ can be computed using the equivariant Riemann-Roch formula:

$$r = \dim x X^m - \dim X + a(m_1, x, X) + a(m_2, x, X) + a(m_3, x, X).$$

8.7 Orbifold quantum cohomology

There is a natural inclusion of the orbifold quantum cohomology of the quotient stack into the group quantum cohomology. This inclusion turns out to be a homomorphism of $\mathbb{Q}$-graded algebras, and it identifies the orbifold quantum cohomology with the $G$-invariant part of the group quantum cohomology.

We review the definition of orbifold quantum cohomology (see [AV04]), which we will need only in the case of a global quotient stack.

**Definition 8.16.** Suppose that a finite group $G$ acts on a smooth projective variety $X$. Let

$$A^*_{orb}([G \backslash X]) := A^*([G \backslash X(G)]) \simeq A^*([G \backslash X(G)]) \simeq A^*(X(G))^G.$$"
has a universal $G$-curve and a universal $G$-morphism, and hence an obstruction theory and a virtual fundamental class. The morphism $\overline{M}_3^G(X) \to \overline{M}_3^G(X)$ constructed in Definition 7.3 is $G((G))$-equivariant and hence induces a morphism $\overline{K}_3^G(X) \to \overline{K}_3^G(X)$ on the quotients, where $\overline{K}_3^G(X) := [G((G)) \backslash \overline{M}_3^G(X)]$. We pull back the virtual fundamental class of $\overline{K}_3^G(X)$ to $\overline{K}_3^G(X)$. By Lemma 7.4, we have a morphism $\overline{N}_3^G(X) \to \overline{M}_3^G(X)$ such that the composition with the quotient $\overline{M}_3^G(X) \to \overline{K}_3^G(X)$ is an isomorphism. We have the following commutative diagram

$$
\overline{N}_3^G(X) \longrightarrow \overline{M}_3^G(X) \longrightarrow \overline{M}_3^G(X) \quad \phi
$$

and the virtual fundamental classes are compatible under any morphism in the above diagram as the universal $G$-curves and $G$-maps pull back. In particular $\phi_*[\overline{N}_3^G(X, \beta)]^{\text{vir}} = \phi_*\phi^*[\overline{M}_3^G(X, \beta)]^{\text{vir}} = [\overline{K}_3^G(X, \beta)]^{\text{vir}}$, as $\phi$ is an isomorphism.

**Definition 8.17.** The evaluation morphism $\text{ev} : \overline{M}_3^G(X) \to (G(\pi))^3$ is $G((G))$-equivariant and induces $\text{ev} : \overline{K}_3^G(X) \to [G((G)) \backslash (X(\pi))]^2 \times X(\pi)$ on the quotients. In particular the diagram

$$
\overline{M}_3^G(X) \longrightarrow (X(\pi))^2 \times X(\pi) \quad \text{ev}
$$

$$
\overline{K}_3^G(X) \longrightarrow [G((G)) \backslash (X(\pi))]^2 \times X(\pi) \quad \text{ev}
$$

is cartesian, where the vertical arrows are quotients by $(G(\pi))^2$.

In [AGV02], the GW invariants for the orbifold quantum cohomology are integrated over the stack $\overline{K}_3^G(X)$, and then multiplied by the order of the monodromy of the third marking. By integrating over $\overline{K}_3^G(X)$ we will obtain the same numbers as $\overline{K}_3^G(X) \to \overline{K}_3^G(X)$ is an etale morphism whose degree is locally equal to the order of the monodromy of the third marking. This follows from the fact that $\overline{M}_3^G(X) \to \overline{M}_3^G(X)$ is a stack quotient by the right action of $(G)$ (by Definition 7.3 and Proposition 7.2).

**Definition 8.18.** The orbifold quantum cohomology of $[G \backslash X]$ is the $\mathbb{Q}$-graded algebra

$$qA^*((G \backslash X)) := Q^*(X, G) \otimes_{\mathbb{Q}} A^*_{\text{orb}}([G \backslash X])$$

with the multiplication defined as follows. Given $\sigma_i \in A^*_{\text{orb}}([G \backslash X])$ for $i \in \{0, 1\}$, define

$$\sigma_0 \ast \sigma_1 := \sum_{\beta \in A^*_{\text{orb}}((X^{\pi}))} q_G(\nu \circ \overline{\pi}_{\infty})_* \left( \overline{\pi}_0 \sigma_0 \cdot \overline{\pi}_1 \sigma_1 \cdot (\text{ev})_* \left[ \overline{K}_3^{G, \beta}(X, \beta) \right]^{\text{vir}} \right).$$

where $\overline{\pi}_i : [G((G)) \backslash (X(\pi))]^2 \times X(\pi) \to [G((G)) \backslash (X(\pi))]$ is the projection on the $i$th factor and $\overline{\pi}_{\infty}$ is the projection on $X(\pi)$. Note that $\sigma_0 \ast \sigma_1$ is $G$-invariant, hence it lies in the isomorphic image of $A^*_{\text{orb}}([G \backslash X])$ in $A^*(X(\pi))$ under $\Phi^*_{X,G}$. Let $\sigma_0 \ast \sigma_1$ be the preimage of $\sigma_0 \ast \sigma_1$ under $\Phi^*_{X,G}$. 


It was stated in [AGV02] that $q\Lambda^*(\big[G\setminus X\big])$ is an associative algebra.

### 8.8 Invariant part of the group quantum cohomology

By tensoring $\Phi^*_{X,G}$ with $Q^*(X, G)$ we get a $Q$-graded morphism of $Q$-modules

$$q\Lambda^*(\big[G\setminus X\big]) \to q\Lambda^*(X, G).$$

which will also be denoted by $\Phi^*_{X,G}$.

**Theorem 8.19.** The morphism $\Phi^*_{X,G}$ is a homomorphism of $Q$-graded algebras identifying $q\Lambda^*(\big[G\setminus X\big])$ with the $G$-invariant part of $q\Lambda^*(X, G)$.

**Proof.** For simplicity write $\overline{\mathcal{K}}(\beta)$ (resp. $\overline{\mathcal{N}}(\beta)$) for $\overline{\mathcal{K}}_3^G(X, \beta)$ (resp. $\overline{\mathcal{N}}_3^G(X, \beta)$). Let $q$ the quotient $(X^{(G)})^{[0,1,\infty]} \to [G(G)\setminus X^{(G)}]^{[0,1]} \times X^{(G)}$. Then

$$((\ell \circ \overline{\pi}_\infty))_* \left( \overline{\mathcal{K}}(\beta) \right) = ((\ell \circ \overline{\pi}_\infty))_* \left( \overline{\mathcal{N}}(\beta) \right)$$

which is by definition the coefficient of $q^G$ in $(\Phi^*_{X,G} \sigma_0) \ast_G (\Phi^*_{X,G} \sigma_1)$. \qed
Chapter 9

Associativity

In this section we address the problem of proving the associativity of the group quantum cohomology. As in the classical quantum cohomology, this is related to numerical equivalences of certain divisors on a moduli space, which in our case is $\mathcal{M}_{0,4}$. We don't know how to prove it for a general $G$, but we can get partial results for cyclic groups and full associativity for $\mathbb{Z}_2$. In the classical quantum cohomology, one considers the space $\mathcal{M}_{0,4}$, which is $\mathbb{P}^1$, and hence any two points are rationally equivalent, providing the famous WDVV equations. In our case, it is not true (as suggested in [JKK03]) that these rational equivalences in $\mathcal{M}_{0,4}$ pullback from $\mathcal{M}_{0,4}$. It is still possible that these divisors are algebraically equivalent, which also has been claimed in [JKK03], but we found the proof unsatisfactory, and hence we will state it as a conjecture.

9.1 Reducing the problem

Suppose that $\sigma_1$, $\sigma_2$ and $\sigma_3$ are classes of $A^*(X, G)$ supported on $\{m_1\} \times X^{m_1}$, $\{m_2\} \times X^{m_2}$ and $\{m_3\} \times X^{m_3}$ respectively, where $m_1$, $m_2$ and $m_3$ are elements of $G$. Then the associativity of the product in $qA^*(X, G)$ can be expressed as the relation

$$(\sigma_1 \ast \sigma_2) \ast \sigma_3 = (\sigma_1 \ast \sigma_3) \ast m_2^{-1} \ast \sigma_2. \tag{9.1}$$

Indeed $(\sigma_1 \ast \sigma_2) \ast \sigma_3 = (\sigma_2 \ast m_2^{-1} \ast \sigma_1) \ast \sigma_3$ by $G$-graded commutativity. By the above relation we have $(\sigma_2 \ast m_2^{-1} \ast \sigma_1) \ast \sigma_3 = (\sigma_2 \ast \sigma_3) \ast m_2^{-1} \ast \sigma_1$ which is $\sigma_1 \ast (\sigma_2 \ast \sigma_3)$ by $G$-graded commutativity again.

We express the triple product $(\sigma_1 \ast \sigma_2) \ast \sigma_3$ as a single integral as follows. Given $\beta_a$ and $\beta_b \in A^+_1(X) \otimes G$ let $\overline{\mathcal{N}}_a^G(X, \beta_a)$ (resp. $\overline{\mathcal{N}}_b^G(X, \beta_b)$) be the moduli space $\overline{\mathcal{N}}_a^G(X, \beta_a)$ (resp. $\overline{\mathcal{N}}_b^G(X, \beta_b)$) where the markings are indexed by the set $\{0_a, 1_a, \infty_a\}$ (resp. $\{0_b, 1_b, \infty_b\}$). Let $\overline{\mathcal{N}}_{\alpha \times \beta}(X, \beta_a, \beta_b)$ be the product of $\overline{\mathcal{N}}_a^G(X, \beta_a)$ and $\overline{\mathcal{N}}_b^G(X, \beta_b)$, and let $\overline{\mathcal{N}}_{\alpha \times \beta}(X, \beta_a, \beta_b)$ be the fiber product

$$
\overline{\mathcal{N}}_{\alpha \times \beta}(X, \beta_a, \beta_b) \longrightarrow \overline{\mathcal{N}}_{\alpha \times \beta}(X, \beta_a, \beta_b)
$$

$$
X^{(G)} \longrightarrow \Delta \longrightarrow X^{(G)} \times X^{(G)}
$$

where $\Delta$ is the twisted diagonal $(x, m) \mapsto ((x, m), (x, m^{-1}))$. Let $[\overline{\mathcal{N}}_{\alpha \times \beta}(X, \beta_a, \beta_b)]^{vir}$ be
the product of $[\mathcal{N}_{a}^{G}(X, \beta_{a})]^{\text{vir}}$ and $[\mathcal{N}_{b}^{G}(X, \beta_{b})]^{\text{vir}}$. Consider the fiber product

$$
\xymatrix{
\left( X(G) \right)^{\{0_a,1_a,*,1_b,\infty_b\}} \ar[r] \ar[d] & \left( X(G) \right)^{\{0_a,1_a,\infty_a,0_b,1_b,\infty_b\}} \\
X(G) \ar[r]^-{\Delta} & X(G) \times X(G)
}
$$

Let

$$
ev_{a \times b} : \mathcal{N}_{a \times b}^{G}(X, \beta_{a}, \beta_{b}) \to \left( X(G) \right)^{\{0_a,1_a,\infty_a,0_b,1_b,\infty_b\}} \quad \text{and} \quad ev_{a|b} : \mathcal{N}_{a|b}^{G}(X, \beta_{a}, \beta_{b}) \to \left( X(G) \right)^{\{0_a,1_a,*,1_b,\infty_b\}}
$$

be the natural evaluation maps, and let $\pi_{4} : \left( X(G) \right)^{\{0_a,1_a,*,1_b,\infty_b\}} \to X(G)$ be the projections. Then the triple product $(\sigma_{1} \ast \sigma_{2}) \ast \sigma_{3}$ is equal to

$$
\sum_{\beta_a, \beta_b} \alpha^{\beta_a+\beta_b}_{G}(\pi_{\infty_b}) \ast [\pi_{0_a}^{*} \sigma_{1} \pi_{1_a}^{*} \sigma_{2} \pi_{1_b}^{*} \sigma_{3} (ev_{a|b})_{*} \Delta_{\pi_{a \times b}^{G}(X, \beta_{a}, \beta_{b})}]^{\text{vir}}
$$

where $\beta_{a}$ and $\beta_{b}$ run over $A_{+}^{G}(X)$. In particular the coefficient of $\alpha^{\beta}_{G}$ is

$$
(\pi_{\infty_b}) \ast [\pi_{0_a}^{*} \sigma_{1} \pi_{1_a}^{*} \sigma_{2} \pi_{1_b}^{*} \sigma_{3} (ev_{a|b})_{*} \Delta_{\pi_{a \times b}^{G}(X, \beta_{a}, \beta_{b})}]^{\text{vir}}
$$

where $\pi_{a \times b}^{G}(X, \beta)$ is the disjoint union of $\pi_{a \times b}^{G}(X, \beta_{a}, \beta_{b})$ over all $\beta_{a}$ and $\beta_{b} \in A_{+}^{G}(X)$ such that $\beta_{a} + \beta_{b} = \beta$. Consider the forgetful morphism $\phi : \left( X(G) \right)^{\{0_a,1_a,*,1_b,\infty_b\}} \to \left( X(G) \right)^{\{0_a,1_a,1_b,\infty_b\}}$, and let $\pi_{4} : \left( X(G) \right)^{\{0_a,1_a,1_b,\infty_b\}} \to X(G)$ be the projection morphisms. Then the above expression is equal to

$$
(\pi_{\infty_b}) \ast [\pi_{0_a}^{*} \sigma_{1} \pi_{1_a}^{*} \sigma_{2} \pi_{1_b}^{*} \sigma_{3} \phi (ev_{a|b})_{*} \Delta_{\pi_{a \times b}^{G}(X, \beta)}]^{\text{vir}}
$$

(9.2)

To show associativity (see Equation 9.1) one has to show that the last expression is equal to

$$
(\pi_{\infty_b}) \ast [\pi_{0_a}^{*} \sigma_{1} \pi_{1_a}^{*} \sigma_{3} \pi_{1_b}^{*} \sigma_{2} \phi (ev_{a|b})_{*} \Delta_{\pi_{a \times b}^{G}(X, \beta)}]^{\text{vir}}
$$

(9.3)

Consider the braid group $B_{4}$ on four strands generated by $\zeta_{0}, \zeta_{1,1}$ and $\zeta_{1,\infty}$. It has a unique action on $(X(G))^{\{0_a,1_a,1_b,\infty_b\}}$ such that

$$
\begin{align*}
\pi_{i} \zeta_{i,j} \cdot x &= \mu_{i}(x) \cdot \pi_{j} x = \pi_{j} \mu_{i}(x) \cdot j x\\
\pi_{j} \zeta_{i,j} \cdot x &= \pi_{i} x \\
\pi_{k} \zeta_{i,j} \cdot x &= \pi_{k} x
\end{align*}
$$

for $k \not\in \{i,j\}$. So for example:

$$
\begin{align*}
\zeta_{1,1} \cdot ((m_{0_a}, x_{0_b}), (m_{1_a}, x_{1_a}), (m_{1_b}, x_{1_b}), (m_{\infty_b}, x_{\infty_b})) &= \\
= ((m_{0_a}, x_{0_b}), (m_{1_a} m_{1_b}^{-1}, m_{1_a} \cdot x_{1_b}), (m_{1_b}, x_{1_b}), (m_{\infty_b}, x_{\infty_b}))
\end{align*}
$$
Chapter 9. Associativity

Lemma 9.1. On intersection theory we have:

$$\zeta^*_{1a,1b} \left( \pi^*_{0a} \sigma_1 \pi^*_{1a} \sigma_2 \pi^*_{1b} \sigma_3 \right) = \pi^*_{0a} \sigma_1 \pi^*_{1a} \sigma_3 \pi^*_{1b} m_3^{-1} \cdot \sigma_2$$

Proof. We compute:

$$\zeta^*_{1a,1b} \left( \pi^*_{0a} \sigma_1 \pi^*_{1a} \sigma_2 \pi^*_{1b} \sigma_3 \right) = \pi^*_{0a} \sigma_1 \pi^*_{1a,1b} \pi^*_{1a} \sigma_2 \zeta^*_{1a,1b} \pi^*_{1b} \sigma_3$$

$$= \pi^*_{0a} \sigma_1 \left( \pi^*_{1a} \zeta^*_{1a,1b} \right)^* \sigma_2 \left( \pi^*_{1b} \zeta^*_{1a,1b} \right)^* \sigma_3$$

$$= \pi^*_{0a} \sigma_1 \left( \pi^*_{1a} \mu_{1a,1b} \right)^* \sigma_2 \left( \pi^*_{1b} \right)^* \sigma_3$$

$$= \pi^*_{0a} \sigma_1 \mu_{1a,1b} \pi^*_{1a} \sigma_2 \pi^*_{1a} \sigma_3$$

$$= \pi^*_{0a} \sigma_1 m_3 \pi^*_{1a} \sigma_2 \pi^*_{1a} \sigma_3$$

$$= \pi^*_{0a} \sigma_1 \pi^*_{1a} \sigma_3 m_3^{-1} \pi^*_{1b} \sigma_2.$$  

Since $\pi_{\infty}$ is invariant for the action of $\zeta^*_{1a,1b}$, to show the associativity it suffices to show that the class $(\phi \circ ev_{a,b})_{x} \Delta^! \left[ N^G_{a \times b}(X, \beta) \right]^{\text{vir}}$ is $\zeta^*_{1a,1b}$-invariant too.

In next sections we will reduce this to the problem of rational equivalences of some divisors on $\mathcal{M}^G_{0,4}$. We proceed as follows. First we study the behavior of the virtual fundamental class under the gluing morphism $\mathcal{N}^G_{a \times b}(X, \beta) \rightarrow \mathcal{M}^G_{a \times b}(X, \beta)$ where $A = \{0_a, 1_a, 1_b, \infty_b\}$. Then we study the behavior of the virtual fundamental class under the cutting edges morphisms $\mathcal{N}^G_{a \times b}(X, \beta) \rightarrow \mathcal{N}^G_{a \times b}(X, \beta)$. These two results together with a conjecture about the rational equivalence of some divisors on $\mathcal{M}^G_{0,4}$ yields the associativity.

9.2 Gluing

Suppose that $E_1$ and $E_2$ are two flat projective $G$-curves over $T$, such that the action on every fiber is admissible and balanced. Let $\Sigma$ be an étale $G$-scheme over $T$ and let $\iota_1 : \Sigma \rightarrow E_1$ and $\iota_2 : \Sigma \rightarrow E_2$ be two $G$-equivariant embeddings of schemes over $T$ such that the images of $\iota_1$ and $\iota_2$ lie in the smooth locus of $E$ over $T$. Let $m_1 : \Sigma \rightarrow G$ and $m_2 : \Sigma \rightarrow G$ be the monodromies of $E_1$ and $E_2$ along the images of $\iota_1$ and $\iota_2$. Suppose that $m_1 = m_2^{-1}$.

Proposition 9.2. There exist a flat projective $G$-curve $E$ over $T$, such that the action on every fiber is admissible and balanced, and $G$-equivariant $T$-morphisms

$$l_1 : E_1 \rightarrow E$$

$$l_2 : E_2 \rightarrow E$$
such that the following diagram of $G$-schemes over $T$

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\ell_1} & E_2 \\
\downarrow{t_1} & & \downarrow{t_2} \\
E & \xrightarrow{i} & E
\end{array}
\]

commutes, and having the following universal property: given two $G$-equivariant morphisms $f_1 : E_1 \to X$ and $f_2 : E_2 \to X$ such that $f_1 \circ t_1 = f_2 \circ t_2$, then there exists a unique $G$-equivariant morphism $f : E \to X$ such that

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\ell_1} & E_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
E & \xrightarrow{f} & X
\end{array}
\]

commutes.

Moreover the construction of $E$ commutes with base change.

**Proof.** Using the gluing of curves (non-equivariant) and its universal property (see [BM96]), one can glue $G$-curves and prove the above universal property. \qed

Let $A_1$ and $A_2$ be two non-empty finite sets. Let $g_1$ and $g_2$ be two non-negative integers. For $i \in \{1,2\}$, let $k_i \in A_i$. Consider the fiber product

\[
\begin{array}{ccc}
\mathcal{M}^G_{A_1, k_1 | k_2, A_2} & \longrightarrow & \mathcal{M}^G_{g_1, A_1} \times \mathcal{M}^G_{g_2, A_2} \\
\downarrow{\mu_{k_1} \times \mu_{k_2}} & & \downarrow{G \times G} \\
G & \xrightarrow{\Delta} & G \times G
\end{array}
\]

where $\Delta(m) = (m, m^{-1})$. By Proposition 9.2, we have a morphism of stacks

\[
\mathcal{M}^G_{A_1, k_1 | k_2, A_2} \longrightarrow \mathcal{M}^G_{g_1 + g_2, A}
\]

where $A = (A_1 \cup A_2) \setminus \{k_1, k_2\}$. This is called the gluing morphism along sections $k_1$ and $k_2$. Using the universal property of gluing, we also have a morphism

\[
\overline{\mathcal{M}}^G_{A_1, k_1 | k_2, A_2}(X, \beta_1, \beta_2) \longrightarrow \overline{\mathcal{M}}^G_{g_1 + g_2, A}(X, \beta_1 + \beta_2)
\]

where the left hand side is defined as the fiber product

\[
\begin{array}{ccc}
\overline{\mathcal{M}}^G_{A_1, k_1 | k_2, A_2}(X, \beta_1, \beta_2) & \longrightarrow & \overline{\mathcal{M}}^G_{g_1, A_1}(X, \beta_1) \times \overline{\mathcal{M}}^G_{g_2, A_2}(X, \beta_2) \\
\downarrow{ev_{k_1} \times ev_{k_2}} & & \downarrow{ev_{k_1} \times ev_{k_2}} \\
X^{(G)} & \xrightarrow{\Delta} & X^{(G)} \times X^{(G)}
\end{array}
\]
9.3 Compatibility of the virtual fundamental class under the cutting edges morphism

Definition 9.3. Define the cutting edges morphism as the inclusion

\[ \mathcal{M}_{A_1, k_1|k_2 A_2}^{G}(X, \beta_1, \beta_2) \to \mathcal{M}_{g_1 A_1}^{G}(X, \beta_1) \times \mathcal{M}_{g_2 A_2}^{G}(X, \beta_2) \]

in the last cartesian diagram.

The general construction of the obstruction theory for stacks of the above form is given in the following.

Definition 9.4. Given an Artin stack \( \mathcal{M} \) with an admissible balanced \( G \)-curve \( \mathcal{E} \to \mathcal{M} \), let \( \mathcal{M}(X) \) be the stable locus of \( \text{Mor}_{\mathcal{M}}^{G}(\mathcal{E}, \mathcal{M} \times X) \). Let \( \pi : \mathcal{E} \to \mathcal{M}(X) \) be the pullback of \( \mathcal{E} \), and let \( f : \mathcal{E} \to X \) be the universal \( G \)-map to \( X \). As in Section 4.1, we define the relative obstruction theory of \( \mathcal{M}(X) \) over \( \mathcal{M} \) as the pair \( (E, \phi_E) \) where

\[ E := R\pi_*^{G}(\omega_\mathcal{E} \otimes f^*\Omega_X)[1] \]

and \( \phi_E : E \to L\mathcal{M}(X)/\mathcal{M} \) is the natural morphism (see Section 4.1).

We've already constructed the obstruction theories and virtual fundamental classes for stack of the form \( \mathcal{M}_{\beta, A}(X, \beta) \) in Section 4.1. Now we can construct them for \( \mathcal{M}_{A_1 \times A_2}(X, \beta_1, \beta_2) \) and \( \mathcal{M}_{A_1, k_1|k_2 A_2}(X, \beta_1, \beta_2) \). Using the above definition we just need to interpret \( \mathcal{M}_{A_1 \times A_2}(X, \beta_1, \beta_2) \) and \( \mathcal{M}_{A_1, k_1|k_2 A_2}(X, \beta_1, \beta_2) \) as stable loci in a stack of \( G \)-equivariant morphisms. The following is straightforward.

Lemma 9.5. Let \( \mathcal{M}_{A_1, k_1|k_2 A_2} \) be the product of \( \mathcal{M}_{A_1} \) and \( \mathcal{M}_{A_2} \). The universal \( G \)-curve \( \mathcal{E}_{A_1 \times A_2} \) over \( \mathcal{M}_{A_1 \times A_2} \) is just the disjoint union of the pullbacks of the universal \( G \)-curves over \( \mathcal{M}_{A_1} \) and \( \mathcal{M}_{A_2} \). The stack \( \mathcal{M}_{G}^{A_1 \times A_2}(X) := \text{Mor}_{\mathcal{M}_{A_1 \times A_2}}^{G}(\mathcal{E}_{A_1 \times A_2}, X) \) decomposes into a disjoint union

\[ \mathcal{M}_{G}^{A_1 \times A_2}(X) = \bigsqcup_{\beta_1, \beta_2} \mathcal{M}_{A_1 \times A_2}(X, \beta_1, \beta_2) \]

where the degree of the morphism is given by \( \beta_1 \) and \( \beta_2 \) on each component. Then \( \mathcal{M}_{A_1 \times A_2}(X, \beta_1, \beta_2) \) is isomorphic to the stable locus of \( \mathcal{M}_{A_1 \times A_2}^{G}(X, \beta_1, \beta_2) \).

Let \( \mathcal{M}_{A_1, k_1|k_2 A_2} \) be the product of \( \mathcal{M}_{A_1} \) and \( \mathcal{M}_{A_2} \). The universal \( G \)-curve \( \mathcal{E}_{A_1, k_1|k_2 A_2} \) over \( \mathcal{M}_{A_1, k_1|k_2 A_2} \) is obtained by gluing the pullbacks of the universal \( G \)-curves over \( \mathcal{M}_{A_1} \) and \( \mathcal{M}_{A_2} \) along \( k_1 \) and \( k_2 \). The stack \( \mathcal{M}_{G}^{A_1, k_1|k_2 A_2}(X) := \text{Mor}_{\mathcal{M}_{A_1, k_1|k_2 A_2}}^{G}(\mathcal{E}_{A_1, k_1|k_2 A_2}, X) \) decomposes into a disjoint union

\[ \mathcal{M}_{G}^{A_1, k_1|k_2 A_2}(X) = \bigsqcup_{\beta_1, \beta_2} \mathcal{M}_{A_1, k_1|k_2 A_2}(X, \beta_1, \beta_2) \]

where the degree of the morphism is given by \( \beta_1 \) and \( \beta_2 \) on each component. Then \( \mathcal{M}_{A_1, k_1|k_2 A_2}(X, \beta_1, \beta_2) \) is isomorphic to the stable locus of \( \mathcal{M}_{A_1, k_1|k_2 A_2}^{G}(X, \beta_1, \beta_2) \).
Chapter 9. Associativity

Using Definition 9.4, we have the virtual fundamental classes

\[ \left[ \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \right]_{\text{vir}} \]
\[ \left[ \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \right]_{\text{vir}}. \]

The following proposition says that the virtual fundamental classes are compatible under cutting edges morphism.

**Proposition 9.6.** We have

\[ \Delta^1 \left[ \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \right]_{\text{vir}} = \left[ \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \right]_{\text{vir}}. \]

**Proof.** The proof is a straightforward generalization of the proof of the axiom III in [Beh97a], where we replace locally free sheaves by locally free G-sheaves, pushforward of sheaves by equivariant pushforward, sections by twisted sections, the target space X by the inertia variety \( X^{(G)} \), etc.

**Corollary 9.7.**

\[ \Delta^1 \left[ \overline{\mathcal{M}}_{a \times b}(X, \beta_1, \beta_2) \right]_{\text{vir}} = \left[ \overline{\mathcal{M}}_{a \times b}(X, \beta_1, \beta_2) \right]_{\text{vir}}. \]

Let

\[ \overline{\mathcal{N}}_{a \times b}^G(X, \beta) = \prod_{\beta_1 + \beta_2 = \beta} \overline{\mathcal{N}}_{a \times b}^G(X, \beta_1, \beta_2) \]

and let \( \overline{\mathcal{N}}_{a \times b}^{G,m}(X, \beta) \) be the closed and open substack of \( \overline{\mathcal{N}}_{a \times b}^G(X, \beta) \) where the monodromies at the sections \( \sigma_{a1}, \sigma_{1a}, \sigma_{1b} \) and \( \sigma_{0a} \) are given by \( m_1, m_2, m_3 \) and \( m_4 \) respectively. The associativity in group quantum cohomology can be now restated as:

**Proposition 9.8.** Let \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) be classes in \( q\mathbb{A}^*(X,G) \) supported on the twisted sectors corresponding to \( m_1, m_2 \) and \( m_3 \) respectively. If the equality

\[ \zeta_{a_1a_2}^* (\phi \circ \text{ev}_{a_1b}^*) \left[ \overline{\mathcal{N}}_{a \times b}^G(X, \beta) \right]_{\text{vir}} = (\phi \circ \text{ev}_{a_1b}^*) \left[ \overline{\mathcal{N}}_{a \times b}^{G,m}(X, \beta) \right]_{\text{vir}} \]

holds, then we have

\[ (\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3). \]

**Proof.** As the group quantum cohomology is a G-graded G-algebra, note that \( (\sigma_1 * \sigma_2) * \sigma_3 \) is supported on the twisted sector corresponding to \( m_4^{-1} = m_1 m_2 m_3 \). It follows that in the equation Equation 9.2 (resp. Equation 9.3) we can restrict the class \( (\phi \circ \text{ev}_{a_1b}^*) \Delta^1 \left[ \overline{\mathcal{N}}_{a \times b}^G(X, \beta) \right]_{\text{vir}} \) to \( X^{m_1} \times X^{m_2} \times X^{m_3} \times X^{m_4} \) (resp. to \( X^{m_1} \times X^{m_3} \times X^{m_3 \times m_2} \times X^{m_4} \)) which is equal to \( (\phi \circ \text{ev}_{a_1b}^*) \left[ \overline{\mathcal{N}}_{a \times b}^{G,m}(X, \beta) \right]_{\text{vir}} \) (resp. to \( (\phi \circ \text{ev}_{a_1b}^*) \left[ \overline{\mathcal{N}}_{a \times b}^{G,m}(X, \beta) \right]_{\text{vir}} \)) by the above corollary. The result follows from Section 9.1. \( \Box \)
9.4 Compatibility of the virtual fundamental class under the gluing morphism

We consider the morphisms
\[ \phi : \overline{\mathcal{M}}_A(X, \beta) \to \mathcal{M}_A \]
\[ \psi' : \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \to \mathcal{M}_{A_1 \times A_2} \]
that forgets the $G$-equivariant map into $X$. We also have the stabilization morphisms
\[ s : \mathcal{M}_A \to \overline{\mathcal{M}}_A \]
\[ s' : \mathcal{M}_{A_1 \times A_2} \to \overline{\mathcal{M}}_{A_1 \times A_2} \]
that stabilizes the $G$-curve. We have the gluing morphisms
\[ gl(X) : \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2) \to \overline{\mathcal{M}}_A(X, \beta_1 + \beta_2) \]
\[ gl : \overline{\mathcal{M}}_{A_1 \times A_2} \to \overline{\mathcal{M}}_A \]
\[ gl : \mathcal{M}_{A_1 \times A_2} \to \mathcal{M}_A. \]

Lemma 9.9. The diagram
\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta) & \xrightarrow{gl(X)} & \overline{\mathcal{M}}_A(X, \beta) \\
\downarrow{\psi'} & & \downarrow{\psi} \\
\mathcal{M}_{A_1 \times A_2} & \xrightarrow{gl} & \mathcal{M}_A
\end{array}
\]
is cartesian, where horizontal (resp. vertical) arrows are gluing (resp. forgetful) morphisms, and where
\[ \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta) := \prod_{\beta_1 + \beta_2 = \beta} \overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta_1, \beta_2). \]
Moreover
\[ gl^! [\overline{\mathcal{M}}_A(X, \beta)]^{vir} = [\overline{\mathcal{M}}_{A_1 \times A_2}(X, \beta)]^{vir}. \]

Proof. We first prove that the diagram
\[
\begin{array}{ccc}
\mathcal{M}_{A_1 \times A_2}(X, \beta) & \longrightarrow & \mathcal{M}_A(X, \beta) \\
\downarrow & & \downarrow{\psi} \\
\mathcal{M}_{A_1 \times A_2} & \xrightarrow{gl} & \mathcal{M}_A
\end{array}
\]
is cartesian, where the horizontal arrows are the gluing morphisms and the vertical arrows are forgetful morphisms. Let $T$ be a scheme and let $\tilde{\mathcal{f}} : T \to \mathcal{M}_A(X, \beta)$ and $(\epsilon_1, \epsilon_2) : T \to \mathcal{M}_{A_1 \times A_2}$ be morphism and let $\theta$ be the natural transformation making the diagram
\[
\begin{array}{ccc}
T & \xrightarrow{\epsilon} & \mathcal{M}_A(X, \beta) \\
\downarrow{\tilde{\mathcal{f}}} & & \downarrow{\psi} \\
\mathcal{M}_{A_1 \times A_2} & \xrightarrow{gl} & \mathcal{M}_A
\end{array}
\]
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commute. We need to show that there exists a unique morphism \( T \to \mathcal{M}_{A_1 \mid k_2 A_2}(X, \beta) \) commuting with the gluing morphism and the forgetful morphism. By definition of \( \mathcal{M}_{A_1 \mid k_2 A_2}(X, \beta) \), we need morphisms \( f_i : T \to \mathcal{M}_{A_i}(X) \) for \( i \in \{1, 2\} \) such that \( \text{ev}_{k_i} \circ f_1 = \iota \circ \text{ev}_{k_2} \circ f_2 \) where \( \iota \) is the involution on \( X(G) \), and such that locally the degrees of \( f_1 \) and \( f_2 \) add up to \( \beta \). Let \( E_{h_i} := E_{\tau_1} \), \( \Sigma_{\epsilon_i} := \Sigma_{\epsilon_i} \) and \( \sigma_{k, k} := \sigma_{\epsilon_i k} \). Define \( f_1 \) by

\[
E_{h_1} = E_{\tau_1} \xrightarrow{\iota} E_{\text{gl}(\tau_1, \tau_2)} \xrightarrow{\sigma} E_{\psi(f)} = E_f
\]

where \( \iota \) are the inclusion into the glued \( G \)-curves. This is the desired morphism and it is unique with the above properties. The first statement follows from the fact that the gluing morphism is representable and hence preserves stability.

The second statement follows from the first and [BF97, Proposition 7.2]. \( \square \)

Consider the diagram

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{A_1 \mid k_2 A_2}(X, \beta) & \xrightarrow{h} & \overline{\mathcal{M}}_{A}(X, \beta) \\
\downarrow{\psi'} & & \downarrow{\psi} \\
\mathcal{M}_{A_1 \mid k_2 A_2} & \xrightarrow{l} & \mathcal{M}_{A} \\
\downarrow{s'} & & \downarrow{s} \\
\overline{\mathcal{M}}_{A_1 \mid k_2 A_2} & \xrightarrow{\text{gl}} & \overline{\mathcal{M}}_{A}
\end{array}
\]

where every square is cartesian (by Lemma 9.9), and where \( h \) and \( l \) are uniquely defined by universal property of fiber products and the commutative diagrams

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{A_1 \mid k_2 A_2}(X, \beta) & \xrightarrow{\text{gl}(X)} & \overline{\mathcal{M}}_{A}(X, \beta) \\
\downarrow{\psi'} & & \downarrow{\psi} \\
\mathcal{M}_{A_1 \mid k_2 A_2} & \xrightarrow{\text{gl}} & \mathcal{M}_{A} \\
\downarrow{s'} & & \downarrow{s} \\
\overline{\mathcal{M}}_{A_1 \mid k_2 A_2} & \xrightarrow{\text{gl}} & \overline{\mathcal{M}}_{A}
\end{array}
\]

Proposition 9.10. We have

\[
h_* \left[ \overline{\mathcal{M}}_{A_1 \mid k_2 A_2}(X, \beta) \right]_{\text{vir}} = \text{gl}^l \left( [\overline{\mathcal{M}}_{A}(X, \beta)]_{\text{vir}} \right).
\]

Proof. As both \( \overline{\mathcal{M}}_{A_1 \mid k_2 A_2} \) and \( \overline{\mathcal{M}}_{A} \) are smooth of pure dimension, the gluing morphism is a local complete intersection in the sense of [Ful98] and [Vis89] and has a well defined orientation \([\text{gl}]\) in bivariant intersection theory (see [Ful98, 17.4] and [Vis89, Section 5]).
We first claim that one has the following relation in the bivariant intersection theory

\[ l_* [\mathcal{I}] = s^* [\mathcal{I}] \]

We adapt the proof of [Beh97a, Proposition 8]. This claim follows from the three facts: \( l \) is birational, \( s \) is flat and \( \mathcal{I} \) is a local complete intersection. We first show that \( l \) is birational. That the restriction of \( l \) to the open dense substack \( \overline{\mathcal{M}}_{\mathcal{A}_1, k_1 | \mathcal{A}_2} \) of \( \mathcal{M}_{\mathcal{A}_1, k_1 | \mathcal{A}_2} \) is an isomorphism onto its image is clear as the stabilization morphism restricted to \( \overline{\mathcal{M}}_{\mathcal{A}_1, k_1 | \mathcal{A}_2} \) is the identity. We only need to show that \( l \) is surjective. An object of \( \overline{\mathcal{M}}_{\mathcal{A}_1, k_1 | \mathcal{A}_2} \) over \( \text{Spec} \, \mathbb{C} \) is a pair \((\epsilon_1, \epsilon_2)\) where for \( i \in \{1, 2\}, \epsilon_i \) is an object of \( \mathcal{M}_{\mathcal{A}_i} \) such that the monodromy of \( E_{\epsilon_i} \) at \( \sigma_{\epsilon_i} \) is inverse of the monodromy of \( E_{\epsilon_2} \) at \( \sigma_{\epsilon_2} \). Let \( \mathfrak{f} \) be an object of \( \mathcal{M}_A \) and suppose we have an isomorphism \( \phi : gl(\epsilon_1, \epsilon_2) \simeq s(\mathfrak{f}) \). Hence we have a diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{s} & E_\mathfrak{f} \\
\downarrow & & \downarrow \\
E_{\epsilon_1} \bigcup E_{\epsilon_2} & \xrightarrow{\iota} & E_\mathfrak{f}
\end{array}
\]

where \( \epsilon \) is some object of \( \overline{\mathcal{M}}_{\mathcal{A}_i} \), and where \( s \) contracts the unstable components and \( \iota \) identifies the orbits \( G \cdot \sigma_{\epsilon_1, \mathfrak{k}_1} \) and \( G \cdot \sigma_{\epsilon_2, \mathfrak{k}_2} \). Let \( p = gl(\sigma_{\epsilon_1, \mathfrak{k}_1}) = gl(\sigma_{\epsilon_2, \mathfrak{k}_2}) \in E_\mathfrak{f} \), and let \( q \in E_\mathfrak{f} \) be such that \( s(q) = p \). There exists \( E_{\mathfrak{f}_1} \) and \( E_{\mathfrak{f}_2} \subseteq E_\mathfrak{f} \) such that

\[
E_{\mathfrak{f}_1} \cup E_{\mathfrak{f}_2} = E_1 \\
E_{\mathfrak{f}_1} \cap E_{\mathfrak{f}_2} = G \cdot q \\
s(E_{\mathfrak{f}_1}) = gl(E_{\mathfrak{f}_1}).
\]

Indeed choose \( E_{\mathfrak{f}_1} \) to be the union of irreducible components of \( E_1 \) lying above \( \iota(E_{\epsilon_1}) \), and let \( E_{\mathfrak{f}_2} \) be the closure of the complement of \( E_{\mathfrak{f}_1} \). Induce the twisted sections and the sections on \( E_{\mathfrak{f}_1} \) from \( E_\mathfrak{f} \). Thus we have produced objects \( \mathfrak{f}_1 \) and \( \mathfrak{f}_2 \) of \( \mathcal{M}_{\mathcal{A}_1} \) and \( \mathcal{M}_{\mathcal{A}_2} \) respectively such that \( \mathfrak{f} \simeq gl(\mathfrak{f}_1, \mathfrak{f}_2) \). Since \( \iota \) is injective when restricted to \( E_{\epsilon_i} \) for each \( i \in \{1, 2\} \), there is a unique morphism \( s_i : E_{\mathfrak{f}_i} \rightarrow E_{\epsilon_i} \) completing the above diagram. It is clear that the \( s_i \)'s are contracting the components to be contracted. This produces an element \((\mathfrak{f}_1, \mathfrak{f}_2)\) of \( \mathcal{M}_{\mathcal{A}_1, k_1 | \mathcal{A}_2} \) such that \( l(\mathfrak{f}_1, \mathfrak{f}_2) \) is isomorphic to the object \((\epsilon_1, \epsilon_2, \phi, \mathfrak{f})\) of \( \mathfrak{B} \). The claim follows from the birationality of \( l \).

The rest of the proof is as in [Beh97a, Axiom V]:

\[
gl' [\overline{\mathcal{M}}_A(X, \beta)]_{\text{vir}} \overset{\text{vir}}{=} (s^* [\mathcal{I}]) [\overline{\mathcal{M}}_A(X, \beta)]_{\text{vir}} \\
= (l_* [\mathcal{I}]) [\overline{\mathcal{M}}_A(X, \beta)]_{\text{vir}} \\
= h_* gl' [\overline{\mathcal{M}}_A(X, \beta)]_{\text{vir}} \\
= h_* [\overline{\mathcal{M}}_{\mathcal{A}_1, k_1 | \mathcal{A}_2}(X, \beta)]_{\text{vir}}
\]

where \( s^* [\mathcal{I}] = l_* [\mathcal{I}] \) was proved above. \( \square \)

The following is a straightforward consequence of the above proposition.
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Corollary 9.11. We have

$$h' \sum_{\beta_1 + \beta_2 = \beta} [\mathcal{N}_{A_1|A_2}(X, \beta_1, \beta_2)]^{vir} = \mathcal{N}(gl)([\mathcal{M}_A(X, \beta)]^{vir})$$

where \( \mathcal{N}(gl) \) is the composition \( \mathcal{N}_{A_1|A_2} \to \mathcal{M}_{A_1|A_2} \to \mathcal{M}_A \), and \( h' : \mathfrak{A}' \to \mathcal{M}_A(X, \beta) \) is the pullback of \( \mathcal{N}(gl) \).

9.5 Associativity in group quantum cohomology

Note that \( \mathcal{N}_{gl} \) maps isomorphically onto the subset of \( G^4 \) consisting of quadruples \( (m_1, m_2, m_3, m_4) \) such that \( m_1m_2m_3m_4 = \text{id} \) by the map

\[
\begin{align*}
(m_0a, m_1a, m_0b), (m_0b, m_1b, m_\infty) & \mapsto (m_0a, m_1a, m_1b, m_\infty) .
\end{align*}
\]

The braid group \( B_4 \) has a natural action on \( G^4 \) defined as follows:

\[
\begin{align*}
\zeta_{0,1} \cdot (m_0a, m_1a, m_1b, m_\infty) &= (m_0a, m_1a, m_1b, m_\infty) \\
\zeta_{1,1} \cdot (m_0a, m_1a, m_1b, m_\infty) &= (m_0a, m_1a, m_1b, m_\infty) \\
\zeta_{1,\infty} \cdot (m_0a, m_1a, m_1b, m_\infty) &= (m_0a, m_1a, m_1b, m_\infty)
\end{align*}
\]

and this action preserves \( \mathcal{N}_{gl} \) under the above inclusion. It also has a natural action on \( \mathcal{M}_A^G \) defined as follows. Given an object \( \epsilon = (E, \sigma_{0a}, \sigma_{1a}, \sigma_{1b}, \sigma_{\infty}) \) of \( \mathcal{M}_A^G \), where \( m = (m_0a, m_1a, m_1b, m_\infty) \in G^4 \), let

\[
\begin{align*}
\zeta_{0,1} \cdot \epsilon &= (E, m_0a \cdot \sigma_{1a}, \sigma_{0a}, \sigma_{1b}, \sigma_{\infty}) \\
\zeta_{1,1} \cdot \epsilon &= (E, \sigma_{0a}, m_1a \cdot \sigma_{1b}, \sigma_{1a}, \sigma_{\infty}) \\
\zeta_{1,\infty} \cdot \epsilon &= (E, \sigma_{0a}, \sigma_{1a}, m_1b, \sigma_{\infty}, \sigma_{1b}) .
\end{align*}
\]

If a morphism \( \mathfrak{f} : \mathfrak{e}_1 \to \mathfrak{e}_2 \) of \( \mathcal{M}_A^G \) is given by a \( G \)-morphism \( \alpha \) of the underlying curves, then the morphism \( \zeta \cdot \mathfrak{f} \) is given by the same \( G \)-morphism \( \alpha \), for any \( \zeta \in B_4 \). There are obvious natural isomorphisms \( \zeta_1(\zeta_2) \simeq (\zeta_1\zeta_2) \epsilon \) for \( \zeta_1, \zeta_2 \in B_4 \) ensuring we have an action of a group on a stack.

Even though \( gl(m) \in \mathcal{M}_A^G \), the gluing morphism \( gl : \mathcal{N}_{gl} \to \mathcal{M}_A^G \) is not \( B_4 \)-equivariant under these actions. This can be seen in the simplest case where the group \( G \) is trivial. The right hand side \( \mathcal{M}_A^G \) is isomorphic to \( \mathcal{M}_0^G \). The braid group \( B_4 \) acts on \( \mathcal{M}_0^G \) by permuting the markings, and has no fixed points. The left hand side \( \mathcal{N}_{gl} \) is just a point, hence the morphism \( gl \) cannot be equivariant. On the other hand, it is equivariant up to rational equivalence as \( \mathcal{M}_0^G \simeq \mathbb{P}^1 \). We can speculate that \( gl \) is equivariant up to rational equivalence for a general \( G \). Unfortunately \( \mathcal{M}_A^G \) is far from being \( \mathbb{P}^1 \) in general. In fact, in general it is not even connected and the connected components aren't necessarily rational curves. To see this, take \( m_0 = (m_1, m_2, m_3, \text{id}) \), then by a Theorem of Abramovich-Vistoli, the forgetful morphism \( \mathcal{M}_A^G \to \mathcal{M}_3^{G, m_1, m_2, m_3} \) is a universal \( G \)-curve, and the fibers are curves whose connected components can be of arbitrary genus.
by the Hurwitz formula. In later sections we will show some examples where \( gl \) is equivariant up to rational equivalence. For now, we will state the conjecture and show how the associativity follows from it.

As before let \( \overline{N}_{al/b}^G,m \) be the locus where the sections have monodromies given by \( m \), which in this case is just the point \( \left( \epsilon_{(m_1,m_2,m_3)}', \epsilon_{(m_2,m_3,m_4)} \right) \) where \( m_{12} = m_1m_2 \), and where \( \epsilon_{(m_1,m_2,m_3)}' \) and \( \epsilon_{(m_2,m_3,m_4)} \) are points of \( \overline{N}_G^3 \) as defined in Section 7.1. Let

\[
\epsilon : \overline{N}_{al/b}^G \to \overline{M}_A^G
\]

\[
m \mapsto \epsilon_m := gl\left( \epsilon_{(m_1,m_2,m_3)}', \epsilon_{(m_2,m_3,m_4)} \right)
\]

**Conjecture 9.12.** The class \( gl, [\overline{N}_{al/b}^G] \) is \( B_4 \)-invariant in \( A_1(\overline{M}_A^G) \). More precisely, given \( m \in G^4 \) such that \( m_1m_2m_3m_4 = id \), we have

\[
\zeta^* \epsilon_* [m] = \epsilon_* \left[ \zeta^{-1}m \right]
\]

for all \( \zeta \in B_4 \).

We will prove this in certain particular cases in the next three sections. For now we will state the important consequence of this conjecture, namely the associativity.

The braid group \( B_4 \) has a natural action on \( \overline{M}_A^G(X, \beta) \), defined in a similar way to the action on \( \overline{M}_A^G \), such that \( \overline{M}_A^G(X, \beta) \to \overline{M}_A^G \) is equivariant. The morphism \( \overline{M}_A^G(X, \beta) \to (X(\mathcal{G}))^{\{0_a,1_a,1_b,\infty_b\}} \) is equivariant as well, where the action on \( (X(\mathcal{G}))^{\{0_a,1_a,1_b,\infty_b\}} \) is given in Section 9.1. The class \( [\overline{M}_A^G(X, \beta)] \) is invariant under the action of \( B_4 \) as it is invariant under permutation and translation of the markings. In fact for any \( \zeta \in B_4 \),

\[
\zeta^* \left[ \overline{M}_A^Gm(X, \beta) \right]_{\text{vir}} = \left[ \overline{M}_A^{G, \zeta^{-1}m}(X, \beta) \right]_{\text{vir}}
\]

as the action of \( \zeta \) restricts to an isomorphism \( \overline{M}_A^{G, \zeta^{-1}m}(X, \beta) \cong \overline{M}_A^Gm(X, \beta) \).

**Proposition 9.13.** The above conjecture implies that \( (\phi \circ ev_{al/b})_* \left[ \overline{N}_{al/b}^G(X, \beta) \right]_{\text{vir}} \) is \( B_4 \)-invariant. More precisely it implies that for any \( \zeta \in B_4 \),

\[
\zeta^* (\phi \circ ev_{al/b})_* \left[ \overline{N}_{al/b}^Gm(X, \beta) \right]_{\text{vir}} = (\phi \circ ev_{al/b})_* \left[ \overline{N}_A^{G, \zeta^{-1}m}(X, \beta) \right]_{\text{vir}}
\]

for any \( m \in G^4 \) such that \( m_1m_2m_3m_4 = id \).

**Proof.** First we note that \( \phi \circ ev_{al/b} = ev_A \circ gl(X) \). Since \( ev_A \) is \( B_4 \)-equivariant, we need to show that \( \zeta^* gl(X)_* \left[ \overline{N}_{al/b}^Gm(X, \beta) \right]_{\text{vir}} = \left[ \overline{N}_{al/b}^{G, \zeta^{-1}m}(X, \beta) \right]_{\text{vir}} \). We have by Corollary 9.11

\[
\begin{align*}
\zeta^* gl(X)_* \left[ \overline{N}_{al/b}^Gm(X, \beta) \right]_{\text{vir}} &= gl'_* h'_* \left[ \overline{N}_{al/b}^Gm(X, \beta) \right]_{\text{vir}} \\
&= gl'_* \left[ \overline{M}_A^Gm(X, \beta) \right]_{\text{vir}} \\
&= \psi^* s^* gl_* \left[ \overline{N}_{al/b}^Gm \right] \cap \left[ \overline{M}_A^Gm(X, \beta) \right]_{\text{vir}}
\end{align*}
\]
where $gl'$ is the pullback of $gl$ under $s \circ \psi$, and the last step is by [Ful98][Example 6.3.4].

We have $\zeta^* gl = \left[ N_{\alpha/\beta}^{G,m} \right] = \zeta^* \epsilon_\alpha \left[ m \right] = \epsilon_\alpha \left[ \zeta^{-1} m \right] = gl \left[ N_{\alpha/\beta}^{G,\zeta^{-1}m} \right]$ by Conjecture 9.12 and $\zeta^* \left[ \mathcal{M}_A^{G,m}(X,\beta) \right]^{vir} = \left[ \mathcal{M}_A^{G,\zeta^{-1}m}(X,\beta) \right]^{vir}$ by the above remark. Moreover $\psi$ and $s$ are $B_4$-equivariant, which proves the result. \hfill \square

By the last proposition and the Section 9.1 we have:

**Theorem 9.14.** The Conjecture 9.12 implies the associativity of the group quantum cohomology. In particular if $m \in G^4$ is such that $m_1 m_2 m_3 = id$ and if $\sigma_1, \sigma_2$ and $\sigma_3$ are classes of $qA^*(X,G)$ supported on the twisted sectors corresponding to $m_1, m_2$ and $m_3$ respectively, then the equality $\zeta_{1a,1b} \cdot [m] = \epsilon_{1a} \left[ \zeta_{1a,1b}^{-1} m \right]$ in Conjecture 9.12 implies $(\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3)$.

### 9.6 Smooth locus of $\mathcal{M}_{g,A}^G \to \mathcal{M}_{g,A}^G$

We begin by studying the ramification locus of the morphism $\mathcal{M}_{g,A}^G \to \mathcal{M}_{g,A}^G$ that sends a $G$-curve to its quotient.

**Lemma 9.15.** Let $E \to \text{Spec} \mathbf{C}$ be a nodal curve with admissible generically free $G$-action, and let $\pi_E : E \to C := \tilde{E}$ be the quotient. Let $\Omega_{E/C} := \Omega_E / \pi_E^* \Omega_C$, and $k = |\text{Stab}_G(p)|$. Then for any $p \in \text{sp}(E)$, we have

$$l(\Omega_{E/C,p}) = k - 1.$$  

if $p$ is a smooth point and

$$l(\Omega_{E/C,p}) = (k - 1)^2.$$  

if $p$ is a node. In particular $\pi_E : E \to C$ is étale except at points with non-trivial stabilizers.

**Proof.** The result is trivial for smooth $p \in \text{sp}(E)$. Suppose $p$ is a nodal point, and $q = \pi_E(p)$ which is also nodal as the action is admissible ( [Miz05]). Let $\tilde{E}$ (resp. $\tilde{C}$) be the local scheme $\text{Spec} \mathcal{O}_{E,p}$ (resp. $\text{Spec} \mathcal{O}_{C,q}$), where $\mathcal{O}_{E,p} \cong \text{Spec} \mathbf{C}[x,y]/(xy)$ is the completion of the local ring $\mathcal{O}_{E,p}$, where $x$ and $y$ are both eigenvectors for the action of $\text{Stab}_G(p)$, and similarly for $\mathcal{O}_{C,q} \cong \text{Spec} \mathbf{C}[u,v]/(uv)$, where $u \mapsto x^k$ and $v \mapsto y^k$ under the quotient $\text{Spec} \mathcal{O}_{E,p} \to \text{Spec} \mathcal{O}_{C,q}$.

Then $\Omega_{\tilde{E}} \cong \mathbf{C}[x,y] \oplus \mathbf{C}[y]$ and the derivation $\mathcal{O}_{\tilde{E}} \to \Omega_{\tilde{E}}$ induces the $\mathbf{C}$-derivation

$$d_E : \mathbf{C}[x,y]/(xy) \to \mathbf{C}[x,y] \oplus \mathbf{C}[y]$$

such that $dx := (1,0)$ and $dy := (0,1)$. Similarly for $\Omega_{\tilde{C}} \cong \mathbf{C}[u,v] \oplus \mathbf{C}[v]$, and $d_C$. Hence the morphism $\Omega_{\tilde{C}} \to \Omega_{\tilde{E}}$ of $\mathcal{O}_{\tilde{C}}$-modules, induces $\mathbf{C}[u] \oplus \mathbf{C}[v] \to \mathbf{C}[x] \oplus \mathbf{C}[y]$, the unique morphism of $\mathbf{C}[u,v]/(uv)$-modules such that $du \mapsto kx^{k-1}dx$ and $dv \mapsto ky^{k-1}dy$. It follows that

$$\Omega_{E/C} := \Omega_{E}/\pi_E^* \Omega_{E} \cong \mathbf{C}[x]/(x^{k-1}) \oplus \mathbf{C}[y]/(y^{k-1})$$

\hfill \square
Theorem 9.16. Let $\mathcal{M}_{g,A}$ be the locus in $\mathcal{M}_{g,A}^G$ of admissible $G$-curves whose nodes (if any) have trivial stabilizers in $G$. Then $\pi : \mathcal{M}_{g,A}^G \to \mathcal{M}_{g,A}$ is étale.

Proof. Let $\tilde{\mathcal{M}} := \mathcal{M}_{g,A}^G$, $\mathcal{M} := \mathcal{M}_{g,A}$ and $\mathcal{M} := \mathcal{M}_{g,A}$. Let $t : T \to T'$ be a square 0 extension by a quasi-coherent ideal $\mathcal{J}$. Let $\epsilon$ be a family of $\mathcal{M}$ over $T$ and $\epsilon$ be a family of $\mathcal{M}$ over $T'$. Suppose there is a morphism $(\alpha, t) : \epsilon \to \epsilon$ over $t$. Consider the diagram

\[
\begin{array}{ccc}
\Sigma_t & \xrightarrow{\alpha} & \Sigma_t \\
\downarrow & & \downarrow \\
E_t & \xrightarrow{\alpha} & E_t \\
\downarrow & & \downarrow \\
\tilde{E}_t & \xrightarrow{p_t} & C_t \\
\downarrow & & \downarrow \\
T & \xrightarrow{t} & T'
\end{array}
\]

where $\Sigma_t$ is the unique deformation of $\Sigma_t$ to $T'$ (the existence and uniqueness follows from étaleness of $\Sigma_t \to T$). We will study the obstruction to completing the above diagram with the dotted lines, such that $f := (E_t, p_t, \Sigma_t, \sigma_t)$ is an object of $\mathcal{M}_{g,A}^G$, and such that $t^*f = \epsilon$.

For simplicity let $E := E_t$ and $\bar{E} := \tilde{E}_t$. The obstructions to extending the above diagram lie in the group $\text{Ext}^{G,2}(\Omega_{E/E}, I)$ where

$I := p_t^* \mathcal{J} \otimes_{\mathcal{O}_{T'}} \mathcal{I}(\Sigma_t)$

and where $\mathcal{I}(\Sigma_t)$ is the ideal sheaf of $\Sigma_t$ in $E_t$. Using the Grothendieck spectral sequence applied to the functor $\text{Hom}^{G} = H^{G,0} \circ \text{Hom}$, to show that the above group is 0, it suffices to show that

$H^{G,0}_E \text{Ext}^2(\Omega_{E/E}, I) = 0$

$H^{G,1}_E \text{Ext}^1(\Omega_{E/E}, I) = 0$

$H^{G,2}_E \text{Hom}(\Omega_{E/E}, I) = 0$.

Using Grothendieck flat base change theorem, it suffices to assume that $T = \text{Spec } \mathbb{C}$. The group $H^{G,2}_E \text{Hom}(\Omega_{E/E}, I)$ vanishes as $E$ is of dimension 1, and $H^{G,1}_E \text{Ext}^1(\Omega_{E/E}, I)$ vanishes as $\Omega_{E/E}$ is concentrated at the nodes and markings. Next we show that $\text{Ext}^2(\Omega_{E/E}, I) = 0$.

The exact sequence of $G$-$\mathcal{O}_E$-modules

$0 \to \pi^* \Omega_E \to \Omega_E \to \Omega_{E/E} \to 0$

where $\pi : E \to \bar{E}$ is the quotient, induces a long exact sequence

$\cdots \to \text{Ext}^1(\pi^* \Omega_E, I) \to \text{Ext}^2(\Omega_{E/E}, I) \to 0$
where $\text{Ext}^2(\Omega_E, \mathcal{I}) = 0$ as $E \to \text{Spec} \mathbb{C}$ is a reduced complete intersection, that is $\Omega_E$ has a two step locally free resolution. If $p \in \text{sp}(E)$ is a node with trivial stabilizer, it follows by Lemma 9.15 that $\Omega_{E/\mathbb{P}} = 0$. If $p \in \text{sp}(E)$ is a smooth point, $q = \pi(p)$ is a smooth point and $\pi_*\Omega_E$ is a free $\mathcal{O}_E$-module in a neighborhood of $p$, and $\text{Ext}^1(\pi^*\Omega_E, \mathcal{I})_p = 0$. The two above cases imply $\text{Ext}^2(\Omega_{E/\mathbb{P}}, \mathcal{I}) = 0$. \qed

9.7 $\mathbb{Z}_k$-covers of $\mathbb{P}^1$ by $\mathbb{P}^1$

Let $G = \mathbb{Z}_k = \langle \tau \rangle$. Suppose $\nu_1 = \tau$, $\nu_2 = \tau^{-1}$ and $\nu_3 = \text{id}$. Then $\overline{\mathcal{M}}_{0,3}^{\mathbb{Z}_k,\nu} \simeq \text{Spec} \mathbb{C}$.

**Proof.** The dimension of $\overline{\mathcal{M}}_{0,3}^{\mathbb{Z}_k,\nu}$ is 0. Moreover the objects of $\overline{\mathcal{M}}_{0,3}^{\mathbb{Z}_k,\nu}$ over $\text{Spec} \mathbb{C}$ are isomorphic to the $k$-fold cover of $\mathbb{P}^1$ with 3-markings where two are totally ramified. This cover has no non-trivial automorphisms, and the result follows. \qed

Suppose $\nu_1 = \tau$, $\nu_2 = \tau^{-1}$ and $\nu_3 = \nu_4 = \text{id}$. We have an isomorphism $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu} \simeq \mathbb{P}^1$ such that the projection $\pi : \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu} \to \mathcal{M}_{0,4} \simeq \mathbb{P}^1$ induces a degree $k$ morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with two totally ramified points.

**Proof.** We know that $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ is a smooth Deligne-Mumford stack of dimension 1. Let $\mathcal{E} := (E, \Sigma, \sigma)$ be an object of $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ over $\mathbb{C}$, and let $\bar{\mathcal{E}} := (\bar{E}, \bar{\sigma})$ be the projection onto $\mathcal{M}_{0,4}$. Since $\bar{E} = E/\mathbb{Z}_k$ is a genus 0 curve, and the quotient $E \to \bar{E}$ has two ramification points $\sigma_1$ and $\sigma_2$ of ramification degree $k$, by the Hurwitz formula for nodal curves we get that $E$ has arithmetic genus 0. The automorphism group of $\mathcal{E}$ is trivial as there is no automorphism of a genus 0 nodal curve with 4-marked points. It follows that $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ has no stacky points and therefore is a smooth curve.

Since each smooth stable curve of $\mathcal{M}_{0,4}$ has $k$ non isomorphic $\mathbb{Z}_k$-covers with monodromies $\nu_1, \ldots, \nu_4$, the morphism $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu} \to \mathcal{M}_{0,4}$ has degree $k$, and it is a degree $k$ cover of $\mathbb{P}^1$.

The action of $\mathbb{Z}_k$ on $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ by translation of the $4^{th}$ section, makes $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ into a $\mathbb{Z}_k$-cover of $\mathbb{P}^1$.

Next we show that this morphism has 2 totally ramified points. Let $\Gamma = (\mathbb{P}^1, 1, 0, \infty)$ be the 3-marked $\mathbb{Z}_k$-cover where $\mathbb{Z}_k$ acts with fixed points 0 and $\infty$ of monodromy $\tau$ and $\tau^{-1}$ respectively. We glue two copies of $\Gamma$ along 0 of the first and $\infty$ of the second, and we can order the markings in two non-isomorphic ways. We get two point of $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu}$ which are fixed under $\mathbb{Z}_k$.

Finally we show that there can't be more ramification points. By Theorem 9.16 the only possible ramifications are at points $\mathbf{c} := (E, \Sigma, \sigma)$ where $E$ has a node with a non-trivial stabilizer. Let $x \in E$ be a node. Let $\bar{E} = \bar{E}_1 \cup \bar{E}_2$ be the decomposition of the quotient $E$ into its two irreducible components, and let $E_1$ (resp. $E_2$) be the closed subvariety of $E$ above $\bar{E}_1$ (resp. $\bar{E}_2$). Suppose $\sigma_i$ and $\sigma_j \in E_1$ for $i \neq j$, then the monodromy of $x$ on $E_1$ must be $(\nu_i \nu_j)^{-1}$. In the case $\nu_i$ or $\nu_j$ is id, we get one of the two points of the last paragraph. Otherwise the monodromy of $x$ is id and the curve $E$ has no nodes with non-trivial stabilizer. It follows that $\overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_k,\nu} \simeq \mathbb{P}^1$. \qed
Chapter 9. Associativity

Theorem 9.17. Let \( \mathbb{Z}_k \) act on a smooth projective variety \( X \). Let \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) be classes of \( q\mathbb{A}^*(X, \mathbb{Z}_k) \) supported on the twisted sectors corresponding to \( \text{id} \), \( \text{id} \) and \( \tau \) respectively. Then \( (\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3) \).

Proof. Let \( m = (\text{id}, \text{id}, \tau, \tau^{-1}) \). Then for any \( \zeta \in B_4, \zeta \cdot e_m \) and \( e_{\zeta m} \) are both points in \( \overline{\mathcal{M}}_{0,4}^{Z_k, m} \cong \mathbb{P}^1 \), so they are rationally equivalent, and the result follows from Theorem 9.14. \( \square \)

9.8 \( \mathbb{Z}_2 \)-covers of \( \mathbb{P}^1 \) by genus 1 curves

Let \( G = \mathbb{Z}_2 = \langle \tau \rangle, \nu_1 = \ldots = \nu_4 = \tau \). The morphism \( \pi : \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \rightarrow \overline{\mathcal{M}}_{0,4} \) is an étale \( \mathbb{Z}_2 \)-gerbe.

Proof. We know that \( \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \) is a smooth Deligne-Mumford stack of dimension 1, and that \( \pi \) is étale over the locus \( \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \) of curves with nodes whose stabilizer is trivial by Theorem 9.16. We will show that \( \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} = \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \).

Let \( \epsilon := (E, \Sigma, \sigma) \) be an object of \( \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \) over \( C \). Let \( \tilde{\epsilon} := (\tilde{E}, \tilde{\sigma}) \) be the projection to \( \overline{\mathcal{M}}_{0,4} \). Suppose that \( \tilde{E} \) has two rational components (otherwise \( E \) itself is smooth and \( \epsilon \) is in \( \overline{\mathcal{M}}_{0,4}^{\mathbb{Z}_2, \nu} \)), and let \( \tilde{E} = \tilde{E}_1 \cup \tilde{E}_2 \) be the decomposition into its two irreducible components. Let \( \tilde{E}_1 \) (resp. \( \tilde{E}_2 \)) be the closed subvarieties of \( E \) above \( \tilde{E}_1 \) (resp. \( \tilde{E}_2 \)), and let \( p \in E \) be a node (necessarily lying above the unique node of \( E \)). Let \( i, j \in \{1, 2, 3, 4\} \) such that \( i \neq j \) and \( \sigma_i, \sigma_j \in E_i \). As \( E_1 \) is rational, the monodromy of \( p \) on \( E_1 \) must be \( (\nu_i \nu_j)^{-1} = \text{id} \), that is the stabilizer of \( p \) is trivial. It follows that \( \pi \) is an étale gerbe. \( \square \)

Theorem 9.18. Let \( \mathbb{Z}_2 \) act on a smooth projective variety \( X \). Then \( q\mathbb{A}^*(X, \mathbb{Z}_2) \) is associative.

Proof. Let \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) be classes of \( q\mathbb{A}^*(X, \mathbb{Z}_2) \) supported on the twisted sectors corresponding to \( \tau, \tau \) and \( \tau \) respectively. Let \( m = (\tau, \tau, \tau, \tau) \). Then for any \( \zeta \in B_4, \zeta \cdot e_m \) and \( e_{\zeta m} = e_m \) are both geometric points in \( \overline{\mathcal{M}}_{0,4}^{Z_2, m} \) which is an étale gerbe over \( \mathbb{P}^1 \), so they are rationally equivalent, and \( (\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3) \) follows from Theorem 9.14.

Let \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) be classes of \( q\mathbb{A}^*(X, \mathbb{Z}_2) \) supported on the twisted sectors corresponding to \( \text{id}, \tau \) and \( \tau \) respectively. Let \( m = (\text{id}, \tau, \tau, \text{id}) \). Then for any \( \zeta \in B_4, \zeta \cdot e_m \) and \( e_{\zeta m} = e_m \) are both geometric points in \( \overline{\mathcal{M}}_{0,4}^{Z_2, m} \cong \mathbb{P}^1 \) by the last section, so they are rationally equivalent, and \( (\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3) \) follows from Theorem 9.14.

The case where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are classes of \( q\mathbb{A}^*(X, \mathbb{Z}_2) \) supported on different sectors is reduced to the case covered in the last section by using the commutativity of \( q\mathbb{A}^*(X, \mathbb{Z}_2) \). \( \square \)
Appendix A

Equivariant Riemann-Roch for Nodal Curves

A.1 Introduction

Let $X$ be a projective $G$-variety over an algebraically closed field $k$. Let $\mathcal{F}$ be a $G-O_X$-module. The $k$-vector spaces $\mathcal{H}^i(X, \mathcal{F})$ have natural structures of $k$-representations of $G$, and they define characters $\chi_{\mathcal{H}^i(X, \mathcal{F})}$ of $G$. Define the $G$-equivariant Euler characteristic of $\mathcal{F}$ to be the virtual character

$$X_{\mathcal{F}} = \sum_{i \geq 0} (-1)^i \chi_{\mathcal{H}^i(X, \mathcal{F})}.$$

Let $E$ be a projective nodal $G$-curve over $k$. Under certain conditions on the $G$-action on $E$, there is a Riemann-Roch type formula expressing the $G$-equivariant Euler characteristic $X_{\mathcal{F}}$ in terms of characters of the representations of stabilizers on the fibers of $\mathcal{F}$.

A.2 Pushforward and pullback

Let $X$ be a $G$-scheme. We denote by $\text{Sh}^G(X)$ (resp. $\text{qCoh}^G(O_X)$, resp. $\text{P}^G(O_X)$) the category of $G$-sheaves of abelian groups (resp. quasi-coherent $G-O_X$-modules, resp. locally free $G-O_X$-modules of finite rank) on $X$. When there is no ambiguity about the structure of ringed space on $X$ we will simply write $\text{qCoh}^G(X)$ (resp. $\text{P}^G(X)$) for $\text{qCoh}^G(O_X)$ (resp. $\text{P}^G(O_X)$).

Suppose $f : X \to Y$ is a $G$-morphism of $G$-schemes. There is a pullback functor for sheaves $f^{-1} : \text{Sh}^G(Y) \to \text{Sh}^G(X)$ and a pullback functor for modules $f^* : \text{qCoh}^G(X) \to \text{qCoh}^G(Y)$ (see [Gro57]). If $X$ is noetherian or $f$ is quasi-compact and separated, then there is a pushforward functor $f_* : \text{qCoh}^G(X) \to \text{qCoh}^G(Y)$. If the $G$-action on $Y$ is trivial, one also has an invariant pushforward functor $f^G_* : \text{qCoh}^G(X) \to \text{qCoh}(Y)$ defined on objects by $f^G_* \mathcal{F}(U) = \mathcal{F}(f^{-1}U)^G$ for an open $U \subseteq X$.

Assume that $G$ is a finite group and $X$ is a $G$-scheme such that the quotient $Y = X/G$ exists as a scheme. Let $\pi : X \to Y$ be the quotient morphism. It is proper, and by the preceding paragraph we have an invariant pushforward

$$\pi_*^G : \text{qCoh}^G(X) \to \text{qCoh}(Y).$$

We will now list without proof some properties of $\pi_*^G$.

Lemma 1.1. The canonical injection $\pi_*^G \mathcal{F} \to \pi_* \mathcal{F}$ and the canonical morphism $\pi^* \pi_* \mathcal{F} \to \mathcal{F}$ induce a morphism of $G-O_X$-modules $\pi^* \pi_*^G \mathcal{F} \to \pi^* \mathcal{F} \to \mathcal{F}$ that is an isomorphism away from points with non trivial stabilizers, thus injective.
Appendix A. Equivariant Riemann-Roch for Nodal Curves

Definition 1.2 ([Bor00], 2.11). A $G$-scheme $X$ is called locally reductive if for each point $P \in \text{sp}(X)$, the order of the stabilizer $G_P$ is invertible in $\mathcal{O}_{X,P}$.

Lemma 1.3 ([Bor00], 2.12 and 2.29). Suppose that $X$ is a locally reductive $G$-scheme such that the quotient $\pi: X \to X/G = Y$ is flat. If $\mathcal{F}$ is a locally free $G-O_X$-sheaf of finite rank, then $\pi_*^G \mathcal{F}$ is a locally free $O_Y$-module of the same rank and $R^i \pi_*^G(\mathcal{F}) = 0$ for $i > 0$.

A.3 Cohomology

Let $\Gamma_X(O_X)$ (resp. $\Gamma_X^G(O_X)$) be the ring of regular functions (resp. $G$-invariant regular functions) on $X$. Note that $\Gamma_X(O_X)$ is naturally a $G$-ring. We consider the following two functors

$$
\Gamma_X : \text{qCoh}^G(O_X) \to \text{qCoh}^G(\Gamma_X(O_X)),
$$

$$
\Gamma_X^G : \text{qCoh}^G(O_X) \to \text{qCoh}(\Gamma_X^G(O_X))
$$

where the first one is the functor of global sections and second is the functor of $G$-invariant global sections. The two functors are left exact and the category $\text{qCoh}^G(O_X)$ has enough injectives ([Gro57]), hence we have the right derived functors

$$
H_X = R^i \Gamma_X : \text{qCoh}^G(O_X) \to \text{qCoh}(\Gamma_X(O_X)),
$$

$$
H_X^G = R^i \Gamma_X^G : \text{qCoh}^G(O_X) \to \text{qCoh}(\Gamma_X^G(O_X)).
$$

Lemma 1.4. Suppose that $X$ is a locally reductive $G$-scheme that admits a quotient $Y = X/G$. Then

$$
H^i_Y(\pi_*^G \mathcal{F}) \cong H_X^G(\mathcal{F})
$$

for any locally free $G-O_X$-module $\mathcal{F}$ on finite rank and for all $i \geq 0$. In fact there is a natural isomorphism between $H^1_X \circ \pi_*^G |_{\pi^G(O_X)}$ and $H_X^G(\mathcal{F})$.

Proof. We can write

$$
\Gamma_X^G = \Gamma_Y \circ \pi_*^G
$$

and the result follows from the Grothendieck spectral sequence ([Gro57] 5.2.3) and from Lemma 1.3. \qed

Lemma 1.5. Suppose that $X$ is a $G$-scheme such that $|G|$ is invertible in $\Gamma_X(O_X)$. Then

$$
H^i_X(\mathcal{F}) \cong H_X^G(\mathcal{F}).
$$

In fact there is a natural transformation between $\Gamma^G \circ H^i_X$ and $H_X^G$.

Proof. We can write

$$
\Gamma_X^G = \Gamma^G \circ \Gamma_X
$$

where $\Gamma^G : \text{qCoh}^G(\Gamma_X(O_X)) \to \text{qCoh}(\Gamma_X^G(O_X))$ (the functor taking $G$-invariants) is exact, and $R^i \Gamma^G = 0$ for $i > 0$. The result follows from the Grothendieck spectral sequence. \qed
A.4 Representations of finite groups and their characters

Let $G$ be a finite group and $k$ be a field whose characteristic does not divide the order of $G$. Denote by $\text{Rep}_k G$ the category of $k$-representations of $G$ and $G$-equivariant $k$-linear maps, and by $\text{Irr}_k G$ the set of irreducible representations of $G$. Given $V$ a representation of $G$, denote by $\chi_V : G \to k$ the corresponding character. The characters of $G$ generate the character ring $\text{Ch}_k G$.

We have a pairing on $\text{Ch}_k G$ given by

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1})$$

for any $\chi_1$ and $\chi_2 \in \text{Ch}_k G$. If $V$ and $W$ are representations of $G$ and $V$ is irreducible, then $\langle \chi_V, \chi_W \rangle_G$ is the multiplicity of $V$ in $W$.

The inverse $i : G \to G$ induces by composition an involution of $\text{Ch}_k G$, denoted by $\chi \mapsto \chi^\vee$. Note that $\langle \chi_1 \chi, \chi_2 \rangle = \langle \chi_1, \chi_1^\vee \chi_2 \rangle$.

Let $H$ be a subgroup of a finite group $G$. Given a representation $V$ of $G$ we denote by $\text{Res}_G^H V$ the natural representation of $H$ on $V$ given by restriction. Given a representation $W$ of $H$, the space $G \times W$ is a right $H$-representaion with action $(g, v) \cdot h = (gh, h^{-1} \cdot v)$ and the quotient is a representation of $G$ denoted by $\text{Ind}_G^H W$. The operations $\text{Res}_G^H$ and $\text{Ind}_G^H$ are functors between the categories $\text{Rep}_k G$ and $\text{Rep}_k H$. They induce group homomorphisms between the underlying groups of character rings of $G$ and $H$, that will be denoted by the same symbols. They are adjoint in the following sense:

$$\langle \chi_1, \text{Ind}_G^H \chi_2 \rangle_G = \langle \text{Res}_G^H \chi_1, \chi_2 \rangle_H$$

(A.1)

for $\chi_1 \in \text{Ch}_k G$ and $\chi_2 \in \text{Ch}_k H$. We will list some properties of characters.

- For any representation $V$ of $G$

$$\chi_V(id) = \dim V.$$  \hspace{1cm}  \text{(A.2)}

- If $k[G]$ denotes the standard representation of $G$, then

$$\sum_{V \in \text{Irr}_k G} \dim V \chi_V = \chi_{k[G]}.$$  \hspace{1cm}  \text{(A.3)}

- For any $\chi \in \text{Ch}_k G$

$$\sum_{V \in \text{Irr}_k G} \dim V \langle \chi_V, \chi \rangle = \chi(id).$$  \hspace{1cm}  \text{(A.4)}

**Lemma 1.6.** If $H$ is an abelian subgroup of $G$ and $\chi \in \text{Ch}_k H$, then

$$\sum_{V \in \text{Irr}_k H} \text{Ind}_G^H(\chi_V \chi) = \chi(id) \chi_{k[G]}.$$
Proof. The irreducible characters of $G$ form an orthogonal basis for $\text{Ch}_k G$ under the pairing $\langle -,- \rangle_G$, hence by Equation A.1 we have

$$\sum_{V \in \text{Irr}_k G} \text{Ind}^G_H (\chi_V \mathcal{A}) = \sum_{W \in \text{Irr}_k G} \sum_{V \in \text{Irr}_k H} \langle \text{Ind}^G_H (\chi_V \mathcal{A}), \chi_W \rangle_G \chi_W$$

$$= \sum_{W \in \text{Irr}_k G} \sum_{V \in \text{Irr}_k H} \langle \chi_V \mathcal{A}, \text{Res}^H_W \chi_W \rangle_H \chi_W$$

$$= \sum_{W \in \text{Irr}_k G} \sum_{V \in \text{Irr}_k H} \langle \chi_V, \chi^\vee \text{Res}^G_W \chi_W \rangle_H \chi_W.$$

The irreducible representations of the abelian group $H$ are one dimensional, hence by Equation A.2 to Equation A.4

$$\sum_{V \in \text{Irr}_k H} \text{Ind}^G_H (\chi_V \mathcal{A}) = \sum_{W \in \text{Irr}_k G} (\chi^\vee \text{Res}^G_W \chi_W) (\text{id}) \chi_W$$

$$= \chi^\vee (\text{id}) \sum_{W \in \text{Irr}_k G} \dim W \chi_W$$

$$= \chi (\text{id}) \chi_{k[G]}.$$

\[\square\]

A.5 $K$-theory

Let $k$ be a field whose characteristic does not divide the order of $G$. Let $X$ be a finite dimensional $G$-scheme, projective over $k$ and such that the structure morphism $X \to \text{Spec } k$ is $G$-invariant. It follows that $X$ is a locally reductive scheme and that $|G|$ is invertible in $\Gamma_X (\mathcal{O}_X) = \Gamma_X (\mathcal{O}_X) = k$.

Given an abelian category $\mathcal{A}$, let $K(\mathcal{A})$ be the Grothendieck group of $\mathcal{A}$. The tensor product of $G$-$\mathcal{O}_X$-modules induces a ring structure on $K(\text{P}^G (\mathcal{X}))$.

Let $\mathcal{F}$ be a locally free $G$-$\mathcal{O}_X$-module of finite rank. By a theorem of Serre ([Gro61b], 2.2.1) $H^i_X (\mathcal{F})$ are finite dimensional vector spaces over $k$. Moreover they vanish for $i \geq \dim X$ by a theorem of Grothendieck ([Gro57], 3.6.5). The same is true for $H^i_G (\mathcal{F})$. It follows that the two functors

$$\Gamma_X : \text{P}^G (\mathcal{X}) \to \text{P}^G (k)$$

$$\Gamma^G_X : \text{P}^G (\mathcal{X}) \to \text{P}(k)$$

induce ring homomorphisms on $K$-theory

$$K(\Gamma_X) : K(\text{P}^G (\mathcal{X})) \to K(\text{P}^G (k)) \approx \text{Ch}_k G$$

$$[\mathcal{F}] \mapsto \sum_{i \geq 0} (-1)^i [H^i_X (\mathcal{F})]$$

$$K(\Gamma^G_X) : K(\text{P}^G (\mathcal{X})) \to K(\text{P}(k)) \approx \mathbb{Z}$$

$$[\mathcal{F}] \mapsto \sum_{i \geq 0} (-1)^i [H^i_G (\mathcal{F})]$$
We define $\mathcal{X}_X$ (resp. $\mathcal{X}_X^G$) to be the composition of $K(\Gamma_X)$ (resp. $K(\Gamma_X^G)$) with the isomorphism $\mathcal{X}: K(P^G(k)) \simeq \text{Ch}_k(G)$ (resp. $\dim : K(P(k)) \simeq \mathbb{Z}$). Given an object $\mathcal{F}$ of $P^G(X)$ we will write $\mathcal{X}_{X,\mathcal{F}}$ (resp. $\mathcal{X}_{X}^G(\mathcal{F})$) for $\mathcal{X}_X([\mathcal{F}])$ (resp. $\mathcal{X}_X^G([\mathcal{F}])$).

In fact we can express $\mathcal{X}_X$ in terms of $\mathcal{X}_X^G$.

**Lemma 1.7.** Let $\mathcal{F}$ be an object of $P^G(X)$. Then

$$\mathcal{X}_{X,\mathcal{F}} = \sum_{V \in \text{Ir}_k G} \mathcal{X}_{X,\mathcal{F} \otimes V^\vee}^G \mathcal{X}_V$$

where $\mathcal{F} \otimes V^\vee$ is by definition $\mathcal{F} \otimes_{\mathcal{O}_X} V^\vee$ where $V$ is the free $G\cdot \mathcal{O}_X$-module corresponding to the constant $G$-vector bundle $X \times V$.

**Proof.**

$$\mathcal{X}_{X,\mathcal{F}} = \sum_{V \in \text{Ir}_k G} \langle \mathcal{X}_{X,\mathcal{F}}, \mathcal{X}_V \rangle \mathcal{X}_V$$

$$= \sum_{V \in \text{Ir}_k G} \langle \mathcal{X}_{X,\mathcal{F}} \mathcal{X}_V, \text{id} \rangle \mathcal{X}_V$$

$$= \sum_{V \in \text{Ir}_k G} \langle \mathcal{X}_{X,\mathcal{F} \otimes V^\vee}, \text{id} \rangle \mathcal{X}_V.$$

We used $H^i_X(\mathcal{F}) \otimes V^\vee = H^i_X(\mathcal{F} \otimes V^\vee)$ where $V$ is a representation of $G$. By Lemma 1.5, we have that

$$\langle \mathcal{X}_{X,\mathcal{F} \otimes V^\vee}, \text{id} \rangle = \langle \sum_{i \geq 0} (-1)^i \mathcal{X}_V^{H^i_X(\mathcal{F} \otimes V^\vee)}, \text{id} \rangle$$

$$= \sum_{i \geq 0} (-1)^i \dim H^i_X(\mathcal{F} \otimes V^\vee)^G$$

$$= \sum_{i \geq 0} (-1)^i \dim H^i_X(\mathcal{F} \otimes V^\vee)^G$$

$$= \sum_{i \geq 0} (-1)^i \dim H^i_X(\mathcal{F} \otimes V^\vee)$$

$$= \mathcal{X}_{X,\mathcal{F} \otimes V^\vee}^G,$$

and the result follows.

**A.6 The smooth case**

Let $G$ be a finite group and $k$ be an algebraically closed field whose characteristic does not divide the order of $G$. Let $E$ be a smooth projective $G$-curve over $k$. In this section, we will compute the homomorphism $\mathcal{X}_E$ just as in [EL80] and in [Bor03].

Let $P \in \text{sp}(E)$ and suppose $\mathcal{F}$ is a $G\cdot \mathcal{O}_E$-module such that $\mathcal{F}_P$ is a free $G_P\cdot \mathcal{O}_{E,p}$-module of finite rank. Define

$$\mathcal{X}_{P,\mathcal{F}} := \mathcal{X}_{P,\mathcal{F}_P} \in \text{Ch}_k G_P$$

$$\mathcal{X}_{P}^G := \dim F_P^G \in \mathbb{Z}$$

where $F_P = k \otimes_{\mathcal{O}_{E,p}} \mathcal{F}_P$, which is naturally a $k$-representation of $G_P$. 
Definition 1.8. If the set of points with trivial stabilizers is dense in $E$ we call the $G$-action generically free.

Suppose $E$ is endowed with a generically free $G$-action. If $P \in E$ is a smooth point, then the stabilizer $G_P$ is cyclic as it has a faithful one dimensional representation on the tangent space of $E$ at $P$.

Lemma 1.9. Let $E$ be a smooth projective curve over $k$ with a generically free $G$-action and let $\pi : E \rightarrow C$ be the quotient. Let $\mathcal{F}$ be a locally free $G_{\mathcal{O}_E}$-module of finite rank. Denote by $\Omega_E$ the sheaf of Kähler differentials of $E$. For each $P \in E$ let $e_P = \lvert G_P \rvert$. Then

$$\deg \pi^* \mathcal{F} = \frac{1}{|G|} \deg \mathcal{F} - \frac{1}{|G|} \sum_{P \in E} \sum_{j=0}^{e_P-1} \chi_{\mathcal{G}_P}^{j}.$$ 

Proof. The canonical morphism $\pi^* \pi^* \mathcal{F} \rightarrow \mathcal{F}$ is injective and we have an exact sequence of $G_{\mathcal{O}_E}$-modules

$$0 \rightarrow \pi^* \pi^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{R} \rightarrow 0. \tag{A.5}$$

where $\mathcal{R}$ is a $G_{\mathcal{O}_E}$-module supported at the ramification points of $\pi : E \rightarrow C$. Thus

$$\deg \pi^* \mathcal{F} = \frac{1}{\deg \pi} \deg \pi^* \pi^* \mathcal{F}$$

$$= \frac{1}{|G|} (\deg \mathcal{F} - \deg \mathcal{R})$$

$$= \frac{1}{|G|} (\deg \mathcal{F} - \sum_{P \in E} \text{l}(\mathcal{R}_P))$$

where $\text{l}(\mathcal{R}_P)$ is the length of the $\mathcal{O}_{P,E}$-module $\mathcal{R}_P$. Let $P \in E$ and $Q = \pi(P)$. By localizing and completing Equation A.5 we have

$$0 \rightarrow \mathcal{O}_{E,P} \otimes_{\mathcal{O}_{C,Q}} \mathcal{F}_P^G \rightarrow \mathcal{F}_P \rightarrow \mathcal{R}_P \rightarrow 0. \tag{A.6}$$

Now $\mathcal{O}_{E,P} \simeq k[[u]]$ where $u$ is an eigenvector for $G_P$, spanning a faithful representation with character $\chi_{P,\Omega_E}$, which implies $\mathcal{O}_{C,Q} \simeq \mathcal{O}_{E,P}^G \simeq k[[u^{e_P}]]$. We have $\mathcal{F}_P \simeq \mathcal{O}_{E,P} \otimes_k F_P \simeq k[[u]] \otimes_k F_P$. The inclusion in Equation A.6 is just the homomorphism $k[[u]] \otimes k[[u^{e_P}]] \otimes_k F_P \rightarrow k[[u]] \otimes_k F_P$ coming from the inclusion $(k[[u]] \otimes_k F_P)^G \rightarrow k[[u]] \otimes_k F_P$ by tensoring by $k[[u]]$.

The group $G_P$ is cyclic, hence the irreducible representations of $G_P$ are all one dimensional and their characters are homomorphisms $G_P \rightarrow k^*$. We can talk about the eigenspace $V_{\chi}$ of some character $\chi$ and a representation $V$ of $G_P$. The representation $\Omega_{E,P}$ of $G_P$ is faithful and thus powers of $\chi_{P,\Omega_E}$ run over all irreducible characters of $G_P$. 


It follows that
\[
(k[[u]] \otimes_k F_P)^{G_P} = \bigoplus_{P \in \text{Irreducible } G_P} k[[u]] \chi^{j}_{P,0} \otimes_k (F_P)_{X^{j}_{P,0} E}^{\vee}
\]
\[
= \bigoplus_{j=0}^{e_{P}-1} k[[u]] \chi^{j}_{P,0} \otimes_k (F_P)_{X^{j}_{P,0} E}^{\vee}
\]
and
\[
\hat{\mathcal{O}}_{E,P} \otimes_{\mathcal{O}_{C,Q}} \mathcal{F}^{G_P}_{P} = \bigoplus_{j=0}^{e_{P}-1} k[[u]] \otimes_{k[[u^{e_{P}}]]} k[[u^{e_{P}}]] u^{j} \otimes_k (F_P)_{X^{j}_{P,0} E}^{\vee}
\]
where \(< u^{j} >\) is the ideal in \(k[[u]]\) generated by \(u^{j}\). It follows that
\[
\text{lp}(\mathcal{R}_P) = \sum_{j=0}^{e_{P}-1} j \cdot \dim(F_P)_{X^{j}_{P,0} E}^{\vee}
\]
\[
= \sum_{j=0}^{e_{P}-1} j \cdot \langle \chi^{j}_{P,0} E, \mathcal{X}_{P,F} \rangle_{G_P}.
\]

Now we compute \(\chi^{G}_{E} \) and \(\chi_{E} \).

**Proposition 1.10.** Let \(E\) be a smooth projective curve over \(k\) with a generically free \(G\)-action and let \(\pi : E \to C\) be the quotient. Let \(\mathcal{F}\) be a locally free \(G-O_E\)-module of finite rank. For each \(P \in E\) let \(e_{P} = |G_P|\). Then
\[
\chi^{G}_{E,F} = \frac{1}{|G|} \deg \mathcal{F} + \chi_{C,O_C} \text{rk } \mathcal{F} - \frac{1}{|G|} \sum_{P \in E} \sum_{j=0}^{e_{P}-1} j \cdot \chi^{G}_{P,0} \mathcal{F}_{P,0}^{\vee}
\]
\[
\chi_{E,F} = \left( \frac{1}{|G|} \deg \mathcal{F} + \chi_{C,O_C} \text{rk } \mathcal{F} \right) \chi_{k[|G|]} - \frac{1}{|G|} \sum_{P \in E} \sum_{j=0}^{e_{P}-1} j \cdot \text{Ind}^{G}_{P,0} \left( \chi^{G}_{P,0} \mathcal{F}_{P,0}^{\vee} \right).
\]

**Proof.** Using Lemma 1.4 and Lemma 1.5 we have \(\chi^{G}_{E,F} = \chi_{C,\pi^{G}_{E,F}}\). Applying the Riemann-Roch and Lemma 1.9 to \(\pi^{G}_{E,F}\) (which is locally free of the same rank as \(\mathcal{F}\) by Lemma 1.3):
\[
\chi_{C,\pi^{G}_{E,F}} = \deg \pi^{G}_{E,F} + \chi_{C,O_C} \text{rk } \mathcal{F}
\]
\[
= \frac{1}{|G|} \deg \mathcal{F} + \chi_{C,O_C} \text{rk } \mathcal{F} - \frac{1}{|G|} \sum_{P \in E} \sum_{j=0}^{e_{P}-1} j \cdot \langle \chi_{P,0}^{G}_{E}, \chi_{P,F} \rangle_{G_P}.
\]
and the first statement follows.

By Lemma 1.7

\[ \chi_{E,\mathcal{F}} = \sum_{\mathcal{V} \in \text{Irr}_G} \chi_{E,\mathcal{F} \otimes \mathcal{V}} \chi_{\mathcal{V}}. \]

We apply the first statement of the Proposition to each of the locally free \( G - \mathcal{O}_E \)-module \( \mathcal{F} \otimes \mathcal{V} \) to get

\[
\begin{align*}
\chi_{E,\mathcal{F}} & = \sum_{\mathcal{V} \in \text{Irr}_G} \left( \frac{1}{|G|} \deg(\mathcal{F} \otimes \mathcal{V}) + \chi_{\mathcal{C},\mathcal{O}_E} \rk(\mathcal{F} \otimes \mathcal{V}) - \frac{1}{|G|} \sum_{j=0}^{e_p-1} \frac{1}{|G|} \sum_{i=0}^{e_p-1} \langle \chi_{P,\Omega^i_E}, \chi_{P,\mathcal{F} \otimes \mathcal{V}} \rangle_{G_P} \right) \chi_{\mathcal{V}} \\
& = \sum_{\mathcal{V} \in \text{Irr}_G} \left( \frac{1}{|G|} \deg \mathcal{F} \chi_{\mathcal{C},\mathcal{O}_E} \rk \mathcal{F} \right) \dim \mathcal{V} - \frac{1}{|G|} \sum_{j=0}^{e_p-1} \frac{1}{|G|} \sum_{i=0}^{e_p-1} \langle \chi_{P,\Omega^i_E}, \chi_{P,\mathcal{F} \otimes \mathcal{V}} \rangle_{G_P} \chi_{\mathcal{V}} \\
& = \left( \frac{1}{|G|} \deg \mathcal{F} \chi_{\mathcal{C},\mathcal{O}_E} \rk \mathcal{F} \right) \chi_{k[G]} - \frac{1}{|G|} \sum_{j=0}^{e_p-1} \frac{1}{|G|} \sum_{i=0}^{e_p-1} j \cdot \text{Ind}_{G_P}^G \left( \chi_{P,\mathcal{F} \otimes \mathcal{O}_E,\Omega^i_E} \right)
\end{align*}
\]

as

\[
\sum_{\mathcal{V} \in \text{Irr}_G} \langle \text{Res}_{G}^G(\chi_{\mathcal{V}}), \chi_{P,\Omega^i_E}^{j} \chi_{P,\mathcal{F}} \rangle_{G_P} \chi_{\mathcal{V}} = \sum_{\mathcal{V} \in \text{Irr}_G} \langle \chi_{\mathcal{V}}, \text{Ind}_{G_P}^G \left( \chi_{P,\Omega^i_E}^{j} \chi_{P,\mathcal{F}} \right) \rangle_{G_P} \chi_{\mathcal{V}}
\]

\[= \text{Ind}_{G_P}^G \left( \chi_{P,\mathcal{F} \otimes \mathcal{O}_E,\Omega^i_E} \right). \]

\[ \square \]

### A.7 Normalization of a \( G \)-curve

Let \( E \) be a curve and \( \eta : \tilde{E} \to E \) be its normalization. Any \( G \)-action on \( E \) induces a unique \( G \)-action on \( \tilde{E} \) making \( \eta \) into a \( G \)-equivariant morphism.

**Lemma 1.11.** Suppose \( E \) is a projective \( G \)-curve and \( \pi : E \to C \) is the quotient. Let \( \eta_E : \tilde{E} \to E \) be a normalization of \( E \). Then the induced morphism \( \tilde{E}/G \to C \) is a normalization of \( C \).

**Proof.** First note that given a smooth dense open subset \( C' \subseteq C \) then the normalization of \( C \) is the unique smooth projective curve \( \tilde{C} \) (up to unique isomorphism) containing \( C' \) as a dense open subset together with a morphism \( \eta_C : \tilde{C} \to C \) whose restriction to \( C' \) is just the inclusion.

Let \( \eta : \tilde{E}/G \to C \) be the induced morphism from the \( G \)-invariant morphism \( \tilde{E} \to E \to C \). Let \( E^\circ \) be the smooth locus of \( E \). Then \( E^\circ/G \) is a smooth and dense subset of \( C \). But this inclusion factors through \( \tilde{E}/G \to C \). And as \( \tilde{E}/G \) is smooth and projective, it is the unique smooth projective curve containing \( E^\circ/G \) as a dense open subset, thus it is the normalization of \( C \). \[ \square \]
It follows that we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\eta_E} & E \\
\downarrow \pi & & \downarrow \\
\tilde{C} & \xrightarrow{\eta_C} & C
\end{array}
\]

where \( \eta_E \) and \( \eta_C \) are normalizations and \( \pi \) and \( \tilde{\pi} \) are quotients.

### A.8 Admissible action

If \( f : Y \to X \) is a \( G \)-morphism of \( G \)-schemes, then it induces a canonical monomorphism of stabilizers \( G_Q \to G_{f(Q)} \) for every \( Q \in Y \).

**Definition 1.12.** A \( G \)-morphism \( f : Y \to X \) is called **fully faithful** if for every point \( Q \in Y \) the canonical monomorphism \( G_Q \to G_{f(Q)} \) is an isomorphism, in which case \( X_Q, f^* = X_{f(Q)}, f \) and

\[ \langle X_Q, f^*, X_{f(Q)}, f \rangle_{G_Q} = \langle X_{f(Q)}, f^*, X_{f(Q)}, f \rangle_{G_{f(Q)}}. \]

**Definition 1.13.** A \( G \)-action on a curve \( E \) is called **admissible** if the normalization \( \eta : \tilde{E} \to E \) is a fully faithful \( G \)-morphism.

**Lemma 1.14.** Let \( G \) be a finite group and \( k \) be an algebraically closed field whose characteristic does not divide the order of \( G \). Suppose \( E \) is a nodal curve with an admissible \( G \)-action defined over \( k \). Then the quotient \( C = E/G \) is a nodal curve, and the projection \( \pi : E \to C \) sends nodes to nodes. Moreover if \( E^o \) and \( C^o \) are the smooth loci of respectively \( E \) and \( C \), then

\[ |C \setminus C^o| = \frac{1}{|G|} \sum_{P \in E \setminus E^o} |G_P|. \]

**Proof.** Let \( P \in E \) be a node and let \( Q_1 \) and \( Q_2 \) be the two points in \( \tilde{E} \) above \( P \). Since \( \eta \) is a dominant morphism of algebraic varieties, we have an injective \( G \)-morphism \( \eta_* \mathcal{O}_{\tilde{E}} \to \mathcal{O}_E \) of \( G \)-\( \mathcal{O}_{\tilde{E}} \)-sheaves. Localizing at \( P \) and taking the completion we get an exact sequence

\[ 0 \to k[[u,v]]/(uv) \xrightarrow{\phi} k[[u]] \oplus k[[v]] \xrightarrow{\delta} k \to 0 \quad (A.7) \]

where \( \tilde{\mathcal{O}}_{E,p} \simeq k[[u,v]]/(uv) \), \( \tilde{\mathcal{O}}_{E,Q_1} \simeq k[[u]] \) and \( \tilde{\mathcal{O}}_{E,Q_2} \simeq k[[v]] \), the homomorphism \( \phi \) is given by \( \phi(u,v) = (p(u,0), p(0,v)) \) and \( \delta \) by \( \delta(p(u), q(v)) = (p(0) - q(0)) \). Since \( G_P \) fixes \( Q_1 \) and \( Q_2 \), we have a \( G_P \)-action on \( k[[u]] \) and \( k[[v]] \) making Equation A.7 \( G_P \)-equivariant, where \( G_P \) acts trivially on \( k \). Since \( G_P \) is finite, there are positive integers \( m \) and \( n \) such that \( k[[u^m]] = k[[u]]^{G_P} \) and \( k[[v^n]] = k[[v]]^{G_P} \). Since taking invariants preserves exactness \( (|G_P| \text{ is invertible in } k) \), we have that \( (k[[u,v]]/(uv))^{G_P} \) is the kernel of the restriction of \( \delta \) to \( k[[u^m]] \oplus k[[v^n]] \). It follows that \( (k[[u,v]]/(uv))^{G_P} \simeq k[[u^m,v^n]]/(u^m v^n) \), which shows that \( \pi(P) \) is nodal since \( \tilde{\mathcal{O}}_{C,\pi(P)} \simeq (\pi_* \mathcal{O}_{\tilde{E}})_{\pi(P)} \simeq \tilde{\mathcal{O}}_{E,P}^{G_P} \).

It follows that \( E \setminus E^o \) is the set of points lying above \( C \setminus C^o \) and the number of points above \( Q \in C \setminus C^o \) is \( \frac{|G|}{|G_P|} \) for any \( P \in E \setminus E^o \) above \( Q \). \( \square \)
A.9 Balanced action

Let $E$ be a possibly singular $G$-curve over $k$. Suppose the $G$-action on $E$ is admissible. Let $Q_i$ be the points on $E$ lying above $P \in E$. Then for any locally free $G\mathcal{O}_E$-module $\mathcal{F}$ of finite rank, $\chi_{Q_i,\mathcal{F}}$ are characters of the same group $G_{Q_i} = G_P$ as $\eta$ is fully faithful.

**Definition 1.15.** An admissible $G$-action on a nodal curve $E$ is balanced at a node $P \in E$ if $\chi_{Q_1,\Omega_E} = \chi_{Q_2,\Omega_E}$. A $G$-action is balanced if it is balanced at all nodes of $E$.

A.10 Equivariant Riemann-Roch for nodal curves

**Theorem 1.16.** Let $G$ be a finite group and $k$ be an algebraically closed field whose characteristic does not divide the order of $G$. Let $E$ be a nodal projective curve over $k$ with a generically free, admissible and balanced $G$-action, and let $\pi : E \to C = E/G$ be the quotient. Let $E^\circ$ be the smooth locus of $E$ and let $\eta : E \to E$ be the normalization. Let $\mathcal{F}$ be a locally free $G\mathcal{O}_E$-module of finite rank. For each $P \in E$ let $e_P = |G_P|$. Then

$$
\chi^G_{E,\mathcal{F}} = \frac{1}{|G|} \deg \eta^* \mathcal{F} + \chi_{G,\mathcal{O}_C} \text{rk } \mathcal{F} - \frac{1}{|G|} \sum_{P \in E^\circ} \sum_{j=0}^{e_P-1} j \cdot \chi^G_{P,\mathcal{F} \otimes \Omega_E} 
$$

$$
\chi_{E,\mathcal{F}} = \left( \frac{1}{|G|} \deg \eta^* \mathcal{F} + \chi_{G,\mathcal{O}_C} \text{rk } \mathcal{F} \right) \chi_{[C]} - \frac{1}{|G|} \sum_{P \in E^\circ} \sum_{j=0}^{e_P-1} j \cdot \text{Ind}^G_{G_P} \left( \chi^G_{P,\mathcal{F} \otimes \Omega_E} \right).
$$

**Proof.** Note that the first statement follows easily from the second. Given a locally free $G\mathcal{O}_E$-module $\mathcal{F}$ of finite rank, the canonical morphism $\mathcal{F} \to \eta_* \eta^* \mathcal{F}$ is injective. Indeed the normalization $\eta : E \to E$ is a dominant morphism of algebraic varieties hence $\mathcal{O}_E \to \eta_* \mathcal{O}_E$ is injective. Tensoring with $\mathcal{F}$ we get an injective morphism $\mathcal{F} \to \eta_* \mathcal{O}_E \otimes \mathcal{F}$ and by the projection formula we have $\eta_* \mathcal{O}_E \otimes \mathcal{F} \simeq \eta_* \eta^* \mathcal{F}$. Thus we have an exact sequence of $G\mathcal{O}_E$-modules

$$
0 \to \mathcal{F} \to \eta_* \eta^* \mathcal{F} \to \mathcal{R} \to 0
$$

where $\mathcal{R}$ is a $G\mathcal{O}_E$-module supported at the singular locus of $E$. We have a long exact sequence of $k$-representations of $G$

$$
0 \to H^0_E(\mathcal{F}) \to H^0_E(\eta^* \mathcal{F}) \to H^0_E(\mathcal{R}) \to 0
$$

$$
\to H^1_E(\mathcal{F}) \to H^1_E(\eta^* \mathcal{F}) \to H^1_E(\mathcal{R}) \to 0
$$

where we used a $G$-equivariant isomorphism $H^0_E(\eta_* \eta^* \mathcal{F}) \simeq H^0_E(\eta^* \mathcal{F})$ which follows from $\eta$ being affine $G$-morphism by a Grothendieck spectral sequence applied to the functor $\Gamma_E \circ \eta_* : \text{qCoh}^G(E) \to \text{qCoh}^G(k)$. Hence we have the following relation between characters

$$
\chi_{E,\mathcal{F}} = \chi_{E,\eta^* \mathcal{F}} - \chi_{E,\mathcal{R}}. \quad (A.8)
$$
Appendix A. Equivariant Riemann-Roch for Nodal Curves

The curve $\tilde{E}$ is smooth and we can apply Proposition 1.10

$$X_{\tilde{E},\eta^*\mathcal{F}} = \left( \frac{1}{|G|} \deg \eta^*\mathcal{F} + X_{\tilde{E}/G,\mathcal{O}_{\tilde{E}/G}} \, \text{rk} \mathcal{F} \right) X_{\mathcal{O}_G}$$

$$- \frac{1}{|G|} \sum_{Q \in \tilde{E}} \sum_{j=0}^{e_Q-1} j \cdot \text{Ind}_{G_Q}^G \left( X_{Q,\eta^*\mathcal{F},X_Q^j,\Omega_{\tilde{E}}} \right)$$

By Lemma 1.11 and Lemma 1.14, we have

$$X_{\tilde{E}/G,\mathcal{O}_{\tilde{E}/G}} = (X_{\mathcal{O}_{\tilde{E}}} + |C \setminus C^0|)$$  \hspace{1cm} (A.9)

$$= X_{\mathcal{O}_{\tilde{E}}} + \frac{1}{|G|} \sum_{P \in E \setminus E^0} e_P.$$  

Let $P \in E \setminus E^0$ and $Q_1$ and $Q_2$ be the two points in $\tilde{E}$ above $P$. Since the $G$-action on $E$ is balanced, $X_{Q_2,\Omega_{\tilde{E}}} = X_{Q_1,\Omega_{\tilde{E}}}$ and it follows that

$$\sum_{j=0}^{e_{Q_1}-1} j \cdot \text{Ind}_{G_{Q_1}}^G \left( X_{Q_1,\eta^*\mathcal{F},X_{Q_1}^j,\Omega_{\tilde{E}}} \right) + \sum_{j=0}^{e_{Q_2}-1} j \cdot \text{Ind}_{G_{Q_2}}^G \left( X_{Q_2,\eta^*\mathcal{F},X_{Q_2}^j,\Omega_{\tilde{E}}} \right)$$

$$- \sum_{j=1}^{e_P-1} j \cdot \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_1}^j,\Omega_{\tilde{E}}} \right) + \sum_{j=1}^{e_P-1} j \cdot \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_2}^j,\Omega_{\tilde{E}}} \right)$$

$$= \sum_{j=1}^{e_P-1} j \cdot \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_1}^j,\Omega_{\tilde{E}}} \right) + \sum_{j=1}^{e_P-1} (e_P - j) \cdot \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_2}^j,\Omega_{\tilde{E}}} \right)$$

By Lemma 1.6, we have

$$e_P \cdot \sum_{j=1}^{e_P-1} \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_1}^j,\Omega_{\tilde{E}}} \right)$$  \hspace{1cm} (A.10)

$$= e_P \cdot \left( \sum_{j=0}^{e_P-1} \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_{Q_1}^j,\Omega_{\tilde{E}}} \right) - \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F}} \right) \right)$$  \hspace{1cm} (A.11)

$$= e_P \cdot \left( \sum_{V \in \text{Irr}_r G_P} \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F},X_V} \right) - \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F}} \right) \right)$$

$$= e_P \cdot \left( X_{P,\mathcal{F},(\text{id})} \cdot X_{\mathcal{O}_G} - \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F}} \right) \right)$$

$$= e_P \cdot \left( \text{rk}(\mathcal{F}) \cdot X_{\mathcal{O}_G} - \text{Ind}_{G_P}^G \left( X_{P,\mathcal{F}} \right) \right)$$

where we used that every irreducible character of $G_P = G_{Q_1}$ is of the form $X_{Q_1,\Omega_{\tilde{E}}}$ for some $0 \leq j \leq e_P - 1$ as the $G$-action on $E$ is generically free. Thus using Equation A.9 to
Appendix A. Equivariant Riemann-Roch for Nodal Curves

Equation A.11 and the fact that $\eta$ is an isomorphism over $E^o$, we have

$$\chi_{E,\eta^*F} = \left( \frac{1}{|G|} \deg \eta^* F + \chi_{\mathcal{C},\mathcal{O}_C} \text{rk} \mathcal{F} \right) \chi_{k[G]}$$  \hspace{1cm} (A.12)

$$- \frac{1}{|G|} \sum_{Q \in \eta^{-1}(E^o)} \sum_{j=0}^{e_Q-1} j \cdot \text{Ind}_{G_Q}^{G} \left( \chi_{Q,\eta^j \mathcal{F}} \chi_{Q,\Omega_E^j} \right)$$

$$+ \frac{1}{|G|} \sum_{P \in E \setminus E^o} e_P \cdot \text{rk}(\mathcal{F}) \cdot \chi_{k[G]}$$

$$- \frac{1}{|G|} \sum_{Q \in E \setminus \eta^{-1}(E^o)} \sum_{j=0}^{e_Q-1} j \cdot \text{Ind}_{G_P}^{G} \left( \chi_{P,\mathcal{F}} \chi_{P,\Omega_E}^j \right)$$

$$= \left( \frac{1}{|G|} \deg \eta^* F + \chi_{\mathcal{C},\mathcal{O}_C} \text{rk} \mathcal{F} \right) \chi_{k[G]}$$

$$- \frac{1}{|G|} \sum_{P \in E^o} \sum_{j=0}^{e_P-1} j \cdot \text{Ind}_{G_P}^{G} \left( \chi_{P,\mathcal{F}} \chi_{P,\Omega_E}^j \right)$$

$$+ \frac{1}{|G|} \sum_{P \in E \setminus E^o} e_P \cdot \text{Ind}_{G_P}^{G} \chi_{P,\mathcal{F}}.$$  

And finally the second statement of Theorem 1.16 follows from Equation A.8, Equation A.12 and

$$\chi_{E,\mathcal{F}} = \chi_{\mathcal{H}_E^0(\mathcal{R})} = \frac{1}{|G|} \sum_{P \in E \setminus E^o} e_P \cdot \text{Ind}_{G_P}^{G} \chi_{P,\mathcal{F}}.$$  \hspace{1cm} (A.13)

To show this, note that since $\mathcal{R}$ is a torsion sheaf $\mathcal{H}_E^0(\mathcal{R}) = 0$, and

$$\mathcal{H}_E^0(\mathcal{R}) = \bigoplus_{P \in E \setminus E^o} k \otimes_{\mathcal{O}_{E,P}} \mathcal{F}_P = \bigoplus_{P \in E \setminus E^o} \mathcal{F}_P.$$  

But for $Q \in E \setminus E^o$ we have

$$\bigoplus_{P \in G \cdot Q} \mathcal{F}_P = \text{Ind}_{G_Q}^{G} \mathcal{F}_Q.$$

And since $\text{Ind}_{G_P}^{G} \mathcal{F}_P = \text{Ind}_{G_Q}^{G} \mathcal{F}_Q$ for $P \in G \cdot Q$, we have

$$\text{Ind}_{G_Q}^{G} \chi_{Q,\mathcal{F}} = \frac{e_P}{|G|} \sum_{P \in G \cdot Q} \text{Ind}_{G_P}^{G} \chi_{P,\mathcal{F}}$$

and Equation A.13 follows.  \hfill \square
Appendix B

Galois Covers

B.1 Some Conventions and Notations

Let \( X \) be a topological space. The fundamental groupoid \( \pi_1(X) \) is the groupoid whose objects are points of \( X \) and whose arrows are homotopy classes of paths whose source is the ending point and the target is the starting point. The composition of arrows in \( \pi_1(X) \) is given by the following rules. Given two paths \( \gamma \) and \( \delta \) such that the source of \( [\gamma] \) is the target of \( [\delta] \) we define \( [\gamma][\delta] \) as the homotopy class of the path \( \gamma \circ \delta(t) = \gamma(2t) \) for \( 0 \leq t \leq \frac{1}{2} \) and \( \delta(2t - 1) \) for \( \frac{1}{2} \leq t \leq 1 \). Given \( x \) and \( y \in X \), denote by \( \pi_1(X,x,y) \) the set of arrows whose target is \( x \) and whose source is \( y \), by \( \pi_1(X,x,-) \) the set of arrows whose target is \( x \), and by \( \pi_1(X,x,x) \) the group \( \pi_1(X,x,x) \). Given \( \alpha \in \pi_1(X,x,y) \), denote by \( c_\alpha : \pi_1(X,y) \to \pi_1(X,x) \) the homomorphism \( \beta \mapsto \alpha \beta \alpha^{-1} \).

Suppose \( p : \tilde{X} \to X \) is a covering. Let \( \tilde{x} \in \tilde{X} \) and let \( x = p(\tilde{x}) \). Given a path \( \gamma \) starting at \( x \) in \( X \), denote by \( \tilde{\gamma} \) the lift of \( \gamma \) to the unique path in \( \tilde{X} \) starting at \( \tilde{x} \). Given \( \alpha = [\gamma] \in \pi_1(X,x,y) \) denote by \( \tilde{x} \cdot \alpha \) the element \( \tilde{\gamma}_\tilde{x}(1) \) of \( \tilde{E} \). This defines a right action of the fundamental groupoid \( \pi_1(X) \) on \( \tilde{X} \) with anchor map \( p \). Note that morphisms of covering spaces of \( X \) are \( \pi_1(X) \)-equivariant.

B.2 Pointed Principal \( G \)- Bundles

Let \( G \) be a finite group. Let \( p : E \to C \) be a principal \( G \)-bundle. Note that \( E \) is a disjoint union of coverings of \( C \). Hence the fundamental groupoid \( \pi_1(C) \) acts on \( E \) with anchor map \( p \). Since \( G \) acts on \( E \) by isomorphisms of covering spaces we have the following.

Lemma 2.1. The left \( G \)-action and the right \( \pi_1(C) \)-action on \( E \) commute, that is for \( \tilde{x} \in E \), \( g \in G \) and \( \alpha \in \pi_1(C) \) such that the target of \( \alpha \) is \( p(\tilde{x}) \) we have

\[
g \cdot (\tilde{x} \cdot \alpha) = (g \cdot \tilde{x}) \cdot \alpha.
\]

It follows that we have a well defined action

\[
\Lambda : G \times \pi_1(C)^{opp} \times_C E \to E
\]

\[
(g, \alpha, \tilde{x}) \mapsto g \cdot \tilde{x} \cdot \alpha.
\]

It is easy to check that \( \Lambda \) is a transitive action if and only if \( C \) is path connected.

A pair \( (E \to C, \tilde{x}) \) consisting of a principal \( G \)-bundle \( E \to C \) and a point \( \tilde{x} \in E \) is called a pointed principal \( G \)-bundle. A morphism of pointed principal \( G \)-bundles is a morphism of principal \( G \)-bundles sending the marked point onto the marked point.

Lemma 2.2. Pointed principal \( G \)-bundles over path connected base space have no non-trivial automorphisms.
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Proof. Let \((p : E \to C, \bar{x})\) be a pointed principal \(G\)-bundle with \(C\) path connected. Let \(\phi\) be an automorphism. Let \(\bar{y} \in E\), and because \(\Lambda\) is transitive there exist \(g \in G\) and \(\beta \in \pi_1(C, p(\bar{x}), p(\bar{y}))\) such that \(g \cdot \bar{x} \cdot \beta = \bar{y}\). Then since \(\phi\) preserves \(\bar{x}\) and is both \(G\)-equivariant and \(\pi_1(C)\)-equivariant, we have \(\phi(\bar{y}) = g \cdot \phi(\bar{x}) \cdot \beta = \bar{y}\) which shows that \(\phi\) is trivial as \(\bar{y}\) was arbitrary.

Let \(x \in C\) and \(\bar{x} \in p^{-1}(x)\). Given \(\alpha \in \pi_1(C, x)\), define \(\theta_{(E, \bar{x})}(\alpha)\) as the unique element of \(G\) such that

\[\bar{x} \cdot \alpha = \theta_{(E, \bar{x})}(\alpha) \cdot \bar{x}.\]

It follows from general properties of group actions that \(\theta_{(E, \bar{x})}\) is a group homomorphism \(\pi_1(C, x) \to G\).

**Lemma 2.3.** Suppose \(C\) is path connected. Then \(E\) is path connected if and only if \(\theta_{(E, \bar{x})}\) is surjective.

Proof. Suppose \(\theta_{(E, \bar{x})}\) is surjective. Let \(\bar{y} \in E\) and set \(y = p(\bar{y})\). Let \(\alpha \in \pi_1(C, x, y)\). Since \(\theta_{(E, \bar{x})}\) is surjective and \(\Lambda\) is transitive, there exists \(\beta \in \pi_1(C, x)\) such that \(\theta_{(E, \bar{x})}(\beta) \cdot \bar{x} \cdot \alpha = \bar{y}\). But \(\theta_{(E, \bar{x})}(\beta) \cdot \bar{x} \cdot \alpha = \bar{x} \cdot \beta \alpha\) which implies that \(\bar{x}\) and \(\bar{y}\) are in the same path connected component. The converse is trivial.

### B.3 \(BG|_C\)

Let \(C\) be a path connected, locally path connected, and locally simply connected space \(C\). Suppose we are given a point \(x \in C\) and a homomorphism \(\theta : \pi_1(C, x) \to G\). Let \(u : \tilde{C} \to C\) be the universal covering space of \(C\), and choose \(x' \in u^{-1}(x)\). Given \(\alpha \in \pi_1(C, x)\), there is a unique isomorphism \(\phi_{x'}(\alpha) : \tilde{C} \to \tilde{C}\) such that \(\phi_{x'}(\alpha)(x') = x' \cdot \alpha\). This defines an action of \(\pi_1(C, x)\) on \(\tilde{C}\). Let \(E_{\theta}\) be the quotient space

\[E_{\theta} = G \times_{\pi_1(C, x)} \tilde{C}\]

of \(G \times \tilde{C}\) by the left action of \(\pi_1(C, x)\) given by \(\alpha \cdot (g, y') = (g\theta(\alpha^{-1}), \phi_{x'}(\alpha)(y'))\). Denote by \([g, y']\) the equivalence class of the pair \((g, y') \in G \times \tilde{C}\) in \(E_{\theta}\). Let \(E_{\theta} \to C\) be the map induced by \(u : \tilde{C} \to C\). We have a \(G\)-action on \(E_{\theta}\) induced by the \(G\)-action on the left factor of \(G \times \tilde{C}\), making \(E_{\theta} \to C\) into a \(G\)-bundle. Clearly \(E_{\theta} \to C\) is a principal \(G\)-bundle. The quotient map \(G \times \tilde{C} \to E_{\theta}\) is a morphism of covering spaces of \(C\), hence it is \(\pi_1(C)\)-equivariant. It follows that the right \(\pi_1(C)\)-action on \(E_{\theta}\) is given by

\[E_{\theta} \times_{\pi_1(C)} \pi_1(C) \to E_{\theta}\]

\([([g, y'], \alpha]) \mapsto [g, y' \cdot \alpha].\]

**Lemma 2.4.** Let \(\theta : \pi_1(C, x) \to G\) be a homomorphism. There exists a pointed principal \(G\)-bundle \(p_{\theta} : (E_{\theta}, \bar{x}_\theta) \to (C, x)\) such that \(\theta_{(E_{\theta}, \bar{x}_\theta)} = \theta\). Moreover the pair \((E_{\theta}, \bar{x}_\theta)\) with the above property is unique up to a unique isomorphism of pointed principal \(G\)-bundles.
Appendix B. Galois Covers

Proof. Let \( E_\theta \) be the bundle constructed above and let \( \bar{x}_\theta = [id, x'] \). Let \( \beta \in \pi_1(C, x) \), then we have

\[
\theta(E_\theta, \bar{x}_\theta)(\beta) \cdot \bar{x}_\theta = \bar{x}_\theta \cdot \beta = [id, x' \cdot \beta] = [id, \phi_{x'}(\beta)(x')] = [\theta(\beta), x'] = \theta(\beta) \cdot \bar{x}_\theta
\]

and it follows that \( \theta(E_\theta, \bar{x}_\theta)(\beta) = \theta(\beta) \).

Let \( p : (F, \tilde{y}) \to (C, x) \) be a pointed principal \( G \)-bundle such that \( \theta(F, \tilde{y}) = \theta \). By the property of universal coverings, there exists a unique morphism \( \Phi' : (U, x') \to (F, \tilde{y}) \) such that \( p \circ \Phi' = u \). Define \( \Phi'' : (G \times U, (id, x')) \to (F, \tilde{y}) \) by \( \Phi''(g, y') = g \cdot \Phi'(y') \). To prove that \( \Phi'' \) descends to a morphism \( \Phi : (E_\theta, \bar{x}_\theta) \to (F, \tilde{y}) \) we must show that \( \Phi'' \) is constant on the \( \pi_1(C, x) \)-orbits. Let \( \alpha \in \pi_1(C, x) \), \( (g, y') \in G \times U \) and \( \beta \in \pi_1(C, x, u(y')) \). We have

\[
\Phi''(g\theta(\alpha^{-1}), \phi_x(\alpha)(y')) = g\theta(\alpha^{-1}) \cdot \Phi'(\phi_{x'}(\alpha)(y')) = g\theta(\alpha^{-1}) \cdot \Phi'(\phi_{x'}(\alpha)(x') \cdot \beta) = g\theta(\alpha^{-1}) \cdot \Phi'(\alpha \cdot \beta) = g \cdot \tilde{y} \cdot \beta
\]

Thus we have constructed \( \Phi : (E_\theta, \bar{x}_\theta) \to (F, \tilde{y}) \). It is clearly \( G \)-equivariant, and thus an isomorphism of pointed principal \( G \)-bundles. It is unique as pointed principal \( G \)-bundles have no nontrivial automorphisms (Lemma 2.2).

Let \( G^{\pi_1(C, x)} \) denote the set \( \text{Hom}(\pi_1(C, x), G) \) of homomorphisms from \( \pi_1(C, x) \) to \( G \). The group \( G \) acts on the left of \( G^{\pi_1(C, x)} \) by conjugation on the target.

Lemma 2.5. Given two homomorphisms \( \theta_1 \) and \( \theta_2 : \pi_1(C, x) \to G \), there exists an isomorphism of principal \( G \)-bundles \( E_{\theta_1} \to E_{\theta_2} \) if and only if \( \theta_1 \) and \( \theta_2 \) lie in the same \( G \)-orbit in \( G^{\pi_1(C, x)} \). More precisely, suppose \( h \in G \) is such that \( \theta_2 = c_h \circ \theta_1 \), then there exists a unique isomorphism of principal \( G \)-bundles \( \Phi_{\theta_1, \theta_2}(h) : E_{\theta_1} \to E_{\theta_2} \) such that \( \Phi_{\theta_1, \theta_2}(h)(\bar{x}_{\theta_1}) = h^{-1} \cdot \bar{x}_{\theta_2} \).

Proof. Suppose first that \( \theta_2 = c_h \circ \theta_1 \) for some \( h \in G \). We know that for \( i = 1 \) or \( 2, E_{\theta_i} \) is the quotient of \( G \times U \) by the right action of \( \pi_1(C, x) \) given by \( (g, y') \cdot \alpha = (g\theta_i(\alpha), \phi_{x'}(\alpha^{-1})(y')) \). Denote by \( [g, y'] \), the equivalence class of \( (g, y') \) in \( E_{\theta_i} \). We define \( \Phi_{\theta_1, \theta_2}(h) : E_{\theta_1} \to E_{\theta_2} \)
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by sending the equivalence class \([g, y']_1\) to \([gh^{-1}, y']_2\). It is well defined as

\[
\Phi_{\theta_1, \theta_2}(h)((\theta_1(\alpha), \phi_{x'}(\alpha^{-1})(y'))_1) = [gh_1(\alpha)h^{-1}, \phi_{x'}(\alpha^{-1})(y')]_2 \\
= [gh^{-1}c_h(\theta_1(\alpha)), \phi_{x'}(\alpha^{-1})(y')]_2 \\
= [gh^{-1}\theta_2(\alpha), \phi_{x'}(\alpha^{-1})(y')]_2 \\
= [gh^{-1}, y']_2 \\
= \Phi_{\theta_1, \theta_2}(h)(([g, y']_1)).
\]

Clearly \(\Phi_{\theta_1, \theta_2}(h)\) is an isomorphism of principal \(G\)-bundles, sending \(\tilde{x}_{\theta_1} = [id, x']_1\) onto \([h^{-1}, x']_2 = h^{-1} \cdot \tilde{x}_{\theta_2}\). Again \(\Phi_{\theta_1, \theta_2}(h)\) is unique such isomorphism as pointed principal \(G\)-bundles have no nontrivial automorphisms.

Conversely suppose that we have an isomorphism of principal \(G\)-bundles \(\Phi : E_{\theta_1} \rightarrow E_{\theta_2}\). Then let \(h \in G\) be the unique element such that \(\Phi(\tilde{x}_{\theta_1}) = h^{-1} \cdot \tilde{x}_{\theta_2}\). Then

\[
\theta_2(\beta) \cdot \tilde{x}_{\theta_2} = \tilde{x}_{\theta_2} \cdot \beta \\
= h \cdot \Phi(\tilde{x}_{\theta_1}) \cdot \beta \\
= h \cdot \Phi(\tilde{x}_{\theta_1} \cdot \beta) \\
= h \cdot \Phi(\theta_1(\beta) \cdot \tilde{x}_{\theta_1}) \\
= h\theta_1(\beta) \cdot \Phi(\tilde{x}_{\theta_1}) \\
= c_h(\theta_1(\beta)) \cdot \tilde{x}_{\theta_2}
\]

and it follows that \(\theta_2 = c_h \circ \theta_1\).

Let \(BG|_C\) be the groupoid whose objects are principal \(G\)-bundles over \(C\) and whose arrows are isomorphisms of principal \(G\)-bundles. We summarize this section in the following.

**Proposition 2.6.** Let \(C\) be path connected, locally path connected and locally simply connected. Choose \(x \in C\). Let \(G\) act on \(G^{x_1(C)}\) by conjugation on the target. Let \(G(G^{x_1(C)})\) denote the transformation groupoid of this action (that is the groupoid whose objects are elements of \(G^{x_1(C)}\) and whose arrows are pairs \((g, \theta)\) where \(g \in G\) and \(\theta \in G^{x_1(C)}\) with source \(\theta\) and target \(c_g \circ \theta\)). Then we have an equivalence of groupoids

\[
\Xi : G(G^{x_1(C)}(C)) \rightarrow BG|_C
\]

that sends a homomorphism \(\theta : \pi_1(C, x) \rightarrow G\) onto the principal \(G\)-bundle \(E_{\theta}\), and sends an arrow \(g : \theta_1 \rightarrow \theta_2\) onto the isomorphism \(\Phi_{\theta_1, \theta_2}(g) : E_{\theta_1} \rightarrow E_{\theta_2}\).

Proof. We first check that \(\Xi\) is a well defined functor. Let \(\theta_i \in G^{x_1(C)}\) for \(i = 1, 2, 3\) and \(g_i \in G\) for \(i = 1, 2\) such that \(\theta_2 = c_{g_1} \circ \theta_1\) and \(\theta_3 = c_{g_2} \circ \theta_2\). Then by Lemma 2.5, \(\Xi(g_2 : \theta_2 \rightarrow \theta_3) \circ \Xi(g_1 : \theta_1 \rightarrow \theta_2)\) is the unique isomorphism \(E_{\theta_1} \rightarrow E_{\theta_3}\) sending \(\tilde{x}_{\theta_1}\) onto \(g_1^{-1} g_2^{-1} \cdot \tilde{x}_{\theta_3}\), that is \(\Xi(g_2 g_1 : \theta_1 \rightarrow \theta_3)\). Thus \(\Xi\) is well defined.

We now show that \(\Xi\) is fully faithful. Let \(\theta_1\) and \(\theta_2 \in G^{x_1(C)}\). We want to show that

\[
\Xi : \text{Hom}_{G(G^{x_1(C)}(C))}(\theta_1, \theta_2) \rightarrow \text{Hom}_{BG|_C}(E_{\theta_1}, E_{\theta_2})
\]

\[g \mapsto \Phi_{\theta_1, \theta_2}(g)\]
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is a bijection. It is injective as $g$ is the unique element of $G$ such that $\Phi_{g_1,g_2}(g)(\tilde{x}_{g_1}) = g^{-1}\tilde{x}_{g_2}$. It is surjective by Lemma 2.5.

To show that $\Xi$ is essentially surjective, let $p : E \to C$ be a principal $G$-bundle, choose $\tilde{x} \in p^{-1}x$, and note that by the Lemma 2.4, $E_{\theta(E,x)}$ is isomorphic to $E$.

**Corollary 2.7.** If $\text{Aut}_G^G(E_\theta)$ denotes the $G$-equivariant automorphisms of $E_\theta$ over $C$, then there is an isomorphism $\Psi_\theta : \text{Aut}_E^G(E_\theta) \to Z_G(\theta(\pi_1(C,x)))$, such that for $\phi \in \text{Aut}_E^G(E_\theta)$ we have

$$\phi(\tilde{x}_\theta) = \Psi_\theta(\phi)^{-1} \cdot \tilde{x}_\theta.$$

**Proof.** Since $\Xi$ is a fully faithfull it induces an isomorphisms of groups

$$\text{Hom}_G(G^1(C,x)) \to \text{Hom}_{B\Gamma_\theta}(E_\theta).$$

The left side is just the centralizer $Z_G(\theta(\pi_1(C,x)))$ and the right side is the automorphism group $\text{Aut}_E^G(E_\theta)$. Let $\phi \in \text{Aut}_E^G(E_\theta)$, and let $g \in G$ be the element mapping onto $\phi$. Then by Lemma 2.5 $\phi(\tilde{x}_\theta) = g^{-1}\tilde{x}_\theta$. 

**B.4 Principal $G$-Bundles Over Curves**

In what follows we assume that “Riemann surfaces” are not necessarily connected. If $C$ is a Riemann surface, then for any (topological) covering $p : E \to C$, the space $E$ has a unique structure of a Riemann surface induced by $p$ from $C$. Thus the results from the last section hold in the category of Riemann surfaces.

Let $C$ be a connected Riemann surface. Let $x_0, \ldots, x_n \in C$ be distinct points and let $C^o = C \setminus \{x_1, \ldots, x_n\}$. Given a finite covering $p : E \to C^o$, there exists a unique branched covering $\tilde{E} \to C$ and a map $E \to \tilde{E}$ such that

$$
\begin{array}{ccc}
E & \longrightarrow & \tilde{E} \\
\downarrow & & \downarrow \\
C^o & \longrightarrow & C
\end{array}
$$

is a pull back diagram. If $E \to C^o$ is a $G$-bundle then $\tilde{E} \to C$ has a unique structure of a $G$-bundle such that the above diagram is a pull back diagram of $G$-bundles (Forster 8.4 and 8.5).

For each $1 \leq i \leq n$ choose a path $d_i$ in $C^o \cup \{x_i\}$ from $x_0$ to $x_i$, such that $d_i$ passes through $x_i$ only once. Given $\tilde{x}_0 \in p^{-1}(x_0)$ we define $\tilde{x}_0 \cdot d_i \in \tilde{E}$ as $\lim_{c \to 1} \tilde{x}_0 \cdot d_i^c$ where $d_i^c$ is the path in $C^o$ given by $t \mapsto d_i(ct)$.

**Lemma 2.8.** For each $1 \leq i \leq n$, there exists an $\alpha_i \in \pi_1(C^o,x_0)$ such that for every finite group $G$ and every homomorphism $\theta : \pi_1(C^o,x_0) \to G$, the element $\theta(\alpha_i)$ fixes $\tilde{x}_0 \cdot d_i \in E_\theta$ and acts on $T_{\tilde{x}_0 \cdot d_i} \tilde{E}_\theta$ by multiplication by $\exp(2\pi \sqrt{-1}/k_i)$ for $k_i = |\text{Stab}_G(\tilde{x}_0 \cdot d_i)|$.

**Proof.** Fix $1 \leq i \leq n$. Let $\phi : D^2 \to C^o \cup \{x_i\}$ be an embedding of the closed disk $D^2 \subseteq \mathbb{C}$ of radius 1, with image $D$, sending $0$ to $x_i$ such that $x_0 \notin \phi(D^2)$. Let $c \in [0,1]$ be the
unique number such that \( d_i(c) \in \partial D \) and such that for all \( c < t \leq 1 \) we have \( d_i(t) \in D' \) where \( D' \) is the interior of \( D \). Let \( \delta \) be the loop:

\[
\delta : [0, 1] \to \partial D \\
\quad t \mapsto \phi(\exp(2\pi \sqrt{-1} t) \cdot \phi^{-1}(d_i(c))).
\]

Define \( \alpha_i \in \pi_1(C^\circ, x_0) \) as the homotopy class of the path \( d_i^* \circ \delta \circ (d_i^*)^{-1} \).

I claim that \( \alpha_i \) depends only on \( d_i \). Let \( \phi' : D^2 \to C^\circ \cup \{ x_i \} \) be another embedding of the closed disk \( D^2 \subseteq \mathbb{C} \) of radius 1, with image \( D' \). Let \( c' \in [0, 1] \) and \( \delta' : [0, 1] \to \partial D' \) be defined as above with \( D \) replaced by \( D' \). Assume that \( D' \subseteq D \). Let \( d_i' : [0, 1] \to D \setminus \{ x_i \} \) be the path given by \( t \mapsto d_i(t(c' - c) + c) \) (the path along \( d_i \) from \( d_i(c) \) to \( d_i(c') \)). Then it is easy to construct a homotopy between \( d_i^* \circ \delta' \circ (d_i^*)^{-1} \) and \( \delta \) in \( D \setminus \{ x_i \} \), thus

\[
[d_i^* \circ \delta' \circ (d_i^*)^{-1}] = [\delta]
\]

and it follows that

\[
[d_i^* \circ \delta \circ (d_i^*)^{-1}] = [d_i^* \circ d_i^* \circ \delta' \circ (d_i^*)^{-1} \circ (d_i^*)^{-1}]
\]

\[
= [d_i^* \circ \delta' \circ (d_i^*)^{-1}]
\]

which shows that \( \alpha_i \) doesn’t depend on the choice of \( D' \subseteq D \).

Let \( G \) be a finite group and \( \theta : \pi_1(C^\circ, x_0) \to G \) a homomorphism. Choose \( \phi : D^2 \to C^\circ \cup \{ x_i \}, D, c \) and \( \delta \) as above. Then \( \tilde{E}_\theta|_D \) is a disjoint union of disks and there exists an embedding \( \tilde{\phi} : D^2 \to \tilde{E}_\theta|_D \) whose image is the component containing \( \tilde{x}_0 \cdot d_i \) and such that

\[
D^2 \xrightarrow{\tilde{\phi}} \tilde{E}_\theta|_D \\
\downarrow \phi \\
D \xrightarrow{p_k} \tilde{E}_\theta|_D
\]

commutes, where \( p_k \) is the map \( z \mapsto z^k \) for \( k = |\text{Stab}_G(\tilde{x}_0)| \) (by Forster 5.10). Hence the element \( [\delta] \in \pi_1(D \setminus \{ x_i \}, d_i(x)) \) fixes \( \tilde{x}_0 \cdot d_i \) and acts on \( T_{\tilde{x}_0 \cdot d_i} \tilde{E}_\theta \) by multiplication by \( \exp(2\pi \sqrt{-1} / k) \), and so does \( \alpha_i \).

For each \( 1 \leq i \leq n \) choose \( \alpha_i \in \pi_1(C^\circ, x_0) \) as in the Lemma. Let \( \prod_i \alpha_i \geq \alpha_1 \times \cdots \times \alpha_n > \) act on the right of \( G^n \times G^{\pi_1(C^\circ, x_0)} \) by

\[
G^n \times G^{\pi_1(C^\circ, x_0)} \times \prod_i \alpha_i \geq \to G^n \times G^{\pi_1(C^\circ, x_0)} \\
(g, \theta, \gamma_i) \mapsto (g \theta(\gamma_i), \theta).
\]

Let \( G \) act on the left of \( G^n \times G^{\pi_1(C^\circ, x_0)} \) by

\[
G \times G^n \times G^{\pi_1(C^\circ, x_0)} \to G^n \times G^{\pi_1(C^\circ, x_0)} \\
(g, g_i, \theta) \mapsto (g_i g^{-1}, c_g \circ \theta).
\]

Note that the two actions commute.
Proposition 2.9. Let $\mathcal{M}^G_{(x_1,\ldots,x_n)}$ be the category whose objects are tuples $(p : E \rightarrow C, x_1,\ldots,x_n)$ where $p : E \rightarrow C$ is a $G$-bundle that restricts to a principal $G$-bundle on $C^o$ and $x_i \in p^{-1}(x_i)$ are marked points, and the arrows are isomorphisms of $G$-bundles preserving the marked points. Denote by $N^G_{(x_1,\ldots,x_n)}$ the transformation groupoid of the $G$-action on the quotient $(G^n \times G^*(G^o,x_0))/\langle \prod_i < \alpha_i > \rangle$. Then we have an equivalence of categories

$$\Theta^G : N^G_{(x_1,\ldots,x_n)} \rightarrow \mathcal{M}^G_{(x_1,\ldots,x_n)}$$

sending an object $[g, \theta] \in (G^n \times G^*(G^o,x_0))/\langle \prod_i < \alpha_i > \rangle$ onto $(E_\theta \rightarrow C, g_1 \cdot \tilde{x}_\theta \cdot d_i)$ and an arrow $g \in G$ from $[g, \theta]$ to $[h_i, \rho]$ onto the unique morphism of $G$-bundles $\Phi_{\theta,\rho}(g) : E_\theta \rightarrow E_\rho$ extending $\Phi_{\theta,\rho}(g)$.

Proof. We start by showing that $\Theta^G$ is well defined. Let $(k_i) \in Z^n$. Then $g_1(\alpha_i^k) \cdot \tilde{x}_\theta \cdot d_i = g_1 \cdot \tilde{x}_\theta \cdot d_i$ by Lemma 2.8. So $\Theta^G$ is well defined on the objects. Given $g \in G$, we need to check that $\Phi_{\theta,\rho}(g)$ preserves the marked points. Note that $h_i = g_1g^{-1}$ and $\rho = c_2 \circ \theta$. Thus we have

$$\Phi_{\theta,\rho}(g)(g_1 \cdot \tilde{x}_\theta \cdot d_i) = g_1 \cdot \Phi_{\theta,\rho}(g)(\tilde{x}_\theta \cdot d_i) = g_1g^{-1} \cdot \tilde{x}_\theta \cdot d_i = h_i \cdot \tilde{x}_\theta \cdot d_i.$$

We now show the essential surjectivity. Let $(p : E \rightarrow C, \tilde{x}_i)$ be an object of $\mathcal{M}^G_{(x_1,\ldots,x_n)}$, and choose $\tilde{x}_0 \in p^{-1}(x_0)$. Let $E^o$ be the restriction of $E$ to $C^o$. Let $\theta = \theta(E^o, \tilde{x}_0)$ and choose $g_1 \in G$ such that $g_1 \cdot \tilde{x}_0 \cdot d_i = \tilde{x}_i$. Let us check that $\Theta^G[g_i, \theta] = (E_\theta \rightarrow C, g_1 \cdot \tilde{x}_0 \cdot d_i)$ is isomorphic to $(E \rightarrow C, \tilde{x}_i)$ as pointed $G$-bundles. By Lemma 2.4, we have an isomorphism $\Phi : E_\theta \rightarrow E^o$ sending $\tilde{x}_\theta$ onto $\tilde{x}_0$, inducing an isomorphism $\bar{\Phi} : E_\theta \rightarrow E$. We need to check that $\bar{\Phi}$ preserves the marked points, which follows from the following computation

$$\bar{\Phi}(g_1 \cdot \tilde{x}_\theta \cdot d_i) = g_1 \cdot \bar{\Phi}(\tilde{x}_\theta \cdot d_i) = g_1 \cdot \tilde{x}_0 \cdot d_i = \tilde{x}_i.$$

We show that $\Theta^G$ is fully faithfull as follows. Let $\theta$ and $\rho : \pi_1(C^o, x_0) \rightarrow G$ be two homomorphisms, and let $g_1, h_i \in G$. Given $\Phi : (E_\theta, g_1 \cdot \tilde{x}_\theta \cdot d_i) ightarrow (E_\rho, h_i \cdot \tilde{x}_\rho \cdot d_i)$, there is a unique $g \in G$ such that $\Phi_{(\theta,\rho)}(g)$ is the restriction of $\Phi$ to $E_\theta$, thus $\Phi_{(\theta,\rho)}(g) = \bar{\Phi}$. We need to check that $g$ is indeed an arrow from $[g_1, \theta]$ to $[h_i, \rho]$. By Lemma 2.5, we know that $\rho = c_2 \circ \theta$. We need to check that $g_1g^{-1} = h_i\rho(\alpha_i^k)$ for some $k \in \mathbb{Z}$. We have

$$h_i \cdot \tilde{x}_\rho \cdot d_i = \Phi(g_1 \cdot \tilde{x}_\theta \cdot d_i) = g_1 \cdot \Phi_{(\theta,\rho)}(g)(\tilde{x}_\theta \cdot d_i) = g_1g^{-1} \cdot \tilde{x}_\rho \cdot d_i$$

that is $h_i^{-1}g_1g^{-1} \in \text{Stab}_G(\tilde{x}_\rho \cdot d_i)$. By Lemma 2.8, we know that Stab$_G(\tilde{x}_\rho \cdot d_i)$ is generated by $\rho(\alpha_i)$, and the result follows.
Corollary 2.10. Let $\text{Aut}_G^G(\tilde{E}_\theta, g_1 \cdot \tilde{x}_\theta \cdot d_1, \ldots, g_n \cdot \tilde{x}_\theta \cdot d_n)$ denote the $G$-equivariant automorphisms of $\tilde{E}_\theta$ over $C$ preserving the marked points, which is clearly a subgroup of $\text{Aut}_G^G(\tilde{E}_\theta)$. Then the isomorphism $\Psi_\theta : \text{Aut}_G^G(\tilde{E}_\theta) \to \mathbb{Z}_G(\theta(\pi_1(C^o, x_0)))$ restricts to an isomorphism

$$\text{Aut}_G^G(\tilde{E}_\theta, g_1 \cdot \tilde{x}_\theta \cdot d_1, \ldots, g_n \cdot \tilde{x}_\theta \cdot d_n) \to \mathbb{Z}_G(\theta(\pi_1(C^o, x_0))) \cap <\theta(\alpha_1) > \cap \ldots \cap <\theta(\alpha_n) >$$

where $<\theta(\alpha_i)>$ denotes the subgroup of $G$ generated by $\theta(\alpha_i)$.

Proof. An element $g \in G$ gives an arrow from $[g_1, \theta]$ to itself if and only if $c_\theta \circ \theta = \theta$ that is $g \in \mathbb{Z}_G(\theta(\pi_1(C^o, x_0)))$, and $g_1 g^{-1} = g_i$ modulo $\mathbb{Z}$ that is $g \in <\theta(\alpha_i)>$. \qed

### B.5 Inflation

Suppose that $H$ is a subgroup of $G$ or more generally that there is a homomorphism $H \to G$. There is a functor $G \times_H -$ from the category of $H$-spaces to the category of $G$-spaces which takes an $H$-space $X$ and sends it to the quotient $G \times_H X$ of the product $G \times X$ by the action $(g, x) \cdot h \mapsto (g \cdot h, h^{-1}x)$. The space $G \times_H X$ has a natural $G$-action induced by the multiplication on $G$. This construction is functorial. Note that there is a natural $H$-morphism $G \times_H : X \to G \times_H X$ induced by the $H$-morphism $H \times X \to G \times X$ and the identification $X \simeq H \times_H X$.

Denote by $\iota$ the inclusion $H \to G$. For simplicity write $\mathcal{M}^H := \mathcal{M}^H_{(C, x_1, \ldots, x_n)}$ and $\mathcal{M}^G := \mathcal{M}^G_{(C, x_1, \ldots, x_n)}$. We will construct a functor $\mathcal{M}^\iota : \mathcal{M}^H \to \mathcal{M}^G$ called the inflation. Given an object $\epsilon := (p : E \to C, \tilde{x}_1, \ldots, \tilde{x}_n)$ of $\mathcal{M}^H_{(C, x_1, \ldots, x_n)}$, let $\mathcal{M}^\iota(\epsilon) := (q : F \to C, \tilde{y}_1, \ldots, \tilde{y}_n)$ be the object of $\mathcal{M}^G_{(C, x_1, \ldots, x_n)}$ constructed as follows: let

$$F := G \times_H E$$

and for each $i$, let

$$\tilde{y}_i := G \times_H (\tilde{x}_i).$$

We use the convention that two functors of groupoids are said to commute if they are naturally isomorphic.

**Proposition 2.11.** Suppose that $\iota : H \to G$ is an injective homomorphism of groups. Then we have a commutative diagram of groupoids

$$\begin{array}{ccc}
\mathcal{N}^H & \xrightarrow{\Omega^H} & \mathcal{M}^H \\
\downarrow N^\iota & & \downarrow M^\iota \\
\mathcal{N}^G & \xrightarrow{\Omega^G} & \mathcal{M}^G
\end{array}$$

where $N^\iota$ is the functor $\mathcal{N}^H \to \mathcal{N}^G$ induced by the inclusion

$$H^n \times H^\pi_1(C^o, x_0) \to G^n \times G^\pi_1(C^o, x_0),$$
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Proof. We have to construct a natural isomorphism \( \tau : \Theta^G \circ \mathcal{N} = \mathcal{M} \circ \Theta^H \). Let \([h_i, \theta] \in (H^n \times H^{\pi_1(C^0, x_0)})/(\prod_i < \alpha_i >)\), and let \(\theta' = \iota \circ \theta\). Taking the quotient of inclusion \(H \times C^0 \to G \times C^0\) by the action of \(\pi_1(C^0, x_0)\), we get \(H\)-equivariant morphism \(\tilde{E}_{\theta} \to \tilde{E}_{\theta'}\), which induces a \(G\)-equivariant isomorphism \(\tau(\theta) : G \times_H \tilde{E}_{\theta} \to \tilde{E}_{\theta'}\) that sends \(G \times_H (\tilde{x}_\theta)\) onto \(\tilde{x}_{\theta'}\), and thus preserves the markings. This is the required natural isomorphism. \(\square\)

Corollary 2.12. The functor \(\mathcal{M} : \mathcal{M}^H \to \mathcal{M}^G\) is faithful and if \(n \geq 1\) it is fully faithful.

Proof. By Corollary 2.10 and Proposition 2.11, the inflation \(\mathcal{M}\) induces a homomorphism of groups

\[
\text{Aut}^H_G(E_{\theta}, h_i \cdot \tilde{x}_\theta \cdot d_i) \longrightarrow \text{Aut}^G_G(E_{\iota \circ \theta}, h_i \cdot \tilde{x}_{\iota \circ \theta} \cdot d_i)
\]

where the bottom arrow is an inclusion and hence \(\mathcal{M}\) is faithful. It is also an isomorphism if \(n \geq 1\), and to the full faithfulness we only need to show that two objects in \(\mathcal{M}^H\) are isomorphic if and only if their images in \(\mathcal{M}^G\) are isomorphic. By Proposition 2.11 it suffices to show this property for \(\mathcal{N}^H\) and \(\mathcal{N}^G\). Let \([h_i, \theta] \in (H^n \times H^{\pi_1(C^0, x_0)})/(\prod_i < \alpha_i >)\), and suppose that for \(g \in G\), we have \([h_i \cdot g^{-1}, c_g \circ \theta] \in (H^n \times H^{\pi_1(C^0, x_0)})/(\prod_i < \alpha_i >)\). Then since \(n \geq 1\), we have \(c_g \circ \theta(\alpha_i) \in H\), which implies that \(h_i \cdot g^{-1} \in H\) and it follows that \(g \in H\). \(\square\)

B.6 Monodromies

Consider the map

\[
\hat{e}^G_{(C, x_1, \ldots, x_n)} : (G^n \times G^{\pi_1(C^0, x_0)})/(\prod_i < \alpha_i >) \to G^n
\]

\[
[g_i, \theta] \mapsto (g_i \theta(\alpha_i) g_i^{-1})
\]

It is invariant under the left \(G\) action, and thus induces a functor

\[
\hat{e}^G_{(C, x_1, \ldots, x_n)}: \mathcal{N}^G_{(C, x_1, \ldots, x_n)} \to G^n
\]

where \(G\) is considered as a category associated to the set \(G\). Given an \(n\)-tuple \((\nu_1, \ldots, \nu_n) \in G^n\), define \(\mathcal{N}^G_{(C, x_1, \ldots, x_n)}(\nu_1, \ldots, \nu_n)\) as the full subcategory of \(\mathcal{N}^G_{(C, x_1, \ldots, x_n)}\) with objects in the fiber of \(\hat{e}^G_{(C, x_1, \ldots, x_n)}\) above \((\nu_1, \ldots, \nu_n)\). Note that \(\mathcal{N}^G_{(C, x_1, \ldots, x_n)}(\nu_1, \ldots, \nu_n)\) is the transformation groupoid of the \(G\)-action on the set \((\hat{e}^G_{(C, x_1, \ldots, x_n)})^{-1}(\nu_1, \ldots, \nu_n)\).

Definition 2.13. Given a \(G\)-bundle \(p : E \to C\) of Riemann surfaces such that it is a principal \(G\)-bundle away from some \(x \in C\). Given \(\bar{x} \in p^{-1}(x)\) we define the monodromy of \(E \to C\) at \(\bar{x}\) as the unique element \(\nu_\bar{x}\) of \(G\) that fixes \(\bar{x}\) and acts on the tangent space \(T_{\bar{x}}E\) by multiplication by \(\exp(2\pi \sqrt{-1}/k)\) where \(k = |\text{Stab}_G(\bar{x})|\).
Proposition 2.14. Given an n-tuple \((\nu_1, \ldots, \nu_n) \in G^n\), the equivalence of categories in Proposition 2.9 restricts to an equivalence of categories

\[ \mathcal{N}_{(C, x_1, \ldots, x_n)}(\nu_1, \ldots, \nu_n) \rightarrow \mathcal{M}_{(C, x_1, \ldots, x_n)}^{G}(\nu_1, \ldots, \nu_n) \]

where \(\mathcal{M}_{(C, x_1, \ldots, x_n)}^{G}(\nu_1, \ldots, \nu_n)\) is the full subcategory of \(\mathcal{M}_{(C, x_1, \ldots, x_n)}^{G}\) whose objects are the marked \(G\)-bundles \((E \to C, x_1, \ldots, x_n)\) whose monodromy at \(x_i\) is \(\nu_i\).

Proof. Given \([g_i, \theta] \in (\mathcal{M}_{(C, x_1, \ldots, x_n)}^{G})^{-1}(\nu_1, \ldots, \nu_n)\), it follows from Lemma 2.8 that the monodromy of \(\tilde{x}_0 \cdot d_i\) is \(\theta(\alpha_i)\). Hence the monodromy of \(g_i \cdot \tilde{x}_0 \cdot d_i\) is \(g_i \theta(\alpha_i) g_i^{-1}\). The result then follows. \(\square\)
Bibliography


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